bon-db/calculus/cont/Z85ECB.json (AoPS)

Problem. Let $f\colon [0,1] \to [0,1]$ be a continuous function which is injective and satisfies

$$f(2x - f(x)) = x \qquad \forall\, x \in [0,1].$$

Prove that $f(x) = x$.

Solution. Plugging $x = 0$ and $x = 1$, we get $f(-f(0)) = 0$ and $f(2 - f(1)) = 1$ respectively. Since the co-domain of f is $[0,1]$, the following inequalities follow: $0 \leq -f(0) \leq 1$ and $0 \leq 2 - f(1) \leq 1$. Now note that as both $f(0)$ and $f(1)$ lie in $[0,1]$, one-sided equality in the previous inequalities is forced which implies $f(0) = 0$ and $f(1) = 1$. Therefore, as f is continuous, it must be surjective. Along with the injectivity, we can conclude that f is bijective.

Fix $x_1 \in [0,1]$ arbitrarily and define a recursive sequence $\{x_n\}_{n\geq 1}$ as follows:

$$x_{n+1} = f(x_n) \quad \forall\, n \in \mathbb{N}$$

which makes sense as the range of f is a subset of $[0,1]$.

Then from our original equation, we get

$$f(2x - f(x)) = x \implies f(x) = 2x - f^{-1}(x).$$

Substituting $x \mapsto x_{n+1}$, we get

$$f(x_{n+1}) = 2x_{n+1} - f^{-1}(x_{n+1})$$

which is equivalent to

$$x_{n+2} = 2x_{n+1} - x_n.$$

Note that

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n = \cdots = x_2 - x_1$$

which implies $\{x_n\}_{n\geq 1}$ is an A.P. If $d = x_2 - x_1 \neq 0$, then $x_n = x_1 + (n-1)d$ diverges as $n \to \infty$. But the sequence is supposed to be bounded as $f(x_n)$ is itself bounded. This implies $x_2 = x_1$, which further implies $f(x_1) = x_1$. Since x_1 was chosen arbitrarily, we conclude that $f(x) = x$ for all $x \in [0,1]$. ■