

**bon-db/calculus/diff/Z9C89A.json (AoPS)**

**Problem.** Let  $f: [a, b] \rightarrow [a, b]$  be a differentiable function such that  $f(a) = a$  and  $f(b) = b$ . Prove that there exist  $u, v \in (a, b)$  such that  $u < v$  and  $f'(u) + f'(v) = 2$ .

*Solution.* By LMVT on  $\left[a, \frac{a+b}{2}\right]$ , we get that there exists  $u \in \left(a, \frac{a+b}{2}\right)$  such that

$$f'(u) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = \frac{2}{b-a} \cdot \left(f\left(\frac{a+b}{2}\right) - f(a)\right).$$

Similarly by LMVT on  $\left[\frac{a+b}{2}, b\right]$ , we get that there exists  $v \in \left(\frac{a+b}{2}, b\right)$  such that

$$f'(v) = \frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} = \frac{2}{b-a} \cdot \left(f(b) - f\left(\frac{a+b}{2}\right)\right).$$

Clearly  $u < v$  and

$$f'(u) + f'(v) = \frac{2}{b-a} \cdot (f(b) - f(a)) = \frac{2}{b-a} \cdot (b-a) = 2. \quad \blacksquare$$