Contents

1 Review of Propositional Logic

Task: Recall enough propositional logic to see how it matches up with set theory.

Definition: A <u>proposition</u> is any declarative sentence that is either true or false.

1.1 Connectives

	Connectives		Notation in Maths
and	$\wedge$		
or	$\vee$	"Inclusive or"	
not	$\neg$	Sometimes denoted $\sim$	
implies	$\rightarrow$	if/then; called implication	$\Rightarrow$
if and only if	$\leftrightarrow$	Called equivalence	$\Leftrightarrow$

1.1.1 Truth Table of the Connectives

Let P, Q be propositions:

P	Q	P$\wedge$Q
F	F	F
F	T	F
T	F	F
T	T	T

P	Q	P$\vee$Q
F	F	F
F	T	T
T	F	T
T	T	T

P	$\neg$P
F	T
T	F

P	Q	P$\rightarrow$Q
F	F	T
F	T	T
T	F	F
T	T	T

P	Q	P$\leftrightarrow$Q
F	F	T
F	T	F
T	F	F
T	T	T

Priority of the Connectives

Highest to Lowest: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$

1.2 Important Tautologies

$$\begin{array}{rcl} (P \rightarrow Q) & \leftrightarrow & (\neg P \vee Q) \\ (P \leftrightarrow Q) & \leftrightarrow & [(P \rightarrow Q) \wedge (Q \rightarrow P)] \\ \neg(P \wedge Q) & \leftrightarrow & (\neg P \vee \neg Q) \\ \neg(P \vee Q) & \leftrightarrow & (\neg P \wedge \neg Q) \end{array} \left.\vphantom{\begin{array}{c} a \\ b \end{array}}\right\} \text{De Morgan Laws}$$

As a result, $\neg$ and $\vee$ together can be used to represent all of $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Less obvious: One connective called the sheffer stroke P|Q (which stands for "not both P and Q" or "P nand Q") can be used to represent all of $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ since $\neg$P $\leftrightarrow$ P|P and P $\vee$ Q $\leftrightarrow$ (P|P) | (Q|Q).

Recall if P$\rightarrow$Q is a given implication, Q$\rightarrow$P is called the <u>converse</u> or P$\rightarrow$Q. $\neg$Q $\rightarrow$ $\neg$P.

1.3 Indirect Arguments/Proofs by Contradiction/Reductio as absurdum

Based on the tautology (P→Q) ↔ (¬Q → ¬P)

Example: Famous argument that $\sqrt{2}$ is irrational.

Proof:

Suppose $\sqrt{2}$ is rational, then it can be expressed is fraction form $\frac{a}{b}$. Let us **assume** that our fraction is in the lowest term, **i.e.** their only common divisor is 1.

Then,

$$\sqrt{2} = \frac{a}{b}$$

Squaring both sides, we have

$$2 = \frac{a^2}{b^2}$$

Multiplying both sides by B^2 yields

$$2b^2 = a^2$$

Since a^2b^2, we can conclude that a^2 is even because whatever the value of b^2 has to be multiplied by 2. If a^2 is even, then a is also even. Since a is even, no matter what the value of a is, we can always find an integer that if we divide a by 2, it is equal to that integer. If we let that integer be k, then $\frac{a}{b} = k$ which means that $a = 2k$.

Substituting the value of $2k$ to a, we have $2b^2 = (2k)^2$ which means that $2b^2 = 4k^2$. dividing both sides by 2 we have $b^2 = 2k^2$. That means that the value b^2 is even, since whatever the value of k you have to multiply it by 2. Again, is b^2 is even, then b is even.

This implies that both a and b are even, which means that both the numerator and the denominator of our fraction are divisible by 2. This contradicts our **assumption** that $\frac{a}{b}$ has no common divisor except 1. Since we found a contradiction, our assumption is, therefore, false. Hence the theorem is true.

qed

2 Predicate logic and Quantifiers

Task: Understand enough predicate logic to make sense of quantified statements.

In predicate logic, propositions depend on variable x, y, z, so their truth value may change depending on which values these variables assume: $P(x), Q(x, y), R(x, y, z)$

2.1 Introduce quantifiers

2.1.1 $\exists$ existential quantifier

Syntax: $\exists x P(x)$

Definition: $\exists x P(x)$ is true if $P(x)$ is true or some value of x; it is false otherwise.

2.1.2 $\forall$ universal quantifier

Syntax: $\forall x P(x)$

Definition: $\forall x P(x)$ is true if $P(x)$ is true for all allowable values of x. It is false otherwise.

2.1.3 $\exists!$ for one and only one

Syntax: $\exists! x P(x)$

Definition: $\exists! x P(x)$ is true if $P(x)$ is true for exactly one value of x and false for all often values of x; otherwise, $\exists! x P(x)$ is false.

2.2 Alternation of Quantifiers

$\forall x \exists y \forall z \quad P(x, y, z)$

NB: The order <u>cannot</u> be exchanged as it might modify the truth values of the statement (think of examples with two quantifiers).

2.3 Negation of Quantifiers

$$\neg(\exists x P(x)) \quad \leftrightarrow \quad \forall x \neg P(x)$$
$$\neg(\forall x P(x)) \quad \leftrightarrow \quad \exists x \neg P(x)$$

3 Set Theory

Task: Understand enough set theory to make sense of other mathematical objects in abstract algebra, graph theory, etc. Set theory started around 1870's $\rightarrow$ late development in mathematics but now taught early in one's maths education due to Bourbaki school.

Definition: A set is a collection of objects. $x \in A$ means the element X is in the set A (**i.e.** belongs to A).

Examples:

1. All students in a class.
2. $\mathbb{N}$ the set of natural numbers starting at 0.

   $\mathbb{N}$ is defined via the following two axioms:

   (a) $0 \in \mathbb{N}$
   (b) if $x \in \mathbb{N}$ then $x + 1 \in \mathbb{N}$ ($x \in \mathbb{N} \to X + A \in \mathbb{N}$)

3. $\mathbb{R}$ set of real numbers also introduced axiomatically

   $\mathbb{R}$ the set of real numbers.

   (a) Additive closure: $\forall x, y \exists z(x + y = z)$
   (b) Multiplicative closure: $\forall x, y, \exists z(x \times y = z)$
   (c) Additive associativity: $x + (y + z) = (x + y) + z$
   (d) Multiplicative associativity: $x \times (y \times z) = (x \times y) \times z$
   (e) Additive commutativity: $x + y = y + x$
   (f) Multiplicative commutativity: $x \times y = y \times x$
   (g) Distributivity: $x \times (y + z) = (x \times y) + (x \times z)$ and $(y + z) \times x = (y \times x) + (z \times x)$
   (h) Additive identity: There is a number, denoted 0, such that or all $x, x + 0 = x$
   (i) Multiplicative identity: There is a number, denoted 1, such that for all $x, x \times 1 = 1 \times x = x$
   (j) Additive inverses: For every x there is a number, denoted $-x$, such that $x + (-x) = 0$
   (k) Multiplicative inverses: For every nonzero x there is a number, denoted x, such that $x \times x^{-1} = x^{-1} \times x = 1$
   (l) $0 \neq 1$
   (m) Irreflexivity of $<$:$\sim (x < x)$
   (n) Transitivity of ¡: If $x < y$ and $y < z$, then $x < z$
   (o) Trichotomy: Either $x < y, y < x$, or $x = y$
   (p) If $x < y$, then $x + y < y + z$
   (q) If $x < y$ and $0 < z$, then $x \times z < y \times z$ and $z \times x < z \times y$
   (r) Completeness: If a nonempty set of real numbers has an upper bound, then it has a *least* upper bound.

4. $\emptyset$ is the empty set (The set with no elements).

Definition: Let A, B be sets. A=b if and only if all elements of A are elements of B and all elements of B are elements of A,
i.e. $A = b \leftrightarrow [\forall x(x \in A \to x \in B)] \cap [\forall y(y \in B \to y \in A)]$

3.1 Two Ways to Describe Sets

1. The enumeration/roster method: list all elements of the set.
   NB: order is irrelevant.
   $A = \{0, 1, 2, 3, 4, 5\} = \{5, 0, 2, 3, 1, 4\}$

2. The formulaic/set builder method: give a formula that generates all elements of the set.
   $A = \{x \in \mathbb{N} \mid 0 \leq x \wedge x \leq 5\} = \{0, 1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : 0 \leq x \wedge x \leq 5\}$

Using $\mathbb{N}$ and the set-builder method, we can define:

$\mathbb{Z} = \{m - n \mid \forall m, n \in \mathbb{N}\}$

$n = 0$ in any natural numbers $\Rightarrow$ we generate all of $\mathbb{N}$

$m = 0$ in any natural number $\Rightarrow$ we generator all negative integers

$\mathbb{Q} = \{\frac{p}{q} \mid p, q \in \mathbb{Z} \wedge q \neq 0\}$

Definition: A set A is called finite if it has a finite number of elements; otherwise it is called infinite.

