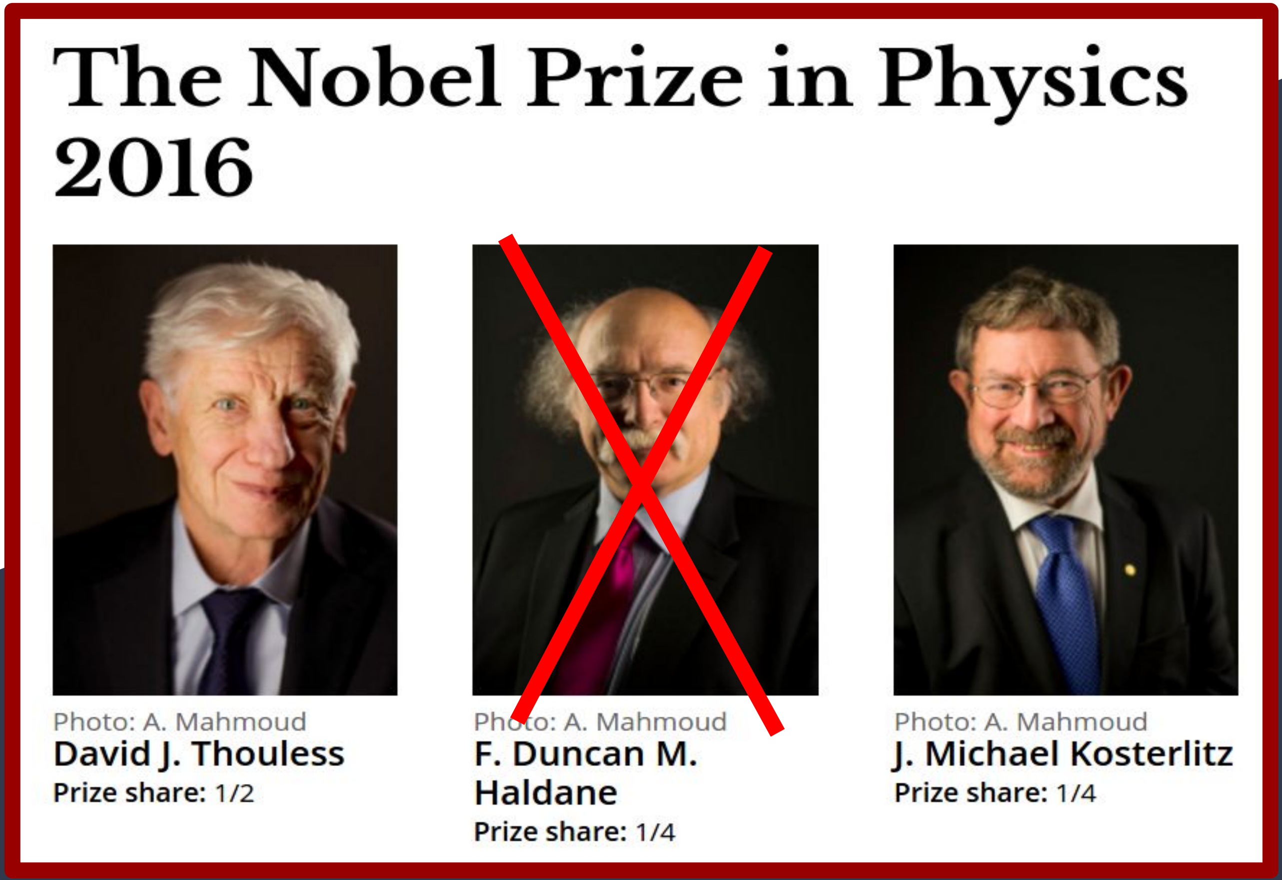
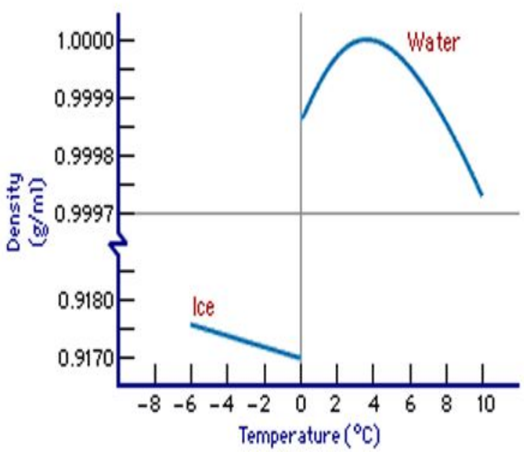
The Kosterlitz–Thouless Transition

Jahan Claes

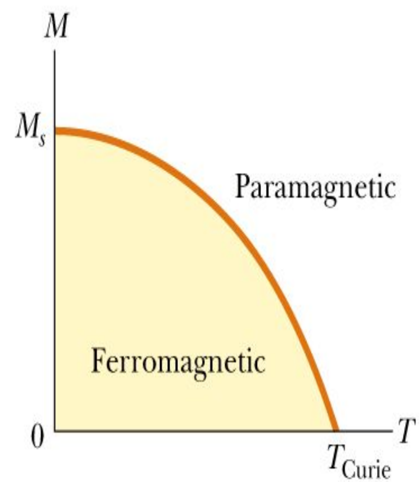


What is a phase transition?

