

bon-db/combi/Z53906.json (Putnam 2002 A3, INMO 2013 P4)

Problem. Let n be an integer greater than 1 and let T_n be the number of non-empty subsets S of $\{1, 2, \dots, n\}$ with the property that the average of the elements of S is an integer. Prove that $T_n - n$ is always even.

Solution. Define a function $f: \mathcal{P}([n]) \rightarrow \mathcal{P}([n])$ by

$$f(A) = \{n + 1 - a \mid a \in A\}.$$

Call the sets satisfying the property to be good. Note that $f(f(A)) = A$. Now it is easy to see that A is good if and only if $f(A)$ is good.

Now we make cases based on the parity of n .

(i) ***n* is even:** We need to show that T_n is even.

If $A \neq f(A)$, we can pair them up. We focus on the case when $f(A) = A$. If $|A|$ is odd, then there must exist an element $a \in A$ such that $a = n + 1 - a$, i.e., $a = \frac{n+1}{2}$, which is impossible as n is even.

So, $|A|$ must be even. Let $|A| = 2k$. Then the sum of all the elements of A is $(n + 1)k$. So, the average is

$$\frac{(n + 1)k}{2k} = \frac{n + 1}{2},$$

which is not an integer.

Thus the condition $f(A) = A$ is not possible. Therefore T_n is even as we can pair up A and $f(A)$, as desired.

(ii) ***n* is odd:** It suffices to show that T_n is odd.

If $A \neq f(A)$, we can pair them up just like in the previous case. We focus on the case when $f(A) = A$.

If $|A|$ is even, note that

$$a \in A \iff n + 1 - a \in A.$$

As

$$\frac{n + 1}{2} = n + 1 - \frac{n + 1}{2},$$

$\frac{n+1}{2} \notin A$ in this case. So, if we create pairs in the fashion $(1, n)$, $(2, n - 1)$, etc., then the presence of any one element in a pair implies the presence of the other.

Suppose k of these pairs are present in A . Then the sum of the elements is

$$\frac{(n+1)k}{2},$$

which is an integer. Therefore, any A constructed from these pairs works.

Here we have $2^{\frac{n-1}{2}} - 1$ ways of choosing which pairs to include.

If $|A|$ is odd, it is easy to see $\frac{n+1}{2} \in A$. Consider the pairs just as in the previous case. Suppose k of these pairs contribute to A . Then the sum of elements of A is

$$\frac{(n+1)k}{2} + \frac{n+1}{2} = \frac{(n+1)(k+1)}{2},$$

which is an integer.

So, there are $2^{\frac{n-1}{2}}$ ways to choose the set A .

Thus, in total, there are $2^{\frac{n-1}{2}+1} - 1$ possible values of A which implies T_n is odd, as desired.

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