4 Set Operations

Task: Understand how to represent sets by Venn diagrams. Understand set union, intersection, complement and difference.

Definition: Let A, B be sets. A is a <u>subset</u> of B. If all elements of A are elements of B, **i.e.** $\forall x(x \in A \rightarrow x \in B)$. We denote that A is a subset of B by $A \subseteq B$

Example: $\mathbb{N} \subseteq \mathbb{Z}$

Definition: Let A, B be sets. A is a <u>proper</u> subset of B if $A \subseteq B \wedge A \neq B$, **i.e.** $A \subseteq B \wedge \exists x \in B s.t. x \notin A$.
A proper subset is always a subset, but a subset is not always a proper subset.

Notation: $A \subset B$

Example: $\mathbb{N} \subset \mathbb{Z}$ since $\exists -1 \in \mathbb{N}$
NB: $\forall A$ a set $\emptyset \subseteq A$

Recall: $B \subseteq C$ means $\forall x(x \in B \rightarrow x \in C)$, but $\emptyset$ has no elements so in $\emptyset \subseteq A$ the quantifier $\forall$ operates on a domain with no elements. Clearly, we need to give meaning to $\exists$ and $\forall$ on empty sets.

<u>Boolean Convention</u>

$$\left.\begin{array}{l}\forall \text{ is true on the empty set} \\ \exists \text{ is false on the empty set}\end{array}\right\} \text{Consistent with common sense}$$

Definition: Let A, B be two sets. The <u>union</u> $A \cup B = \{x \mid x \in A \vee x \in B\}$

Definition: Let A, B be two sets. The <u>intersection</u> $A \cap B = \{x \mid x \in A \wedge x \in B\}$

Definition: Let A, B be sets. A and B are called <u>disjoint</u> is $A \cap B = \emptyset$

Definition Let A, B be two sets. $A - B = A \backslash B = \{a \mid x \in A \wedge x \notin B\}$

Examples:

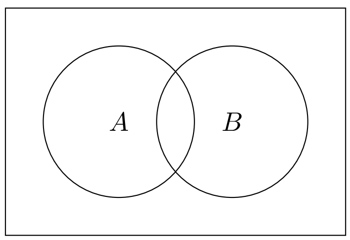
$$\begin{array}{ll} A = \{1, 2, 5\} & B = \{1, 3, 6\} \\ A \cup B = \{1, 2, 3, 5, 6\} & A \cap B = \{1\} \\ A \backslash B = \{2, 5\} & B \backslash A = \{3, 6\} \end{array}$$

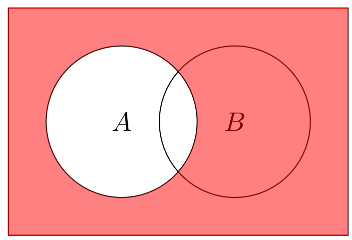
Definition: Let A, U be sets s.t. $A \subseteq U$. The <u>complement</u> of A in U = $U \setminus A = A^C = \{x \mid x \in U \wedge x \notin A\}$

Remark: The notation A^C is unambiguous only if the universe U is clearly defined or understood.

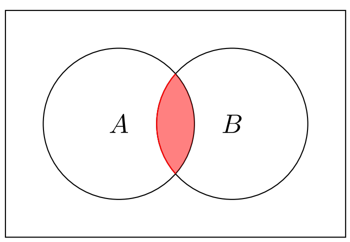
4.1 Venn Diagrams

Schematic representation of set operations.

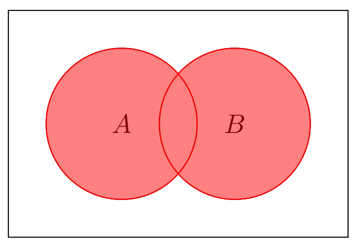




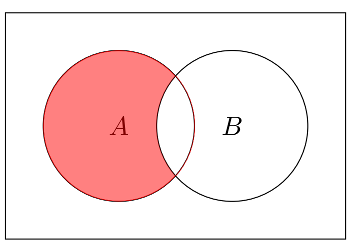
A^C



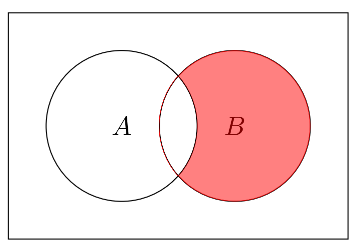
$A \cap B$



$A \cup B$



$A \setminus B$



$B \setminus A$

4.2 Properties of Set Operations

Correspondence between Logic and Set Theory

Logical Connective	Set operation
$\wedge$	intersection $\cap$
$\vee$	union $\cup$
$\neg$	complement $(\ )^C$

As a result, various properties of set operations become obvious:

- Commutativity
  - $A \cap B = B \cap A$
  - $A \cup B = B \cup A$
- Associativity
  - $(A \cup B) \cup C = A \cup (B \cup C)$
  - $(A \cap B) \cap C = A \cap (B \cap C)$
- Distributivity
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup B)$
- De Morgan Laws in Set Theory
  - $(A \cap B)^C = A^C \cup B^C$
  - $(A \cup B)^C = A^C \cap B^C$
- Involutivity of the Complement
  - $(A^C)^C = A$

NB: An involution is a map such that applying it twice gives the identity. Familiar examples: reflecting across the x-axis, the y-axis, or the origin in the plane.

- Transitivity of Inclusion
  - $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$
- Criterion for proving equality of sets
  - $A = B \leftrightarrow A \subseteq C \wedge B \subseteq A$
- Criterion for proving non-equality of sets
  - $A \neq B \leftrightarrow (A \backslash B) \cup (B \backslash A) \neq 0$

4.3 Example Proof in Set Theory

Proposition: $\forall A, B$ sets. $(A \cap B) \cup (A \backslash B) = A$

Proof: Use the criterion for proving equality of sets from above, **i.e.** inclusion in both directions.

Show $(A \cap B) \cup (A \backslash B) \subseteq A$: $\forall x \in (A \cap B) \cup (A \backslash B), x \in (A \cap B)$ or $x \in A \backslash B$. If $x \in (A \cap B)$ then clearly $x \in A$ as $A \cap B \subseteq A$ by definition. If $x \in A \backslash B$, then by definition $x \in A$ and $x \notin B$ so definitely $x \in A$. In both cases, $x \in A$ as needed.

Show $A \subseteq (A \cap B) \cup (A \backslash B)$: $\forall x \in A$, we have two possibilities, namely $x \in B$ or $x \notin B$. If $x \in B$, then $x \in A$ and $x \in B$, so $x \in A \cap B$. If $x \notin B$, then $x \in A$ and $x \notin B$, so $x \notin A \backslash B$. In both cases, $x \in (A \cap B)$ or $x \in (A \backslash B)$ so $x \in (A \cap B) \cup (A \backslash B)$ as needed.

qed

5 The Power Set

Task: Understand what the power set of a set A is.

Definition: Let A be a set. The power set of A denoted $P(A)$ is the collection of all the subsets of A.

Recall: $\emptyset \subseteq A$. It is also clear from the definition of a subset that $A \subseteq A$.

Examples:

1. $A = \{0, 1\}$
   $P(A) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
2. $A = \{a, b, c\}$
   $P(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$
3. $A = \emptyset$
   $P(A) = \{\emptyset\}$
   $P(P(A)) = \{\emptyset, \{\emptyset\}\}$

NB: $\emptyset$ and $\{\emptyset\}$ are different objects. $\emptyset$ has no elements, whereas $\{\emptyset\}$ has one element.

Remark: $P(A)$ and A are viewed as living in separate worlds to avoid phenomena like Russell' paradox.

Q: If A has n elements, how many elements does $P(A)$ have?

A: 2^n

Theorem: Let A be a set with n elements, then $P(A)$ contains 2^n elements.

Proof: Based on the on/off switch idea.

