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# 1 Review of Propositional Logic

**Task:** Recall enough propositional logic to see how it matches up with set theory.

**Definition:** A proposition is any declarative sentence that is either true or false.

## 1.1 Connectives

	<u>Connectives</u>	<u>Notation in Maths</u>
and	$\wedge$	
or	$\vee$	"Inclusive or"
not	$\neg$	Sometimes denoted $\sim$
implies	$\rightarrow$	if/then; called implication $\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence $\Leftrightarrow$

### 1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	$P \wedge Q$	P	Q	$P \vee Q$	P	$\neg P$	P	Q	$P \rightarrow Q$	P	Q	$P \leftrightarrow Q$
F	F	F	F	F	F	F	T	F	F	T	F	F	T
F	T	F	F	T	T	F	T	F	T	T	F	T	F
T	F	F	T	F	T	T	F	T	F	F	T	F	F
T	T	T	T	T	T	T	T	T	T	T	T	T	T

### Priority of the Connectives

**Highest to Lowest:**  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

## 1.2 Important Tautologies

$$\begin{array}{ll}
 (P \rightarrow Q) & \leftrightarrow (\neg P \vee Q) \\
 (P \leftrightarrow Q) & \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\
 \neg(P \wedge Q) & \leftrightarrow (\neg P \vee \neg Q) \\
 \neg(P \vee Q) & \leftrightarrow (\neg P \wedge \neg Q)
 \end{array}
 \left. \vphantom{\begin{array}{l} (P \rightarrow Q) \\ (P \leftrightarrow Q) \\ \neg(P \wedge Q) \\ \neg(P \vee Q) \end{array}} \right\} \text{De Morgan Laws}$$

As a result,  $\neg$  and  $\vee$  together can be used to represent all of  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ .

**Less obvious:** One connective called the sheffer stroke  $P|Q$  (which stands for "not both P and Q" or "P nand Q") can be used to represent all of  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$  since  $\neg P \leftrightarrow P|P$  and  $P \vee Q \leftrightarrow (P|P) | (Q|Q)$ .

**Recall** if  $P \rightarrow Q$  is a given implication,  $Q \rightarrow P$  is called the converse or  $P \rightarrow Q$ .  
 $\neg Q \rightarrow \neg P$ .

### 1.3 Indirect Arguments/Proofs by Contradiction/Reductio as absurdum

Based on the tautology  $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$

**Example:** Famous argument that  $\sqrt{2}$  is irrational.

**Proof:**

**Suppose**  $\sqrt{2}$  is rational, then it can be expressed as fraction form  $\frac{a}{b}$ . Let us **assume** that our fraction is in the lowest term, **i.e.** their only common divisor is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by  $b^2$  yields

$$2b^2 = a^2$$

Since  $a^2 = 2b^2$ , we can conclude that  $a^2$  is even because whatever the value of  $b^2$  has to be multiplied by 2. If  $a^2$  is even, then  $a$  is also even. Since  $a$  is even, no matter what the value of  $a$  is, we can always find an integer that if we divide  $a$  by 2, it is equal to that integer. If we let that integer be  $k$ , then  $\frac{a}{2} = k$  which means that  $a = 2k$ .

Substituting the value of  $2k$  to  $a$ , we have  $2b^2 = (2k)^2$  which means that  $2b^2 = 4k^2$ . dividing both sides by 2 we have  $b^2 = 2k^2$ . That means that the value  $b^2$  is even, since whatever the value of  $k$  you have to multiply it by 2. Again, if  $b^2$  is even, then  $b$  is even.

This implies that both  $a$  and  $b$  are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that  $\frac{a}{b}$  has no common divisor except 1. Since we found a contradiction, our assumption is, therefore, false. Hence the theorem is true.

qed

## 2 Predicate logic and Quantifiers

**Task:** Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variable  $x, y, z$ , so their truth value may change depending on which values these variables assume:  
 $P(x), Q(x, y), R(x, y, z)$

## 2.1 Introduce quantifiers

### 2.1.1 $\exists$ existential quantifier

**Syntax:**  $\exists xP(x)$

**Definition:**  $\exists xP(x)$  is true if  $P(x)$  is true for some value of  $x$ ; it is false otherwise.

### 2.1.2 $\forall$ universal quantifier

**Syntax:**  $\forall xP(x)$

**Definition:**  $\forall xP(x)$  is true if  $P(x)$  is true for all allowable values of  $x$ . It is false otherwise.

### 2.1.3 $\exists!$ for one and only one

**Syntax:**  $\exists!xP(x)$

**Definition:**  $\exists!xP(x)$  is true if  $P(x)$  is true for exactly one value of  $x$  and false for all other values of  $x$ ; otherwise,  $\exists!xP(x)$  is false.

## 2.2 Alternation of Quantifiers

$$\forall x\exists y\forall z \quad P(x, y, z)$$

**NB:** The order cannot be exchanged as it might modify the truth values of the statement (think of examples with two quantifiers).

## 2.3 Negation of Quantifiers

$$\begin{aligned}\neg(\exists xP(x)) &\leftrightarrow \forall x\neg P(x) \\ \neg(\forall xP(x)) &\leftrightarrow \exists x\neg P(x)\end{aligned}$$

## 3 Set Theory

**Task:** Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc. Set theory started around 1870's  $\rightarrow$  late development in mathematics but now taught early in one's maths education due to Bourbaki school.

**Definition:** A set is a collection of objects.  $x \in A$  means the element  $x$  is in the set  $A$  (**i.e.** belongs to  $A$ ).

**Examples:**

1. All students in a class.
2.  $\mathbb{N}$  the set of natural numbers starting at 0.

$\mathbb{N}$  is defined via the following two axioms:

- (a)  $0 \in \mathbb{N}$
- (b) if  $x \in \mathbb{N}$  then  $x + 1 \in \mathbb{N}$  ( $x \in \mathbb{N} \rightarrow X + A \in \mathbb{N}$ )
- 3.  $\mathbb{R}$  set of real numbers also introduced axiomatically
  - $\mathbb{R}$  the set of real numbers.
  - (a) Additive closure:  $\forall x, y \exists z (x + y = z)$
  - (b) Multiplicative closure:  $\forall x, y, \exists z (x \times y = z)$
  - (c) Additive associativity:  $x + (y + z) = (x + y) + z$
  - (d) Multiplicative associativity:  $x \times (y \times z) = (x \times y) \times z$
  - (e) Additive commutativity:  $x + y = y + x$
  - (f) Multiplicative commutativity:  $x \times y = y \times x$
  - (g) Distributivity:  $x \times (y + z) = (x \times y) + (x \times z)$  and  $(y + z) \times x = (y \times x) + (z \times x)$
  - (h) Additive identity: There is a number, denoted 0, such that or all  $x, x + 0 = x$
  - (i) Multiplicative identity: There is a number, denoted 1, such that for all  $x, x \times 1 = 1 \times x = x$
  - (j) Additive inverses: For every  $x$  there is a number, denoted  $-x$ , such that  $x + (-x) = 0$
  - (k) Multiplicative inverses: For every nonzero  $x$  there is a number, denoted  $x^{-1}$ , such that  $x \times x^{-1} = x^{-1} \times x = 1$
  - (l)  $0 \neq 1$
  - (m) Irreflexivity of  $<$ :  $\sim (x < x)$
  - (n) Transitivity of  $<$ : If  $x < y$  and  $y < z$ , then  $x < z$
  - (o) Trichotomy: Either  $x < y, y < x$ , or  $x = y$
  - (p) If  $x < y$ , then  $x + y < y + z$
  - (q) If  $x < y$  and  $0 < z$ , then  $x \times z < y \times z$  and  $z \times x < z \times y$
  - (r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.
- 4.  $\emptyset$  is the empty set (The set with no elements).

**Definition:** Let A, B be sets.  $A=B$  if and only if all elements of A are elements of B and all elements of B are elements of A,  
 i.e.  $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \cap [\forall y(y \in B \rightarrow y \in A)]$

### 3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.  
**NB:** order is irrelevant.  
 $A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$
2. The formulaic/set builder method: give a formula that generates all elements of the set.  
 $A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using  $\mathbb{N}$  and the set-builder method, we can define:

$$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$$

$n = 0$  in any natural numbers  $\Rightarrow$  we generate all of  $\mathbb{N}$

$m = 0$  in any natural number  $\Rightarrow$  we generate all negative integers

$$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$$

**Definition:** A set  $A$  is called finite if it has a finite number of elements; otherwise it is called infinite.

## 4 Set Operations

**Task:** Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement and difference.

**Definition:** Let  $A, B$  be sets.  $A$  is a subset of  $B$ . If all elements of  $A$  are elements of  $B$ , **i.e.**  $\forall x(x \in A \rightarrow x \in B)$ . We denote that  $A$  is a subset of  $B$  by  $A \subseteq B$

**Example:**  $\mathbb{N} \subseteq \mathbb{Z}$

**Definition:** Let  $A, B$  be sets.  $A$  is a proper subset of  $B$  if  $A \subseteq B \wedge A \neq B$ , **i.e.**  $A \subseteq B \wedge \exists x \in B \text{ s.t. } x \notin A$ .

A proper subset is always a subset, but a subset is not always a proper subset.

**Notation:**  $A \subset B$

**Example:**  $\mathbb{N} \subset \mathbb{Z}$  since  $\exists -1 \in \mathbb{Z}$

**NB:**  $\forall A$  a set  $\emptyset \subseteq A$

**Recall:**  $B \subseteq C$  means  $\forall x(x \in B \rightarrow x \in C)$ , but  $\emptyset$  has no elements so in  $\emptyset \subseteq A$  the quantifier  $\forall$  operates on a domain with no elements. Clearly, we need to give meaning to  $\exists$  and  $\forall$  on empty sets.

Boolean Convention

$\forall$  is true on the empty set  
 $\exists$  is false on the empty set

} Consistent with common sense

**Definition:** Let  $A, B$  be two sets. The union  $A \cup B = \{x \mid x \in A \vee x \in B\}$

**Definition:** Let  $A, B$  be two sets. The intersection  $A \cap B = \{x \mid x \in A \wedge x \in B\}$

**Definition:** Let  $A, B$  be sets.  $A$  and  $B$  are called disjoint if  $A \cap B = \emptyset$

**Definition** Let  $A, B$  be two sets.  $A - B = A \setminus B = \{a \mid x \in A \wedge x \notin B\}$

**Examples:**

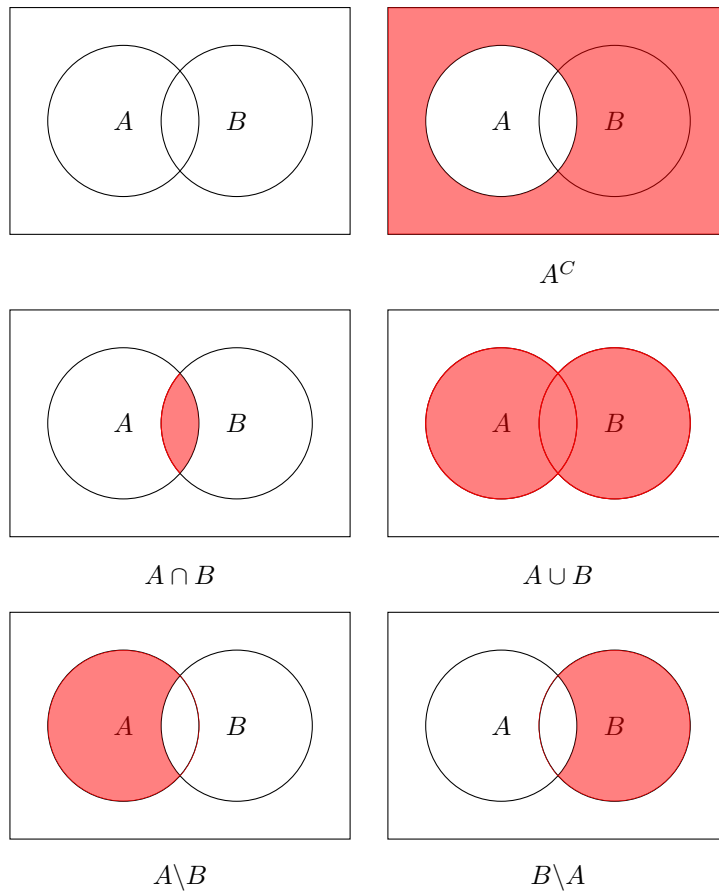
$A = \{1, 2, 5\}$	$B = \{1, 3, 6\}$
$A \cup B = \{1, 2, 3, 5, 6\}$	$A \cap B = \{1\}$
$A \setminus B = \{2, 5\}$	$B \setminus A = \{3, 6\}$

**Definition:** Let  $A, U$  be sets s.t.  $A \subseteq U$ . The complement of  $A$  in  $U = U \setminus A = A^C = \{x \mid x \in U \wedge x \notin A\}$

**Remark:** The notation  $A^C$  is unambiguous only if the universe  $U$  is clearly defined or understood.

## 4.1 Venn Diagrams

Schematic representation of set operations.





## 4.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $( )^C$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$
  - $(A \cup B)^C = A^C \cap B^C$
- Involutivity of the Complement
  - $(A^C)^C = A$

**NB:** An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- Criterion for proving equality of sets
  - $A = B \leftrightarrow A \subseteq C \wedge B \subseteq A$
- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \setminus B) \cup (B \setminus A) \neq \emptyset$

### 4.3 Example Proof in Set Theory

**Proposition:**  $\forall A, B$  sets.  $(A \cap B) \cup (A \setminus B) = A$

**Proof:** Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

Show  $(A \cap B) \cup (A \setminus B) \subseteq A$ :  $\forall x \in (A \cap B) \cup (A \setminus B), x \in (A \cap B)$  or  $x \in A \setminus B$ .

If  $x \in (A \cap B)$  then clearly  $x \in A$  as  $A \cap B \subseteq A$  by definition. If  $x \in A \setminus B$ , then by definition  $x \in A$  and  $x \notin B$  so definitely  $x \in A$ . In both cases,  $x \in A$  as needed.

Show  $A \subseteq (A \cap B) \cup (A \setminus B)$ :  $\forall x \in A$ , we have two possibilities, namely  $x \in B$

or  $x \notin B$ . If  $x \in B$ , then  $x \in A$  and  $x \in B$ , so  $x \in A \cap B$ . If  $x \notin B$ , then  $x \in A$  and  $x \notin B$ , so  $x \in A \setminus B$ . In both cases,  $x \in (A \cap B)$  or  $x \in (A \setminus B)$  so  $x \in (A \cap B) \cup (A \setminus B)$  as needed.

qed

## 5 The Power Set

**Task:** Understand what the power set of a set  $A$  is.

**Definition:** Let  $A$  be a set. The power set of  $A$  denoted  $P(A)$  is the collection of all the subsets of  $A$ .

**Recall:**  $\emptyset \subseteq A$ . It is also clear from the definition of a subset that  $A \subseteq A$ .

**Examples:**

1.  $A = \{0, 1\}$   
 $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2.  $A = \{a, b, c\}$   
 $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3.  $A = \emptyset$   
 $P(A) = \{\emptyset\}$   
 $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

**NB:**  $\emptyset$  and  $\{\emptyset\}$  are different objects.  $\emptyset$  has no elements, whereas  $\{\emptyset\}$  has one element.

**Remark:**  $P(A)$  and  $A$  are viewed as living in separate worlds to avoid phenomena like Russell' paradox.

**Q:** If  $A$  has  $n$  elements, how many elements does  $P(A)$  have?

**A:**  $2^n$

**Theorem:** Let  $A$  be a set with  $n$  elements, then  $P(A)$  contains  $2^n$  elements.

**Proof:** Based on the on/off switch idea.

$\forall x \in A$ , we have two choices: either we include  $x$  in the subset or we don't (on vs off switch).  $A$  has  $n$  elements  $\Rightarrow$  we have  $2^n$  subsets of  $A$ .

**qed**

**Alternate Proof:** Using mathematical induction.

**NB:** It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could limit, else what would its power set be?

## 6 Cartesian Products

**Task:** Understand sets like  $\mathbb{R}^1$  in a more theoretical way.

**Recall from Calculus:**

$$\mathbb{R} = \mathbb{R}^1 \ni x$$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_1)$$

$\vdots$

$$\underbrace{\mathbb{R} \times \mathbb{R}}_{n \text{ times}} = \mathbb{R}^n \ni (x_1, x_2, \dots, x_n)$$

These are examples of Cartesian products.

**Definition:** Let  $A, B$  be sets. The Cartesian product denoted by  $A \times B$  consists of all ordered pairs  $(x, y)$  s.t.  $x \in A \wedge y \in B$ , i.e.  $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

**Further Examples:**

$$1. A = \{1, 3, 7\}$$

$$B = \{1, 5\}$$

$$A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$$

**NB:** The order in which elements in a pair matters:  $(7, 1)$  is different from  $(1, 7)$ . This is why we call  $(x, y)$  an ordered pair.

$$2. A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \leftarrow \text{circle of radius 1}$$

$$B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = [-2, 2] \leftarrow \text{closed interval}$$

$$A \times B \leftarrow \text{cylinder of radius 1 and height 4}$$

### 6.1 Cardinality (number of elements) in a Cartesian product

If  $A$  has  $n$  elements and  $B$  has  $p$  elements,  $A \times B$  has  $np$  elements.

**Example:**

1.  $\#(A) = 3$        $A = \{1, 3, 7\}$   
     $\#(B) = 2$        $B = \{1, 5\}$   
     $\#(A \times B) = 3 \times 2 = 6$
2. Both  $A$  and  $B$  are infinite sets, so  $A \times B$  is infinite as well.

**Remark:** We can define Cartesian products of any length, **e.g.**  $A \times A \times B \times A$ ,  $B \times A \times B \times A \times B$ , etc. If all sets are finite, the number of elements is the product of the numbers of elements of each factor. If  $\#(A) = 3$  and  $\#(B) = 2$  as above,  $\#(A \times B \times A) = 3 \times 3 \times 3 = 18$  and  $\#(B \times A \times B) = 2 \times 3 \times 2 = 12$ .

## 7 Relations

**Task:** Define subsets of Cartesian products with certain properties. Understand the predicates " $=$ " (equality) and other predicates in predicate logic in a more abstract light.

Start with  $x = y$ . The elements  $x$  is some notation  $R$  to  $y$  (equality in this case). We can also denote it as  $xRy$  or  $(x, y) \in E$

Let  $x, y$  in  $\mathbb{R}$ , then  $E = \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$ .

The "diagonal" in  $\mathbb{R} \times \mathbb{R}$  gives exactly the elements equal to each other.

More generally:

**Definition:** Let  $A, B$  be sets. A subset of the Cartesian product  $A \times B$  is called a relations between  $A$  and  $B$ . A subset of the Cartesian product  $A \times A$  is called a relations on  $A$ .

**Remark:** Note how general this definition is. To make it useful for understanding predicates, we will need to introduce key properties relations can satisfy.

**Example:**  $A = \{1, 3, 7\}$        $B = \{1, 2, 5\}$

We can define a relation  $S$  on  $A \times B$  by  $S = \{(1, 1), (1, 5), (3, 2)\}$ . This means  $1S1$ ,  $1S5$  and  $3S2$  and no other ordered pairs in  $A \times B$  satisfy  $S$ .

**Remark:** The relations we defined involve 2 elements, so they are often called binary relations in the literature.

## 8 Equivalence Relations

**Task:** Define the most useful kind of relation.

**Definition:** A relation  $R$  on a set  $A$  is called

1. reflexive iff (if and only if)  $\forall x \in A, xRx$
2. symmetric iff  $\forall x, y \in A, xRy \rightarrow yRx$
3. transitive iff  $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$

An equivalence relation on  $A$  is a relation that is reflexive, symmetric and transitive.

