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Lüscher equation with long-range forces

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Abstract:
We derive the modified Lüscher equation in the presence of the long-range force caused by the exchange of a light particle. It is shown that the use of this equation enables one to circumvent the problems related to the strong partial-wave mixing and the t-channel sub-threshold singularities. It is also demonstrated that the present method is intrincically linked to the so-called modified effective-range expansion (MERE) in the infinite volume. A detailed comparison with the two recently proposed alternative approaches is provided.

Contents

1 Introduction

In recent decades, the Lüscher method [1] has become a standard tool for the extraction of the scattering phase shifts from the finite-volume energy levels, measured in lattice QCD. The method has been generalized to the case of moving frames, particles with spin and coupled two-body channels (see here [2–13] for a representative list of references). In all cases, the formalism is based on fundamental assumptions, namely:

- The interactions between particles are short-range. The relation $R/L \ll 1$ holds, where R is the characteristic range of interaction and L denotes a size of the cubic box (the spatial extension of the lattice) in which the system is placed. The quantity R is typically given by the inverse of the lightest mass in the theory, $R \sim M^{-1}$.

- Owing to the condition $R/L \ll 1$, the polarization corrections, proportional to $\exp(-L/R)$, are strongly suppressed and can be neglected. This allows one to write down an equation (referred to as the Lüscher equation or the two-body quantization condition), which determines the finite-volume spectrum in terms of the observables (S-matrix elements) only. The details of the short-range interactions do not matter.

- Again, owing to the condition $R/L \ll 1$, the partial-wave mixing is small, and it is possible to truncate higher partial waves in the Lüscher equation.

Obviously, the condition $R/L \ll 1$ is violated, when the scattering in the presence of the electromagnetic interactions is considered (QCD+photons on the lattice). Moreover, at physical quark masses, the pions are rather light that leads to the problems in the study of the nucleon-nucleon interactions on the lattice. Namely, as explicitly demonstrated in the recent paper [14], the partial-wave mixing at the physical point in the Lüscher equation for the NN scattering is indeed substantial. A closely related problem is the appearance of the so-called t-channel (left-hand) cut in the NN scattering amplitude running from negative infinity till $s = (2m_N)^2 - M_\pi^2$ along the real axis in the complex s-plane, see, e.g., [15, 16] (Here, m_N and M_π denote the nucleon and the pion masses, respectively.). Since in Nature $M_\pi \ll m_N$, the gap between the t-channel cut and the right-hand unitarity cut is very small. On the other hand, as seen in the latest studies of the NN scattering on the lattice, many energy levels are located on the t-channel cut (see, e.g., Fig. 3 in Ref. [17]). The standard Lüscher approach is obviously not applicable in this case. Note also that NN scattering is not only the physically interesting process where these problems emerge. For example, in the description of the $T_{cc}(3875)^+$ state the same problem shows up in full glory. Moreover, it can be seen that the structure of the singularities is completely different in two cases $M_\pi < M_{D^*} - M_D$ and $M_\pi < M_{D^*} - M_D$, and hence reveals a critical dependence on the values of quark masses used in the lattice calculations [18].

As a remedy to the above problems, the authors of Ref. [14] have advocated solving the quantization condition in the three-dimensional plane-wave basis, in order to determine the parameters of the effective chiral Lagrangian directly from the fit to the lattice energy levels. Note also that recently a similar method has been successfully applied to the analysis of lattice data that aimed at the extraction of the $T_{cc}(3875)^+$ pole [18]. Albeit this proposal solves the problem in principle, it implicitly rests on the assumption of a rapid convergence of the chiral expansion (or the effective field theory expansion, in general) – otherwise, the number of the independent fitting parameters blows up. Furthermore, these parameters depend on the momentum cutoff used in the Lippmann-Schwinger equation for the NN amplitude and, in order the results to be valid, an (approximate) cutoff independence of the extracted physical observables should hold in a reasonable window of the cutoffs used. As a side remark, note that similar problems arise in all formulations of the three-body quantization condition [19–24], where no other solution has been found so far. However, in a much simpler two-body case, one is tempted to look further for the alternatives which directly express the finite-volume two-body spectrum in term of the physical observables.

An alternative solution, which has been suggested recently [15, 16], is based on splitting the hadron interactions into the long-range and short-range components that are then treated separately. In this respect, the approach described in these papers is conceptually close to the one pursued in the present work. While a detailed comparison of all existing approaches will be given at the end of this work, we still mention here the most important difference between Refs. [15, 16] and our paper. Namely, in Refs. [15, 16] an auxiliary on-shell K-matrix $\bar{K}^{\mathsf{os}}$ has been introduced, which has to be determined from the fit to the lattice data. Once this is done, one should solve the integral equations, in order to arrive at the physical amplitudes. So, it is essentially a two-step process. In our paper an analog of this auxiliary K-matrix is introduced as well. However, its relation to the physical amplitude has an algebraic form and there is no need to solve integral equations. From this point of view, our approach is closer to the original Lüscher single-step formalism than the approach described in Refs. [15, 16].

For completeness, we also note that, to the best of our knowledge, Ref. [25] is the only place in the literature where the two-body quantization condition in the presence of the Coulomb force has been considered. The latter is however treated perturbatively, and only the non-derivative strong coupling is taken into account. In the following we shall see that including the derivative couplings is a rather subtle issue.

The aim of the present paper is to address the problem of the long-range force in the Lüscher equation in a fully general fashion and to derive a modified Lüscher equation, which has a much larger domain of applicability than the original one. To simplify life as much as possible, in this paper we do not consider the theories with massless particles – the inclusion of QED is relegated to future publications. Furthermore, we ignore purely technical issues like the inclusion of spin, relativistic kinematics or moving frames. The key observation that allows one to achieve the stated objective is that the long-range part of the potential, which gives rise to all above problems, is usually well known and can be expressed in terms of few parameters that can be accurately measured on the lattice (like the axial-vector coupling constant g_A or the pion decay constant F_π in case of the one-pion exchange). The short-range part of potential is unknown and should be fitted to the lattice data on the two-body energy levels by using the modified Lüscher equation.

We shall see below that the method to achieve the above goal is to reformulate the so-called modified effective-range expansion (MERE) [26] in a finite volume. To this end, in Sect. 2 we invest a certain effort to relate MERE to the non-relativistic effective theory (NREFT) framework along the lines described in Ref. [27] and discuss, in particular, the inclusion of the non-derivative couplings which were omitted in Ref. [27]. The latter framework can be directly recast in a finite volume, as done in Sect. 3, and leads to a modified quantization condition with the long-range part split off. Section 4 is dedicated to the comparison of our approach to alternative ones known in the literature. The numerical implementation of the proposed framework constitutes a separate piece of work and will not be considered here.

2 Modified effective-range expansion in the effective field theory framework

2.1 Modified effective range expansion

In Ref. [26], van Haeringen and Kok consider a non-relativistic scattering problem on a sum of two local, rotationally invariant potentials:

$$V(r) = V_L(r) + V_S(r)\,. \tag{2.1}$$

Here, $V_L(r), V_S(r)$ denote the long-range and short-range parts of the potential, respectively. Due to the long-range nature of the full potential, the effective-range expansion in the partial-wave with the angular momentum ℓ,

$$q^{2\ell+1}\cot\delta_\ell(q) = -\frac{1}{a_\ell} + \frac{1}{2}\,r_\ell q^2 + \cdots\,, \tag{2.2}$$

has a very small radius of convergence. In other words, the effective range r_ℓ and the subsequent coefficients (shape parameters) are unnaturally large.

Further, the authors define the function

$$K_\ell^M(q^2) = M_\ell(q) + \frac{q^{2\ell+1}}{|f_\ell(q)|^2}\left(\cot(\delta_\ell(q)-\sigma_\ell(q)) - i\right)\,. \tag{2.3}$$

Here, $\delta_\ell(q), \sigma_\ell(q)$ denote, respectively, the full phase shift and the phase shift in the problem with the long-range potential $V_L(r)$ only (i.e., setting $V_S(r)=0$). Furthermore,

$$M_\ell(q) = \frac{1}{\ell!}\left(-\frac{iq}{2}\right)^\ell \lim_{r\to 0}\frac{d^{2\ell+1}}{dr^{2\ell+1}}\,r^\ell\,\frac{f_\ell(q,r)}{f_\ell(q)}\,, \tag{2.4}$$

where $f_\ell(q,r)$ is the Jost solution in the case $V_S(r)=0$, and

$$f_\ell(q) = \frac{q^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!}\lim_{r\to 0} r^\ell f_\ell(q,r)\,. \tag{2.5}$$

The main result of Ref. [26] consists in demonstrating the fact that the quantity $K_\ell^M(q^2)$, defined by Eq. (2.3), is a polynomial in the variable q^2, with a radius of convergence much larger than the original version of the effective-range expansion, displayed in Eq. (2.2).

The derivation given in Ref. [26] has however a caveat that has been briefly mentioned already in the same paper and was discussed in more detail in Ref. [28]. Namely, the quantity $M_\ell(q)$ is well-defined, if and only if the potential $V_L(r)$ is regular enough at the origin, so that $r^{2\ell}V_L(r)$ stays analytic at $r=0$. The class of such potentials is termed "superregular" in Ref. [28]. Usual Coulomb or Yukawa potentials do not belong to this class, even for the S-wave scattering.

