Why Does a Flame Front Flutter?

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Motivation

Flame fronts in combustion systems often develop irregular, wavy, and time dependent distortions known as **flutter**. Despite flames being such a simple physical system, the behaviour of fluttering is inherently nonlinear and cannot be explained by classical, steady-state stability alone.

The **Kuramoto-Sivashinsky (KS) equation** is one of the simplest PDEs that captures:

- the onset of long-wavelength **linear instability**,
- the formation of regular **spatial patterns**,
- their subsequent **breakup through nonlinear interactions**,
- and the emergence of **spatiotemporal chaos**.

Studying the KS equation offers a dynamical systems explanation of why flame fronts flutter, how unstable modes grow, interact, transfer energy across scales, and ultimately lead to chaotic motion.

The Kuramoto-Sivashinsky Equation

The 1D Kuramoto-Sivashinsky equation is a 4th order nonlinear PDE describing the evolution of a flame front velocity $u(x,t)$ on a periodic domain

$$u_t = -u_{xx} - u_{xxxx} - u\,u_x, \qquad x \in [0, L]$$

with periodic boundary conditions $u(x+L,t) = u(x,t)$. The domain length L is an important parameter that determines the system's behaviour. In many studies, the equation is analysed in a dimensionless form, where the domain length is scaled as $L = 2\pi\tilde{L}$ with $\tilde{L}$ being a dimensionless parameter. The equation combines:

- **long-wavelength instability** from the 'anti-diffusive' term u_{xx},
- **short-wavelength damping** from the 'hyperviscous' term u_{xxxx},
- **nonlinear advection** that transfers energy from low to high wave numbers uu_x.

Linearisation and Dispersion Relation (Part A)

Equilibrium and linearisation

The flat front $u(x,t) \equiv 0$ is a spatially uniform equilibrium. To analyse its stability, we perturb it by a small disturbance

$$u(x,t) = 0 + \varepsilon\tilde{u}(x,t), \qquad 0 < \varepsilon \ll 1.$$

Substituting into the PDE

$$(\varepsilon\tilde{u})_t = -(\varepsilon\tilde{u})_{xx} - (\varepsilon\tilde{u})_{xxxx} - (\varepsilon\tilde{u})(\varepsilon\tilde{u})_x.$$

Dividing by ε and keeping only terms of order ε (the nonlinear term is $O(\varepsilon^2)$ and is neglected) gives the **linearised equation**

$$\tilde{u}_t = -\tilde{u}_{xx} - \tilde{u}_{xxxx}.$$

Normal modes and dispersion relation

We look for separated solutions of the form

$$\tilde{u}(x,t) = e^{\lambda t} e^{jkx},$$

where k is a spatial wavenumber and λ is a (complex) growth rate.

Then

$$\tilde{u}_x = jk\tilde{u}, \quad \tilde{u}_{xx} = -k^2\tilde{u}, \quad \tilde{u}_{xxxx} = k^4\tilde{u}.$$

After substituting into the linearised equation and cancelling $\tilde{u} \neq 0$ gives the **dispersion relation**

$$\lambda(k) = k^2 - k^4.$$

Linearisation and Dispersion Relation (Part B)

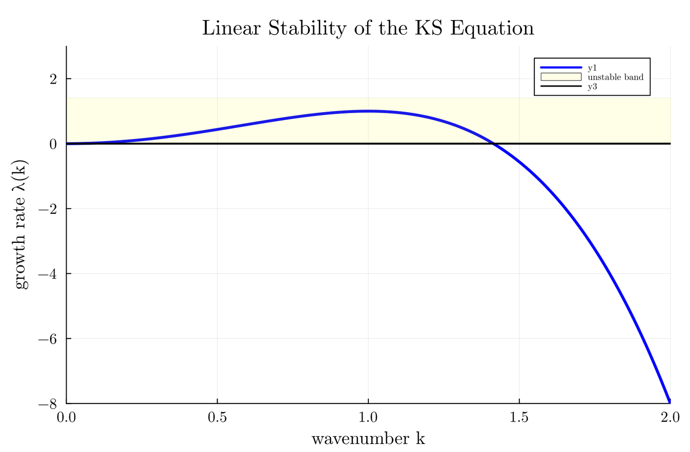

Figure 1. Growth rate $\lambda(k)$ for KS. Modes with $\lambda(k) > 0$ form the unstable band.

