bon-db/combi/ZD0096.json (ISI 2021 P1)

Problem. There are three cities each of which has exactly the same number of citizens, say n. Every citizen in each city has exactly a total of $(n+1)$ friends in the other two cities. Show that there exist three people, one from each city, such that they are friends. We assume that friendship is mutual (that is, a symmetric relation).

Solution by **Sammy27** (#16 on the thread).

Solution. Equivalently, we prove that a simple $(n+1)$-regular tripartite graph with n vertices in each partition set A, B and C must contain a triangle.

Assume not. For $a \in A$, define $B\deg(a)$ as the degree of a after deleting C and $C\deg(a)$ as the degree of a after deleting B.

Clearly, $B\deg(a) + C\deg(a) = n+1$. Similar definitions follow symmetrically. Let

$$
\begin{aligned}
S_a &= \{(a, B\deg(a)) : a \in A\} \cup \{(a, C\deg(a)) : a \in A\},\\
S_b &= \{(b, C\deg(b)) : b \in B\} \cup \{(b, A\deg(b)) : b \in B\},\\
S_c &= \{(c, A\deg(c)) : c \in C\} \cup \{(c, B\deg(c)) : c \in C\}.
\end{aligned}
$$

Choose $(x, y) \in S_a \cup S_b \cup S_c$ such that y is maximized.

WLOG, say $x \in A$ and $y = B\deg(x)$. Because $B\deg(x) \le n$, we must have $C\deg(x) \ge 1$, which implies that there exists $z \in C$ such that there is an edge between x and z.

Note that $B\deg(z) \le n - B\deg(x)$ is forced, otherwise, by PHP, we would have a triangle. However, this suggests that $A\deg(z) \ge n + 1 - (n - B\deg(x)) = B\deg(x) + 1$. This contradicts the choice of y as $(z, A\deg(z)) \in S_a \cup S_b \cup S_c$ with $A\deg(z) > B\deg(x)$.

Hence, the graph must contain a triangle, and we are done. ■