**Notation:** Instead of  $xRy$ , an equivalence relation is often denoted by  $x \equiv y$  or  $x \sim y$ .

**Examples:**

1. "=" equality is an equivalence relation.
  - (a)  $x = x$  reflexive
  - (b)  $x = y \Rightarrow y = x$  symmetric
  - (c)  $x = y \wedge y = z \Rightarrow x = z$  transitive
2.  $A = \mathbb{N}$   
 $x \equiv y \pmod{3}$  is an equivalence relation.  $x \equiv y \pmod{3}$  means  $x - y = 3m$  for some  $m \in \mathbb{Z}$ , **i.e.**  $x$  and  $y$  have the same remainder when divided by 3. The set of all possible remainders is  $\{0, 1, 2\}$   
**NB:** In correct logic notation,  $x \equiv y \pmod{3}$  if  $\exists m \in \mathbb{Z} \text{ s.t. } x - y = 3m$ 
  - (a)  $x \equiv x \pmod{3}$  since  $x - x = 0 = 3 \times 0 \rightarrow$  reflexive
  - (b)  $x \equiv y \pmod{3} \Rightarrow y \equiv x \pmod{3}$  because  $x \equiv y \pmod{3}$  means  $x - y = 3m$  for some  $m \in \mathbb{Z} \Rightarrow y - x = -3m = 3 \times (-m) \Rightarrow y \equiv x \pmod{3} \rightarrow$  symmetric
  - (c) Assume  $x \equiv y \pmod{3}$  and  $y \equiv z \pmod{3}$   
 $x \equiv y \pmod{3} \Rightarrow \exists m \in \mathbb{Z} \text{ s.t. } x - y = 3m \Rightarrow y = x - 3m$   
 $y \equiv z \pmod{3} \Rightarrow \exists p \in \mathbb{Z} \text{ s.t. } y - z = 3p \Rightarrow y = z + 3p$   
Therefore,  $x - 3m = z + 3p \Leftrightarrow x - z = 3p + 3m = 3(p + m)$   
Since  $p, m \in \mathbb{Z}, p + m \in \mathbb{Z} \Rightarrow x \equiv z \pmod{3} \rightarrow$  transitive.
3. Let  $f : A \rightarrow A$  be any function on a non empty set  $A$ . We define the relation  $R = \{(x, y) \mid f(x) = f(y)\}$ 
  - (a)  $\forall x \in A, f(x) = f(x) \Rightarrow (x, x) \in R \rightarrow$  reflexive
  - (b) If  $(x, y) \in R$ , then  $f(x) = f(y) \Rightarrow f(y) = f(x)$ , **i.e.**  $(y, x) \in R \rightarrow$  symmetric
  - (c) If  $(x, y) \in R$  and  $(y, z) \in R$ , then  $f(x) = f(y)$  and  $f(y) = f(z)$ , which by the transitivity of equality implies  $f(x) = f(z)$ , **i.e.**  $(x, z) \in R$  as needed, so  $R$  is transitive as well.  
 $f(x)$  can be  $e^x, \sin x, (x)$ , etc.



4. Let  $\lambda$  be the set of all triangles in the plane.  $ABC \sim A'B'C'$  if  $ABC$  and  $A'B'C'$  are similar triangles, **i.e.** have equal angles.

(a)  $\forall ABC \in \lambda, ABC \sim ABC$  so  $\sim$  is reflexive

(b)  $ABC \sim A'B'C' \Rightarrow A'B'C' \sim ABC$  so  $\sim$  is symmetric

(c)  $ABC \sim A'B'C'$  and  $A'B'C' \sim A''B''C'' \Rightarrow ABC \sim A''B''C''$ ,  
so  $\sim$  is transitive

Clearly (a), (b), (c) use the fact that equality of angles is an equivalence relation.

**Exercise:** For various predicates you've encountered, check whether reflexive, symmetric or transitive. Examples of predicates include  $\neq, <, >, \leq, \geq, \subseteq$

## 9 Equivalence Relations and Partitions

**Task:** Understand how equivalence relations divide sets.

**Definition:** Let  $A$  be a set. A partition of  $A$  is a collection of non empty sets, any two of which are disjoint such that their union is  $A$ , **i.e.**  $\lambda = \{A_\alpha \mid \alpha \in I\}$  s.t.  $\forall \alpha, \alpha' \in I$  satisfy  $\alpha \neq \alpha', A_\alpha \cap A_{\alpha'} = \emptyset$  and  $\bigcup_{\alpha \in I} A_\alpha = A$

Here  $I$  is an indexing act (may be infinite).  $A_\alpha$  is the union of all the  $A_\alpha$ 's  
(possibly an infinite union)

**Example**  $\{(n, n+1) \mid n \in \mathbb{Z}\}$  is a partition of  $\mathbb{R}$



$$\bigcup_{n \in \mathbb{Z}} (n, n+1] = \mathbb{R}$$

$$(n, n+1] \cap (m, m+1] = \emptyset \text{ if } n \neq m$$

**Definition:** If  $R$  is an equivalence relations on a set  $A$  and  $x \in A$ , the equivalence class of  $x$  denoted  $[x]_R$  is the set  $\{y \mid xRy\}$ . The collection of all equivalence classes is called  $A$  modulo  $R$  and denoted  $A/R$ .

**Examples:**

1.  $A = \mathbb{N} \quad x \equiv y \pmod{3}$

We have the equivalence classes  $[0]_R, [1]_R$  and  $[2]_R$  given by the then possible remainders under division by 3.

$$[0]_R = \{0, 3, 6, 9, \dots\}$$

$$[1]_R = \{1, 4, 7, 10, \dots\}$$

$$[2]_R = \{2, 5, 8, 11, \dots\}$$

Clearly  $[0]_R \cup [1]_R \cup [2]_R = \mathbb{N}$  and they are mutually disjoint  $\Rightarrow R$  gives a partition of  $\mathbb{N}$ .

2.  $ABC \sim A'B'C'$

$$[ABC] = \{\text{The set of all triangles with angles of magnitude } \angle ABC, \angle BAC, \angle ACB\}$$

The union over the set of all  $[ABC]$  is the set of all triangles and

$[ABC] \cap [A'B''] = \emptyset$  if  $ABC \neq^* A'B'C'$  since it means these triangles have at least one angle that if difference.

\* In the original notes, not  $\sim$  is used (a tilde with a slash going through it) but I couldn't find this symbol in latex.

3.  $A = \mathbb{C} \quad x \cap y \text{ if } |x| = |y| \quad \text{equivalence relation}$   
 $[x] = \{y \in \mathbb{C} \mid |x| = |y|\} = [r] \text{ for } r \in [0, +\infty) \wedge (r \geq 0)$

circle of radius  $|x|$



$$\bigcup_{r \in [0, +\infty)} [r] = \mathbb{C}$$

$[r_1] \cap [r_2] \neq \emptyset$  if  $r_1 \neq r_2$  since two distinct circles in  $\mathbb{C} \simeq \mathbb{R}^2$  with empty intersection.

circles  $r_1 \wedge r_2$



**Theorem:** For any equivalence relation  $R$  on a set  $A$ , its equivalence classes form a partition of  $A$ , **i.e.**

1.  $\forall x \in A, \exists y \in A$  s.t.  $x \in [y]$  (every element of  $A$  sits somewhere)
2.  $xRy \Leftrightarrow [x] = [y]$  (all elements related by  $R$  belong to the same equivalence class)
3.  $\neg(xRy) \Leftrightarrow [x] \cap [y] = \emptyset$  (if two elements are not related by  $R$ , they belong to disjoint equivalence classes)

**Proof:**

1. Trivial. Let  $y = x$ .  $x \in [x]$  because  $R$  is an equivalence relation. Hence reflexive, so  $xRx$  holds.
2. We will prove  $xRy \Leftrightarrow [x] \subseteq [y]$  and  $[y] \subseteq [x]$   
 $\Rightarrow$  Fix  $x \in A, [x] = \{z \in A \mid xRz\} \Rightarrow \forall y \in A$  s.t.  $xRy, y \in [x]$ .  
Furthermore,  $[y] = \{w \in A \mid yRw\}$   
 $\Rightarrow \forall w \in [y], yRw$  but  $xRy \Rightarrow xRw$  by transitivity. Therefore,  $w \in [x]$ . We have shown  $[y] \subseteq [x]$ .  
Since  $R$  is an equivalence relation, it is also symmetric. **i.e.**  $xRy \Leftrightarrow yRx$ . So by the same argument with  $x$  and  $y$  swapped  $yRx \Rightarrow [x] \subseteq [y]$ . Thus  $xRy \Rightarrow [x] = [y]$ .  
 $\Rightarrow [x] = [y] \Rightarrow y \in [x]$  but  $[x] = \{y \in A \mid xRy\}$
3.  $\Rightarrow$  We will prove the contrapositive. Assume  $[x] \cap [y] \neq \emptyset \Rightarrow \exists z \in [x] \cap [y]. z \in [x]$  means  $xRz$ , whereas  $z \in [y]$  means  $yRz \Leftrightarrow zRy$  by symmetric of  $R$ . We thus have  $xRz$  and  $zRy \Rightarrow xRy$  by transitivity of  $R$ .  $xRy$  contradicts  $\neg(xRy)$  so indeed  $\neg(xRy) \Rightarrow [x] \cap [y] = \emptyset$   
 $\Leftarrow$  Once again we use the contrapositive.  
Assume  $\neg(\neg(xRy)) \Leftrightarrow xRy$ . By part (b)  $xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap [y] \neq \emptyset$



$[y] \neq \emptyset$  since  $x \in [x]$  and  $y \in [y]$ , **i.e.** These equivalence classes are non empty. We have obtained the needed contradiction.

qed

**Q:** What partition does " $=$ " impose on  $\mathbb{R}$ ?

**A:**  $[x] = \{x\}$  since  $E = \{(x, x) \mid x \in \mathbb{R}\}$  the diagonal.

The one element equivalence class is the smallest equivalence class possible (by definition, an equivalence class cannot be empty as it contains  $x$  itself).

We call such a partition the finest possible partition.

**Remark:** The theorem above shows how every equivalence relations partitions a set. It turns out every partition of a set can be used to define an equivalence relation:  $xRy$  is  $x$  and  $y$  belong to the same subset of the partition (check this is indeed an equivalence relations!). Therefore, there is a 1-1 correspondence between partitions and equivalence relations: to each equivalence relation there corresponds a partition and vice versa.

## 10 Partial Orders

**Task:** Understand another type of relation with special properties.

**Definition:** Let  $A$  be a set. A relation  $R$  on  $A$  is called anti-symmetric if  $\forall x, y \in A$  s.t.  $xRy \wedge yRx$ , then  $x = y$ .

**Definition:** A partial order is a relation on a set  $A$  that is reflexive, anti-symmetric, and transitive.

**Examples:**

1.  $A = \mathbb{R}$   $\leq$  "less than or equal to" is a partial order
  - (a)  $\forall x \in \mathbb{R} x \leq x \rightarrow$  reflexive
  - (b)  $\forall x, y \in \mathbb{R}$  s.t.  $x \leq y \wedge y \leq x \implies x = y \rightarrow$  anti-symmetric
  - (c)  $\forall x, y, z \in \mathbb{R}$  s.t.  $x \leq y \wedge y \leq z \implies x \leq z \rightarrow$  transitive
 Same conclusion if  $A = \mathbb{Z} \vee \mathbb{N}$
2.  $A$  is a set. Consider  $P(A)$ , the power set of  $A$ . The relation  $\subseteq$  "being a subset of" is a partial order.
  - (a)  $\forall B \in P(A), B \subseteq B \rightarrow$  reflexive.
  - (b) *forall*  $B, C \in P(A), B \subseteq C \wedge C \subseteq B \implies B = C$  (recall the criterion for proving equality of sets)  $\rightarrow$  anti-symmetric
  - (c)  $\forall B, C, D \in P(A)$  s.t.  $B \subseteq C \wedge C \subseteq D \implies B \subseteq D \rightarrow$  transitive

The most important example of a partial order is example (2) "being a subset of".

**Q:** Why is "being a subset of" a partial order as opposed to a total order?

**A:** There might exist products  $B, C$  of  $A$  s.t. neither  $B \subseteq C$  nor  $C \subseteq B$  holds, **i.e.** where  $B \wedge C$  are not related via inclusion.

## 11 Functions

**Task:** Define a function rigorously and make sense of terminology associated to functions.

**Definition:** Let  $A, B$  be sets. A function  $f : A \rightarrow B$  is a rule that assigns to every element of  $A$  one and only one elements of  $B$ , **i.e.**  $\forall x \in A \exists! y \in B$  s.t.  $f(x) = y$ .  $A$  is called the domain of  $f$  and  $B$  is called the codomain.

**Examples:**

1.  $A = \{1, 3, 7\}$   
 $B = \{1, 2, 5\}$

Is a function.



Not a function; 3 sent to both 1 and 5



Is a function.



2.  $A = B = \mathbb{R}$   $F : \mathbb{R} \rightarrow \mathbb{R}$  given by  $f(x) = x$  is called the identity function.

**Definition:** Let  $A, B$  be sets and let  $f : A \rightarrow B$  be a function. The range of  $f$  denoted by  $f(A)$  if the subset of  $B$  defined by  $f(A) = \{y \in B \mid \exists x \in A \text{ s.t. } f(x) = y\}$ .

**Definition:** Let  $A$  be a set. A Boolean function on  $A$  is a function  $F : A \rightarrow \{T, F\}$  which has  $A$  as its domain and the set of truth values  $\{T, F\}$  as its codomain.  $f : A \rightarrow \{T, F\}$  thus assigns truth values to the elements of  $A$ .

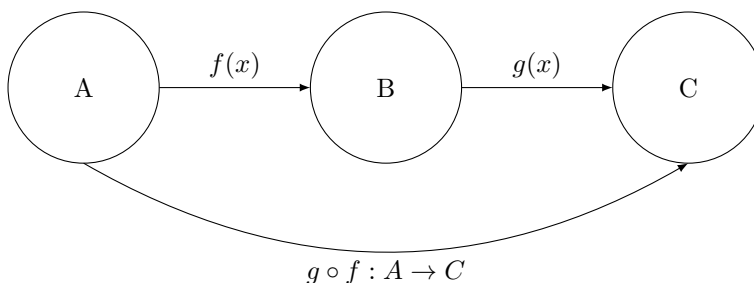
Functions are often represented by graphs. If  $f : A \rightarrow B$  is a function, the graph of  $f$  denoted  $\Gamma(f)$  is the subset of the Cartesian product  $A \times B$  given by  $\{(x, f(x)) \mid x \in A\}$ .

**Q:** Is it possible to obtain every subset of  $A \times B$  as the graph of some function?

**A:** No! For  $f : A \rightarrow B$  to be a function  $\forall x \in A \exists! y \in B$  s.t.  $f(x) = y$ , so for  $\Gamma \subseteq A \times B$  to be the graph of some function,  $\Gamma$  must satisfy that  $\forall x \in A \exists! y \in B$  s.t.  $(x, y) \in \Gamma$ . Then we can define  $f$  by letting  $y = f(x)$ .

## 12 Composition of Functions

**Task:** Understand the natural operation that allows us to combine functions.



**Example:**

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} & f(x) &= 2x \\ g : \mathbb{R} &\rightarrow \mathbb{R} & g(x) &= \cos x \\ g \circ f(x) &= g(f(x)) = g(2x) = \cos(2x) \\ f \circ g(x) &= f(g(x)) = f(\cos x) = 2(\cos x) = 2\cos x \end{aligned}$$

## 13 Inverting Functions

**Task:** Figure out which properties a function has to satisfy so that its action can be undone, **i.e.** when we can define an inverse to the original function.

Given  $f : A \rightarrow B$ , want  $f^{-1} : B \rightarrow A$  s.t.  $f^{-1} \circ f : A \rightarrow A$  is the identity  $f^{-1} \circ f(x) = f^{-1}(f(x)) = x$

$$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$$

It turns out  $f$  has to satisfy two properties for  $f^{-1}$  to exist.

1. Injective

## 2. Surjective

**Definition:** A function  $f : A \rightarrow B$  is called injective or an injection (sometimes called one to one) if  $f(x) = f(y) \Rightarrow x = y$

**Examples:**

$\sin x : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$  is injective

$\sin x : \mathbb{R} \rightarrow \mathbb{R}$  is not injective because  $\sin x = \sin \pi = 0$

**Definition:** A function  $f : A \rightarrow B$  is called surjective or a surjection (sometimes called onto) if  $\forall z \in B \exists x \in A$  s.t.  $f(x) = z$ .

**Remark:**  $f$  assigns a value to each element of  $A$  by its definition as a function, but it is not required to cover all of  $B$ .  $f$  is surjective if its range is all of  $B$ .

**Examples:**

$\sin x : \mathbb{R} \rightarrow [-1, 1]$  is surjective

$\sin x : \mathbb{R} \rightarrow \mathbb{R}$  is not surjective since  $\nexists x \in \mathbb{R}$  s.t.  $\sin x = 2$ . We know  $|\sin x| \leq 1 \forall x \in \mathbb{R}$

**Definition:** A function  $f : A \rightarrow B$  is called bijjective or a bijection if  $f$  is both injective and surjective.

**Example:**  $f : \mathbb{R} \rightarrow \mathbb{R}$   $f(x) = 2x + 1$  is bijective.

- Check injectivity  $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$  as needed.
- Check surjectivity  $\forall z \in \mathbb{R}. f(x) = z$  means  $2x + 1 = z$ .  
Solve for  $x$ :  $2x = z - 1 \Rightarrow x = \frac{z-1}{2} \in \mathbb{R} \Rightarrow f$  is surjective.

**Remark:** All bijective functions have inverses because we can define the inverse of a bijection and it will be a function:

- Surjectivity ensures  $f_{-1}$  assigns an element to every element of  $B$  (its domain).
- Injectivity ensures  $f_{-1}$  assigns to each elements of  $B$  one and only one elements of  $A$ .

**Conclusion:**  $f : A \rightarrow B$  bijective  $\Rightarrow f_{-1}$  exists, **i.e.**  $f_{-1}$  is a function. It turns out (reverse the arguments above) that  $f_{-1}$  exists  $\Rightarrow f : A \rightarrow B$  is bijective.

Altogether we get the following theorem:

**Theorem:** Let  $f : A \rightarrow B$  be a function.  $f_{-1}$  exists  $\Leftrightarrow f : A \rightarrow B$  is bijective.

**Q:** How do we find the inverse function  $f_{-1}$  given  $f : A \rightarrow B$ ?

**A:** If  $f(x) = y$ , solve for  $x$  as a function of  $y$  since  $f_{-1}(f(x)) = f_{-1}(y) = x$  s  $f_{-1} \circ f$  is the identity.

**Example:**  $f(x) = 2x + 1 = y$ . Solve for  $x$  in terms of  $y$ .

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ 2x = y - 1 &\quad x = \frac{y-1}{2} \end{aligned}$$

## 14 Functions Defined on Finite Sets

**Task:** Derive conclusions about a function given the number of elements of the domain and codomain if finite; understand the pigeonhole principle.

**Proposition:** Let  $A, B$  be sets and let  $f : A \rightarrow B$  be a function. Assume  $A$  is finite. Then  $f$  is injective  $\Leftrightarrow f(A)$  has the same number of elements as  $A$ .

