bon-db/calculus/int/Z82BA2.json (AoPS)

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function satisfying

$$\int_0^1 f(t)\mathrm{d}t = \int_0^1 tf(t)\mathrm{d}t.$$

Prove that there exists $d \in (0,1)$ such that $f(d) = \int_0^d f(t)\mathrm{d}t$.

Solution. Define $g\colon [0,1] \to \mathbb{R}$ such that

$$g(x) = x\int_0^x f(t)\mathrm{d}t - \int_0^x tf(t)\mathrm{d}t.$$

Since f is continuous on $[0,1]$, g is differentiable on $[0,1]$ by FTC. Note that $g(0) = g(1) = 0$. So, by Rolle's theorem, there exists $c \in (0,1)$ such that $g'(c) = 0$. Computing g', we get

$$g'(x) = \int_0^x f(t)\mathrm{d}t$$

by using FTC. So $g'(c) = 0$ implies $\int_0^c f(t)\mathrm{d}t = 0$.

Now define $h\colon [0,c] \to \mathbb{R}$ such that

$$h(x) = e^{-x} \cdot \int_0^x f(t)\mathrm{d}t.$$

Similarly, due to FTC, h is differentiable on $[0,c]$. Also, $h(0) = h(c) = 0$ and hence, by Rolle's theorem, there exists $d \in (0,c)$ such that $h'(d) = 0$. Computing h', we get

$$h'(x) = e^{-x} \cdot (f(x) - \int_0^x f(t)\mathrm{d}t).$$

Thus $h'(d) = 0$ implies $f(d) = \int_0^d f(t)\mathrm{d}t$, as desired. ■