# bon-db/alg/ineq/Z2EC09.json (IMOSL 2000 A1)

**Problem.** Let $a$, $b$, $c$ be positive real numbers such that $abc = 1$. Prove that,

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \le 1.$$

*Solution.* Substitute $a = \frac{y}{z}$, $b = \frac{z}{x}$, $c = \frac{x}{y}$.

Then we have to prove that,

$$(x + y - z)(y + z - x)(z + x - y) \le xyz.$$

WLOG assume $x = \max\{x, y, z\}$. Then note that $(x + y - z)$ and $(z + x - y)$ are both positive. If $(y + z - x)$ is negative, we are clearly done.

Otherwise, $(y + z - x)$ is also positive. Then $x$, $y$, $z$ are sides of a triangle. Let $p = x + y - z$, $q = y + z - x$, $r = z + x - y$.

Then note that the inequality that we have to prove changes to,

$$\left(\frac{p+q}{2}\right)\left(\frac{q+r}{2}\right)\left(\frac{r+p}{2}\right) \ge pqr$$

which is indeed true using AM-GM. Equality holds if $a = b = c = 1$. ■