The derivative of $f(x) = a^x$

We use the definition of derivatives here

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \to 0} \frac{a^x \left(a^h - 1\right)}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h} \\ &= a^x \cdot \lim_{h \to 0} \frac{e^{h \ln a} - 1}{h} \end{aligned}$$

Then, we apply L'Hôpital's rule to the second part

$$\begin{aligned} \lim_{h \to 0} \frac{e^{h \ln a} - 1}{h} &= \lim_{h \to 0} \frac{\left(e^{h \ln a} - 1\right)'}{h'} \\ &= \lim_{h \to 0} \frac{\ln a \cdot e^{h \ln a}}{1} \\ &= \lim_{h \to 0} \left(\ln a \cdot a^h\right) \\ &= \ln a \end{aligned}$$

So

$$(a^x)' = a^x \cdot \ln a$$

The derivative of $f(x) = \log_a x$

Since this function is the inverse function of $g(x) = a^x$, that is, $f^{-1}(x) = a^x$

$$\begin{aligned} f'(x) &= \frac{1}{(f^{-1})'(f(x))} \\ &= \frac{1}{(f^{-1})'(\log_a x)} \\ &= \frac{1}{a^{\log_a x} \cdot \ln a} \\ &= \frac{1}{x \cdot \ln a} \end{aligned}$$

So

$$(\log_a x)' = \frac{1}{x \cdot \ln a}$$