

## bon-db/calculus/diff/Z9C89A.json (AoPS)

**Problem (L).** Let  $f: [a, b] \rightarrow [a, b]$  be a differentiable function such that  $f(a) = a$  and  $f(b) = b$ . Prove that there exist  $u, v \in (a, b)$  such that  $u < v$  and  $f'(u) + f'(v) = 2$ .

*Solution.* By LMVT on  $[a, \frac{a+b}{2}]$ , we get that there exists  $u \in (a, \frac{a+b}{2})$  such that

$$f'(u) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = \frac{2}{b-a} \cdot \left(f\left(\frac{a+b}{2}\right) - f(a)\right).$$

Similarly by LMVT on  $[\frac{a+b}{2}, b]$ , we get that there exists  $v \in (\frac{a+b}{2}, b)$  such that

$$f'(v) = \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = \frac{2}{b-a} \cdot \left(f(b) - f\left(\frac{a+b}{2}\right)\right).$$

Clearly  $u < v$  and

$$f'(u) + f'(v) = \frac{2}{b-a} \cdot (f(b) - f(a)) = \frac{2}{b-a} \cdot (b - a) = 2.$$

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