bon-db/combi/ZF1E42.json (JBMOSL 2021 N1)

Problem. Suppose a, b, and c are natural numbers such that $ab + 1$, $bc + 1$ and $ca + 1$ are factorials of positive integers. Find all possible triplets (a, b, c).

Solution. Let $ab + 1 = p!, \ bc + 1 = q!, \ ca + 1 = r!$.

If $\min\{p, q, r\} \geq 3$, then $ab \equiv -1$, $bc \equiv -1$, $ca \equiv -1$ modulo 3. Multiplying these three, we get

$$(abc)^2 \equiv -1 \pmod{3},$$

which is not possible as perfect squares are either 0 or 1 modulo 3.

So $\min\{p, q, r\} \leq 2$. WLOG assume p is the minimum. As $p = ab + 1 \geq 2$, we have $p \geq 2$. This forces $p = 2$, which further forces $a = b = 1$. So $c = q! - 1 = r! - 1$.

It is easy to see that all triplets of the form $(1, 1, n! - 1)$, $n \geq 2$, and their permutations work. ∎