

## bon-db/combi/Z53906.json (Putnam 2002 A3, INMO 2013 P4)

**Problem.** Let  $n$  be an integer greater than 1 and let  $T_n$  be the number of non-empty subsets  $S$  of  $\{1, 2, \dots, n\}$  with the property that the average of the elements of  $S$  is an integer. Prove that  $T_n - n$  is always even.

*Solution.* Define a function  $f: \mathcal{P}([n]) \rightarrow \mathcal{P}([n])$  by

$$f(A) = \{n + 1 - a \mid a \in A\}.$$

Call the sets satisfying the property to be good. Note that  $f(f(A)) = A$ . Now it is easy to see that  $A$  is good if and only if  $f(A)$  is good.

Now we make cases based on the parity of  $n$ .

(i)  **$n$  is even:** We need to show that  $T_n$  is even.

If  $A \neq f(A)$ , we can pair them up. We focus on the case when  $f(A) = A$ . If  $|A|$  is odd, then there must exist an element  $a \in A$  such that  $a = n + 1 - a$ , i.e.,  $a = \frac{n+1}{2}$ , which is impossible as  $n$  is even.

So,  $|A|$  must be even. Let  $|A| = 2k$ . Then the sum of all the elements of  $A$  is  $(n + 1)k$ . So, the average is

$$\frac{(n + 1)k}{2k} = \frac{n + 1}{2},$$

which is not an integer.

Thus the condition  $f(A) = A$  is not possible. Therefore  $T_n$  is even as we can pair up  $A$  and  $f(A)$ , as desired.

(ii)  **$n$  is odd:** It suffices to show that  $T_n$  is odd.

If  $A \neq f(A)$ , we can pair them up just like in the previous case. We focus on the case when  $f(A) = A$ .

If  $|A|$  is even, note that

$$a \in A \iff n + 1 - a \in A.$$

As

$$\frac{n + 1}{2} = n + 1 - \frac{n + 1}{2},$$

$\frac{n+1}{2} \notin A$  in this case. So, if we create pairs in the fashion  $(1, n)$ ,  $(2, n - 1)$ , etc., then the presence of any one element in a pair implies the presence of the other.

Suppose  $k$  of these pairs are present in  $A$ . Then the sum of the elements is

$$\frac{(n+1)k}{2},$$

which is an integer. Therefore, any  $A$  constructed from these pairs works.

Here we have  $2^{\frac{n-1}{2}} - 1$  ways of choosing which pairs to include.

If  $|A|$  is odd, it is easy to see  $\frac{n+1}{2} \in A$ . Consider the pairs just as in the previous case. Suppose  $k$  of these pairs contribute to  $A$ . Then the sum of elements of  $A$  is

$$\frac{(n+1)k}{2} + \frac{n+1}{2} = \frac{(n+1)(k+1)}{2},$$

which is an integer.

So, there are  $2^{\frac{n-1}{2}}$  ways to choose the set  $A$ .

Thus, in total, there are  $2^{\frac{n-1}{2}+1} - 1$  possible values of  $A$  which implies  $T_n$  is odd, as desired.

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