bon-db/alg/poly/Z7D495.json (AoPS)

Problem. Find all real polynomials $P(x)$ with $P(0)=0$ such that there exists a function $F(x)$ satisfying $F(x)>x$ for all $x\geq 0$ and

$$F(P(x))=P(F(x)) \quad \forall x\in\mathbb{R}.$$

Solution. Plugging $x=0$ gives
$$P(F(0))=F(0).$$
Composing F on both sides, we get

$$P(F^2(0))=F(P(F(0)))=F^2(0).$$

Similarly,

$$P(F^n(0))=F^n(0) \implies P(F^{n+1}(0))=F(P(F^n(0)))=F^{n+1}(0).$$

Therefore,
$$P(F^n(0))=F^n(0) \quad \forall\, n\in\mathbb{N}.$$

Let $Q(x)=P(x)-x$. Clearly $Q(x)$ is a real polynomial.

As $F(0)>0$ and $F(x)>x$, we get

$$F(0)>0 \implies F^2(0)>F(0) \implies \cdots \implies F^n(0)>F^{n-1}(0).$$

Hence, $\{F^n(0)\}_{n\geq 1}$ is a strictly increasing sequence. Therefore,

$$Q(F^n(0))=0 \quad \forall\, n\in\mathbb{N},$$

and since there are infinitely many zeroes of Q, this forces $Q\equiv 0$ which implies $P\equiv x$. For $P(x)=x$, we can pick $F(x)=x+1$, which clearly satisfies both conditions. ■