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1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A proposition is any declarative sentence that is either true or false.

1.1 Connectives

	Connectives	Notation in Maths
and	\wedge	
or	\vee	"Inclusive or"
not	\neg	Sometimes denoted \sim
implies	\rightarrow	if/then; called implication \Rightarrow
if and only if	\leftrightarrow	Called equivalence \Leftrightarrow

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	$P \wedge Q$	P	Q	$P \vee Q$	P	$\neg P$	P	Q	$P \rightarrow Q$	P	Q	$P \leftrightarrow Q$
F	F	F	F	F	F	F	T	F	F	T	F	F	T
F	T	F	F	T	T	F	T	F	T	T	F	T	F
T	F	F	T	F	T	T	F	T	F	F	T	F	F
T	T	T	T	T	T	T	T	T	T	T	T	T	T

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{ll}
 (P \rightarrow Q) & \leftrightarrow (\neg P \vee Q) \\
 (P \leftrightarrow Q) & \leftrightarrow [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\
 \neg(P \wedge Q) & \leftrightarrow (\neg P \vee \neg Q) \\
 \neg(P \vee Q) & \leftrightarrow (\neg P \wedge \neg Q)
 \end{array}
 \left. \vphantom{\begin{array}{l} (P \rightarrow Q) \\ (P \leftrightarrow Q) \\ \neg(P \wedge Q) \\ \neg(P \vee Q) \end{array}} \right\} \text{De Morgan Laws}$$

As a result, \neg and \vee together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the sheffer stroke $P|Q$ (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ since $\neg P \leftrightarrow P|P$ and $P \vee Q \leftrightarrow (P|P) | (Q|Q)$.

Recall is $P \rightarrow Q$ is a given implication, $Q \rightarrow P$ is called the converse or $P \rightarrow Q$.
 $\neg Q \rightarrow \neg P$.

1.3 Indirect Arguments/Proofs by Contradiction/Reductio as absurdum

Based on the tautology $(P \rightarrow Q) \leftrightarrow (\neg Q \rightarrow \neg P)$

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed as fraction form $\frac{a}{b}$. Let us **assume** that our fraction is in the lowest term, **i.e.** their only common divisor is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by b^2 yields

$$2b^2 = a^2$$

Since $a^2 = 2b^2$, we can conclude that a^2 is even because whatever the value of b^2 has to be multiplied by 2. Is a^2 is even, then a is also even. Since a is even, no matter what the value of a is, we can always find an integer that if we divide a by 2, it is equal to that integer. If we let that integer be k , then $\frac{a}{b} = k$ which means that $a = 2k$.

Substituting the value of $2k$ to a , we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. dividing both sides by 2 we have $b^2 = 2k^2$. That means that the value b^2 is even, since whatever the value of k you have to multiply it by 2. Again, is b^2 is even, then b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that $\frac{a}{b}$ has no common divisor except 1. Since we found a contradiction, our assumption is, therefore, false. Hence the theorem is true.

qed

2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variable x, y, z , so their truth value may change depending on which values these variables assume:
 $P(x), Q(x, y), R(x, y, z)$

2.1 Introduce quantifiers

2.1.1 \exists existential quantifier

Syntax: $\exists xP(x)$

Definition: $\exists xP(x)$ is true if $P(x)$ is true for some value of x ; it is false otherwise.

2.1.2 \forall universal quantifier

Syntax: $\forall xP(x)$

Definition: $\forall xP(x)$ is true if $P(x)$ is true for all allowable values of x . It is false otherwise.

2.1.3 $\exists!$ for one and only one

Syntax: $\exists! xP(x)$

Definition: $\exists! xP(x)$ is true if $P(x)$ is true for exactly one value of x and false for all other values of x ; otherwise, $\exists! xP(x)$ is false.

2.2 Alternation of Quantifiers

$$\forall x \exists y \forall z \quad P(x, y, z)$$

NB: The order cannot be exchanged as it might modify the truth values of the statement (think of examples with two quantifiers).

2.3 Negation of Quantifiers

$$\begin{aligned}\neg(\exists xP(x)) &\leftrightarrow \forall x\neg P(x) \\ \neg(\forall xP(x)) &\leftrightarrow \exists x\neg P(x)\end{aligned}$$

3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc. Set theory started around 1870's \rightarrow late development in mathematics but now taught early in one's maths education due to Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element x is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. \mathbb{N} the set of natural numbers starting at 0.