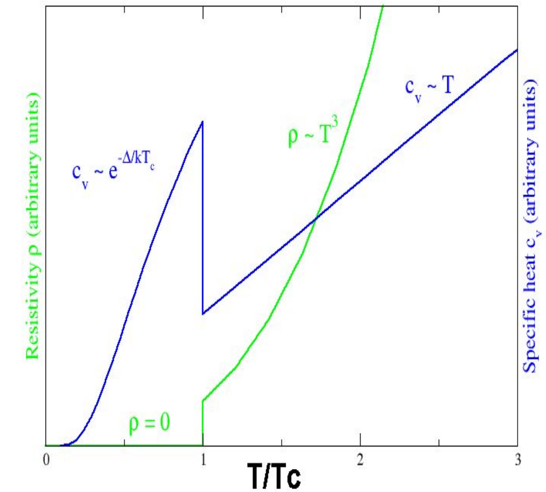
Ice-to-Water



Ferromagnet-to-Paramagnet

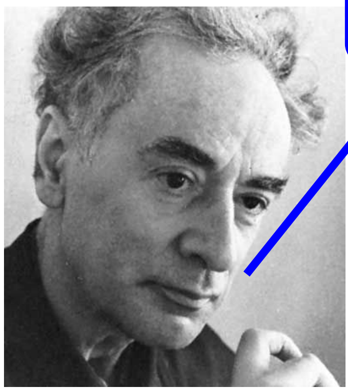


Superconductor-to-Normal



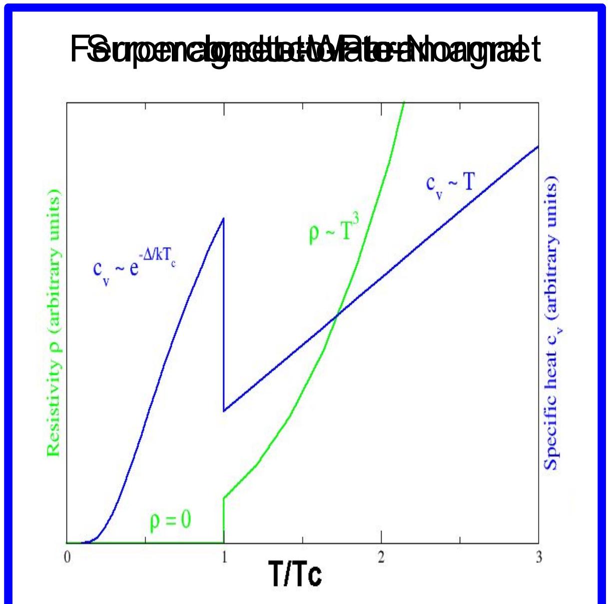
A **phase transition** is a discontinuous change in some property of a material as an external parameter (e.g. temperature) is varied.

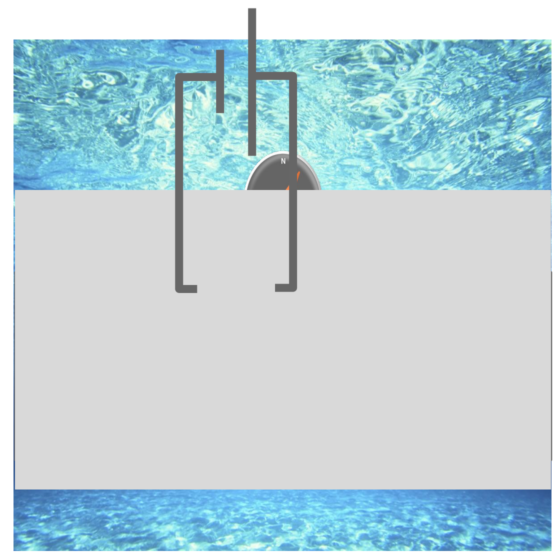
What is a phase transition… according to Landau?



Lev Landau

“A phase transition can always be described by a **local** order parameter.”





The KT transition is a phase transition with **no local order parameter**.

A quick review of Statistical Mechanics

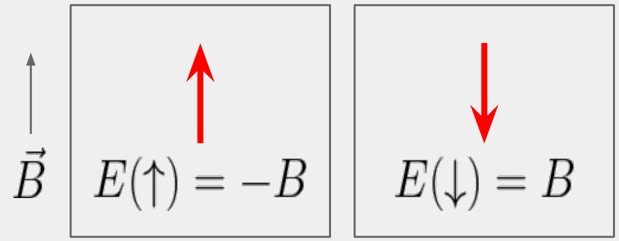
In Statistical Mechanics, everything is determined by the Boltzmann distribution:

$$P(x) \propto e^{-E(x)/T}$$

To calculate a quantity, you average over the distribution:

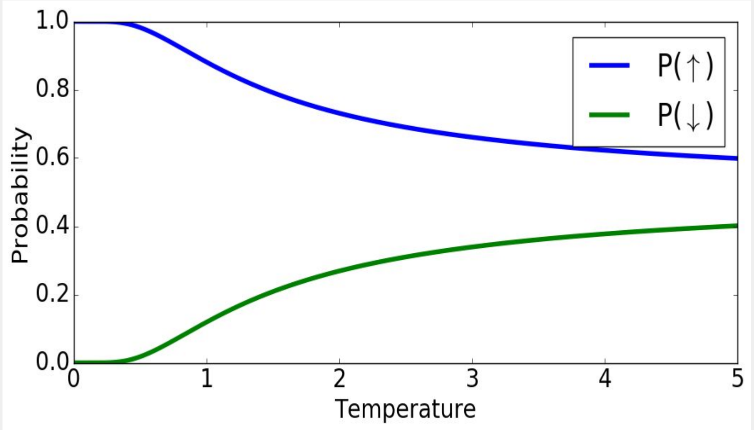
$$\langle O \rangle = \frac{\sum_x O(x) e^{-E(x)/T}}{\sum_x e^{-E(x)/T}}$$

Example: Spin in a magnetic field



$$P(\uparrow) \propto e^{B/T}$$

$$P(\downarrow) \propto e^{-B/T}$$

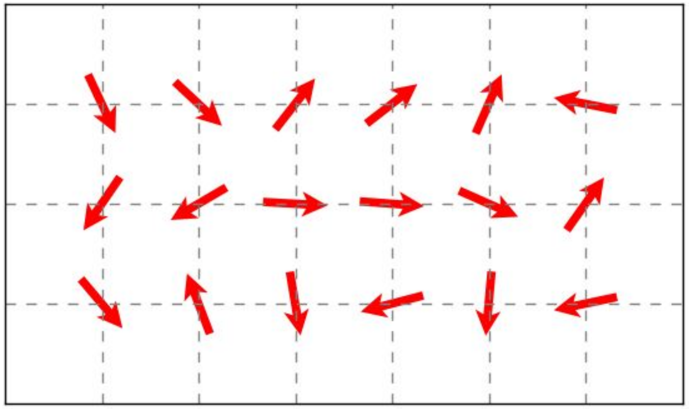


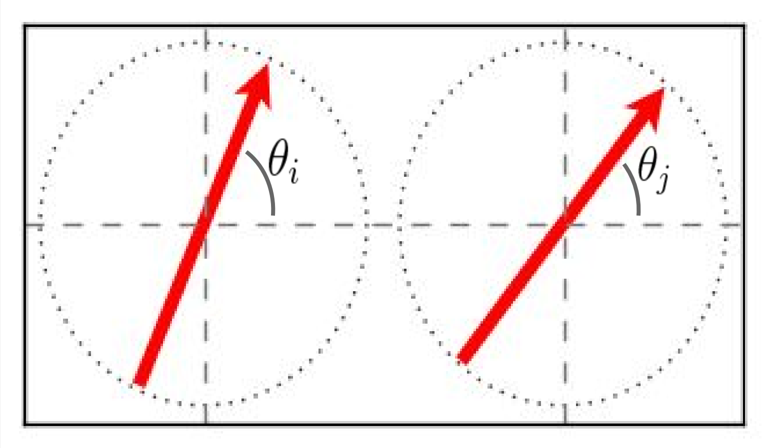
Punchline: In Statistical Mechanics, all you need is E

