bon-db/calculus/seq/ZE8A6E.json (AoPS)

Problem. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$. Then find

$$\lim_{n\to\infty} \frac{e^{{a_n}^2} + a_n}{4}.$$

Solution. As $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{2}$, so for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left|\frac{a_{n+1}}{a_n} - \frac{1}{2}\right| < \varepsilon.$$

This gives us the following inequality,

$$\frac{1}{2} - \varepsilon < \frac{a_{N+i}}{a_{N+(i-1)}} < \frac{1}{2} + \varepsilon$$

for all $1 \leq i \leq k$ where k is some fixed natural number ≥ 1. Now note that if we take $\varepsilon < \frac{1}{2}$, then everything on both the sides of the inequality is positive. So we can multiply them which gives,

$$\left(\frac{1}{2} - \varepsilon\right)^k < \frac{a_{N+k}}{a_N} < \left(\frac{1}{2} + \varepsilon\right)^k.$$

Limiting $k \to \infty$, by Sandwich theorem, we get that $\lim_{n\to\infty} a_n = 0$.

Therefore,

$$\lim_{n\to\infty} \frac{e^{a_n^2} + a_n}{4} = \frac{1}{4}. \qquad \blacksquare$$