Amplification and Losses in Optical Channel

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Introduction

Amplified spontaneous emission (ASE) — is a process of photon's spontaneous by atoms of the medium which generates signals that are uncorrelated in phase with original signal. Leads to noise in experiments.

The main task of the research is to describe the propagation of coherent states through an optical channel with loss and amplifiers and to evaluate impact of ASE on measurement results.

Wave equation

Let us describe a theory of electromagnetic field in linear medium which is enclosed in cavity between two separated walls parallel to x-y plane by firstly considering Maxwell's equations in integral form:

$$\oint_s \mathbf{D} \cdot d\mathbf{s} = \rho, \tag{1}$$

$$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0, \tag{2}$$

$$\oint_l \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_s \mathbf{B} \cdot d\mathbf{s}, \tag{3}$$

$$\oint_l \mathbf{H} \cdot d\mathbf{l} = \frac{\partial}{\partial t} \int_s \mathbf{D} \cdot d\mathbf{s}. \tag{4}$$

We consider an electric field vector which is polarized in the x-axis. Thus, the magnetic field is polarized in the y-direction.

Choosing the little section l_1, let us calculate the left integral in Eqs.(3):

$$\oint_{l_1} \mathbf{E} \cdot d\mathbf{l} = (E_x(z+dz) - E_x(z))dx = \frac{\partial E_x}{\partial z} dxdz. \tag{5}$$

Similarly for the right integral:

$$-\frac{\partial}{\partial t} \int_s \mathbf{B} \cdot d\mathbf{s} = -\frac{\partial B_y}{\partial t} dxdz. \tag{6}$$

Due to Eqs.(3):

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}. \tag{7}$$

Utilizing Eqs.(4), one can notice that

$$\frac{\partial H_y}{\partial z} = -\frac{\partial D_x}{\partial t} \tag{8}$$

holds.

Noticing that medium is linear, one can match Eqs.(7) and (8) and get a wave equation for the field:

$$\varepsilon\mu\frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial z^2} \tag{9}$$

and get a specific equality between two values E_x and H_y:

$$E_x = \sqrt{\frac{\mu}{\varepsilon}} H_y. \tag{10}$$

One of the solutions of this equation is

$$E_x(z,t) = A\exp(i(kz-\omega t)) + A^* \exp(-i(kz-\omega t)), \tag{11}$$

$$H_y(z,t) = \sqrt{\frac{\mu}{\varepsilon}}(A\exp(i(kz-\omega t)) + A^* \exp(-i(kz-\omega t))), \tag{12}$$

which obeys a harmonic oscillator with frequency ω.

Let us define two real numbers p and q:

$$p = \sqrt{\varepsilon V}(A + A^*), \tag{13}$$

$$q = i\frac{\sqrt{\varepsilon V}}{\omega}(A - A^*), \tag{14}$$

where V denotes the volume of the cavity. After that we can rewrite the energy of the field in terms of two numbers p and q:

$$W = \frac{1}{2}\int (\varepsilon E_x^2(z,t) + \mu H_y^2(z,t))\, dV = \frac{1}{2}(p^2 + \omega^2 q^2). \tag{15}$$

Indeed, this energy obeys the harmonic oscillator with frequency ω.

Annihilation and creation operators

Now let replace two numbers p and q by observables $\hat{p}$ and $\hat{q}$ and assume that two operators satisfy the following equation for commutator:

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar. \tag{16}$$

As it has been mentioned in [2], the observable which represents the energy of system W can be represented as

$$\hat{W} = \frac{1}{2}(\hat{p}^2 + \omega^2 \hat{q}^2). \tag{17}$$

Now we can define the annihilation operator $\hat{a}$ and creation operator which is conjugated with the first:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} + i\hat{p}), \tag{18}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega}}(\omega\hat{q} - i\hat{p}). \tag{19}$$

It is easy to show that for these operators

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{20}$$

holds.