**Proof:**

$A$  is finite so we can write it as  $A = \{a_1, a_2, \dots, a_p\}$  for some  $p$ . Then  $f(A) = \{f(a_1), f(a_2), \dots, f(a_p)\} \subseteq B$ . A priori, some  $f(a_i)$  might be the same as some  $f(a_j)$ . However,  $f$  injective  $\Leftrightarrow f(a_i) \neq f(a_j)$  whenever  $i \neq j \Leftrightarrow f(A)$  has exactly  $p$  elements just like  $A$ .

qed

**Corollary 1** Let  $A, B$  be finite sets such that  $\#(A) = \#(B)$ . Let  $f : A \rightarrow B$  be a function.  $f$  is injective  $\Leftrightarrow f$  is bijective.

**Proof:**

$\Rightarrow$  Suppose  $f : A \rightarrow B$  is injective. Since  $A$  is finite, by the previous proposition,  $f(A)$  has the same number of elements as  $A$ , but  $f(A) \subseteq B$  and  $B$  has the same number of elements as  $A \Rightarrow \#(A) = \#(f(A)) = \#(B)$ , which means  $f(A) = B$ , i.e.  $f$  is also surjective  $\Rightarrow f$  is bijective.

$\Leftarrow f$  is bijective  $\Leftarrow f$  is injective.

qed

**Corollary 2 (The Pigeonhole Principle)** Let  $A, B$  be finite sets. If  $\#(B) < \#(A)$ , and let  $f : A \rightarrow B$  be a function.  $\exists a, a' \in A$  where  $a \neq a'$ .  $f(a) = f(a')$

**Remark:** The name pigeonhole principle is due to Paul Erdős and Richard Rado. Before it was known as the principle of the drawers of Dirichlet. It has a simple statement, but it's a very powerful result in both mathematics and computer science.

**Proof:** Since  $f(A) \subseteq B$  and  $\#(B) < \#(A)$ ,  $f(A)$  cannot have as many elements as  $A$ , so by the proposition,  $f$  cannot be injective, i.e.  $\exists a, a' \in A$  where  $a \neq a'$  (i.e. distinct elements) s.t.  $f(a) = f(a')$

qed

**Examples:**

1. You have 8 friends. At least two of them were born the same day of the week.  $\#(\text{days of the week}) = 7 < 8$ .
2. A family of five gives each other presents for Christmas. There are 12 presents under the tree. We conclude at least one person for three presents or more.
3. In a list of 30 words in English, at least two will begin with the same letter.  $\#(\text{Letter in the English alphabet}) = 26 \leq 30$ .

**14.1 Behaviour of Functions on Infinite Sets**

Let  $A$  be a set and  $f : A \rightarrow A$  be a function. If  $A$  is finite, the corollary 1 tells us  $f$  injective  $\Leftrightarrow f$  bijective. What if  $A$  is not finite?

**14.1.1 Hilbert's Hotel problem (jazzier name: Hilbert's paradox of the Grand Hotel)**

A fully occupied hotel with infinitely many rooms can always accommodate an additional guest as follows: The person in Room 1 moves to Room 2. The person in Room 2 moves to Room 3 and so on, **i.e.** if the rooms at  $x_1, x_2, x_3, \dots$  define the function  $f(x_1) = x_2, f(x_2) = x_3, \dots, f(x_m) = x_{m+1}$ .

**Claim:** As defined  $f$  is injective but not surjective (hence not bijective!). Let  $H = \{x_1, x_2, \dots\}$  the hotel consisting of infinitely many rooms.  $f : H \rightarrow H$  is given by  $f(x_n) = x_{n+1}$ .  $f(H) = H \setminus \{x_1\}$ . We can use this idea to prove:

**Proposition:** A set  $A$  is finite  $\Leftrightarrow \forall f : A \rightarrow A$  an injective function is also bijective.

**Proof:** If the set  $X$  is finite then it follows immediately that every injective function  $f : X \rightarrow X$  is bijective.

Suppose that the set  $X$  is infinite. Then there exists some infinite sequence  $x_1, x_2, x_3, \dots$  of distinct elements of  $X$  (where an element of  $X$  occurs at most once in this list). Then there exists a function  $f : X \rightarrow X$  defined such that  $f(x_n) = x_{n+1}$  for all positive integers of  $n$ , and  $f(x) = x$  for all elements of  $x$  of  $X$ . If  $x$  is not a member of the infinite sequence  $x_1, x_2, x_3, \dots$  then the only elements of  $X$  that gets mapped to  $x$  is the element  $x$  itself; if  $x = x_n$ , where  $n > 1$ , then the only element of  $X$  gets mapped to  $x$ . It follows that the function  $f$  is injective. However it is not surjective, since  $x_1$  does not belong to the range of the function. This function  $f$  is thus an example of a function from the set  $X$  to itself which is injective but not bijective.

## 15 Mathematical Induction

**Task:** Understand how to construct a proof using mathematical induction.

$\mathbb{N} = \{0, 1, 2, \dots\}$  set of natural numbers.

Recall that  $\mathbb{N}$  is constructed using 2 axioms:

1.  $0 \in \mathbb{N}$
2. If  $n \in \mathbb{N}$ , then  $n + 1 \in \mathbb{N}$

**Remarks:**

1. This is exactly the process of counting.
2. If we start at 1, then we construct  $\mathbb{N}^* = \{1, 2, 3, 4, \dots\} = \mathbb{N} \setminus \{0\}$

via the axioms

1.  $1 \in \mathbb{N}^*$
2. if  $n \in \mathbb{N}^*$ , then  $n + 1 \in \mathbb{N}^*$

$\mathbb{N}$  or  $\mathbb{N}^*$  is used for mathematical induction.

### 15.1 Mathematical Induction Consists of Two Steps:

**Step 1** Prove statements  $P(1)$  called the base case.

**Step 2** For any  $n$ , assume  $P(n)$  and prove  $P(n+1)$ . This is called the inductive step.

In other words, step 2 proves the statement  $\forall n P(n) \rightarrow P(n+1)$

**Remark:** Step 2 is not just an implication but infinitely many! In logic notation, we have:

**Step 1**  $P(1)$

**Step 2**  $\forall n (P(n) \rightarrow P(n+1))$

Therefore,  $\forall n P(n)$

Let's see how the argument proceeds:

1.  $P(1)$                       Step 1 (base case)
2.  $P(1) \rightarrow P(2)$                       by Step 2 with  $n = 1$
3.  $P(2)$                       by 1 & 2
4.  $P(2) \rightarrow P(3)$                       by Step 2 with  $n = 2$
5.  $P(3)$                       by 3 & 4
6.  $P(3) \rightarrow P(4)$                       by Step 2 with  $n = 3$
7.  $P(4)$                       by 5 & 6
- $\vdots$

8.  $P(n)$  for any  $n$ .

This is like a row of dominos: knocking over the first one in a row makes all the others fall. Another idea is climbing a ladder.

**Examples:**

1. Prove  $1 + 3 + 5 + \dots + (2n - 1) = n^2$  by induction.

**Base Case:** Verify statement for  $n = 1$

When  $n = 1$ ,  $2n - 1 = 2 \times 1 - 1 = 1^2$

**Inductive Step:** Assume  $P(n)$ , i.e.  $1 + 3 + 5 + \dots + (2n - 1) = n^2$  and seek to prove  $P(n + 1)$ , i.e. the statement  $1 + 3 + 5 + \dots + (2(n + 1) - 1) = (n + 1)^2$

We start with LHS:  $1 + \underbrace{3 + 5 + \dots + (2n - 1)}_{n^2} + (2(n + 1) - 1) = n^2 + 2n + 2 - 1 = n^2 + 2n + 1 = (n + 1)^2$

2. Prove  $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$  by induction.

**Base Case:** Verify statement for  $n = 1$

When  $n = 1$ ,  $1 = \frac{1 \times (1+1)}{2} = \frac{1 \times 2}{2} = 1$

**Inductive Step:** Assume  $P(n)$ , i.e.  $1 + 2 + 3 + \dots + n = \frac{n \times (n+1)}{2}$  and seek to prove  $1 + 2 + 3 + \dots + n = \frac{(n+1)(n+2)}{2}$

$\underbrace{1 + 2 + 3 + \dots + n}_{\frac{n(n+1)}{2}} + n + 1 = \frac{n(n+1)}{2} + n + 1 = (n + 1)(\frac{n}{2} + 1) = (n + 1)\frac{n+2}{2} = \frac{(n+1)(n+2)}{2}$  as needed.

**Remarks:**

1. For some argument by induction, it might be necessary to assume not just  $P(n)$  at the inductive step but also  $P(1), P(2), \dots, P(n - 1)$ . This is called strong induction.

**Base Case:** Prove  $P(1)$

**Inductive Step:** Assume  $P(1), P(2), \dots, P(n)$  and prove  $P(n + 1)$ .

An example of result requiring the use of strong induction is the Fundamental Theorem of Arithmetic:  $\forall n \in \mathbb{N}, n \geq 2, n$  can be expressed as a product of one or more prime numbers.

2. One has to be careful with argument involving induction. Here is an illustration why:

Polya's argument that all horses are the same colour:

**Base Case:**  $P(1)$  There is only one horse, so it has a colour.

**Inductive Step** Assume any  $n$  horses are the same colour.

Consider a group of  $n + 1$  horses. Exclude the first horse and look at the other  $n$ . All of these are the same colour by our assumption. Now exclude the last horse. The remaining  $n$  horses are the same



colour by our assumption. Therefore, the first horse, the horses in the middle, and the last horse are all of the same colour. We have established the inductive step.

**Q:** Where does the argument fail?

**A:** For  $n = 2$ ,  $P(2)$  is false because there are no middle horses to compare to.

item[] The Grand Hotel Cigar Mystery

Recall Hilbert's hotel - the grand Hotel. Suppose that the Grand Hotel does not allow smoking and no cigars may be taken into the hotel. In spite of the rules, the guest in Room 1 goes to Room 2 to get a cigar. The guest in Room 2 goes to Room 3 to get 2 cigars (one for him and one for the person in room 1), etc. In other words, guest in Room  $N$  goes to Room  $N+1$  to get  $N$  cigars. They will each get back to their rooms, smoke one cigar, and give the result to the person in Room  $N-2$ .

**Q:** Where is the fallacy?

**A:** This is an induction argument without a base case. No cigars are allowed in the hotel so no guests have cigars. An induction cannot get off the ground without a base case.

## 16 Abstract Algebra

**Task:** Understand binary operators, semigroups, monoids, and groups as well as their properties.

### 16.1 Binary Operations

**Definition:** Let  $A$  be a set. A binary operation  $*$  on  $A$  is an operation applied to any two elements  $x, y \in A$  that yields on elements  $x * y$  in  $A$ . In other words,  $*$  is a binary operation on  $A$  if  $\forall x, y \in A, x * y \in A$ .

**Examples:**

1.  $\mathbb{R}, +$  addition on  $\mathbb{R} : \forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$
2.  $\mathbb{R}, -$  subtraction on  $\mathbb{R} : \forall x, y \in \mathbb{R}, x - y \in \mathbb{R}$
3.  $\mathbb{R}, \times$  multiplication on  $\mathbb{R} : \forall x, y \in \mathbb{R}, x \times y \in \mathbb{R}$
4.  $\mathbb{R}, /$ , division on  $\mathbb{R}$  is NOT a binary operation because  $\forall x \in \mathbb{R} \exists o \in \mathbb{R}$  s.t.  $\frac{x}{o}$  is undefined (not an element of  $\mathbb{R}$ )

**Definition:** A binary operation  $*$  on a set  $A$  is called commutative if  $\forall x, y \in A, x * y = y * x$

**Examples:**

1.  $\mathbb{R}, +$  is commutative since  $\forall x, y \in \mathbb{R}, x + y = y + x$

2.  $\mathbb{R}, \times$  is commutative since  $\forall x, y \in \mathbb{R}, x \times y = y \times x$
3.  $\mathbb{R}, -$  is not commutative since  $\forall x, y \in \mathbb{R}, x - y \neq y - x$  in general.  $x - y = y - x$  only if  $x = y$
4. Let  $M_n$  be the set of  $n$  by  $n$  matrices with entries in  $\mathbb{R}$  and let  $*$  be matrix multiplication.  $\forall A, B \in M_n, A * B \in M_n$ , so  $*$  is a binary operation, but  $AB \neq BA$  in general. Therefore  $*$  is not commutative.

**Definition:** A binary operation  $*$  on a set  $A$  is called associative if  $\forall x, y, z (x * y) * z = x * (y * z)$

**Examples:**

1.  $\mathbb{R}, +$  is associative since  $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$
2.  $\mathbb{R}, \times$  is associative since  $\forall x, y, z \in \mathbb{R}, (x \times y) \times z = x \times (y \times z)$
3. Intersection  $\cap$  on sets is associative since  $\forall A, B, C$  sets  $(A \cap B) \cap C = A \cap (B \cap C)$ .
4. Union  $\cup$  on sets is associative since  $\forall A, B, C$  sets  $(A \cup B) \cup C = A \cup (B \cup C)$
5.  $\mathbb{R}, -$  is not associative since  $(1 - 3) - 5 = -2 - 5 = -7$  but  $1 - (3 - 5) = 1 - (-2) = 1 + 2 = 3$

**Remark:** When we are dealing with associative binary operations we can drop the parentheses, i.e.  $(x * y) * z$  can be written  $x * y * z$ .

## 16.2 Semigroups

**Definition:** A semigroup is a set endowed with an associative binary operation. We denote the semigroup  $(A, *)$

**Examples:**

1.  $(\mathbb{R}, +)$  and  $(\mathbb{R}, -)$  are semigroups.
2. Let  $A$  be a set and let  $P(A)$  be its power set.  $(P(A), \cap)$  and  $(P(A), \cup)$  are both semigroups.
3.  $(M_n, *)$ , the set of  $n \times n$  matrices with entries in  $\mathbb{R}$  with the operation of matrix multiplication (which is associative  $\rightarrow$  a bit gory to prove) forms a semigroup.

Since  $*$  is associative on a semigroup, we can define  $a^n$  :

$$a^1 = a$$

$$a^2 = a * a$$

$$a^3 = a * a * a$$

$$\vdots$$

Recursively,  $a^1 = 1$  and  $a^n = a * a^{n-1}, \forall n > 1$

**NB:** In  $(\mathbb{R}, \times), \forall a \in \mathbb{R}, a^n = \underbrace{a \times a \times \dots \times a}_{n \text{ times}}$ , whereas in  $(\mathbb{R}, +), \forall a \in \mathbb{R}, a^n =$

$$\underbrace{a + a + \dots + a}_{n \text{ times}} = na. \text{ Be careful what } * \text{ stands for!}$$

**Theorem:** Let  $(A, *)$  be a semigroup.  $\forall a \in A, a^m * a^n = a^{m+n}, \forall m, n \in \mathbb{N}^*$ .

**Proof:** By induction on  $m$ .

**Base Case:**  $m = 1$   $a^1 * a^n = a * a^n = a * 1 + n$

**Inductive Step:** Assume the result is true for  $m = p$ , **i.e.**  $a^p * a^n = a^{p+n}$   
and seek to prove that  $a^{p+1} * a^n = a^{p+1+n}$

$$a^p + 1 * a^n = (a * a^p) * a^n = a * (a^p * a * n) = a * a^{p+n} = a^{p+1+n}$$

**Theorem:** Let  $(A, *)$  be a semigroup.  $\forall a \in A, (a^m)^n = a^{mn}, \forall m, n \in \mathbb{N}^*$

**Proof:** By induction on  $n$ .

**Base Case:**  $n = 1$   $(a^m)^1 = a^m = a^{m \times 1}$

**Inductive Step:** Assume the result if true for  $n = p$ , **i.e.**  $(a^m)^p = a^{mp}$   
and seek to prove that  $(a^m)^{p+1} = a^{m(p+1)}$

$$(a^m)^{p+1} = (a^m)^p * a^m = a^{mp} * a^m = a^{mp+m} = a^{m(p+1)}$$

### 16.2.1 General Associative Law

Let  $(A, *)$  be a semigroup.  $\forall a_1, \dots, a_s \in A, a_1 * a_2 * \dots * a_s$  has the same value regardless of how the product is bracketed.

**Proof** Use associativity of  $*$ .

qed

**NB:** Unless  $(A, *)$  has a commutative binary operation,  $a_1 * a_2 * \dots * a_s$  does depend on the ORDER in which the  $a_j$ 's appear in  $a_1 * a_2 * \dots * a_s$

### 16.2.2 Identity Elements

**Definition:** Let  $(A, *)$  be a semigroup. An element  $e \in A$  is called an identity element for the binary operation  $*$  if  $e * x = x * e = x, \forall x \in A$ .

**Examples:**

1.  $(\mathbb{R}, +)$  has 0 as the identity element.
2.  $(\mathbb{R}, \times)$  has 1 as the identity element.
3. Given a set  $A$ ,  $(P(A), \cup)$  has  $\emptyset$  (the empty set) as its identity elements, whereas  $(P(A), \cap)$  does NOT have an identity element.
4.  $(Mn, *)$  has  $In$ , the identity matrix as its identity element.

**Theorem** A binary operation on a set cannot have more than one identity elements, **i.e.** if an identity element exists, then it is unique.

**Proof:** Assume not (proof by contradiction). Let  $e$  and  $e'$  both be identity elements for a binary operation on a set  $A$ .  $e = e * e' = e'$

qed

## 16.3 Monoids

**Definition:** A monoid is a set  $A$  endowed with an associative binary operation  $*$  that has an identity element  $e$ . In other words, a monoid is a semigroup  $(A, *)$  where  $*$  has an identity element  $e$ .

**Definition:** A monoid  $(A, *)$  is called commutative (or Abelian) if the binary operation  $*$  is commutative.

**Example:**

1.  $(\mathbb{R}, +)$  is a commutative monoid with  $e = 0$ .
2.  $(\mathbb{R}, \times)$  is a commutative monoid with  $e = 1$ .
3. Given a set  $A$ ,  $(P(A), \cup)$  is a commutative monoid with  $e = \emptyset$ .
4.  $(M, n*)$  is a monoid since  $e = In$ , but it is not commutative since matrix multiplication is not commutative.
5.  $(\mathbb{N}, +)$  is a commutative monoid with  $e = 0$ , whereas  $(\mathbb{N}*, +)$  is merely a semigroup (recall  $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ )

**Theorem:** Let  $(A, *)$  be a monoid and let  $a \in A$ . Then  $a^m * a^n = a^{m+n}$ ,  $\forall m, n \in \mathbb{N}$

**Remark:** Recall that we proved this theorem for semigroups if  $m, n \in \mathbb{N}^*$ . We now need to extend that result.

**Proof:** A monoid is a semigroup  $\implies \forall a \in A, a^m * a^n = a^{m+n}$  whenever  $m, n \in \mathbb{N}^*$ , i.e.  $m > 0$  and  $n > 0$ . Now let  $m = 0$ .  $a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$   
If  $n = 0$ ,  $a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$

qed

**Theorem:** Let  $(A, *)$  be a monoid,  $\forall a \in A \forall m, n \in \mathbb{N}, (a^m)^n = a^{mn}$

**Remark:** Once again, we had this result for semigroups when  $m > 0$  and  $n > 0$

**Proof:** By the remark, we only need to prove the result when  $m = 0$  or  $n = 0$ . If  $m = 0$ ,  $(a^0)^n = (e)^n = e = a^0 = a^{0 \times n}$ . If  $n = 0$  then  $(a^m)^0 = e = a^0 = a^{0 \times m}$

## 17 Inverses

**Task:** Understand what an inverse is and what formal properties it satisfies.

**Definition:** Let  $(A, *)$  be a monoid with identity element  $e$  and let  $a \in A$ . An element  $y$  of  $A$  is called the inverse of  $x$  if  $x * y = y * x = e$ . If an element  $a \in A$  has an inverse, then  $a$  is called invertible.

