bon-db/alg/Z1480F.json (AoPS)

Problem. M is a finite subset of $\mathbb{R}$ with $|M| \geq 3$ which has the property that for all a, $b \in M$, $a^2 + b\sqrt{2} \in \mathbb{Q}$. Then prove that $a\sqrt{2} \in \mathbb{Q}$ for all $a \in M$.

Solution. Pick distinct a, b, $c \in M$.

Then we note that $a^2 + b\sqrt{2} \in \mathbb{Q}$ and $a^2 + c\sqrt{2} \in \mathbb{Q}$. Subtracting implies $(b - c)\sqrt{2} \in \mathbb{Q}$. This also implies that $(b - c) \in \mathbb{Q}\sqrt{2}$.

Now, $b^2 + b\sqrt{2} \in \mathbb{Q}$ and $c^2 + c\sqrt{2} \in \mathbb{Q}$. Subtracting implies $(b^2 - c^2) + (b - c)\sqrt{2} \in \mathbb{Q}$ which further implies $(b^2 - c^2) \in \mathbb{Q}$. Now note that as $(b - c) \in \mathbb{Q}\sqrt{2}$, we also get that $(b + c) \in \mathbb{Q}\sqrt{2}$.

Adding $(b + c)$ and $(b - c)$, we get that $b \in \mathbb{Q}\sqrt{2}$ which implies that $b\sqrt{2} \in \mathbb{Q}$.

Similarly $c\sqrt{2} \in \mathbb{Q}$. Since b was arbitrarily fixed from M, we can conclude that for all $a \in M$, $a\sqrt{2} \in \mathbb{Q}$. ■