# bon-db/alg/Z0FF1F.json (Putnam 2017 A2)

**Problem.** Let $Q_0(x) = 1$, $Q_1(x) = x$, and

$$Q_n(x) = \frac{(Q_{n-1}(x))^2 - 1}{Q_{n-2}(x)}$$

for all $n \geq 2$. Show that, whenever $n$ is a positive integer, $Q_n(x)$ is equal to a polynomial with integer coefficients.

*Solution.* We use a lemma which absolutely obliterates this problem.

**Lemma (L)**

et $P(x)$, $Q(x) \in \mathbb{Z}[x]$ be two monic non-constant polynomials and $R(x) \in \mathbb{R}[x]$ be another polynomial such that it satisfies,

$$P(x) = Q(x) \cdot R(x).$$

Then $R(x) \in \mathbb{Z}[x]$ and is monic.

*Proof.* Let $\deg(R) = n$. Pick any $n+1$ points where $Q(x)$ is non-zero. Therefore, for these points, $R(x) = P(x)/Q(x)$ is rational. By Lagrange Interpolation, we can conclude that $R(x)$ has rational coefficients.

Now we show that the coefficients must all be integers.

Let $R(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0 x^0$ where $a_i$ are rational numbers $\frac{p_i}{q_i}$ for integers $p_i$, $q_i$ with $\gcd(p_i, q_i) = 1$.

Let $L = \operatorname{lcm}(q_1, q_2, \ldots, q_n)$. If $L = 1$, we are done.

Suppose not. Then there exists a prime $u$ such that $u \mid L$. We multiply both sides by $L$. This helps us clearing out the denominator of $R(x)$.

Clearly the polynomial $L \cdot P(x)$ is divisible by $u$. So by working in $\mathbb{F}_u$ we can deduce that $L \cdot P(x) \equiv 0$ in $\mathbb{F}_u$. Now as $L \cdot P(x) = Q(x) \cdot (L \cdot R(x))$, we must have $Q(x) \equiv 0$ or $L \cdot R(x) \equiv 0$ in $\mathbb{F}_u$. But clearly $Q(x) \not\equiv 0$ as it is monic. Thus $L \cdot R(x) \equiv 0$ in $\mathbb{F}_u$.

This means that $u \mid p_i \cdot \frac{L}{q_i}$ for all $i$.

Now pick $q_m$ such that $\nu_u(q_m) = \nu_u(L)$. Clearly $\nu_u(q_m) \geq 1$.

Therefore $\nu_u\left(\frac{L}{q_m}\right) = 0$. Also, as $\gcd(p_m, q_m) = 1$, so $\gcd(p_m, u) = 1$. This means that $\gcd\left(p_m \cdot \frac{L}{q_m}, u\right) = 1$ which contradicts the fact that $u \mid p_i \cdot \frac{L}{q_i}$ for all $i$. $\square$

Now lets use this bazooka to kill our fly.

Note that $Q_2(x) = x^2 - 1$. Therefore by using induction on $n$ with the base case starting from $n = 1$, we can say that $Q_n(x) \cdot Q_{n-2}(x) = Q_{n-1}(x)^2 - 1$ along with the fact that $Q_{n-2}(x)$ and $Q_{n-1}(x)^2 - 1$ are monic integer polynomials implies that $Q_n(x)$ is a monic integer polynomial. ■