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CSE 2331
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Homework 5

1. a) $T(n) = 6cn + 4T(n/4)$ W.C. $K \le n/4$ every time

$\approx cn + 4T(n/4)$ $T(n/4) = cn/4 + 4T(n/4^2)$

$= cn + 4[cn/4 + 4T(n/4^2)] = cn + cn + 4^2 T(n/4^2)$

$= cn + cn + 4^2[cn/4^2 + 4T(n/4^3)]$

$= cn + cn + cn + 4^3 T(n/4^3)$ $T(n/4^2) = cn/4^2 + 4T(n/4^3)$

$\vdots$

$= Kcn + 4^K T(n/4^K)$ $\frac{n}{4^K} = 1$ $K = \log_4(n)$

$= c\log_4(n)\, n + nc$ $\in \Theta(n \log_2(n))$

b) $Prob(K \le n/4) = 1/4$

$ET(n) = cn + 4\left(\left(\frac{1}{4}\right) ET(n/4)\right) = cn + ET(n/4)$

$= cn + cn/4 + ET(n/4^2)$

$= cn + cn/4 + cn/4^2 + ET(n/4^3)$

$\vdots$

$= cn\left[1 + \frac{1}{4} + \frac{1}{4^2} \ldots + \frac{1}{4^K}\right] + ET(n/4^K)$ $K = \log_4(n)$

$\frac{1}{1-\frac{1}{4}} cn + T(1) = \frac{4}{3} cn + c$

$ET(n) \in \Theta(n)$

2. a) W.C. $K = n/4$ every time

$T(n) = cn + 9T(n/3)$

$= cn + 9[cn/3 + 9T(n/3^2)] = cn + \frac{9}{3}cn + 9^2 T(n/3^2)$

$= cn + 9/3\, cn + 9^2[cn/3^2 + 9T(n/3^3)] = cn + 9/3\, cn + \frac{9^2}{3^2} cn + 9^3 T(n/3^3)$

$\vdots$

$= cn\left[1 + \frac{9}{3} + \left(\frac{9}{3}\right)^2 + \ldots + \left(\frac{9}{3}\right)^K\right] + 9^K T(n/3^K)$ $K = \log_3(n)$

$= cn[1 + 3 + 9 + \ldots + n] + 9^{\log_3(n)} c$

LB: $\ge cn^2 \in \Omega(n^2)$ UB: $\le cn^2 + cn^2 \in O(n^2)$

$T(n) \in \Theta(n^2)$

b) $Prob(K < n/4) = \frac{1}{4}$

$ET(n) = cn + 9\left(\frac{1}{4}\right) ET(n/3) = cn + 9/4\, ET(n/3)$

$\vdots$

$= cn + \frac{3}{4}cn + 9/16\, cn + \ldots + (9/4)^K ET(n/3^K)$

$K = \log_3(n)$

$$= cn\left[1+\frac{3}{4}+\frac{9}{16}+\dots+\left(\frac{3}{4}\right)^{\log_3(n)}\right] + (9/4)^{\log_3(n)} \cancel{T(1)}^{\,c}$$

$$\approx \frac{1}{1-\frac{3}{4}} cn + (9/4)^{\log_3 n} c \in \Theta(n)$$

$ET(n) \in \Theta(n)$

3. $$\sum_{j=1}^{n}\sum_{k=1}^{j} c = \sum_{j=1}^{n} cj$$

UB: $\le \sum_{j=1}^{n} cn = cn^2$

LB: $\le \sum_{j=n/2}^{n} c\frac{n}{2} = \frac{c}{4}n^2$

$= c\,n(n+1)/2$

start $i=1$ after r^{th} $i = 1\cdot 2^r$

stop $i=n$ $n = 2^r$ $r = \log_2(n)$

$cn^2 \cdot \log_2(n) \in \Theta(n^2 \log_2(n))$

W.C. $K \bmod 7 \neq 3$ and while loop runs to completion

b) $\text{Prob}(K \bmod 7 = 3) = \frac{1}{7}$ $\text{Prob}(k \bmod 7 \neq 3) = 6/7$

*FOR LARGE N

$$E(x) = \sum_{i=1}^{\infty}\left(\frac{6}{7}\right)^i = \frac{1}{1-6/7} - 1 = 6$$

$$ET(n) = cn^2 \cdot E(X) + cn^2$$
$$= cn^2 \cdot 6 + cn^2 = 7cn^2$$

$ET(n) \in \Theta(n^2)$

Homework 5 cont.

4. a) W.C. k is $n-1$ every time

$$T(n) = c\sqrt{n} + T(n-1)$$
$$= c\sqrt{n} + c\sqrt{n-1} + T(n-2)$$
$$= c\sqrt{n} + c\sqrt{n-1} + c\sqrt{n-2} + T(n-3)$$

$$T(n-1) = c\sqrt{n-1} + T(n-2)$$
$$T(n-2) = c\sqrt{n-2} + T(n-3)$$

$\vdots$

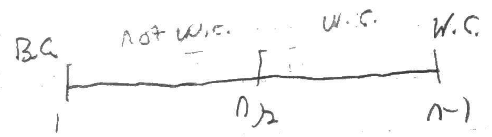
$$= c\sqrt{n} + c\sqrt{n-1} + c\sqrt{n-2} + \dots + T(n-k) \qquad n-k=0 \quad k=n$$
$$= c\sqrt{n} + c\sqrt{n-1} + c\sqrt{n-2} + \dots + c\sqrt{n-(k-1)} + c$$

UB: $\le \underbrace{c\sqrt{n} + c\sqrt{n} + c\sqrt{n} + \dots + c\sqrt{n}}_{k} = c\sqrt{n}\cdot n + c\sqrt{n} \in \Theta(n^{3/2})$

