bon-db/calculus/diff/Z406B7.json (AoPS)

Problem (L). et $f\colon [a,b] \to \mathbb{R}$ be a continuous function which is differentiable on (a,b). Given that $f(a) = f(b)$ and $f'(a) = f'(b)$, prove that for all $\lambda \in \mathbb{R}$,

$$f''(x) - \lambda f'(x)^2 = 0$$

has at least one real root.

Solution. Define $g(x) = \frac{f'(x)}{e^{\lambda f(x)}}$. Clearly $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b). Then note that

$$g(a) = \frac{f'(a)}{e^{\lambda f(a)}} = \frac{f'(b)}{e^{\lambda f(b)}} = g(b).$$

Therefore, by LMVT, there exists $c \in (a,b)$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0.$$

Note

$$g'(x) = \frac{f''(x)e^{\lambda f(x)} - f'(x)^2 e^{\lambda f(x)}}{e^{2\lambda f(x)}}.$$

So

$$g'(c) = 0 \implies f''(c) - \lambda f'(c)^2 = 0,$$

as desired. ■