bon-db/calculus/diff/Z65DB6.json (ISI 2023 P8)

Problem. Let $f\colon [0,1] \to \mathbb{R}$ be a continuous function which is differentiable on $(0,1)$. Prove that either $f(x) = ax + b$ for all $x \in [0,1]$ for some constants a, $b \in \mathbb{R}$ or there exists $t \in (0,1)$ such that $|f(1) - f(0)| < |f'(t)|$.

Solution. Suppose $|f(1) - f(0)| \geq |f'(t)|$ for all $t \in (0,1)$. Then it suffices to show that f is linear. Note that f satisfies the conditions if and only if $-f$ satisfies them. So, WLOG assume that $f(0) < f(1)$.

Fix $t \in (0,1)$ arbitrarily. By LMVT on $(0,t)$, we know there exists $c_1 \in (0,t)$ such that

$$\frac{f(t) - f(0)}{t - 0} = f'(c_1)$$

which implies

$$\left|\frac{f(t) - f(0)}{t}\right| = |f'(c_1)| \leq |f(1) - f(0)|$$

and hence

$$|f(t) - f(0)| \leq |f(1) - f(0)|t.$$

Similarly, by LMVT on $(t,1)$, we get

$$|f(1) - f(t)| \leq |f(1) - f(0)|(1 - t).$$

Adding these two gives

$$\begin{aligned} |f(1) - f(t)| + |f(t) - f(0)| &\leq |f(1) - f(0)| \\ &= |(f(1) - f(t)) - (f(t) - f(0))| \\ &\leq |f(1) - f(t)| + |f(t) - f(0)|. \end{aligned}$$

Due to the equality condition of the triangle equality, it suggests that $f(t) \in (\min\{f(0), f(1))\}, \max\{f(0),$
$(f(0), f(1))$. Furthermore, the equality condition in the final inequality forces the equality to hold in all the preceding inequalities which implies

$$\left|\frac{f(t) - f(0)}{t}\right| = |f(1) - f(0)|$$

that is

$$f(t) - f(0) = (f(1) - f(0))t.$$

This shows that f is linear on $(0,1)$ and hence, by continuity, it is linear on $[0,1]$, as desired. ■