- Modes grow exponentially when $\lambda(k) > 0$.
- Instability occurs only for $0 < |k| < 1$, leading to a *finite band* of long-wave modes that drive the dynamics.
- The most unstable mode satisfies
$$\frac{d\lambda}{dk} = 2k - 4k^3 = 0 \quad \Rightarrow \quad |k| = \frac{1}{\sqrt{2}}.$$
Preferred wavelength of the instability associated with the mode:
$$\lambda_{\text{space}} \approx 2\pi\sqrt{2}.$$

- On a periodic domain of length L, allowed wavenumbers are
$$k_n = \frac{2\pi n}{L}, \qquad n \in \mathbb{Z}.$$

In summary, the domain size controls the spectrum of unstable modes, and thus increasing the length leads to more complex and chaotic spatiotemporal dynamics.

Spatiotemporal Chaos

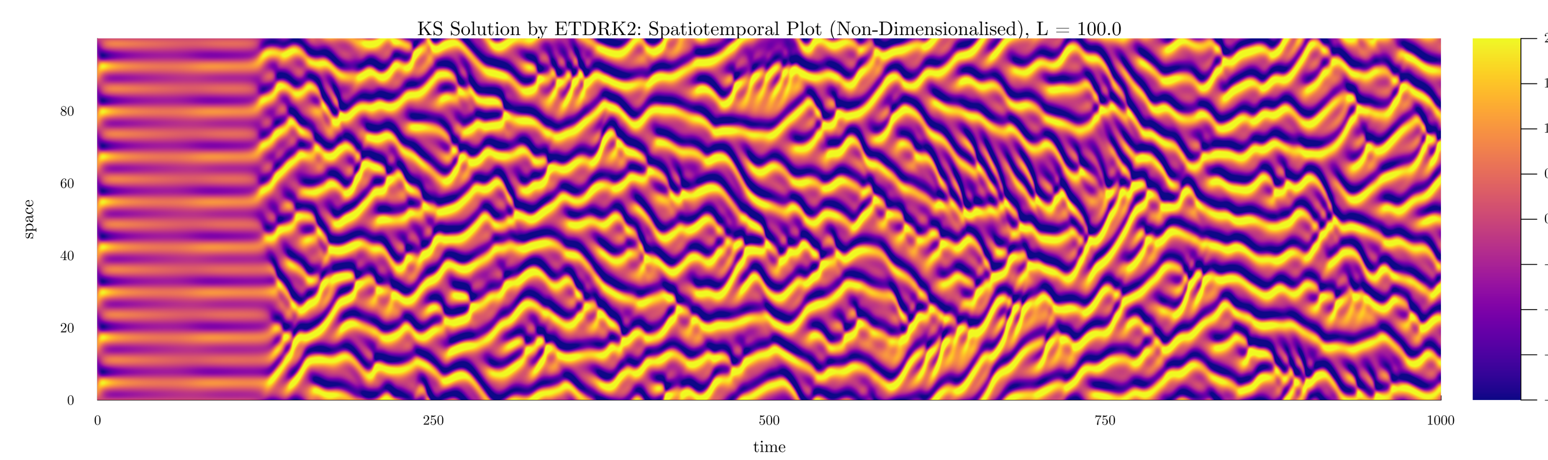

Figure 2. Spatiotemporal evolution of $u(x,t)$ for $L = 100$, $N = 2048$. Colours represent turbulent displacement.

How this solution was obtained

- **Fourier pseudo–spectral method**: spatial derivatives computed in Fourier space; nonlinearity evaluated in real space and transformed via FFT.
- **ETDRK2 time–stepping**: exponential integration of the stiff linear part $(u_{xx} + u_{xxxx})$, and explicit second order Runge-Kutta for the nonlinear term (uu_x).
- **Dealiasing**: The quadratic nonlinearity (uu_x) generates higher harmonics that can "wrap around" the discrete grid (aliasing), causing numerical instability. The $2/3$ rule is applied that explicitly zeroes out the highest $1/3$ of wavenumbers at every time step.
- **Parameters**: $L = 100$ (chosen as highly unstable regime supporting many active modes), $N = 2048$, $\Delta t = 0.1$, and an initial perturbation $u(x,0) = \sin(16\pi x/L)$.

