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Homework 2

1. $T(n) = \sum_{i=12}^{n^3} \sum_{j=7}^{\sqrt{i}\log_2(i)} c \qquad \sum_{i=12}^{n^3} c\left(\log_2(i)\sqrt{i} - 7 + 1\right)$

UB: $\leq \sum_{i=12}^{n^3} c\log_2(i)\sqrt{i} \qquad \leq \sum_{i=1}^{n^3} c\log_2(i)\sqrt{i} \leq \sum_{i=1}^{n^3} c\log_2(n^3)\sqrt{n^3}$

$\leq c\log_2(n^3)\sqrt{n^3}\, n^3 = 3c\log_2(n)\, n^{9/2} \in \Theta\left(n^{9/2}\log_2 n\right)$

LB: $\geq \sum_{i=12}^{n^3} \sum_{j=\frac{\sqrt{i}\log_2(i)}{2}}^{\sqrt{i}\log_2(i)} c \geq \sum_{j=12}^{n^3} \sqrt{i}\log_2(i)\frac{c}{2} \geq \sum_{i=\frac{n^3}{2}}^{n^3} \sqrt{i}\log_2(i)\frac{c}{2} \geq \sum_{i=\frac{n^3}{2}}^{n^3} \sqrt{\frac{n^3}{2}}\log_2\left(\frac{n^3}{2}\right)\frac{c}{2}$

$\geq \left(n^3 - \frac{n^3}{2} + 1\right)\sqrt{n^3/2}\,\log_2\left(\frac{n^3}{2}\right)\frac{c}{2} \geq \frac{n^3}{2}\cdot\frac{n^{3/2}}{\sqrt{2}}\cdot\frac{3c}{2}\left(\log_2(n) - 1\right) \geq \frac{n^{9/2}}{4\sqrt{2}}\cdot 3c\log_2(n) - \frac{3cn^{9/2}}{4\sqrt{2}}$

LB $\in \Theta\left(n^{9/2}\log_2 n\right)$

$T(n) \in \Theta\left(n^{9/2}\log_2 n\right)$

2. $T(n) = \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} \sum_{j=i}^{\lfloor n\sqrt{n}\rfloor} \sum_{k=6j^2}^{6j^2+25} c = \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} \sum_{j=i}^{\lfloor n\sqrt{n}\rfloor} c\left(6j^2 + 25 - 6j^2 + 1\right) = \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} \sum_{j=i}^{\lfloor n\sqrt{n}\rfloor} c\, 26$

$= \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} 26c\left(n^{3/2} - i + 1\right)$

UB: $\leq \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} 26c\left(n^{3/2} - i + 1\right) \leq \sum_{i=4n}^{\lfloor n\sqrt{n}\rfloor} 26c\left(n^{3/2} - 4n + 1\right)$

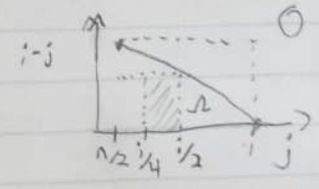
$\leq \left(n\sqrt{n} - 4n\right)\cdot 26c\left(n^{3/2} - 4n + 1\right) \leq c\left(n\sqrt{n}\right)\left(26n^{3/2} + 26\right)$

$\leq c\left(26n^3 + 26n^{3/2}\right) \in \Theta\left(n^3\right)$.

LB: $\geq \sum_{i=4n}^{n^{3/2}/2} 26c\left(n^{3/2} - i\right) \geq \sum_{i=4n}^{n\sqrt{n}/2} 26c\left(\frac{n^{3/2}}{2}\right) \geq \left(\frac{n^{3/2}}{2} - 4n\right) 13c\, n^{3/2}$

$\geq \frac{13}{2}c\,n^3 - 13\cdot 4\cdot c\cdot n^{3/2} \in \Theta\left(n^3\right)$

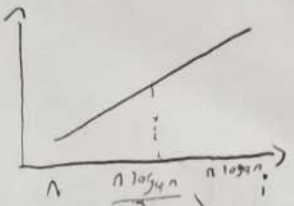
$T(n) \in \Theta\left(n^3\right)$



3. $\sum_{i=n}^{n\log_4(n)} \sum_{j=\lceil n/3\rceil}^{i} \sum_{k=j}^{i} c = \sum_{i=n}^{n\log_4(n)} \sum_{j=\lceil n/3\rceil}^{i} c\left(i - j + 1\right)$

UB: $\leq \sum_{i=n}^{n\log_4(n)} \sum_{j=1}^{i} c\left(i - 1 + 1\right) \leq \sum_{i=n}^{n\log_4(n)} \left(c\, i\right)\cdot\left(i - 1 + 1\right)$

