bon-db/alg/ineq/Z6548B.json (ISI 2023 P6)

Problem. Let $\{u_n\}_{n\geq 1}$ be a sequence of real numbers defined as $u_1 = 1$ and

$$u_{n+1} = u_n + \frac{1}{u_n} \text{ for all } n \geq 1.$$

Prove that $u_n \leq \frac{3\sqrt{n}}{2}$ for all n.

Solution. Note that $u_1 \leq \frac{3}{2}\sqrt{1}$.

Also $u_2 = u_1 + \frac{1}{u_1} > 0$. Similarly, if $u_i > 0$, then $u_{i+1} = u_i + \frac{1}{u_i} > 0$. So $u_1 > 0 \implies u_2 > 0 \implies \cdots$ and so on. This gives us that $u_i > 0$ for all $i \geq 1$.

Now note that for all $n \geq 2$, by AM-GM we get that,

$$u_n = u_{n-1} + \frac{1}{u_{n-1}} \geq 2.$$

Claim () — $u_n \leq \frac{3}{2}\sqrt{n}$ for all $n \geq 2$.

Proof. We proceed by using induction on n.

- **Base Case:** For $n = 2$, $u_2 = 2 \leq \frac{3}{2}\sqrt{2}$.
- **Induction Hypothesis:** Assume that our claim is true for some $n = k \geq 2$.
- **Inductive Step:** We prove the claim for $n = k + 1$.

$$\begin{aligned} u_{k+1}^2 &= u_k^2 + \frac{1}{u_k^2} + 2 \\ &\leq \frac{9k}{4} + \frac{1}{2} + 2 \\ &= \frac{9(k+1)}{4}. \end{aligned}$$

This gives us that $a_{k+1} \leq \frac{3}{2}\sqrt{k+1}$ and thus our induction is complete. □

Hence with the proof of our claim and the case $n = 1$, the problem is solved. ■