bon-db/calculus/int/Z61F93.json (AoPS)

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function. Find

$$\lim_{n\to\infty} n \int_0^1 f(x)x^n \mathrm{d}x.$$

Solution. By using continuity of f at $x = 1$, we get that for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x \in [0,1]$,

$$|x-1| < \delta \implies |f(x) - f(1)| < \varepsilon$$

that is

$$f(1) - \varepsilon \le f(x) \le f(1) + \varepsilon \quad \forall x \in (1-\delta, 1+\delta).$$

Let $M = \max_{t\in[0,1]} f(t)$ and $m = \min_{t\in[0,1]} f(t)$, which exist as f is continuous. Then we have

$$\begin{aligned}\int_0^1 f(x)x^n \mathrm{d}x &= \int_0^{1-\delta} f(x)x^n \mathrm{d}x + \int_{1-\delta}^1 f(x)x^n \mathrm{d}x \\ &\le \int_0^{1-\delta} Mx^n \mathrm{d}x + \int_{1-\delta}^1 (f(1)+\varepsilon)x^n \mathrm{d}x \\ &= \frac{M(1-\delta)^{n+1}}{n+1} + \frac{(f(1)+\varepsilon)(1-(1-\delta)^{n+1})}{n+1}\end{aligned}$$

which implies

$$n\int_0^1 f(x)x^n \mathrm{d}x \le \frac{n}{n+1}\cdot M(1-\delta)^{n+1} + \frac{n}{n+1}(f(1)+\varepsilon)(1-(1-\delta)^{n+1}).$$

Similarly, we also get

$$\begin{aligned}\int_0^1 f(x)x^n \mathrm{d}x &= \int_0^{1-\delta} f(x)x^n \mathrm{d}x + \int_{1-\delta}^1 f(x)x^n \mathrm{d}x \\ &\ge \int_0^{1-\delta} mx^n \mathrm{d}x + \int_{1-\delta}^1 (f(1)-\varepsilon)x^n \mathrm{d}x \\ &= \frac{m(1-\delta)^{n+1}}{n+1} + \frac{(f(1)-\varepsilon)(1-(1-\delta)^{n+1})}{n+1}\end{aligned}$$

which implies

$$n\int_0^1 f(x)x^n \mathrm{d}x \ge \frac{n}{n+1}\cdot m(1-\delta)^{n+1} + \frac{n}{n+1}(f(1)-\varepsilon)(1-(1-\delta)^{n+1})$$

Let

$$a_n = \frac{n}{n+1}\cdot m(1-\delta)^{n+1} + \frac{n}{n+1}(f(1)-\varepsilon)(1-(1-\delta)^{n+1})$$

and

$$b_n = \frac{n}{n+1}\cdot M(1-\delta)^{n+1} + \frac{n}{n+1}(f(1)+\varepsilon)(1-(1-\delta)^{n+1}).$$

Therefore, we now have

$$a_n \leq n \int_0^1 f(x) x^n \mathrm{d}x \leq b_n.$$

Note that $\lim_{n\to\infty} a_n = f(1) - \varepsilon$ and $\lim_{n\to\infty} b_n = f(1) + \varepsilon$. This suggests that the sequences $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ are bounded. Therefore,

$$\left\{ n \int_0^1 f(x) x^n \mathrm{d}x \right\}_{n\geq 1}$$

is also bounded and hence

$$\liminf_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x$$

and

$$\limsup_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x$$

both exist.

Using the following limits in our preceding inequality, we get

$$f(1) - \varepsilon \leq \liminf_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x \leq f(1) + \varepsilon$$

and

$$f(1) - \varepsilon \leq \limsup_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x \leq f(1) + \varepsilon$$

for any arbitrary $\varepsilon > 0$. Considering small enough ε, we get that

$$\liminf_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x = \limsup_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x = f(1)$$

which implies $\lim_{n\to\infty} n \int_0^1 f(x) x^n \mathrm{d}x$ exists and is equal to $f(1)$. ■