bon-db/alg/ZF199C.json (AoPS)

Problem. Find all values of x for which

$$6^x + 1 = 8^x - 27^{x-1}.$$

Solution. Let $a = 3^x$ and $b = 6^x$. Then our equation changes to

$$b + 1 = \frac{b^3}{a^3} - \left(\frac{a}{3}\right)^3.$$

This factorizes cleanly as

$$\left(\frac{b}{a}\right)^3 + \left(-\frac{a}{3}\right)^3 + (-1)^3 - 3\left(\frac{b}{a}\right)\left(-\frac{a}{3}\right)(-1) = 0$$

which implies

$$\left(\frac{b}{a} - \frac{a}{3} - 1\right)\left(\left(\frac{b}{a} + \frac{a}{3}\right)^2 + \left(1 - \frac{a}{3}\right)^2 + \left(\frac{b}{a} + 1\right)^2\right) = 0.$$

Therefore, either $\frac{b}{a} - \frac{a}{3} - 1 = 0$ or $\frac{b}{a} = -\frac{a}{3} = -1$.

The second condition clearly has no solution. We focus on the first condition. Note that

$$\frac{b}{a} - \frac{a}{3} - 1 = 0 \implies 2^x - 3^{x-1} - 1 = 0.$$

Define $g(x) = 2^x - 3^{x-1} - 1$. It is easy to check that $g(x)$ attains a maxima at $x_0 = \log_{2/3}\left(\frac{1}{3}\log_2 3\right)$ and that $g(x)$ is strictly increasing on $(-\infty, x_0)$ and strictly decreasing on (x_0, ∞). We also get that

$$\lim_{x\to-\infty} g(x) = -1, \quad \lim_{x\to\infty} g(x) = -\infty, \quad g(x_0) > 0$$

which implies, by IVT and strict monotonicity, that g has at exactly two roots, one in $(-\infty, x_0)$ and the other in (x_0, ∞).

Plugging $x = 1$ and $x = 2$, we get that both 1 and 2 are roots of g which implies $x = 1$ and $x = 2$ are the only possible solutions of our primary equation. ■