The Countability of Sets

Task: Understand what it means for a set to be countable, countably infinite and uncountably infinite. Show that the set of all languages over a finite alphabet is uncountably infinite, wheras the set ofa ll regular languages over a finite alphabet is countably infinite.

We want to understand sizes of sets. In the unit on functions last term, when we looked at functions defined on finite sets, we wrote down a set A with m elements as $A = \{a_1, \ldots, a_n\}$. This notation mimics the process of counting: a_1 is the first element of A, a_2 is the second element of A, and so on up to a_n is the $n^t h$ element of A. In other words, another way of saying A in a set of n elements is that there exists a bijective function $f : A \to \{1, 2, \ldots, n\}$.

Definition: A set A has n elements $\leftrightarrow \exists f : A \to J_n$ a bijection.

NB: This definition works $\forall n \geq 1, n \in \mathbb{N}^*$

Notation: $\exists f : A \to J_n$ a bijection is denoted as $A \sim J_n$. More generally, $A \sim B$ means $\exists f : A \to B$ a bijection, and it is a relation on sets. In fact, it is an equivalence relation (check!). $[J_n]$ is the equivalence class of all sets A of size n, i.e. $\#(A) = n$.

Definition: A set A is finite if $A \sim J_n$ for some $n \in \mathbb{N}^*$ or $A = \emptyset$.

Definition: A set A is infinite if A is not finite.

Examples: $\mathbb{N}, \mathbb{Q}, \mathbb{R}$, etc.

To understand sizes of infinite sets, generalize the construction above. Let $J = \mathbb{N}^* = \{1, 2, \ldots\}$

Definition: A set A is uncountably infinite if A is neither finite nor countably infinite. In fact, we often treat together the cases A is finite or A is countably infinite since in both of these cases the counting mechanism that is so familiar to us works. Therefore, we have the following definition:

Definition: A set A is countable if A is finite ($A \sim J_n$ or $A = \emptyset$) or A is countably infinite ($A \sim J$).

There is a difference in the terminology regarding countability between CS sources (textbooks, articles, etc.) and maths sources. This is the dictionary:

CS	Maths
countable	at most countable
countably infinite	countable
uncountably infinite	uncountable

It's best to double check which terminology a source is using.

Goal: Characterize [$\mathbb{N}$], the equivalence class of countably infinite set, and [$\mathbb{R}$], the equivalence class of uncountably infinite sets the same size as $\mathbb{R}$.

Bad news: Both [$\mathbb{N}$] and [$\mathbb{R}$] consist of ininite sets.

Good news: We only care about these two equivalence classes.

NB: There are uncountably infinite sets of size bigger than [$\mathbb{R}$] that can be obtained from the power set construction, but it is unlikely you will see them in your CS coursework.

To characterize [$\mathbb{N}$], we need to recall the notion of a sequence:

Definition: A <u>sequence</u> is a set of elements $\{x_1, x_2, \ldots\}$ indexed by J, i.e. $\exists f :\to \{x_1, x_2, \ldots\}$ s.t. $f(n) = x_n \forall n \in J$.

Recall that sequences and their limits are used to define various notions in calculus (differentiation, integration, etc.). Also, calculators use sequences in order to compute with various rational and irrational numbers.