bon-db/calculus/diff/ZF4877.json (Putnam 2003 A3)

Problem. Find the minimum value of

$$|\sin x+\cos x+\tan x+\cot x+\sec x+\operatorname{cosec} x|$$

for real numbers x.

Solution by **Kent Merryfield** (#2 on the thread).

Solution. Let $u=\cos x+\sin x=\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$.

We will represent everything in terms of u, which ranges over the interval $\left[-\sqrt{2},\sqrt{2}\right]$.

We note that

$$u^2=\sin^2 x+2\sin x\cos x+\cos^2 x=1+2\sin x\cos x,$$

so $\sin x\cos x=\frac{u^2-1}{2}$. Then:

$$\sin x+\cos x=u.$$

$$\tan x+\cot x=\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}=\frac{2}{u^2-1}.$$

$$\sec x+\operatorname{cosec} x=\frac{\cos x+\sin x}{\sin x\cos x}=\frac{2u}{u^2-1}.$$

So the function we are trying to minimize is the absolute value of

$$u+\frac{2}{u^2-1}+\frac{2u}{u^2-1}=\frac{u^3+u+2}{u^2-1}=\frac{(u+1)(u^2-u+2)}{(u+1)(u-1)}=\frac{u^2-u+2}{u-1}.$$

This last identity holds except at the removable singularities at $u=-1$ ($x=\pi$ or $3\pi/2$) and except for the poles at $u=1$ ($x=0$ or $\pi/2$). Near the removable singularities, the absolute value of the function is near 2; near the poles it is large.

If $f(u)=\frac{u^2-u+2}{u-1}$ then $f'(u)=\frac{-u^2+2u+1}{(u-1)^2}$.

For $|f|$ to have a minimum, either f must have a critical point or f must be zero. But $f(u)$ is never zero, because u^2-u+2 is an irreducible quadratic. Potential critical points happen at the roots of $-u^2+2u+1=0$, namely $u=1\pm\sqrt{2}$. But the positive root lies outside the domain of f.

Hence our critical points happen when $u=1-\sqrt{2}$, which corresponds to values of x in the second and fourth quadrants.

The value of f at that point is $1 - 2\sqrt{2}$, so the absolute value is $2\sqrt{2} - 1$, which is smaller than 2 (about 1.828.) One last check from the calculus point of view: what does $f(u)$ equal when u is $\pm\sqrt{2}$?

This corresponds to $x = \pi/4$ or $5\pi/4$, and the values of f are $2 + 3\sqrt{2}$ and $2 - 3\sqrt{2}$, both of which are bigger than 2 in absolute value.

Thus, the minimum of the absolute value is $2\sqrt{2} - 1$, which happens twice in each period. ∎