LB: $\ge \underbrace{c\sqrt{\frac{n}{2}} + c\sqrt{\frac{n}{2}} + c\sqrt{\frac{n}{2}} + c\sqrt{\frac{n}{2}} + \dots + c\sqrt{\frac{n}{2}}}_{k/2}$ $\quad \frac{k}{2} c\sqrt{\frac{n}{2}} = \frac{c}{\sqrt{2}\,2} n^{3/2} \in \Theta(n^{3/2})$

$$T(n) \in \Theta(n^{3/2})$$

b)



$Pr(k < n/2) = 1/2 \qquad ET(n \mid k < n/2) \le c\sqrt{n} + ET(n/2) \qquad k = n/2$

$Pr(k \ge n/2) = 1/2 \qquad ET(n \mid k \ge n/2) \le c\sqrt{n} + ET(n-1) \qquad k = n-1$

$$ET(n) = Pr(k < n/2)\,E(n \mid k < n/2) + Pr(k \ge n/2)\,E(n \mid k \ge n/2)$$
$$\le 1/2\,(c\sqrt{n} + ET(n/2)) + \tfrac{1}{2}(c\sqrt{n} + ET(n-1))$$
$$\le c\sqrt{n} + \tfrac{1}{2}ET(n/2) + \tfrac{1}{2}ET(n-1) \qquad ET(n-1) \le ET(n)$$

$ET(n) \le c\sqrt{n} + \frac{1}{2}ET(n/2) + \frac{1}{2}ET(n)$

$ET(n) \le c'\sqrt{n} + ET(n/2) \qquad c' = 2c$

$$\le c'\sqrt{n} + c'\sqrt{\tfrac{n}{2}} + c'\sqrt{\tfrac{n}{4}} + \dots + T(1)$$
$$\le \frac{1}{1-\frac{1}{\sqrt{2}}} c'\sqrt{n} + c \in \Theta(\sqrt{n})$$

LB: B.C. $k=1$

$$T(n) = cn + T(1) \in \Theta(n)$$

$$ET(n) \in \Theta(n)$$

$k = \frac{n}{2}$ every time

$Cn + 2T(\frac{n}{2})$ $\qquad k = \log_2(n)$

$\vdots \; Cn + Cn + Cn + \dots + 2^k T(\frac{n}{2^k})$

$\in \Theta(n \log_2(n))$

5. W.C. $k = n-1$ every time AND/OR $k = 1$ every time

$T(n) = Cn + T(n-1) + T(1)$ $\qquad T(n) = Cn + T(1) + T(n-1)$

$= C + Cn + T(n-1)$ $\qquad \vdots$

$\qquad \in \Theta(n^2)$

UB: $\leq C'n + T(n-1) \qquad C' = 2C$

$\leq C'n + C'(n-1) + T(n-2)$

$\leq C'n + C'(n-1) + C'(n-2) + T(n-3)$

$\leq C'n + C'(n-1) + C'(n-2) + \dots + C'(n-(k-1)) + T(n-k)$ $\qquad n-k=0 \quad n=k$

$\leq k C'n + T(0) \quad \in \Theta(n^2)$

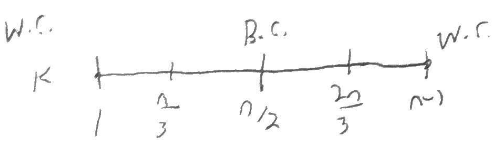
LB: $\geq Cn + T(n-1)$

$\geq Cn + C(n-1) + C(n-2) + \dots + C(n-(k-1)) + T(n-k)$

$\geq \frac{k}{2} C(\frac{n}{2}) \geq \frac{C}{8} n^2 \quad \in \Theta(n^2)$

$T(n) \in \Theta(n^2)$

b)



$P(k < n/3) = 1/3$

$P(\frac{n}{3} \leq k \leq \frac{2n}{3}) = \frac{1}{3}$

$P(k > 2n/3) = \frac{1}{3}$

$E(n \mid k < n/3) = Cn + ET(n-1)$

$E(n \mid \frac{n}{3} \leq k \leq \frac{2n}{3}) = Cn + ET(n/3) + ET(\frac{2n}{3})$

$E(n \mid k > 2n/3) = Cn + ET(n-1)$

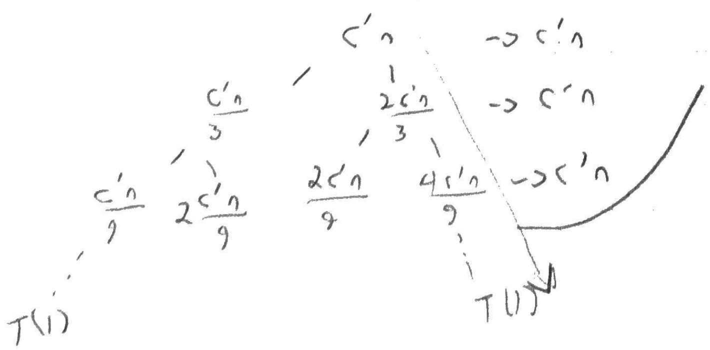
LB: B.C. $k = n/2$

$\in \Theta(n \log_2(n))$

UB: $ET(n) \leq Cn + \frac{2}{3} ET(n-1) + \frac{1}{3} ET(\frac{n}{3}) + ET(2n/3)$

$\leq C'n + ET(\frac{n}{3}) + ET(2n/3) \qquad C' = 3C$

$ET(n-1) \leq ET(n)$



$k = \log_{3/2}(n)$

$(\frac{2}{3})^k n = 1$

$ET(n) \leq C'n \log_{3/2}(n)$

$\in \Theta(n \log_2(n))$

$ET(n) \in \Theta(n \log_2(n))$