bon-db/calculus/diff/Z1CB3F.json (ISI B.Stat. 2011 P4)

Problem. Let f be a twice differentiable function on the open interval $(-1, 1)$ such that $f(0) = 1$. Suppose f also satisfies $f(x) \geq 0$, $f'(x) \leq 0$, and $f''(x) \leq f(x)$, for all $x \geq 0$. Show that $f'(0) \geq -\sqrt{2}$.

Solution by **Aditi Chakraborty**.

Solution. Define $g(x) = e^{-x}(f'(x) + f(x))$ and $h(x) = e^{x}(f'(x) - f(x))$. Then note that $g'(x) = e^{-x}(f''(x) - f(x)) \leq 0$ which implies g is non-increasing and hence, for all $x \in [0, 1)$,

$$g(x) \leq g(0) \implies e^{-x}(f'(x) + f(x)) \leq f'(0) + 1,$$

that is,

$$f'(x) + f(x) \leq e^{x}(f'(0) + 1).$$

Similarly, note that $h'(x) = e^{x}(f''(x) - f(x)) \leq 0$ which implies h is non-increasing and hence, for all $x \in [0, 1)$,

$$h(x) \leq h(0) \implies e^{x}(f'(x) - f(x)) \leq f'(0) - 1,$$

that is,

$$f'(x) - f(x) \leq e^{-x}(f'(0) + 1).$$

Adding these two conditions, we get

$$2f'(x) \leq e^{x}(f'(0) + 1) + e^{-x}(f'(0) - 1).$$

As f is twice differentiable, f' is continuous and hence integrable. So integrating the preceding inequality on the interval $[0, t]$, we get

$$2\int_0^t f'(x)\mathrm{d}x \leq (f'(0) + 1)\int_0^t e^{x}\mathrm{d}x + (f'(0) - 1)\int_0^t e^{-x}\mathrm{d}x$$

which implies

$$2f(t) - 2 \leq f'(0)(e^{t} - e^{-t}) + e^{t} + e^{-t} - 2.$$

Using the fact that $f(x) \geq 0$ for all $x \in [0, 1)$, we get

$$f'(0) \geq -\left(\frac{e^{t} + e^{-t}}{e^{t} - e^{-t}}\right) = -\left(\frac{e^{2t} + 1}{e^{2t} - 1}\right)$$

for all $t \in [0, 1)$. Limiting $t \to 1^{-}$, we get

$$f'(0) \geq -\left(\frac{e^{2} + 1}{e^{2} - 1}\right) > -\sqrt{2}.$$

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