If one is dealing with the potentials which are not superregular, one has to use certain convention on top of Eq. (2.4) in order to define the quantity $M_\ell(q)$. This is nothing but the renormalization prescription that has to be imposed on the ultraviolet-divergent loop containing an arbitrary number of instantaneous "exchanges" corresponding to the potential $V_L(r)$. A non-trivial problem consists in a mathematically consistent formulation of the renormalization prescription, using the same language as used in the derivation of Eqs. (2.3) and (2.4). There exists a well-known exact solution of the problem for any ℓ in case of Coulomb interaction that is given in the textbooks, see, e.g., [29]. The solution for a general potential which is less singular than $r^{-3/2}$ is discussed in Ref. [28] albeit only in the case $\ell=0$ and, roughly speaking, boils down to a subtraction of the q-independent (divergent) constant from $M_\ell(q)$. We are not, however, aware of the discussion of the case $\ell\neq 0$ in the literature. The term, divergent at $r\to 0$ can be identified in this case as well. However, its coefficient is q-dependent, in general, and the challenge consists in showing that this coefficient is a low-energy polynomial in q^2 in the wide region determined by the heavy scale. For this reason, in the present paper we adopt a different strategy. Namely, we truncate the partial-wave expansion $\ell \leq \ell_{\sf max}$ from the beginning and regularize the potential $V_L(r)$ in order to render it superregular. For example, a kind of the Pauli-Villars regularization will perfectly do the job in case of Yukawa potential we shall be primarily dealing with:

$$V_L(r) = \frac{ge^{-M_\pi r}}{r} \to \frac{ge^{-M_\pi r}}{r} - \sum_{i=1}^{2\ell_{\sf max}+1} c_i \frac{ge^{-M_i r}}{r}\,. \tag{2.6}$$

Here, $M_i = n_i M$, where M denotes a typical heavy scale of the theory (determined, for instance, by the inverse range in $V_S(r)$), whereas n_i are numbers of order unity. The requirement that the first $2\ell_{\sf max}+1$ terms in the Laurent expansion vanish leads to the following linear system of equations:

$$\begin{aligned} 1 &= \sum_{i=1}^{2\ell_{\sf max}+1} c_i\,, \\ \frac{M_\pi}{M} &= \sum_{i=1}^{2\ell_{\sf max}+1} c_i n_i\,, \\ &\cdots\,, \\ \left(\frac{M_\pi}{M}\right)^{2\ell_{\sf max}} &= \sum_{i=1}^{2\ell_{\sf max}+1} c_i n_i^{2\ell_{\sf max}}\,. \end{aligned} \tag{2.7}$$

It is straightforwardly seen that the regularized potential is indeed superregular for all $\ell \leq \ell_{\sf max}$. Furthermore, for generic values of n_i, all c_i obtained from the solution of the above system of equations are quantities of the order one, and so no unnaturally large coefficients emerge.

Finally, in the splitting $V(r) = V_L(r) + V_S(r)$, one could modify both $V_L(r)$ and $V_S(r)$, adding and subtracting the same string of the short-range Yukawa terms. This will render $V_L(r)$ superregular and will not change the interpretation of $V_S(r)$ as a short-range potential.

2.2 NREFT framework

In the literature, there have been several attempts to reformulate the modified effective-range expansion in the effective field theory language [25, 27, 30, 31]. We shall mainly follow the path outlined in these papers and derive an analog of Eq. (2.3) in the effective field theory setting. In order to do this, we recall that, in the non-relativistic effective field theory, the scattering amplitude is merely a solution of the Lippmann-Schwinger (LS) equation with the potential determined by a matrix element of the interaction Lagrangian between the free two-particle states. We still assume that the potential is a sum of a long-range and short-range parts, but do not assume anymore that the potential is local. The short-range potential in the momentum space is a familiar low-energy polynomial. Its partial-wave expansion can be written in the following form

$$\langle\mathbf{p}|V_S|\mathbf{q}\rangle = 4\pi\sum_{\ell m} Y_{\ell m}(\hat{p})V_S^\ell(p,q)Y_{\ell m}^*(\hat{q})\,. \tag{2.8}$$

Here,

$$V_S^\ell = (pq)^\ell \sum_{a=0}^{\infty}\sum_{b=0}^{a} C_\ell^{ab}(p^2)^b(q^2)^{a-b}\,, \qquad C_\ell^{ab} = C_\ell^{ba}\,. \tag{2.9}$$

Furthermore, $p = |\mathbf{p}|$, $\hat{p}$ denotes a unit vector in direction of $\mathbf{p}$, and $Y_{\ell m}(\hat{p})$ are the spherical functions. Writing down explicitly the first few terms in the potential, one gets

$$\langle\mathbf{p}|V_S|\mathbf{q}\rangle = C_0^{00} + 3C_1^{00}\mathbf{p}\mathbf{q} + C_0^{10}(\mathbf{p}^2+\mathbf{q}^2) + \cdots\,, \tag{2.10}$$

which in the position space corresponds to a sum of a δ-like potential and the derivatives thereof. The long-range potential might be taken to be the regularized Yukawa potential, corresponding to an exchange of a light particle, see above. In any case, it is assumed to be local. Furthermore, ultraviolet divergences will be present in the LS equation, in general. We assume that these divergences are regularized and renormalized in a standard fashion (say, the power-divergence subtraction (PDS) scheme, bearing the case of NN scattering in mind). Since the presence of a long-range force non-trivially affects only the infrared behavior of the theory, it is expected that the issue of renormalization is inessential in the present context. To simplify things, one could also merely assume that the momentum cutoff is performed at a very large value Λ, and the Λ-dependent effective couplings are adjusted order by order to reproduce the behavior of the S-matrix elements at low momenta.

The fully off-shell LS equation for the scattering amplitude T in momentum space is given by (the regularization is implicit)

$$T(\mathbf{p},\mathbf{q};q_0^2+i\varepsilon) = V(\mathbf{p},\mathbf{q}) + \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{V(\mathbf{p},\mathbf{k})T(\mathbf{k},\mathbf{q};q_0^2+i\varepsilon)}{\mathbf{k}^2-q_0^2+i\varepsilon}\,. \tag{2.11}$$

The partial-wave amplitudes are defined as follows:

$$T(\mathbf{p},\mathbf{q};q_0^2+i\varepsilon) = 4\pi\sum_{\ell m} Y_{\ell m}(\hat{p})T_\ell(p,q;q_0^2+i\varepsilon)Y_{\ell m}^*(\hat{q})\,, \tag{2.12}$$

The phase shift is related to the on-shell scattering amplitude:

$$T_\ell(q_0,q_0;q_0^2+i\varepsilon) \doteq T_\ell(q_0) = \frac{4\pi}{q_0\cot\delta_\ell(q_0)-iq_0}\,. \tag{2.13}$$

Splitting now the full potential into the long-and short range parts, $V = V_L + V_S$, and defining the scattering amplitude and the Green function for the long-range potential only

$$\begin{aligned} T_L(q_0^2+i\varepsilon) &= V_L + V_LG_0(q_0^2+i\varepsilon)T_L(q_0^2+i\varepsilon)\,,\\ G_L(q_0^2+i\varepsilon) &= G_0(q_0^2+i\varepsilon) + G_0(q_0^2+i\varepsilon)T_L(q_0^2+i\varepsilon)G_0(q_0^2+i\varepsilon)\,, \end{aligned} \tag{2.14}$$

one gets

$$\begin{aligned}T(q_0^2+i\varepsilon) &= T_L(q_0^2+i\varepsilon)\\&+ (1+T_L(q_0^2+i\varepsilon)G_0(q_0^2+i\varepsilon))T_S(q_0^2+i\varepsilon)(G_0(q_0^2+i\varepsilon)T_L(q_0^2+i\varepsilon)+1)\,,\end{aligned} \tag{2.15}$$

where

$$T_S(q_0^2+i\varepsilon) = V_S + V_S G_L(q_0^2+i\varepsilon)T_S(q_0^2+i\varepsilon)\,. \tag{2.16}$$

In the above equations, $G_0(q_0^2+i\varepsilon)$ stands for the free Green function

$$\langle\mathbf{p}|G_0(q_0^2+i\varepsilon)|\mathbf{q}\rangle = \frac{(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})}{\mathbf{p}^2-q_0^2-i\varepsilon}\,. \tag{2.17}$$

The above expressions are of course familiar from the theory of scattering on two potentials.

In order to simplify life further, we assume that the long-range potential is repulsive and does not create bound states. Then, the spectral representation of the Green function takes the form:

$$\langle\mathbf{p}|G_L(q_0^2+i\varepsilon)|\mathbf{q}\rangle = \int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\langle\mathbf{p}|\psi_{\mathbf{k}}^{(\pm)}\rangle\langle\psi_{\mathbf{k}}^{(\pm)}|\mathbf{q}\rangle}{\mathbf{k}^2-q_0^2+i\varepsilon}\,, \tag{2.18}$$

where $\psi_{\mathbf{k}}^{(\pm)}$ denote the eigenfunctions of the Hamiltonian $H_L = H_0 + V_L$, corresponding to the eigenvalue $\mathbf{k}^2$, and $(\pm)$ specifies outgoing/ingoing boundary conditions on the wave function. These wave functions can be constructed with the use of the Møller operators:

$$\begin{aligned}|\psi_{\mathbf{k}}^{(\pm)}\rangle &= (1+G_0(k^2\pm i\varepsilon)T_L(k^2\pm i\varepsilon))|\mathbf{k}\rangle \doteq \Omega(k^2\pm i\varepsilon)|\mathbf{k}\rangle\,,\\ \langle\psi_{\mathbf{k}}^{(\pm)}| &= \langle\mathbf{k}|(1+T_L(k^2\mp i\varepsilon)G_0(k^2\mp i\varepsilon)) \doteq \langle\mathbf{k}|\Omega^\dagger(k^2\pm i\varepsilon)\,.\end{aligned} \tag{2.19}$$

Now, let us consider Born series for the quantity

$$\begin{aligned}&\langle\mathbf{p}|\Omega^\dagger(q_0^2-i\varepsilon)T_S(q_0^2+i\varepsilon)\Omega(q_0^2+i\varepsilon)|\mathbf{q}\rangle\\ &= \int\langle\mathbf{p}|\Omega^\dagger(q_0^2-i\varepsilon)|\mathbf{k}_1\rangle\frac{d^3\mathbf{k}_1}{(2\pi)^3}\,\langle\mathbf{k}_1|V_S|\mathbf{k}_2\rangle\frac{d^3\mathbf{k}_2}{(2\pi)^3}\,\langle\mathbf{k}_2|\Omega(q_0^2+i\varepsilon)|\mathbf{q}\rangle\\ &+ \int\langle\mathbf{p}|\Omega^\dagger(q_0^2-i\varepsilon)|\mathbf{k}_1\rangle\frac{d^3\mathbf{k}_1}{(2\pi)^3}\,\langle\mathbf{k}_1|\psi_{\mathbf{l}}^{(+)}\rangle\frac{d^3\mathbf{l}}{(2\pi)^3(\mathbf{l}^2-q_0^2-i\varepsilon)}\,\langle\psi_{\mathbf{l}}^{(+)}|\mathbf{k}_2\rangle\\ &\quad\times\langle\mathbf{k}_2|V_S|\mathbf{k}_3\rangle\frac{d^3\mathbf{k}_3}{(2\pi)^3}\,\langle\mathbf{k}_3|\Omega(q_0^2+i\varepsilon)|\mathbf{q}\rangle+\cdots\,.\end{aligned} \tag{2.20}$$

