bon-db/combi/Z18EFC.json (IMOSL 2008 C2)

Problem. Let $n \in \mathbb{N}$ and A_n set of all permutations $(a_1, \dots, a_n)$ of the set $\{1, 2, \dots, n\}$ for which

$$k \mid 2(a_1 + \cdots + a_k) \quad \forall\, 1 \le k \le n.$$

Find the number of elements of the set A_n.

Solution. We can check $n = 1,\ 2,\ 3$ by hand. This gives $|A_1| = 1$, $|A_2| = 2$ and $|A_3| = 6$. Now we proceed by induction.

For the base case, assume that we know the value of $|A_n|$ for some $n \ge 3$, and so, we want to find the value of $|A_{n+1}|$. We divide this into cases.

(i) **n is odd:** In this case, if the last digit of the tuple is $n + 1$ itself, then we simply get $|A_n|$ cases by using induction since the sum of all the numbers is clearly divisible by $n + 1$. Now, suppose the last digit (say ℓ) is not $n + 1$. Then note that by setting $k = n$, we get,

$$n \mid 2\left(\frac{(n+1)(n+2)}{2} - \ell\right) \equiv 2 - 2k = 2(1 - \ell).$$

Now since n is odd, we can simply ignore the 2 from which we get $\ell \equiv 1 \pmod{n}$. This means $\ell = 1$. So the first n elements of the tuple are just a rearrangement of $(2, 3, \dots, n + 1)$. Now note that even if we decrease all the $(a_1, a_2, \dots, a_n)$ by 1, the problem condition stays equivalent due to divisibility. Thus we can decrease all the first n elements by 1 which gives us a rearrangement of $(1, 2, \dots, n)$. The number of such tuples is just $|A_n|$ again. So in total, we get $|A_{n+1}| = 2|A_n|$ when n is odd.

(ii) **n is odd:** In this case, if the final digit is $n+1$, we get $|A_n|$ tuples in a similar fashion as seen previously.

Otherwise, similarly, we again get $n \mid 2(1 - \ell)$. But this time, we cannot ignore the 2 anymore. So in this case, we get $k \equiv 1 \pmod{\frac{n}{2}}$.

Now if the final element is 1, we again get that there are $|A_n|$ such tuples.

The only other case that remains is when the final element is $\frac{n}{2} + 1 = \frac{n+2}{2}$. Let the second last element be g. Now we set $k = n - 1$ to get,

$$n - 1 \mid 2\left(\frac{(n+1)(n+2)}{2} - \left(\frac{n+2}{2}\right) - g\right) \equiv 2 \cdot 3 - 3 - 2g = 3 - 2g.$$

This gives us that $2g \equiv 3 \pmod{n-1}$. Thus in this case we have $2g = (n-1)x + 3$ for some integer x. Due to parity, this forces x to be odd. Now if $x \ge 3$, then $g = \frac{(n-1)x+3}{2} \ge \frac{(n-1)\cdot 3+3}{2} = \frac{3n}{2}$. But $\frac{3n}{2} > n+1$ for all $n > 2$ which gives a contradiction. Thus we now get $x = 1$ which gives $g = \frac{n+2}{2}$. But this gives a further contradiction as our final element was $\frac{n+2}{2}$ itself. Thus there are no possible tuples when the final element is $\frac{n+2}{2}$.

Thus, in total, this case also gives us $|A_{n+1}| = 2|A_n|$ when n is even.

Combining these two cases, we get that $|A_{n+1}| = 2|A_n|$ for all $n \geq 3$. Thus our final answer looks like $|A_1| = 1$, $|A_2| = 2$ and $|A_n| = 3 \cdot 2^{n-2}$ for all $n \geq 3$. ■