Entropic Uncertainty Relation

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Moscow, 2023

Introduction

Let A and B be arbitrary observables in Hilbert space with eigenvectors $|a_i\rangle$ and $|b_j\rangle$ respectively. Uncertainty relations in most studies are represented by Robertson relation:

$$\Delta_\psi A \Delta_\psi B \geq \frac{1}{2}|\langle[A,B]\rangle_\psi| \tag{1}$$

This uncertainty relation formulation is not complete, because the right side of the inequality depends on the chosen state $|\psi\rangle$. The entropic uncertainty relation, based on Shannon entropy, was offered to solve this problem.

The main task of this research is to derive this relation and to consider them on the Gaussian beam.

Discrete probability distribution

Let us consider two probability distributions $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_n)$, where $p_i = |\langle a_i|\psi\rangle|^2$, $q_j = |\langle b_j|\psi\rangle|^2$.

Shannon entropy can be represented as follows:

$$H(P) = -\sum_j P_j \log(P_j) \tag{2}$$

Let us introduce the following quantities:

$$c = \max_{i,j} |\langle a_i|b_j\rangle| \tag{3}$$

$$M_r(P) = (\sum_i (P_i)^{1+r})^{1/r} \tag{4}$$

Basic properties of $M_r(P)$:

1) $M_0(P) = \exp(-H(P))$

2) $M_\infty(P) = \max_j(P_j)$

3) $M_{-1}(P) = \frac{1}{N}$, where N is the number of events.

Riesz theorem. $x = (x_1, \dots, x_n)^T$ — an arbitrary vector, Let $T = (T_{jk})$ be a linear operator for any j and k such that $(Tx)_j = \sum_k T_{jk}x_k$ and $\sum_j |(Tx)_j|^2 = \sum_k |x_k|^2$. $c = \max_{j,k}|T_{jk}|$. Then for all a and a' such that $1 \leq a \leq 2$, $1/a + 1/a' = 1$, holds:

$$c^{1/a'}(\sum_j |(Tx)_j|^{a'})^{1/a'} \leq c^{1/a}(\sum_k |x_k|^a)^{1/a} \tag{5}$$

Taking $x_k = |\langle a_k|\psi\rangle|$, $T_{jk} = |\langle a_k|b_j\rangle|$, $(Tx)_j = |\langle b_j|\psi\rangle|$, $a = 2$, $a' = 2$, we obtain:

$$M_0(p)M_0(q) \leq c^2 \tag{6}$$

It can be represented in terms of entropy as follows:

$$H(p) + H(q) \geq -2\log(c) \tag{7}$$

Absolutely continuous probability distribution

Let us introduce

$$M_r(P) = \Big(\int_{-\infty}^{+\infty} |P(x)|^{1+r}dx\Big)^{1/r} \tag{8}$$

Similarly to discrete probability distribution,

$$H(P) = -\log(M_0(P)) = -\int_{-\infty}^{+\infty}, P(x)\log(P(x))dx \tag{9}$$

and

$$M_0(P(x))M_0(P(y)) \leq c^2 \tag{10}$$

Or equivalently,

$$H(P(x)) + H(P(y)) \geq -2\log(c) \tag{11}$$

It was obtained by Hirschman that $c = (2\pi)^{-1/2}$. Therefore,

$$H(P(x)) + H(P(y)) \geq 2\pi \tag{12}$$

It is clear that the expression in the right side of the Eq.(12) is the fixed lower bound.

Gaussian beam

Gaussian beam's coordinate and momentum distribution is given as follows:

$$P(x) = \frac{1}{\sqrt{2\pi}\sigma_x}\exp(-\frac{x^2}{2\sigma_x^2}) \tag{13}$$

$$P(p) = \frac{1}{\sqrt{2\pi}\sigma_p}\exp(-\frac{p^2}{2\sigma_p^2}) \tag{14}$$

Let us find a Shannon entropy for the coordinate distribution:

$$H(P(x)) = -\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma_x}\exp(-\frac{x^2}{2\sigma_x^2})\log(\frac{1}{\sqrt{2\pi}\sigma_x}\exp(-\frac{x^2}{2\sigma_x^2}))dx =$$

$$= \frac{1}{2}\log(e)\mathrm{erf}(+\infty) + \log(\sqrt{2\pi}\sigma_x)\mathrm{erf}(+\infty) = \frac{1}{2}\log(e) + \log(\sqrt{2\pi}\sigma_x).$$

Similarly, for the momentum distribution:

$$H(P(p)) = \frac{1}{2}\log(e) + \log(\sqrt{2\pi}\sigma_p). \tag{15}$$

Utilizing relations between σ_x and σ_p in critical case ($\sigma_p = 1/(2\sigma_x)$), we obtain:

$$H(P(x)) + H(P(p)) = \log(\pi e) \approx 2.14 > \log(2\pi) \tag{16}$$

Conclusion

The incompleteness of the Heisenberg principle as a formulation of the uncertainty relation is revealed. The entropic uncertainty relation is derived, which is a more complete formulation of the uncertainty principle, in which the lower bound for the sums of the Shannon entropies of two distributions of physical observables is a fixed value. The relation is verified using the example of Gaussian beam, the sum of two Shannon entropies equals to $\log(\pi e)$, which satisfies the lower bound of entropic uncertainty relations.

References

[1] Hans Maassen, J.B.M. Uffink, *Phys. Rev. Lett.* (1988).

[2] I.I. Hirschman, *Am. J. Math.* (1957).