bon-db/alg/ineq/Z54AC7.json (AoPS)

Problem. Let a_2, a_3, $\ldots$, a_n be positive reals, $n \geq 3$, satisfying

$$a_2a_3 \cdots a_n = 1.$$

Prove that

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n.$$

Solution. Note that

$$\begin{aligned}
&(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n \\
&= (1+a_2)^2 \left(\frac{1}{2}+\frac{1}{2}+a_3\right)^3 \cdots \left(\frac{1}{n-1}+\cdots+\frac{1}{n-1}+a_n\right)^n \\
&\geq (2^2 \times 3^3 \times \cdots \times n^n)\left(\frac{1}{2^2} \times \frac{1}{3^3} \times \cdots \times \frac{1}{(n-1)^{n-1}}\right) a_2a_3 \cdots a_n \\
&= n^n.
\end{aligned}$$

For equality to hold, we need

$$a_2 = 1, a_3 = \frac{1}{2}, \ldots, a_n = \frac{1}{n-1}.$$

But then

$$1 = a_2a_3 \cdots a_n = 1 \times \frac{1}{2} \times \cdots \times \frac{1}{n-1} < 1$$

which is not possible. Hence,

$$(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n > n^n,$$

as desired. ■