bon-db/alg/poly/Z4467B.json (Putnam 2023 A2)

Problem. Let n be an even positive integer. Let p be a monic, real polynomial of degree $2n$; that is to say, $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1 x + a_0$ for some real coefficients $a_0, \ldots, a_{2n-1}$. Suppose that $p(1/k) = k^2$ for all integers k such that $1 \le |k| \le n$. Find all other real numbers x for which $p(1/x) = x^2$.

Solution by **chakrabortyahan** (#15 on the thread).

Solution. $Q(x) = x^2 p(x) - 1$ has zeroes $\left\{\pm\frac{1}{k}\right\}_{k=1}^{n}$. So,

$$Q(x) = \left(\prod_{k=1}^{n}[x^2 - 1/k^2]\right)(x^2 + bx + c)$$

and say α, β are the zeroes of the quadratic. As $[x^1]Q(x) = 0$ and $[x^0]Q(x) = -1$ so sum of reciprocals of the zeroes of the polynomial is equal to 0 By this we can say $\frac{1}{\alpha} + \frac{1}{\beta} = 0 \implies \alpha = -\beta$ and note that $\alpha \cdot \beta \cdot \frac{1}{(n!)^2} = -1$ and so the zeroes of the quadratic are $\pm(n!)$ and the other roots of $p\left(\frac{1}{x}\right) = x^2$ are $\boxed{\pm\frac{1}{n!}}$. ■