Unit 2: Probability and distributions

3. Normal distribution

Sta 101 - Spring 2015

Duke University, Department of Statistical Science

February 2, 2015

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Slides posted at http://bitly.com/sta101sp15

1. Housekeeping

2. Main ideas

    1. Two types of probability distributions: discrete and continuous

    2. Normal distribution is unimodal, symmetric, and follows the 69-95-99.7 rule

    3. Z scores serve as a ruler for any distribution

    4. Z distribution is normal with $\mu = 0$ and $\sigma = 1$

    5. Normally distributed data plot as a straight line on the normal probability plot

3. Summary

4. Exercise [time permitting]

- Peer evaluation 1 by Friday 11:59pm
- Office hours:
  - currently MTWR 3-4pm
  - propose changing to TR 3-5pm, is this better?

Clicker question

(a) No, keep OH at MTWR 3-4pm
(b) Change to TR 3-5pm

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- A *discrete probability distribution* lists all possible events and the probabilities with which they occur
  - The events listed must be disjoint
  - Each probability must be between 0 and 1
  - The probabilities must total 1

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- A *discrete probability distribution* lists all possible events and the probabilities with which they occur
  - The events listed must be disjoint
  - Each probability must be between 0 and 1
  - The probabilities must total 1
- A *continuous probability distribution* differs from a discrete probability distribution in several ways:
  - The probability that a continuous random variable will equal to any specific value is zero.
  - As such, they cannot be expressed in tabular form.
  - Instead, we use an equation or a formula to describe its distribution via a probability density function (pdf).
  - We can calculate the probability for ranges of values the random variable takes (area under the curve).

Discrete:
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Outcome	X	P(X)
Win \$10 (black aces)	10	$\frac{2}{52}$
Win \$8 (red aces: 10 - 2)	8	$\frac{2}{52}$
Lose \$2 (non-ace reds)	-2	$\frac{24}{52}$
No win / loss	0	$\frac{24}{52}$
		$\frac{52}{52} = 1$

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Continuous:
Distribution of weekly expenditures of entertainment for a family is right skewed with median of \$70.

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- *68-95-99.7 Rule:*
  - about 68% of the distribution falls within 1 SD of the mean
  - about 95% falls within 2 SD of the mean
  - about 99.7% falls within 3 SD of the mean
  - it is possible for observations to fall 4, 5, or more standard deviations away from the mean, but this is very rare if the data are nearly normal

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- While most variables are nearly normal, but none are exactly normal

Clicker question

Speeds of cars on a highway are normally distributed with mean 65 miles / hour. The minimum speed recorded is 48 miles / hour and the maximum speed recorded is 83 miles / hour. Which of the following is most likely to be the standard deviation of the distribution?

(a) -5

(b) 5

(c) 10

(d) 15

(e) 30

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(a) -5 → *SD cannot be negative*

(b) *5* $\rightarrow 65 \pm (3 \times 5) = (50, 80)$

(c) 10 $\rightarrow 65 \pm (3 \times 10) = (35, 95)$

(d) 15 $\rightarrow 65 \pm (3 \times 15) = (20, 110)$

(e) 30 $\rightarrow 65 \pm (3 \times 30) = (-25, 155)$

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- Observations with $|Z| > 2$ are usually considered *unusual*

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4. Z distribution is normal with $\mu = 0$ and $\sigma = 1$

- Linear transformations of normally distributed random variable will also be normally distributed.
- Hence, if

$$\mathrm{Z} = \frac{\mathrm{X} - \mu}{\sigma}, \text{ where } \mathrm{X} \sim \mathrm{N}(\mu, \sigma),$$

then

$$\mathrm{Z} \sim \mathrm{N}(0, 1)$$

- Z distribution is a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$ (unit normal distribution)

Clicker question

Scores on a standardized test are normally distributed with a mean of 100 and a standard deviation of 20. If these scores are converted to standard normal Z scores, which of the following statements will be correct?

(a) The mean will equal 0, but the median cannot be determined.
(b) The mean of the standardized Z-scores will equal 100.
(c) The mean of the standardized Z-scores will equal 5.
(d) Both the mean and median score will equal 0.
(e) A score of 70 is considered unusually low on this test.

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Application exercise: 2.3 Normal distribution

See the course website for instructions.

Clicker question

Which of the following is false?

(a) Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

(b) Majority of Z scores in a right skewed distribution are negative.

(c) In a normal distribution, Q1 and Q3 are more than one SD away from the mean.

(d) Regardless of the shape of the distribution (symmetric vs. skewed) the Z score of the mean is always 0.

