bon-db/calculus/diff/Z2EA73.json (IMC 2002 D1 P2)

Problem. Does there exist a continuously differentiable function $f\colon \mathbb{R} \to \mathbb{R}$ such that for every $x \in \mathbb{R}$ we have $f(x) > 0$ and $f'(x) = f(f(x))$?

Solution. For the sake of contradiction, assume such a function exists. Note that as $f(x)$ is differentiable, $f(f(x))$ is also differentiable and so is $f'(x) = f(f(x))$.

Note that $f'(x) = f(f(x)) > 0$ implies f is strictly increasing. This along with $f > 0$ implies that $\lim_{x\to-\infty} f(x)$ exists. (Details are left to the reader.)

Let $\ell = \lim_{x\to-\infty} f(x)$. By LMVT, there exists $c_x \in (x, x+1)$ such that

$$f'(c_x) = \frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x) \qquad \forall\, x \in \mathbb{R}.$$

Then note $\lim_{x\to-\infty} c_x = -\infty$. So,

$$\lim_{x\to-\infty} f'(c_x) = \ell - \ell = 0.$$

Differentiating our original equation, we get

$$f''(x) = f'(f(x)) \cdot f'(x) > 0.$$

Therefore $f'(x)$ is also strictly increasing. Recall that we derived $f'(x) > 0$ for all $x \in \mathbb{R}$. So, by an analogous logic, $\lim_{x\to-\infty} f'(x)$ also exists. Thus

$$\lim_{x\to-\infty} f'(x) = \lim_{x\to-\infty} f'(c_x) = 0.$$

Plugging this back into our original equation, we get

$$0 = \lim_{x\to-\infty} f'(x) = \lim_{x\to-\infty} f(f(x)) = f(\lim_{x\to-\infty} f(x)) = f(\ell)$$

which contradicts the fact that $f(x) > 0$ for all $x \in \mathbb{R}$.

Thus our initial claim must have been false and we are done. ■