bon-db/calculus/int/ZF8633.json (IMC 2022 D1 P1)

Problem. Let $f : [0, 1] \to (0, \infty)$ be an integrable function such that $f(x)f(1-x) = 1$ for all $x \in [0, 1]$. Prove that $\int_0^1 f(x)\mathrm{d}x \geq 1$.

Solution by **Levieee** (#25 on the thread).

Solution. By the Cauchy-Schwarz inequality:

$$\int_0^1 f(x)\mathrm{d}x \int_0^1 f(1-x)\mathrm{d}x \geq \left(\int_0^1 \sqrt{f(x)f(1-x)}\mathrm{d}x\right)^2.$$

Since

$$\int_0^1 \sqrt{f(x)f(1-x)}\mathrm{d}x = \int_0^1 1\mathrm{d}x = 1,$$

we have

$$\int_0^1 f(x)\mathrm{d}x \int_0^1 f(1-x)\mathrm{d}x \geq 1.$$

By King's rule:

$$\int_0^1 f(x)\mathrm{d}x \int_0^1 f(1-x)\mathrm{d}x = \int_0^1 f(x)\mathrm{d}x \int_0^1 f(x)\mathrm{d}x = \left(\int_0^1 f(x)\mathrm{d}x\right)^2.$$

Thus:

$$\left(\int_0^1 f(x)\mathrm{d}x\right)^2 \geq 1.$$

Taking the square root:

$$\int_0^1 f(x)\mathrm{d}x \geq 1. \qquad \blacksquare$$