Let us define photon number operator as

$$\hat{n} = \hat{a}^\dagger \hat{a}. \tag{21}$$

One can notice that this operator is Hermitian and the energy observable can be represented via the photon number operator as follows:

$$\hat{W} = \hbar\omega(\hat{n} + \frac{1}{2}). \tag{22}$$

Annihilation and creation operators map the set of Fock states to itself as follows:

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \tag{23}$$

$$\hat{a}^\dagger\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle. \tag{24}$$

It has been clarified in [**?**] that the Fock state $|n\rangle$ is the eigenstate of photon number operator with eigenvalue $n \in 0 \cup \mathbb{N}$. Therefore, it is the eigenstate of energy observable $\hat{W}$ with eigenvalue $\hbar\omega(n + \frac{1}{2})$. That means that energy of system takes only a discrete set of values. This fact correlates with the reality.

Coherent states

A coherent state with complex amplitude α is the linear combination of Fock states:

$$\left|\alpha\right\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{|\alpha|^2}{2}} \left|n\right\rangle, \tag{25}$$

where $\alpha \in \mathbb{C}$.

It is relatively easy to show that the coherent state α is the eigenstate of the annihilation operator with the eigenvalue α and it is the eigenstate of creation operator with the eigenvalue α^*. The coherent state is normalized but it does not have the orthogonality and it implies a coherent light with the complex amplitude α.

Description of medium atoms

Boson operators b and $b^\dagger$ — operators acting in space of states $|m\rangle_b$ which correspond to the number of atoms in ground state.

$$\hat{b}\,|m\rangle_b = \sqrt{m}\,|m-1\rangle_b \tag{26}$$

$$\hat{b}^\dagger\,|m\rangle_b = \sqrt{m+1}\,|m+1\rangle_b \tag{27}$$

Let us consider a medium with N atoms which are maintained in an excited state. There is a coherent light which propagates in the medium and interacts with the medium atoms. Interaction between the light and medium can be described with the following Hamiltonian:

$$\hat{H} = i\kappa \sum_{n=1}^{N} (\hat{a}^\dagger\,|0\rangle\,\langle 1|_n - \hat{a}\,|1\rangle\,\langle 0|_n) = i\kappa(\hat{a}^\dagger \hat{S}_- - \hat{a}\hat{S}_+) \tag{28}$$

Utilizing Holstein-Primakoff transformation:

$$\hat{S}_+ = \sqrt{N}\sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{N}}\hat{b},\ \ \hat{S}_- = \sqrt{N}\hat{b}^\dagger\sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{N}} \tag{29}$$

Assuming that atoms are kept in excited state ($\langle b^\dagger b\rangle \ll N$):

$$\hat{S}_+ \approx \sqrt{N}\hat{b},\ \ \hat{S}_- \approx \sqrt{N}\hat{b}^\dagger \tag{30}$$

Thus,

$$\hat{H} = i\kappa\sqrt{N}(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b}) \tag{31}$$

Propagating photon's evolution operator can be represented as follows:

$$\hat{U}_v = e^{-i\hat{H}t/\hbar} = e^{v(\hat{a}^\dagger \hat{b}^\dagger - \hat{a}\hat{b})},\ \ v = \kappa\sqrt{N}t/\hbar \tag{32}$$

Annihilation operator's transformation in conjugated channel:

$$\mathrm{Amp}^*_G[\hat{a}] = \hat{U}_v^\dagger \hat{a}\hat{U}_v. \tag{33}$$

Hadamard lemma. Let λ be a complex number, J, A, C_n — arbitrary operators, then:

$$e^{\lambda J} A e^{-\lambda J} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} C_n,\ \ C_0 = A,\ \ C_n = [J, C_{n-1}] \tag{34}$$

Thus, for Eq.(33): $C_{\text{even}} = \hat{a},\ C_{\text{odd}} = \hat{b}^\dagger$.

