bon-db/calculus/diff/Z6749A.json (RMCS 43)

Problem. Let $f : [a, b] \to \mathbb{R}$ be a differentiable function with $f(a) = b$ and $f(b) = a$. Prove that there exist two distinct real numbers c, $d \in (a, b)$ such that $f'(c)f'(d) = 1$.

Solution. If there exists $t \in (a, b)$ such that $f(t) = t$, then LMVT on (a, t) and (t, b) finishes. Otherwise, assume that $f(x) \neq x$ for all $x \in (a, b)$.

Define $g(x) = f(f(x)) - x$. Then $g(a) = g(b) = 0$. Clearly g is differentiable and by LMVT, there exists $c \in (a, b)$ such that $g'(c) = 0$. Note that $g'(x) = f'(f(x)) \cdot f'(x)$. Moreover, since $f(x) \neq x$ for all $x \in (a, b)$, we get $f(c) \neq c$. Let $d = f(c)$. Then $g'(c) = 0$ implies $f'(c) \cdot f'(d) = 1$ as desired. ■