

bon-db/alg/poly/Z391CC.json (AoPS)

Problem. Find all polynomials $P(x)$ with degree $\leq n$ and non-negative coefficients such that

$$P(x)P\left(\frac{1}{x}\right) \leq P(1)^2$$

for all positive x . Here n is a natural number.

Solution. Let $P(x) = a_0 + a_1x^1 + \cdots + a_nx^n$. Then for all $x > 0$, by Cauchy–Schwarz inequality, we get

$$\begin{aligned} P(x)P\left(\frac{1}{x}\right) &= ((\sqrt{a_0})^2 + \cdots + (\sqrt{a_nx^n})^2)((\sqrt{a_0})^2 + \cdots + (\sqrt{a_n/x^n})^2) \\ &\geq (a_0 + \cdots + a_n)^2 = P(1)^2. \end{aligned}$$

Thus the equality holds which forces the equality in CS, i.e.,

$$\frac{\sqrt{a_0}}{\sqrt{a_0}} = \cdots = \frac{\sqrt{a_nx^n}}{\sqrt{\frac{a_n}{x^n}}},$$

which implies $x^n = x^{n-1} = \cdots = 1$ which is absurd unless $n = 1$. Therefore $P(x)$ must be a constant polynomial and it is easy to see that all $P(x) \equiv c$, $c \geq 0$, work. ■