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Homework 3

1. $T(<10) = C$ $\quad$ $n<10 \in \Theta(1)$, so using 1 as base case as opposed to 10

line 2: C $\quad$ $\left[\left(\frac{5}{8}\right)^k n = 1\right]\left(\frac{8}{5}\right)^k$

line 3: $T(5n/8)$ $\quad$ $n = \left(\frac{8}{5}\right)^k$ $\quad$ $k = \log_{8/5}(n)$

Aside:

$T(5/8\, n) = c + T\left(\left(\frac{5}{8}\right)^2 n\right)$

$T\left(\left(\frac{5}{8}\right)^2 n\right) = c + T\left(\left(\frac{5}{8}\right)^3 n\right)$

$$
\begin{aligned}
T(n) &= c + T(5n/8) \\
&= c + \left(c + T\left(\left(\tfrac{5}{8}\right)^2 n\right)\right) = 2c + T\left(\left(\tfrac{5}{8}\right)^2 n\right) \\
&= 2c + \left(c + T\left(\left(\tfrac{5}{8}\right)^3 n\right)\right) = 3c + T\left(\left(\tfrac{5}{8}\right)^3 n\right) \\
&\vdots \\
&= \underbrace{c+c+c+\ldots+c}_{k} + T\left(\left(\tfrac{5}{8}\right)^k n\right) = kc + T\left(\left(\tfrac{5}{8}\right)^k n\right) \\
&= \log_{8/5}(n)\, c + T\left(\left(\tfrac{5}{8}\right)^{\log_{8/5}(n)} n\right) = \log_{8/5}(n)\, c + \cancel{T(n/n)}^{\,c} \\
&\in \Theta(\log_2(n))
\end{aligned}
$$

2. $T(<20) = c$ $\quad$ $T(1) = C$

$T_{inner} = \sum_{j=1}^{n/2} c_1 = \frac{n}{2} c_1$ $\quad$ $T_{outer}: \sum_{i=1}^{5} T\left(\frac{n}{5}\right) + \frac{n}{2} c_1$

Aside:

$T\left(\frac{n}{5}\right) = \frac{5}{2\cdot 5} cn + 5T\left(\frac{n}{5^2}\right)$

$T\left(\frac{n}{5^2}\right) = \frac{5}{2\cdot 5^2} cn + 5T\left(\frac{n}{5^3}\right)$

$$
\begin{aligned}
T(n) &= \tfrac{5}{2} cn + 5T\left(\tfrac{n}{5}\right) \\
&= \tfrac{5}{2} cn + 5\left(\tfrac{5}{2\cdot 5} cn + 5T\left(\tfrac{n}{5^2}\right)\right) = \tfrac{2\cdot 5}{2} cn + 5^2 T(n/5^2) \\
&= 2\cdot 5/2\, cn + 5^2\left(5/2\cdot 5^2\, cn + 5T(n/5^3)\right) = 3\cdot\tfrac{5}{2} cn + 5^3 T(n/5^3) \\
&\vdots \qquad n/5^k = 1 \quad k = \log_5(n) \\
&= k\cdot\tfrac{5}{2} cn + 5^k T(n/5^k) \\
&= \log_5(n)\cdot\tfrac{5}{2} cn + 5^{\log_5(n)} T\left(n/5^{\log_5(n)}\right) = \log_5(n)\cdot\tfrac{5}{2} cn + n\,\cancel{T(1)}^{\,c} \\
&\in \Theta(\log_2(n)\, n)
\end{aligned}
$$

3. $T(<20) = C$ $\quad$ $T(1) = c$ $\quad$ loop: $\sum_{i=1}^{\sqrt{n}} c = C\sqrt{n}$

Aside:

