bon-db/alg/poly/ZFF44E.json (USAMO 1976 P5)

Problem. Let $P(x)$, $Q(x)$, $R(x)$, $S(x)$ be polynomials with real coefficients such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + \cdots + 1)S(x)$$

for all $x \in \mathbb{C}$. Prove that $x - 1 \mid S(x)$.

Solution. Define

$$G(x) = x^2R(1) + xQ(1) + P(1).$$

Let ω be a fifth root of unity. Plugging in ω^i for $1 \le i \le 4$ into the original equation gives

$$P(1) + \omega^i Q(1) + \omega^{2i} R(1) = 0.$$

So ω, ω^2, ω^3, ω^4 are roots of $G(x)$.

But note $\deg(G(x)) \le 2$ which forces $G \equiv 0$ implying $P(1) = Q(1) = R(1) = 0$.

Therefore, putting $x = 1$ into the original equation gives

$$P(1) + Q(1) + R(1) = 5S(1)$$

which implies

$$S(1) = 0 \implies x - 1 \mid S(x). \qquad \blacksquare$$