The 2-Dimensional XY Model

Square Lattice of spins

$$E = -\sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j$$



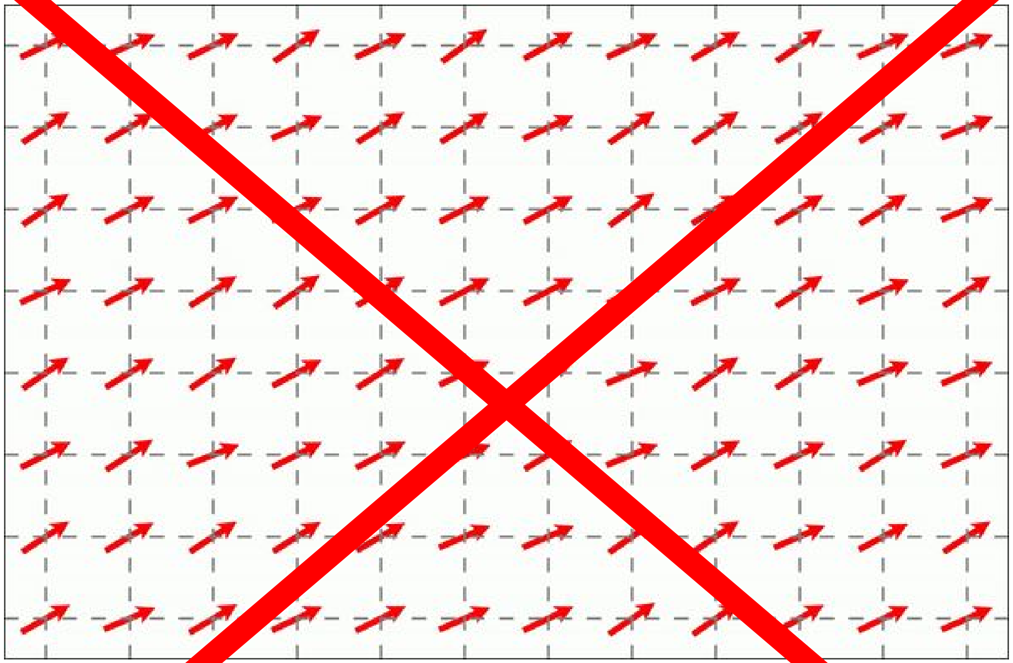


$$\begin{aligned} \vec{S}_i \cdot \vec{S}_j &= |\vec{S}_i||\vec{S}_j| \cos(\theta_i - \theta_j) \\ &= \cos(\theta_i - \theta_j) \end{aligned}$$

$$E = -\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

Low-Temperature XY Model

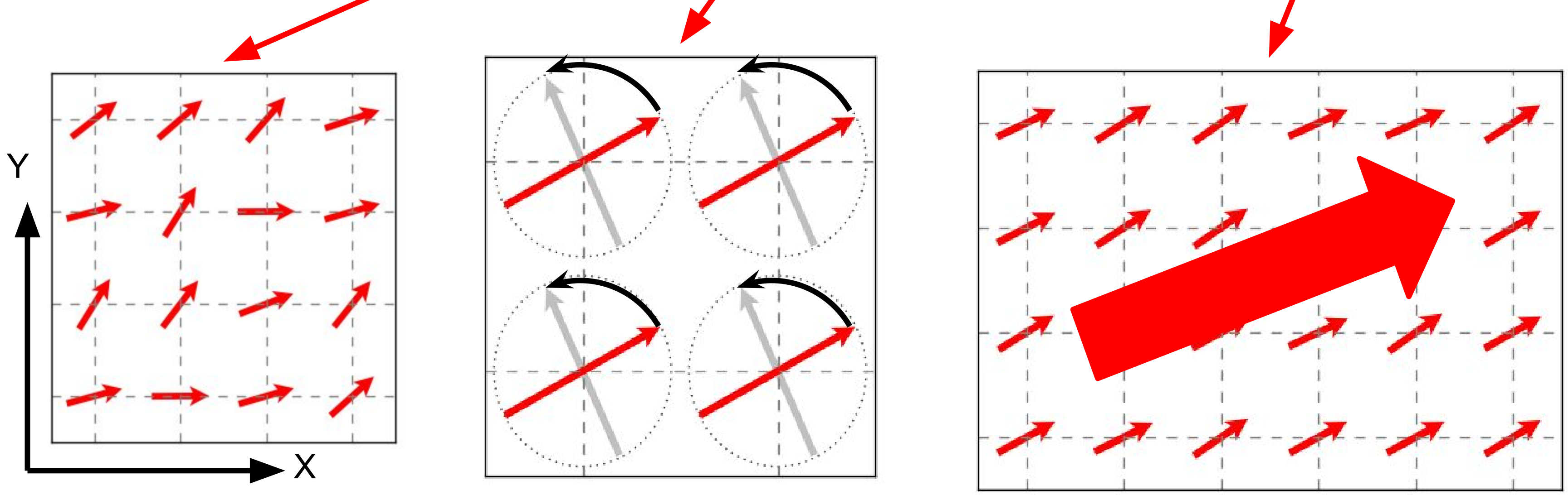
$$E = -\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