Consequently,

$$\text{Amp}^*_G[\hat{a}] = \cosh(v)\hat{a} + \sinh(v)\hat{b}^\dagger \tag{35}$$

Glauber-Sudarshan P representation

Glauber-Sudarshan P-representation — quasiprobability distribution, gives the $\hat{\rho}$ quantum state's decomposition in optical phase space:

$$\hat{\rho} = \int d^2\alpha P(\alpha) \left|\alpha\right\rangle \left\langle\alpha\right|,\ d^2\alpha = d\text{Re}(\alpha)d\text{Im}(\alpha), \tag{36}$$

$$P(\alpha) = \text{Tr}[:\delta(\hat{a}-\alpha):\hat{\rho}], \tag{37}$$

$$:\delta(\hat{a}-\alpha) := \frac{1}{\pi^2}\int d^2\beta e^{\alpha\beta^*-\alpha^*\beta} e^{\beta\hat{a}^\dagger} e^{-\beta^*\hat{a}} \tag{38}$$

$$\text{Amp}_{G=\cosh^2(v)}[\left|\gamma\right\rangle\left\langle\gamma\right|] = \text{Tr}_b[\hat{U}_v \left|\gamma\right\rangle\left\langle\gamma\right| \otimes \left|0\right\rangle\left\langle0\right|_b \hat{U}_v^\dagger] \tag{39}$$

After amplification in the medium:

$$P(\alpha,\gamma,G) = \text{Tr}\big[:\delta(\hat{a}-\alpha):\text{Amp}_G[\left|\gamma\right\rangle\left\langle\gamma\right|]\big] \tag{40}$$

Baker-Campbell-Hausdorff formula (BCH)

$$\exp(-c\hat{a}^\dagger)\exp(c^*\hat{a}) = \exp(c^*\hat{a} - c\hat{a}^\dagger)\exp(\frac{|c|^2}{2}) \tag{41}$$

Utilizing BCH for Eq.(40):

$$P(\alpha,\gamma,G) = \frac{1}{\pi^2}\int d^2\beta \left\langle\gamma\right| \left\langle0\right| \exp((G-1)|\beta|^2)\exp(G\hat{a}^\dagger\beta)\times$$

$$\times\exp(-\sqrt{G}\hat{a}\beta^*)\exp(-\sqrt{G-1}\beta^*\hat{b}^\dagger)\exp(\sqrt{G-1}\beta\hat{b})e^{\alpha\beta^*-\alpha^*\beta}\left|\gamma\right\rangle\left|0\right\rangle$$

Utilizing Hadamard lemma:

$$\left\langle0\right|\exp(-\sqrt{G-1}\beta^*\hat{b}^\dagger)\exp(\sqrt{G-1}\beta\hat{b})\left|0\right\rangle = 1$$

$$\left\langle\gamma\right|\exp(G\hat{a}^\dagger\beta)\exp(-\sqrt{G}\hat{a}\beta^*)\left|\gamma\right\rangle = \sum_{n=0}\sum_{m=0}\frac{G^{\frac{n+m}{2}}(\gamma^*)^m(\gamma)^n(\beta)^m(-\beta^*)^n}{n!m!}$$

Thus,

$$P(\alpha,\gamma,G) = \frac{1}{\pi^2}\int d^2\beta\exp(-(G-1)|\beta|^2)e^{-\beta(\alpha^*-\gamma^*\sqrt{G})}e^{\beta^*(\alpha-\gamma\sqrt{G})} =$$

$$= \frac{1}{\pi^2(G-1)} \exp\left(-\frac{|\alpha - \sqrt{G}\gamma|^2}{G-1}\right) \int d^2\beta \exp\left(\theta - \beta\right)(\theta^* + \beta^*),$$

$$\text{where } \theta = \frac{\alpha - \sqrt{G}\gamma}{\sqrt{G-1}}$$

$$P(\alpha, \sqrt{T}\gamma, G) = \frac{1}{\pi(G-1)} \exp\left(-\frac{|\alpha - \sqrt{GT}\gamma|^2}{G-1}\right), \tag{42}$$

where G — amplification coefficient, T — transmission coefficient.

