bon-db/calculus/diff/ZF51A6.json

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a function which is $n+1$ times differentiable and satisfies

$$f(1) = f(0) = f'(0) = \cdots = f^{(n)}(0) = 0.$$

Prove that there exists $\varepsilon \in (0,1)$ such that $f^{(n+1)}(\varepsilon) = 0$.

Solution. As $f(1) = f(0) = 0$, by LMVT we get $\varepsilon_1 \in (0,1)$ such that $f'(\varepsilon_1) = 0$. Now as $f'(\varepsilon_1) = f'(0) = 0$, again by LMVT we get $\varepsilon_2 \in (0, \varepsilon_1)$ such that $f''(\varepsilon_2) = 0$. Suppose we have $\varepsilon_k \in (0,1)$ with $k \le n$ such that $f^{(k)}(\varepsilon_k) = 0$. Now, as $f^{(k)}(x)$ is differentiable, by LMVT, we get $\varepsilon_{k+1} \in (0, \varepsilon_k)$ such that $f^{(k+1)}(\varepsilon_{k+1}) = 0$. Continuing as above, we get ε_{n+1} such that $f^{(n+1)}(\varepsilon_{n+1}) = 0$, as desired. ∎