$\forall x \in A$, we have two choices: either we include x in the subset or we don't (on vs off switch). A has n elements $\Rightarrow$ we have 2^n subsets of A.

qed

Alternate Proof: Using mathematical induction.

NB: It is an axiom of set theory (in the ZFC standard system) that every set has a power set, which implies no set consisting of all possible sets could limit, else what would its power set be?

6 Cartesian Products

Task: Understand sets like $\mathbb{R}^1$ in a more theoretical way.

Recall from Calculus:

$$\mathbb{R} = \mathbb{R}^1 \ni x$$

$$\mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \ni (x_1, x_1)$$

$$\vdots$$

$$\underbrace{\mathbb{R} \times \mathbb{R}}_{\text{n times}} = \mathbb{R}^2 \ni (x_1, x_2, ..., x_n)$$

These are examples of Cartesian products.

Definition: Let A, B be sets. The Cartesian product denoted by $A]timesB$ consists of all ordered pairs $(x, y) stx \in A \wedge y \in B$, **i.e.** $A \times B = \{(x, y) \mid x \in A \wedge y \in B\}$

Further Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 5\}$
   $A \times B = \{(1, 1), (1, 5), (3, 1), (3, 5), (7, 1), (7, 5)\}$
   NB: The order in which elements in a pair matters: $(7, 1)$ is different from $(1, 7)$. This is why we call (x, y) an <u>ordered</u> pair.
2. $A = \{(x, y) \in \mathbb{R}^| x^2 + y^2 = 1\} \leftarrow$ circle of radius 1
   $B = \{z \in \mathbb{R} \mid -2 \leq z \leq 2\} = \{-2, 2\} \leftarrow$ closed interval
   $A \times B \leftarrow$ cylinder of radius 1 and height 4

6.1 Cardinality (number of elements) in a Cartesian product

If A has n elements and B has p elements, $A \times B$ has np elements.

Example:

1. $\#(A) = 3 \qquad A = \{1, 3, 7\}$
   $\#(B) = 2 \qquad B = \{1, 5\}$
   $\#(A) \times B) = 3 \times 2 = 6$
2. Both A and B are infinite sets, so $A \times B$ is infinite as well.

Remark: We can define Cartesian products of any length, **e.g.** $A \times A \times B \times A$, $B \times A \times B \times A \times B$, etc. If all sets are finite, the number of elements is the product of the numbers of elements of each factor. If $\#(A) = 3$ and $\#(B) = 2$ as above, $\#(A \times B]timesA) = 3 \times 3 \times 3 = 18$ and $\#(B \times A \times B) = 2 \times 3 \times 2 = 12$.

7 Relations

Task: Define subsets of Cartesian products with certain properties. Understand the predicates " = " (equality) and other predicates in predicate logic in a more abstract light.

Start with $x = y$. The elements x is some notation R to y (equality in this case). We can also denote it as xRy or $(x, y) \in E)$

Let x, y in $\mathbb{R}$, then $E = \{(x, x) \mid x \in \mathbb{R}\} \subset \mathbb{R} \times \mathbb{R}$.

The "diagonal" in $\mathbb{R} \times \mathbb{R}$ gives exactly the elements equal to each other.

More generally:

Definition: Let A, B be sets. A subset of the Cartesian product $A \times B$ is called a relations between A and B. A subset of the Cartesian product $A \times A$ is called a relations on A.

Remark: Note how general this definition is. To make it useful for understanding predicates, we will need to introduce key properties relations can satisfy.

Example: $A = \{1, 3, 7\} \qquad B = \{1, 2, 5\}$

We can define a relation S on $A \times B$ by $S = \{(1, 1), (1, 5), (3, 2)\}$. This means $1S1$, $1S5$ and $3S2$ and no other ordered pairs in $A \times B$ satisfy S.

Remark: The relations we defined involve 2 elements, so they are often called binary relations in the literature.

8 Equivalence Relations

Task: Define the most useful kind of relation.

Definition: A relation R on a set A is called

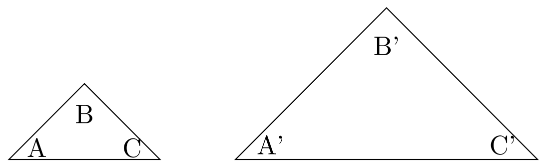
1. reflexive iff (if and only if) $\forall x \in A, xRx$
2. symmetric iff $\forall x, y \in A, xRy \to yRx$
3. transitive iff $\forall x, y, z \in A, xRy \wedge yRz \to xRz$

An equivalence relation on A is a relation that is reflexive, symmetric and transitive.

Notation: Instead of xRy, an equivalence relation is often denoted by $x \equiv y$ or $x \sim y$.

Examples:

1. "=" equality is an equivalence relation.
   (a) $x = x$ reflexive
   (b) $x = y \Rightarrow y = x$ symmetric
   (c) $x = y \wedge y = z \Rightarrow x = z$ transitive
2. $A = \mathbb{N}$
   $x \equiv y \bmod 3$ is an equivalence relation. $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z}$, **i.e.** x and y have the same remainder when divided by 3. The set of all possible remainders is $\{0, 1, 2\}$
   NB: In correct logic notation, $x \equiv y \bmod 23$ if $\exists m \in \mathbb{Z} s.t. x - y = 3m$
   (a) $x \equiv x \bmod 3$ since $x - x = 0 = 3 \times 0 \to$ reflexive
   (b) $x \equiv y \bmod 3 \Rightarrow y \equiv x \bmod 3$ because $x \equiv y \bmod 3$ means $x - y = 3m$ for some $m \in \mathbb{Z} \Rightarrow y - x = -3m = 3 \times (-m) \Rightarrow y \equiv x \bmod 3 \to$ symmetric
   (c) Assume $x \equiv y \bmod 3$ and $y \equiv z \bmod 3$
   $x \equiv y \bmod 3 \Rightarrow \exists m \in \mathbb{Z}$ s.t. $x - y = 3m \Rightarrow y = x - 3m$
   $y \equiv z \bmod 3 \Rightarrow \exists p \in \mathbb{Z}$ s.t. $y - z = 3p \Rightarrow y = z + 3p$
   Therefore, $x - 3m = z + 3p \Leftrightarrow x - z = 3p + 3m = 3(p + m)$
   Since $p, m \in \mathbb{Z}, p + m \in \mathbb{Z} \Rightarrow x \equiv z \bmod 3 \to$ transitive.
3. Let $f : A \to A$ be any function on a non empty set A. We define the relation $R = \{(x, y) \mid f(x) = f(y)\}$
   (a) $\forall x \in A, f(x) = f(x) \Rightarrow (x, x) \in R \to reflexive$
   (b) If $(x, y) \in R$, then $f(x) = f(y) \Rightarrow f(y) = f(x)$, **i.e.** $(y, x) \in R \to$ symmetric
   (c) If $(x, y) \in R$ and $(y, z) \in R$, then $f(x) = f(y)$ and $f(y) = f(z)$, which by the transitivity of equality implies $f(x) = f(z)$, **i.e.** $(x, z) \in R$ as needed, so R is transitive as well.
   $f(x)$ can be $e^x, sin\, x, (x)$, etc.



4. Let λ be the set of all triangles in the plane. $ABC \sim A'B'C'$ if ABC and $A'B'C'$ are similar triangles, **i.e.** have equal angles.
   (a) $\forall ABC \in \lambda, ABC \sim ABC$ so $\sim$ is reflexive
   (b) $ABC \sim A'B'C' \Rightarrow A'B'C' \sim ABC$ so $\sim$ is symmetric
   (c) $ABC \sim A'B'C'$ and $A'B'C' \sim A"B"C" \Rightarrow ABC \sim A"B"C"$, so $\sim$ is transitive

Clearly (a), (b), (c) use the fact that equality of angles is an equivalence relation.

Exercise: For various predicates you've encountered, check whether reflexive, symmetric or transitive. Examples of predicates include $\neq, <, >, \leq, \geq, \subseteq$

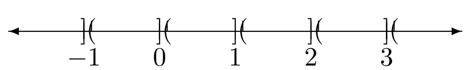
9 Equivalence Relations and Partitions

Task: Understand how equivalence relations divide sets.