Optical line with loss and amplifier

Consider a QKD protocol in which the legitimate sender (Alice) generates a random bit string and then encodes the information about the string into a sequence of coherent states: The bit value 0 get encoded into a pure coherent state $|\gamma\rangle$ and the bit 1 gets encoded into a pure coherent state $|\eta\rangle$. Here, γ and η are complex numbers. Let $|\eta| > |\gamma|$. Alice utilizes the phase randomization procedure before sending a bit-encoding state to the receiver (Bob). Analytically, phase randomization can be described in the following way

$$|\gamma\rangle\langle\gamma| \to \hat{\rho}_A = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, |\gamma e^{i\phi}\rangle\langle\gamma e^{i\phi}| = e^{-|\gamma|^2} \sum_{n=0}^{\infty} \frac{|\gamma|^{2n}}{n!} |n\rangle\langle n| \tag{43}$$

The sum in the right-hand side of the equation is taken over Fock states (the quantum states containing a specific number of photons).

After the state sent by Alice passes the simplest channel consisting of a piece of optical fibre with transmission coefficient T and amplifier characterized by coefficient G (amplification coefficient), according to the Eq.(42), it gets transformed into:

$$\hat{\rho}_B = \int_{\mathbb{C}} d^2\alpha P(\alpha) \cdot \frac{1}{2\pi} \int_0^{2\pi} d\phi \, |\alpha e^{i\phi}\rangle\langle\alpha e^{i\phi}|, \tag{44}$$

$$\text{where } P(\alpha) = \frac{1}{\pi(G-1)} \exp\left(-\frac{|\alpha - \sqrt{GT}\gamma|^2}{G-1}\right)$$

Bob's detector measures the mean energy received during the time interval $[-\Delta t/2, \Delta t/2]$. It has been shown in [3], that the operator of the electromagnetic field takes the form

$$\hat{\phi}(\tau) = \sqrt{\frac{\hbar\omega}{2}} \left[\hat{a}e^{-i\omega\tau} + \hat{a}^\dagger e^{i\omega\tau}\right], \tag{45}$$

where ω is the propagating mode frequency. The operator of the energy contained in the aforementioned time interval is then defined as

$$\hat{W} = \frac{1}{\Delta t} \int_{-\Delta t/2}^{\Delta t/2} d\tau \, \hat{\phi}^2(\tau). \tag{46}$$

It is relatively easy to show that

$$\hat{W} = \frac{\hbar\omega}{2} \left[\left(\hat{a}\hat{a} + \hat{a}^\dagger \hat{a}^\dagger \right) \frac{\sin(\omega\Delta t)}{(\omega\Delta t)} + 2\hat{a}^\dagger \hat{a} \right] + \frac{\hbar\omega}{2} \tag{47}$$

We are particularly interested in case of $\omega\Delta t \gg 1$:

$$\hat{W} = \hbar\omega \cdot \hat{Q} + \frac{\hbar\omega}{2}, \quad \hat{Q} = \hat{a}^\dagger \hat{a}. \tag{48}$$

Now we are interested in finding the mean value and variance of operator $\hat{Q}$ with respect to Bob's state $\hat{\rho}_B$.

The mean value $\langle \hat{Q} \rangle_{\hat{\rho}_B}$ and the variance $\Delta Q^2_{\hat{\rho}_B}$ can be found from the following equations:

$$\langle \hat{Q} \rangle_{\hat{\rho}_B} = \mathrm{Tr}[\hat{Q}\hat{\rho}_B], \tag{49}$$

$$\Delta Q^2_{\hat{\rho}_B} = \langle \hat{Q}^2 \rangle_{\hat{\rho}_B} - \langle \hat{Q} \rangle^2_{\hat{\rho}_B} = \mathrm{Tr}[\hat{Q}^2 \hat{\rho}_B] - (\mathrm{Tr}[\hat{Q}\hat{\rho}_B])^2. \tag{50}$$

