

**bon-db/calculus/int/Z61F93.json (AoPS)**

**Problem.** Let  $f: [0, 1] \rightarrow \mathbb{R}$  be a continuous function. Find

$$\lim_{n \rightarrow \infty} n \int_0^1 f(x) x^n dx.$$

*Solution.* By using continuity of  $f$  at  $x = 1$ , we get that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that for all  $x \in [0, 1]$ ,

$$|x - 1| < \delta \implies |f(x) - f(1)| < \varepsilon$$

that is

$$f(1) - \varepsilon \leq f(x) \leq f(1) + \varepsilon \quad \forall x \in (1 - \delta, 1 + \delta).$$

Let  $M = \max_{t \in [0, 1]} f(t)$  and  $m = \min_{t \in [0, 1]} f(t)$ , which exist as  $f$  is continuous. Then we have

$$\begin{aligned} \int_0^1 f(x) x^n dx &= \int_0^{1-\delta} f(x) x^n dx + \int_{1-\delta}^1 f(x) x^n dx \\ &\leq \int_0^{1-\delta} M x^n dx + \int_{1-\delta}^1 (f(1) + \varepsilon) x^n dx \\ &= \frac{M(1 - \delta)^{n+1}}{n + 1} + \frac{(f(1) + \varepsilon)(1 - (1 - \delta)^{n+1})}{n + 1} \end{aligned}$$

which implies

$$n \int_0^1 f(x) x^n dx \leq \frac{n}{n + 1} \cdot M(1 - \delta)^{n+1} + \frac{n}{n + 1} (f(1) + \varepsilon)(1 - (1 - \delta)^{n+1}).$$

Similarly, we also get

$$\begin{aligned} \int_0^1 f(x) x^n dx &= \int_0^{1-\delta} f(x) x^n dx + \int_{1-\delta}^1 f(x) x^n dx \\ &\geq \int_0^{1-\delta} m x^n dx + \int_{1-\delta}^1 (f(1) - \varepsilon) x^n dx \\ &= \frac{m(1 - \delta)^{n+1}}{n + 1} + \frac{(f(1) - \varepsilon)(1 - (1 - \delta)^{n+1})}{n + 1} \end{aligned}$$

which implies

$$n \int_0^1 f(x) x^n dx \geq \frac{n}{n + 1} \cdot m(1 - \delta)^{n+1} + \frac{n}{n + 1} (f(1) - \varepsilon)(1 - (1 - \delta)^{n+1})$$

Let

$$a_n = \frac{n}{n + 1} \cdot m(1 - \delta)^{n+1} + \frac{n}{n + 1} (f(1) - \varepsilon)(1 - (1 - \delta)^{n+1})$$

and

$$b_n = \frac{n}{n + 1} \cdot M(1 - \delta)^{n+1} + \frac{n}{n + 1} (f(1) + \varepsilon)(1 - (1 - \delta)^{n+1}).$$

Therefore, we now have

$$a_n \leq n \int_0^1 f(x)x^n dx \leq b_n.$$

Note that  $\lim_{n \rightarrow \infty} a_n = f(1) - \varepsilon$  and  $\lim_{n \rightarrow \infty} b_n = f(1) + \varepsilon$ . This suggests that the sequences  $\{a_n\}_{n \geq 1}$  and  $\{b_n\}_{n \geq 1}$  are bounded. Therefore,

$$\left\{ n \int_0^1 f(x)x^n dx \right\}_{n \geq 1}$$

is also bounded and hence

$$\liminf_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx$$

and

$$\limsup_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx$$

both exist.

Using the following limits in our preceding inequality, we get

$$f(1) - \varepsilon \leq \liminf_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx \leq f(1) + \varepsilon$$

and

$$f(1) - \varepsilon \leq \limsup_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx \leq f(1) + \varepsilon$$

for any arbitrary  $\varepsilon > 0$ . Considering small enough  $\varepsilon$ , we get that

$$\liminf_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx = \limsup_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx = f(1)$$

which implies  $\lim_{n \rightarrow \infty} n \int_0^1 f(x)x^n dx$  exists and is equal to  $f(1)$ . ■