

**bon-db/calculus/diff/Z2EA73.json (IMC 2002 D1 P2)**

**Problem.** Does there exist a continuously differentiable function  $f: \mathbb{R} \rightarrow \mathbb{R}$  such that for every  $x \in \mathbb{R}$  we have  $f(x) > 0$  and  $f'(x) = f(f(x))$ ?

*Solution.* For the sake of contradiction, assume such a function exists. Note that as  $f(x)$  is differentiable,  $f(f(x))$  is also differentiable and so is  $f'(x) = f(f(x))$ .

Note that  $f'(x) = f(f(x)) > 0$  implies  $f$  is strictly increasing. This along with  $f > 0$  implies that  $\lim_{x \rightarrow -\infty} f(x)$  exists. (Details are left to the reader.)

Let  $\ell = \lim_{x \rightarrow -\infty} f(x)$ . By LMVT, there exists  $c_x \in (x, x+1)$  such that

$$f'(c_x) = \frac{f(x+1) - f(x)}{(x+1) - x} = f(x+1) - f(x) \quad \forall x \in \mathbb{R}.$$

Then note  $\lim_{x \rightarrow -\infty} c_x = -\infty$ . So,

$$\lim_{x \rightarrow -\infty} f'(c_x) = \ell - \ell = 0.$$

Differentiating our original equation, we get

$$f''(x) = f'(f(x)) \cdot f'(x) > 0.$$

Therefore  $f'(x)$  is also strictly increasing. Recall that we derived  $f'(x) > 0$  for all  $x \in \mathbb{R}$ . So, by an analogous logic,  $\lim_{x \rightarrow -\infty} f'(x)$  also exists. Thus

$$\lim_{x \rightarrow -\infty} f'(x) = \lim_{x \rightarrow -\infty} f'(c_x) = 0.$$

Plugging this back into our original equation, we get

$$0 = \lim_{x \rightarrow -\infty} f'(x) = \lim_{x \rightarrow -\infty} f(f(x)) = f(\lim_{x \rightarrow -\infty} f(x)) = f(\ell)$$

which contradicts the fact that  $f(x) > 0$  for all  $x \in \mathbb{R}$ .

Thus our initial claim must have been false and we are done. ■