bon-db/calculus/seq/Z3402A.json (AoPS)

Problem. Two sequences $\{a_i\}_{i=1}^m \subset \mathbb{R}^+$ and $\{b_i\}_{i=1}^k \subset \mathbb{R}^+$ are given satisfying,

$$a_1^{1/n} + \cdots + a_m^{1/n} = b_1^{1/n} + \cdots + b_k^{1/n} \quad \forall\, n \in \mathbb{N}.$$

Then prove that

(i) $m = k$

(ii) $a_1 a_2 \cdots a_m = b_1 b_2 \cdots b_k$.

Solution. We first solve part (i) and then part (ii).

(i) Taking $n \to \infty$, we get,

$$m = \lim_{n\to\infty} (a_1^{1/n} + \cdots + a_m^{1/n}) = \lim_{n\to\infty} (b_1^{1/n} + \cdots + b_k^{1/n}) = k.$$

(ii) Note that from $m = k$, we can say that,

$$\frac{a_1^{1/n} - 1}{n} + \cdots + \frac{a_m^{1/n} - 1}{n} = \frac{b_1^{1/n} - 1}{n} + \cdots + \frac{b_k^{1/n} - 1}{n}.$$

Now taking $n \to \infty$, we get,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\sum_{i=1}^{m}\left(\frac{a_i^{1/n}-1}{n}\right)\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{k}\left(\frac{b_i^{1/n}-1}{n}\right)\right)\\
\implies & \sum_{i=1}^{m}\lim_{n\to\infty}\left(\frac{a_i^{1/n}-1}{n}\right) = \sum_{i=1}^{k}\lim_{n\to\infty}\left(\frac{b_i^{1/n}-1}{n}\right)\\
\implies & \sum_{i=1}^{m}\ln(a_i) = \sum_{i=1}^{k}\ln(b_i)\\
\implies & a_1 a_2 \cdots a_m = b_1 b_2 \cdots b_k.
\end{aligned}$$

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