bon-db/calculus/diff/ZBCC82.json (Putnam 2010 A2)

Problem (L). et $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function which satisfies

$$f'(x+n) = \frac{f(x+n) - f(x)}{n} \quad \forall\, n \in \mathbb{N}.$$

Find all such possible functions $f(x)$.

Solution. Clearly $f(x)$ is infinitely differentiable. Putting $n = 1$ in our primary equation, we get

$$f'(x+1) = f(x+1) - f(x).$$

By substituting $x \mapsto x + 1$, we get

$$f'(x+2) = f(x+2) - f(x+1).$$

Adding these two equations and then dividing by 2 gives

$$\frac{f'(x+1) + f'(x+2)}{2} = \frac{f(x+2) - f(x)}{2}.$$

Now, we put $n = 2$ in our primary equation to get

$$f'(x+2) = \frac{f(x+2) - f(x)}{2}.$$

Substituting this into our result gives

$$\frac{f'(x+1) + f'(x+2)}{2} = f'(x+2) \implies f'(x+1) = f'(x+2).$$

Then note that $f'(x) = f'(x+1) = \cdots = f'(x+n)$.

To finish, note that differentiating the primary equation gives

$$f''(x+n) = \frac{f'(x+n) - f'(x)}{n} = 0.$$

Substituting $x \mapsto x - n$ implies $f''(x) = 0$. Since $f(x)$ is infinitely differentiable, $f''(x)$ is continuous. Integrating $f''(x) = 0$ twice gives $f(x) = ax + b$.

Clearly a linear function satisfies our original equation and we are done. ∎