bon-db/calculus/diff/Z889BD.json (Putnam 2015 B1)

Problem. Let f be a three times differentiable function (defined on $\mathbb{R}$ and real-valued) such that f has at least five distinct real zeros. Prove that $f + 6f' + 12f'' + 8f'''$ has at least two distinct real zeros.

Solution by **ZETA_in_olympiad** (#30 on the thread).

Solution. Let $g(x) = e^{x/2}f(x)$. Then g has at least five distinct zeroes. By Rolle's Theorem, g', g'', and g''' have at least four, three, and two distinct real zeroes, respectively. Since

$$g'''(x) = \frac{1}{8}e^{x/2}(f(x) + 6f'(x) + 12f"(x) + 8f'''(x))$$

and $e^{x/2}$ is never zero, we're done. ■