bon-db/calculus/diff/ZAFC5E.json (AoPS)

Problem. Let $f \colon [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = 0$ and $|f'(x)| \le |f(x)|$ for all $x \in [0,1]$. Prove that $f(x) = 0$ for all $x \in [0,1]$.

Solution. Since $|f(x)|$ is continuous over $[0,1]$, it must be bounded. Let $M = \max_{x \in [0,1]} |f(x)|$. Also let $S = \{x \in [0,1] \mid |f(x)| = M\}$. Since $x \in [0,1]$, S must be bounded. Let $c = \inf\{S\}$. By continuity, $|f(c)| = M$.

> **Claim —** $c = 0$.

Proof. For the sake of contradiction, assume that $c > 0$. Then by the LMVT on $[0, c]$, we get $t \in (0, c)$ such that
$$f'(t) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c}.$$
So,
$$|f(c)| = |c| \cdot |f'(t)|.$$
Clearly $0 < c \le 1$, so
$$|f(c)| = c \cdot |f'(t)| \le |f'(t)| \le |f(t)|.$$
But then
$$M = |f(c)| \le |f(t)| \le M \implies |f(t)| = M.$$
Since $t < c$, this contradicts the minimality of c and our claim is proved. $\qquad\square$

Therefore
$$M = |f(c)| = |f(0)| = 0$$
which implies
$$|f(x)| \le M = 0 \implies f(x) = 0 \quad \forall\, x \in [0,1].$$

$\blacksquare$