Definition: Let A be a set. A partition of A is a collection of non empty sets, any two of which are disjoint such that their union is A, **i.e.** $\lambda = \{A_\alpha \mid \alpha \in I\}$ s.t. $\forall \alpha, \alpha' \in I$ satisfy $\alpha \neq \alpha', A_\alpha \cap A_{\alpha'} \neq \emptyset$ and $\underset{\alpha\in I}{\cup} A_\alpha = A$

Here I is an indexing act (may be infinite). $\underset{\alpha\in I}{A_\alpha}$ is the union of all the A_α's (possibly an infinite union)

Example $\{(n, n+1) \mid n \in \mathbb{Z}\}$ is a partition of $\mathbb{R}$



$\underset{n\in\mathbb{Z}}{\cup}(n, n+1] = \mathbb{R}$

$(n, n+1] \cap (m, m+1] = \emptyset$ if $n \neq m$

Definition: If R is an equivalence relations on a set A and $x \in A$, the equivalence class of x denoted $[x]_R$ is the set $\{y \mid xRy\}$. The collection of all equivalence classes is called A modulo R and denoted A/R.

Examples:

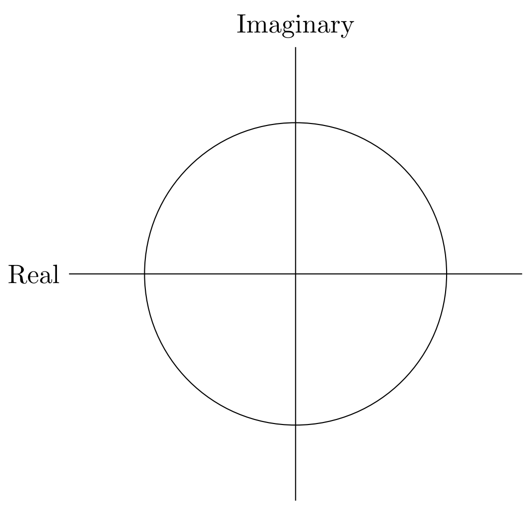
1. $A = \mathbb{N} \qquad x \equiv y \bmod 3$
   We have the equivalence classes $[0]_R, [1]_R$ and $[2]_R$ given by the then possible remainders under division by 3.
   $[0]_R = \{0, 3, 6, 9, ...\}$
   $[1]_R = \{1, 4, 7, 10, ...\}$
   $[2]_R = \{2, 5, 8, 11, ...\}$
   Clearly $[0]_R \cup [1]_R \cup [2]_R = \mathbb{N}$ and they are mutually disjoint $\Rightarrow R$ gives a partition of $\mathbb{N}$.

2. $ABC \sim A'B'C'$
   $[ABC] = \{$The set of all triangles with angles of magnitude $\measuredangle ABC, \measuredangle BAC, \measuredangle ACB\}$
   The union over the set of all $[ABC]$ is the set of all triangles and $[ABC] \cap [A'B''] = \emptyset$ if $ABC \neq^* A'B'C'$ since it means these triangles have at least one angle that if difference.

* In the original notes, not $\sim$ is used (a tilde with a slash going through it) but I couldn't find this symbol in latex.

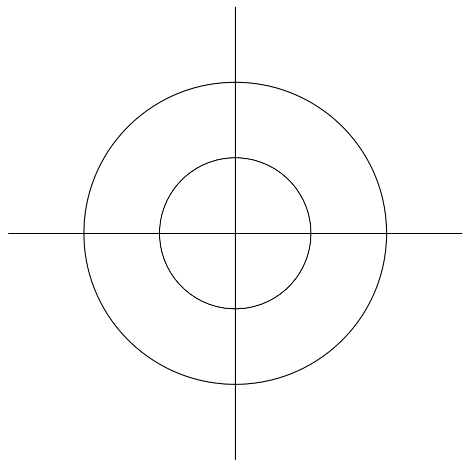
3. $A = \mathbb{C} \qquad x \cap y$ if $|x| = |y| \qquad$ equivalence relation
   $[x] = \{y \in \mathbb{C} \mid |x| = |y|\} = [r]$ for $r \in [0, +\infty) \wedge (r \geq 0)$

circle of radius $|x|$



$\bigcup_{r \in [0,+\infty)} [r] = \mathbb{C}$
$[r_1] \cap [r_2] \neq \emptyset$ if $r_1 \neq r_2$ since two distinct circles in $\mathbb{C} \simeq \mathbb{R}^2$ with empty intersection.

circles $r_1 \wedge r_2$



Theorem: For any equivalence relation R on a set A, its equivalence classes form a partition of A, **i.e.**

1. $\forall x \in A, \exists y \in A$ s.t. $x \in [y]$ (every element of A sits somewhere)
2. $xRy \Leftrightarrow [x] = [y]$ (all elements related by R belong to the same equivalence class)
3. $\neg(xRy) \Leftrightarrow [x] \cap [y] = \emptyset$ (if two elements are not related by R, the they belong to disjoint equivalence classes)

Proof:

1. Trivial. Let $y = x$. $x \in [x]$ because R is an equivalence relation. Hence reflexive, so xRx holds.
2. We will prove $xRy \Leftrightarrow [x] \subseteq [y]$ and $[y] \subseteq [x]$
$\Rightarrow$ Fix $x \in A, [x] = \{z \in A \mid xRz\} \Rightarrow \forall y \in A$ s.t. $xRy, y \in [x]$. Furthermore, $[y] = \{w \in A \mid yRw\}$
$\Rightarrow \forall w \in [u], yRw$ but $xRy \Rightarrow xRw$ by transitivity. Therefore, $w \in [x]$. We have shown $[y] \subseteq [x]$.
Since R is an equivalence relation, it is also symmetric. **i.e.** $xRy \Leftrightarrow yRx$. So by the same argument with x and y swapped $yRx \Rightarrow [x] \subseteq [y]$. Thus $xRy \Rightarrow [x] = [y]$.
$\Rightarrow [x] = [y] \Rightarrow y \in [x]$ but $[x] = \{y \in A \mid xRy\}$
3. $\Rightarrow$ We will prove the contrapositive. Assume $[x] \cap [y] \neq \emptyset \Rightarrow \exists z \in [x] \cap [y]. z \in [x]$ means xRy, whereas $z \in [y]$ means $yRx \Leftrightarrow zRy$ by symmetric of R. We thus have xRz and $zRy \Rightarrow xRy$ by transitivity of $R. xRy$ contradicts $\neg(xRy)$ so indeed $\neg(xRy) \Rightarrow [x] \cap [y] = \emptyset$
$\Leftarrow$ Once again we use the contrapositive.
Assume $\neg(\neg(xRy)) \Leftrightarrow xRy$. By part (b) $xRy \Rightarrow [x] = [y] \Rightarrow [x] \cap$

$[y] \neq \emptyset$ since $x \in [x]$ and $y \in [y]$, **i.e.** These equivalence classes are non empty. We have obtained the needed contradiction.

qed

Q: What partition does "=" impose on $\mathbb{R}$?

A: $[x] = \{x\}$ since $E = \{(x, x) \mid x \in \mathbb{R}\}$ the diagonal.
The one element equivalence class is the smallest equivalence class possible (by definition, an equivalence class cannot be empty as it contains x itself). We call such a partition the finest possible partition.

Remark: The theorem above shows how every equivalence relations partitions a set. It turns out every partition of a set can be used to define an equivalence relation: xRy is x nd y belong to the same subset of the partition (check this is indeed an equivalence relations!). Therefore, there is a 1-1 correspondence between partitions and equivalence relations: to each equivalence relation there corresponds a partition and vice versa.

10 Partial Orders

Task: Understand another type of relation with special properties.

Definition: Let A be a set. A relation R on A is called anti-symmetric if $\forall x, y \in A$ s.t. $xRy \wedge yRx$, then $x = y$.

Definition: A partial order is a relation on a set A that is reflexive, anti-symmetric, and transitive.

Examples:

1. $A = \mathbb{R}$ $\leq$ "less than or equal to" is a partial order
   (a) $\forall x \in \mathbb{R} x \leq x \rightarrow$ reflexive
   (b) $\forall x, y \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq x \implies x = y \rightarrow$ anti-symmetric
   (c) $\forall x, y, z \in \mathbb{R}$ s.t. $x \leq y \wedge y \leq z \implies x \leq z \rightarrow$ transitive
   Same conclusion if $A = \mathbb{Z} \vee A\mathbb{N}$
2. A is a set. Consider $P(A)$, the power set of A. The relation $\subseteq$ "being a subset of" is a partial order.
   (a) $\forall B \in P(A), B \subseteq B \rightarrow$ reflexive.
   (b) $forall B, C \in P(A), B \subseteq C \wedge C \subseteq B \implies B = C$ (recall the criterion for proving equality of sets) $\rightarrow$ anti-symmetric
   (c) $\forall B, C, D \in P(A)$ s.t. $B \subseteq C \wedge C \subseteq D \implies B \subseteq B \rightarrow$ transitive

The most important example of a partial order is example (2) "being a subset of".