**Examples:**

1.  $(\mathbb{R}, +)$  has identity element 0.  $\forall x \in \mathbb{R}, (-x)$  is the inverse of  $x$  since  $x + (-x) = (-x) + x = 0$ .
2.  $(\mathbb{R}, \times)$  has identity element 1.  $x \in \mathbb{R}$  is invertible only if  $x \neq 0$ . If  $x \neq 0$ , the inverse of  $x$  is  $\frac{1}{x}$  since  $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$ .
3.  $(Mn, *)$  the identity element is  $In$ .  $A \in Mn$  is invertible if  $\det(A) \neq 0$ .  $A^{-1}$  the inverse is exactly the one you computed in linear algebra. If  $\det(a) = 0$ ,  $A$  is NOT invertible.
4. Given a set  $A, (P(A), \cup)$  has  $\emptyset$  as its identity element of all the elements of  $P(A)$  only  $\emptyset$  is invertible and has itself as its inverse:  $\emptyset \cup \emptyset = \emptyset \cup \emptyset = \emptyset$

**Theorem:** Let  $(A, *)$  be a monoid. If  $a \in A$  has an inverse, then that inverse is unique.

**Proof:** By contradiction: Assume not, then  $\exists a \in A$  s.t. both  $b$  and  $c$  in  $A$  are its inverses, **i.e.**  $a * b = b * a = e$ , the identity element of  $(A, *)$  and  $a * c = c * a = e$  and  $b \neq c$ , then  $b = b * e = b * (a * c) = (b * a) * c = e * c = c$ .

qed

Since every invertible element  $a$  for  $(A, *)$  a monoid has a unique inverse, we can denote the inverse by the more standard notation  $a^{-1}$ .

Next, we need to understand inverses of elements obtained via the binary operation:

**Theorem:** Let  $(A, *)$  be a monoid and let  $a, b$  be invertible elements of  $A$ .  $a * b$  is also invertible and  $a * b^{-1} = b^{-1} * a^{-1}$ .

**Remark:** You might remember this formula from linear algebra when you looked at the inverse of a product of matrices  $AB$ .

**Proof:** Let  $e$  be the identity element of  $(A, *)$ .  $a * a^{-1} = a^{-1} * a = e$  and  $b * b^{-1} = b^{-1} * b = e$ . We would like to show  $b^{-1} * a * a^{-1}$  is the inverse of  $a * b$  by computing  $(a * b) * (b^{-1} * a^{-1})$  and  $(b^{-1} * a^{-1}) * (a * b)$  and showing both are  $e$ .

$$(a * b) * (a^{-1} * b^{-1}) = a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$$

$$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b = b^{-1} * e * b = (b^{-1} * e) * b = b^{-1} * b = e$$

Thus  $b^{-1} * a^{-1}$  satisfies the conditions needed for it to be the inverse of  $a * b$ . Since an inverse of unique,  $a * b$  is invertible and  $b^{-1} * a^{-1}$ .

**Theorem:** Let  $(A, *)$  be a monoid, and let  $a, b \in A$ . Let  $x \in A$  be invertible.  $a = b * x \Leftrightarrow b = a * x^{-1}$ . Similarly,  $a = x * b \Leftrightarrow b = x^{-1} * a$

**Proof:** Let  $e$  be the identity element of  $(A, *)$ .

First  $a = b * x \Leftrightarrow b = a * x^{-1}$ :

$\Rightarrow$  Assume  $a = b * x * a * x^{-1} = (b * x) * x^{-1} = b * x * x^{-1} = b * e = b$  as needed.

$\Leftarrow$  Assume  $b = a * x^{-1}$ . Then  $b * x = (a * x^{-1}) * x = a * (x^{-1} * x) = a * e = 1$  as needed.

Apply the same type of argument to show  $a = x * b \Leftrightarrow b = x^{-1} * a$ .

qed

Let  $(A, *)$  be a monoid. We can now make sense of  $a^n$  for  $n \in \mathbb{Z}, n < -$  for every  $n \in A$  invertible. Since  $n$  is a negative integer,  $\exists p \in \mathbb{N}$  s.t.  $n = -1$ . Set  $a^n = a^{-p} = (a^p)^{-1}$ .

**Theorem:** Let  $(A, *)$  be a monoid and let  $a \in A$  be invertible. Then  $a^n * a^m = a^{m+n} \quad \forall m, n \in \mathbb{Z}$ .

**Proof:** When  $m \geq 0 \wedge n \geq 0$  we have already proven this result. The rest of the proof splits into cases.

**Case 1:**  $m = n \vee n = 0$

If  $m = 0, n \in \mathbb{Z}, a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$  as needed.

If  $m \in \mathbb{Z}, n = 0, a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$  as needed.

**Case 2:**  $m < 0 \wedge n < 0$

$m < 0 \Rightarrow \exists p \in \mathbb{N}$  s.t.  $p = -m. n < 0 \Rightarrow \exists q \in \mathbb{N}$  s.t.  $q = -n$ .

$a^m = a^{-p} = (a^p)^{-1} \wedge a^n = a^{-q} = (a^q)^{-1}$

$a^m * a^n = (a^p)^{-1} * (a^q)^{-1} = (a^q * a^p)^{-1} = (a^{p+q})^{-1} = a^{-(p+q)} = a^{-q-p} = a^{m+n} = a^{n+m}$

**Case 3:**  $m \wedge n$  have opposite signs.

Without loss of generality, assume  $m < 0 \wedge n > 0$  (the case  $m > 0 \wedge n < 0$  is handled by the same argument). Since  $m < 0, \exists p \in \mathbb{N}$  s.t.  $p = -m$ . This case splits into two subcases:

**Case 3.1:**  $m + n \geq 0$

Set  $q = m + n$ . Then  $a^{m+n} = a^q = e * a^q = (a^p)^{-1} * a^p * a^q = (a^p)^{-1} * a^{p+q} = a^{-p} * a^{p+q} = a^m * a^{-m+m+n} = a^m * a^n$

**Case 3.2:**  $m + n < 0$

Set  $q = -(m+n) = -m-n \in \mathbb{N}^*$ . Then  $a^{m+n} = a^{-q} = (a^q)^{-1} * e = (a^q)^{-1} * (a^{-n} * a^n) = (a^q)^{-1} * (a^n)^{-1} * a^n = (a^n * a^q)^{-1} * a^n = (a^{n+p})^{-1} * a^n = (a^{n-m-n})^{-1} * a^n = (a^{-m})^{-1} * a^n = (a^p)^{-1} * a^n = a^m * a^n$

**Theorem:** Let  $(A, *)$  be a monoid, and let  $a$  be an invertible element of  $A$ .  $\forall m, n \in \mathbb{Z}, (a^m)^n = a^{mn}$ .

**Proof:** We consider 3 cases:

**Case 1:**  $n > 0$ , i.e.  $n \in \mathbb{N}^*$ .  $m \in \mathbb{Z}$  with no additional restrictions we proceed by induction on  $m$ .

**Base Case:**  $n = 1$   $(a^m)^1 = a^m = a^{m \times 1}$

**Inductive Step:** We assume  $(a^m)^n = a^{mn}$  and seek to prove  $(a^m)^{n+1} = a^{m(n+1)}$ . Start with  $(a^m)^{n+1} = (a^m)^n * (a^m)^1 = a^{mn} * a^m = a^{mn+m} = a^{m(n+1)}$

**Case 2:**  $n = 0$ ; no restriction on  $m \in \mathbb{Z}$

$$(a^m)^n = (a^m)^0 = e = a^0 = a^{m \times 0} = a^{mn}$$

**Case 3:**  $n < 0$ ; no restriction on  $m \in \mathbb{Z}$ .

Since  $n < 0$ ,  $\exists p \in \mathbb{N}$  s.t.  $p = -n$ . By case 1,  $(a^m)^p = a^{mp}$

$$(a^m)^n = (a^m)^{-p} = ((a^m)^p)^{-1} = (a^{mp})^{-1} = a^{-mp} = a^{m(-p)} = a^{mn}$$

## 17.1 Groups

A notion formally defined in the 1870's even though theorems about groups proven as early as a century before that.

**Definition:** A group is a monoid in which every element is invertible. In other words, a group is a set  $A$  endowed with a binary operation  $*$  satisfying the following properties:

1.  $*$  is associative, **i.e.**  $\forall x, y, z \in A, (x * y) * z = x * (y * z)$
2. There exists an identity element  $e \in A$ , **i.e.**  $\exists e \in A$  s.t.  $\forall a \in A, a * e = e * a = a$
3. Every element of  $A$  is invertible, **i.e.**  $\forall a \in A \exists a^{-1} \in A$  s.t.  $a * a^{-1} = a^{-1} * a = e$

**Notation for Groups:**  $(A, *) \vee (\underbrace{A}_{\text{set}}, \underbrace{*}_{\text{operation}}, \underbrace{e}_{\text{identity}})$

**Remark:** Closure under the operation  $*$  is implicit in the definition **i.e.**  $\forall a, b \in A, a * b \in A$

**Definition:** A group  $(A, *, e)$  is called commutative or Abelian if its operation  $*$  is commutative.

**Examples:**

1.  $(\mathbb{R}, +, 0)$  is an Abelian group.  
 $-x$  is the inverse of  $x, \forall x \in \mathbb{R}$
2.  $(\mathbb{Q}^*, \times, 1)$   $\mathbb{Q}^* = \mathbb{Q}^* \setminus \{0\}$   $(\mathbb{Q}^*, \times, 1)$  is Abelian  
 $\forall q \in \mathbb{Q}^*, q^{-1} = \frac{1}{q}$  is the inverse.
3.  $(\mathbb{R}^3, +, 0)$  vectors in  $\mathbb{R}^3$  with vector addition forms an Abelian group.  
 $(x, y, z) + (x', y', z') = (x + x', y + y', z + z')$  vector addition.  
 $0 = (0, 0, 0)$  is the identity.  $(-x, -y, -z) = -(x, y, z)$  is the inverse of  $(x, y, z)$ .

4.  $(\tilde{M}m, *, In)$   $n \times n$  invertible matrices with real coefficients under matrix multiplication with  $In$  as the identity elements forms a group which is NOT Abelian.
5. Set  $A = \mathbb{Z}$  and recall the equivalence relation  $x \equiv y \pmod{3}$  i.e.  $x \wedge y$  have the same remainder under the division by 3. Recall that  $\mathbb{Z}/N = \{0, 1, 2\}$ , i.e. the set of equivalence classes under the partition determined by this equivalence relation. We denote  $\mathbb{Z}/N = \{0, 1, 2\} = \mathbb{Z}_3$

Consider  $(\mathbb{Z}_3, \oplus_3, 0)$  where  $\oplus_3$  is the operation of addition modulo 3, i.e.  $1 + 0 = 1, 1 + 1 = 2, 1 + 2 = 3 \equiv 0 \pmod{3}$ .

**Claim:**  $(\mathbb{Z}_3, \oplus_3, 0)$  is an Abelian group.

**Proof of Claim:** Associativity of  $\oplus_3$  follows from the associativity of  $+$ , addition of  $\mathbb{Z}$ . Clearly, 0 is the identity (don't forget 0 stands for all elements with remainder 0 under division by 3, i.e.  $\{0, 3, -3, 6, -3, \dots\}$ ). To compute inverses recall that  $a \oplus_3 a^{-1} = 0$ , 0 is the inverse of 0 because  $0 + 0 = 0$ . 2 is the inverse of 1 because  $1 + 2 = 3 \equiv 0 \pmod{3}$ , and 1 is the inverse of 2 because  $2 + 1 = 3 \equiv 0 \pmod{3}$ .

More generally, consider the equivalence relation on  $\mathbb{Z}$  given by  $x \equiv y \pmod{n}$  for  $n \geq 1$ .  $\mathbb{Z}/N = \{0, 1, \dots, n-1\} = \mathbb{Z}_n$ . All possible remainders under division by  $n$  are the equivalence classes. Let  $\oplus_n$  be addition mod  $n$ . By the same argument as above,  $(\mathbb{Z}_n, \oplus_n, 0)$  is an Abelian group.

**Q:** What if we consider multiplication mod  $n$ , i.e.  $\otimes_n$ . Is  $(\mathbb{Z}_n, \otimes_n, 1)$  a group?

**A:** No!  $(\mathbb{Z}_n, \otimes_n, 1)$  is not even a monoid because  $1 \otimes_n 0 = 0 \otimes_n 1 = 0$ , so 1 is not an identity element for  $\otimes_n$  on  $\mathbb{Z}_n$ .

**Q:** Can this be fixed?

**A:** Troubleshoot how to get rid of 0.

Consider  $\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{0\} = \{1, 2, \dots, n-1\}$  all non-zero elements in  $\mathbb{Z}_n$ . This eliminates 0 as an element, but can 0 arise any other way from the binary operation? It turns out the answer depends on  $n$ . If  $n$  is not prime, say  $n = 6$ , we get two divisors, i.e. elements that yield 0 when multiplied by precisely the factors of  $n$ , for  $n = 6$ ,  $\mathbb{Z}_6^* = \{1, 2, 3, 4, 5\}$  but  $2 \otimes_6 3 = 6 \equiv 0 \pmod{6}$ , so  $2 \wedge 3$  are two divisors.

**Claim:** If  $n$  is prime, then  $(\mathbb{Z}_n^*, \otimes_n, 1)$  is an Abelian group.

Used in cryptology  $\rightarrow$  you will see next semester.

As an example, let us look at the multiplication table for  $\mathbb{Z}_5^*$  to see the inverse of various elements:  $\mathbb{Z}_5^* = \mathbb{Z}_5 \setminus \{0\} = \{0, 1, 2, 3, 4\} \setminus \{0\} = \{1, 2, 3, 4\}$

	1	2	3	4
1	1	2	3	4
2	2	4	1	2
3	3	1	4	2
4	4	3	2	1

$$\begin{aligned}
 1^{-1} &= 1 & 1 \otimes_5 1 &= 1 \\
 2^{-1} &= 3 & 2 \otimes_5 3 &= 6 \equiv 1 \pmod{5} \\
 3^{-1} &= 2 & 3 \otimes_5 2 &= 6 \equiv 1 \pmod{5} \\
 4^{-1} &= 4 & 4 \otimes_5 4 &= 16 \equiv 1 \pmod{5}
 \end{aligned}$$



The fact that  $\mathbb{Z}_n^*, \otimes_n, 1$  is Abelian follows from the commutativity of multiplication on  $\mathbb{Z}$ .

6. Let  $(A, *, e)$  be any group and let  $a \in A$ .

Consider  $A' = \{a^m \mid m \in \mathbb{Z}\}$  all powers of  $a$ . It turns out  $(A', *, e)$  is a group called the cyclic group determined by  $a$ .  $(A', *, e)$  is Abelian regardless of whether the original group was Abelian or not because of the theorem we proved on powers of  $a$ :  $\forall m, n \in \mathbb{Z} \quad a^m * a^n = a^{m+n} = a^{n+m} = a^n * a^m$ .

Cyclic groups come in two flavours: finite ( $A'$  is a finite set) and infinite ( $A'$  is an infinite set).

For example, let  $(A, *, e) = (\mathbb{Q}^*, \cdot, 1)$

If  $a = -1$   $A' = \{(-1)^m \mid m \in \mathbb{Z}\} = \{-1, 1\}$  is finite.

If  $a = 2$   $A' = \{2^m \mid m \in \mathbb{Z}\} = \{1, 2, \frac{1}{2}, 4, \frac{1}{4}, \dots\}$  is infinite.

## 18 Homomorphisms and Isomorphisms

**Task:** Understand the most natural functions between objects in abstract algebra such as semigroups, monoids or groups.

**Definition:** Let  $(A, *)$  and  $(B, *)$  be vitg semigroups, monoids or groups. A function  $f : A \rightarrow B$  is called a homomorphism if  $f(x * y) = f(x) * f(y) \forall x, y \in A$ . In other words, if  $f$  is a function that respects (behaves well with respect) to the binary operation.

**Examples:**

1. Consider  $(\mathbb{Z}, +, 0)$  and  $(\mathbb{R}^*, \cdot, 1)$ .

Pick  $a \in \mathbb{R}^*$ , then  $f(n) = a^n$  is a homomorphism between  $(\mathbb{Z}, +, 0)$  and  $(\mathbb{R}^*, \cdot, 1)$  because  $(\mathbb{R}^*, \cdot, 1)$  is a group, and we proved for groups that  $a^{m+n} = f(m+n) = a^m * a^n = f(m) * f(n) \quad \forall m, n \in \mathbb{Z}$ .

2. More generally,  $\forall a \in A$  invertible, where  $(A, *)$  is a monoid with identity element  $e$ ,  $f(n) = a^n$  gives a homomorphism between  $(\mathbb{Z}, +, 0)$  and  $(A', *, e)$ , where as before  $A' = \{a^m \mid m \in \mathbb{Z}\} \subset A$ .

We get even better behaviour if we require  $f : A \rightarrow B$  to be bijective.

**Definition:** Let  $(A, *)$  and  $(B, *)$  both be semigroups, monoids or groups. A function  $f : A \rightarrow B$  is called an isomorphism if  $f : A \rightarrow B$  is both bijective AND a homomorphism.

**Examples:**

1. Let  $A' = \{2^m \mid m \in \mathbb{Z}\} = \{1, 2, \frac{1}{2}, 4, \frac{1}{4}, \dots\}$   
 $f(m) = 2^m$  from  $(\mathbb{Z}, +, 0)$  to  $(A', \cdot, 1)$  is an isomorphism since  $2^m \neq 2^n$  if  $m \neq n$ .
2. Let  $A' = \{(-1)^m \mid m \in \mathbb{Z}\} = \{-1, 1\}$   
 $f(m) = (-1)^m$  from  $(\mathbb{Z}, +, 0)$  to  $(A', \cdot, 1)$  is NOT an isomorphism since it's not injective  $(-1)^2 = (-1)^4 = 1$ .

**Theorem:** Let  $(A, *)$  and  $(B, *)$  both be semigroups, monoids or groups. The inverse  $f^{-1} : B \rightarrow A$  of any isomorphism  $f : A \rightarrow B$  from  $A$  to  $B$  is itself an isomorphism.

**Proof:** If  $f : A \rightarrow B$  is an isomorphism  $\Rightarrow f : A \rightarrow B$  is bijective  $\Rightarrow f^{-1} : B \rightarrow A$  is bijective (proven when we discussed functions).

To show  $f^{-1} : B \rightarrow A$  is a homomorphism, let  $u, v \in B$ .  $\exists x, y \in A$  s.t.  $x = f^{-1}(u)$  and  $y = f^{-1}(v)$ , but then  $u = f(x)$  and  $v = f(y)$ .

Since  $f : A \rightarrow B$  is a homomorphism,  $f(x * y) = f(x) * f(y) = u * v$ . Then  $f^{-1}(u * v) = f^{-1}(f(x * y)) = x * y = f^{-1}(u) * f^{-1}(v)$  as needed.

qed

**Definition:** Let  $(A, *)$  and  $(B, *)$  both be semigroups, monoids or groups. If  $\exists f : A \rightarrow B$  an isomorphism between  $A$  and  $B$ , then  $(A, *)$  and  $(B, *)$  are said to be isomorphic.

