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Homework 4

1. a) W.C.: $K = n$

$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{\sqrt{n}} c = \sum_{i=1}^{n} \sqrt{n}\, c = cn^{3/2} \in \Theta(n^{3/2})$

b) $ET(n) = \Sigma_Q \Pr(Q)\, t(Q) = \Sigma_Q \Pr(K=n)\, t(K=n)$

$\Pr(K=n) = \frac{1}{n}$

$ET(n) = \sum_{q=1}^{n} \left(\frac{1}{n}\right) t(K=q) = \sum_{q=1}^{n} \left[\sum_{i=1}^{q} \sum_{j=1}^{\sqrt{q}} c\right] \cdot \frac{1}{n}$

$= \sum_{q=1}^{n} \frac{c}{n} \cdot q^{3/2}$ UB: $\leq \sum_{q=1}^{n} \frac{c}{n} \cdot n^{3/2} \leq cn^{3/2} \in \Theta(n^{3/2})$

LB: $\geq \sum_{q=n/2}^{n} \frac{c}{n} \left(\frac{n}{2}\right)^{3/2} \geq \frac{c\, n^{3/2}}{n \cdot 2^{3/2}} \left(\frac{n}{2}\right) = \frac{c\, n^{3/2}}{2 \cdot 2^{3/2}} \in \Theta(n^{3/2})$

$ET(n) \in \Theta(n^{3/2})$

2. a) W.C. : $K < \sqrt{n}$

$T(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} c = cn^2$

$\Pr(K < \sqrt{n}) = \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$

$\Pr(K \geq \sqrt{n}) = 1 - \frac{1}{\sqrt{n}}$

b) $ET(n) = \Sigma_Q \Pr(Q)\, t(Q)$

$= \text{Prob}(K < \sqrt{n})\, t(K < \sqrt{n}) + \text{Prob}(K \geq \sqrt{n})\, t(+ \geq \sqrt{n})$

$= \frac{1}{\sqrt{n}} \cdot cn^2 + \left(1 - \frac{1}{\sqrt{n}}\right) c = cn^{3/2} + c\left(\frac{\sqrt{n}-1}{\sqrt{n}}\right)$

$\in \Theta(n^{3/2})$

3.

a) W.C. $n > 10$ and K is even and $K \le n/2$

$T(n) = c + T(n-3) + T(n-9)$

LB: $\ge c + 2T(n-9) \ge 2T(n-9) \ge 2 \cdot 2T(n-18) \ge 2 \cdot 2 \cdot 2\, T(n-27)$

$\ge \underbrace{2 \cdot 2 \cdot 2 \cdots 2}_{K} T(n-9K)$ $\quad 2^{n/9}\, T(0)$ $\quad n-9K=0 \quad K = n/9$

$\in \Omega(2^{n/9})$

b) $\text{Prob}(K \text{ is even}) = \frac{1}{2}$ $\qquad ET(n) = ET_1(n) + ET_2(n) + ET_3(n)$

$\text{Prob}(K \le n/2) = \frac{1}{2}$ $\qquad$ Steps: 1-3, 4, 5

$ET_1(n) = c$

$ET_2(n) = \text{Prob}(K \text{ is even})\, t(K \text{ is even}) + \text{Prob}(K \text{ is odd})\, t(K \text{ is odd})$

$= \frac{1}{2}(c + ET(n-3)) + \frac{1}{2}c = c + \frac{1}{2}ET(n-3)$

$ET_3(n) = \text{Prob}(K \le n/2)\, t(K \le n/2) + \text{Prob}(K > n/2)\, t(K > n/2)$

$= \frac{1}{2}(c + ET(n-9)) + \frac{1}{2}c = c + \frac{1}{2}ET(n-9)$

$ET(n) = c + \frac{1}{2}ET(n-3) + c + \frac{1}{2}ET(n-9) + c = 3c + \frac{1}{2}ET(n-3) + \frac{1}{2}ET(n-9)$

