bon-db/alg/Z87387.json (ISI B.Stat. 2006 P3)

Problem. Prove that $n^4 + 4^n$ is composite for all values of n greater than 1.

Solution. We divide the solution into two cases based on the parity of n.

- **If n is even:** Then $4 \mid n^4$ and $4 \mid 4^n$ which implies $4 \mid n^4 + 4^n$ which makes it composite.
- **If n is odd:** Then let $n = 2k + 1$ where $k \geq 1$ as $n \geq 3$.

  Then $n^4 + 4^n = n^4 + 4 \cdot 2^{4k}$. Let $2^k = m$.

  $$n^4 + 4^n = n^4 + 4 \cdot m^4 = (n^2 + 2m^2 - 2nm)(n^2 + 2m^2 + 2nm).$$

  To finish, note that,

  $$n^2 + 2m^2 + 2mn > n^2 + 2m^2 - 2mn = (n - m)^2 + m^2 \geq 4.$$

  Thus both the two numbers are ≥ 4 which means that their product is composite.

■