On the energy shell $p = q = q_0^2$, the above expression simplifies to

$$\begin{aligned}&\langle\mathbf{p}|\Omega^\dagger(q_0^2-i\varepsilon)T_S(q_0^2+i\varepsilon)\Omega(q_0^2+i\varepsilon)|\mathbf{q}\rangle\\ &= \int\langle\psi_{\mathbf{p}}^{(-)}|\mathbf{k}_1\rangle\frac{d^3\mathbf{k}_1}{(2\pi)^3}\,\langle\mathbf{k}_1|V_S|\mathbf{k}_2\rangle\frac{d^3\mathbf{k}_2}{(2\pi)^3}\,\langle\mathbf{k}_2|\psi_{\mathbf{q}}^{(+)}\rangle\end{aligned}$$

$$+\int \langle\psi_{\mathbf{p}}^{(-)}|\mathbf{k}_1\rangle \frac{d^3\mathbf{k}_1}{(2\pi)^3}\,\langle\mathbf{k}_1|\psi_{\mathbf{l}}^{(+)}\rangle \frac{d^3\mathbf{l}}{(2\pi)^3(\mathbf{l}^2-q_0^2-i\varepsilon)}\,\langle\psi_{\mathbf{l}}^{(+)}|\mathbf{k}_2\rangle$$
$$\times\,\langle\mathbf{k}_2|V_S|\mathbf{k}_3\rangle \frac{d^3\mathbf{k}_3}{(2\pi)^3}\,\langle\mathbf{k}_3|\psi_{\mathbf{q}}^{(+)}\rangle+\cdots\,. \tag{2.21}$$

The partial-wave expansion of the asymptotic wave functions is defined as follows:

$$\langle\mathbf{p}|\psi_{\mathbf{k}}^{(\pm)}\rangle=\sum_{\ell m}Y_{\ell m}(\hat{p})\psi_\ell^{(\pm)}(k,p)Y_{\ell m}^*(\hat{k})\,. \tag{2.22}$$

Furthermore,

$$\left(\psi_\ell^{(-)}(k,p)\right)^*=e^{2i\sigma_\ell(k)}\left(\psi_\ell^{(+)}(k,p)\right)^*, \tag{2.23}$$

where $\sigma_\ell(k)$ denotes the scattering phase shift in case of the long-range potential only. Now, the partial-wave expansion of the quantity defined in Eq. (2.21) is given by

$$\langle\mathbf{p}|\Omega^\dagger(q_0^2-i\varepsilon)T_S(q_0^2+i\varepsilon)\Omega(q_0^2+i\varepsilon)|\mathbf{q}\rangle=4\pi\sum_{\ell m}Y_{\ell m}(\hat{p})e^{2i\sigma_\ell(q_0)}B_\ell(q_0)Y_{\ell m}^*(\hat{q})\,, \tag{2.24}$$

where

$$B(\mathbf{p},\mathbf{q};q_0^2+i\varepsilon)=\tilde{V}_S(\mathbf{p},\mathbf{q})+\int\frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{\tilde{V}_S(\mathbf{p},\mathbf{k})B(\mathbf{k},\mathbf{q};q_0^2+i\varepsilon)}{\mathbf{k}^2-q_0^2+i\varepsilon}\,, \tag{2.25}$$

$$\tilde{V}_S(\mathbf{p},\mathbf{q})=\langle\psi_{\mathbf{p}}^{(+)}|V_S|\psi_{\mathbf{q}}^{(+)}\rangle\,, \tag{2.26}$$

and $B_\ell(q_0)$ is equal to the partial-wave amplitude $B_\ell(p,q;q_0^2)$ on the energy shell $p=q=q_0^2$. In analogy to Eq. (2.13), one may write

$$B_\ell(q_0)=\frac{4\pi}{q_0\cot\tilde{\delta}_\ell(q_0)-iq_0}\,. \tag{2.27}$$

Using Eqs. (2.15) and (2.24), one finally gets:

$$\tilde{\delta}_\ell(q_0)=\delta_\ell(q_0)-\sigma_\ell(q_0)\,. \tag{2.28}$$

2.3 Non-derivative interactions

Let us first restrict ourselves to $\ell=0$ and assume that only the coupling C_0^{00} is different from zero. Then, the potential $\tilde{V}_S$ is separable:

$$\tilde{V}_S(\mathbf{p},\mathbf{q})=\left(\tilde{\psi}_{\mathbf{p}}^{(+)}(0)\right)^*C_0\tilde{\psi}_{\mathbf{q}}^{(+)}(0)\,. \tag{2.29}$$

Here, $\tilde{\psi}_{\mathbf{q}}^{(+)}(\mathbf{r})$ stands for the wave function in the coordinate space. Then, on the energy shell $|\mathbf{q}|=q_0$, the S-wave amplitude takes the form

$$B_0(q_0)=\frac{|\tilde{\psi}_{\mathbf{q}}^{(+)}(0)|^2}{(C_0^{00})^{-1}-\langle G_L^0(q_0)\rangle}\,, \tag{2.30}$$

where

$$\langle G_L^0(q_0)\rangle = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{|\tilde\psi_{\mathbf{k}}^{(+)}(0)|^2}{\mathbf{k}^2-q_0^2-i\varepsilon}\,. \tag{2.31}$$

Let us now assume that the conditions of Ref. [26] are fulfilled and, namely, the long-range potential V_L is local. It is convenient to define the partial-wave expansion of the Green function in the momentum/coordinate spaces as follows:

$$\begin{aligned}\langle\mathbf{p}|G_L(q_0^2+i\varepsilon)|\mathbf{q}\rangle &= 4\pi\sum_{\ell m}\mathscr{Y}_{\ell m}(\mathbf{p})G_L^\ell(p,q;q_0^2+i\varepsilon)\mathscr{Y}^*_{\ell m}(\mathbf{q})\,,\\ \langle\mathbf{r}|G_L(q_0^2+i\varepsilon)|\mathbf{w}\rangle &= 4\pi\sum_{\ell m}\mathscr{Y}_{\ell m}(\mathbf{r})\tilde G_L^\ell(r,w;q_0^2+i\varepsilon)\mathscr{Y}^*_{\ell m}(\mathbf{w})\,,\end{aligned} \tag{2.32}$$

with $\mathscr{Y}_{\ell m}(\mathbf{z})=z^\ell Y_{\ell m}(\hat z)$. Then,

$$\langle G_L^\ell(q_0)\rangle = \lim_{r,w\to 0}\tilde G_L^\ell(r,w;q_0^2+i\varepsilon)\,. \tag{2.33}$$

2.4 Calculating the loop

The Green function in the coordinate space can be expressed through the Møller operator

$$\langle\mathbf{r}|G_L(q_0^2+i\varepsilon)|\mathbf{w}\rangle = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\langle\mathbf{r}|\Omega(q_0^2+i\varepsilon)|\mathbf{p}\rangle\frac{e^{-i\mathbf{p}\mathbf{w}}}{\mathbf{p}^2-q_0^2-i\varepsilon}\,. \tag{2.34}$$

In Ref. [32], Fuda and Whiting defined the off-shell scattering wave function (cf. with Eq. (2.19)):

$$\langle\mathbf{r}|\Omega(q_0^2+i\varepsilon)|\mathbf{p}\rangle = 4\pi\sum_{\ell m}Y_{\ell m}(\hat r)i^\ell\frac{\phi_\ell(q_0,p,r)}{pr}\,Y^*_{\ell m}(\hat p)\,. \tag{2.35}$$

This scattering wave function obeys the equation

$$\left(q_0^2+\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-V_L(r)\right)\phi_\ell(q_0,p,r)=(q_0^2-p^2)u_\ell(pr)\,, \tag{2.36}$$

where

$$u_\ell(z)=zj_\ell(z)\,,\qquad v_\ell(z)=zn_\ell(z)\,,\qquad w_\ell^{(\pm)}(z)=zh^{(\pm)}(z)=-v_\ell(z)\pm iu_\ell(z) \tag{2.37}$$

are expressed through the spherical Bessel, Neumann and Hankel functions, respectively. The familiar on-shell wave function is given by $\phi_\ell(p,r)\doteq\phi_\ell(p,p,r)$.

Using the expansion of the plane wave into spherical functions in Eq. (2.34), we obtain

$$G_L^\ell(r,w;q_0^2+i\varepsilon)=4\pi\int_0^\infty\frac{p^2dp}{(2\pi)^3}\,\frac{\phi_\ell(q_0,p,r)}{pr^{\ell+1}}\,\frac{j_\ell(pw)}{w^\ell}\,\frac{1}{p^2-q_0^2-i\varepsilon}\,. \tag{2.38}$$

Performing limit $w\to 0$, one gets:

$$G_L^\ell(r,0;q_0^2+i\varepsilon)=\frac{4\pi}{(2\ell+1)!!}\,\frac{(2i)^\ell l!}{(2l+1)!}\int_0^\infty\frac{pdp}{(2\pi)^3}\,\frac{D^{2\ell+1}\phi_\ell(q_0,p,r)}{p^2-q_0^2-i\varepsilon}\,, \tag{2.39}$$

where

$$D^{2\ell+1}\phi_\ell(q_0,p,r) \doteq \lim_{r\to 0}\left(\frac{-ip}{2}\right)^\ell \frac{1}{l!}\frac{d^{2\ell+1}}{dr^{2\ell+1}}\, r^\ell\phi_\ell(q_0,p,r)\,. \tag{2.40}$$

In order to perform the integral over p, we rewrite the wave function ϕ_ℓ in terms of the off-shell functions f_ℓ [32]:

$$\begin{aligned}\phi_\ell(q_0,p,r) &= -\frac{\pi p}{2}\left(\frac{q_0}{p}\right)^\ell \frac{f_\ell(q_0,p)-f_\ell(q_0,-p)}{i\pi p f_\ell(q_0)}\, e^{-i\ell\pi/2}f_\ell(q_0,r)\\ &+ \frac{1}{2i}\left(e^{-i\ell\pi/2}f_\ell(q_0,p,r)-e^{i\ell\pi/2}f_\ell(q_0,-p,r)\right)\,.\end{aligned} \tag{2.41}$$

Here, the function f_ℓ obeys the equation

$$\left(q_0^2+\frac{d^2}{dr^2}-\frac{\ell(\ell+1)}{r^2}-V_L(r)\right)f_\ell(q_0,p,r) = (q_0^2-p^2)e^{i\ell\pi/2}w_\ell^{(+)}(pr)\,, \tag{2.42}$$

and has the asymptotic normalization

$$f_\ell(q_0,p,r)\sim e^{ipr}\,,\qquad \text{as } r\to\infty\,. \tag{2.43}$$

The off-shell Jost functions are defined as

$$f_\ell(q_0,p) = \frac{p^\ell e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!}\,\lim_{r\to 0} r^\ell f_\ell(q_0,p,r)\,, \tag{2.44}$$

and the usual Jost functions are obtained from the off-shell Jost functions according to $f_\ell(q_0)=f_\ell(q_0,q_0)$.