Q: Why is "being a subset of" a partial order as opposed to a total order?

A: There might exist products B, C of A s.t. neither $B \subseteq C$ nor $C \subseteq B$ holds, **i.e.** where $B \wedge C$ are not related via inclusion.

11 Functions

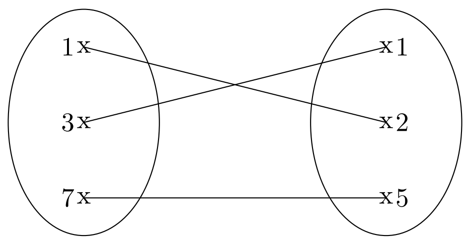
Task: Define a function rigorously and make sense of terminology associated to functions.

Definition: Let A, B be sets. A function $f : A \to B$ is a rule that assigns to every element of A one and only one elements of B, **i.e.** $\forall x \in A \exists! y \in B$ s.t. $f(x) = y$. A is called the domain of f and B is called the codomain.

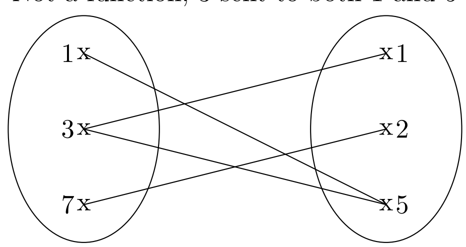
Examples:

1. $A = \{1, 3, 7\}$
   $B = \{1, 2, 5\}$

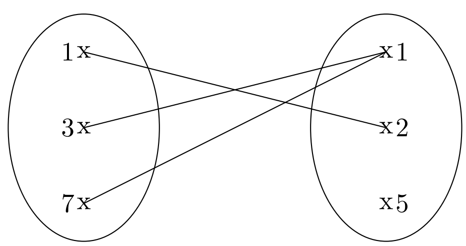
Is a function.



Not a function; 3 sent to both 1 and 5



Is a function.



2. $A = B = \mathbb{R}$ $F : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ is called the identity function.

Definition: Let A, B be sets and let $f : A \to B$ be a function. The range of f denoted by $f(A)$ if the subset of B defined by $f(A) = \{y \in B \mid \exists x \in A$ s.t. $f(x) = y\}$.

Definition: Let A be a set. A Boolean function on A is a function $F : A \to \{T, F\}$ which has A as its domain and the set of truth values $\{T, F\}$ as is codomain. $f : A \to \{T, F\}$ thus assigns truth values to the elements of A.

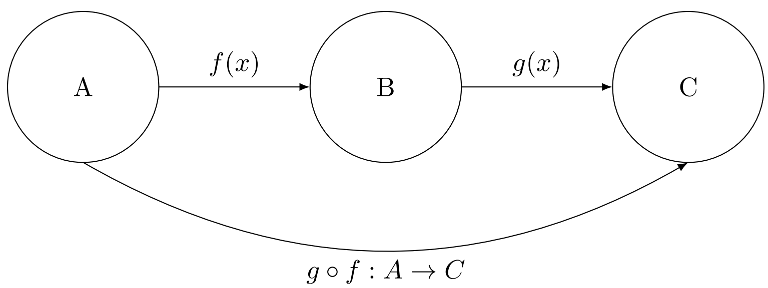
Function are often represented by graphs. If $f : A \to B$ is a function, the graph of f denoted $\Gamma(f)$ is the subset of the Cartesian product $A \times B$ given by $\{(x, f(x)) \mid x \in A\}$.

Q: Is it possible to obtain every subset of $A \times B$ as the graph of some function?

A: No! For $f : A \to B$ to be a function $\forall x]inA\exists!y \in B$ s.t. $f(x) = y$, so for $\Gamma \subseteq A \times B$ to be the graph of some function, Γ must satisfy that $\forall x \in A \exists! y \in B$ s.t. $(x, y) \in \Gamma$. Then we can define f by letting $y = f(x)$.

12 Composition of Functions

Task: Understand the natural operation that allows us to combine functions.



Example:

$f : \mathbb{R} \to \mathbb{R}$ $\quad$ $f(x) = 2x$
$g : \mathbb{R} \to \mathbb{R}$ $\quad$ $cos\ x$
$g \circ f(x) = g(f(x)) = g(2x = cos(2x))$
$f \circ g(x) = f(g(x)) = f(cos\ x) = 2(cos\ x) = 2cos\ x$

13 Inverting Functions

Task: Figure out which properties a function has to satisfy so that its action can be undone, **i.e.** when we can define an inverse to the original function.

Given $f : A \to B$, want $f^{-1} : B \to A$ s.t. $f^{-1} \circ f : A \to A$ is the identity $f^{-1} \circ f(x) = f^{-1}(f(x)) = x$
$A \xrightarrow{f} B \xrightarrow{f^{-1}} A$
It turns out f has to satisfy two properties for f^{-1} to exist.

1. Injective

2. Surjective

Definition: A function $f : A \to B$ is called injective or an injection (sometimes called one to one) if $f(x) = f(y) \Rightarrow x = y$

Examples:
$\sin x : [0, \frac{\pi}{2}] \to \mathbb{R}$ is injective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not injective because $\sin x = \sin \pi = 0$

Definition: A function $f : A \to B$ is called surjective or a surjection (sometimes called onto) if $\forall z \in B \exists x \in A$ s.t. $\overline{f(x) = z.}$

Remark: f assigns a value to each element of A by its definition as a function, but it is not required to cover all of B. f is surjective if its range is all of B.

Examples:
$\sin x : \mathbb{R} \to [-1, 1]$ is surjective
$\sin x : \mathbb{R} \to \mathbb{R}$ is not surjective since $\nexists x \in \mathbb{R}$ s.t. $\sin x = 2$. We know $|\sin x| \leq 1 \forall x \in \mathbb{R}$

Definition: A function $f : A \to B$ is called bijective or a bijection if f is both injective and surjective.

Example: $f : \mathbb{R} \to \mathbb{R} \qquad f(x) = 2x + 1$ is bijective.

- Check injectivity $f(x_1) = f(x_2) \Rightarrow 2x_1 + 1 = 2x_2 + 1 \Leftrightarrow 2x_1 = 2x_2 \Leftrightarrow x_1 = x_2$ as needed.
- Check surjectivity $\forall z \in \mathbb{R}. f(x) = z$ means $2x + 1 = z$.
Solve for x: $2x = z - 1 \Rightarrow x = \frac{z-1}{2} \in \mathbb{R} \Rightarrow f$ is surjective.

Remark: All bijective functions have inverses because we can define the inverse of a bijection and it will be a function:

- Surjectivity ensures f_{-1} assigns an element to every element of B (its domain).
- Injectivity ensures f_{-1} assigns to each elements of B one and only one elements of A.

Conclusion: $f : A \to B$ bijective $\Rightarrow f_{-1}$ exists, **i.e.** f_{-1} is a function. It turns out (reverse the arguments above) that f_{-1} exists $\Rightarrow f : A \to B$ is bijective.
Altogether we get the following theorem:

Theorem: Let $f : A \to B$ be a function. f_{-1} exists $\Leftrightarrow f : A \to B$ is bijective.

Q: How do we find the inverse function f_{-1} given $f : A \to B$?

A: If $f(x) = y$, solve for x as a function of y since $f_{-1}(f(x)) = f_{-1}(y) = x$ s $f_{-1} \circ f$ is the identity.

Example: $f(x) = 2x + 1 = y$. Solce for x in terms of y.

$f : \mathbb{R} \to \mathbb{R}$

$2x = y - 1 \qquad x = \frac{y-1}{2}$

14 Functions Defined on Finite Sets

Task: Derive conclusions about a function given the number of elements of the domain and codomain if finite; understand the pigeonhole principle.

Proposition: Let A, B be sets and let $f : A \to B$ be a function. Assume A is finite. Then f is injective $\Leftrightarrow$ $f(A)$ has the same number of elements as A.

