bon-db/calculus/diff/Z93F66.json (AoPS)

Problem. The function $f \colon [0,1] \to \mathbb{R}$ is continuous over $[0,1]$ and differentiable over $(0,1)$. It is given that $f(0) = 1$ and $f(1)^3 + 2f(1) = 5$. Then prove that there exists $c \in (0,1)$ such that,

$$f'(c) = \frac{2}{2 + 3f(c)^2}.$$

Solution. Consider the function $g(x) = f(x)^3 + 2f(x) - 2x - 3$.

It is easy to check that $g(x)$ is continuous over $[0,1]$ and differentiable over $(0,1)$.

Also, $g(0) = g(1) = 0$. Therefore by Rolle's Theorem, there exists a $c \in (0,1)$ such that $g'(c) = 0$.

Note that $g'(x) = 3f(x)^2 \cdot f'(x) + 2f'(x) - 2$.

From here, $g'(c) = 0$ implies that,

$$f'(c) = \frac{2}{2 + 3f(c)^2}. \qquad \blacksquare$$