Substituting Eq. (2.41) into Eq. (2.39), it is seen that the integration can be extended from $-\infty$ to $+\infty$, owing to the symmetry of the integrand:

$$\begin{aligned}G_L^\ell(r,0;q_0^2+i\varepsilon) &= \frac{1}{(2\ell+1)!!}\,\frac{(2i)^\ell l!}{(2l+1)!}\,2\pi i e^{-i\ell\pi/2}\int_{-\infty}^{\infty}\frac{pdp}{(2\pi)^3(p^2-q_0^2-i\varepsilon)}\\ &\times\left(\frac{f_\ell(q_0,p)}{f_\ell(q_0)}\,D^{2\ell+1}f_\ell(q_0,r)-D^{2\ell+1}f_\ell(q_0,p,r)\right)\,.\end{aligned} \tag{2.45}$$

Note that the factor $(q_0/p)^\ell$ has disappeared, since $D^{2\ell+1}f_\ell(q_0,r)$ contains the factor q_0^ℓ instead of p^ℓ, cf. with Eq. (2.40).

In order to perform the integral by using Cauchy's theorem, it is important to show that the Jost solutions do not have singularities in the upper complex plane of the variable p. To this end, we define the functions

$$\begin{aligned}g_\ell(q_0,p,r) &= \frac{e^{-i\ell\pi/2}(2\ell+1)}{(2\ell+1)!!}\,(pr)^\ell f_\ell(q_0,p,r)\,,\\ z_\ell(pr) &= \frac{(2\ell+1)}{(2\ell+1)!!}\,(pr)^\ell w^{(+)}(pr)\,.\end{aligned} \tag{2.46}$$

Using Eq. (2.42) and the asymptotic condition, it can be shown that the function g_ℓ obeys the following integral equation

$$g_\ell(q_0,p,r) = z_\ell(pr) - \frac{1}{q_0}\int_0^\infty dw\theta(w-r)\left(\frac{r}{w}\right)^\ell (u_\ell(q_0r)v_\ell(q_0w) - v_\ell(q_0r)u_\ell(q_0w)) \\ \times V_L(w)g_\ell(q_0,p,w)\,. \tag{2.47}$$

Solving this equation iteratively, one arrives at

$$g_\ell(q_0,p,r) = z_\ell(pr) + \int_0^\infty dwK_\ell(r,w;q_0)z_\ell(pw)\,. \tag{2.48}$$

An exact form of the kernel K_ℓ is not important. It suffices to know that the kernel does not depend on p and vanishes at $w<r$. Furthermore, assuming $r\to 0$, we get

$$f_\ell(q_0,p) = z_\ell(0) + \int_0^\infty dwK_\ell(0,w;q_0)z_\ell(pw)\,, \qquad z_\ell(0) = \frac{(2\ell+1)!}{l!2^\ell(2\ell+1)!!}\,. \tag{2.49}$$

Acting now with the operator $D^{2\ell+1}$ on Eq. (2.48) and taking the limit $r\to 0$, one gets:

$$\lim_{r\to 0} D^{2\ell+1}f_\ell(q_0,p,r) = p^{2\ell+1}\tilde{z}_\ell(0) + p^\ell\int_0^\infty dw\tilde{K}_\ell(0,w;q_0)z_\ell(pw)\,. \tag{2.50}$$

Again, $\tilde{z}_\ell$, $\tilde{K}_\ell$ are independent of p. Performing now Cauchy integrals, one gets:

$$\begin{aligned}\int_{-\infty}^\infty \frac{pdp}{2\pi i}\,\frac{f_\ell(q_0,p)}{p^2-q_0^2-i\varepsilon} &= z_\ell(0)\int_{-\infty}^\infty \frac{pdp}{2\pi i}\,\frac{1}{p^2-q_0^2-i\varepsilon} + \frac{1}{2}\int_0^\infty K_\ell(0,w;q_0)z_\ell(q_0w) \\ &= -\frac{1}{2}\,z_\ell(0) + \frac{1}{2}\,f_\ell(q_0)\,.\end{aligned} \tag{2.51}$$

Here, one has used the fact that the integral, multiplying $z_\ell(0)$, vanishes in the symmetric boundaries. Furthermore,

$$\begin{aligned}\int_{-\infty}^\infty \frac{pdp}{2\pi i}\,\frac{D^{2\ell+1}f_\ell(q_0,p)}{p^2-q_0^2-i\varepsilon} &= \tilde{z}_\ell(0)\int_{-\infty}^\infty \frac{pdp}{2\pi i}\,\frac{p^{2\ell+1}}{p^2-q_0^2-i\varepsilon} + \frac{q_0^\ell}{2}\int_0^\infty dw\tilde{K}_\ell(0,w;q_0)z_\ell(q_0w) \\ &= \tilde{z}_\ell(0)\int_{-\infty}^\infty \frac{dp(p^{2\ell+2}-q_0^{2\ell+2})}{p^2-q_0^2+i\varepsilon} + \frac{1}{2}\,D^{2\ell+1}f_\ell(q_0) \\ &= \tilde{z}_\ell(0)X_\ell(q_0^2) + \frac{1}{2}\,D^{2\ell+1}f_\ell(q_0)\,.\end{aligned} \tag{2.52}$$

Here, $X_\ell(q_0^2)$ denotes a polynomial of order ℓ in the variable q_0^2. The coefficients of this polynomial are ultraviolet-divergent and can be regularized, e.g., introducing a momentum cutoff on the integration momenta, $|p|\leq\Lambda$.

Collecting all factors together, we obtain

$$\langle G_L^\ell(q_0)\rangle = \frac{1}{4\pi\left((2\ell+1)!!\right)^2}\,M_\ell(q_0) + \text{real polynomial in } q_0^2\,, \tag{2.53}$$

where $M_\ell(q_0)$ is given by Eq. (2.4). The real polynomial can be safely dropped as it corresponds to the choice of the renormalization prescription. Furthermore, using

$$|\tilde{\psi}_{\mathbf{k}}(0)| = \lim_{r\to 0}\left|\frac{\phi_0(k,r)}{kr}\right| = \frac{1}{|f_0(k)|} \tag{2.54}$$

along with Eqs. (2.27) and (2.30), for the S-wave phase shift one obtains:

$$\frac{4\pi}{C_0^{00}} = M_0 + \frac{q_0}{|f_0(q_0)|^2}\,(\cot\tilde{\delta}_0(q_0) - i)\,. \tag{2.55}$$

The essence of the modified effective-range expansion is now made crystal clear: it refers to the short-range potential only and, consequently, has a larger radius of convergence.

One more remark is in order. It should be pointed out that the final result crucially depends on the validity of Eqs. (2.51) and (2.52). Using Cauchy's theorem straightforwardly is not allowed, since the integrand does not vanish sufficiently fast at the infinity. The result given above corresponds to the choice of symmetric boundary conditions $-\Lambda \le p \le \Lambda$ and $\Lambda \to \infty$, which follows from extending the initial integration area by using the fact that the integrand is even under the interchange $p \leftrightarrow -p$. The terms containing the potential are exponentially vanishing on a large semicircle in the complex plane, and so the Cauchy's theorem can be used there without further ado.

2.5 Derivative interactions

Consider now the situation when the matrix element of the potential V_S is a generic low-energy polynomial defined in Eq. (2.10). This is no more true for the potential $\tilde{V}_S(\mathbf{p},\mathbf{q})$, defined in Eq. (2.26). Here, we wish to address the structure of the latter in more detail. The partial-wave expansion of the V_S has the form:

Convoluting Eq. (2.8) with the wave functions, the integrals of the following type emerge

$$A_\ell^a = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\mathscr{Y}^*_{\ell m}(\mathbf{p})(\mathbf{p}^2)^a\langle\mathbf{p}|\psi^{(+)}_{\mathbf{k}}\rangle\,. \tag{2.56}$$

One can now use the identity

$$(\mathbf{p}^2)^a = (\mathbf{p}^2-\mathbf{k}^2+\mathbf{k}^2)^a = (\mathbf{p}^2-\mathbf{k}^2)^a + a(\mathbf{p}^2-\mathbf{k}^2)^{a-1}\mathbf{k}^2 + \cdots\,, \tag{2.57}$$

and rewrite Eq. (2.56) as

$$A_\ell^a = \lim_{\mathbf{r}\to 0}\mathscr{Y}^*_{\ell m}(i\nabla)\sum_{b=0}^{a}\frac{(-1)^{a-b}a!}{b!(a-b)!}\,(\mathbf{k}^2)^b(\mathbf{k}^2+\triangle)^{a-b}\tilde{\psi}^{(+)}_{\mathbf{k}}(\mathbf{r})\,. \tag{2.58}$$

Here, as in Ref. [26], it is assumed that the long-range potential $V_L(r)$ is local and spherically symmetric. Furthermore, consider the case $\ell = 0$ first. Using Schrödinger equation, one then gets

$$(\mathbf{k}^2+\triangle)\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) = \text{const}\cdot V_L(r)\psi^{(+)}_{\mathbf{k}}(\mathbf{r}) \to \text{const}\cdot V_L(0)\psi^{(+)}_{\mathbf{k}}(0)\,,\quad \text{as } \mathbf{r}\to 0\,. \tag{2.59}$$

In case of regularized Yukawa coupling, the quantity $V_L(0)$ is finite.