$\leq \sum_{i=n}^{n\log_4(n)} c\, i^2$



$$\le \sum_{i=n}^{n\log_4(n)} c(n\log_4(n))^2 \le c(n^2\log_4^2(n))(n\log_4(n)-n+1)$$

$$\le cn^3\log_4^3 n + cn^2\log_4^2(n) \qquad \in \Theta(n^3\log_2^3(n))$$

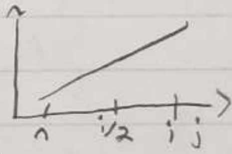
$$LB: \ge \sum_{i=n}^{n\log_4(n)}\sum_{j=\lfloor n/3 \rfloor}^{i/2} c(i-j) \ge \sum_{i=n}^{n\log_4(n)}\sum_{j=i/4}^{i/2}\frac{ci}{2} \ge \sum_{i=n}^{n\log_4(n)}\frac{ci}{2}\left(\frac{i}{2}-\frac{i}{4}+1\right)$$

$$\ge \sum_{i=n}^{n\log_4(n)}\frac{ci^2}{8} \ge \sum_{i=\frac{n\log_4(n)}{2}}^{n\log_4(n)}\frac{ci^2}{8} \ge \sum_{i=\frac{n\log_4(n)}{2}}^{n\log_4(n)}\frac{c}{8}\cdot\frac{n^2\log_4^2(n)}{4}$$

$$\ge \frac{c}{32}n^2\log_4^2(n)\left(\frac{n\log_4(n)}{2}\right) \ge \frac{c}{64}n^3\log_4^3(n) \in \Theta(n^3\log_2^3(n))$$

$$T(n)\in\Theta(n^3\log_2^3(n))$$

4. $$\sum_{i=n}^{n^2}\sum_{j=n}^{i}\sum_{k=j^2}^{j^2+\sqrt{i}} c = \sum_{i=n}^{n^2}\sum_{j=n}^{i} c(j^2+\sqrt{i}-j^2+1)$$



$$UB: \le \sum_{i=n}^{n^2}\sum_{j=1}^{i} c(\sqrt{i}+1) \le \sum_{i=n}^{n^2} ci^{1/2}+c(i-1+1) \le \sum_{i=n}^{n^2} ci^{3/2}+ci$$

$$\le \sum_{i=1}^{n^2} c(n^2)^{3/2}+cn^2 \le cn^3+cn^2(n^2-1+1) \le cn^5+cn^4 \in \Theta(n^5)$$

$$LB: \ge \sum_{i=n}^{n^2}\sum_{j=i/2}^{i} ci^{1/2} \ge \sum_{i=n}^{n^2} c\sqrt{i}\left(\frac{i}{2}\right) \ge \sum_{i=n}^{n^2}\frac{c}{2}i^{3/2}$$

$$\ge \sum_{i=n^2/2}^{n^2}\frac{c}{2\cdot 2^{3/2}}n^3 \ge \frac{c}{2\cdot 2^{3/2}}n^3\left(\frac{n^2}{2}\right) \ge \frac{c}{2\cdot 2^{3/2}\cdot 2}n^5 \in \Theta(n^5)$$

5.

inner loop: init $j = n^3$ after γ^{th} iteration $j = n^3 / 5^\gamma$

stop when $j = n$

$n^3 / 5^\gamma = n \quad 5^\gamma = n^2 \quad \gamma = \log_5(n^2) = 2\log_5(n)$

$T_{inner}(n) \in \Theta(\log_5(n)) \quad T_{inner}(n) = 2C\log_5(n)$

Outer loop: init $i = n$ after γ^{th} iteration $i = n + \gamma\sqrt{n}$

stop when $i = n\log_6(n)$

$n\log_6(n) = n + \gamma\sqrt{n} \quad \gamma = \sqrt{n}\log_6(n) - \sqrt{n}$

$T_{outer}(n) \in \Theta(\sqrt{n}\log_6(n)) \quad T_{outer}(n) = C_2(\sqrt{n}\log_6(n) - \sqrt{n})$

$T(n) = C_2(\sqrt{n}\log_6(n) - \sqrt{n})\, C_1 \cdot 2(\log_5(n)) = C'(\sqrt{n}\log_2^2(n) - \sqrt{n}\log_2(n))$

$\in \Theta(\sqrt{n}\log_2^2(n))$

6. inner loop: init $j = 17$ after γ^{th} iteration $j = 17 + 8\gamma$

stop when $j = 6i$

$6i = 8\gamma + 17 \quad \gamma = (6i - 17)/8$

$T_{inner}(n) = \frac{C}{8}(6i - 17)$

outer loop:

$T(n) = \frac{C}{8}(6 \cdot 6 - 17) + \frac{C}{8}(6 \cdot 6 \cdot 4^1 - 17) + \frac{C}{8}(6 \cdot 6 \cdot 4^2 - 17) + \dots +$

$\frac{C}{8}\left(6 \cdot \frac{n^{5/2}}{4^2} - 17\right) + \frac{C}{8}\left(6 \cdot \frac{n^{5/2}}{4} - 17\right) + \frac{C}{8}(6n^{5/2} - 17)$

UB: $\leq \frac{C}{8}6n^{5/2} + \frac{C}{8}6 \cdot \frac{n^{5/2}}{4} + \frac{C}{8} \cdot 6 \cdot \frac{n^{5/2}}{4^2} + \dots + \frac{C}{8}(6^2 \cdot 4^2) + \frac{C}{8}(6^2 \cdot 4) + \frac{C}{8}(6^2)$