Due to Eq.(44):

$$\langle \hat{Q} \rangle_{\hat{\rho}_B} = G - 1 + |\gamma|^2, \tag{51}$$

$$\Delta Q^2_{\hat{\rho}_B} = (G-1)^2 + 2(G-1)|\gamma|^2 + < \hat{Q} >_{\hat{\rho}_B} . \tag{52}$$

Effective optical line

It has been shown in [1] that the sequence of the N amplifiers G_0 and N losses T_0 can be represented as loss T that is followed by amplifier G such that

$$G = G_0(N(1-T_0) + T_0), \; T = \frac{T_0}{N(1-T_0) + T_0}, \tag{53}$$

in case of $G_0 \to \frac{1}{T_0}$.

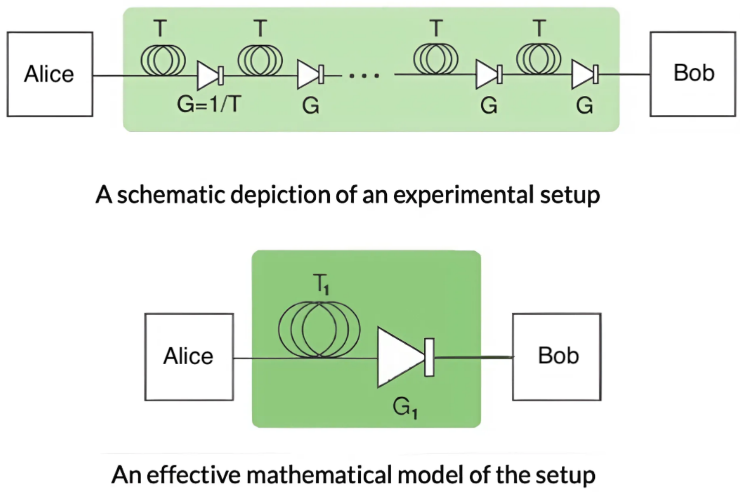

Figure 1: Amplifier and loss sequence representation as one loss and one amplifier

In practice, $T_0 = 10^{-0.02d}$, $[d] = \text{km}$, d — distance between two adjacent amplifiers. The G_0 value is chosen close to $\frac{1}{T_0}$ to preserve the mean value of propagating photons.

We are particularly interested in finding the dependence of the SNR (Signal-to-noise ratio) on the distance between amplifiers d:

$$\text{SNR}(d) = \frac{\langle \hat{Q} \rangle_{\rho_B}}{\sqrt{\Delta \hat{Q}^2}}(d) \tag{54}$$

For considered case:

$$\text{SNR}(d) = \frac{\frac{D_{AB}}{d}(10^{0.02d} - 1) + |\gamma|^2}{\sqrt{(\frac{D_{AB}}{d}(10^{0.02d} - 1))^2 + 2|\gamma|^2(\frac{D_{AB}}{d}(10^{0.02d} - 1)) + < \hat{Q} >_{\hat{\rho}_B}(d)}}, \tag{55}$$

where D_{AB} is the distance between Alice and Bob.

For the initial average number of photons $|\gamma|^2 = 10^4$ and distances D_{AB} of 10, 50, 100, 500 and 1000 km were plotted graphs of the SNR dependence on distance between two adjacent amplifiers d. And a horizontal line was plotted for each graph, corresponding to the SNR value in the absence of any amplifiers in fiber, it is relatively easy to show that for this case $\langle \hat{Q} \rangle_{\hat{\rho}_B} = T|\gamma|^2$, $\Delta \hat{Q}^2_{\hat{\rho}_B} = T|\gamma|^2$, $T = 10^{-0.02 D_{AB}}$