Acting now with the operator $(\mathbf{k}^2+\triangle)$ on both sides of this equation once more, one gets a term, containing V_L^2, as well as terms with the space derivatives acting on $V_L(r)$. Continuing this operation, we get a string of terms, containing $\nabla_{i_1}\ldots\nabla_{i_k}V_L(r)$. Furthermore, owing to the rotational symmetry,

$$\lim_{r\to 0}\nabla_{i_1}\ldots\nabla_{i_k}V_L(r)=\begin{cases}(\delta_{i_1i_2}\cdots\delta_{i_{k-1}i_k}+\text{perm})V_k\,, & \text{even } k\,,\\ 0 & \text{odd } k\,.\end{cases} \tag{2.60}$$

One could stick to the dimensional regularization here, in which all V_k are finite. Furthermore, in the dimensional regularization, $V_k\sim\mu^k$, where μ denotes the small mass scale of the long-distance potential (the pion mass M_π, in case of Yukawa interactions). We remind the reader that we are dealing here with the long-distance (infrared) problems, for which the details of the ultraviolet renormalization should not matter.

The Kronecker δ-symbols, which are present in Eq. (2.60), can be further contracted with $\nabla_{i_1}\cdots$ acting on the wave function, turning them into the Laplacians $\triangle$ that can be again eliminated with the use of the Schrödinger equation. At the end of the day, for $\ell=0$,

$$A_0^a=\sum_{b=0}^{a}(\mathbf{k}^2)^b h_0^{a-b}\tilde\psi_{\mathbf{k}}^{(+)}(\mathbf{0})\,, \tag{2.61}$$

where the coefficients h_0^{a-b} are expressed through $V_L(0)$ and the derivatives of the potential at the origin. It is important to mention that the mass scale in the derivatives is set by the long-range potential and, therefore, the expansion in the derivatives is converging fast.

Next, consider the case $\ell\neq 0$ and restore the factor $\mathscr{Y}^*_{\ell m}(i\nabla)$ in the expression for A_ℓ^a. This factor contains exactly ℓ derivatives which should be commuted through all potentials to the right. Performing the limit $\mathbf{r}\to 0$, it is straightforward to ensure that

$$\begin{aligned}A_\ell^a &= \sum_{b=0}^{a}(\mathbf{k}^2)^b h_\ell^{a-b}\lim_{\mathbf{r}\to 0}\mathscr{Y}^*_{\ell m}(i\nabla)\tilde\psi_{\mathbf{k}}^{(+)}(\mathbf{r})\\ &\doteq 4\pi\sum_{b=0}^{a}c_\ell(\mathbf{k}^2)^{a-b}h_\ell^b\lim_{r\to 0}i^\ell\frac{\phi_\ell(k,r)}{kr^{\ell+1}}\,Y^*_{\ell m}(\hat k)\,,\end{aligned} \tag{2.62}$$

where

$$c_\ell\delta_{mm'}=\lim_{\mathbf{r}\to 0}\mathscr{Y}^*_{\ell m}(i\nabla)\mathscr{Y}_{\ell m'}(\mathbf{r})=\frac{i^\ell\ell!(2\ell+1)}{4\pi}\delta_{mm'}\,. \tag{2.63}$$

Furthermore, using

$$\lim_{r\to 0}\frac{\phi_\ell(k,r)}{kr^{\ell+1}}=\frac{k^\ell}{f_\ell(k)(2\ell+1)!!}\,, \tag{2.64}$$

one obtains

$$\tilde V_S^\ell(p,q)=\frac{1}{[(2\ell+1)!!]^2 f_\ell^*(p)f_\ell(q)}\,(pq)^\ell\bar V_S^\ell(p,q)\,,$$

$$\bar{V}_S^\ell(p,q) = \sum_{a=0}^{\infty}\sum_{b=0}^{a} \tilde{C}_\ell^{ab}(p^2)^b(q^2)^{a-b}\,. \tag{2.65}$$

In the above expression, the couplings $\tilde{C}_\ell^{ab}$ are expressed through C_ℓ^{ab} in form of the series in the small scale M. In other words, no unnaturally large couplings emerge. This property is crucial for arguing that the sum, given in the above equation, still represents a low-energy polynomial. To summarize, $\tilde{V}_S^\ell(p,q)$ unlike $V_S^\ell(p,q)$, is *not* a low-energy polynomial. The difference is however minimal and boils down to the Jost functions that enter the expression as a multiplicative factor.

At the next step, we carry out the partial-wave expansion in Eq. (2.25) and use the following ansatz for the partial-wave amplitude:

$$B_\ell(p,q;q_0^2+i\varepsilon) = \frac{1}{[(2\ell+1)!!]^2 f_\ell^*(p)f_\ell(q)}\,(pq)^\ell \bar{B}_\ell(p,q;q_0^2+i\varepsilon)\,. \tag{2.66}$$

This gives

$$\bar{B}_\ell(p,q;q_0^2+i\varepsilon) = \bar{V}_S^\ell(p,q) + \int \frac{k^2dk}{(2\pi)^2}\,\frac{k^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\,\frac{\bar{V}_S^\ell(p,k)\bar{B}_\ell(k,q;q_0^2+i\varepsilon)}{k^2-q_0^2-i\varepsilon}\,. \tag{2.67}$$

Let us now define a new amplitude that obeys an integral equation with a regular kernel:

$$\begin{aligned} R_\ell(p,q;q_0^2) &= \bar{V}_S^\ell(p,q) \\ &+ \int \frac{k^2dk}{(2\pi)^2}\,\frac{k^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\,\frac{\bar{V}_S^\ell(p,k)R_\ell(k,q;q_0^2)-\bar{V}_S^\ell(p,q_0)R_\ell(q_0,q;q_0^2)}{k^2-q_0^2}\,. \end{aligned} \tag{2.68}$$

These two amplitudes on the energy shell are related by

$$\begin{aligned} \bar{B}_\ell(q_0,q_0;q_0^2+i\varepsilon) &= R_\ell(q_0,q_0;q_0^2) + \bar{B}_\ell(q_0,q_0;q_0^2+i\varepsilon)R_\ell(q_0,q_0;q_0^2) \\ &\times \int \frac{k^2dk}{(2\pi)^2}\,\frac{k^{2\ell}}{[(2\ell+1)!!]^2|f_\ell(k)|^2}\,\frac{1}{k^2-q_0^2-i\varepsilon}\,. \end{aligned} \tag{2.69}$$

Note that $R_\ell(p,q;q_0^2)$, like $\bar{V}_S^\ell(p,q)$, is a low-energy polynomial. Identifying $K_\ell^M(q_0^2) = \big[R_\ell(q_0,q_0;q_0^2)\big]^{-1}$, we finally get:

$$B_\ell(q_0) = \frac{q_0^{2\ell}}{|f_\ell(q_0)|^2[(2\ell+1)!!]^2}\left\{K_\ell^M(q_0,q_0) - \langle G_L^\ell(q_0)\rangle\right\}^{-1}. \tag{2.70}$$

Finally, using Eqs. (2.27) and (2.53), one arrives at the modified effective-range expansion as given in Eq. (2.3), with $K_\ell^M(q_0^2)$ being by a low-energy polynomial.

To summarize, using effective field theory methods, we have rederived the modified effective range expansion formula of Ref. [26], where the effects of the long-range interactions are separated and included in the functions $f_\ell(q)$ and $M_\ell(q)$ that do not depend on the short-range potential V_S. This neat separation is however based on the assumption that the long-range potential $V_L(r)$ is local. The most important cases of the long-range force:

the one-pion exchange as well as Coulomb interactions are exactly of this type. It can be further expected that, with some effort, the method could be generalized to the case of a finite sum $V_1(r) + (\triangle V_2(r) + V_2(r)\triangle) + \nabla V_3(r)\nabla + \ldots$, albeit the final formula, probably, takes a more complicated form (Here, the couplings in front of $V_1(r), V_2(r), V_3(r), \ldots$ are assumed to be of the natural size.). In this paper, we are not pursuing this idea further. On the other hand, a most generic non-local long-range potential (say, the separable potential with a very smooth cutoff) is, most probably, is not amenable for this kind of treatment at all. Putting things differently, in general, there are two mass scales present in the potential V_L – the ones associated with the momentum transfer and to the relative momentum in the CM frame, respectively. The long-range potential, in which the former scale is small whereas the latter scale is of a natural size, can be treated with the method, similar to the described above.

3 Modified Lüscher equation

3.1 Derivation of the quantization condition

In a finite box, The Green function $G_L(q_0^2)$, which enters in the equation for $T_S(q_0^2)$, can be expanded in a sum over all eigenvectors of the pertinent Hamiltonian in a finite volume:

$$\langle \mathbf{p}|G_L(q_0^2)|\mathbf{q}\rangle = \sum_n \frac{\langle \mathbf{p}|\psi_n\rangle\langle\psi_n|\mathbf{q}\rangle}{q_n^2 - q_0^2}\,. \tag{3.1}$$

One may further define

$$|\psi_n\rangle = (1 + G_0(q_n^2)T_L(q_n^2))|\phi_n\rangle\,. \tag{3.2}$$

Since the states $|\psi_n\rangle$ are eigenvectors, one gets

$$(q_n^2 - \mathbf{K}^2 - V_L)|\psi_n\rangle = (q_n^2 - \mathbf{K}^2)|\phi_n\rangle = 0\,, \tag{3.3}$$

where $\mathbf{K}$ stands for the three-momentum *operator*. The solution of this equation is given by

$$|\phi_n\rangle = \frac{1}{L^3}\sum_{\mathbf{k}} c_{\mathbf{k}}^n|\mathbf{k}\rangle\,, \qquad \sum_{\mathbf{k}} \mathbf{k}^2 c_{\mathbf{k}}^n = q_n^2 \sum_{\mathbf{k}} c_{\mathbf{k}}^n\,. \tag{3.4}$$

The coefficients $c_{\mathbf{k}}^n$ obey the following conditions

$$\frac{1}{L^3}\sum_{\mathbf{k}} c_{\mathbf{k}}^n c_{\mathbf{k}}^m = \delta^{mn}\,, \qquad \sum_n c_{\mathbf{k}}^n c_{\mathbf{q}}^n = L^3\delta_{\mathbf{pq}}\,, \qquad |\mathbf{k}\rangle = \sum_n c_{\mathbf{k}}^n|\phi_n\rangle\,. \tag{3.5}$$

In the infinite volume, the states $|\phi_n\rangle$ are plane waves with the same energy as the energy of the scattering state. Hence, the sum over n contains exactly one term corresponding to this plane wave. In a finite volume, the states $|\phi_n\rangle$ are always superpositions of the plane waves, but the distribution is strongly peaked around $\mathbf{k}^2 = q_n^2$.