**Remark:** "Isomorphic" comes from "iso" same + "morphé" form same abstract algebra structure on both  $(A, *)$  and  $(B, *)$  given to you in two different guises. As the French would say: "Même Marie, autre chapeau" same Mary, different hat.

## 19 Formal Languages

**Task:** Use what we learned about structures in abstract algebra in order to make sense of formal languages and grammars.

Let  $A$  be a finite set. When studying formal languages, we call  $A$  an alphabet and the elements of  $A$  letters.

**Examples:**

1.  $A = \{0, 1\}$                       binary digits
2.  $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$                       decimal digits
3.  $A =$  letters of the English alphabet

**Definition:**  $\forall n \in \mathbb{N}^*$ , we define a word of length  $n$  in the alphabet  $A$  as being any string of the form  $a_1, a_2, \dots, a_n$  s.t.  $a_i \in A \quad \forall i, 1 \leq i \leq n$ . Let  $A^n$  be the set of all words of length  $n$  over the alphabet  $A$ .

**Remark:** There is a one-to-one correspondence between the string  $a_1 a_2 \dots a_n$  and the ordered  $n$ -tuple  $(a_1, a_2, \dots, a_n) \in A^n = \underbrace{A \times \dots \times A}_{n \text{ times}}$  the Cartesian

product of  $n$  copies of  $A$ .

**Definition:** Let  $A^+ = \bigcup_{n=1}^{\infty} A^n = A^1 \cup A^2 \cup A^3 \cup \dots$   $A^+$  is the set of all words of positive length over the alphabet  $A$ .

**Examples:**

1.  $A = \{0, 1\}$ ,  $A^+$  is the set of all binary strings of finite length that is at least on, **i.e.** 0, 1, 01, 10, 00, 11, etc.
2. If  $A$  = letters of the English alphabet, then  $A^+$  consists of all non-empty strings of finite lengths of letters from the English alphabet. It is useful to also have the empty for  $\varepsilon$  in our set of strings.  $\varepsilon$  has length 0. Define  $A^0 = \{\varepsilon\}$  and then adjoin the empty word  $\varepsilon$  to  $A^+$ . We get  $A^* = \{\varepsilon\} \cup A^+ = A^0 \cup \bigcup_{n=1}^{\infty} A^n = \bigcup_{n=0}^{\infty} A^n$ .

**Notation:** We denote the length of a word  $w$  by  $|w|$ . Next introduce an operation on  $A^*$ .

**Definition:** Let  $A$  be a finite set and let  $w_1 \wedge w_2$  be words in  $A^*$ .  $w_1 = a_1 a_2 \dots a_m \wedge w_2 = b_1 b_2 \dots b_n$ . The concatenation of  $w_1 \wedge w_2$  is the word  $w_1 \circ w_2$ , where  $w_1 \circ w_2 = a_1 a_2 \dots a_m b_1 b_2 \dots b_n$ . Sometimes  $w_1 \circ w_2$  is denoted as just  $w_1 w_2$ . Note that  $|w_1 \circ w_2| = |w_1| + |w_2|$ . Concatenation of words is:

1. associative
2. NOT commutative if  $A$  has more than one element.

**Proof of (1):** Let  $w_1, w_2, w_3 \in A^*$ .  $w_1 = a_1 a_2 \dots a_m$  for some  $m \in \mathbb{N}$ ,  $w_2 = b_1 b_2 \dots b_n$  for some  $n \in \mathbb{N}$  and  $w_3 = c_1 c_2 \dots c_p$  for some  $p \in \mathbb{N}$ .  $w_1 \circ (w_2 \circ w_3) = a_1 a_2 \dots a_m b_1 b_2 \dots b_n c_1 c_2 \dots c_p$ .

qed

**Proof of (2):** Since  $A$  has at least two elements,  $\exists a, b \in A$  s.t.  $a \neq b$ .

$$a \circ b = ab \neq ba = b \circ a.$$

qed

$A^*$  is closed under the operation of concatenation  $\Rightarrow$  concatenation is a binary operation of  $A^*$  as  $\forall w_1, w_2 \in A^*, w_1 \circ w_2 \in A^*$ .

**Theorem** Let  $A$  be a finite set.  $(A^*, \circ)$  is a monoid with identity element  $\varepsilon$ .

**Proof:** Concatenation  $\circ$  is an associative binary operation on  $A^*$  as we showed above. Moreover,  $\forall w \in A^*, \varepsilon \circ w = w \circ \varepsilon = w$ , so  $\varepsilon$  is the identity element of  $A^*$ .

qed

**Definition:** Let  $A$  be a finite set. A language over  $A$  is a subset of  $A^*$ . A language  $L$  over  $A$  is called a formal language if  $\exists$  a finite set of rules of algorithm that generates exactly  $L$ , **i.e.** all words that belong to  $L$  and no other words.

**Theorem:** Let  $A$  be a finite set.

1. If  $L_1$  and  $L_2$  are languages over  $A$ ,  $L_1 \cup L_2$  is a language over  $A$ .
2. If  $L_1$  and  $L_2$  are languages over  $A$ ,  $L_1 \cap L_2$  is a language over  $A$ .
3. If  $L_1$  and  $L_2$  are languages over  $A$ , the concatenation of  $L_1 \wedge L_2$  given by  $L_1 \circ L_2 = \{w_1 \circ w_2 \in A^* \mid w_1 \in L_1 \wedge w_2 \in L_2\}$  is a language over  $A$ .
4. Let  $L$  be a language over  $A$ . Define  $L^1 = L$  inductively for and  $n \geq 1$   $L^n = L \circ L^{n-1}$ .  $L^n$  is a language over  $A$ . Furthermore,  $L^* = \{\varepsilon\} \cup L^1 \cup L^2 \cup L^3 \cup \dots = \bigcup_{n=0}^{\infty} L^n$  is a language over  $A$ .

**Proof:** By definition, a language over  $A$  is a subset of  $A^*$ . Therefore, if  $L_1 \subseteq A^* \wedge L_2 \subseteq A^*$ , then  $L_1 \cup L_2 \subseteq A^* \wedge L_1 \cap L_2 \subseteq A^*$ .  $\forall w_1 \circ w_2 \in L_1 \circ L_2$ ,  $w_1 \circ w_2 \in A^*$  becomes  $w_1 \in A^n$  for some  $n$  and  $w_2 \in A^m$  for some  $m$  so  $w_1 \circ w_2 \in A^{m+n} \subseteq A^* = \bigcup_{n=1}^{\infty} A^n$ .

Applying the same reasoning inductively, we see that  $L \subseteq A^* \Rightarrow L^* \subseteq A^*$  as  $L^n \subseteq A^* \forall n \geq 0$ .

qed

**Remark:** This theorem gives us a theoretic way of building languages, but we need a practical way. The practical way of building a language is through the notion of a grammar.

**Definition:** A (formal) grammar is a set of production rules for strings in a language.

To generate a language we use:

1.  $A$  the set, which is the alphabet of the language.
2. A start symbol  $\langle s \rangle$
3. A set of production rules.

**Example:**  $A = \{0, 1\}$ ; start symbol  $\langle s \rangle$ ; 2 production rules given by:

1.  $\langle s \rangle \rightarrow 0\langle s \rangle 1$
2.  $\langle s \rangle \rightarrow 01$

Let's see what we generate: via rule 2  $\langle s \rangle \rightarrow 01$ , so we get  $\langle s \rangle \Rightarrow 01$

Via rule 1  $\langle s \rangle \rightarrow 0\langle s \rangle 1$ , then via rule 2,  $0\langle s \rangle 1 \rightarrow 0011$ . We write the process as  $\langle s \rangle \rightarrow 0\langle s \rangle 1 \Rightarrow 0011$ .

Via rule 1,  $\langle s \rangle \rightarrow 0\langle s \rangle 1$ , then via rule 1 again  $0\langle s \rangle 1 \rightarrow 00\langle s \rangle 11$ , then via rule 2,  $00\langle s \rangle 11 \rightarrow 000111$ .

We got  $\langle s \rangle \Rightarrow 0\langle s \rangle 1 \Rightarrow 00\langle s \rangle 11 \Rightarrow 000111$ .

The language  $L$  we generated thus consists of all strings of the form  $0^m 1^m$  ( $m$  0's followed by  $m$  1's) for all  $m \geq 1, m \in \mathbb{N}$

We saw 2 types of strings that appeared in this process of generating  $L$ :

1. terminals, **i.e.** the elements of  $A$
2. nonterminals, **i.e.** strings that don't consist solely of 0's and 1's such as  $\langle s \rangle$ ,  $0\langle s \rangle 1$ ,  $00\langle s \rangle 11$ , etc.

The production rules then have the form:

nonterminal  $\rightarrow$  word over the alphabet  $V = \text{terminals, non-terminals}$

$\langle T \rangle \rightarrow w$

In our notation, the set of nonterminals is  $V \setminus A$ , so  $\langle T \rangle \in V \setminus A \wedge w \in V^* = \bigcup_{n=0}^{\infty} V^n$ . To the production rule  $\langle T \rangle \rightarrow w$ .

We can associate the ordered pair  $(\langle T \rangle, w) \in (V \setminus A) \times V^*$ , so the set of production rules, which we will denote by  $P$ , is a subset of the Cartesian product  $(V \setminus A) \times V^*$ .

Grammars come in two flavours:

1. Context-free grammars where we can replace any occurrence of  $\langle T \rangle$  by  $w$  if  $\langle T \rangle \rightarrow w$  is one of our production rules.
2. Context-sensitive grammars only certain replacements of  $\langle T \rangle$  by  $w$  are allowed, which are governed by the syntax of our language  $L$ .

The example we have was of a context free grammar. We can now finally define context free grammars.

**Definition:** A context free grammar  $(V, A, \langle s \rangle, P)$  consists of a finite set  $V$ , a subset  $A$  of  $V$ , an element  $\langle s \rangle$  of  $V \setminus A$ , and a finite subset  $P$  of the Cartesian product  $V \setminus A \times V^*$ .

**Notation:**  $(\begin{matrix} V \\ \text{set of terminals and non terminals} \end{matrix}, \begin{matrix} A \\ \text{set of terminals} \end{matrix}, \begin{matrix} \langle s \rangle \\ \text{start symbol} \end{matrix}, \begin{matrix} P \\ \text{set of production rules} \end{matrix})$

**Example:**  $A = \{0, 1\}$ ; start symbol  $\langle s \rangle$ ; 3 production rules given by:

1.  $\langle s \rangle \rightarrow 0\langle s \rangle 1$
2.  $\langle s \rangle \rightarrow 01$
3.  $\langle s \rangle \rightarrow 0011$

We notice here that the word 0011 can be generated in 2 ways in this context free grammar:

By rule 3,  $\langle s \rangle \rightarrow 0011$  so  $\langle s \rangle \Rightarrow 0011$

$\vee$

By rule 1,  $\langle s \rangle \rightarrow 0\langle s \rangle 1$  and by rule 2,  $0\langle s \rangle 1 \rightarrow 0011$ . Therefore,  $\langle s \rangle \Rightarrow 0\langle s \rangle 1 \Rightarrow 0011$ .

**Definition:** A grammar is called ambiguous if it generates the same string in more than one way.

Obviously, we prefer to have unambiguous grammars, else we waste computer operations.

Next, we need to spell out how words relate to each other in the production of our language via the grammar:

**Definition:** Let  $w'$  and  $w''$  be words over the alphabet  $V = \text{terminals, non-terminals}$ . We say that  $w'$  directly yields  $w''$  if  $\exists$  words  $u \wedge v$  over the alphabet  $V$  and a production rule  $\langle T \rangle \rightarrow w$  of the grammar s.t.  $w' = u \langle T \rangle \wedge w'' = uwv$ , where either or both of the words  $u$  and  $v$  may be the empty word.

In other words,  $w'$  directly yields  $w'' \Leftrightarrow \exists$  production rule  $\langle T \rangle \rightarrow w$  in the grammar s.t.  $w''$  may be obtained from  $w'$  by replacing a simple occurrence of the nonterminal  $\langle T \rangle$  within the word  $w'$  by the word  $w$ .

**Notation:**  $w'$  directly yields  $w''$  is denoted by  $w' \Rightarrow w''$

**Definition:** Let  $w' \wedge w''$  be words over the alphabet  $V$ . We say that  $w'$  yields  $w''$  if either  $w' = w''$  or else  $\exists$  words  $w_0, w_1, \dots, w_n$  over the alphabet  $V$  s.t.  $w_0 = w', w_n = w'', w_{i-1} \Rightarrow w_i$  for all  $i, 1 \leq i \leq n$ . In other words,  $w_0 \Rightarrow w_1 \Rightarrow w_2 \Rightarrow \dots \Rightarrow w_n - 1 \Rightarrow w_n$

**Notation:**  $w'$  yields  $w''$  is denoted by  $w' \Rightarrow^* w''$ .

**Definition:** Let  $(V, A, \langle s \rangle, P)$  be a context free grammar. The language generated by this grammar is the subset  $L$  or  $A^*$  defined by  $L = \{w \in A^* \mid \langle s \rangle \Rightarrow^* w\}$

In other words, the language  $L$  generated by a context free grammar  $(V, A, \langle s \rangle, P)$  consists of the set of all finite strings consisting entirely of terminals that may be obtained from the start symbol  $\langle s \rangle$  by applying a finite sequence of production rules of the grammar where the application of one production rule causes one and only one nonterminal to be replaced by the string in  $V^*$  corresponding of the right hand side of the production rule.

## 19.1 Phrase Structure Grammars

**Definition:** A phrase structure grammar  $(V, A, \langle s \rangle, P)$  consists of a finite set  $V$ , a subset  $A$  of  $V$ , an element  $\langle s \rangle$  of  $V \setminus A$ , and a finite subset  $P$  of  $(V^* \setminus A^*) \times V^*$

In a context free grammar, the set of production rules  $P \subset (V \setminus A) \times V^*$ .

In a phrase structure grammar,  $P \subset (V^* \setminus A^*) \times V^*$ . In other words, a production rule in a phrase structure grammar  $r \rightarrow w$  has a left hand side  $n$  that may contain more than one nonterminal. It is required to contain at least one nonterminal.

For example, if  $A\{0, 1\}$  and  $\langle s \rangle$  is the start symbol in a phrase grammar

grammar,  $0 < s > 0 < s > 0 \rightarrow 00010$  would be an acceptable production rule in a phrase structure grammar but not in a context free grammar.

The notions  $w' \Rightarrow w''$  ( $w'$  directly yields  $w''$ ) and  $w' \Rightarrow^* w''$  ( $w'$  yields  $w''$ ) are defined the same way as for context free grammars except that our production rules may, of course, be more general as we saw in the example above.

**Definition:** Let  $(V, A, <s>, P)$  be a phrase structure grammar. The language generated by this grammar is the subset  $L$  or  $A^*$  defined by  $L = \{w \in A^* \mid <s> \Rightarrow^* w\}$

**Remark:** The term phrase structure grammars was introduced by Noam Chowsky.

**Definition:** A language  $L$  generated by a context-free grammar is called a context-free language.

We now want to understand a particularly important subclass of context free languages called regular languages.

## 20 Regular Languages

**Task:** Understand when a language is regular and how regular languages are produced. Understand basics of automata theory.

**History:** The term regular languages was introduced by Stephen Kleene in 1951. A more descriptive name is finite-state languages as we will see that a language is regular  $\Leftrightarrow$  it can be recognised by a finite state acceptor, which is a type of finite state machine.

The definition of a regular language is very abstract, though. First, describe what operations the collection of regular languages is closed under: Let  $A$  be a finite set, and let  $A^*$  be the set of all words over the alphabet  $A$ . The regular language over the alphabet  $A$  constitute the smallest collection  $C$  of subsets of  $A^*$  satisfying that:

1. All finite subsets of  $A^*$  belong to  $C$ .
2.  $C$  is closed under the Kleene start operation (if  $M \subseteq A^*$  is inside  $C$ , i.e.  $M \subseteq C$ , then  $M^* \subseteq C$ )
3.  $C$  is closed under concatenation (if  $M \subseteq A^*, N \subseteq A^*$  satisfy that  $M \subseteq C \wedge N \subseteq C$ , then  $M \circ N \subseteq C$ )
4.  $C$  is closed under union (if  $M \subseteq A^* \wedge N \subseteq A^*$  satisfy that  $M \subseteq C \wedge N \subseteq C$ , then  $M \cup N \subseteq C$ )

**Definition:** Let  $A$  be a finite set, and let  $A^*$  be the set of words over the alphabet  $A$ . A subset  $L$  of  $A^*$  is called a regular language over the alphabet  $A$  if  $L = L_m$  for some finite sequence  $L_1, L_2, \dots, L_m$  of subsets of  $A^*$  with the property that  $\forall i, 1 \leq i \leq m, L_i$  satisfies one of the following:

1.  $L_i$  is a finite set
2.  $L_i = L_j^*$  for some  $j, 1 \leq j \leq i$  (the Klenne star operation applied to one of the previous  $L'_j$ s)
3.  $L_i = L_j \circ L_k$  for some  $j, k$  such that  $1 \leq j, k < i$  ( $L_i$  is a concatenation of previous  $L'_j$ s)
4.  $L_i = L_j \cup L_k$  for some  $j, k$  such that  $1 \leq k, j < i$  ( $L_i$  is a union of previous  $L'_j$ s)

**Example 1:** Let  $A = \{0, 1\}$ . Let  $L = \{0^m 1^n \mid m, n \in \mathbb{N} \quad m \geq 0, n \geq 0\}$ .  $L$  is a regular language. Note that  $L$  consists of all strings of first 0's, then 1's or the empty string  $\varepsilon$ .  $0^m 1^n$  stands for  $m$  0's followed by  $n$  1's, **i.e.**  $0^m \circ 1^n$ . Let us examine  $L' = \{0^m \mid m \in \mathbb{N}, m \geq 0\}$  and  $L'' = \{1^n \mid n \in \mathbb{N}, n \geq 0\}$

**Q:** Can we obtain them via operations listed among 1-4?

**A:** Yes! Let  $M = \{0\}$   $M \subseteq A \subseteq A^*$  and  $M^* = L' = \{0^m \mid m \in \mathbb{N} \quad m \geq 0\}$ . Let  $N = \{1\}$   $N \subseteq A \subseteq A^*$  and  $N^* = L'' = \{1^n \mid n \in \mathbb{N}, n \geq 0\}$ . In other words, we can do  $L_1 = \{0\}, L_2 = \{1\}, L_3 = L_1^*, L_4 = L_2^*, L_5 = L_4 \circ L_3 = L$ . Therefore,  $L$  is a regular language.

**Example 2** Let  $A = \{0, 1\}$ . Let  $L = \{0^m 1^m \mid m \in \mathbb{N}, m \geq 1\}$ .  $L$  is the language we used as an example earlier. It turns out  $L$  is NOT regular. This language consists of strings of 0's followed by an equal number of strings of 1's. For a machine to decide that the string  $0^m 1^m$  is inside the language is must store the number of 1's, as it examines the number of 0's or vice versa. The number of strings of the type  $0^m 1^m$  is not finite, however, so a finite-state machine cannot recognise this language. Heuristically, regular languages correspond to problems that can be solved with finite memory, **i.e.** we only need to remember one of finitely many things. By contrast, nonregular languages correspond to problems that cannot be solved with finite memory.