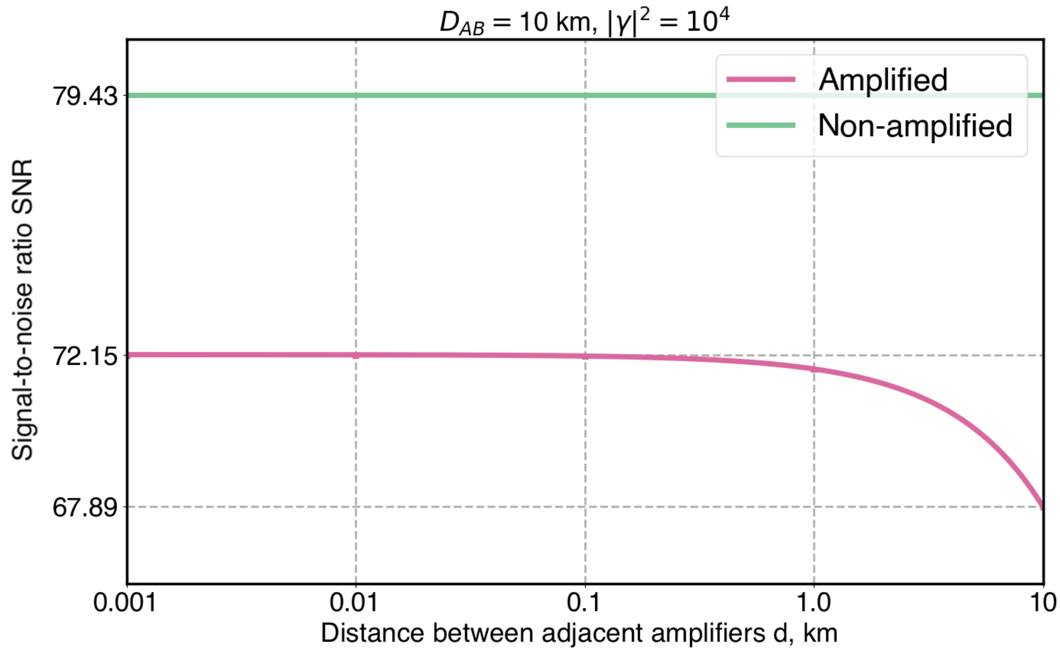

Figure 2: SNR(d) graph.

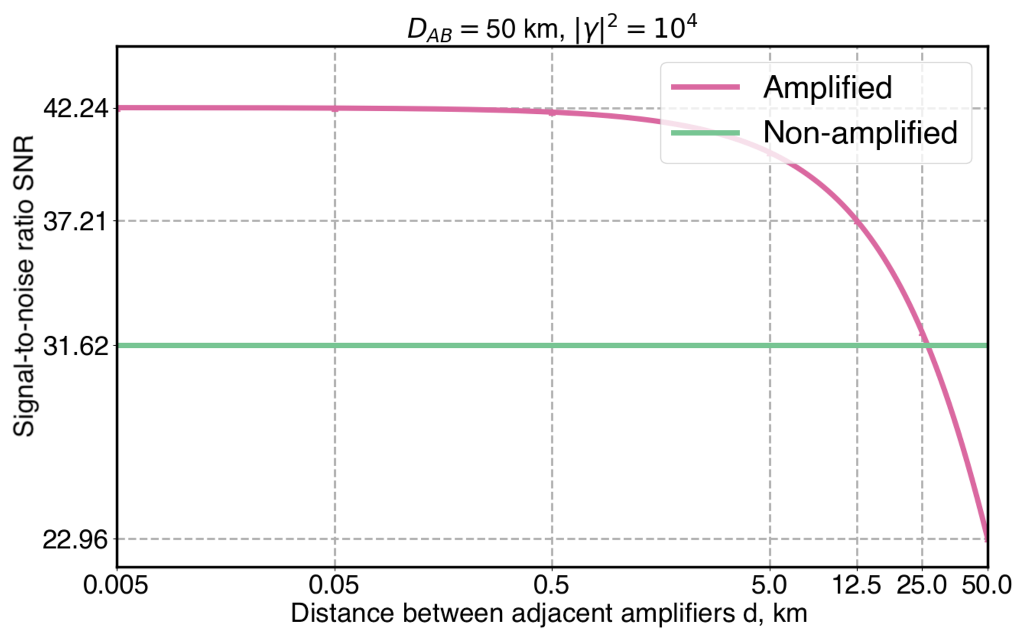

Figure 3: SNR(d) graph.

