

**bon-db/calculus/diff/Z406B7.json (AoPS)**

**Problem.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a continuous function which is differentiable on  $(a, b)$ . Given that  $f(a) = f(b)$  and  $f'(a) = f'(b)$ , prove that for all  $\lambda \in \mathbb{R}$ ,

$$f''(x) - \lambda f'(x)^2 = 0$$

has at least one real root.

*Solution.* Define  $g(x) = \frac{f'(x)}{e^{\lambda f(x)}}$ . Clearly  $g(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then note that

$$g(a) = \frac{f'(a)}{e^{\lambda f(a)}} = \frac{f'(b)}{e^{\lambda f(b)}} = g(b).$$

Therefore, by LMVT, there exists  $c \in (a, b)$  such that

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0.$$

Note

$$g'(x) = \frac{f''(x)e^{\lambda f(x)} - f'(x)^2 e^{\lambda f(x)}}{e^{2\lambda f(x)}}.$$

So

$$g'(c) = 0 \implies f''(c) - \lambda f'(c)^2 = 0,$$

as desired. ■