bon-db/alg/poly/ZC8DCE.json (ISI 2021 P7)

Problem. Let a, b, c be three real numbers which are roots of a cubic polynomial, and satisfy $a+b+c=6$ and $ab+bc+ca=9$. Suppose $a<b<c$. Show that

$$0<a<1<b<3<c<4.$$

Solution by **polarLines** (#7 on the thread).

Solution. Note that

$$P(x)=(x-a)(x-b)(x-c)=x^3-\left(\sum a\right)x^2+\left(\sum ab\right)x-abc=x^3-6x^2+9x-abc.$$

Now $P'(x)=3x^2-12x+9=3(x-3)(x-1)$. Since the extrema occur at $x=1,3$ and we know that P has three distinct real roots, it is clear that $a<1<b<3<c$. Also, this tells us that $P(1)>0$, $P(3)<0$.

$P(3)<0 \implies -abc=3^3-6\times 3^2+9\times 3-abc<0 \overset{b,c>0}{\implies} a>0.$

We can observe that $P(4)=4-abc=P(1)>0$ as noted earlier. This gives $c<4$ by IVP. ■