The energy spectrum can be determined by a finite-volume analog of a quantity that is present in Eq. (2.15).

$$B^{nm} = \langle\phi_n|B(q_0^2)|\phi_m\rangle = \langle\phi_n|(1+T_L(q_0^2)G_0(q_0^2))T_S(q_0^2)(G_0(q_0^2)T_L(q_0^2)+1)|\phi_m\rangle\,. \quad (3.6)$$

On the mass shell, $q_0^2 = q_n^2 = q_m^2$, this quantity can be obtained from the solution of the equation

$$B^{nm} = \tilde{V}_S^{nm} + \sum_l \tilde{V}_S^{nl} \frac{1}{q_l^2 - q_0^2} B^{lm}\,. \quad (3.7)$$

where

$$\tilde{V}_S^{nm} = \langle\psi_n|V_S|\psi_m\rangle\,. \quad (3.8)$$

Using the partial-wave expansion

$$v_{\ell m}^n = \frac{1}{L^3}\sum_{\mathbf{p}} \mathscr{Y}_{\ell m}^*(\mathbf{p})\langle\mathbf{p}|\psi_n\rangle\,, \quad (3.9)$$

we get

$$\langle\psi_n|V_S|\psi_m\rangle = 4\pi \sum_{\ell m,\ell' m'} (v_{\ell m}^n)^*\, \bar{V}_S^\ell(q_n,q_m)\delta_{\ell\ell'}\delta_{mm'} v_{\ell' m'}^m\,. \quad (3.10)$$

The coefficients in the low-energy polynomial $\bar{V}_S^\ell(q_n,q_m)$ in the above expression differ from a similar quantity in Eq. (2.65) by exponential corrections. For the time being, we neglect all such contributions and discuss them only very briefly at the end of this section.

The next steps in the derivation repeat those in the infinite volume. We use following ansatz for the matrix B

$$B^{nm} = 4\pi \sum_{\ell m} (v_{\ell m}^n)^*\, \bar{B}_{\ell m,\ell' m'}(q_n,q_m;q_0^2) v_{\ell' m'}^m\,, \quad (3.11)$$

and get

$$\begin{aligned} B_{\ell m,\ell' m'}(q_n,q_m;q_0^2) &= \bar{V}_S^\ell(q_n,q_m)\delta_{\ell\ell'}\delta_{mm'} \\ &+ \sum_k \sum_{\ell'' m''} \bar{V}_S^\ell(q_n,q_k) \frac{4\pi v_{\ell m}^k \left(v_{\ell'' m''}^k\right)^*}{q_k^2 - q_0^2} B_{\ell'' m'',\ell' m'}(q_k,q_m;q_0^2)\,. \end{aligned} \quad (3.12)$$

Define again

$$\begin{aligned} R_{\ell m,\ell' m'}(q_n,q_m;q_0^2) &= \bar{V}_S^\ell(q_n,q_m)\delta_{\ell\ell'}\delta_{mm'} + \sum_k \sum_{\ell'' m''} \frac{4\pi v_{\ell m}^k \left(v_{\ell'' m''}^k\right)^*}{q_k^2 - q_0^2} \\ &\times \left(\bar{V}_S^\ell(q_n,q_k) R_{\ell'' m'',\ell' m'}(q_k,q_m;q_0^2) - \bar{V}_S^\ell(q_n,q_0) R_{\ell'' m'',\ell' m'}(q_0,q_m;q_0^2)\right)\,. \end{aligned} \quad (3.13)$$

The infinite-volume limit in this (subtracted) equation can be performed, and the quantity $R_{\ell m,\ell' m'}(p,q;q_0^2)$ tends to $\delta_{\ell\ell'}\delta_{mm'}R_\ell(p,q;q_0^2)$ in this limit. On the mass shell, with $q_n^2 = q_m^2 = q_0^2$ we, therefore, obtain

$$
\begin{aligned}
& B_{\ell m,\ell' m'}(q_0,q_0;q_0^2) = \delta_{\ell\ell'}\delta_{mm'}R_\ell(q_0,q_0;q_0^2) \\
+ & \sum_{\ell'' m''} R_\ell(q_0,q_0;q_0^2) H_{\ell m,\ell'' m''}(q_0) B_{\ell'' m'',\ell' m'}(q_0,q_0;q_0^2)\,,
\end{aligned} \tag{3.14}
$$

where

$$
H_{\ell m,\ell' m'}(q_0) = 4\pi \sum_k \frac{v^k_{\ell m}\left(v^k_{\ell' m'}\right)^*}{q_k^2 - q_0^2} = \frac{4\pi}{L^6}\sum_{\mathbf{p},\mathbf{q}} \mathscr{Y}^*_{\ell m}(\mathbf{p})\langle\mathbf{p}|G_L(q_0^2)|\mathbf{q}\rangle \mathscr{Y}_{\ell' m'}(\mathbf{q})\,. \tag{3.15}
$$

The modified quantization condition then takes the form $\det \mathscr{A} = 0$, where

$$
\mathscr{A}_{\ell m,\ell' m'}(q_0) = \delta_{\ell\ell'}\delta_{mm'}K^M_\ell(q_0^2) - H_{\ell m,\ell' m'}(q_0)\,. \tag{3.16}
$$

3.2 Calculation of the function $H_{\ell m,\ell' m'}(q_0)$

Owing to our choice of the superregular long-range potential, the quantity $H_{\ell m,\ell m}(q_0)$ is free of the ultraviolet divergences for $\ell,\ell' \leq \ell_{\sf max}$. However, one still needs a finite renormalization, in order to ensure that the definition of the function $H_{\ell m,\ell' m'}(q_0)$ in a finite volume is consistent with its infinite-volume counterpart. Below, we shall consider the cases $q_0^2 < 0$ and $q_0^2 > 0$ separately.

3.2.1 The case $q_0^2 < 0$

In this case, a consistent definition of the loop function is given by

$$
H_{\ell m,\ell' m'}(q_0) = (H_{\ell m,\ell' m'}(q_0) - H^\infty_{\ell m,\ell' m'}(q_0)) + 4\pi\delta_{\ell\ell'}\delta_{mm'}M_\ell(q_0)\,. \tag{3.17}
$$

It should be mentioned here that the functions $f_\ell(p,r)$ and hence $M_\ell(p)$ are analytic in the upper half of the complex p-plane.[1] Consequently, $M_\ell(q_0)$ is well-defined for negative values of q_0^2, taking into account the presence of the infinitesimal positive imaginary part in q_0^2. Furthermore, using Eqs. (2.14), (3.16) and applying the Poisson formula, one gets:

$$
H_{\ell m,\ell' m'}(q_0) - H^\infty_{\ell m,\ell' m'}(q_0) = H^{(1)}_{\ell m,\ell' m'}(q_0) + H^{(2)}_{\ell m,\ell' m'}(q_0) + H^{(3)}_{\ell m,\ell' m'}(q_0)\,, \tag{3.18}
$$

where

$$
\begin{aligned}
H^{(1)}_{\ell m,\ell' m'}(q_0) &= 4\pi\sum_{\mathbf{n}}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathscr{Y}^*_{\ell m}(\mathbf{p})(e^{i\mathbf{p}\mathbf{n}L}-1)\mathscr{Y}_{\ell' m'}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\,,\\
H^{(2)}_{\ell m,\ell' m'}(q_0) &= 4\pi\sum_{\mathbf{n},\mathbf{s}}\int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{d^3\mathbf{p}}{(2\pi)^3}\,(e^{i\mathbf{n}\mathbf{p}L-i\mathbf{s}\mathbf{q}L}-1)\,\frac{\mathscr{Y}^*_{\ell m}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\,\langle\mathbf{p}|T_L(q_0^2)|\mathbf{q}\rangle\,\frac{\mathscr{Y}_{\ell' m'}(\mathbf{q})}{\mathbf{q}^2-q_0^2}\,,
\end{aligned}
$$

[1] Considering the Born series of the Green function $G_L(q_0^2)$, it is easy to get convinced that $M_\ell(q_0)$ is real everywhere below threshold.

$$H^{(3)}_{\ell m,\ell' m'}(q_0) = 4\pi \int \frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{\mathscr{Y}^*_{\ell m}(\mathbf{p})}{\mathbf{p}^2-q_0^2}\left(\langle \mathbf{p}|T_L(q_0^2)|\mathbf{q}\rangle - \langle \mathbf{p}|T_L^\infty(q_0^2)|\mathbf{q}\rangle\right)\frac{\mathscr{Y}_{\ell' m'}(\mathbf{q})}{\mathbf{q}^2-q_0^2}\,. \quad (3.19)$$

The first term here is the standard Lüscher zeta-function. In order to calculate the remaining two terms, let us consider the Lippmann-Schwinger equation for $T_L(q_0^2)$ (a finite-volume counterpart of Eq. (2.14)). Carrying out the partial-wave expansion

$$\langle \mathbf{p}|V_L|\mathbf{q}\rangle = 4\pi \sum_{\ell m} Y_{\ell m}(\hat p) V_L^\ell(p,q) Y^*_{\ell m}(\hat q)\,,$$
$$\langle \mathbf{p}|T_L(q_0^2)|\mathbf{q}\rangle = 4\pi \sum_{\ell m,\ell' m'} Y_{\ell m}(\hat p) T_L^{\ell m,\ell' m'}(p,q;q_0^2) Y^*_{\ell' m'}(\hat q)\,, \quad (3.20)$$

one gets

$$T_L^{\ell m,\ell' m'}(p,q;q_0^2) = \delta^{\ell\ell'}\delta^{mm'} V_L^\ell(p,q)$$
$$+\, 4\pi \sum_{\ell'' m'} \int_0^\infty \frac{k^2 dk}{(2\pi)^3}\, V_L^\ell(p,k)\, \frac{f_{\ell m,\ell'' m''}(k) - f_{\ell m,\ell'' m''}(-k)}{k^2-q_0^2}\, T_L^{\ell'' m'',\ell' m'}(p,q;q_0^2)\,, \quad (3.21)$$

where

$$f_{\ell m,\ell'' m''}(k) - f_{\ell m,\ell'' m''}(-k) = \int d\Omega\, Y^*_{\ell m}(\hat k) \sum_{\mathbf{n}} e^{-i\mathbf{k}\mathbf{n}L} Y_{\ell'' m''}(\hat k)\,. \quad (3.22)$$

The quantity $f_{\ell m,\ell'' m''}(k)$ is analytic in the upper half-plane of the variable k.