\mathbb{N} is defined via the following two axioms:

- (a) $0 \in \mathbb{N}$
- (b) if $x \in \mathbb{N}$ then $x + 1 \in \mathbb{N}$ ($x \in \mathbb{N} \rightarrow X + A \in \mathbb{N}$)
- 3. \mathbb{R} set of real numbers also introduced axiomatically
 - \mathbb{R} the set of real numbers.
 - (a) Additive closure: $\forall x, y \exists z (x + y = z)$
 - (b) Multiplicative closure: $\forall x, y, \exists z (x \times y = z)$
 - (c) Additive associativity: $x + (y + z) = (x + y) + z$
 - (d) Multiplicative associativity: $x \times (y \times z) = (x \times y) \times z$
 - (e) Additive commutativity: $x + y = y + x$
 - (f) Multiplicative commutativity: $x \times y = y \times x$
 - (g) Distributivity: $x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$
 - (h) Additive identity: There is a number, denoted 0, such that or all $x, x + 0 = x$
 - (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$
 - (j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$
 - (k) Multiplicative inverses: For every nonzero x there is a number, denoted x^{-1} , such that $x \times x^{-1} = x^{-1} \times x = 1$
 - (l) $0 \neq 1$
 - (m) Irreflexivity of $<$: $\sim (x < x)$
 - (n) Transitivity of $<$: If $x < y$ and $y < z$, then $x < z$
 - (o) Trichotomy: Either $x < y, y < x$, or $x = y$
 - (p) If $x < y$, then $x + y < y + z$
 - (q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$
 - (r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.
- 4. \emptyset is the empty set (The set with no elements).

Definition: Let A, B be sets. $A=B$ if and only if all elements of A are elements of B and all elements of B are elements of A,
 i.e. $A = B \leftrightarrow [\forall x(x \in A \rightarrow x \in B)] \cap [\forall y(y \in B \rightarrow y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
NB: order is irrelevant.
 $A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$
2. The formulaic/set builder method: give a formula that generates all elements of the set.
 $A = \{x \in \mathbb{N} \mid 0 \leq x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using \mathbb{N} and the set-builder method, we can define:

$$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$$

$n = 0$ in any natural numbers \Rightarrow we generate all of \mathbb{N}

$m = 0$ in any natural number \Rightarrow we generate all negative integers

$$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$$

Definition: A set A is called finite if it has a finite number of elements; otherwise it is called infinite.

4 Set Operations

Task: Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement and difference.

Definition: Let A, B be sets. A is a subset of B . If all elements of A are elements of B , **i.e.** $\forall x(x \in A \rightarrow x \in B)$. We denote that A is a subset of B by $A \subseteq B$

Example: $\mathbb{N} \subseteq \mathbb{Z}$

Definition: Let A, B be sets. A is a proper subset of B if $A \subseteq B \wedge A \neq B$, **i.e.** $A \subseteq B \wedge \exists x \in B \text{ s.t. } x \notin A$.

A proper subset is always a subset, but a subset is not always a proper subset.

Notation: $A \subset B$

Example: $\mathbb{N} \subset \mathbb{Z}$ since $\exists -1 \in \mathbb{Z}$

NB: $\forall A$ a set $\emptyset \subseteq A$

Recall: $B \subseteq C$ means $\forall x(x \in B \rightarrow x \in C)$, but \emptyset has no elements so in $\emptyset \subseteq A$ the quantifier \forall operates on a domain with no elements. Clearly, we need to give meaning to \exists and \forall on empty sets.

Boolean Convention

\forall is true on the empty set
 \exists is false on the empty set

} Consistent with common sense

Definition: Let A, B be two sets. The union $A \cup B = \{x \mid x \in A \vee x \in B\}$

Definition: Let A, B be two sets. The intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be sets. A and B are called disjoint if $A \cap B = \emptyset$

Definition Let A, B be two sets. $A - B = A \setminus B = \{a \mid x \in A \wedge x \notin B\}$

Examples:

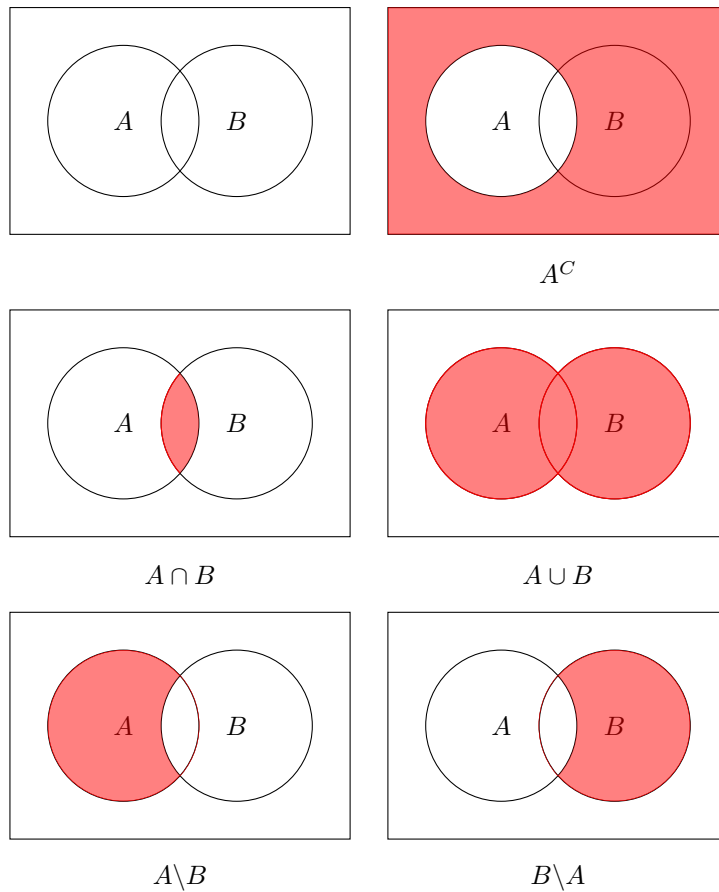
$A = \{1, 2, 5\}$	$B = \{1, 3, 6\}$
$A \cup B = \{1, 2, 3, 5, 6\}$	$A \cap B = \{1\}$
$A \setminus B = \{2, 5\}$	$B \setminus A = \{3, 6\}$

Definition: Let A, U be sets s.t. $A \subseteq U$. The complement of A in $U = U \setminus A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Remark: The notation A^C is unambiguous only if the universe U is clearly defined or understood.

4.1 Venn Diagrams

Schematic representation of set operations.



4.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation
\wedge	intersection \cap
\vee	union \cup
\neg	complement $()^C$

As a result, various properties of set operations become obvious:

- Commutativity
 - $A \cap B = B \cap A$
 - $A \cup B = B \cup A$
- Associativity
 - $(A \cup B) \cup C = A \cup (B \cup C)$
 - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity
 - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
 - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- De Morgan Laws in Set Theory
 - $(A \cap B)^C = A^C \cup B^C$
 - $(A \cup B)^C = A^C \cap B^C$
- Involutivity of the Complement
 - $(A^C)^C = A$

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
 - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- Criterion for proving equality of sets
 - $A = B \leftrightarrow A \subseteq C \wedge B \subseteq A$
- Criterion for proving non-equality of sets
 - $A \neq B \leftrightarrow (A \setminus B) \cup (B \setminus A) \neq \emptyset$

4.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \setminus B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

Show $(A \cap B) \cup (A \setminus B) \subseteq A$: $\forall x \in (A \cap B) \cup (A \setminus B), x \in (A \cap B)$ or $x \in A \setminus B$.

If $x \in (A \cap B)$ then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \setminus B$, then by definition $x \in A$ and $x \notin B$ so definitely $x \in A$. In both cases, $x \in A$ as needed.

Show $A \subseteq (A \cap B) \cup (A \setminus B)$: $\forall x \in A$, we have two possibilities, namely $x \in B$

or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \in A \setminus B$. In both cases, $x \in (A \cap B)$ or $x \in (A \setminus B)$ so $x \in (A \cap B) \cup (A \setminus B)$ as needed.

qed

5 The Power Set

Task: Understand what the power set of a set A is.

Definition: Let A be a set. The power set of A denoted $P(A)$ is the collection of all the subsets of A .

Recall: $\emptyset \subseteq A$. It is also clear from the definition of a subset that $A \subseteq A$.

