Problem Statement

We have a γ/Z decaying into two leptons (either e^+e^- or $\mu^+\mu^-$) in the lab frame. The more forward lepton is labeled ℓ_1, and is at an angle θ_1 with the forward beam axis. The other lepton is labeled ℓ_2, and is at an angle θ_2.

In the rest frame of the γ/Z, the two leptons decay back to back, with equal energy and momentum. ℓ_1 forms an angle $\theta^{'}$ with the forward x axis.

We need to find what range of angles $\theta^{'}$ will produce leptons in the lab frame with momentum above p_{min}.

Rest Frame

$$E_1^{'} = E_2^{'} \equiv E_\ell^{'} = \frac{m_Z}{2}$$

$$\left|p_1^{'}\right| = \left|p_2^{'}\right| \equiv \left|p_\ell^{'}\right| = \sqrt{E_\ell^{'2} - m_\ell^2} = \sqrt{\frac{m_Z^2}{4} - m_\ell^2}$$

$$\left|p_{\ell,x}^{'}\right| = \left|p_\ell^{'}\right| \cos\theta^{'}$$

$$\left|p_{\ell,y}^{'}\right| = \left|p_\ell^{'}\right| \sin\theta^{'}$$

Boost

$$E_Z = \sqrt{p_Z^2 + m_Z^2}$$

$$\gamma = \frac{E_Z}{m_Z}$$

$$\beta = \frac{|p_Z|}{E_Z}$$

Lab Frame

$$|p_{\ell,y}| = \left|p_{\ell,y}^{'}\right| = \left|p_\ell^{'}\right| \sin\theta^{'}$$

$$|p_{\ell,x,1}| = \gamma(\left|p_{\ell,x}^{'}\right| + \beta E_\ell^{'})$$

$$|p_{\ell,x,2}| = \gamma(-\left|p_{\ell,x}^{'}\right| + \beta E_\ell^{'})$$

Here, $|p_{\ell,2}|$ will always correspond to the lepton with lower momentum, so we want to find the range of $\theta^{'}$ where $|p_{\ell,2}| \geq |p_{min}|$.

Solution

$$\begin{aligned}
p_{\ell,2}^2 &= p_{\ell,x,2}^2 + p_{\ell,y,2}^2 \\
&= \gamma^2(-\left|p_\ell^{'}\right| \cos\theta^{'} + \beta E_\ell^{'})^2 + \left|p_\ell^{'}\right|^2 \sin^2\theta^{'} \\
&= \gamma^2\left|p_\ell^{'}\right|^2 \cos^2\theta^{'} + \gamma^2\beta^2 E_\ell^{'2} - 2\gamma^2\beta\left|p_\ell^{'}\right| \cos\theta^{'} E_\ell^{'} + \left|p_\ell^{'}\right|^2 \sin^2\theta^{'} \\
&= \gamma^2\left|p_\ell^{'}\right|^2 \cos^2\theta^{'} + \gamma^2\beta^2 E_\ell^{'2} - 2\gamma^2\beta\left|p_\ell^{'}\right| \cos\theta^{'} E_\ell^{'} + \left|p_\ell^{'}\right|^2 (1-\cos^2\theta^{'}) \\
&\geq p_{min}^2
\end{aligned}$$

$$\begin{aligned}
0 &\leq [p_\ell^{'2} - p_{min}^2 + \gamma^2\beta^2 E_\ell^{'2}] - [2\gamma^2\beta\left|p_\ell^{'}\right| E_\ell^{'}]\cos\theta^{'} + [\gamma^2 p_\ell^{'2} - p_\ell^{'2}]\cos^2\theta^{'} \\
&= [\frac{m_Z^2}{4} - m_\ell^2 - p_{min}^2 + \frac{p_Z^2}{4}] - [\frac{p_Z}{m_Z}\sqrt{p_Z^2+m_Z^2}\sqrt{\frac{m_Z^2}{4} - m_\ell^2}]\cos\theta^{'} + [\frac{p_Z^2}{m_Z^2}(\frac{m_Z^2}{4} - m_\ell^2)]\cos^2\theta^{'}
\end{aligned}$$

$$\begin{aligned}
&A \equiv \frac{p_Z}{m_Z}\sqrt{\frac{m_Z^2}{4} - m_\ell^2} \\
&E_Z \equiv \sqrt{p_Z^2 + m_Z^2} \\
&x \equiv \cos\theta^{'} \\
&[\frac{E_Z^2}{4} - m_\ell^2 - p_{min}^2] - [AE_Z]x + [A^2]x^2 \geq 0 \\
&x = \frac{AE_Z \pm \sqrt{A^2E_Z^2 - A^2(E_Z^2 - 4m_\ell^2 - 4p_{min}^2)}}{\frac{E_Z^2}{2} - 2m_\ell^2 - 2p_{min}^2} \\
&\ = \frac{2A(E_Z \pm 2\sqrt{m_\ell^2 + p_{min}^2})}{E_Z^2 - 4(m_\ell^2 + p_{min}^2)} \\
&\ = \frac{2A}{E_Z \mp 2\sqrt{m_\ell^2 + p_{min}^2}}
\end{aligned}$$

The positive sign solution is the correct one to use here.

$$\theta^{'} = \cos^{-1}(\frac{2A}{E_Z + 2\sqrt{m_\ell^2 + p_{min}^2}})$$