bon-db/calculus/diff/ZD143D.json (AoPS)

Problem. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function which satisfies

$$f\left(\frac{1}{n}\right) = 1 \quad \forall n \in \mathbb{N}.$$

Prove that $f'(0) = f''(0) = 0$.

Solution. By continuity of $f(x)$, we get $\lim_{n\to\infty} f\left(\frac{1}{n}\right) = f(0) = 1$.

Now note $f\left(\frac{1}{n}\right) = f(0) = 1$. So by LMVT, there exists $c_n \in \left(0, \frac{1}{n}\right)$ such that

$$f'(c_n) = \frac{f\left(\frac{1}{n}\right) - f(0)}{\frac{1}{n} - 0} = \frac{1-1}{\frac{1}{n}} = 0.$$

As $f(x)$ is twice differentiable, $f'(x)$ is differentiable and thus continuous. This implies

$$f'(0) = \lim_{n\to\infty} f'(c_n) = 0.$$

So now, by the definition of derivative,

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x - 0} = \lim_{n\to\infty} \frac{f'(c_n) - f'(0)}{c_n - 0} = 0. \qquad \blacksquare$$