

**bon-db/alg/poly/Z4467B.json (Putnam 2023 A2)**

**Problem.** Let  $n$  be an even positive integer. Let  $p$  be a monic, real polynomial of degree  $2n$ ; that is to say,  $p(x) = x^{2n} + a_{2n-1}x^{2n-1} + \cdots + a_1x + a_0$  for some real coefficients  $a_0, \dots, a_{2n-1}$ . Suppose that  $p(1/k) = k^2$  for all integers  $k$  such that  $1 \leq |k| \leq n$ . Find all other real numbers  $x$  for which  $p(1/x) = x^2$ .

**Solution** by [chakrabortyahan](#) (#15 on the thread).

*Solution.*  $Q(x) = x^2p(x) - 1$  has zeroes  $\left\{\pm\frac{1}{k}\right\}_{k=1}^n$ . So,

$$Q(x) = \left(\prod_{k=1}^n [x^2 - 1/k^2]\right) (x^2 + bx + c)$$

and say  $\alpha, \beta$  are the zeroes of the quadratic. As  $[x^1]Q(x) = 0$  and  $[x^0]Q(x) = -1$  so sum of reciprocals of the zeroes of the polynomial is equal to 0 By this we can say  $\frac{1}{\alpha} + \frac{1}{\beta} = 0 \implies \alpha = -\beta$  and note that  $\alpha \cdot \beta \cdot \frac{1}{(n!)^2} = -1$  and so the zeroes of the quadratic are  $\pm(n!)$  and the other roots of  $p\left(\frac{1}{x}\right) = x^2$  are  $\boxed{\pm\frac{1}{n!}}$ . ■