A Tiny Tip in Fraction2Polynomial with Taylor Series Base

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Contents

1 Inspiration - Learning from a defination

Definition 1.1. *Define the* ***Bernoulli number*** *as*

$$\frac{x}{e^x - 1} = \Sigma_{n=0}^{\infty} \frac{B_n}{n!} x^n \tag{1}$$

Expanding e^x with Taylor series $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$,
then we have

$$\begin{aligned} \frac{x}{e^x - 1} &= \frac{x}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots} = \frac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots} \\ &= 1 - \left(\frac{x}{2!} + \frac{x^2}{3!} + \cdots\right) + \left(\frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)^2 - \cdots \end{aligned} \tag{2}$$

and then

$$\begin{aligned} &B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_4 = -\frac{1}{30}, B_6 = \frac{1}{42}, \\ &B_8 = -\frac{1}{30}, B_{10} = \frac{5}{66}, B_{12} = -\frac{691}{2730}, B_{14} = \frac{7}{6} \end{aligned} \tag{3}$$

2 Tip or Trick - Tool-kit from it

Actually, the step (2) can be abstracted as a mathematical tip as follow:

$$\frac{1}{1 + A} = 1 - A + A^2 - A^3 + \cdots = \Sigma_{n=0}^{\infty} A^n \cdot (-1)^n \tag{4}$$

Proof.

$$\begin{aligned} (1 + A)\cdot&\left(1 - A + A^2 - A^3 + \cdots\right) \\ &= 1 - A + A^2 - A^3 + \cdots + A - A^2 + A^3 - A^4 + \cdots \\ &= 1 \end{aligned} \tag{5}$$

Application: Simplify fractions $\frac{1}{1+A}$ to polynomials.

3 Essence - Taylor Series

The essencial of this transformation is Taylor Series of function $f(x) = \frac{1}{1+x}$ in $x = 0$.

$$\begin{aligned} &f(x) = \Sigma_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \\ &In \quad x_0 = 0, \\ &f(x) = \frac{1}{1 + x} \\ &\quad\;\; = 1 - x + x^2 - x^3 + \cdots = \Sigma_{n=0}^{\infty} (-1)^n x^n \end{aligned} \tag{6}$$

4 Think twice - exercise questions

Taylor Series is an approximation in $x = x_0\,(= 0 \quad here)$, but why it "seemly" fits all x values in $\mathbb{R}$ in this tip? Is there any other examples like that, or just to say the Taylor series is always accurate?