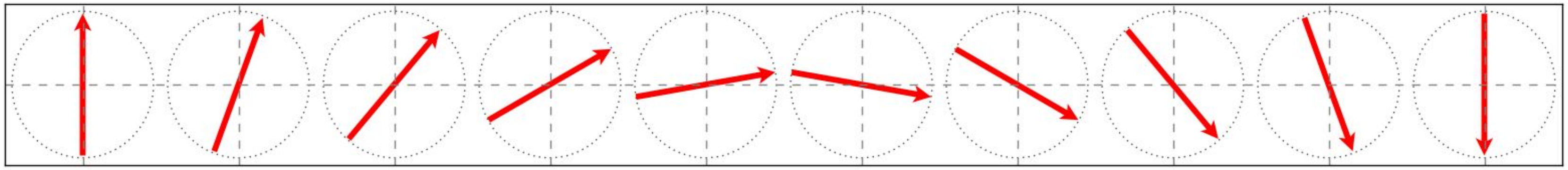


$$\langle \vec{S}(x) \rangle = \vec{S}_0$$

Mermin-Wagner-Berenzinskii-Hohenberg-Coleman-Skywalker-J.Jonah.Jameson Theorem

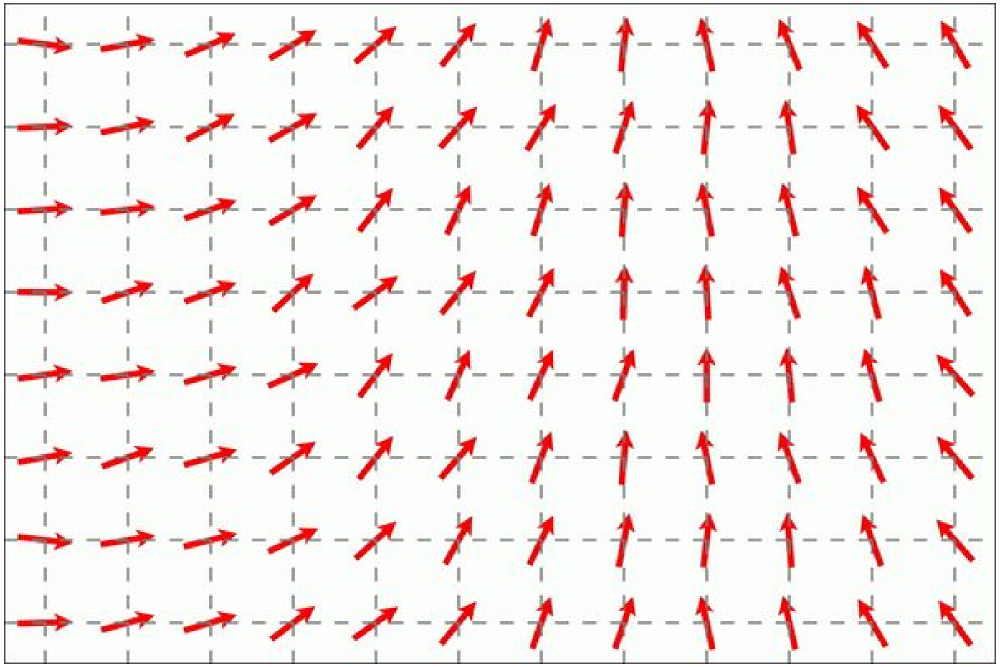
"At finite temperature in $\boxed{d \leq 2}$, a $\boxed{\text{continuous symmetry}}$ cannot be $\boxed{\text{spontaneously broken.}}$"





Low-Temperature XY Model

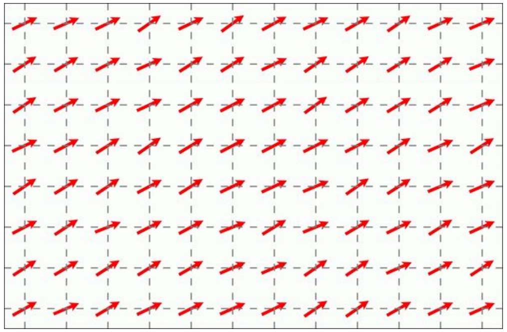
$$E = -\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

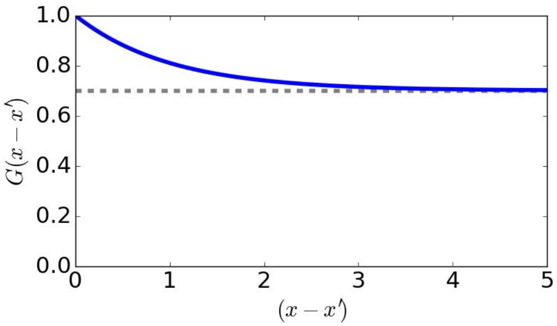


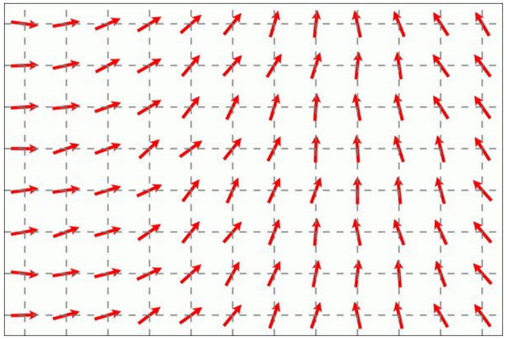
$$\langle \vec{S}(x) \rangle = 0$$

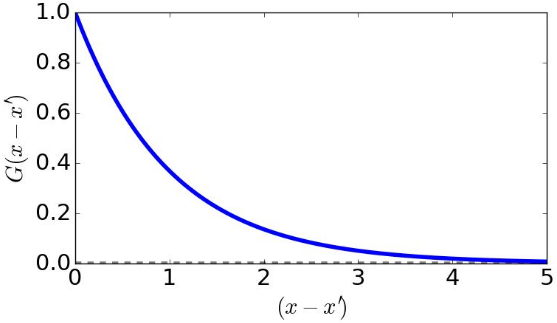
The correlation function

$$G(x - x') = \langle \vec{S}(x) \cdot \vec{S}(x') \rangle$$





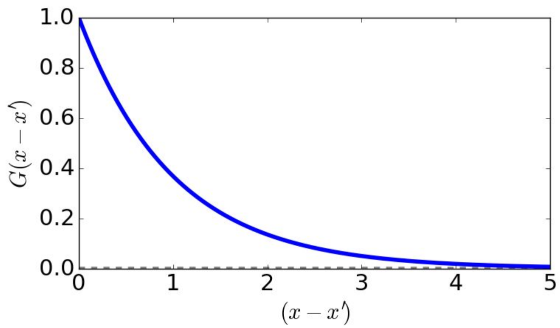


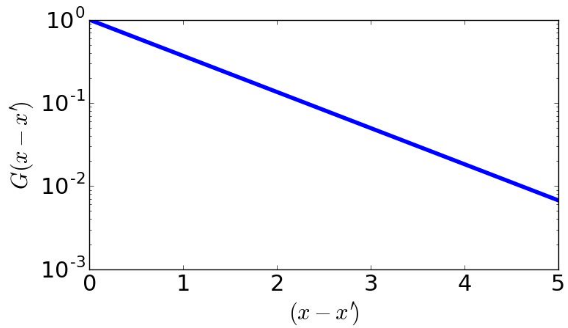


Decay of correlations

Exponential

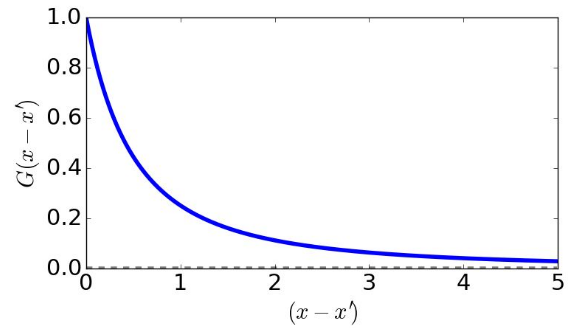
$$G(x-x') \propto e^{-\frac{x-x'}{\xi}}$$