Furthermore, one may straightforwardly ensure that

$$V_L^\ell(p,q) = V_L^\ell(-p,q) = V_L^\ell(p,-q) = V_L^\ell(-p,-q)\,, \quad (3.23)$$

and, hence, from Eq. (3.21) one concludes that

$$T_L^{\ell m,\ell' m'}(p,q;q_0^2) = T_L^{\ell m,\ell' m'}(-p,q;q_0^2) = T_L^{\ell m,\ell' m'}(p,-q;q_0^2) = T_L^{\ell m,\ell' m'}(-p,-q;q_0^2)\,. \quad (3.24)$$

This means that the integration over the variable k can be extended over the whole real axis, from -∞ to $+\infty$:

$$T_L^{\ell m,\ell' m'}(p,q;q_0^2) = \delta^{\ell\ell'}\delta^{mm'} V_L^\ell(p,q)$$
$$+\, 4\pi \sum_{\ell'' m'} \int_{-\infty}^\infty \frac{k^2 dk}{(2\pi)^3}\, V_L^\ell(p,k)\, \frac{f_{\ell m,\ell'' m''}(k)}{k^2-q_0^2}\, T_L^{\ell'' m'',\ell' m'}(p,q;q_0^2)\,, \quad (3.25)$$

Using now the fact that $f_{\ell m,\ell'' m''}(k)$ is analytic in the upper half-plane, one may shift the variables $p,q,k \to p,q,k+i\sigma$. The value of σ is restricted by the singularities appearing in the free Green function as well as in the potential $V_L^\ell(p,q)$. Namely, σ must fulfill the condition $\sigma < |q_0|$, in order the free Green function stays regular. The restriction coming from the potential does not depend on q_0. For instance, in case of Yukawa interaction, we have $\sigma < M_\pi/2$. The quantity σ should obey both conditions. Performing the contour shift in the second and third lines of Eq. (3.19) as well, one sees that the finite-volume corrections to $H_{\ell m,\ell' m'}(q_0)$ are suppressed by the factor $e^{-\sigma L}$.

3.2.2 The case $q_0^2 > 0$

The infinite-volume limit of the quantity $H_{\ell m,\ell m}(q_0)$ in this case implies using the principal value prescription. Furthermore, from unitarity one can straightforwardly conclude that

$$((2\ell+1)!!)^2 \langle G_L^\ell(q_0)\rangle = ((2\ell+1)!!)^2 \langle G_L^\ell(q_0)\rangle_{\mathsf{p.v.}} + i\frac{q_0^{2\ell+1}}{4\pi}\,\frac{(1+F^\ell(q_0))^2}{1-iq_0 R_L^\ell(q_0,q_0;q_0^2)/(4\pi)}\,, \tag{3.26}$$

where

$$F_\ell(q_0) = \frac{1}{q_0^\ell}\,\mathsf{p.v.}\int_0^\infty \frac{p^2dp}{2\pi^2}\,\frac{p^\ell R_L^\ell(p,q_0;q_0^2)}{p^2-q_0^2}\,. \tag{3.27}$$

Here, R_L^ℓ denotes the K-matrix for the scattering on the long-range potential. Furthermore, since, by definition, $\mathrm{Im}\,\langle G_L^\ell(q_0)\rangle_{\mathsf{p.v.}} = 0$, with the use of Eqs. (2.3) and (2.53) one obtains, on the one hand,

$$\mathrm{Im}\,\langle G_L^\ell(q_0)\rangle = \frac{q_0^{2\ell+1}}{4\pi((2\ell+1)!!)^2}\,\frac{1}{|f_\ell(q_0)|^2}\,, \tag{3.28}$$

and, on the other hand,

$$\begin{aligned}\mathrm{Im}\,\langle G_L^\ell(q_0)\rangle &= \frac{q_0^{2\ell+1}}{4\pi((2\ell+1)!!)^2}\,\frac{(1+F^\ell(q_0))^2}{1+q_0^2 R_L^\ell(q_0,q_0;q_0^2)/(4\pi)^2}\\ &= \frac{q_0^{2\ell+1}}{4\pi((2\ell+1)!!)^2}\left|1+\frac{1}{q_0^\ell}\int_0^\infty \frac{p^2dp}{2\pi^2}\,\frac{p^\ell T_L^\ell(p,q_0;q_0^2)}{p^2-q_0^2-i\varepsilon}\right|^2 .\end{aligned} \tag{3.29}$$

Using now Eq. (2.19) and performing the partial-wave expansion of the on-shell function in analogy with Eq. (2.35), we obtain:

$$\frac{\phi_\ell(q_0,r)}{q_0 r} = j_\ell(q_0 r) + \int_0^\infty \frac{p^2dp}{2\pi^2}\,\frac{j_\ell(pr)T_L^\ell(p,q_0;q_0^2)}{p^2-q_0^2-i\varepsilon}\,, \tag{3.30}$$

where $\phi_\ell(q_0,r)$ is the on-shell wave function. Performing the limit $r\to 0$ in this equation, we get:

$$\lim_{r\to 0}(2\ell+1)!!\,\frac{\phi_\ell(q_0 r)}{(q_0 r)^{\ell+1}} = 1+\frac{1}{q_0^\ell}\int_0^\infty \frac{p^2dp}{2\pi^2}\,\frac{p^\ell T_L^\ell(p,q_0;q_0^2)}{p^2-q_0^2-i\varepsilon}\,. \tag{3.31}$$

Finally, from Eq. (2.64), one concludes that Eqs. (3.28) and (3.29) are consistent. This represents a nice check of our approach.

A consistent definition of the quantity $H_{\ell m,\ell' m'}(q_0)$ is given by

$$H_{\ell m,\ell' m'}(q_0) = (H_{\ell m,\ell' m'}(q_0) - H^\infty_{\ell m,\ell' m'}(q_0)) + \delta_{\ell\ell'}\delta_{mm'}((2\ell+1)!!)^2\langle G_L^\ell(q_0)\rangle_{\mathsf{p.v.}}\,, \tag{3.32}$$

where $H^\infty_{\ell m,\ell' m'}(q_0)$ is defined by a counterpart of Eq. (3.15), with sums replaced by integrals with the principal-value prescription everywhere. Note also that, since the potential is superregular, no ultraviolet divergences arise except in the free loop containing no potential exchange. There, it can be handled, as usual, by using dimensional regularization and cancels anyway in the difference of the finite-volume and the infinite-volume contributions.

Neglecting exponentially suppressed contributions from the long-range interactions, one could reduce the calculation of the function $H_{\ell m,\ell' m'}(q_0)$ to the solution of the linear equation in the angular momentum basis. This equation has the following form:

$$H_{\ell m,\ell' m'}(q_0) = H^0_{\ell m,\ell' m'}(q_0) + \sum_{\ell m,\ell'' m''} H^0_{\ell m,\ell'' m''}(q_0) q_0^{2\ell} R_L^{\ell''}(q_0) H_{\ell'' m'',\ell' m'}(q_0) \,. \tag{3.33}$$

Here, $R_L^\ell(q_0) = 4\pi \tan\sigma_\ell(q_0)/q_0$ are the partial-wave on-shell K-matrices, corresponding to the long-range potential, and

$$H^0_{\ell m,\ell' m'}(q_0) = \frac{4\pi}{L^3} \sum_{\mathbf{k}} \frac{\mathscr{Y}^*_{\ell m}(\mathbf{k})\mathscr{Y}_{\ell' m'}(\mathbf{k})}{\mathbf{k}^2 - q_0^2} \,. \tag{3.34}$$

This quantity can be expressed through a linear combination of the Lüscher zeta-functions. The quantity $H^\infty_{\ell m,\ell' m'}(q_0)$ should be taken equal to zero in this case.

Note however that neglecting exponential corrections coming from the long-range potential might be dangerous. This is seen, for example, from the fact that, below threshold, $q_0^2 < 0$, the quantity $R_L^\ell(q_0)$ develops the t-channel singularity that was mentioned earlier, whereas the exact function $H_{\ell m,\ell' m'}(q_0)$ is of course regular there. The derivation of the modified quantization condition, which was presented above, nicely demonstrates the origin of the problem and a way to circumvent it. In fact, the problem is handmade and is not present in Eq. (3.16). It emerges first, when one tries to evaluate $H_{\ell m,\ell' m'}(q_0)$ from Eq. (3.33) and continue analytically below threshold. All this is perfectly consistent with the discussion in the recent paper [16].

3.3 Partial-wave mixing

Above threshold, the modified zeta-function is determined from Eq. (3.33) or from the pertinent equation in the plane-wave basis. Since V_L is a long-range potential, it is expected that many partial waves will contribute to this expression. However, this is not a problem, since V_L is a well-known function, with the parameters that are determined very precisely elsewhere (e.g., the pion mass and the pion axial-vector coupling, in case of the one-pion exchange potential). Hence, the solution of Eq. (3.33) does not imply a fit to lattice data. On the other hand, the short-range interaction, encoded in the function $K^M_\ell(q_0^2)$, is determined from the fit. One expects that the partial-wave mixing effect in the modified Lüscher equation is small, exactly because of the short-range nature of these interactions.

3.4 Exponentially suppressed effects

Up to now, we have consistently dropped the exponentially suppressed effects. However, as mentioned already in the introduction, these effects can turn relatively large, owing to the small mass scale. In the case of, say, NN scattering, one may indirectly estimate the size of the exponential effects, comparing the finite-volume spectra obtained in the plane-wave basis with the solutions of the modified Lüscher equation with the same input. A simpler method to estimate the size of the exponential effects is the comparison of the modified Lüscher functions $H_{\ell m,\ell' m'}(q_0)$, calculated in the plane wave basis and in the angular momentum basis. This comparison does not involve any parameters that characterize the short-range interactions.

There is one place, however, where one already knows that the exponential effects are important. We remind the reader that the energy levels, which lie on the t-channel cut, are indeed observed in the NN system on the lattice. Physical bound states cannot be present there and, hence, the infinite-volume limit of the Lüscher equation does not predict a pole in this region. The observed poles can only emerge because of the exponential contributions.

