

**bon-db/alg/poly/ZFF44E.json (USAMO 1976 P5)**

**Problem.** Let  $P(x)$ ,  $Q(x)$ ,  $R(x)$ ,  $S(x)$  be polynomials with real coefficients such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + \cdots + 1)S(x)$$

for all  $x \in \mathbb{C}$ . Prove that  $x - 1 \mid S(x)$ .

*Solution.* Define

$$G(x) = x^2R(1) + xQ(1) + P(1).$$

Let  $\omega$  be a fifth root of unity. Plugging in  $\omega^i$  for  $1 \leq i \leq 4$  into the original equation gives

$$P(1) + \omega^i Q(1) + \omega^{2i} R(1) = 0.$$

So  $\omega$ ,  $\omega^2$ ,  $\omega^3$ ,  $\omega^4$  are roots of  $G(x)$ .

But note  $\deg(G(x)) \leq 2$  which forces  $G \equiv 0$  implying  $P(1) = Q(1) = R(1) = 0$ .

Therefore, putting  $x = 1$  into the original equation gives

$$P(1) + Q(1) + R(1) = 5S(1)$$

which implies

$$S(1) = 0 \implies x - 1 \mid S(x). \quad \blacksquare$$