bon-db/calculus/cont/ZE7E4B.json (AoPS)

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a bounded function satisfying

$$f(2x) = 3f(x) \quad \forall\, x \in [0, 1/2].$$

Prove that $\lim_{x\to 0^+} f(x) = 0$.

Solution. Note that $f(2x) = 3f(x)$ for all $x \in \left[0, \frac{1}{2}\right]$ which implies $f(x) = 3f\left(\frac{x}{2}\right)$ for all $x \in [0,1]$. Similarly,

$$f(x) = 3f\left(\frac{x}{2}\right) = 3^2 f\left(\frac{x}{2^2}\right) = \cdots = 3^n f\left(\frac{x}{2^n}\right) \quad \forall\, x \in [0,1]$$

which implies $f(x) = \frac{f(2^n x)}{3^n}$ for all $x \in \left[0, \frac{1}{2^n}\right]$ and any $n \in \mathbb{N}$.

Also, plugging $x = 0$ implies $f(x) = 0$. Since f is bounded, there exists $B > 0$ such that $|f(x)| < B$ for all $x \in [0,1]$.

Fix $\varepsilon > 0$ arbitrarily. We pick $\delta = \frac{1}{2^N}$ where $N = \left\lfloor \log_3\left(\frac{B}{\varepsilon}\right)\right\rfloor + 1$. Then for all $x \in (0, \delta)$ we have

$$|f(x)| = \left|\frac{f(2^N x)}{3^N}\right| \le \frac{B}{3^N} < \varepsilon$$

which implies $\lim_{x\to 0^+} f(x) = 0 = f(0)$, as desired. ■