

**bon-db/calculus/diff/Z71DE5.json (IMC 2009 D2 P2)**

**Problem.** Suppose  $f: \mathbb{R} \rightarrow \mathbb{R}$  is a two times differentiable function satisfying  $f(0) = 1$ ,  $f'(0) = 0$  and for all  $x \in [0, \infty)$ , it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that, for all  $x \in [0, \infty)$ ,

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

*Solution.* Define  $g(x) = e^{-2x}f(x) + 2e^x - 3$ . Then note that since  $f(x)$  is twice differentiable,  $g(x)$  is twice differentiable too.

Note that

$$g'(x) = \frac{e^{2x}f'(x) - 2e^{2x}f(x)}{e^{4x}} + 2e^x.$$

Define  $h(x) = e^{-x}g'(x) = e^{-3x}f'(x) - 2e^{-3x}f(x) + 2$ . As  $f(x)$  is twice differentiable,  $h(x)$  is differentiable. Now note

$$\begin{aligned} h'(x) &= e^{-3x}f''(x) - 3e^{-3x}f'(x) - 2(e^{-3x}f'(x) - 3e^{-3x}f(x)) \\ &= e^{-3x}(f''(x) - 5f'(x) + 6f(x)). \end{aligned}$$

We know that  $f''(x) - 5f'(x) + 6f(x) \geq 0$  for all  $x \geq 0$  which implies  $h'(x) \geq 0$  for all  $x \geq 0$ . This means that  $h(x)$  is non-decreasing on  $[0, \infty)$ . Therefore  $e^{-x}g'(x) \geq e^0g'(0) = 0$  which implies  $g'(x) \geq 0$  for all  $x \geq 0$ . This further implies that  $g(x)$  is non-decreasing on  $[0, \infty)$ . Then  $g(x) \geq g(0) = 0$  for all  $x \in [0, \infty)$ . Now note  $g(x) = e^{-2x}f(x) + 2e^x - 3 \geq 0$  is equivalent to  $f(x) \geq 3e^{2x} - 2e^{3x}$  for all  $x \geq 0$  and we are done. ■