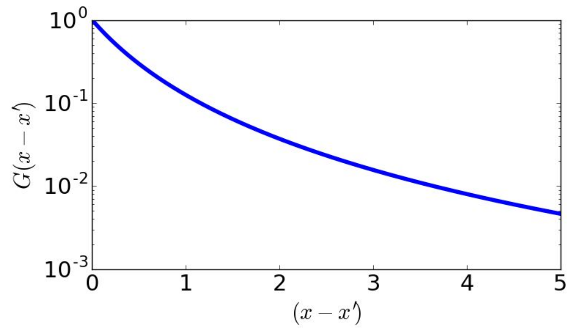




~~Not Exponential~~ Algebraic

$$G(x-x') \propto \frac{1}{(x-x')^{\eta}}$$





Here is some math

$$E = -\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$$\approx -\sum_{\langle i,j \rangle} \left[1 - \frac{1}{2}(\theta_i - \theta_j)^2 \right]$$

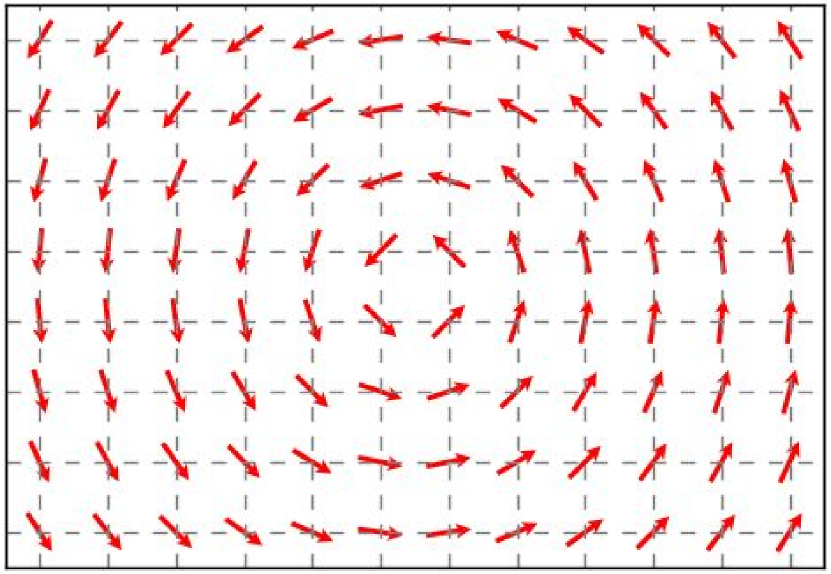
$$\langle \vec{S}(x) \cdot \vec{S}(x') \rangle = \langle \cos\left(\theta(x) - \theta(x')\right) \rangle$$

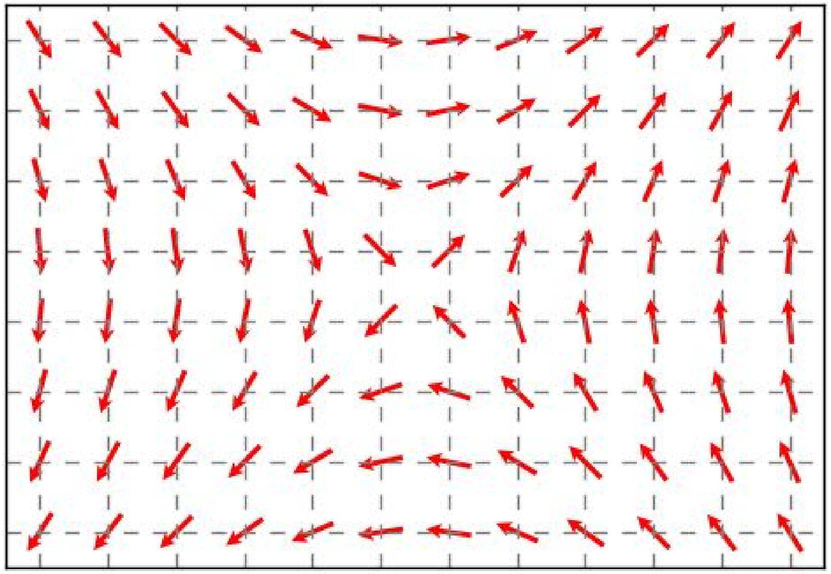
$$\propto \int \mathcal{D}\theta \cos\left(\theta(x) - \theta(x')\right) e^{-\frac{E(\theta)}{T}}$$

$$\propto \frac{1}{(x - x')^{\frac{T}{2\pi}}}$$

Where can this go wrong?

The vortex and anti-vortex states

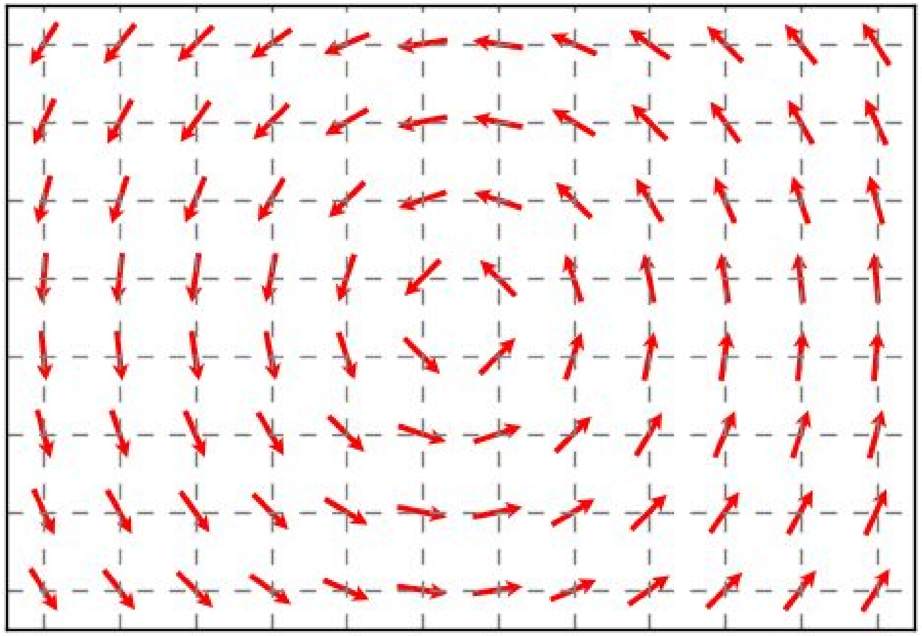




$$E = -\sum_{\langle i,j \rangle} \cos(\theta_i - \theta_j)$$

$$\approx -\sum_{\langle i,j \rangle} \left[1 - \frac{1}{2}(\theta_i - \theta_j)^2 \right]$$

Do vortices occur?



