bon-db/alg/ineq/ZB6732.json (Peru TST 2007 D1 P2)

Problem. Let a, b, c be positive real numbers, such that:

$$a+b+c \geq \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

Prove that:

$$a+b+c \geq \frac{3}{a+b+c}+\frac{2}{abc}.$$

Solution. Note that

$$\frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \geq \frac{3}{a+b+c}.$$

Therefore,

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)^2 \geq 3\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \geq \frac{3(a+b+c)}{abc} \geq \frac{3\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{abc}.$$

This gives that

$$\frac{2(a+b+c)}{3} \geq \frac{2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}{3} \geq \frac{2}{abc}.$$

Adding $\frac{a+b+c}{3} \geq \frac{3}{a+b+c}$ and $\frac{2(a+b+c)}{3} \geq \frac{2}{abc}$, we get,

$$a+b+c \geq \frac{3}{a+b+c}+\frac{2}{abc}.$$

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