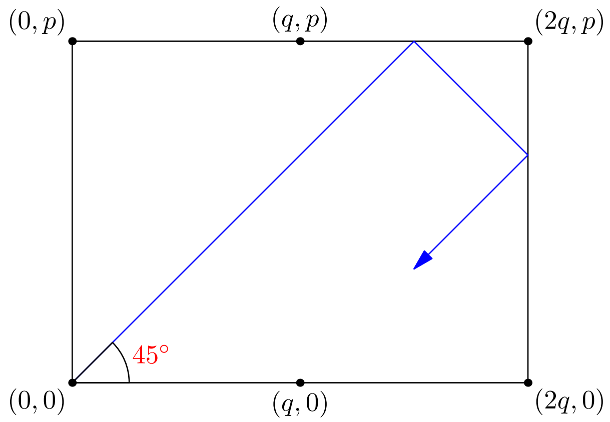
bon-db/combi/ZAE705.json (AoPS)

Problem. A rectangular $p \times 2q$ pool table has pockets in every corner and in the middle of each $2q$-long side. A ball is rolled from a corner pocket at a $45°$ angle with respect to the side rails. Find necessary and sufficient conditions on the real numbers p and q for the ball to eventually get into a pocket (angle of incidence is equal to the angle of reflection).

Solution. We lay the board on the cartesian plane as shown in the diagram.



We extend the grid over the entire plane by drawing lines $y = 2nq$ and $x = np$ for all $n \in \mathbb{N}_0$. The holes will be present at $(x, y) = (mq, np)$ for m, $n \in \mathbb{N}_0$.

When the ball rebounds after hitting an edge, instead of returning back, we let it pass thought the edges.

If the ball goes into a hole after hitting the edges multiple times, then the ray $y = x$ with slops of $45°$ with the X-axis from origin must pass through a hole and vice-versa.

Let's say that the ball enters the hole (mq, np). Here note that both m and n are non-zero as the hole in which the ball enters obviously does not lie on the axis.

Then the hole must lie on the line $y = x$ which implies $mq = np \implies \frac{p}{q} = \frac{m}{n} \implies \frac{p}{q} \in \mathbb{Q}$.

Therefore, for the ball to enter a hole, $\frac{p}{q}$ must be rational. Also the ball enters a hole if $\frac{p}{q}$ is rational because the grid condition is equivalent to the ball reflecting after hitting the side as we have already proved. ■