bon-db/calculus/diff/Z8A2F8.json (ISI B.Stat. 2006 P6)

Problem. Solve the following:

(a) Let $f(x) = x - xe^{-\frac{1}{x}}, \quad x > 0$. Show that $f(x)$ is an increasing function on $(0, \infty)$, and $\lim_{x\to\infty} f(x) = 1$.

(b) Using part (a) or otherwise, draw graphs of $y = x - 1, y = x, y = x + 1$, and $y = xe^{-\frac{1}{|x|}}$ for $-\infty < x < \infty$ using the same X and Y axes.

Solution. We first solve part (a) and use it to draw the graph in part (b).

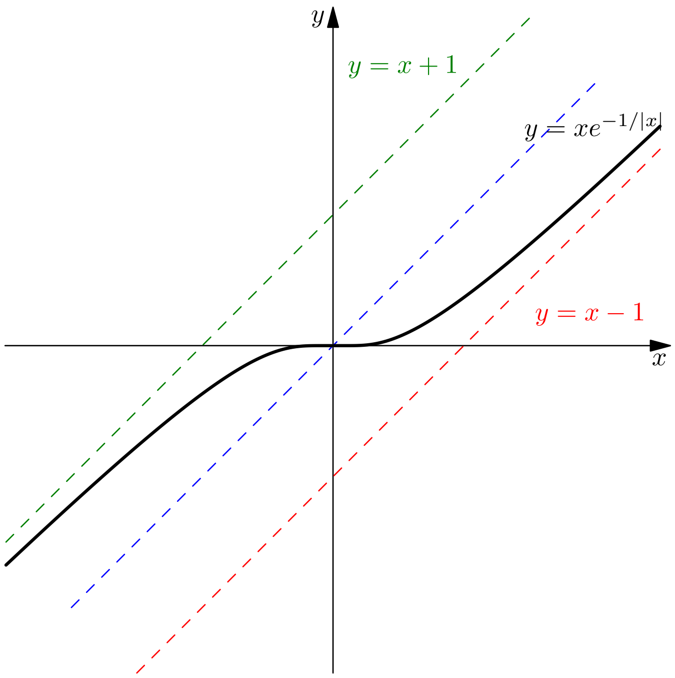
(a) We have $f(x) = x - xe^{-\frac{1}{x}}$. So,

$$f'(x) = 1 - e^{-\frac{1}{x}}\left(\frac{x+1}{x}\right).$$

Define $g(x) = e^x - x - 1$. Then $g'(x) = e^x - 1$. Note that $g'(x) = e^x - 1 \geq 0$ for all $x \geq 0$ (equality holds iff $x = 0$). Now note that there does not exist any $x_0 > 0$ such that $g(x_0) = 0$ (this would otherwise give a contradiction to the fact that $g'(x) > 0$ for all $x > 0$ by Rolle's Theorem on $[0, x_0]$). Thus $g(x) > 0$ for all $x > 0$. Substituting $x \to \frac{1}{x}$, we get that $f'(x) > 0$ for all $x > 0$.

Now $\lim_{x\to\infty} f(x) = \lim_{y\to 0^+} f\left(\frac{1}{y}\right) = \lim_{y\to 0^+} \frac{e^y - 1}{ye^y} = 1$.

(b) The graph is given below:



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