

## bon-db/combi/Z69E82.json (AoPS)

**Problem (L).** et  $\{a_n\}_{n \geq 0}$  be a sequence defined by  $a_0 = 5$ ,  $a_1 = 27$ , and

$$a_n = 7a_{n-1}^2 + a_{n-2} \quad \forall n \geq 2.$$

Find the remainder when  $a_{2025!}$  is divided by 43.

*Solution.* Consider all the tuples  $(a_n, a_{n+1})$  under modulo 43, i.e.,  $(b_n, b_{n+1})$  where  $b_n$  is the remainder when  $a_n$  is divided by 43. Clearly, there are at most  $43^2$  different possibilities for  $(b_n, b_{n+1})$  considering all the remainder pairs.

Let

$$S = \{(b_0, b_1), (b_1, b_2), \dots, (b_{43^2}, b_{43^2+1})\}.$$

As  $|S| = 43^2 + 1$ , by the pigeonhole principle, there exist at least two tuples in  $S$  such that

$$(b_i, b_{i+1}) = (b_j, b_{j+1}), \quad i < j.$$

Now note  $b_i = b_j$  and  $b_{i+1} = b_{j+1}$  implies

$$b_{i+2} \equiv 7b_{i+1}^2 + b_i \equiv 7b_{j+1}^2 + b_j \equiv b_{j+2} \pmod{43},$$

which implies  $b_{i+2} = b_{j+2}$ . Therefore

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i+1}, b_{i+2}) = (b_{j+1}, b_{j+2}).$$

Similarly,

$$(b_{i+1}, b_{i+2}) = (b_{j+1}, b_{j+2}) \implies (b_{i+2}, b_{i+3}) = (b_{j+2}, b_{j+3}) \implies \dots \text{ and so on.}$$

Also,

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i-1}, b_i) = (b_{j-1}, b_j)$$

as

$$b_{i-1} \equiv b_{i+1} - 7b_i^2 \equiv b_{j+1} - 7b_j^2 \equiv b_{j-1} \pmod{43}.$$

Similarly,

$$(b_i, b_{i+1}) = (b_j, b_{j+1}) \implies (b_{i-1}, b_i) = (b_{j-1}, b_j) \implies \dots \text{ and so on.}$$

Therefore we can conclude that

$$b_{n+(j-i)} = b_n \quad \forall n \geq 0$$

Note that  $j - i \leq 43^2 - 0 = 43^2 \leq 2025$  which implies  $j - i \mid 2025!$ . To finish, note

$$a_{2025!} \equiv b_{2025!} = b_0 = 5 \pmod{43}. \quad \blacksquare$$