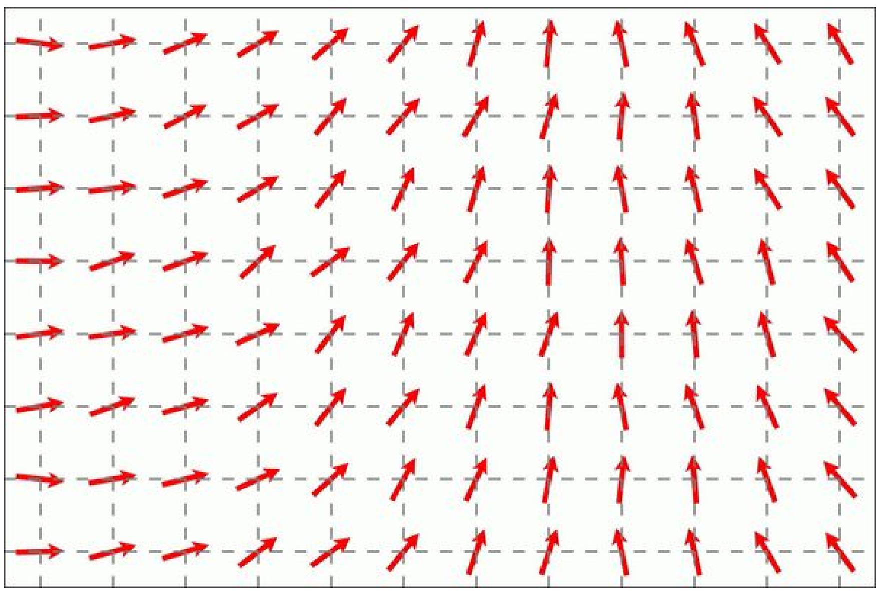
$$E = \pi \ln(L)$$

$$P(\text{vortex}) \propto L^2 e^{-E(\text{vortex})/T}$$
$$= L^2 e^{-\pi \ln(L)/T}$$
$$= L^2 L^{-\frac{\pi}{T}}$$

