bon-db/alg/ineq/Z4051D.json (ISI 2017 P8)

Problem. Let k, n and r be positive integers.

(a) Let $Q(x) = x^k + a_1x^{k+1} + \cdots + a_nx^{k+n}$ be a polynomial with real coefficients. Show that the function $\frac{Q(x)}{x^k}$ is strictly positive for all real x satisfying

$$0 < |x| < \frac{1}{1 + \sum\limits_{i=1}^{n} |a_i|}.$$

(b) Let $P(x) = b_0 + b_1x + \cdots + b_rx^r$ be a non-zero polynomial with real coefficients. Let m be the smallest number such that $b_m \neq 0$. Prove that the graph of $y = P(x)$ cuts the x-axis at the origin (i.e., P changes signs at $x = 0$) if and only if m is an odd integer.

Solution by **lifeismathematics** (#15 on the thread).

Solution. We first solve part (a) and then use it to solve part (b).

(a) Since $|x| < \frac{1}{1+\sum_{i=1}^{n} a_i} < 1$ we have $x^i \leq |x|^i \leq |x| \, \forall \, i \in \{1, 2, \cdots, n\}$.

Now this implies $a_ix^i \geq -|a_i||x| \, \forall \, i \in \{1, 2, \cdots, n\}$, now this gives,

$$\frac{Q(x)}{x^k} \geq 1 - |x| \left(\sum_{i=1}^{n} |a_i| \right) \geq 1 - \frac{\sum_{i=1}^{n} |a_i|}{1 + \sum_{i=1}^{n} |a_i|} = \frac{1}{1 + \sum_{i=1}^{n} |a_i|} > 0,$$

hence $\frac{Q(x)}{x^k} > 0$ in the prescribed interval of x.

(b) We have $P(x) = x^m(\mathcal{H}(x))$, where $\mathcal{H}(x) := b_m + b_{m+1}x + \cdots + b_rx^{r-m}$, now $\mathcal{H}(0) \neq 0$, and since $\mathcal{H}(x)$ is continuous at 0, we have $\mathcal{H}(x)$ has a fixed sign on $(-\delta, \delta)$ for some $\delta > 0$. But as $P(x)$ changes sign about 0, we must have x^m to change sign about $0 \iff m$ must be odd.

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