bon-db/nt/ZF0572.json (AoPS)

Problem. The sequence $(a_1, a_2, \ldots, a_n)$ is a permutation of $(1, 2, \ldots, n)$ where n is an even natural number.

(i) Prove that there exist a_i, a_j in the sequence such that $n \mid (a_i + i) - (a_j + j)$.

(ii) Will the result still be true if n is odd?

Solution. We first solve part (i) and then part (ii).

(i) FTSOC assume that there does not exist any a_i and a_j such that $a_i + i \equiv a_j + j \pmod{n}$.

Therefore $\{a_i + i\}_{i=1}^{n}$ is the CRS modulo n. Therefore, summing the elements of the two sets, of the CRS and the set $\{a_i + i\}_{i=1}^{n}$, we get that $\sum_{i=1}^{n} a_i \equiv 0 \pmod{n}$. This implies that $n \mid \frac{n}{2} \cdot (n+1)$ which gives a contradiction as n is even.

(ii) No, the result is not true for odd n. Consider the sequence $\{a_i\}_{i=1}^{n}$ where $a_i = i$.

Then if $a_i + i \equiv a_j + j \pmod{n}$ for distinct i and j, then $n \mid 2(i - j)$ which implies $n \mid i - j$ as $\gcd(n, 2) = 1$. Moreover, note that as $i \neq j$, $n \leq |i - j| \leq n - 1$ which gives a contradiction.

Thus all $(a_i + i)$ are distinct modulo n.

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