bon-db/calculus/int/Z4B131.json (ISI B.Stat. 2007 P3)

Problem. Let $f(u)$ be a continuous function and, for any real number u, let $\lfloor u \rfloor$ denote the greatest integer less than or equal to u. Show that for any $x > 1$,

$$\int_1^x \lfloor u \rfloor \left(\lfloor u \rfloor + 1\right) f(u)\mathrm{d}u = 2\sum_{i=1}^{\lfloor x \rfloor} i \int_i^x f(u)\mathrm{d}u.$$

Solution. Follow this calculations below:

$$\begin{aligned}
&\int_1^x \lfloor t \rfloor \left(\lfloor t \rfloor + 1\right) f(t)\mathrm{d}t \\
&= \int_1^{\lfloor x \rfloor} \lfloor t \rfloor \left(\lfloor t \rfloor + 1\right) f(t)\mathrm{d}t + \int_{\lfloor x \rfloor}^x \lfloor t \rfloor \left(\lfloor t \rfloor + 1\right) f(t)\mathrm{d}t \\
&= \sum_{i=1}^{\lfloor x \rfloor - 1} \int_i^{i+1} \lfloor t \rfloor \left(\lfloor t \rfloor + 1\right) f(t)\mathrm{d}t + \lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= \sum_{i=1}^{\lfloor x \rfloor - 1} i(i+1) \int_i^{i+1} f(t)\mathrm{d}t + \lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= 2\sum_{i=1}^{\lfloor x \rfloor - 1} \sum_{j=1}^{i} j \int_i^{i+1} f(t)\mathrm{d}t + \lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= 2\sum_{j=1}^{\lfloor x \rfloor - 1} \sum_{i=j}^{\lfloor x \rfloor - 1} j \int_i^{i+1} f(t)\mathrm{d}t + \lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= 2\sum_{j=1}^{\lfloor x \rfloor - 1} j \int_j^{\lfloor x \rfloor} f(t)\mathrm{d}t + \lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= 2\left(\sum_{j=1}^{\lfloor x \rfloor - 1} j \int_j^{\lfloor x \rfloor} f(t)\mathrm{d}t + j\int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t\right) + \left(\lfloor x \rfloor \left(\lfloor x \rfloor + 1\right) - \left(\lfloor x \rfloor - 1\right)\lfloor x \rfloor\right) \int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t \\
&= 2\sum_{j=1}^{\lfloor x \rfloor - 1} j \int_j^x f(t)\mathrm{d}t + \left(\int_{\lfloor x \rfloor}^x f(t)\mathrm{d}t\right) \cdot 2\lfloor x \rfloor \\
&= 2\sum_{j=1}^{\lfloor x \rfloor} j \int_j^x f(t)\mathrm{d}t.
\end{aligned}$$

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