4 Comparison with the existing approaches

In the recent paper [16] a modified two-body quantization condition has been derived in the presence of both the long- and short-range forces. The authors present their central result in two different forms. Namely, Eq. (3.63) of that paper is written down in a plane wave *and* angular momentum basis. From this point of view, it bears strong resemblance with the approach of Ref. [14], however, with a significant conceptual difference. Namely, all short-range interactions in Ref. [16] are summed up and enter the quantization condition through an auxiliary *on-shell* K-matrix $\bar{K}^{\mathsf{os}}$ that can be parameterized in terms of few low-energy couplings. On the contrary, in Ref. [14] the long- and short-range interactions are treated on equal footing. Hence, as pointed out already in the introduction, the validity of the approach of Ref. [14] rests on the assumption of the convergence of the chiral expansion, whereas the approach of Ref. [14] is less restrictive in this sense.

Furthermore, in Eqs. (5.3) and (5.4) of Ref. [16] the authors recast their central result in the angular momentum basis. This result bears close analogy to our modified quantization condition. For example, the quantity F^T from their Eq. (5.4) is similar to our modified Lüscher function $H_{\ell m,\ell' m'}(q_0)$. The main difference, as already noted in the introduction, is that the authors of Ref. [16] propose a two-step procedure for the analysis of lattice data. Namely, at the first step, an auxiliary matrix $\bar{K}^{\mathsf{os}}$ is determined from data. At the next step, $\bar{K}^{\mathsf{os}}$ is substituted into the integral equation which is solved to obtain the physical K-matrix. We propose to unite these two steps in one – in our approach, the auxiliary K-matrix is related to the physical one at the same CM energy through a simple algebraic expression.

5 Conclusions

i) In this paper, we have derived a modified Lüscher equation in the presence of both the long-range and short-range interactions. The presence of the former leads to the (interrelated) conceptual difficulties. Namely,

  – Partial-wave expansion may converge slowly, and hence there could be a significant admixture of the higher partial waves in the Lüscher equation that complicates the analysis of data.

– The long-range interactions lead to a t-channel cut in the scattering amplitude that moves very close to the threshold, if the range of the interactions increases. Using lattice energy levels that lie below the t-channel threshold in the Lüscher equation is inconsistent.

– The exponentially suppressed contributions could be still significant for not so large values of L.

Our approach which, loosely speaking, represents a re-formulation of the modified effective-range expansion of Ref. [26] in a finite volume, is capable to address all above challenges.

ii) The alternative approaches appeared recently in the literature. In the paper, a detailed comparison to these approaches are given. We argue that with the use of the newly proposed approach, which is conceptually the closest to the original Lüscher framework, it should be possible to carry out the analysis of data with a substantially lesser effort.

iii) The modified Lüscher function, which incorporates the long-range interaction, is a central ingredient of our approach. In the present paper we consider the evaluation of this function in great detail, paying particular attention to the issues of the ultraviolet divergences and renormalization. Once this function, which does not depend on the unknown parameters of the short-range force, is calculated and tabulated, the analysis of data exactly follows the standard pattern. An explicit calculation of this function is however a rather challenging enterprise and will be discussed in a separate publication.

iv) Note that in this paper we deliberately ignored all issues related with spin of particles, moving frames, relativistic effects, etc. All this in inessential in the context of the problems considered and would only blur the discussion.

v) It remains to be seen, whether Coulomb interaction can be consistently treated in the same manner, and whether the results would add something substantial to the findings of Ref. [25]. Here, it should be also mentioned that, due to the removal of the zero mode of the Coulomb field in case of periodic boundary conditions, the resulting Lagrangian is no more local. This, in its turn, might cause problems in the matching of the non-relativistic effective field theory, which is used for the derivation of the Lüscher equation, to its relativistic counterpart (see, e.g., Ref. [33]). In this context, it would be interesting to explore the possibility of using different boundary conditions.

vi) The major challenge consists in using the same method in the three-particle problem. For instance, it remains to be seen, whether the long-range one-pion exchange force in the three-nucleon system can be separated as neatly from the short-range interactions as done in case of the nucleon-nucleon scattering.

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References

[1] Martin Lüscher. Two particle states on a torus and their relation to the scattering matrix. *Nucl. Phys. B*, 354:531–578, 1991.

[2] K. Rummukainen and Steven A. Gottlieb. Resonance scattering phase shifts on a nonrest frame lattice. *Nucl. Phys. B*, 450:397–436, 1995.

[3] Michael Lage, Ulf-G. Meißner, and Akaki Rusetsky. A Method to measure the antikaon-nucleon scattering length in lattice QCD. *Phys. Lett. B*, 681:439–443, 2009.

[4] V. Bernard, M. Lage, U.-G. Meißner, and A. Rusetsky. Scalar mesons in a finite volume. *JHEP*, 01:019, 2011.

[5] Song He, Xu Feng, and Chuan Liu. Two particle states and the S-matrix elements in multi-channel scattering. *JHEP*, 07:011, 2005.

[6] Chuan Liu, Xu Feng, and Song He. Two particle states in a box and the S-matrix in multi-channel scattering. *Int. J. Mod. Phys. A*, 21:847–850, 2006.

[7] Maxwell T. Hansen and Stephen R. Sharpe. Multiple-channel generalization of Lellouch-Luscher formula. *Phys. Rev. D*, 86:016007, 2012.

[8] Raul A. Briceno and Zohreh Davoudi. Moving multichannel systems in a finite volume with application to proton-proton fusion. *Phys. Rev. D*, 88(9):094507, 2013.

[9] Ning Li and Chuan Liu. Generalized Lüscher formula in multichannel baryon-meson scattering. *Phys. Rev. D*, 87(1):014502, 2013.

[10] Peng Guo, Jozef Dudek, Robert Edwards, and Adam P. Szczepaniak. Coupled-channel scattering on a torus. *Phys. Rev. D*, 88(1):014501, 2013.

[11] Luka Leskovec and Sasa Prelovsek. Scattering phase shifts for two particles of different mass and non-zero total momentum in lattice QCD. *Phys. Rev. D*, 85:114507, 2012.

[12] C. h. Kim, C. T. Sachrajda, and Stephen R. Sharpe. Finite-volume effects for two-hadron states in moving frames. *Nucl. Phys. B*, 727:218–243, 2005.

[13] M. Gockeler, R. Horsley, M. Lage, U.-G. Meißner, P. E. L. Rakow, A. Rusetsky, G. Schierholz, and J. M. Zanotti. Scattering phases for meson and baryon resonances on general moving-frame lattices. *Phys. Rev. D*, 86:094513, 2012.

[14] Lu Meng and E. Epelbaum. Two-particle scattering from finite-volume quantization conditions using the plane wave basis. *JHEP*, 10:051, 2021.

[15] André Baião Raposo and Maxwell T. Hansen. The Lüscher scattering formalism on the t-channel cut. *PoS*, LATTICE2022:051, 2023.

[16] André Baião Raposo and Maxwell T. Hansen. Finite-volume scattering on the left-hand cut. 11 2023.

[17] Jeremy R. Green, Andrew D. Hanlon, Parikshit M. Junnarkar, and Hartmut Wittig. Weakly bound H dibaryon from SU(3)-flavor-symmetric QCD. *Phys. Rev. Lett.*, 127(24):242003, 2021.

[18] Meng-Lin Du, Arseniy Filin, Vadim Baru, Xiang-Kun Dong, Evgeny Epelbaum, Feng-Kun Guo, Christoph Hanhart, Alexey Nefediev, Juan Nieves, and Qian Wang. Role of Left-Hand Cut Contributions on Pole Extractions from Lattice Data: Case Study for Tcc(3875)+. *Phys. Rev. Lett.*, 131(13):131903, 2023.

[19] Maxwell T. Hansen and Stephen R. Sharpe. Relativistic, model-independent, three-particle quantization condition. *Phys. Rev.*, D90(11):116003, 2014.

[20] Maxwell T. Hansen and Stephen R. Sharpe. Expressing the three-particle finite-volume spectrum in terms of the three-to-three scattering amplitude. *Phys. Rev.*, D92(11):114509, 2015.

[21] Hans-Werner Hammer, Jin-Yi Pang, and A. Rusetsky. Three-particle quantization condition in a finite volume: 1. The role of the three-particle force. *JHEP*, 09:109, 2017.

[22] H. W. Hammer, J. Y. Pang, and A. Rusetsky. Three particle quantization condition in a finite volume: 2. General formalism and the analysis of data. *JHEP*, 10:115, 2017.

[23] M. Mai and M. Döring. Three-body Unitarity in the Finite Volume. *Eur. Phys. J.*, A53(12):240, 2017.

[24] Maxim Mai and Michael Döring. Finite-Volume Spectrum of $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$ Systems. *Phys. Rev. Lett.*, 122(6):062503, 2019.

[25] Silas R. Beane and Martin J. Savage. Two-Particle Elastic Scattering in a Finite Volume Including QED. *Phys. Rev. D*, 90(7):074511, 2014.

[26] H. van Haeringen and L. P. Kok. MODIFIED EFFECTIVE RANGE FUNCTION. *Phys. Rev. A*, 26:1218–1225, 1982.

[27] Xinwei Kong and Finn Ravndal. Coulomb effects in low-energy proton proton scattering. *Nucl. Phys. A*, 665:137–163, 2000.

[28] A. M. Badalian, L. P. Kok, M. I. Polikarpov, and Yu. A. Simonov. Resonances in Coupled Channels in Nuclear and Particle Physics. *Phys. Rept.*, 82:31–177, 1982.

[29] Marvin L. Goldberger and Kenneth M. Watson. *Collision Theory.* Dover Publications, 10 2004.

[30] Michael C. Birse, Judith A. McGovern, and Keith G. Richardson. A Renormalization group treatment of two-body scattering. *Phys. Lett. B*, 464:169–176, 1999.

[31] James V. Steele and R. J. Furnstahl. Removing pions from two nucleon effective field theory. *Nucl. Phys. A*, 645:439–461, 1999.

[32] Michael G. Fuda and James S. Whiting. Generalization of the Jost Function and Its Application to Off-Shell Scattering. *Phys. Rev. C*, 8:1255–1261, 1973.

[33] Zohreh Davoudi, James Harrison, Andreas Jüttner, Antonin Portelli, and Martin J. Savage. Theoretical aspects of quantum electrodynamics in a finite volume with periodic boundary conditions. *Phys. Rev. D*, 99(3):034510, 2019.