UB: $\le 3c + ET(n-3) = c + c + c + c\; ET(n-3 \cdot 2) = 3c + c + c + ET(n-3 \cdot 3)$

$\vdots$

$= 3c + \underbrace{c + c + c + \dots + c}_{K-1} + ET(n-3K)$ $\qquad n-3K=0 \qquad K = n/3$

$\le 3c + \frac{cn}{3} + \cancel{T(0)} \in \Theta(n)$

LB: $\le 3c + ET(n-9) \le 3c + c + ET(n-9 \cdot 2) \le 3c + c + c + ET(n-9 \cdot 3)$

$\vdots = 3c + \underbrace{c + c + c + \dots + c}_{K-1} + ET(n-9K)$ $\qquad n-9K=0 \quad \frac{n}{9} = k$

$= 3c + c\left(\frac{n}{9} - 1\right) + T(0) \in \Theta(n)$

$ET(n) \in \Theta(n)$

4.

a) W.C. $(n > 10)$ $\quad (c = \text{heads})$

$i = n \qquad \text{after } \gamma^{th} \qquad i = n \cdot \left(\frac{1}{2}\right)^{\gamma} \qquad \text{stop when } i = 5$

$5 = n \cdot \left(\frac{1}{2}\right)^{\gamma} \qquad \frac{n}{5} = 2^{\gamma} \qquad \gamma = \log_2(n) - \log_2(5)$

$T_{while}(n) = c\left(\log_2(n) - \log_2(5)\right) \approx c\log_2(n)$

$T(n) = c\log_2(n) + T(n-3) \qquad\qquad T(n-3) = c\log_2(n-3) + T(n-6)$

$= c\log_2(n) + c\log_2(n-3) + T(n-6)$

$= c\log_2(n) + c\log_2(n-3) + c\log_2(n-3\cdot 2) + T(n-3\cdot 3)$

$\vdots$

$= \underbrace{c\log_2(n) + c\log_2(n-3) + \dots + c\log_2(n-k-1)}_{k} + T(n-3k)$

$n - 3k = 0$

$k = n/3$

UB: $\leq c\log_2(n) + c\log_2(n) + \dots + c\log_2(n) + c$

$\leq c\log_2(n)\left[\underbrace{1+1+1+\dots+1}_{k}\right] + c \quad \leq \frac{n}{3}c\log_2(n) + c \quad \in O(n\log_2(n))$

LB: $\geq \underbrace{c\log_2(n/2) + c\log_2(n/2) + \dots + c\log_2(n/2)}_{k/2} + c$

$\geq \frac{n}{6}c\log_2(n/2) + c \quad \in \Omega(n\log_2(n))$

$T(n) \in \Theta(n\log_2(n))$

b) $\text{Prob}(c = \text{heads}) = 1/2$

$ET(n) = \text{Prob}(c=\text{heads})\, ET(c=\text{heads}) + \text{Prob}(c \neq \text{heads})\, ET(c \neq \text{heads})$

$= \frac{1}{2}\left(c\log_2(n) + ET(n-3)\right) + \frac{1}{2}\left(c\log_2(n)\right) = c\log_2(n) + \frac{1}{2}ET(n-3)$

$= c\log_2(n) + \frac{1}{2}\cdot c \cdot \log_2(n-3) + \left(\frac{1}{2}\right)^2 ET(n-2\cdot 3)$

$= c\log_2(n) + \frac{1}{2}(c)\log_2(n-3) + \left(\frac{1}{2}\right)^2 (c)\log_2(n-6) + \left(\frac{1}{2}\right)^3 ET(n-3\cdot 3)$

$\vdots$

$= c\log_2(n) + \frac{1}{2}(c)\log_2(n-3) + \left(\frac{1}{2}\right)^2(c)\log_2(n-6) + \dots + \left(\frac{1}{2}\right)^k ET(n-3\cdot k)$

UB: $\leq c\log_2(n) + \frac{1}{2}(c)\log_2(n) + \dots + \left(\frac{1}{2}\right)^{n/3}\log_2(n) + \left(\frac{1}{2}\right)^{n/3} T(0)$ $\qquad n-3k=0$

