bon-db/calculus/int/Z9EA19.json (Romania 2025 G12 P3)

Problem. Solve the following:

(a) Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a strictly monotonous function such that $\int_a^b f(x)\mathrm{d}x = 0$. Show that $f(a) \cdot f(b) < 0$.

(b) Find all convergent sequences $(a_n)_{n\geq 1}$ for which there exists a strictly monotonous function $f : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{a_{n-1}}^{a_n} f(x)\mathrm{d}x = \int_{a_n}^{a_{n+1}} f(x)\mathrm{d}x, \text{ for all } n \geq 2.$$

Solution. We first solve part (a) and then use it to solve part (b).

(a) Note that the conditions remain same if we consider $-f$ instead of f. So WLOG assume f is strictly increasing. Therefore $f(a) < f(b)$.

(i) **Case 1:** $f(a) = 0$. Then

$$\begin{aligned} \int_a^b f(x)\mathrm{d}x &= \int_a^{\frac{a+b}{2}} f(x)\mathrm{d}x + \int_{\frac{a+b}{2}}^b f(x)\mathrm{d}x \\ &\geq \int_a^{\frac{a+b}{2}} f(a)\mathrm{d}x + \int_{\frac{a+b}{2}}^b f\left(\frac{a+b}{2}\right)\mathrm{d}x \\ &= \frac{b-a}{2} \cdot f(a) + \frac{b-a}{2} \cdot f\left(\frac{a+b}{2}\right) > 0. \end{aligned}$$

But we know that $\int_a^b f(x)\mathrm{d}x = 0$ which gives us our contradiction.

(ii) **Case 2:** $f(b) = 0$. This follows similarly like the previous case.

(iii) **Case 3:** $f(a), f(b) \neq 0$.

$$0 = \int_a^b f(x)\mathrm{d}x \geq \int_a^b \mathrm{d}x = f(a) \cdot \left(\frac{b-a}{2}\right).$$

Similarly, we also get that,

$$f(b) \cdot \left(\frac{b-a}{2}\right) \geq 0.$$

Multiplying these two, we get,

$$f(a)f(b)\left(\frac{b-a}{2}\right)^2 \leq 0 \implies f(a) \cdot f(b) < 0.$$

(b) Let

$$\lambda = \int_{a_1}^{a_2} f(x)\mathrm{d}x = \int_{a_2}^{a_3} f(x)\mathrm{d}x = \cdots.$$

Therefore

$$n\lambda = \int_{a_1}^{a_2} f(x)\mathrm{d}x + \cdots + \int_{a_n}^{a_{n+1}} f(x)\mathrm{d}x = \int_{a_1}^{a_{n+1}} f(x)\mathrm{d}x.$$

Define $b_n = \int_{a_1}^{a_{n+1}} f(x)\mathrm{d}x$ for all $n \geq 1$.

Note, $b_{n+1} - b_n = \int_{a_{n+1}}^{a_{n+2}} f(x)\mathrm{d}x = 0 \implies b_{n+1} = b_n$.

This implies that b_n converges which further implies that b_n is bounded. So there exists $M > 0$ such that $|b_n| \leq M$.

So,

$$|\lambda| \leq \left|\frac{b_{n+1}}{n}\right| \leq \left|\frac{M}{n}\right| \implies |\lambda| \leq \lim_{n\to\infty} \left|\frac{M}{n}\right| = 0 \implies \lambda = 0.$$

> **Claim** () — $\{a_n\}_{n\geq 1}$ has at most two different terms.

Proof. FTSOC assume that $\{a_n\}_{n\geq 1}$ has at least 3 different terms.

Let a_i, a_j, a_k be three such that in order, i.e., $a_i < a_j < a_k$.

Firstly note that for $m < n$,

$$0 = \int_{a_m}^{a_{m+1}} f(x)\mathrm{d}x = \cdots = \int_{a_{n-1}}^{a_n} f(x)\mathrm{d}x.$$

Summing them all, we get that $\int_{a_m}^{a_n} f(x)\mathrm{d}x = \int_{a_n}^{a_m} f(x)\mathrm{d}x = 0$ for all m, $n \in \mathbb{N}$.

So using this, we can state that,

$$\int_{a_i}^{a_j} f(x)\mathrm{d}x = \int_{a_j}^{a_k} f(x)\mathrm{d}x = \int_{a_i}^{a_k} f(x)\mathrm{d}x = 0.$$

Using part (a) and our previous finding, we get that,

$$f(a_i) \cdot f(a_j) < 0, f(a_j) \cdot f(a_k) < 0, f(a_i) \cdot f(a_k) < 0.$$

Multiplying these three, we get a contradiction as,

$$0 > (f(a_i) \cdot f(a_j) \cdot f(a_k))^2 \geq 0.$$

□

So we can further divide this claim into two cases.

(i) **$\{a_n\}_{n\geq 1}$ is a constant sequence:** Consider $f(x) = x$. A bit of checking shows that this works.

(ii) **$\{a_n\}_{n\geq 1}$ has exactly two different terms in it:** Consider $f(x) = \frac{2(x-b)}{b-a} + 1$. Then note that, $\int_a^b f(x)\mathrm{d}x = \int_b^a f(x)\mathrm{d}x = 0$.

If $(a_{n-1}, a_n) = (a, b)$ or (b, a) or (a, a) or (b, b), then we have,

$$\int_{a_{n-1}}^{a_n} f(x)\mathrm{d}x = 0.$$

So such an f works.

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