bon-db/calculus/diff/ZFACC2.json (AoPS)

Problem. Suppose that $f\colon [0,\infty) \to \mathbb{R}$ is continuous, f' exists on $(0,\infty)$, $f(0)=0$, and f' is monotonically increasing. Show that

$$g(x) = \frac{f(x)}{x}$$

is monotonically increasing on $(0,\infty)$

Solution. Clearly $g(x)$ is differentiable on $(0,\infty)$. Note that

$$g'(x) = \frac{f'(x)x - f(x)}{x^2}.$$

It suffices to show that $g'(x) \geq 0$ which is equivalent to showing $f'(x) \geq \frac{f(x)}{x}$. But note that by LMVT,

$$\frac{f(x)}{x} = \frac{f(x)-f(0)}{x-0} = f'(c_x) \leq f'(x)$$

where $c_x \in (0,x)$ and the last inequality follows by monotonicity of f'. This finishes our proof. ■

Solution. Fix $0 < x < y$ arbitrarily. By LMVT on $(0,x)$, we get

$$\frac{f(x)-f(0)}{x-0} = f'(c_x)$$

for some $c_x \in (0,x)$. Again, by LMVT on (x,y), we get

$$\frac{f(y)-f(x)}{y-x} = f'(d_{x,y})$$

for some $d_{x,y} \in (x,y)$. Now, as $c_x < x < d_{x,y}$, we get $f'(c_x) \leq f'(d_{x,y})$ which implies

$$\frac{f(x)-f(0)}{x-0} \leq \frac{f(y)-f(x)}{y-x}.$$

By cross-multiplying terms and simplifying, we get

$$\frac{f(y)}{y} \geq \frac{f(x)}{x}$$

which finishes our proof. ■

Solution. From the condition that f' is monotonically increasing, it is straightforward to prove that f is convex. (Left as an exercise to the reader.)

Fix $0 < x < y$ arbitrarily. By convexity,

$$f\left(\left(\frac{x}{y}\right) \cdot y + \left(1 - \frac{x}{y}\right) \cdot 0\right) \leq \frac{x}{y} \cdot f(y) + \left(1 - \frac{x}{y}\right) f(0)$$

which implies

$$\frac{f(x)}{x} \leq \frac{f(y)}{y},$$

as desired. ■