$\leq \frac{C}{8}6n^{5/2}\left[1 + 1/4 + 1/4^2 + \dots + 6 \cdot 4^2/n^{5/2} + 6 \cdot 4/n^{5/2} + 6/n^{5/2}\right]$

$\approx \frac{1}{1 - \frac{1}{4}} \cdot \frac{C}{8}6n^{5/2} \in \Theta(n^{5/2})$

LB: $\geq \frac{C}{8}(6n^{5/2} - 17) + \frac{-17}{8}C - \frac{17}{8}C + \dots -17/8\,C$

$\geq \frac{C}{8}6n^{5/2}\left[1 - 17/6n^{5/2} + \dots + -17/6n^{5/2}\right] \in \Theta(n^{5/2})$

$T(n) \in \Theta(n^{5/2})$

7.

$T_{inner}(n)$ init $j=3$ after r^{th} iteration $j = 3 \cdot 1.5^r$

stop when $j = 2i^2$

$2i^2 = 3 \cdot 1.5^r$ $\frac{2}{3} i^2 = 1.5^r$

$r = \log_{1.5}(\frac{2}{3} i^2) = 2\log_{1.5}(\frac{2}{3} i)$

$T_{inner}(n) = C\, 2 \log_{1.5}(\frac{2}{3} i)$

$\sum_{i=6}^{n^2} 2c \log_{1.5}(\frac{2}{3} i)$ UB: $\leq \sum_{i=1}^{n^2} 2c \log_{1.5}(2i) \leq \sum_{i=1}^{n^2} 4c \log_{1.5}(2n)$

$\leq (n^2 - 1 + 1)\, 4c \log_{1.5}(2n) \leq 4c\, n^2 \log_{1.5}(2n) \in \Theta(n^2 \log_2(n))$

LB:

$\geq \sum_{i=\frac{n^2}{2}}^{n^2} 2c \log_{1.5}(\frac{2}{3} i) \geq \sum_{i=n^2/2}^{n^2} 2c \log_{1.5}(n^2/3)$

$\geq (n^2 - n^2/2 + 1)\, 2c \cdot 2\log_{1.5}(n/3) \geq 2c\, n^2 \log_{1.5}(n/3) \in \Theta(n^2 \log_2(n))$

$T(n) \in \Theta(n^2 \log_2(n))$

8. $T_{inner} = c\, i^{1/2}$ $\sum_{j=1}^{i^{1/2}} c$

$T(n) = c\sqrt{5} + c\sqrt{15} + c\sqrt{45} + \ldots + c\sqrt{\frac{n^3}{3^2}} + c\sqrt{\frac{n^3}{3}} + c\sqrt{n^3}$

$= c\sqrt{n^3}\left[1 + 1/\sqrt{3} + 1/\sqrt{3^2} + \ldots + \sqrt{45}/\sqrt{n^3} + \sqrt{15}/\sqrt{n^3} + \sqrt{5}/\sqrt{n^3}\right]$

$\approx \frac{1}{1 - 1/\sqrt{3}}\, c\, n^{1.5} \in \Theta(n^{1.5})$

9.

$T_{inner}(n)$: init $j = n^2$ after γ^{th} iteration $j = n^2 - \gamma i$

stop when $j = 6$

$6 = n^2 - \gamma i \qquad \gamma = (n^2-6)/i$

outer:

$$T(n) = c\,\frac{n^2-6}{4} + c\,\frac{n^2-6}{4\cdot 5^1} + c\,\frac{n^2-6}{4\cdot 5^2} + \ldots + c\,\frac{(n^2-6)}{7n/5^2} + c\,\frac{(n^2-6)}{7n/5} + c\,\frac{(n^2-6)}{7n}$$

$$= c\,\frac{(n^2-6)}{4}\left[1 + \frac{1}{5} + {}^1/_{5^2} + \ldots + \frac{4\cdot 5^2}{7n} + \frac{4\cdot 5^1}{7n} + \frac{4}{7n}\right]$$

$$\approx \frac{1}{1-{}^1/_5}\, c\left(\frac{n^2-6}{4}\right) \quad \in \Theta(n^2)$$

10.

$T_{inner}(n)$: init $j = n^{1.5}$ after γ^{th} iteration $j = n^{1.5}/7^{\gamma}$

stop when $j = n^{.5}$

$n^{.5} = n^{1.5}/7^{\gamma} \qquad 7^{\gamma} = n \qquad \gamma = \log_7(n)$

$T_{outer}(n)$: init $i = 1$ after γ^{th} iteration $i = 1 \cdot 3.5^{\gamma}$

stop when $i = n^2$

$n^2 = 3.5^{\gamma} \qquad \gamma = 2\log_{3.5}(n)$

$T(n) = ?$

$c_2\, 2\log_{3.5}(n) \cdot c_1 \log_7(n)$

$\in \Theta(\log_2^2(n))$