

bon-db/alg/poly/Z7D495.json (AoPS)

Problem. Find all real polynomials $P(x)$ with $P(0) = 0$ such that there exists a function $F(x)$ satisfying $F(x) > x$ for all $x \geq 0$ and

$$F(P(x)) = P(F(x)) \quad \forall x \in \mathbb{R}.$$

Solution. Plugging $x = 0$ gives

$$P(F(0)) = F(0).$$

Composing F on both sides, we get

$$P(F^2(0)) = F(P(F(0))) = F^2(0).$$

Similarly,

$$P(F^n(0)) = F^n(0) \implies P(F^{n+1}(0)) = F(P(F^n(0))) = F^{n+1}(0).$$

Therefore,

$$P(F^n(0)) = F^n(0) \quad \forall n \in \mathbb{N}.$$

Let $Q(x) = P(x) - x$. Clearly $Q(x)$ is a real polynomial.

As $F(0) > 0$ and $F(x) > x$, we get

$$F(0) > 0 \implies F^2(0) > F(0) \implies \dots \implies F^n(0) > F^{n-1}(0).$$

Hence, $\{F^n(0)\}_{n \geq 1}$ is a strictly increasing sequence. Therefore,

$$Q(F^n(0)) = 0 \quad \forall n \in \mathbb{N},$$

and since there are infinitely many zeroes of Q , this forces $Q \equiv 0$ which implies $P \equiv x$. For $P(x) = x$, we can pick $F(x) = x + 1$, which clearly satisfies both conditions. ■