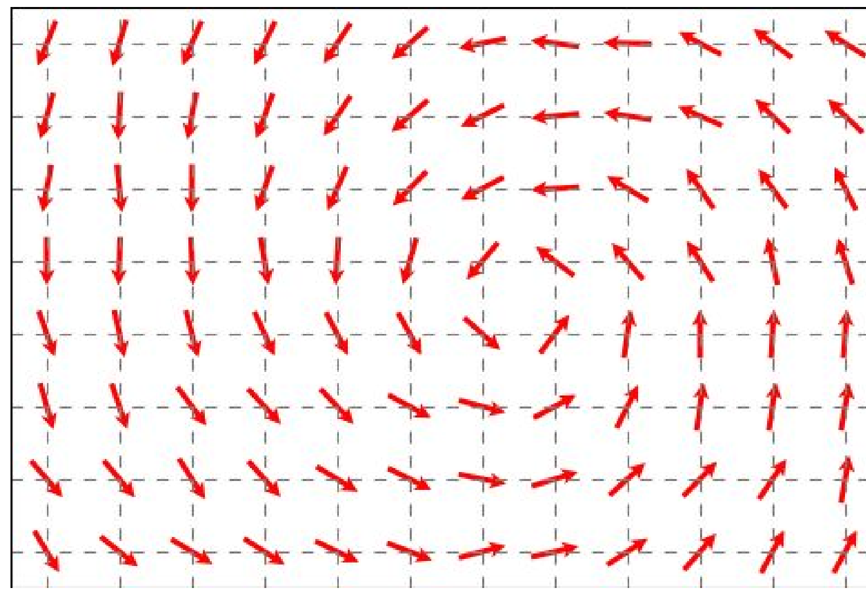
$$\boxed{T > \frac{\pi}{2}}$$

Low-Temperature XY model

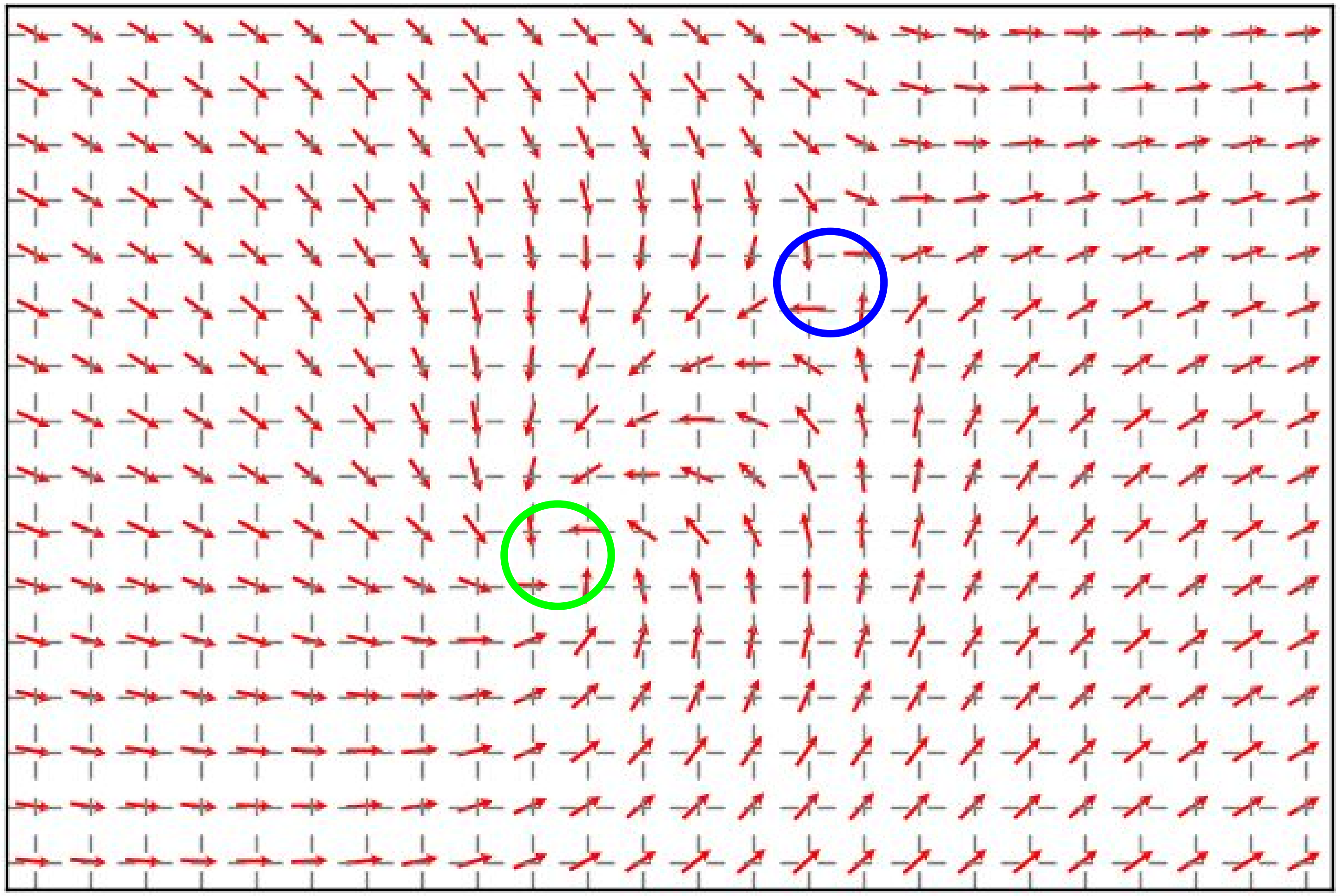
$$\boxed{T < \frac{\pi}{2}}$$



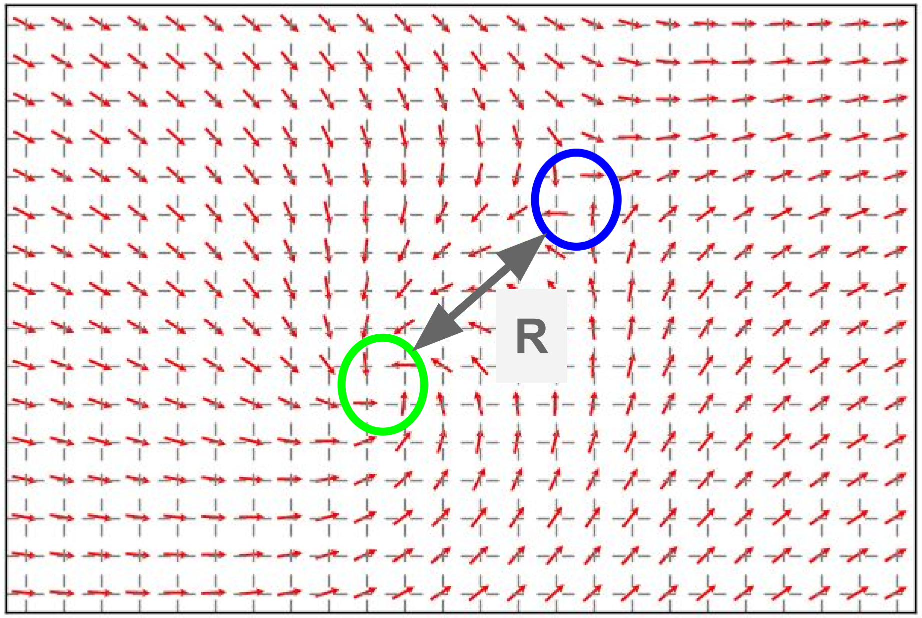
$$\boxed{T > \frac{\pi}{2}}$$



Vortex-Antivortex pairs



Do vortex/antivortex pairs occur?



$$P(R) \propto e^{-\frac{\pi}{2T}\ln(R)} 2\pi R$$
$$= 2\pi R^{1-\frac{\pi}{2T}}$$

$$\boxed{T < \frac{\pi}{2}}$$

Pairs exist for small R

$$\boxed{T > \frac{\pi}{2}}$$

Pairs exist for large R

$$E = \frac{\pi}{2}\ln(R)$$

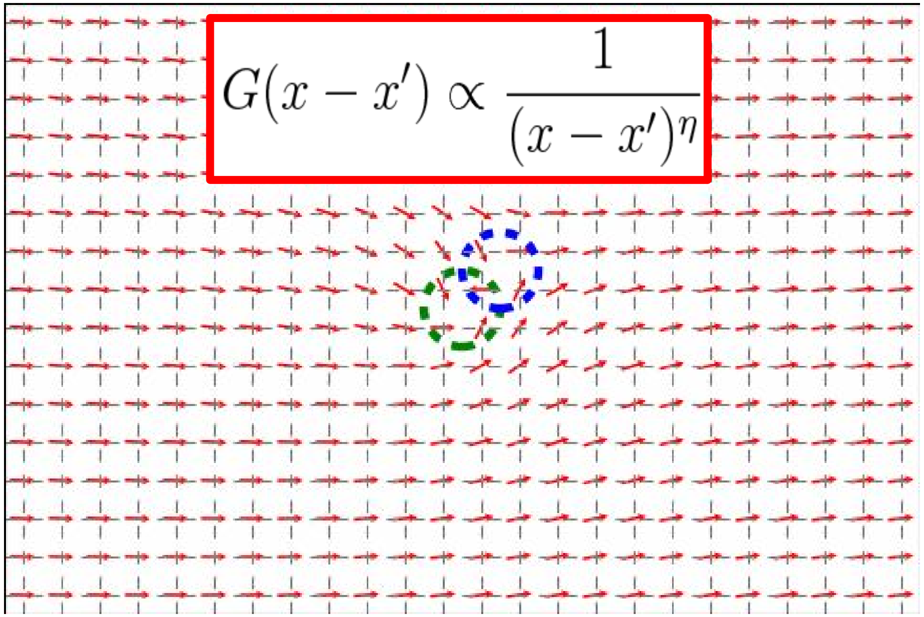
Low-Temperature XY model

$$T < \frac{\pi}{2}$$

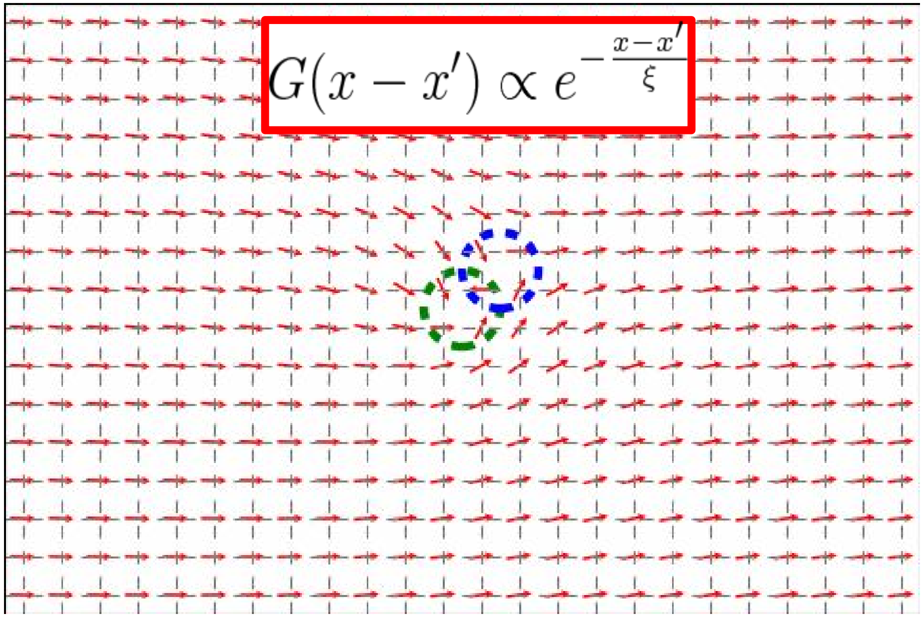
"Unbinding Transition"

