bon-db/combi/Z69E82.json (**AoPS**)

Problem. Let $\{a_n\}_{n\geq 0}$ be a sequence defined by $a_0 = 5$, $a_1 = 27$, and

$$a_n = 7a_{n-1}^2 + a_{n-2} \quad \forall\, n \geq 2.$$

Find the remainder when $a_{2025!}$ is divided by 43.

Solution. Consider all the tuples (a_n, a_{n+1}) under modulo 43, i.e., (b_n, b_{n+1}) where b_n is the remainder when a_n is divided by 43. Clearly, there are at most 43^2 different possibilities for (b_n, b_{n+1}) considering all the remainder pairs.

Let

$$S = \{(b_0, b_1), (b_1, b_2), \ldots, (b_{43^2}, b_{43^2+1})\}.$$

As $|S| = 43^2 + 1$, by the pigeonhole principle, there exist at least two tuples in S such that

$$(b_i, b_{i+1}) = (b_j, b_{j+1}), \quad i < j.$$

Now note $b_i = b_j$ and $b_{i+1} = b_{j+1}$ implies

$$b_{i+2} \equiv 7b_{i+1}^2 + b_i \equiv 7b_{j+1}^2 + b_j \equiv b_{j+2} \pmod{43},$$

which implies $b_{i+2} = b_{j+2}$. Therefore

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i+1}, b_{i+2}) = (b_{j+1}, b_{j+2}).$$

Similarly,

$$(b_{i+1}, b_{i+2}) = (b_{j+1}, b_{j+2}) \implies (b_{i+2}, b_{i+3}) = (b_{j+2}, b_{j+3}) \implies \cdots \text{ and so on.}$$

Also,

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i-1}, b_i) = (b_{j-1}, b_j)$$

as

$$b_{i-1} \equiv b_{i+1} - 7b_i^2 \equiv b_{j+1} - 7b_j^2 \equiv b_{j-1} \pmod{43}.$$

Similarly,

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i-1}, b_i) = (b_{j-1}, b_j) \implies \cdots \text{ and so on.}$$

Therefore we can conclude that

$$b_{n+(j-i)} = b_n \quad \forall\, n \geq 0$$

Note that $j - i \leq 43^2 - 0 = 43^2 \leq 2025$ which implies $j - i \mid 2025!$. To finish, note

$$a_{2025!} \equiv b_{2025!} = b_0 = 5 \pmod{43}. \qquad \blacksquare$$