**Theorem:** The collection of regular languages  $L$  is also closed under the following two operations:

1. Intersection, **i.e.** if  $L', L''$  are regular languages (**i.e.**  $L' \cup L'' \in C$ ) then their intersection  $L' \cap L''$  is a regular language.
2. Complement, **i.e.** if  $L$  is a regular language (**i.e.**  $L \in C$ ), then  $A^* \setminus L$  is a regular language ( $A^* \setminus L \in C$ ).

**Remark:** These two properties did not come into the definition of a regular language, but they are true and often quite useful.



## 20.1 Finite State Acceptors and Automata Theory

**Definition:** An automation is a mathematical model of a computing device.

Plural of automation is automata.

**Basic idea:** Reason about computability without having to worry about the complexity of actual implementation.

It is most reasonable to consider at the beginning just finite states automata, **i.e.** machines with a finite number of internal states. The data entered discretely, and each datum causes the machine to either remain in the same internal state or else make the transition to some other state determined solely by 2 pieces of information:

1. The current state
2. The input datum

In other words, if  $S$  is the finite set of all possible states of our finite state machine, then the transition mapping  $t$  that tells us how the internal state of the machine changes on inputting a datum will depend on the current state  $s \in S$  and the input datum  $a$ , **i.e.** the machine will enter a (potentially) new state  $s' = t(s, a)$ .

**Want** to use finite state machines to recognise languages over some alphabet  $A$ . Let  $L$  be our language.

Since our finite state machine accepts (**i.e.** returns yes to)  $w$  if  $w \in L$ ,

<u>Input</u>	<u>Output</u>
Word $w = a_1 \dots a_n, a_i \in A \forall i$	Yes if $w \in L$
	No if $w \notin L$

we call our machine a finite state acceptor. We want to give a rigorous definition of a finite state acceptor. To check  $w = a_1 \dots a_n$ , we input each  $a_i$  starting with  $a_1$  and trace how the internal state of the machine changes.  $S$  is our set of states of the machine (a finite set). The transition mapping  $t$  takes the pair  $(s, a)$  and returns the new state  $s' = t(s, a)$  (where  $s \in S \wedge a \in A$ ) that the machine has reached so  $t : S \times A \rightarrow S$ .

Some elements and subsets of  $S$  are important to understand:

1. The initial state  $i \in S$  where the machine starts
2. The subset  $F \subseteq S$  of finishing states

It turns out that knowing  $S, F, i, t, A$  satisfies a finite state acceptor completely.

**Definition:** A finite state acceptor  $(S, A, i, t, F)$  consists of a finite set  $S$  of states, a finite set  $A$  that is the input alphabet, a starting state  $i \in S$ , a transition mapping  $t : S \times A \rightarrow S$ , and a set  $F$  of finishing states, where  $F \subseteq S$ .

**Definition:** Let  $(S, A, i, t, F)$  be a finite state acceptor, and let  $A^*$  denote the set of words over the input alphabet  $A$ . A word  $a_1, a_2 \dots a_n$  of length  $n$  over the alphabet  $A$  is said to be recognised or accepted by the finite state acceptor if  $\exists s_0, s_1, \dots, s_n \in S$  states s.t.  $s_0 = i$  (the initial state),  $s_n \in F$ , and  $s_i = t(s_{i-1}, a_i) \forall i \quad 1 \leq i \leq n$ .

**Definition:** Let  $(S, A, i, t, F)$  be a finite state acceptor. A language  $L$  over the alphabet  $A$  is said to be recognised or accepted by the finite state acceptor.

In the definition of a finite state acceptor,  $t$  is the transition mapping, which may or may not be a function (hence the careful terminology). This is because finite state acceptors come in 2 flavours:

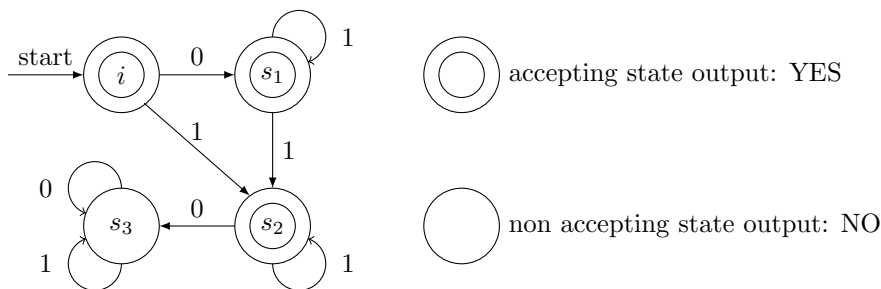
1. Deterministic: every state has exactly one transition for each possible input, **i.e.**  $\forall (s, a) \in S \times A \exists! t(s, a) \in S$ . In other words, the transition mapping is a function.
2. Non-deterministic: an input can lead to one, more than one or no transition for a given state. Some  $(s, a) \in S \times A$  might be assigned to more than one element of  $S$ , **i.e.** the transition mapping is not a function.

**Surprisingly**  $\exists$  algorithm that transforms a non deterministic (thought more complex one) using the powerset construction.

As a result, we have the following theorem:

**Theorem:** A language  $L$  over som alphabet  $A$  is a regular language  $\Leftrightarrow L$  is recognised by a deterministic finite state acceptor with input alphabet  $A \Leftrightarrow L$  is recognised by a nondeterministic finite state acceptor with input alphabet  $A$ .

**Example:** Build a deterministic finite state acceptor for the regular language  $L = \{0^m 1^n \mid m, n \in \mathbb{N}, m \geq 0, n \geq 0\}$



Accepting states in this examples:  $i, s_1, s_2$

Non accepting states:  $s_3$

Start states:  $i$

Here  $S = \{i, s_1, s_2, s_3\}$   $F = \{i, s_1, s_2\}$   $A = \{0, 1\}$   $t : S \times A \rightarrow S$   
 $t(i, 0) = s_1$   $t(i, 1) = s_2$   $t(s_1, 0) = s_1$   $t(s_1, 1) = s_2$   $t(s_2, 0) = s_3$   $t(s_2, 1) = s_2$

Let's process some strings:

String	$\varepsilon$ (empty string)	String	0	0	1	1	1
State i	i	State i	$s_1$	$s_1$	$s_2$	$s_2$	$s_2$
Output	YES	Output	YES				

String	1	1	String	1	String	0	1	0	1
State i	$s_2$	$s_2$	State i	$s_2$	State i	$s_1$	$s_2$	$s_3$	$s_3$
Output	YES		Output	YES	Output	NO			

Now that we really understand what a finite state acceptor is, we can develop a criterion for recognised regular languages called the Myhill-Nerode theorem based on an equivalence relation we can set up on words in our language over the alphabet  $A$ .

**Definition:** Let  $x, y \in L$ , a language over the alphabet  $A$ . We call  $x$  and  $y$  equivalent over  $L$  denoted by  $x \equiv_L y$  if  $\forall w \in A^*, xw \in L \Leftrightarrow yw \in L$ .

**Note:**  $xw$  means the concatenation  $x \circ w$ , and  $yw$  is the concatenation  $y \circ w$ .

**Idea:** If  $x \equiv_L y$ , then  $x$  and  $y$  place our finite state acceptor into the same state  $s$ .

**Notation:** Let  $L/N$  be the set of equivalence classes determined by the equivalence relation  $\equiv_L$ .

**The Myhill-Nerode Theorem:** Let  $L$  be a language over the alphabet  $A$ . If the set  $L/N$  of equivalence classes in  $L$  is infinite, then  $L$  is not a regular language.

**Stretch of Proof:** All element of one equivalence class in  $L/N$  place our automation into the same state  $s$ . Elements of distinct equivalence classes place the automation into distinct state, **i.e.** if  $[x], [y] \in L/N$  and  $[x] \neq [y]$ , then all elements of  $[x]$  place the automation into some state  $s$ , while all elements of  $[y]$  place the automation into some state  $s'$ , with  $s \neq s' \Rightarrow$  an automation that can recognise  $L$  has as many states at the number of equivalence classes in  $L/N$ , but  $L/N$  is NOT finite  $\Rightarrow L$  cannot be recognised by a finite state automation  $\Rightarrow L$  is not regular by the theorem above.

qed

## 21 Regular Grammars

**Task:** Understand what is the form of the production rules of a grammar that generates a regular language.

**Recall:** that a context-free grammar is given by  $(V, A\langle s \rangle, P)$  where every production rule  $\langle T \rangle \rightarrow w$  in  $P$  causes one and only one nonterminal to be replaced by a string in  $V^*$ .

**Definition:** A context-free grammar  $(V, A\langle s \rangle, P)$  is called a regular grammar if every production rule in  $P$  is of one of the three forms:

- (i)  $\langle A \rangle \rightarrow b\langle B \rangle$
- (ii)  $\langle A \rangle \rightarrow b$
- (iii)  $\langle A \rangle \rightarrow \varepsilon$

where  $\langle A \rangle$  and  $\langle B \rangle$  are nonterminals,  $b$  is a terminal, and  $\varepsilon$  is the empty word. A regular grammar is said to be in normal form if all its production rules are of types (i) and (iii).

**Remark:** In the literature, you often see this definition labelled left-regular grammar as opposed to right-regular grammar, where the production rules of types 1 have the form  $\langle A \rangle \rightarrow \langle B \rangle b$ , (**i.e.** the terminal is on the right of the nonterminal). This distinction is not really important as long as we stick to one type throughout since both left regular grammars and right regular grammars generate regular languages.

**Lemma:** Any language generated by a regular grammar may be generated by a regular grammar in normal form.

**Proof:** Let  $\langle A \rangle \rightarrow b$  be a rule of type (ii). Replace it by two rules:  $\langle A \rangle \rightarrow b\langle F \rangle$  and  $\langle F \rangle \rightarrow \varepsilon$ , where  $\langle F \rangle$  is a new nonterminal. Add  $\langle F \rangle$  to the set  $V$ . We do the same for every rule of type (ii) obtaining a bigger set  $V$ , but now our production rules are only of type (i) and (iii) and we are generating the same language.

qed

**Example:** Recall the regular language  $L = \{0^m 1^n \mid m, n \in \mathbb{N}, m \geq 0, n \geq 0\}$ . We can generate it from the regular grammar in normal form given by production rules:

1.  $\langle s \rangle \rightarrow 0\langle A \rangle$
2.  $\langle A \rangle \rightarrow 0\langle A \rangle$
3.  $\langle A \rangle \rightarrow \varepsilon$
4.  $\langle s \rangle \rightarrow \varepsilon$
5.  $\langle A \rangle \rightarrow 1\langle B \rangle$

6.  $\langle B \rangle \rightarrow 1 \langle B \rangle$

7.  $\langle s \rangle \rightarrow 1 \langle B \rangle$

8.  $\langle B \rangle \rightarrow \varepsilon$

Rules (1), (2), (5), (6), (7) are of type (i), where rules (3), (4) and (8) are of types (iii).

(1) and (3) gives 0. (1), (2) applied  $m - 1$  times and (3) gives  $0^m$  for  $m \geq 2$ .

(7) and (8) give 1. (7), (6) applied  $n - 1$  times and (8) give  $1^n$  for  $n \geq 2$ .

(1), (5) and (8) give 01. (1), (5), (6) applied  $n - 1$  times and (8) gives  $01^n$  for  $n \geq 2$ .

(1), (2) applied  $m - 1$  times, (5) and (8) gives  $0^m 1$  for  $m \geq 2$ .

(1), (2) applied  $m - 1$  times, (5), (6) applied  $n - 1$  times, and (8) gives  $0^m 1^n$  for  $m \geq 2, n \geq 2$ .

Rule (4) gives the empty word  $\varepsilon = 0^0 1^0$ .

**Q:** Why does a regular grammar yield a regular language, **i.e.** one recognised by a finite state acceptor?

**A:** Not obvious from the definition, but we can construct the finite state acceptor from the regular grammar as follows: our regular grammar is given by  $(V, A, \langle s \rangle, P)$ . Want a finite state acceptor  $(S, A, i, t, F)$ . Immediately, we see the alphabet  $A$  is the same and  $i = \langle s \rangle$ . This gives us the idea of associating to every nonterminal symbol in  $V \setminus A$  a state.  $\langle s \rangle \in V \setminus A$ , so that's good. Next we ask:

**Q:** Is it sufficient for  $S = V \setminus A$ ?

**A:** No! Our set  $F$  of finishing/accepting states should be nonempty. So we add an element  $\{f\}$  to  $V \setminus A$ , where our acceptor will end up when a word in our language. Thus,  $S = (V \setminus A) \cup \{f\}$  and  $F = \{f\}$ .  $F \subseteq S$  as needed.

**Q:** How do we define  $t$ ?

**A:** Use the production rules in  $P$ ! For every rule of type (i), which is of the form  $\langle A \rangle \rightarrow b \langle B \rangle$  set  $t(\langle A \rangle, b) = \langle B \rangle$ . This works out well because our nonterminals  $\langle A \rangle$  and  $\langle B \rangle$  are states of the acceptor and the terminal  $b \in A$  for  $t$  takes an element of  $S \times A$  to an element of  $S$  as needed. Now look at production rules of type (ii),  $\langle A \rangle \rightarrow b$  and of types (iii),  $\langle A \rangle \rightarrow \varepsilon$ . Those are applied when we finish constructing a word  $w$  in our language  $L$ , **i.e.** at the very last step, so our acceptor should end up in the finishing state  $f$  whenever a production rule of type (ii) or (iii) is applied. Write a production rule of type (ii) or (iii) as  $\langle A \rangle \rightarrow w$ , then we can set  $t(\langle A \rangle, w) = f$ . We have finished constructing  $t$  as well. Technically,  $t : S \times (A \cup \{\varepsilon\}) \rightarrow S$  instead of  $t : S \times A \rightarrow S$ , but we can easily fix the transition function  $t$  by combining the last two transitions for each accepted word.

**Remark:** The same general principals as we used above allow us to go from a finite state acceptor to a regular grammar. This gives us the following theorem:

**Theorem:** A language  $L$  is regular  $\Leftrightarrow L$  is recognised by a finite state acceptor  $\Leftrightarrow L$  is generated by a regular grammar.

## 21.1 Applications of Formal Languages and Grammars as well as Automata Theory

1. Compiler architecture uses context-free grammars
2. Parsers - recognise if commands comply with the syntax of a language
3. Pattern matching and data mining - guess the language from a given set of words (applied in CS, etc)
4. Natural language processing - example in David Wilkins' notes pp40-44
5. Checking proofs by computers/automatic theorem proving - simpler example of this kind in David Wilkins' notes pp45-57 that pertains to propositional logic
6. Theory of regular expressions (related to the definition of a regular language that we gave) enables
  - (a) grep/awk/sed in Unix
  - (b) More efficient coding (avoiding unnecessary detours in your code)
7. Biology - John Conway's game of life is a cellular automation
8. Modelling of AI characters in games uses the finite state automation idea. Our character can choose among different behaviours based on stimuli - like a finite state automation reacting to input
9. Strategy and tactics in games - teach the opposition to recognise certain patterns, then suddenly change them to gain an advantage and score - used in football, fencing, etc.
10. Learning a sport/a numerical instrument/a new field or subject - split the information into blocks and learn how to combine them into meaningful patterns - uses notions from context-sensitive grammars.
11. Finite state automata and probability - chaos theory, financial mathematics.

etc...

## 22 Graph Theory

**Task:** Introduce terminology related to graphs; understand different types of graphs; learn how to put together arguments involving graphs.

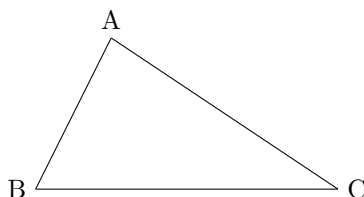
An undirected graph consists of:

1. A finite set of points  $V$  called vertices
2. A finite set  $E$  of edges joining two distinct vertices of the graph.

**Understand the meaning of an edge better:** Let  $V$  be the set of vertices. Consider  $P(V)$ , the power set of  $V$ . Let  $V_2 \subseteq P(V)$  consist of all subsets of  $V$  containing exactly 2 points, **i.e.**  $V_2 = \{A \in P(V) \mid \#(A) = 2\}$ . Identify each element in  $V_2$  with the edge joining the two points. In other words, if  $\{a, b\} \in V_2$ , then we can let  $ab$  be the edge corresponding to  $\{1, b\}$ .

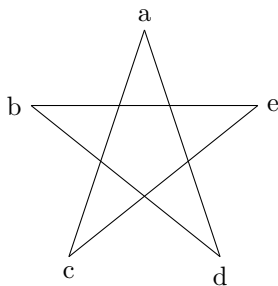
**Examples:**

1. A triangle is an undirected graph.  
 $V = \{A, B, C\}$



3 possible 2 element subsets of  $V$ :  $\{A, B\} \rightarrow AB$   
 $\{A, C\} \rightarrow AC$   
 $\{B, C\} \rightarrow BC$   
 $E = \{AB, AC, BC\}$

2. A pentagram is an example of an undirected graph.  
 $V = a, b, c, d, e$



$E = \{ac, ad, be, ce, bd\}$

**Convention:** The set of vertices cannot be empty, **i.e.**  $V \neq \emptyset$ .

**Q:** If  $V \neq \emptyset$ , what is the simplest possible undirected graph?

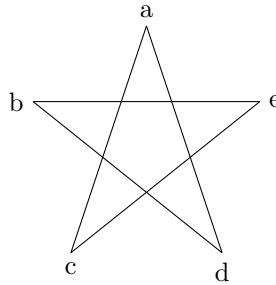
**A:** A graph consisting of a single point, **i.e.** with one vertex and two edges.

**Definition:** A graph is called trivial if it consists of one vertex and zero edges.  
Next, study how vertices and edges relate to each other.

**Definition:** If  $v$  is a vertex of some graph, if  $e$  is an edge of that graph, and it  $e = vv'$  for  $v'$  another vertex, then the vertex  $v$  is called incident to the edge  $e$  and the edge  $e$  is called incident to the vertex  $v$ .

**Example:**

$b$  is incident to edges  $be$  and  $bd$   
 $be$  is incident to vertices  $b$  and  $e$



**Definition:** Let  $(V, E)$  be an undirected graph. Two vertices  $A, B \in V$   $A \neq B$  are called adjacent if  $\exists$  edge  $AB \in E$ .

We represent the incidence relations among the vertices  $V$  and edges  $E$  of an undirected graph via:

1. An incidence table
2. An incidence matrix

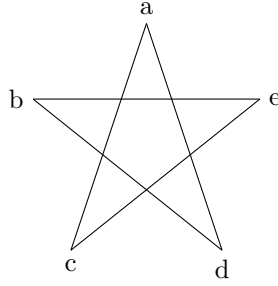
**Legend:**

- 1 an incidence relation holds
- 0 an incidence relation does not hold

From the pentagram:

$$V = \{a, b, c, d, e\}$$
$$E = \{ac, ad, be, bd, ce\}$$





The incidence table is:

	ac	ad	be	bd	ce
a	1	1	0	0	0
b	0	0	1	1	0
c	1	0	0	0	1
d	0	1	0	1	0
e	0	0	1	0	1

Correspondingly, the incidence matrix is:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Note that for the incidence matrix to make sense, we need to know that vertices were considered in the order  $\{a, b, c, d, e\}$  and edges in the order  $\{ac, ad, be, bd, ce\}$ . If we shuffle either set, the incidence matrix changes. With this in mind, we can now rigorously define the incidence matrix:

**Definition:** Let  $(V, E)$  be an undirected graph with  $m$  vertices and  $n$  edges. Let vertices be ordered as  $v_1, v_2, \dots, v_m$ , and let the edges be ordered

$e_1, e_2, \dots, e_n$ . The incidence matrix for such a graph is given by  $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ ,

where the entry  $a_{ij}$  in row  $i$  and column  $j$  has the value 1 if the  $i^{th}$  vertex is incident to the  $j^{th}$  edge and has value 0 otherwise.

