

## The derivative of $f(x) = a^x$

We use the definition of derivatives here

$$\begin{aligned}f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\&= \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\&= \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} \\&= a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h} \\&= a^x \cdot \lim_{h \rightarrow 0} \frac{e^{h \ln a} - 1}{h}\end{aligned}$$

Then, we apply L'Hôpital's rule to the second part

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{e^{h \ln a} - 1}{h} &= \lim_{h \rightarrow 0} \frac{(e^{h \ln a} - 1)'}{h'} \\&= \lim_{h \rightarrow 0} \frac{\ln a \cdot e^{h \ln a}}{1} \\&= \lim_{h \rightarrow 0} (\ln a \cdot a^h) \\&= \ln a\end{aligned}$$

So

$$(a^x)' = a^x \cdot \ln a$$

## The derivative of $f(x) = \log_a x$

Since this function is the inverse function of  $g(x) = a^x$ , that is,  $f^{-1}(x) = a^x$

$$\begin{aligned}f'(x) &= \frac{1}{(f^{-1})'(f(x))} \\&= \frac{1}{(f^{-1})'(\log_a x)} \\&= \frac{1}{a^{\log_a x} \cdot \ln a} \\&= \frac{1}{x \cdot \ln a}\end{aligned}$$

So

$$(\log_a x)' = \frac{1}{x \cdot \ln a}$$