$T\left(\frac{n}{3}\right) = C\sqrt{\frac{n}{3}} + T(n/3^2)$

$T(n/3^2) = C\sqrt{n/3^2} + T(n/3^3)$

$n/3^k = 1$ $\quad$ $k = \log_3(n)$

$$
\begin{aligned}
T(n) &= C\sqrt{n} + T(n/3) \\
&= C\sqrt{n} + C\sqrt{\tfrac{n}{3}} + T(n/3^2) \\
&= C\sqrt{n} + C\sqrt{n/3} + C\sqrt{n/3^2} + T(n/3^3) \\
&\vdots \\
&= C\sqrt{n} + C\sqrt{n/3} + \ldots + C\sqrt{n/3^{k-1}} + T(n/3^k) \\
&= C\sqrt{n}\left[1 + \tfrac{1}{\sqrt{3}} + \tfrac{1}{\sqrt{3}} + \ldots + \tfrac{1}{3^{k-1}}\right] + T(n/3^k) \approx \tfrac{1}{1-1/\sqrt{3}} C\sqrt{n} + \cancel{T(1)}^{\,c} \in \Theta(n^{.5})
\end{aligned}
$$

4. $T(<20) = c \quad T(1) = c \qquad \sum_{i=1}^{6}\left(\frac{n}{3}c' + T(\tfrac{n}{6})\right) \qquad 2nc' + 6\,T(\tfrac{n}{6})$

$T_{inner} = \sum_{i=1}^{\lceil n/3 \rceil} c' = \frac{n}{3}c' \qquad$ Let $2c' = c$

Aside:

$T(\tfrac{n}{6}) = c\cdot\frac{n}{6} + 6\,T(n/6^2)$

$T(n/6^2) = cn/6^2 + 6\,T(n/6^3)$

$T(n) = cn + 6\,T(n/6)$

$= cn + 6[\,c\cdot n/6 + 6\,T(n/6^2)] = cn + cn + 6^2 T(n/6^2)$

$= 2cn + 6^2[\,cn/6^2 + 6\,T(n/6^3)] = 3cn + 6^3 T(n/6^3)$

$\vdots \qquad n/6^k = 1 \qquad k = \log_6(n)$

$= kcn + 6^k T(n/6^k)$

$= \log_6(n)\,cn + 6^{\log_6(n)}\,T(n/6^{\log_6(n)}) = \log_6(n)\,cn + n\,\cancel{T(1)}^{\,c}$

$\in \Theta(\log_2(n)\cdot n)$

5. $T(<10) = c \rightarrow T(1) = c$

$T(n) = c + T(\tfrac{2}{3}n) \qquad T(\tfrac{2n}{3}) = c + T((\tfrac{2}{3})^2 n)$

$= c + c + T((\tfrac{2}{3})^2 n)$

$= c + c + c + T((\tfrac{2}{3})^3 n) \qquad (\tfrac{2}{3})^k n = 1 \qquad (\tfrac{3}{2})^k = n$

$\vdots \qquad k = \log_{3/2} n$

$= kc + T((\tfrac{2}{3})^k n)$

$= \log_{3/2}(n)\cdot c + \cancel{T(1)}^{\,c} \qquad \in \Theta(\log_2(n))$

6. $T(<20) = c \rightarrow T(1) = c \rightarrow T(0) = c$

$i = 8$ after γ^{th} $i = 8\cdot 5^{\gamma}$ stop when $i = n$

$n = 8\cdot 5^{\gamma} \qquad \gamma = \log_5(n/8)$

$T_{loop}(n) = c\log_5(n/8) \in \Theta(\log_2(n))$

$T(n-4) = c\log_2(n-4) + T(n-4\cdot 2)$

$T(n) = c\log_2(n) + T(n-4)$

$T(n-4\cdot 2) = c\log_2(n-4\cdot 2) + T(n-4\cdot 3)$

$= c\log_2(n) + c\log_2(n-4) + T(n-4\cdot 2)$

$= c\log_2(n) + c\log_2(n-4) + c\log_2(n-4\cdot 2) + T(n-4\cdot 3) \qquad T(n-4\cdot 3) = c\log_5(\frac{n-4\cdot 3}{8}) + T(n-4\cdot 4)$

