

Tech Tools in Functional Analysis

—A Note of My Functional Analysis Course

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Contents

1	Introduction	3
2	2 Important Inequalities-Schwartz and Minkowski	3

1 Introduction

This article is for recording my thoughts and notes in Functional Analysis Course. Most of them are content that I personally think is important. Hope it's helpful for you:).

2 2 Important Inequalities-Schwartz and Minkowski

In 1st section, I want to introduce 2 important inequalities - also conclusion or tech tools - because they are common and useful in much proofs of propositions in my textbook. Besides, there are not unique method to prove this 2 inequalities, so I'll choose the most brief and elegant ones to show. Then I'll show you their contact with Bessel series of inequality and equality and Holder's inequality.

Theorem 2.1 (Basic). ***Schwartz Inequality:** Let X be a inner product space, then for any $x, y \in X$, we have*

$$|\langle x, y \rangle| \leq \|x\| \|y\| \quad (1)$$

***Minkowski Inequality:** Let X be a normed space, then for any $x, y \in X$, we have*

$$\|x + y\| \leq \|x\| + \|y\| \quad (2)$$

Theorem 2.2 (Tech). ***Bessel Series of Inequality and Equality:***

$$\|f - \sum_{k=1}^N c_k \phi_k\|^2 \geq \|f - \sum_{k=1}^N \langle f, \phi_k \rangle \phi_k\|^2 = \|f\|^2 - \sum_{k=1}^N |\langle f, \phi_k \rangle|^2 \geq 0 \quad (3)$$

While $N \rightarrow \infty$, we have $\{\phi_{k=1}^\infty\}$ is complete orthonormal system, then we have

Theorem 2.3 (Tech). ***Pasevarll Identity:** $\forall f, f_1, f_2 \in L^2$*

$$\begin{aligned} 1^\circ. \|f\|^2 &= \sum_{k=1}^\infty |\langle f, \phi_k \rangle|^2 \\ 2^\circ. \|f_1, f_2\| &= \sum_{k=1}^\infty |\langle f_1, \phi_k \rangle \overline{\langle f_2, \phi_k \rangle}| \end{aligned} \quad (4)$$

Theorem 2.4 (Tech). ***Holder's Inequality:** Let X be a measure space, $p, q \in [1, \infty]$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then for any $f, g \in L^p$, we have*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q \quad (5)$$