bon-db/calculus/diff/Z9C89A.json (AoPS)

Problem. Let $f\colon [a,b] \to [a,b]$ be a differentiable function such that $f(a) = a$ and $f(b) = b$. Prove that there exist u, $v \in (a,b)$ such that $u < v$ and $f'(u) + f'(v) = 2$.

Solution. By LMVT on $\left[a, \frac{a+b}{2}\right]$, we get that there exists $u \in \left(a, \frac{a+b}{2}\right)$ such that

$$f'(u) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = \frac{2}{b-a} \cdot \left(f\left(\frac{a+b}{2}\right) - f(a)\right).$$

Similarly by LMVT on $\left[\frac{a+b}{2}, b\right]$, we get that there exists $v \in \left(\frac{a+b}{2}, b\right)$ such that

$$f'(v) = \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = \frac{2}{b-a} \cdot \left(f(b) - f\left(\frac{a+b}{2}\right)\right).$$

Clearly $u < v$ and

$$f'(u) + f'(v) = \frac{2}{b-a} \cdot (f(b) - f(a)) = \frac{2}{b-a} \cdot (b-a) = 2. \qquad \blacksquare$$