bon-db/calculus/diff/ZAFC5E.json (AoPS)

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $|f'(x)| \le |f(x)|$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Solution. Since $|f(x)|$ is continuous over $[0,1]$, it must be bounded. Let $M = \max_{x\in[0,1]} |f(x)|$. Also let $S = \{x \in [0,1] \mid |f(x)| = M\}$. Since $x \in [0,1]$, S must be bounded. Let $c = \inf\{S\}$. By continuity, $|f(c)| = M$.

Claim — $c = 0$.

Proof. For the sake of contradiction, assume that $c > 0$. Then by the LMVT on $[0,c]$, we get $t \in (0,c)$ such that

$$f'(t) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c}.$$

So,

$$|f(c)| = |c| \cdot |f'(t)|.$$

Clearly $0 < c \le 1$, so

$$|f(c)| = c \cdot |f'(t)| \le |f'(t)| \le |f(t)|.$$

But then

$$M = |f(c)| \le |f(t)| \le M \implies |f(t)| = M.$$

Since $t < c$, this contradicts the minimality of c and our claim is proved. □

Therefore

$$M = |f(c)| = |f(0)| = 0$$

which implies

$$|f(x)| \le M = 0 \implies f(x) = 0 \quad \forall\, x \in [0,1].$$

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