$$T > \frac{\pi}{2}$$

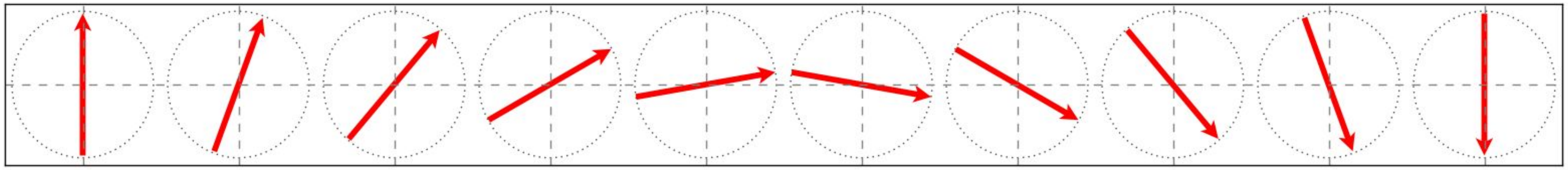
$$G(x - x') \propto \frac{1}{(x - x')^{\eta}}$$



$$G(x - x') \propto e^{-\frac{x - x'}{\xi}}$$



Experimental observation of the KT transition



$$\rho = \frac{\text{Energy to rotate N spins an amount } \delta\theta}{N(\delta\theta)^2}$$

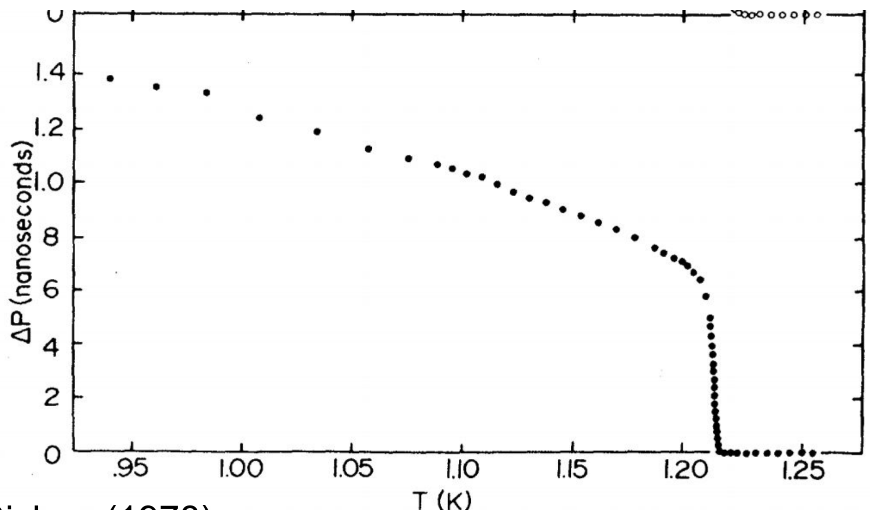
$$G(x - x') = \langle \vec{S}(x) \cdot \vec{S}(x') \rangle$$

$$\boxed{T < \frac{\pi}{2}} \longleftrightarrow \boxed{T > \frac{\pi}{2}}$$

Abrupt drop in ρ

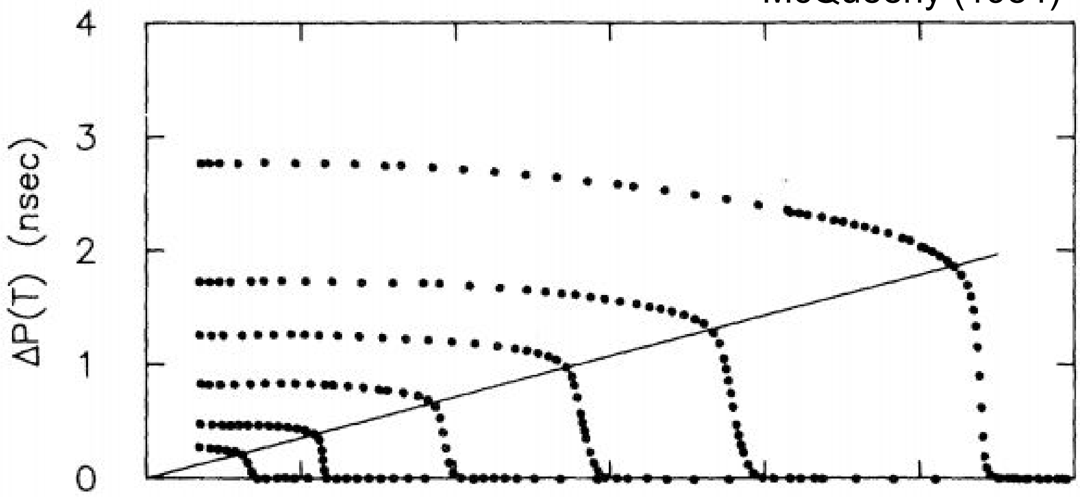
$$\boxed{G(x - x') \propto \frac{1}{(x - x')^\eta}} \qquad \boxed{G(x - x') \propto e^{-\frac{x - x'}{\xi}}}$$

Experimental observation of the KT transition



Bishop (1978)

McQueeny (1984)



J.M. Kosterlitz and D.J. Thouless, *Metastability and Phase Transitions in Two-Dimensional Systems*, (1973)

H.J. Jenson, *The Kosterlitz-Thouless Transition*

D. J. Bishop and J. D. Reppy, *Study of the Superfluid Transition in Two-Dimensional ^{4}He Films*, (1978)

D. McQueeney, G. Agnolet and J.D. Reppy, *Surface Superfluidity in Dilute ^{4}He-^{3}He Mixtures*, (1984)