Examples:

1. $A = \{0, 1\}$
 $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2. $A = \{a, b, c\}$
 $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3. $A = \emptyset$
 $P(A) = \{\emptyset\}$
 $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

NB: \emptyset and $\{\emptyset\}$ are different objects. \emptyset has no elements, whereas $\{\emptyset\}$ has one element.

Remark: $P(A)$ and A are viewed as living in separate worlds to avoid phenomena like Russell' paradox.

Q: If A has n elements, how many elements does $P(A)$ have?

A: 2^n

Theorem: Let A be a set with n elements, then $P(A)$ contains 2^n elements.

Proof: Based on the on/off switch idea.

$\forall x \in A$, we have two choices: either we include x in the subset or we don't (on vs off switch). A has n elements \Rightarrow we have 2^n subsets of A .

qed

Alternate Proof: Using mathematical induction.

NB: It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could limit, else what would its power set be?

6 Cartesian Products

Task: Understand sets like \mathbb{R}^1 in a more theoretical way.

Recall from Calculus:

$$\mathbb{R} = \mathbb{R}^1 \ni x$$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_1)$$

\vdots

$$\underbrace{\mathbb{R} \times \mathbb{R}}_{n \text{ times}} = \mathbb{R}^n \ni (x_1, x_2, \dots, x_n)$$

These are examples of Cartesian products.

Definition: Let A, B be sets. The Cartesian product denoted by $A \times B$ consists of all ordered pairs (x, y) s.t. $x \in A \wedge y \in B$, i.e. $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Further Examples:

$$1. A = \{1, 3, 7\}$$

$$B = \{1, 5\}$$

$$A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$$

NB: The order in which elements in a pair matters: $(7, 1)$ is different from $(1, 7)$. This is why we call (x, y) an ordered pair.

$$2. A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \leftarrow \text{circle of radius 1}$$

$$B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = [-2, 2] \leftarrow \text{closed interval}$$

$$A \times B \leftarrow \text{cylinder of radius 1 and height 4}$$

6.1 Cardinality (number of elements) in a Cartesian product

If A has n elements and B has p elements, $A \times B$ has np elements.

Example:

1. $\#(A) = 3$ $A = \{1, 3, 7\}$
 $\#(B) = 2$ $B = \{1, 5\}$
 $\#(A \times B) = 3 \times 2 = 6$
2. Both A and B are infinite sets, so $A \times B$ is infinite as well.

Remark: We can define Cartesian products of any length, **e.g.** $A \times A \times B \times A$, $B \times A \times B \times A \times B$, etc. If all sets are finite, the number of elements is the product of the numbers of elements of each factor. If $\#(A) = 3$ and $\#(B) = 2$ as above, $\#(A \times B \times A \times A) = 3 \times 3 \times 3 \times 3 = 81$ and $\#(B \times A \times B) = 2 \times 3 \times 2 = 12$.

7 Relations

Task: Define subsets of Cartesian products with certain properties. Understand the predicates " $=$ " (equality) and other predicates in predicate logic in a more abstract light.

Start with $x = y$. The elements x is some notation R to y (equality in this case). We can also denote it as xRy or $(x, y) \in E$

Let x, y in \mathbb{R} , then $E = \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

The "diagonal" in $\mathbb{R} \times \mathbb{R}$ gives exactly the elements equal to each other.

More generally:

Definition: Let A, B be sets. A subset of the Cartesian product $A \times B$ is called a relations between A and B . A subset of the Cartesian product $A \times A$ is called a relations on A .

Remark: Note how general this definition is. To make it useful for understanding predicates, we will need to introduce key properties relations can satisfy.

Example: $A = \{1, 3, 7\}$ $B = \{1, 2, 5\}$

We can define a relation S on $A \times B$ by $S = \{(1, 1), (1, 5), (3, 2)\}$. This means $1S1$, $1S5$ and $3S2$ and no other ordered pairs in $A \times B$ satisfy S .

Remark: The relations we defined involve 2 elements, so they are often called binary relations in the literature.

8 Equivalence Relations

Task: Define the most useful kind of relation.

Definition: A relation R on a set A is called

1. reflexive iff (if and only if) $\forall x \in A, xRx$
2. symmetric iff $\forall x, y \in A, xRy \rightarrow yRx$
3. transitive iff $\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$

An equivalence relation on A is a relation that is reflexive, symmetric and transitive.