bon-db/calculus/cont/ZCD7C4.json (IMC 2023 P1)

Problem. Find all functions $f : \mathbb{R} \to \mathbb{R}$ that have a continuous second derivative and for which the equality $f(7x+1) = 49f(x)$ holds for all $x \in \mathbb{R}$.

Solution. Differentiating the equation twice, we get that

$$f''(7x+1) = f''(x).$$

Fix $x_0 \in \mathbb{R}$ arbitrarily. Then define the sequence $\{x_i\}_{i\geq 0}$ by $x_{n+1} = \frac{x_n - 1}{7}$. Then note that

$$|x_{n+1} - x_n| = \frac{1}{7}|x_n - x_{n-1}|.$$

Therefore, by the condition of contractive sequence, $\{x_n\}_{n\geq 0}$ converges. Checking gives that it converges to $-\frac{1}{6}$. (If you want to avoid contractive sequence, then you can define the sequence $y_n = x_n + \frac{1}{6}$ and then work with it.)

Note that as $f''(7x+1) = f''(x)$, $f''(x_0) = f''(x_n)$. Taking limits on both sides, we get that $f''(x_0) = f''(-\frac{1}{6})$. Since x_0 was fixed arbitrarily, $f''(x) = c$ for some constant $c \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Since $f''(x)$ is continuous, integrating twice and checking gives that $f(x) = 3tx^2 + tx + \frac{t}{12}$ is the only solution where t is a fixed real constant.

It is easy to check that this function indeed works and we are done. ■