Proof:

A is finite so we can write it as $A = \{a_1, a_2, ..., a_p\}$ for some p. Then $f(A) = \{f(a_1), f(a_2), ..., f(a_p)\} \subseteq B$. A priori, some $f(a_i)$ might be the same as some $f(a_j)$. However, f injective $\Leftrightarrow f(a_i) \neq f(a_j)$ whenever $i \neq j \Leftrightarrow f(A)$ has example p elements just like A.

qed

Corollary 1 Let A, B be finite sets such that $\#(A) = \#(B)$. Let $f : A \to B$ be a function. f is injective $\Leftrightarrow$ f is bijective.

Proof:

$\Rightarrow$ Suppose $f : A \to B$ is injective. Since A is finite, by the previous proposition, $f(A)$ has the same number of elements as A, but $f(A) \subseteq B$ and B has the same number of elements as $A \Rightarrow \#(A) = \#(f(A)) = \#(B)$, which means $f(A) - B$, **i.e.** f is also surjective $\Rightarrow f$ is bijective.

$\Leftarrow$ f is bijective $\Leftarrow$ f is injective.

qed

Corollary 2 (The Pigeonhole Principle) Let A, B be finite sets. If $\#(B) < \#(A)$, and let $f : A \to B$ be a function. $\exists a, a' \in A$ where $a \neq a'. f(a) = f(a')$

Remark: The name pigeonhole principle is due to Paul Erdös and Richard Rado. Before is was known as the principle of the drawers of Dirichlet. It has a simple statement, but it's a very powerful result in both mathematics and computer science.

Proof: Since $f(A) \subseteq B$ and $\#(B) < \#(A), f(A)$ cannot hve as many elements as A, so by the proposition, f cannot be injective, **i.e.** $\exists a, a' \in A$ where $a \neq a'$ (**i.e.** distinct elements) s.t. $f(a) = f(a')$

qed

Examples:

1. You have 8 friends. At least two of them were born the same day of the week. #(days of the week) $= 7 < 8$.
2. A family of five fives each other presents for Christmas. There are 12 presents under the tree. We conclude at least one person for three presents or more.
3. In a list of 30 words in English, at least two will begin with the same letter. #(Letter in the English alphabet) = 26 ¡ 30.

14.1 Behaviour of Functions on Infinite Sets

Let A be a set and $f : A \to A$ be a function. If A is finite , the corollary 1 tells us f injective $\Leftrightarrow f$ bijective. What if A is not finite?

14.1.1 Hilbert's Hotel problem (jazzier name: Hilbert's paradox of the Grand Hotel)

A fully occupied hotel with infinitely many rooms can always accommodate an additional guest as follows: The person in Room 1 removes to Room 2. The person in Room 2 moves to Room 3 and so on, **i.e.** if the rooms at $x_1, x_2, x_3...$ define the function $f(x_1) = x_2, f(x_2) = x_3, ..., f(x_m) = x_{m+1}$.

Claim: As defined f is injective but not surjective (hence not bijective!). Let $H = \{x_1, x_2, ...\}$ the hotel consisting of infinitely many rooms. $f : H \to H$ is given by $f(x_n) = f(x_{n+1})$. $f(H) = H \backslash \{x_1\}$. We cn use this idea to prove:

Proposition: A set A is finite $\Leftrightarrow \forall f : A \to A$ an injective function is also bijective.

Proof: If the set X is finite then it follows immediately that every injective function $f : X \to X$ is bijective.
Suppose that the set X is infinite. Then there exists some infinite sequence $x_1, x_2, x_3, ...$ of distinct elements of X (where an element of X occurs at most once in this list). Then there exists a function $f : X \to X$ defined such that $f(x_n) = x_{n+1}$ for all positive integers of n, and $f(x) = x$ for all elements of x of X. If x is not a member of the infinite sequence $x_1, x_2, x_3, ...$ then the only elements of X that gets mapped to x is the element x itself; if $x = x_n$, where $x > 1$, then the only element of X gets mapped to x. It follows that the function f is injective. However it is not surjective, since x_1 does not belong to the range of the function. This function f is thus an example of a function from the set X to itself which is injective but not bijective.

15 Mathematical Induction

Task: Understand how to construct a proof using mathematical induction.

$\mathbb{N} = \{0, 1, 2, ...\}$ set of natural numbers.
Recall that $\mathbb{N}$ is constructed using 2 axioms:

1. $0 \in \mathbb{N}$
2. If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$

Remarks:

1. This is exactly the process of counting.
2. If we start at 1, then we construct $\mathbb{N}* = \{1, 2, 3, 4, ...\} = \mathbb{N} \backslash \{0\}$

via the axioms

1. $1 \in \mathbb{N}*$
2. if $n \in \mathbb{N}*$, then $n + 1 \in \mathbb{N}*$

$\mathbb{N}$ or $\mathbb{N}*$ is used for mathematical induction.

15.1 Mathematical Induction Consists of Two Steps:

Step 1 Prove statements $P(1)$ called <u>the base case</u>.

Step 2 For any n, assume $P(n)$ and prove $P(n+1)$. This is called <u>the inductive step</u>.
In other words, step 2 proves the statement $\forall n P(n) \rightarrow P(n + 1)$

Remark: Step 2 is not just an implication but infinitely many! In logic notation, we have:

Step 1 $P(1)$

Step 2 $\forall n(P(n) \rightarrow P(n + 1))$
Therefore, $\forall n P(n)$
Let's see how the argument proceeds:

1. $P(1)$ Step 1 (base case)
2. $P(1) \rightarrow P(2)$ by Step 2 with $n = 1$
3. $P(2)$ by 1 & 2
4. $P(2) \rightarrow P(3)$ by Step 2 with $n = 2$
5. $P(3)$ by 3 & 4
6. $P(3) \rightarrow P(4)$ by Step 2 with $n = 3$
7. $P(4)$ by 5 & 6

$\vdots$

8. $P(n)$ for any n.
   This is like a row of dominos: knocking over the first one in a row makes all the others fall. Another idea is climbing a ladder.

Examples:

1. Prove $1+3+5+...+(2n-1)=n^2$ by induction.

   Base Case: Verify statement for $n=1$

   When $n=1$, $2n-1=2\times 1-1=1^2$

   Inductive Step: Assume $P(n)$, **i.e.** $1+3+5+...+(2n-1)=n^2$ and seek to prove $P(n+1)$, **i.e.** the statement $1+3+5...+(2n-1+2(n+1)-1)=(n+1)^2$

   We start with LHS: $1+\underbrace{3+5+...+(2n-1)}_{n^2}+(2(n+1)-1)=$

   $n^2+2n+2-1=n^2+2n+1=(n+1)^2$

2. Prove $1+2+3++n=\frac{n(n+1)}{2}$ by induction.

   Base Case: Verify statement for $n=1$

   When $n=1, 1=\frac{1\times(1+1)}{2}=\frac{1\times 2}{2}=1$

   Inductive Step: Assume $P(n)$, **i.e.** $1+2+3+...+n=\frac{n\times(n+1)}{2}$ and seek to prove $1+2+3+...+n=\frac{(n+1)(n+2)}{2}$

   $\underbrace{1+2+3+...+n}_{\frac{n(n+1)}{2}}+n+1=\frac{n(n+1)}{2}+n+1=(n+1)(\frac{n}{2}+1)=$

   $(n+1)\frac{n+2}{2}=\frac{(n+1)(n+2)}{2}$ as needed.

Remarks:

1. For some argument by induction, it might be necessary to assume not just $P(n)$ at the inductive step but also $P(1),P(2)P,...mP(n-1)$. This is called strong induction.

   Base Case: Prove $P(1)$

   Inductive Step: Assume $P(a),P(2),...,P(n)$ and prove $P(n+1)$.

   An example of result requiring the use of strong induction is the Fundamental Theorem of Arithmetic: $\forall n\in\mathbb{N}, n\geq 2, n$ can be expressed as a product of one or more prime numbers.

2. One has to be careful with argument involving induction. Here is an illustration why:

   Polya's argument that all horses are the same colour:

   Base Case: $P(1)$ There is only one horse, sot hat has a colour.

   Inductive Step Assume any n horses are the same colour.

   Consider a group of $n+1$ horses. Exclude the first horse and look at the other n. All of these are the same colour by our assumption. Now exclude the last horse. The remaining n horses are the same colour by our assumption. Therefore, the first horse, the horses in the middle, and the last horse are all of the same colour. We have established the inductive step.

Q: Where does the argument fail?

