bon-db/calculus/seq/Z65914.json (AoPS)

Problem. Define a sequence of real numbers $\{a_m(j)\}_{m\geq 1, j\geq 0}$ with $a_m(0) = \frac{d}{2^m}$ where $d > 0$ and,

$$a_m(j+1) = a_m(j)^2 + 2a_m(j).$$

Find the limit $\lim_{n\to\infty} a_n(n)$.

Solution. Define $b_m(j) = a_m(j) + 1$.

Then note that $b_m(j+1) = b_m(j)^2$. Hence $b_m(0) = 1 + \frac{d}{2^m}$. It is easy to derive that $b_m(j) = \left(1 + \frac{d}{2^m}\right)^{2^j}$ for all $j \geq 0$.

Therefore $b_m(m) = \left(1 + \frac{d}{2^m}\right)^{2^m}$. So,

$$\lim_{m\to\infty} a_m(m) = \lim_{m\to\infty} b_m(m) - 1 = \lim_{m\to\infty} \left(1 + \frac{d}{2^m}\right)^{\frac{2^m}{d}\cdot d} - 1 = e^d - 1. \qquad \blacksquare$$