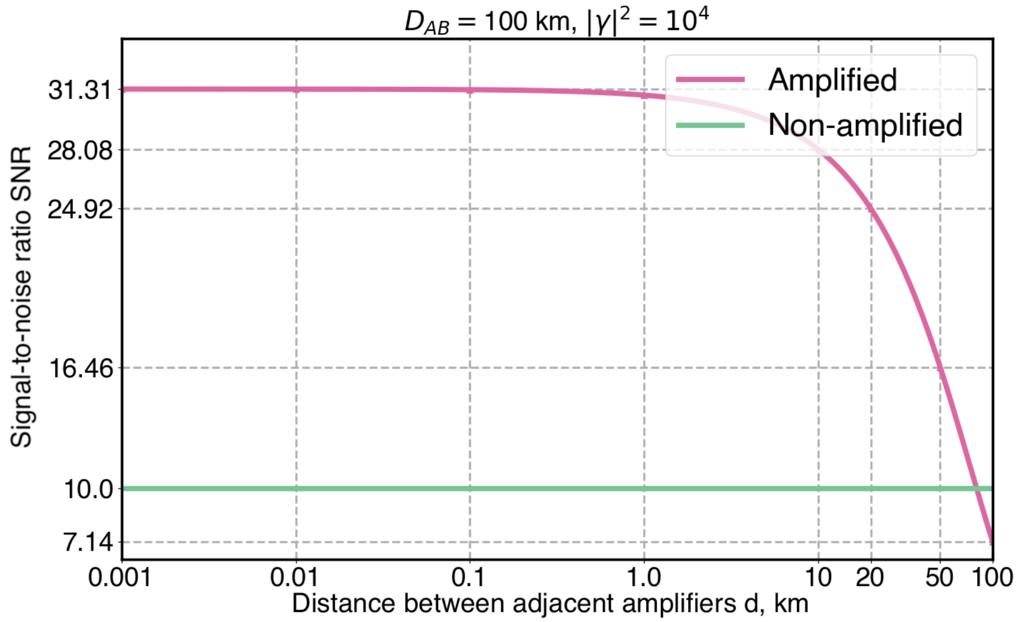

Figure 4: SNR(d) graph.