$\vdots$

$= c\log_2(n) + c\log_2(n-4) + c\log_2(n-4\cdot 2) + T(n-4k)$

UB: $\leq c\log_2(n)[1+1+1+1] + T(0) = \frac{n}{4}c\log_2(n) \in O(n\log_2(n))$

LB: $\geq c\log_2(\frac{n}{2}) + c\log_2(\frac{n}{2}) + \dots + c\log_2(\frac{n}{2}) + T(0) = \frac{n}{8}\log_2(n)\,c \in \Omega(n\log_2(n)) \qquad n - 4k = 0$

$k = n/4$

$T(n) \in \Theta(n\log_2(n))$

$\frac{n-8}{4} = 24$

7. $T(<20) = c \rightarrow T(1) = c$

start $i = n-3$ after γ^{th} iteration $i = n-3-4\gamma$

Stop when $i = 5$ $\quad 5 = n-3-4\gamma \quad \gamma = (n-8)/4$

$$T(n) = \sum_{\gamma=1}^{(n-8)/4} T(i) = \sum_{\gamma=1}^{(n-8)/4} T(n-3-4\gamma) = \underbrace{T(n-7) + T(n-11)} + \ldots + T(5) + T$$

LB: $\geq T(n-7) + T(n-11) \geq 2\,T(n-11)$

$\geq 2 \cdot 2\, T(n-11 \cdot 2) \geq 2 \cdot 2 \cdot 2\, T(n-11 \cdot 3)$

$\vdots$ $\quad n - 11k = 0 \quad \frac{n}{11} = k$

$\geq 2^k\, T(n-11 \cdot k)$

$2^{n/11}\, \cancel{T(0)}^{c} \in \Omega(2^{n/11})$

8 $\quad T(<20) = c \rightarrow T(1) = c$

$$\sum_{i=1}^{\lfloor n/2 \rfloor} c' = \frac{n}{2} c' \qquad \text{let } \frac{c'}{2} = c$$

$T(n) = cn + T(\frac{n}{5}) + T(\frac{n}{5})$

$= cn + 2T(\frac{n}{5})$ $\qquad T(\frac{n}{5}) = c \cdot \frac{n}{5} + 2T(n/5^2)$

$= cn + 2[c \cdot \frac{n}{5} + 2T(n/5^2)] = cn + 2cn/5 + 2^2 T(n/5^2)$

$= cn + 2cn/5 + 2^2[c \cdot \frac{n}{5^2} + 2T(n/5^3)]$ $\qquad T(\frac{n}{5^2}) = c \cdot \frac{n}{5^2} + 2T(n/5^3)$

$\hookrightarrow = cn + \frac{2cn}{5} + 2^2 cn/5^2 + 2^3 T(n/5^3)$ $\qquad n/5^k = 1 \quad k = \log_5(n)$

$\vdots$

$= \frac{2^0 cn}{5^0} + \frac{2^1 cn}{5^1} + \frac{2^2 cn}{5^2} + \ldots + \frac{2^{k-2} cn}{5^{k-2}} + \frac{2^{k-1} cn}{5^{k-1}} + 2^k T(n/5^k)$

$cn[1 + \frac{2}{5} + (\frac{2}{5})^2 + \ldots + (\frac{2}{5})^{k-2} + (\frac{2}{5})^{k-1}] + 2^k T(n/5^k)$

$\approx \frac{1}{1-\frac{2}{5}} cn + 2^{\log_5(n)} \cancel{T(1)}^{c} = \frac{5}{3} cn + 2^{\log_5(n)} c$

$2^{\log_5(n)} = n^{\log_5(2)}$

$\in \Theta(n)$

9. $T(< 20) = c \rightarrow T(1) = c$ $\qquad T\left(\frac{2n}{7}\right) \qquad T(5n/7)$

