bon-db/combi/ZAEDFC.json (ISI B.Stat. 2007 P6)

Problem. Let $S = \{1, 2, \cdots, n\}$ where n is an odd integer. Let f be a function defined on $\{(i, j) \mid i \in S, j \in S\}$ taking values in S such that

(i) $f(s, r) = f(r, s)$ for all $r, s \in S$

(ii) $\{f(r, s) : s \in S\} = S$ for all $r \in S$

Show that $\{f(r, r) : r \in S\} = S$.

Solution. Let S_i denote the set of all the tuples (a, b) such that $f(a, b) = i$. Now note that for each $a \in 1, 2, \ldots, n$, there is exactly one k such that $f(a, k) = i$. Thus the set S_i has at exactly n elements.

Now note that if (a, b) is in S_i, (b, a) is also in S_i. Now if $a \neq b$, then we get a pair of two tuples that are present in S_i. But the cardinality of S_i is odd. Thus there must be some m such that (m, m) is also in S_i.

Denote this tuple as (m_i, m_i) for each S_i. Now note that a tuple (m_i, m_i) cannot be present in two sets. This is because otherwise, $f(m_i, m_i) = i$ and $f(m_i, m_i) = j$ which implies $i = j$, contradiction.

So each S_i has at least one (m_i, m_i) such that this (m_i, m_i) tuple is not present in any other S_j.

Now note that we have exactly n many (m_i, m_i) tuples and there is at least one distinct (m_i, m_i) present in each of the n sets of S_i. Thus the equality holds and there is exactly one (m_i, m_i) in each S_i.

Thus we have that there is exactly one tuple (m_i, m_i) in each S_i and we are done, because this forces $f(r, r) \neq f(s, s)$. ■