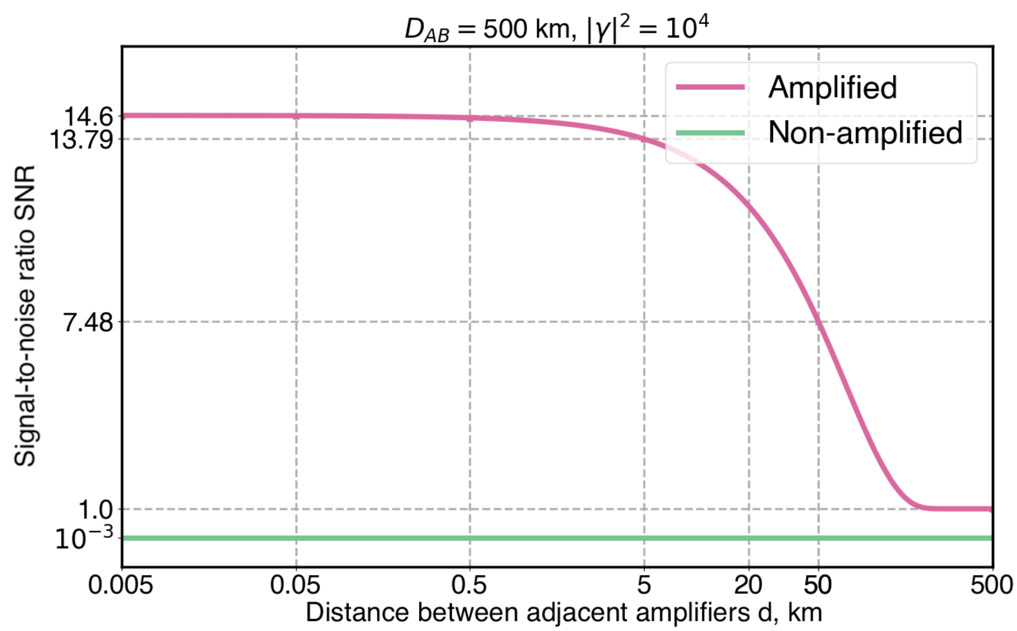

Figure 5: SNR(d) graph.

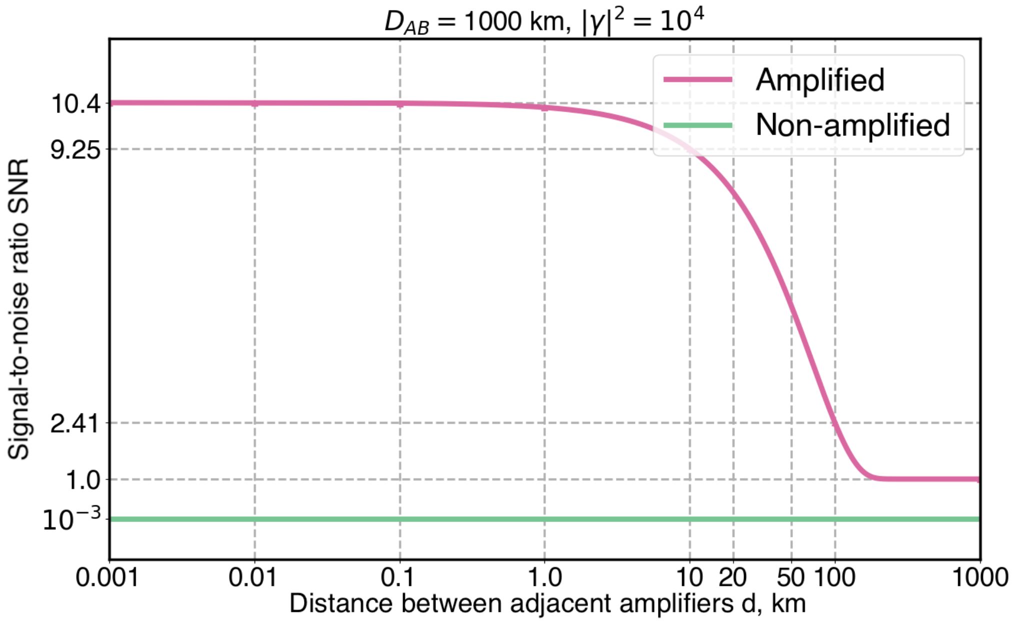

Figure 6: SNR(d) graph.

According to the graphs it is clear that for the distances of $D_{AB} = 10$ km and $D_{AB} = 50$ km it is unprofitable to install amplifiers in the system. For the distance of $D_{AB} = 10$ km it is more profitable to install two or more amplifiers. For the distances of $D_{AB} = 500$ km and $D_{AB} = 1000$ km it is necessary to install amplifiers in the channel. This result is the consequence of amplified spontaneous emission.

The critical value of D_{AB}, such that it is the first value for which it is profitable to install amplifiers in the channel, was obtained. It accounts to $D_{ABcr} = (189.5 \pm 0.5)$ km.

Conclusion

In this work, a research was conducted on the transformation of coherent states when propagating through an optical channel with amplifiers and losses. For the case of one mode, SNR(d) graphs were plotted for several lengths of optical fiber. Particular attention was paid to the case of no amplifiers, emphasizing the contribution of ASE to the results of measurements. Thus, the results of this work emphasize the importance of taking into account ASE effects when designing optical fiber systems.

References

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