Q.3 Show that if the $n \times n$ matrix A is tridiagonal then a solution to the system of linear equation $Ax = b$ can be obtained in $O(n)$ step.

→ Consider the tridiagonal matrix as

$$A = \begin{bmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & \ddots & \ddots & \ddots \\ & 0 & & a & & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

Then the equation $Ax = b$, becomes.

$$\begin{bmatrix} b_1 & c_1 & & 0 \\ a_2 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ 0 & & & c_{n-1} \\ & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

→ Now Doing the 1st steps of Gaussian elimination we obtain

$$\begin{bmatrix} 1 & \frac{c_1}{b_1} & \cdots & 0 \\ a_2 & b_2 & c_2 & \\ & \ddots & \ddots & \ddots \\ 0 & & & c_{n-1} \\ & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d_1}{b_1} \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{c_1}{b_1} & & \\ \cancel{0} & 1 & \frac{c_2}{b_2 - a_2 \cdot \frac{c_1}{b_1}} & \\ & & & \\ & & & c_{n-1} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{c_1}{b_1} & & & 0 \\ 0 & 1 & \frac{c_2}{b_2 - a_2 \cdot \frac{c_1}{b_1}} & & \\ & \ddots & \ddots & \ddots & \\ & 0 & & & c_{n-1} \\ & & & a_n & b_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \frac{d_1}{b_1} \\ \frac{d_2 - a_2 \cdot \frac{d_1}{b_1}}{b_2 - a_2 \cdot \frac{c_1}{b_1}} \\ \vdots \\ d_n \end{bmatrix}$$

After doing All steps of Gaussian elimination We obtain

$$\begin{bmatrix} 1 & c_1' & & 0 & \\ 0 & 1 & c_2' & & \\ 0 & 0 & 1 & c_3' & \\ & & \ddots & \ddots & c_{n-1}' \\ & 0 & & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} d_1' \\ d_2' \\ \vdots \\ d_n' \end{bmatrix}$$

Where

$$c_i' = \frac{c_i}{b_i} \quad \text{for } i = 1$$

$$= \frac{c_i}{b_i - a_i \cdot c_{i-1}'} \quad \text{for } i = 2, \ldots n-1$$

and

$$d_i' = \frac{d_i}{b_i} \quad \text{for } i = 1$$

$$= \frac{d_i - a_i d_{i-1}'}{b_i - a_i \cdot c_{i-1}'} \quad \text{for } i = 2, 3, \ldots n$$

Then in back-substitution we obtain the solution as -

$$x_n = d_n'$$

$$x_i = d_i' - c_i' \cdot x_{i+1} \quad \text{for } i = n-1, n-2, \ldots 1$$

Complexcity

① Gaussian elimination-

a) Multiplication/ division :-

$\cancel{1}\ \cancel{1} + n - \cancel{1} + \cancel{1} + n + n - \cancel{1} + 2n$

$= 5n$

b) Addition / subtraction :-

$\cancel{1}\ n - 1 + 2n$

$= (3n-1)$.

② back subsitution:-

a) Multipilication/ division :-

$(n-1)$

b) addition / subtraction :- $(n-1)$

So, Total complexcity $= 5n + 3n - 1 + 2(n-1)$

$= 5n + 3n - 1 + 2n - 2$

$= (10n - 3)$

$\sim O(n)$

So, for a tridiagonal Matrix A, the solution ~~of~~ to the system of linear equation is ~~order~~ $O(n)$.

This particulari ~~apt~~ algorithm is called "Thomas alogorithm"