

## bon-db/calculus/cont/Z85ECB.json (AoPS)

**Problem.** Let  $f: [0, 1] \rightarrow [0, 1]$  be a continuous function which is injective and satisfies

$$f(2x - f(x)) = x \quad \forall x \in [0, 1].$$

Prove that  $f(x) = x$ .

*Solution.* Plugging  $x = 0$  and  $x = 1$ , we get  $f(-f(0)) = 0$  and  $f(2 - f(1)) = 1$  respectively. Since the co-domain of  $f$  is  $[0, 1]$ , the following inequalities follow:  $0 \leq -f(0) \leq 1$  and  $0 \leq 2 - f(1) \leq 1$ . Now note that as both  $f(0)$  and  $f(1)$  lie in  $[0, 1]$ , one-sided equality in the previous inequalities is forced which implies  $f(0) = 0$  and  $f(1) = 1$ . Therefore, as  $f$  is continuous, it must be surjective. Along with the injectivity, we can conclude that  $f$  is bijective.

Fix  $x_1 \in [0, 1]$  arbitrarily and define a recursive sequence  $\{x_n\}_{n \geq 1}$  as follows:

$$x_{n+1} = f(x_n) \quad \forall n \in \mathbb{N}$$

which makes sense as the range of  $f$  is a subset of  $[0, 1]$ .

Then from our original equation, we get

$$f(2x - f(x)) = x \implies f(x) = 2x - f^{-1}(x).$$

Substituting  $x \mapsto x_{n+1}$ , we get

$$f(x_{n+1}) = 2x_{n+1} - f^{-1}(x_{n+1})$$

which is equivalent to

$$x_{n+2} = 2x_{n+1} - x_n.$$

Note that

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n = \dots = x_2 - x_1$$

which implies  $\{x_n\}_{n \geq 1}$  is an A.P. If  $d = x_2 - x_1 \neq 0$ , then  $x_n = x_1 + (n - 1)d$  diverges as  $n \rightarrow \infty$ . But the sequence is supposed to be bounded as  $f(x_n)$  is itself bounded. This implies  $x_2 = x_1$ , which further implies  $f(x_1) = x_1$ . Since  $x_1$  was chosen arbitrarily, we conclude that  $f(x) = x$  for all  $x \in [0, 1]$ . ■