A: For $n = 2, P(2)$ is false because there are no middle horses to compare to.

item[] The Grand Hotel Cigar Mystery

Recall Hilbert's hotel - the grand Hotel. Suppose that the Grand Hotel does not allow smoking and no cigars may be taken into the hotel. In spite of the rules, the guest in Room 1 goes to Room 2 to get a cigar. The guest in Room 2 goes to Room 3 to get 2 cigars (one for him and one for the person in room 1), etc. In other words, guest in Room N goes to Room N+1 to get N cigars. They will each get back to their rooms, smoke one cigar, and give the result to the person in Room N-2.

Q: Where is the fallacy?

A: This is an induction argument without a base case. No cigars are allowed in the hotel so no guests have cigars. An induction cannot get off the ground without a base case.

16 Abstract Algebra

Task: Understand binary operators, semigroups, monoids, and groups as well as their properties.

16.1 Binary Operations

Definition: Let A be a set. A binary operation $*$ on A is an operation applied to any two elements $x, y \in A$ that yields on elements $x * y$ in A. In other words, $*$ us a binary operation on A if $\forall x, y \in A, x * y \in A$.

Examples:

1. $\mathbb{R}, +$ addition on $\mathbb{R} : \forall x, y \in \mathbb{R}, x + y \in \mathbb{R}$
2. $\mathbb{R}, -$ subtraction on $\mathbb{R} : \forall x, y \in \mathbb{R}, x - y \in \mathbb{R}$
3. $\mathbb{R}, \times$ multiplication on $\mathbb{R} : \forall x, y \in \mathbb{R}, x \times y \in \mathbb{R}$
4. $\mathbb{R}, /$, division on $\mathbb{R}$ is NOT a binary operation because $\forall x \in \mathbb{R} \exists o \in \mathbb{R}$ s.t. $\frac{x}{o}$ is undefined (not an element of $\mathbb{R}$)

Definition: A binary operation $*$ on a set A is called commutative if $\forall x, y \in A, x * y = y * x$

Examples:

1. $\mathbb{R}, +$ is commutative since $\forall x, y \in \mathbb{R}, x + y = y + x$

2. $\mathbb{R}, \times$ is commutative since $\forall x, y \in \mathbb{R}, x \times y = y \times x$
3. $\mathbb{R}, -$ is not commutative since $\forall x, y \in \mathbb{R}, x - y \neq y - x$ in general. $x - y = y - x$ only if $x = y$
4. Let Mn be the set of n by n matrices with entries in $\mathbb{R}$ and let $*$ be matrix multiplication. $\forall A, B \in Mn, A * B \in Mn$, so $*$ is a binary operation, but $AB \neq BA$ in general. Therefore $*$ is not commutative.

Definition: A binary operation $*$ on a set A is called associative if $\forall x, yz(x * y) * z = x * (y * z)$

Examples:

1. $\mathbb{R}, +$ is associative since $\forall x, y, z \in \mathbb{R}, (x + y) + z = x + (y + z)$
2. $\mathbb{R}, \times$ is associative since $\forall x, y, z \in \mathbb{R}, (x \times y) \times z = x \times (y \times z)$
3. Intersection $\cap$ on sets is associative since $\forall A, B, C$ sets $(A \cap B) \cap C = A \cap (B \cap C)$.
4. Union $\cup$ on sets is associative since $\forall A, B, C$ sets $(A \cup B) \cup C = A \cup (B \cup C)$
5. $\mathbb{R}, -$ is not associative since $(1-3)-5 = -2-5 = -7$ but $1-(3-5) = 1-(-2) = 1+2 = 3$

Remark: When we are dealing with associative binary operations we can drop the parentheses, **i.e.** $(x * y) * z$ can be written $x * y * z$.

16.2 Semigroups

Definition: A semigroup is a set endowed with an associative binary operation. We denote the semigroup $(A, *)$

Examples:

1. $(\mathbb{R}, +)$ and $(\mathbb{R}, -)$ are semigroups.
2. Let A be a set and let $P(A)$ be its power set. $(P(A), \cap)$ and $(P(A), \cup)$ are both semigorups.
3. $(Mn, *)$, the set of $n \times n$ matrices with entries in $\mathbb{R}$ with the operation of matrix multiplication (which is associative $\rightarrow$ a bit gory to prove) forms a semigroup.

   Since $*$ is associative on a semigroup, we can define a^n :
   $a^1 = a$
   $a^2 = a * a$
   $a^3 = a * a * a$
   $\vdots$
   Recursively, $a^1 = 1$ and $a^n = a * a^{n-1}, \forall n > 1$

NB: In $(\mathbb{R}, \times), \forall a \in \mathbb{R}, a^n = \underbrace{a \times a \times ... \times a}_{n\ times}$, whereas in $(\mathbb{R}, +), \forall a \in \mathbb{R}, a^n = \underbrace{a + a + ... + a}_{n\ times} = na$. Be careful what $*$ stands for!

Theorem: Let $(A, *)$ be a semigroup. $\forall a \in A, a^m * a^n = a^{m+n}, \forall m, n \in \mathbb{N}*$.

Proof: By induction on m.

Base Case: $m = 1$ $\qquad a^1 * a^n = a * a^n = a * 1 + n$

Inductive Step: Assume the result is true for $m = p$, **i.e.** $a^p * a^n = a^{p+n}$ and seek to prove that $a^{p+1} * a^n = a^{p+1+n}$

$a^p + 1 * a^n = (a * a^p) * a^n = a * (a^p * a * n) = a * a^{p+n} = a^{p+1+n}$

Theorem: Let$(A, *)$ be a semigroup. $\forall a \in A, (a^m)^n = a^(mn), \forall m, n \in \mathbb{N}*$

Proof: By induction on n.

Base Case: $n = 1$ $\qquad (a^m)^1 = a^m = a^{m \times 1}$

Inductive Step: Assume the result if true for $n = p$, **i.e.** $(a^m)^p = a^{mp}$ and seek to prove that $(a^m)^{p+1} = a^{m(p+1)}$

$(a^m)^{p+1} = (a^m)^p * a^m = a^{mp} * a^m = a^{mp+m} = a^{m(p+1)}$

16.2.1 General Associative Law

Let $(A, *)$ be a semigroup. $\forall a_1, ..., a_s \in A, a_1 * a_2 * ... * a_s$ has the same value regardless of how the product is bracketed.

Proof Use associativity of $*$.

qed

NB: Unless $(A, *)$ has a commutative binary operation, $a_1 * a_2 * ... * a_s$ does depend on the <u>ORDER</u> in which the $a_j's$ appearin $a_1 * a_2 * ... * a_s$

16.2.2 Identity Elements

Definition: Let $(A, *)$ be a semigroup. An element $e \in A$ is called an identity element for the binary operation $*$ if $e * x = x * e = x, \forall x \in A$.

Examples:

1. $(\mathbb{R}, +)$ has 0 as the identity element.
2. $(\mathbb{R}, \times)$ has 1 as the identity element.
3. Given a set $A, (P(A), \cup)$ has $\emptyset$ (the empty set) as its identity elements, whereas $(P(A), \cap)$ does <u>NOT</u> have an identity element.
4. $(Mn, *)$ has In, the identity matrix as its identity element.

Theorem A binary operation on a set cannot have more than one identity elements, **i.e.** if an identity element exists, then it is unique.

Proof: Assume not (proof by contradiction). Let e and e' both be identity elements for a binary operation on a set A. $e = e * e' = e'$

qed

16.3 Monoids

Definition: A monoid is a set A endowed with an associative binary operation $*$ that has an identity element e. In other words, a monoid is a semigroup $(A, *)$ where $*$ has an identity element e.

Definition: A monoid $(A, *)$ is called commutative (or Abelian) if the binary operation $*$ is commutative.

Example:

1. $(\mathbb{R}, +)$ is a commutative monoid with $e = o$.
2. $(\mathbb{R}, \times)$ is a commutative monoid with $e = 1$.
3. Given a set A, $(P(A), \cup)$ is a commutative monoid with $e = \emptyset$.
4. $(M, n*)$ is a monoid since $e = In$, but it is not commutative since matrix multiplication is not commutative.
5. $(\mathbb{N}, +)$ is a commutative monoid with $e = o$, whereas $(\mathbb{N}*, +)$ is merely a semigroup (recall $\mathbb{N}* = \mathbb{N}\backslash\{0\}$)

Theorem: Let $(A, *)$ be a monoid and let $a \in A$. Then $a^m * a^n = a^{m+n}, \forall m, n \in \mathbb{N}$

Remark: Recall that we proved this theorem for semigroups if $m, n \in \mathbb{N}*$. We now need to extend that result.