What the plot shows

- The system begins with a regular, periodic horizontal stripes which shows how the transient phase is dominated by the sinusoidal initial conditions before the nonlinear interactions fully take over.
- After the transient phase, the plot exhibits fully developed spatiotemporal chaos. At any fixed point in space, the value fluctuates unpredictably over time. At any fixed time, the spatial profile is aperiodic and complex.
- These stripes propagate, merge and split. This creation and annihilation of these coherent structures is the signature behaviour of the KS system in the chaotic regime.

Bifurcation Structure of Stationary KS Solutions

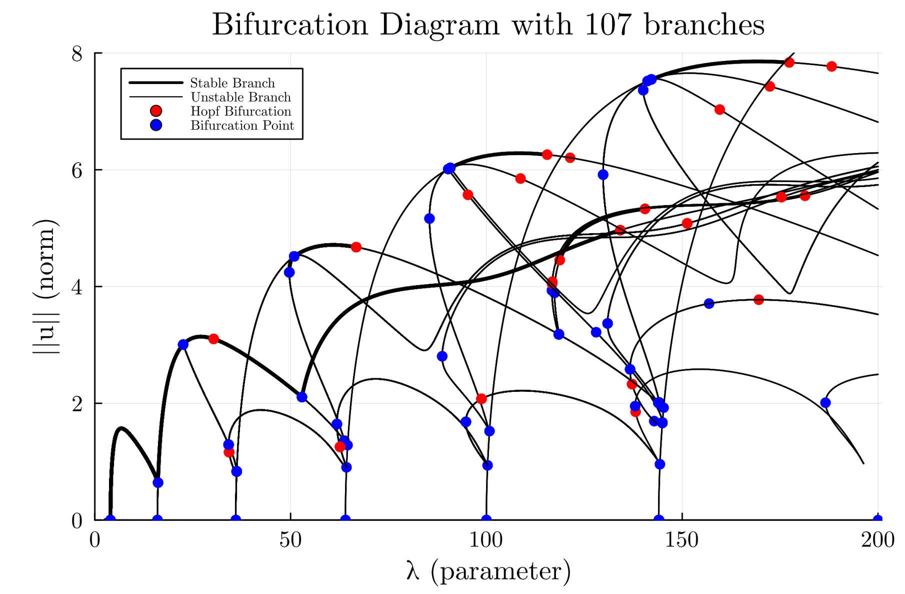

Figure 3. Continuation of stationary solutions of the KS equation using a truncated Fourier expansion (following Evstigneev & Ryabkov, 2021). Branches show steady states; symbols mark bifurcation points.

- The KS equation was projected onto the first $N = 30$ sine modes:
$$u(x) = \sum_{k=1}^{N} a_k \sin(kx),$$
giving a system of N nonlinear algebraic equations for the stationary states.
- Used **BifurcationKit.jl** with pseudo-arclength continuation in control parameter λ.
- Automatically detected **steady-state** bifurcations (folds, pitchforks) and **Hopf** bifurcations (oscillatory onset).

What the diagram shows

- The trivial flat state $u = 0$ loses stability via a series of *steady-state bifurcations* as predicted from the dispersion relation.
- Multiple equilibria coexist for large λ which creates a dense and interconnected skeletal structure.
- *Hopf bifurcations* indicate the transition where stable stationary points lose stability and lead to *travelling wave* solutions.
- Complexity grows with λ as expected; the intertwining branches reflects the increasing complexity and available phase space as the domain size increases.

So Why Does a Flame Front Flutter?

Although the details differ between models, the KSE captures the essential reason why flame fronts never remain flat: *small perturbations amplify, interact, and continuously reshape the interface*. No single mode dominates for long; patterns propagate, merge and split, producing an irregular motion that will never settle.

The "flutter" is therefore the outcome of a flame front that is *perpetually reorganising itself* through instability, bifurcations and chaos. The result is thus a dynamic, constantly renewing structure that is characteristic of unstable propagating fronts.

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