bon-db/calculus/diff/Z71DE5.json (IMC 2009 D2 P2)

Problem (S). uppose $f\colon \mathbb{R} \to \mathbb{R}$ is a two times differentiable function satisfying $f(0) = 1$, $f'(0) = 0$ and for all $x \in [0, \infty)$, it satisfies

$$f''(x) - 5f'(x) + 6f(x) \geq 0.$$

Prove that, for all $x \in [0, \infty)$,

$$f(x) \geq 3e^{2x} - 2e^{3x}.$$

Solution. Define $g(x) = e^{-2x}f(x) + 2e^x - 3$. Then note that since $f(x)$ is twice differentiable, $g(x)$ is twice differentiable too.

Note that

$$g'(x) = \frac{e^{2x}f'(x) - 2e^{2x}f(x)}{e^{4x}} + 2e^x.$$

Define $h(x) = e^{-x}g'(x) = e^{-3x}f'(x) - 2e^{-3x}f(x) + 2$. As $f(x)$ is twice differentiable, $h(x)$ is differentiable. Now note

$$\begin{aligned} h'(x) &= e^{-3x}f''(x) - 3e^{-3x}f'(x) - 2(e^{-3x}f'(x) - 3e^{-3x}f(x)) \\ &= e^{-3x}(f''(x) - 5f'(x) + 6f(x)). \end{aligned}$$

We know that $f''(x) - 5f'(x) + 6f(x) \geq 0$ for all $x \geq 0$ which implies $h'(x) \geq 0$ for all $x \geq 0$. This means that $h(x)$ is non-decreasing on $[0, \infty)$. Therefore $e^{-x}g'(x) \geq e^0 g'(0) = 0$ which implies $g'(x) \geq 0$ for all $x \geq 0$. This further implies that $g(x)$ is non-decreasing on $[0, \infty)$. Then $g(x) \geq g(0) = 0$ for all $x \in [0, \infty)$. Now note $g(x) = e^{-2x}f(x) + 2e^x - 3 \geq 0$ is equivalent to $f(x) \geq 3e^{2x} - 2e^{3x}$ for all $x \geq 0$ and we are done. ■