Similarly, we can define the adjacency table and the adjacency matrix of a graph:

**Definition:** Let  $(V, E)$  be an undirected graph with  $m$  vertices, and let these vertices be ordered as  $v_1, v_2, \dots, v_m$ . The adjacency matrix for this graph

is given by  $\begin{pmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mm} \end{pmatrix}$  where  $b_{ij} = 1$  if  $v_i$  and  $v_j$  are adjacent to each other and  $b_{ij} = 0$  if  $v_i$  and  $v_j$  are not adjacent to each other.

**Remark:** "Being adjacent to" is a symmetric relation on the set of vertices  $V$ , so the adjacency matrix is symmetric, **i.e.**  $b_{ij} = b_{ji} \quad \forall i, j \quad 1 \leq i, j \leq m$ . It is not reflexive so all the entries on the diagonal are zero.

## 22.1 Complete graphs

**Definition:** A graph  $(V, E)$  is called complete if  $\forall v, v' \in V$  s.t.  $v \neq v'$ , the edge  $vv' \in E$ . In other words, a complete graph has the highest number of edges possible given its number of vertices.

**Examples:**

1. The triangle is a complete graph.
2. The pentagram is not a complete graph.

**Notation:** A complete graph with  $n$  vertices is denoted by  $K_n$ .

**Q:** How does the adjacency matrix of a complete graph look like?

**A:** All entries are 1 except on the diagonal, where they are all zero.

**Definition:** A graph  $(V, E)$  is called bipartite if  $\exists$  subsets  $V_1$  and  $V_2$  s.t.

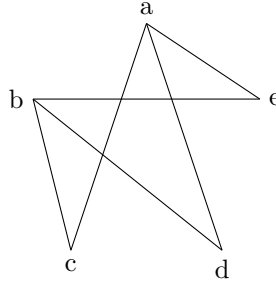
1.  $V_1 \cup V_2 = V$
2.  $V_1 \cap V_2 = \emptyset$
3. Every edge in  $E$  is of the form  $vw$  with  $v \in V_1$  and  $w \in V_2$ .

A bipartite graph is called a complete bipartite graph if  $\forall v \in V_1 \quad \forall w \in V_2 \quad \exists vw \in E$ .

**Notation:** A complete bipartite graph where the set  $V_1$  has  $p$  elements and the set  $V_2$  has  $q$  elements is denoted by  $K_{p,q}$ .

**Example:**

$V_1 = \{a, b\}$   
 $V_2 = \{c, d, e\}$   
 $V = \{a, b, c, d, e\}$   
 $E = \{ac, ad, ae, bc, bd, be\}$   
 is a complete bipartite graph.



Next, relate graph to each other via functions with special properties.

## 23 Isomorphism of Graphs

**Definition:** An isomorphism between two graphs  $(V, E)$  and  $(V', E')$  is a bijective function  $\varphi : V \rightarrow V'$  satisfying that  $\forall a, b \in V$  with  $a \neq b$  the edge  $ab \in E \Leftrightarrow$  the edge  $\varphi(a)\varphi(b) \in E'$ .

**Recall:** A function  $\varphi : V \rightarrow V'$  is bijective  $\Leftrightarrow$  it has an inverse  $\varphi^{-1} : V' \rightarrow V$ . The bijection  $\varphi : V \rightarrow V'$  that gives the isomorphism between  $(V, E)$  and  $(V', E')$  thus sets up the following:

1. A 1-1 correspondence of the vertices  $V$  of  $(V, E)$  with the vertices  $V'$  of  $(V', E') \rightarrow$  comes from  $\varphi : V \rightarrow V'$  being bijective.
2. A 1-1 correspondence of the edges  $E$  of  $(V, E)$  with the edges  $E'$  of  $(V', E') \rightarrow$  comes from the additional property in the definition of an isomorphism that  $\forall a, b \in V$  with  $a \neq b, ab \in E \Leftrightarrow \varphi(a)\varphi(b) \in E'$ .

**Remark:** Just like an isomorphism of groups discussed earlier in the course, an isomorphism of graphs means  $(V, E)$  and  $(V', E')$  have the same "iso" from "morphé". "Being isomorphic" is an equivalence relations, so we get classes of graphs that have the same form as our equivalence classes.

**Definition:** If there exists an isomorphism  $\varphi : V \rightarrow V'$  between two graphs  $(V, E)$  and  $(V', E')$ , then  $(V, E)$  and  $(V', E')$  are called isomorphic.

## 24 Subgraphs

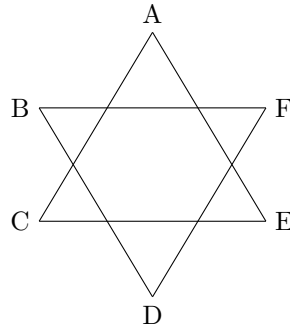
**Task:** Understand subobjects of a graph.

**Definition:** Let  $(V, E)$  and  $(V', E')$  be graphs. The graph  $(V', E')$  is called a subgraph of  $(V, E)$  if  $V' \subseteq V$  and  $E' \subseteq E$ , **i.e.** if  $(V', E')$  consists of a subset  $V'$  of the vertices of  $(V, E)$  and a subset  $E'$  of edges  $(V, E)$  between vertices in  $V'$ .

**Example:** Star of David on the flag of Israel

$$V = \{a, b, c, d, e, f\}$$

$$E = \{ac, ce, ae, bf, fd, bd\}$$



2 triangle subgraphs of the star of David:

$$V' = \{a, c, e\} \quad E' = \{ac, ce, ae\}$$

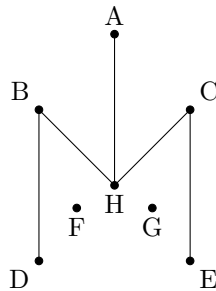
$$V'' = \{b, f, d\} \quad E'' = \{bf, fd, bd\}$$

## 25 Vertex Degrees

**Task:** Use numbers to understand incidence relationships.

**Definition:** Let  $(V, E)$  be a graph. The degree  $\deg v$  of a vertex  $v \in V$  is defined as the number of edges of the graph that are incidence to  $v$ , **i.e.** the number of edge with  $v$  as one of their endpoints.

**Example:**



$$\deg f = \deg g = 0$$

$$\deg d = \deg e = \deg a = 1$$

$$\deg b = \deg c = 2$$

$$\deg h = 3$$

**Definition:** A vertex of degree 0 is called an isolated vertex.

**Definition:** A vertex of degree 1 is called a pendent vertex.

**Theorem:** Let  $(V, E)$  be a graph. Then  $\sum_{v \in V} \deg v = 2\#(E)$ , where  $\sum_{v \in V} \deg v$  is the sum of the degrees of all the vertices of the graph, and  $\#(E)$  is the number of edges of the graph.

**Proof:**  $\sum_{v \in V} \deg v$  is the sum of all the entries in the adjacency matrix. Every edge  $vv' \in E$  contributes 2 to the sum  $\sum_{v \in V} \deg v$ , 1 for the vertex  $v$  and 1 for the vertex  $v' \Rightarrow$  each edge must be counted twice, so  $\sum_{v \in V} \deg v = 2\#(E)$ .

qed

**Corollary:**  $\sum_{v \in V} \deg v$  is an even integer.

**Proof:** Since  $\sum_{v \in V} \deg v = 2\#(E)$ , and  $\#(E) \in \mathbb{N}$ , the result follows.

qed

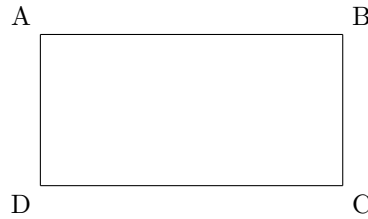
**Corollary:** In any graph, the number of vertices of odd degrees must be even.

**Proof:** Assume not, then  $\sum_{v \in V} \deg v$  is an odd integer as  $odd + even = odd \Rightarrow \Leftarrow$  to the previous corollary.

qed

**Definition:** A graph is called  $k$ -regular for some non-negative integer  $k$  if every vertex of the graph has degree equal to  $k$ .

**Example:** A rectangle is 2-regular.  
 $\deg a = \deg b = \deg c = \deg d = 2$ .



**Definition:** A graph  $(V, E)$  is called regular is  $\exists k \in \mathbb{N}$  s.t.  $(V, E)$  is  $k$ -regular.

**Corollary:** Let  $(V, E)$  be a  $k$ -regular graph. Then  $k\#(V) = 2\#(E)$  where  $\#(V)$  is the number of vertices and  $\#(E)$  is the number of edges.

**Proof:** By the theorem,  $\sum_{v \in V} \deg v = 2\#(E)$ , but  $(V, E)$  is  $k$ -regular  $\Rightarrow \deg v = k \forall v \in V$ . Therefore  $\sum_{v \in V} \deg v = \#(V) \cdot k = 2\#(E)$ .

qed

**Example:** Consider a complete graph  $(V, E)$  with  $n$  vertices.  $(V, E)$  is  $(n - 1)$ -regular because every vertex is adjacent to all the remaining  $(n - 1)$  vertices.

**Corollary:** A complete bipartite graph  $k_{p,q}$  is regular  $\Leftrightarrow p = q$

**Proof:** Recall that  $V = V_1 \cup V_2$   $V_1 \cap V_2 = \emptyset$  for a bipartite graph.  
 $\Leftarrow$  is  $p = q, \forall v \in V_1$  satisfies that  $\deg v = p = q$  and  $\forall v \in V_2$  satisfies that  $\deg v = p = q$  since the graph is complete  $\Rightarrow (V, E)$  is  $p$ -regular.  
 $\Rightarrow (V, E)$  is regular  $\Rightarrow \forall v \in V_1$  and  $\forall v' \in V_2$ ,  $\deg v = \deg v'$ , but  $(V, E)$  is complete  $\Rightarrow v$  is adjacent to all vertices in  $V_2$ , **i.e.**  $\deg v = \#(V_2)$  and  $v'$  is adjacent to all vertices in  $V_1$ , **i.e.**  $\deg v' = \#(V_1) \rightarrow \#(V_1) = \#(V_2)$

## 26 Walks, trails and paths

**Task:** Make rigorous the notion of traversing parts of a graph in order to understand its structure better.

**Definition:** Let  $(V, E)$  be a graph. A walk  $v_0 v_1 v_2 \dots v_n$  of length  $n$  in the graph from vertex  $a$  to vertex  $b$  is determined by a finite sequence  $v_0, v_1, v_2, \dots, v_n$  of vertices of the graph s.t.  $v_0 = a, v_n = b$  and  $v_{i-1} v_i$  is an edge of the graph for  $i = 1, 2, \dots, n$ .

**Definition:** A walk  $v_0 v_1 v_2 \dots v_n$  in a graph is said to traverse the edges  $v_{i-1} v_i$  and to pass through the vertices  $v_0, v_1, \dots, v_n$ . Length of walk = # of edges traversed  $\rightarrow$  the smallest possible number is two edges. As a result, we have the following definition:

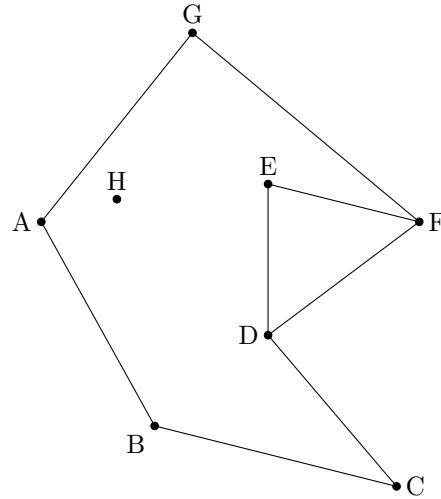
**Definition:** A walk that consists of a single vertex  $v \in V$  and has length two is called trivial.

**Definition:** Let  $(V, E)$  be a graph. A trail  $v_0 v_1 v_2 \dots v_n$  of length  $n$  in the graph from some vertex  $a$  to some vertex  $b$  is a walk of length  $n$  from  $a$  to  $b$  with the property that edges  $v_{i-1} v_i$  are distinct for  $i = 1, 2, \dots, n$ . In other words, a trail is a walk in the graph, which traverses edges of the graph at most once.

**Definition:** A walk, trail or path in a graph is called trivial if it is a walk of length two consist of a single vertex  $v \in V$ ; otherwise, the walk, trail, or path is called non-trivial.

**Example:**

1.  $h$  is a trivial walk/trail/path
2.  $defd$  is a trail, but not a path because we pass through the vertex  $d$  twice.
3.  $def$  is a path
4.  $gfdefdc$  is a walk but not a trail of a path

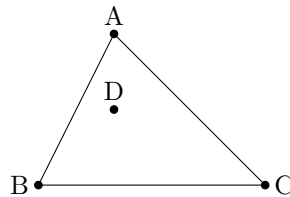


## 27 Connected Graphs

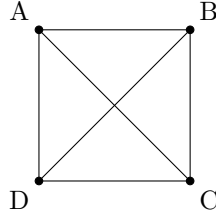
**Task:** Use the ideas above related to traversing parts of a graph in order to define a particularly important category of graphs.

**Definition:** An undirected graph  $(V, E)$  is called connected if  $\forall u, v \in V$  vertices,  $\exists$  path in the graph from  $u$  to  $v$ .

**Examples:** 1. Is not connected as  $d$  is not connected to any other vertex.



2. Is connected.  $\exists$  path between any two of the vertices.



**Theorem:** Let  $(V, E)$  be an undirected graph, and let  $u, v \in V$ .  $\exists$  path between  $u$  and  $v$  in the graph  $\Leftrightarrow \exists$  walk in the graph between  $u$  and  $v$ .

**Proof:**  $\Rightarrow$  trivial: A path is a walk.

$\Leftarrow \exists$  walk between  $u$  and  $v$ . Choose the walk of least length between  $u$  and  $v$ , (**i.e.**  $\nexists$  a walk of lower length than this one) and prove it is a path. Let this walk be  $a_0 a_1 \dots a_n$  with  $a_0 = u$  and  $a_n = v$ . Assume  $\exists j, k$  with  $a \leq j, k \leq n$  s.t.  $j < k$  and  $a_j = a_k$ , but then  $a_0 a_1 \dots a_j a_{j+1} \dots a_n$  would be a walk from  $u$  to  $v$  of strictly smaller length than  $a_0 a_1 \dots a_n$ .  $\Rightarrow \Leftarrow$  as we chose  $a_0 a_1 \dots a_n$  to be of minimal length  $\Rightarrow a_i \neq a_j \forall i, j$  s.t.  $0 \leq i, j \leq n \Rightarrow a_0 a_1 \dots a_n$  is a path between  $u$  and  $v$ .

qed

**Corollary:** An undirected graph  $(V, E)$  is connected  $\Leftrightarrow \forall u, v \in V \exists$  walk in the graph between  $u$  and  $v$ .

## 28 Components of a graph

**Task:** Divide a graph into subgraphs that are isolated from each other.

Let  $(V, E)$  be an undirected graph. We define a relation  $\sim$  on the set of vertices  $V$ , where  $a, b \in V$  satisfy  $a \sim b$  iff  $\exists$  walk in the graph from  $a$  to  $b$ .

**Lemma:** Let  $(V, E)$  be an undirected graph. The relation  $a \sim b$  or  $a, b \in V$ , which holds iff  $\exists$  walk in the graph between  $a$  and  $b$  is an equivalence relation.

**Proof:** We must show  $\sim$  is reflexive, symmetric, and transitive.

**Reflexive:**  $\forall v \in V, v \sim v$  since the trivial walk is a walk from  $v$  to itself.

**Symmetric:** If  $a \sim b$  for  $a, b \in V$ , then  $\exists$  walk  $v_0 v_1 \dots v_n$  where  $v_0 = a$  and  $v_n = b$ . This walk can be reversed to  $v_n v_{n-1} \dots v_1 v_0$ , which now goes from  $v_n = b$  to  $v_0 = a$ . Therefore,  $b \sim a$  as needed.



**Transitive:** If  $a \sim b$  and  $b \sim c$ , for  $a, b, c \in V$ , there  $\exists$  walk  $av_1v_2...v_{n-1}b$  from  $a$  to  $b$  and  $\exists$  walk  $bw_1w_2...w_{m-1}c$  from  $b$  to  $c$ . We put these two walks together (concatenate them) to yield the walk  $av_1v_2...v_{n-1}bw_1w_2...w_{m-1}c$  from  $a$  to  $c$ . Therefore  $a \sim c$ .

qed

The equivalence relation  $\sim$  on  $V$  partitions it into disjoint subsets  $v_1, v_2, \dots, v_p$ , where

1.  $v_1 \cup v_2 \cup \dots \cup v_p = V$
2.  $v_i \cap v_j = \emptyset$  if  $i \neq j$
3. Two vertices  $a, b \in v_i \Leftrightarrow a \sim b$ , i.e.  $\exists$  walk in  $(V, E)$  from  $a$  to  $b$

Note that an edge is a walk of length 1, so if  $a, b \in V$  satisfy that  $\exists ab \in E$ , then  $a$  and  $b$  belong to the same  $v_i$ . As a result, we can partition the set of edges as follows:

$$E_i \{ab \in E \mid a, b \in v_i\}$$

Clearly,  $E_1 \cup E_2 \cup \dots \cup E_p = E$  and  $E_i \cap E_j = \emptyset$  if  $i \neq j$ . Furthermore,  $(V_1, E_1), (V_2, E_2), \dots, (V_p, E_p)$  are subgraphs of  $(V, E)$ , and these subgraphs are disjoint since  $v_i \cap v_j = \emptyset$  and  $E_i \cap E_j = \emptyset$  if  $i \neq j$ . The subgraphs  $(V_i, E_i)$  are called the components (or connected components) of the graph  $(V, E)$ .

**Lemma:** The vertices and edges of any walk in an undirected graph are all contained in a single component of that graph.

**Proof:** Let  $v_0v_1...v_n$  be a walk in a graph  $(V, E)$ , then  $v_0v_1...v_r$  is a walk in  $(V, E) \forall r \quad 1 \leq r \leq n \Rightarrow v_0 \sim v_r \forall r \quad 1 \leq r \leq n \Rightarrow v_r$  belongs to the same component of the graph as  $v_0$ . The same is true for all the edges  $v_{i-1}v_i$  for  $1 \leq i \leq n$

qed

**Lemma:** Each component of an undirected graph is connected.

**Proof:** Let  $(V, E)$  be a graph and let  $(V_i, E_i)$  be any component of  $(V, E)$ .  $\forall u, v \in V_i$ , by definition  $\exists$  walk in  $(V, E)$  between  $u$  and  $v$ . By previous lemma, however, all vertices and edges of this walk are in  $(V_i, E_i) \Rightarrow$  the walk between  $u$  and  $v$  is a walk in  $(V_i, E_i)$ , but this assertion is true  $\forall u, v \in V_i \Rightarrow (V_i, E_i)$  is connected.

qed

## 28.1 Moral of the story

Any undirected graph can be represented as a disjoint union of connected subgraphs, namely its components  $\Rightarrow$  the study of undirected graphs reduces to the study of connected graphs, as components don't share either vertices or edges.

