bon-db/calculus/diff/Z15E7F.json (AoPS)

Problem (L). et $f\colon [a,b] \to [a,b]$ be a differentiable function. Consider a point (α, β) on the line joining $(a, f(a))$ and $(b, f(b))$ such that $\alpha \notin [a,b]$. Prove that there exists a tangent to the curve passing through (α, β).

Solution. Define

$$g(x) = \frac{f(x) - \beta}{x - \alpha}.$$

Clearly $g(x)$ is differentiable. Then note that

$$g(a) = \frac{f(a) - \beta}{a - \alpha} \quad \text{and} \quad g(b) = \frac{f(b) - \beta}{b - \alpha}.$$

Since the points $(a, f(a))$, $(b, f(b))$, and (α, β) lie on a straight line, the slope of line joining $(a, f(a))$ and (α, β) is equal to that of $(b, f(b))$ and (α, β). Therefore, we have $g(a) = g(b)$. So by LMVT on (a, b), we get $c \in (a, b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0.$$

Note

$$g'(x) = \frac{f'(x)(x - \alpha) - (f(x) - \beta)}{(x - \alpha)^2}.$$

Therefore

$$g'(c) = 0 \implies f'(c)(c - \alpha) = f(c) - \beta \implies f'(c) = \frac{f(c) - \beta}{c - \alpha}.$$

Choosing c as our point on the curve through which the tangent passes, we are done. ■