$\approx 2c\log_2(n)$ $\qquad k = n/3$

LB: $\geq c\log_2(n)$

$ET(n) \in \Theta(\log_2(n))$

5.

a) W.C. $(n > 10)$ $(k < 2n/3$ every time$)$

$T(n) = cn + 3T(n/3)$ $\qquad \frac{n}{3^k} = 1$

$= cn + cn + 3^2 T(n/3^2) = ckn + 3^k T(n/3^k)$ $\qquad k = \log_3(n)$

$= cn\log_3(n) + nc \in \Theta(n\log_2(n))$

b) $\text{Prob}(k \le 2n/3) = 2/3$ $\qquad ET(n) = ET_0(n) + ET_1(n) + ET_2(n) + ET_3(n)$

\# times $k < 2/3\, n$

$ET(n) = cn + 3 \cdot ET(\text{steps } 8\text{-}9)$

$= cn + 3\left(\left(\frac{2}{3}\right) ET(n/3)\right) = cn + 2ET(n/3)$

$= cn + 2\left[cn/3 + 2ET(n/3^2)\right] = cn + 2cn/3 + 2^2 ET(n/3^2)$

$= cn + 2cn/3 + 2^2\left[cn/3^2 + 2ET(n/3^3)\right] = cn + 2cn/3 + 2^2 cn/3^2 + 2^3 ET(n/3^3)$

$= \underbrace{cn + 2cn/3 + 2^2 cn/3^2 + \dots +}_{\log_3(n)} 2^k ET(n/3^k)$ $\qquad k = \log_3(n)$

UB: $\le cn + \frac{2}{3}cn + \frac{2^2}{3^2}cn + \dots + 2^{\log_3(n)} \overset{c}{T(1)}$

$cn\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^{\log_3(n)-1} + 2^{\log_3(n)}/n\right] \le cn \frac{1}{1-\frac{2}{3}} = 3cn$

LB: $\ge cn$

$ET(n) \in \Theta(n)$

6. W.C. $(n > 10)$ $\quad k \le n/4$ every time

$$\sum_{i=1}^{9} 6cn + T(n/3) = 54cn + 9T(n/3)$$

$T(n) = 54cn + 9T(n/3) = 54cn + 9 \cdot cn/3 + 9^2 T(n/3^2)$

$= 54cn + 9 \cdot cn/3 + 9^2 cn/3^2 + 9^3 T(n/3^3)$

$= 54cn + 9 \cdot cn/3 + 9^2 cn/3^2 + \dots + 9^k T(n/3^k)$ $\qquad k = \log_3(n)$

LB: $\ge cn^2 \in \Omega(n^2)$

UB: $\le cn^2[54 + 9/3 + 9^2/3^2 + \dots + 1] \in O(n^2)$

$T(n) \in \Theta(n^2)$

b) $prob(k \le n/4) = 1/4$

$ET(n) = cn + 9((1/4)ET(n/3)) = cn + 9/4\, ET(n/3)$

$= cn + 9/4[cn/3 + 9/4\, ET(n/3^2)] = cn + 9/12\, cn + (9/4)^2 ET(n/3^2)$

$= cn + 3/4\, cn + (9/4)^2 [cn/3^2 + 9/4\, ET(n/3^3)]$

$= cn + 3/4\, cn + 9/16\, cn + (9/4)^3 ET(n/3^3)$

$\vdots$ $\qquad k = \log_3(n)$

$= cn + 3/4\, cn + 9/16\, cn + \dots + (9/4)^k ET(n/3^k)$

$= cn[1 + 3/4 + (3/4)^2 + \dots + (3/4)^{\log_3 n - 1}] + (9/4)^{\log_3(n)} \cancel{T(1)}^{c}$

$\approx 4cn + (9/4)^{\log_3(n)} c \in \Theta(n)$

$ET(n) \in \Theta(n)$