bon-db/alg/ineq/ZF235B.json (AoPS)

Problem. Let a_1, a_2, ..., a_n be positive reals and let

$$m = \min\left\{a_1 + \frac{1}{a_2}, a_2 + \frac{1}{a_3}, \dots, a_n + \frac{1}{a_1}\right\}.$$

Show that

$$\sqrt[n]{a_1 a_2 \cdots a_n} + \frac{1}{\sqrt[n]{a_1 a_2 \cdots a_n}} \geq m.$$

Solution. Let

$$G = \sqrt[n]{a_1 a_2 \cdots a_n},$$

and let k be the index such that

$$a_k = \min_{1 \leq i \leq n} \{a_i\}.$$

So, $G \geq a_k$. If $G \leq a_{k+1}$, then

$$G + \frac{1}{G} \geq a_k + \frac{1}{a_{k+1}} \geq m.$$

Otherwise $G > a_{k+1}$. Similarly, we can continue using this pattern until we reach an index j such that

$$G \geq a_j \quad \text{and} \quad G \leq a_{j+1}.$$

If no such index exists, then $G > a_i$ for all $1 \leq i \leq n$, which implies

$$nG > \sum_{i=1}^{n} a_i \geq nG,$$

where the last inequality follows by AM–GM. This however is a clear contradiction and hence, such an index j must exist and we are done. ∎