Proof: A monoid is a semigroup $\implies \forall a \in A, a^m * a^n = a^{m+n}$ whenever $m, n \in \mathbb{N}*$, **i.e.** $m > 0$ and $n > 0$. Now let $m = 0$. $a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$
If $n = 0, a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$

qed

Theorem: Let $A, *)$ be a monoid, $\forall a \in A \forall m, n \in \mathbb{N}, (a^m)^n = a^{mn}$

Remark: Once again, we had this result for semigroups when $m > 0$ and $n > 0$

Proof: By the remark, we only need to prove the result when $m = 0$ or $n = 0$. If $m = 0, (a^0)^n = (e)^n = e = a^0 = a^{0\times n}$. If $n = 0$ then $(a^m)^0 = e = a^0 = a^{0\times m}$

17 Inverses

Task: Understand what an inverse is and what formal properties it satisfies.

Definition: Let $(A, *)$ be a monoid with identity element e and let $a \in A$. An element y of A is called the inverse of x if $x * y = y * x = e$. If an element $a \in A$ has an inverse, then a is called invertible.

Examples:

1. $(\mathbb{R}, +)$ has identity element 0. $\forall x \in \mathbb{R}, (-x)$ is the inverse of x since $x + (-x) = (-x) = x = 0$.
2. $(\mathbb{R}, \times)$ has identity element 1. $x \in \mathbb{R}$ is invertible only if $x \neq 0$. If $x \neq 0$, the inverse of x is $\frac{1}{x}$ since $x \times \frac{1}{x} = \frac{1}{x} \times x = 1$.
3. $(Mn, *)$ the identity element is In. $A \in Mn$ is invertible if $det(A) \neq 0$. A^{-1} the inverse is exactly the one you computed in linear algebra. If $det(a) = 0, A$ is <u>NOT</u> invertible.
4. Given a set $A, (P(A), \cup)$ has $\emptyset$ as its identity element of all the elements of $P(A)$ only $\emptyset$ is invertible and has itself asits inverse: $\emptyset \cup \emptyset = \emptyset \cup \emptyset = \emptyset$

Theorem: Let $(A, *)$ be a monoid. If $a \in A$ has an inverse, then that inverse is unique.

Proof: By contradiction: Assume not, then $\exists a \in A$ s.t. both b and c in A are its inverses, **i.e.** $a * b = b * a = e$, the identity element of $(A, *)$ and $a * c = c * a = e$ and $b \neq c$, then $b = b * e = b * (a * c) = (b * a) * c = e * c = c$.

qed

Since every invertible element of a for $(A, *)$ a monoid has a unique inverse, we can denote the inverse by the more standard notation a^{-1}.

Next, we need to understand inverses of elements obtained via the binary operation:

Theorem: Let $(A, *)$ be a monoid and let a, b be invertible elements of A. $a * b$ is also invertible and $a * b^{-1} = b^{-1} * a^{-1}$.

Remark: You might remember this formula from linear algebra when you looked at the inverse of a product of matrices AB.

Proof: Let e be the identity element of $(A, *) a * a^{-1} = a^{-1} * a = e$ and $b * b^{-1} = b^{-1} * b = e$. We would like to show $b^{-1} * a * -1$ is the inverse of $a * b$ by computing $(a * b) * (b^{-1} * a^{-1})$ and $(b^{-1} * a^{-1}) * (a * b)$ and showing both are e.

$(a * b) * (a^{-1} * b^{-1}) = a * (b * b^{-1}) * a^{-1} = a * e * a^{-1} = a * a^{-1} = e$

$(b^{-1} * a^{-1}) * (a * b) = b^{-1} * (a^{-1} * a) * b = b^{-1} * e * b = (b^{-1} * e) * b = b^{-1} * b = e$

Thus $b^{-1} * a^{-1}$ satisfies the conditions needed for it to be the inverse of $a * b$.Since an inverse of unique, $a * b$ is invertible and $b^{-1} * a^{-1}$.

Theorem: Let $(A, *)$ be a monoid, and let $a, b \in A$. Let $x \in A$ be invertible. $a = b * x \Leftrightarrow b = a * x^{-1}$. Similarly, $a = x * b \Leftrightarrow b = x^{-1} * a$

Proof: Let e be the identity element of $(A, *)$.

First $a = b * x \Leftrightarrow b = a * x^{-1}$:

$\Rightarrow$ Assume $a = b * x . a * x^{-1} = (b * x) * x^{-1} = b * x * x^{-1} = b * e = b$ as needed.

$\Leftarrow$ Assume $b = a * x^{-1}$. Then $b * x = (a * x^{-1}) = a * (x^{-1} * x) = a * e = 1$ as needed.

Apply the same type of argument to show $a = x * b \Leftrightarrow b = x^{-1} * a$.

qed

Let $(A, *)$ be a monoid. We can now make sense of a^n for $n \in \mathbb{Z}, n < -$ for every $n \in A$ invertible. Since n is a negative integer, $\exists p \in \mathbb{N}$ s.t. $n = -1$. Set $a^n = a^{-p} = (a^p)^{-1}$.

Theorem: Let $(A, *)$ be a monoid and let $a \in A$ be invertible. Then $a^n * a^m = a^{m+n} \qquad \forall m, n \in \mathbb{Z}$.

Proof: When $m \geq 0 \wedge n \geq n$ we have already proven this result. The rest of the proof splits into cases.

Case 1: $m = n \vee n = 0$

If $m = 0, n \in \mathbb{Z}, a^m * a^n = a^0 * a^n = e * a^n = a^n = a^{0+n}$ as needed.

If $m \in \mathbb{Z}, n = 0, a^m * a^n = a^m * a^0 = a^m * e = a^m = a^{m+0}$ as needed.

Case 2: $m < 0 \wedge n < 0$

$m < 0 \Rightarrow \exists p \in \mathbb{N}\, s.t.\, p = -m$. $n < 0 \Rightarrow \exists q \in \mathbb{N}\, s.t.\, q = -n$.

$a^m = a^{-p} = (a^p)^{-1} \wedge a^n = a^{-q} = (a^q)^{-1}$

$a^m * a^n = (a^p)^{-1} * (a^q)^{-1} = (a^q * a^p)^{-1} = (a^{p+q})^{-1} = a^{-(p+q)} = a^{-q-p} = a^{m+n} = a^{n+m}$

Case 3: $m \wedge n$ have opposite signs.

Without loss of generality, assume $m < 0 \wedge n > 0$ (the case $m > 0 \wedge n < 0$ is handled by the same argument). Since $m < 0, \exists p \in \mathbb{N}\, s.t.\, p = -m$. This case splits into two subcases:

Case 3.1: $m + n \geq 0$

Set $q = m + n$. Then $a^{m+n} = a^q = e * a^q = (a^p)^{-1} * a^p * a^q = (a^p)^{-1} * a^{p+q} = a^{-p} * a^{p+q} = a^m * a^{-m+m+n} = a^m * a^n$

Case 3.2: $m + n < 0$

Set $q = -(m+n) = -m-n \in \mathbb{N}^*$. Then $a^{m+n} = a^{-q} = (a^q)^{-1} * e = (a^q)^{-1} * (a^{-n} * a^n) = (a^q)^{-1} * (a^n)^{-1} * a^n = (a^n * a^q)^{-1} * a^n = (a^{n+p})^{-1} * a^n = (a^{n-m-n})^{-1} * a^n = (a^{-m})^{-1} * a^n = (a^p)^{-1} * a^n = a^m * a^n$

Theorem: Let $(A, *)$ be a monoid, and let a be an invertible element of A. $\forall m, n \in \mathbb{Z}, (a^m)^n = a^{mn}$.

Proof: We consider 3 cases:

Case 1: $n > 0$, **i.e.** $n \in \mathbb{N}^*$. $m \in \mathbb{Z}$ with no additional restrictions we proceed by induction on m.

Base Case: $n = 1$ $(a^m)^1 = a^m = a^{m\times 1}$

Inductive Step: We assume $(a^m)^n = a^{mn}$ and seek to prove $(a^m)^{n+1} = a^{m(n+1)}$. Start with $(a^m)^{n+1} = (a^m)^n * (a^m)^1 = a^{mn} * a^m = a^{mn+m} = a^{m(n+1)}$

Case 2: $n = 0$; no restriction on $m \in \mathbb{Z}$

$(a^m)^n = (a^m)^0 = e = a^0 = a^{m\times 0} = a^{mn}$