$\sum_{i=1}^{\lfloor n/2 \rfloor} c' = \frac{n}{2} c' \qquad \frac{c'}{2} = c$

$T(n) =$ $\qquad cn \qquad \rightarrow cn$

$T\left(\frac{2n}{7}\right) = \frac{2}{7}cn \qquad T\left(\frac{5n}{7}\right) = \frac{5}{7}cn \qquad \rightarrow cn$

$T\left(\left(\frac{2}{7}\right)^2 n\right) = c\,4n/49 \qquad T\left(\frac{10}{49}n\right) = 10/49 \cdot cn \qquad T\left(\frac{10n}{49}\right) = \frac{10cn}{49} \qquad T\left(\frac{25}{49}n\right) = \frac{25}{49}cn \qquad \rightarrow cn$

$T(1) \qquad T(1)$

$\left(\frac{5}{7}\right)^k n = 1$

$k = \log_{7/5}(n)$

Shortest path to $T(1)$: $\log_{7/2}(n)$

height: $\log_{7/5}(n)$

$cn \log_{7/2}(n) \leq T(n) \leq cn \log_{7/5}(n) \qquad \in \Theta(n \log_2(n))$

10. $T(< 20) = c \rightarrow T(1) = c$

$T_{inner}: \sum_{j=1}^{\lfloor n/3 \rfloor} c' = n/3\, c' \qquad c'' = \frac{c'}{3} \qquad T_{outer}: \sum_{i=1}^{\lfloor n/2 \rfloor} c'' n = \frac{c''}{2} n^2$

$c''/2 = c$

$T(n) = cn^2 + T\left(\frac{2}{3}n\right) + T\left(\frac{2}{3}n\right) = cn^2 + 2\,T\left(\frac{2}{3}n\right)$

$= cn^2 + 2\left[c\left(\frac{2}{3}n\right)^2 + 2T\left(\left(\frac{2}{3}\right)^2 n\right)\right] = cn^2 + 2\cdot\left(\frac{2}{3}\right)^2 cn^2 + 2^2 T\left(\left(\frac{2}{3}\right)^2 n\right) \qquad T\left(\frac{2}{3}n\right) = c\left(\frac{2}{3}n\right)^2 + 2T\left(\left(\frac{2}{3}\right)^2 n\right)$

$= cn^2 + 2\cdot\left(\frac{2}{3}\right)^2 cn^2 + 2^2\left[c\left(\left(\frac{2}{3}\right)^2 n\right)^2 + 2T\left(\left(\frac{2}{3}\right)^3 n\right)\right] \qquad T\left(\left(\frac{2}{3}\right)^2 n\right) = c\left(\left(\frac{2}{3}\right)^2 n\right)^2 + 2T\left(\left(\frac{2}{3}\right)^3 n\right)$

$\hookrightarrow = cn^2 + 2\cdot\left(\frac{2}{3}\right)^2 cn^2 + 2^2\left(\frac{2}{3}\right)^{2\cdot 2} cn^2 + 2^3 T\left(\left(\frac{2}{3}\right)^3 n\right)$

$\vdots$

$2^0 \cdot cn^2 + 2^1 \cdot \left(\frac{2}{3}\right)^2 cn^2 + 2^2 \cdot \left(\frac{2}{3}\right)^{2\cdot 2} cn^2 + \ldots + 2^{(k-1)} \cdot \left(\frac{2}{3}\right)^{2\cdot(k-1)} cn^2 + 2^k T\left(\left(\frac{2}{3}\right)^k n\right)$

$cn^2\left[2^0 \cdot 1 + 2\cdot\frac{4}{9} + 2^2\cdot\frac{16}{81} + \ldots + 2^{\log_{3/2}(n)-1}\left(\frac{2}{3}\right)^{2\cdot(\log_{3/2}(n)-1)}\right] + 2^{\log_{3/2}(n)} T(1)$

$\approx \frac{1}{1 - 8/9} cn^2 + 2^{\log_{3/2}(n)} c$

$\frac{2}{3}^k n = 1$

$\in \Theta(n^2)$

$k = \log_{3/2}(n)$