bon-db/calculus/int/Z768E4.json (AoPS)

Problem. Let $f\colon [0,1] \to \mathbb{R}^+$ be a continuous function. Find

$$\lim_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n}.$$

Solution. As f is continuous, by EVT, there exists $c \in [0,1]$ such that $f(x) \leq f(c)$ for all $x \in [0,1]$. Therefore,

$$\int_0^1 f(x)^n \mathrm{d}x \leq \int_0^1 f(c)^n \mathrm{d}x = f(c)^n$$

which implies

$$\left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n} \leq f(c).$$

Now by continuity of f at $x = c$, we get that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in [0,1]$,

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon,$$

that is, $f(x) \in (f(c) - \varepsilon, f(c) + \varepsilon)$ for all $x \in (c - \delta, c + \delta) \cap [0,1]$. Fix $\varepsilon > 0$. Let $\mathcal{I} = (c - \delta, c + \delta) \cap [0,1]$ and let ℓ be the length of the interval $\mathcal{I}$. Therefore,

$$\int_0^1 f(x)^n \mathrm{d}x \geq \int_{\mathcal{I}} f(x)^n \mathrm{d}x \geq \int_{\mathcal{I}} (f(c) - \varepsilon)^n \mathrm{d}x = (f(c) - \varepsilon)^n \cdot \ell$$

which implies

$$\left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n} \geq (f(c) - \varepsilon) \cdot \ell^{1/n}.$$

Clearly $\left\{\ell^{1/n}\right\}_{n\geq 1}$ is bounded below. Therefore,

$$\left\{\left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n}\right\}_{n\geq 1}$$

is bounded both above and below. This implies that both

$$\liminf_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n} \quad \text{and} \quad \limsup_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n}$$

exist.

Clearly $\lim_{n\to\infty} (f(c) - \varepsilon) \cdot \ell^{1/n} = f(c) - \varepsilon$. Considering limit superior and limit inferior in the following inequality:

$$(f(c) - \varepsilon)^{1/n} \leq \left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n} \leq f(c)$$

we get

$$f(c) - \varepsilon \leq \liminf_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x\right)^{1/n} \leq f(c)$$

and

$$f(c) - \varepsilon \leq \limsup_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x \right)^{1/n} \leq f(c)$$

for any arbitrary $\varepsilon > 0$. Considering small ε, we get

$$\liminf_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x \right)^{1/n} = \limsup_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x \right)^{1/n} = f(c)$$

which implies

$$\limsup_{n\to\infty} \left(\int_0^1 f(x)^n \mathrm{d}x \right)^{1/n} = f(c).$$

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Remark (Motivation)**.** From where does one motivate the fact that the limit should be the maximum value of the function? Well, if this were a finite discrete sum instead of an integral, that is,

$$\left(\frac{a_1^n + a_2^n + \cdots + a_d^n}{n} \right)^{1/n},$$

then the limit simply would've been $\max_i a_i$. (Left as an exercise to the reader.) Since an integral is basically a continuous sum instead of a discrete one, it suggests that the limit might be the maximum/supremum value that f attains on it's entire domain.