## 29 Circuits

**Task:** Use closed walks to understand the structure of graphs better.

**Definition:** Let  $(V, E)$  be a graph. A walk  $v_0v_1\dots v_n$  in  $(V, E)$  is called closed if  $v_0 = v_n$ , **i.e.** if it starts and ends at the same vertex.

**Definition:** Let  $(V, E)$  be a graph. A circuit is a nontrivial closed trail in  $(V, E)$ , **i.e.** a closed walk with no repeated edges passing through at least two vertices.

**Definition:** A circuit is called simple if the vertices  $v_0, v_1, v_2, \dots, v_{n-1}$  are distinct.

**NB:** This is the strangest condition regarding vertices that we can impose since  $v_0 = v_n$ .

**Alternative terminology:** Some authors use cycle to denote a simple circuit, which for others cycle denotes a circuit regardless of whether it is simple or not.

**Q:** When does a graph have simple circuits?

**A:** We can give 2 criteria for the existence of simple circuits:

1. Every vertex has degree  $\geq 2$ .
2.  $\forall u, v \in V$  s.t.  $\exists$  2 distinct paths from  $u$  to  $v$ .

**Theorem:** If  $(V, E)$  has no isolated or pendant vertices, **i.e.**  $\forall v \in V$   $\deg v \geq 2$ , then  $(V, E)$  contains at least one simple circuit.

**Proof:** Consider all paths in  $(V, E)$ . The maximum length of a path is  $|V| - 1$  since a path of length  $p$  passed through  $p+1$  vertices. Take a path  $v_0v_1\dots v_m$  in  $(V, E)$  of maximum length, **i.e.** any other path in  $(V, E)$  has length  $\leq m = \text{length of } v_0v_1\dots v_m$ . Now consider the vertex  $v_m$ .  $\deg v_m \geq 2$  by assumption. We know  $v_{m-1}$  is adjacent to  $v_m$  since the edge  $v_{m-1}v_m$  is part of the path  $v_0v_1\dots v_m$ , but  $\deg v_m \geq 2$  means  $\exists w \in V$  s.t.  $wv_m \in E$ . If  $w \neq v_i$  for  $0 \leq i \leq m-2$ , then  $v_0v_1\dots v_mv_w$  is a path in  $(V, E)$  longer than  $v_0v_1\dots v_m \Rightarrow$  to the fact that  $v_0v_1\dots v_m$  was chosen of maximal length. Therefore,  $w = v_i$  for some  $0 \leq i \leq m-2$ , but then  $v_iv_{i+1}\dots v_mv_i$  is a simple circuit in the graph.

qed

**Theorem:** Let  $(V, E)$  be an undirected graph and let  $u, v \in V$  be vertices s.t.  $u \neq v$  and  $\exists$  at least two distinct paths in  $(V, E)$  from  $u$  to  $v$ . Then the graph contains at least one.

**Theorem:** If  $(V, E)$  has no isolated or pendant vertices, **i.e.**  $\forall v \in V$   $\deg v \geq 2$ , then  $(V, E)$  contains at least one simple circuit.

**Proof:** Consider all paths in  $(V, E)$ . The maximum length of a path is  $\#(V) - 1$  since a path of length  $p$  passes through  $p+1$  vertices. Take a path  $v_0v_1\dots v_m$  in  $(V, E)$  of maximum length, **i.e.** any other path in  $(V, E)$  has length  $\leq m = \text{length of } v_0v_1\dots v_m$ . Now consider the vertex  $v_m$ .  $\text{def } v_m \geq 2$  by assumption. We know  $v_{m-1}$  is adjacent to  $v_m$  since the edge  $v_{m-1}v_m$  is part of the path  $v_0v_1\dots v_m$ , but  $\text{def } v_m \geq 2$  means  $\exists w \in V$  s.t.  $ww_m \in E$ . If  $w \neq v_i$  for  $0 \leq i \leq m-2$ , then  $v_0v_1\dots v_mw$  is a path in  $(V, E)$  longer than  $v_0v_1\dots v_m \Rightarrow \Leftarrow$  to the fact that  $v_0v_1\dots v_m$  was chosen of maximal length. Therefore,  $w = v_i$  for some  $0 \leq i \leq m-2$ , but then  $v_iv_{i+1}\dots v_mv_i$  is a simple circuit in the graph.

qed

**Theorem:** Let  $(V, E)$  be an undirected graph and let  $u, v \in V$  be vertices s.t.  $u \neq v$  and  $\exists$  at least two distinct paths in  $(V, E)$  from  $u$  to  $v$ . Then the graph contains at least one simple circuit.

**Proof:** Let  $a_0a_1a_2\dots a_m$  and  $b_0b_1\dots b_n$  be the two distinct paths in the graph between  $u$  and  $v$ , **i.e.**  $a_0 = b_0 = u$  and  $a_m = b_n = v$ . Let  $m \leq n$ . Since the paths are distinct  $\exists i$  with  $0 \leq i \leq m$  s.t.  $a_1 \neq b_i$ . Choose the smallest  $i$  for which  $a_i \neq b_i$ , **i.e.**  $a_0 = b_0, a_1 = b_1, \dots, a_{i-1} = b_{i-1}$ , but  $a_i \neq b_i$ . We have thus eliminated the redundencies at the beginning of the paths. We now need to eliminate redundencies at the other end of the paths. We know  $a_m = b_n$  so  $a_j \in \{b_k \mid i-1 < k \leq n\}$  is certainly satisfied for  $j = m$ , but we want to choose the smallest index for which this condition is satisfied. Let this index be  $p \Rightarrow a_p \in \{b_k \mid i-1 < k \leq n\}$ , **i.e.**  $a_p = b_s$  for some  $s$  s.t.  $i-1 < s \leq n$ . Since  $p$  is the smallest index satisfying  $a_p \in \{b_k \mid i-1 < k \leq n\}$ ,  $a_i, a_{i+1}, \dots, a_{p-1} \notin \{b_k \mid i-1 < k \leq n\} \Rightarrow$

$$\underbrace{a_{i-1}a_i\dots a_p}_{\text{indices running in increasing order}} \quad (= b_s) \quad \underbrace{b_{s-1}\dots b_i}_{\text{indices running in decreasing order}} \quad a_{i-1} (=$$

$b_{i-1})$  is a simple circuit in  $(V, E) \Rightarrow (V, E)$  has at least one simple circuit.

qed

## 30 Eulerian trails and circuits

**Task:** Look at trails and circuits that traverse every edge of a graph. Derive criteria when such trails and circuits exist.

**Definition:** An Eulerian trail in a graph is a trail that traverses every edge of that graph. In other words, an Eulerian trail is a walk that traverses every edge of the graph exactly once.

Trail  $\Rightarrow$  an edge is traversed at most once.

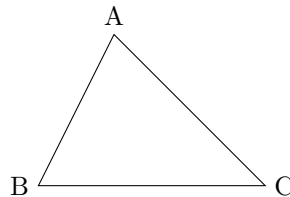
Eulerian  $\Rightarrow$  every edge is traversed.

**Definition:** An Eulerian circuit in a graph is a circuit that traverses every edge of the graph.

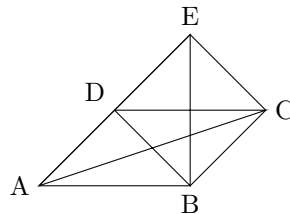
**Origin of the terminology:** Eulerian comes from the swiss mathematician Leonhard Euler (1707-1783) who solved the problem of the seven bridges of Königsberg/Kaliningrad (Then Prussia, now Russia) over the river Prugel in 1736. His negative solution is considered the beginning of graph theory as a subfield of mathematics. We will rederive Euler's results shortly. Google to see the configuration of the bridges on the river Prugel.

**Examples:**

1.  $abc$  is an Eulerian trail and an Eulerian circuit. The triangle is  $k_3$ .



2. Consider  $k_5$ , the complete graph with 5 vertices.  
 $eabecdbcade$  is an Eulerian circuit.



In both cases, the degree of the vertices is even for all vertices. We'll see this property is important and derive other necessary and sufficient conditions for the existence of Eulerian trails and circuits.

**Theorem:** Let  $(V, E)$  be a graph, and let  $v_0v_1\dots v_m$  be a trail in  $(V, E)$ . Let  $v \in V$  be a vertex, then the number of edges of the trail incident to  $v$  is even except when the trail is not closed and the trail starts and finishes at  $v$ , in which case the number of edges of the trail incident to the vertex  $v$  is odd.

**Proof:** Note that 0 is an even integer as  $0 = 2 \times 0$ .

**Case 1:**  $v \neq v_0 \wedge v \neq v_m$ . If the trail does not pass through  $v$ , the number of edges incident to  $v$  belonging to the trail is 0, which is even.

If the trail passes through  $v$ , then edges of the trail incident to  $v$  are of the form  $v_{i-1}v_i$  and  $v_iv_{i+1}$  with  $v = v_i$  and  $0 < i < m$ . Therefore, the number of edges of the trail incident to  $v$  equals twice the number of integers  $i$  among  $1, 2, \dots, m-1$  ( $0 < i < m$ ) s.t.  $v = v_i \Rightarrow$  the number of even.

**Case 2:**  $v = v_0$  and the trail is not closed, **i.e.**  $v_m \neq v_0$ . The edges incident to  $v$  are  $v_0v_1$  along with  $v_{i-1}v_i$  and  $v_iv_{i+1}$  whenever  $v = v_i$ , hence  $1+2 \cdot \#(\text{instances when } v = v_i)$ , which is odd.

**Case 3:**  $v = v_m$  and the trail is not closed, **i.e.**  $v_m \neq v_0$ . Repeat the argument in case 2 with  $v_{m-1}v_m$  replacing  $v_0v_1$  to get that the number of edges incident to  $v$  is odd.

**Case 4:** The trail is closed and  $v = v_0 = v_m$ . The edges incident to  $v$  are  $v_0v_1, v_{m-1}v_m$  as well as  $v_{i-1}v_i$  and  $v_iv_{i+1}$  for each  $i$  s.t.  $v = v_i \Rightarrow$  once again, the number of edges incident to  $v$  is even.

qed

**Corollary 1:** Let  $v$  be a vertex of the graph. Given any circuit in the graph, the number of edges incident to  $v$  traversed by that circuit is even.

**Proof:** Apply the theorem to  $v_0v_1\dots v_m$  s.t.  $v_0 = v_m$ . We deduce that the number of edges incident to  $v$  is even.

**Corollary 2:** If a graph admits an Eulerian circuit, then the degree of every vertex of that graph must be even.

**Proof:** Let  $(V, E)$  be the graph.  $\forall v \in V$ , the number of edges of any Eulerian circuit incident to  $v$  is even by the previous corollary. Since an Eulerian circuit by definition traverses every edge of the graph, every edge incident to  $v$  is an edge of the Eulerian circuit  $\Rightarrow \deg v$  is even  $\forall v \in V$  (**NB:**  $\deg v$  could be zero if  $v$  is an isolated vertex).

**Example:** By the previous corollary,  $K_4$ , the complete graph on four vertices, cannot have an Eulerian circuit since  $\forall v$  in  $K_4$ ,  $\deg v = 3$  ( $K_4$  is 3-regular as we observed in a previous lecture).

**Corollary 3:** If a graph admits an Eulerian trail that is not a circuit, then the degrees of exactly two vertices of the graph must be odd, and the degrees of the remaining vertices must be even. The vertices with odd degrees are exactly the initial and final vertices of the Eulerian trail.

**Proof:** By the theorem, the initial and final vertices of the Eulerian trail have odd degree, whereas all vertices in between have even degrees.

qed

Next, prove the converse of corollary 2: A non-trivial connected graph has an Eulerian circuit if the degree of each of its vertices is even. The proof is carried out in a series of lemmas:

**Lemma A:** If the degree of each vertex is even, then  $\exists$  a circuit.

**Lemma B:** If the degree of each vertex is even, if  $\exists$  circuit, and if  $\exists$  edges not in the circuit incident to a vertex in the circuit, we can construct another circuit.

**Lemma C:** If we have two circuits with at least one vertex in common, we can combine them.

**Lemma D:** A criterion for when a trail is Eulerian in a connected graph.

**Lemma A:** Let  $vw$  be an edge of a graph in which the degree of every vertex is even, then  $\exists$  circuit of the graph that traverses the edge  $vw$ .

**Proof:** We construct the circuit starting with the edge  $vw$ . Let  $v_0 = v$  and  $v_1 = w$ . Let  $v_0v_1\dots v_k$  be any trail of length  $k \geq 1$  traversing the edge  $vw$ . Suppose  $v_k \neq v = v_0$ . As we proved in the previous theorem, since  $v_k$  is an endpoint of a non-closed trail, then the number of edges of the trail incident to  $v_k$  is odd, but  $\deg v_k$  is even  $\Rightarrow \exists$  edge of the graph incident to  $v_k$  that is not traversed by the trail  $v_0v_1\dots v_k$ . Let  $v_kv_{k+1}$  be this edge, then  $v_0v_1\dots v_kv_{k+1}$  is a trail of length  $k+1$  that starts at  $v$  and traverses  $vw$ . Since every edge of the graph is traversed at most once by a trail, the length of any trail in the graph cannot be greater than the number of edges of the graph  $\#(E)$ . We have shown above that if our trail is not closed, then it can be extended. By successive extensions, we will eventually have constructed a trail that cannot be extended (in at most  $\#(E) - 1$  steps). Therefore, that trail must be closed. As the edge  $vw$  is traversed, this trail is nontrivial  $\Rightarrow$  it is a circuit.

qed

**Lemma B:** Let  $(V, E)$  be a connected graph s.t.  $\forall v \in V$ ,  $\deg v$  is even, and let some circuit  $v_0v_1\dots v_{m-1}v_0$ , then  $\exists$  another circuit in  $(V, E)$  passing through  $v_i$  that does not traverse any edge traversed by  $v_0v_1\dots v_{m-1}v_0$ .

**Proof:** Let  $E'$  be the set of edges not traversed by  $v_0v_1\dots v_{m-1}v_0$ .  $(V, E')$  is a subgraph of  $(V, E)$ .  $\forall v \in V$ ,  $\#$  of edges of  $v_0v_1\dots v_{m-1}v_0$  incident to  $v = d(v) - d'(v)$ , where  $d(v) = \deg(v) = \#$  of edges in  $(V, E)$  incident to  $v$  and  $d'(v) = \#$  of edges in  $(V, E')$  incident to  $v$ . By Corollary 1,  $d(v) - d'(v)$  is even, but by assumption  $d(v) = \deg v$  is even  $\Rightarrow d'(v)$  is even  $\Rightarrow$  the degree of every vertex in the subgraph  $(V, E')$  is even. Now consider the vertex  $v_i$  in the statement of Lemma B. Some but not all edges incident to  $v_i$  are traversed by  $v_0v_1\dots v_{m-1}v_0 \Rightarrow d'(v_i) > 0$ , i.e. at least one edge incident to  $v_i$  is in the subgraph  $(V, E')$ . We are now exactly in the scenario described by Lemma A  $\Rightarrow$  by Lemma A,  $\exists$  circuit in  $(V, E')$  passing through  $v_i$ . This circuit is also a circuit in  $(V, E)$  as  $(V, E')$  is a subgraph of  $(V, E)$ , and since all of its edges are in  $E'$ , this other circuit does not traverse any edge traversed by  $v_0v_1\dots v_{m-1}v_0$ .

qed

**Lemma C:** Suppose that a graph contains a circuit of length  $m$  and a circuit of length  $n$ . Suppose also that no edges of the graph is traversed by both circuits, and that at least one vertex of the graph is common to both circuits, then the graph contains a circuit of length  $m + n$ .

**Proof:** Let  $v$  be a vertex of the graph that is common to both circuits. We assume both circuits start and finish at the vertex  $v$ . Let the first circuit be  $vv_1 \dots v_{m-1}v$ , and let the second circuit be  $vw_1w_2 \dots w_{n-1}v$ . We concatenate the two circuits obtaining a circuit  $vv_1 \dots v_{m-1}vw_1w_2 \dots w_{n-1}v$  of length  $m + n$

qed

**Lmma D:** Let  $(V, E)$  be a connected graph, and let some trail in this graph be given. Suppose that no vertex of the graph has the property that not all the edges of the graph incident to that vertex are traversed by the trail. Then the given trail is an Eulerian trail.

**Proof:** Let  $v_1$  be the set of vertices through which the trail passes, and let  $v_2$  be the set of vertices through which the trail does not pass.  $v = v_1 \cup v_2 \wedge v_1 \cap v_2 = \emptyset$ . The conclusion of Lemma D amounts to showing  $v_2 = \emptyset$ .  $\forall u \in v_1, u$  is incident to at least one edge traversed by the trail.  $\Rightarrow$  all edges incident to the vertices in  $v_1$  are traversed by the trail, but then every vertex in  $v$  adjacent to a vertex in  $v_1$  must belong to  $v_1 \Rightarrow$  no edge can join a vertex in  $v_1$  to a vertex in  $v_2$ . If  $v_2 \neq \emptyset$ , then  $\exists w \in v_2$ , but then  $w$  cannot be joined by a path to any vertex in  $v_1 \Rightarrow v_1 \wedge v_2$  are in different connected components of the graph  $\Rightarrow \Leftarrow$  since the graph is connected  $\Rightarrow$  it has only one connected component. Therefore,  $v_2 = \emptyset$ .

qed

Finally, we can prove Euler's theorem:

**Theorem** A nontrivial connected graph contains an Eulerian circuit if the degree of every vertex of the graph is even.

**Proof:** Let  $(V, E)$  be a non-trivial connected graph s.t.  $\forall v \in V$ ,  $\deg v$  is even. By Lemma A,  $(V, E)$  contains at least one circuit. It therefore contains a circuit of maximal length (i.e. at least as long as any other circuit in the graph). We seek to prove that this circuit of maximal length is indeed Eulerian.

If the graph contains some vertex  $v$  s.t. some but not all of the edges of the graph incident to  $v$  are traversed by the circuit of maximal length, and  $v$  is a vertex on the circuit of maximal length, then by Lemma B  $\exists$  a second circuit in  $(V, E)$  passing through  $v$ , which would not traverse any edge traversed by the circuit of maximal length. By Lemma C, however, we can concatenate the two circuits, obtaining a circuit of length strictly greater than the length of the circuit of maximal length  $\Rightarrow \Leftarrow$  we conclude no vertex that belongs to the circuit of maximal length has the property

that not all edges incident to it are traversed by the circuit of maximal length. Since  $(V, E)$  is connected, by Lemma D, the circuit of maximal length must be Eulerian.

qed

Corollary 2 along with this theorem together gives us:

**Theorem:** A non-trivial connected graph has an Eulerian circuit  $\Leftrightarrow$  the degree of each of its vertices is even.

**Corollary:** Suppose a connected graph has exactly two vertices whose degree is odd.  $\exists$  an Eulerian trail in the graph joining the two vertices with odd degrees.

**Proof:** We reduce this case to the previous one by embedding the graph  $(V, E)$  with vertices  $v, w$  that have odd degree into a graph  $(V', E')$  s.t.  $v' = v \cup \{u\}$  for  $u \notin V$  and  $E' = E \cup \{uv, uw\}$ .  $(V, E)$  is a subgraph of  $(V', E')$  and  $(V', E')$  is connected and each one of its vertices has even degree by construction. By the theorem we just proved,  $(V', E')$  has an Eulerian circuit. We reorder the vertices so that the final two edges are the two added edges  $wu$  and  $uv$ . We not delete the edges  $wu$  and  $uv$  to obtain an Eulerian trail in the original graph  $(V, E)$  from  $v$  to  $w$ .

qed