

**bon-db/calculus/diff/ZD143D.json (AoPS)**

**Problem.** Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function which satisfies

$$f\left(\frac{1}{n}\right) = 1 \quad \forall n \in \mathbb{N}.$$

Prove that  $f'(0) = f''(0) = 0$ .

*Solution.* By continuity of  $f(x)$ , we get  $\lim_{n \rightarrow \infty} f\left(\frac{1}{n}\right) = f(0) = 1$ .

Now note  $f\left(\frac{1}{n}\right) = f(0) = 1$ . So by LMVT, there exists  $c_n \in \left(0, \frac{1}{n}\right)$  such that

$$f'(c_n) = \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n} - 0} = \frac{1 - 1}{\frac{1}{n}} = 0.$$

As  $f(x)$  is twice differentiable,  $f'(x)$  is differentiable and thus continuous. This implies

$$f'(0) = \lim_{n \rightarrow \infty} f'(c_n) = 0.$$

So now, by the definition of derivative,

$$f''(0) = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{n \rightarrow \infty} \frac{f'(c_n) - f'(0)}{c_n - 0} = 0. \quad \blacksquare$$