bon-db/calculus/seq/Z85ECB.json

Problem. Let $f\colon [0,1] \to [0,1]$ be a continuous function which is injective and satisfies

$$f(2x - f(x)) = x.$$

Prove that $f(x) = x$.

Solution. Put $x = 1$ to get $f(2 - f(1)) = 1$ and $x = 0$ to get $f(-f(0)) = 0$. So, we get u and $v \in [0,1]$ such that $f(u) = 0$ and $f(v) = 1$.

Now using the continuity of $f(x)$, we can conclude that $f(x)$ is surjective. Since f is also injective, we conclude that f is bijective. This implies that $f(x)$ is invertible.

Now fix $x_1 \in [0,1]$ arbitrarily and define a recursive sequence $\{x_n\}_{n\geq 1}$ as follows:

$$x_{n+1} = f(x_n) \quad \forall\, n \in \mathbb{N}.$$

Then from our original equation, we get

$$f(2x - f(x)) = x \implies f(x) = 2x - f^{-1}(x).$$

Substituting $x \mapsto x_{n+1}$, we get

$$f(x_{n+1}) = 2x_{n+1} - f^{-1}(x_{n+1})$$

which is equivalent to

$$x_{n+2} = 2x_{n+1} - x_n.$$

Note that

$$x_{n+2} - x_{n+1} = x_{n+1} - x_n = \cdots = x_2 - x_1$$

which implies $\{x_n\}_{n\geq 1}$ is an A.P. If $d = x_2 - x_1 \neq 0$, then $x_n = x_1 + (n-1)d$ diverges as $n \to \infty$. But the sequence is supposed to be bounded as $f(x_n)$ is itself bounded. This implies $x_2 = x_1$, which further implies $f(x_1) = x_1$. Since x_1 was chosen arbitrarily, we conclude that $f(x) = x$ for all $x \in [0,1]$. ■