bon-db/alg/ZD3EA4.json (AoPS)

Problem. The real numbers $x_1, x_2, \dots, x_{2025}$ satisfy,

$$x_1 + x_2 = 2x_{\sigma(1)}, x_2 + x_3 = 2x_{\sigma(2)}, \dots, x_{2025} + x_1 = 2x_{\sigma(2025)}.$$

Where $(\sigma(1), \sigma(2), \dots, \sigma(2025))$ is a permutation of $(1, 2, \dots, 2025)$. Prove that $x_1 = x_2 = \cdots = x_{2025}$.

Solution. Square all the identities and sum them up. This gives

$$\sum_{i=1}^{2025} (x_i + x_{i+1})^2 = 4 \sum_{i=1}^{2025} x_{\sigma(i)}^2$$

which on expanding, simplifying, and re-factorizing gives

$$\sum_{i=1}^{2025} (x_i - x_{i+1})^2 = 0.$$

Thus all x_i's are equal and we are done. ■

Solution. We use arrows to solve this problem. Draw a graph with the x_i as the vertices and draw a directed edge from $x_i \to x_{\sigma(i)}$. It is well known that this graph can be decomposed into disjoint cycles (including 1-cycles).

> **Claim** (L) — et $M = \max\{x_1, x_2, \dots, x_{2025}\}$. Then all the elements in a cycle containing M must be equal to M.

Proof. FTSOC assume not. This means that there is another element x_i in the same cycle as that of M such that $x_i < M$.

Firstly note that $2x_{\sigma(i)} = x_i + x_{i+1} < M + x_{i+1} \le 2M \implies x_{\sigma(i)} < M$. Similarly, $x_{\sigma(i)} < M \implies x_{\sigma(\sigma(i))} < M$. We can continue this pattern till we get back to the element with value M which implies $M < M$ giving our desired contradiction. □

Now pick a cycle containing M. Therefore all the elements in this cycle are equal to M.

Let the elements of this cycle be denoted as $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$. Here $\sigma(i_1) = i_2$, $\sigma(i_2) = i_3$ and so on up to $\sigma(i_k) = i_1$.

Now, $x_{i_1-1} + x_{i_1} = 2x_{i_2} \implies x_{i_1-1} = M$. Similarly, we can also derive that $x_{i_2-1} = x_{i_3-1} = \cdots = x_{i_k-1} = M$.

Note that from the indices $\{i_1, i_2, \dots, i_k\}$, we just traversed to $\{i_1 - 1, i_2 - 1, \dots, i_k - 1\}$. We can continue doing so till we cover all the indices modulo 2025.

This shows that all the elements $x_i = M$ and we are done. ■