bon-db/calculus/seq/ZA2689.json (AoPS)

Problem. Let a_0 and b_0 be any two positive integers. Define a_n, b_n for $n \geq 1$ using the relations

$$a_n = a_{n-1} + 2b_{n-1}, \quad b_n = a_{n-1} + b_{n-1},$$

and let $c_n = \frac{a_n}{b_n}$ for $n = 0, 1, 2, \ldots$.
Prove that the limit $\lim_{n\to\infty} c_n$ exists and find the limit.

Solution. We have,

$$\begin{aligned} c_n = \frac{a_n}{b_n} &= \frac{a_{n-1} + 2b_{n-1}}{a_{n-1} + b_{n-1}} \\ &= 1 + \frac{b_{n-1}}{a_{n-1} + b_{n-1}} \\ &= 1 + \frac{1}{c_{n-1} + 1}. \end{aligned}$$

Therefore,

$$\sqrt{2} - c_{n+1} = \sqrt{2} - \left(1 + \frac{1}{c_n + 1}\right) = (\sqrt{2} - 1) + \frac{1}{(\sqrt{2}+1) - (\sqrt{2} - c_n)}.$$

Note that,

$$a_0, b_0 > 0 \implies a_1 = a_0 + 2b_0 > 0, b_1 = a_0 + b_0 > 0 \implies a_1, b_1 > 0 \implies a_2, b_2 > 0 \implies \cdots.$$

So, a_i, $b_i > 0$ which gives that $c_i > 0$ for all $i \geq 0$.

Let $t = \sqrt{2} - c_n$. Now,

$$\begin{aligned} |\sqrt{2} - c_{n+1}| &= \left|(\sqrt{2} - 1) - \frac{1}{(\sqrt{2}+1) - (\sqrt{2} - c_n)}\right| \\ &= \left|\frac{(\sqrt{2}-1)t}{(\sqrt{2}+1) - t}\right| \\ &= \frac{1}{\sqrt{2}+1} \cdot \left|\frac{t}{1 + c_n}\right| \\ &< \frac{1}{\sqrt{2}+1} \cdot |t| \\ &= \frac{1}{\sqrt{2}+1} \cdot |\sqrt{2} - c_n|. \end{aligned}$$

$$|\sqrt{2} - c_{n+1}| < \frac{1}{\sqrt{2}+1} \cdot |\sqrt{2} - c_n|$$

$$\begin{aligned} &\vdots \\ &= \frac{1}{(\sqrt{2}+1)^{n+1}} \cdot c_0. \end{aligned}$$

Therefore, by Sandwich Theorem, $\lim_{n\to\infty} \sqrt{2} - c_n = 0$ which implies $\lim_{n\to\infty} c_n = \sqrt{2}$. ■