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- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis

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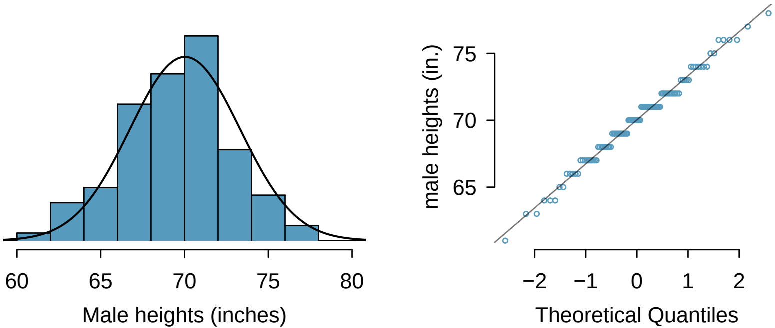
- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis
- If there is a one-to-one relationship between the data and the theoretical quantiles, then the data follow a nearly normal distribution
- Since a one-to-one relationship would appear as a straight line on a scatter plot, the closer the points are to a perfect straight line, the more confident we can be that the data follow the normal model
- Constructing a normal probability plot requires calculating percentiles and corresponding Z-scores for each observation, which is tedious. Therefore we generally rely on software when making these plots

A histogram and *normal probability plot* of a sample of 100 male heights.



Why do the points on the normal probability have jumps?

We construct a normal probability plot for the heights of a sample of 100 men as follows:

1. Order the observations.
2. Determine the percentile of each observation in the ordered data set.
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4. Create a scatterplot of the observations (vertical) against the Z scores (horizontal)

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Observation i	1	2	3	$\cdots$	100
x_i	61	63	63	$\cdots$	78
Percentile , $i/(n+1)$	0.99%	1.98%	2.97%	$\cdots$	99.01%
z_i	-2.33	-2.06	-1.89	$\cdots$	2.33

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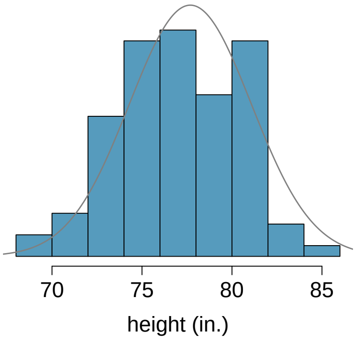
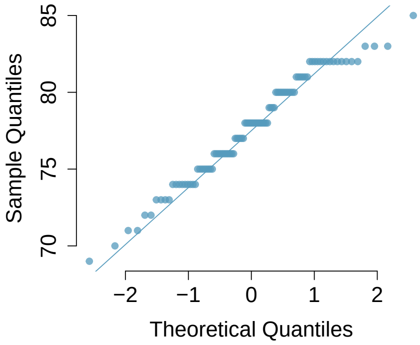
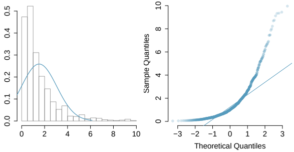
How are the Z scores corresponding to each percentile determined?

Below is a histogram and normal probability plot for the heights of Duke men's basketball players (from 1990s and 2000s). Do these data appear to follow a normal distribution?

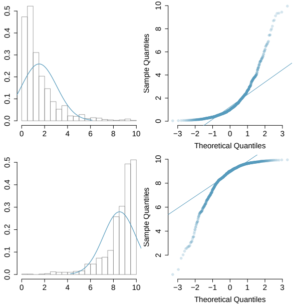




Source: GoDuke.com



Right Skew - Points bend up and to the left



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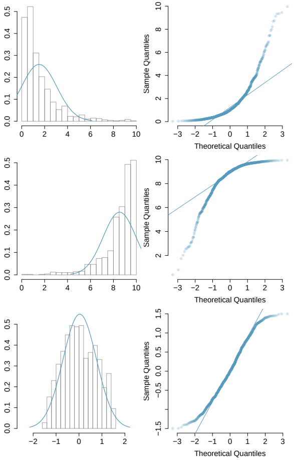
Left Skew - Points bend down and to the right

Normal probability plot and skewness



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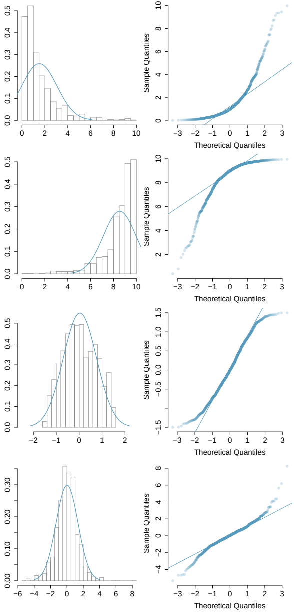
Skinny Tails - S shaped-curve indicating shorter than normal tails (narrower, less variable, than expected)



Right Skew - Points bend up and to the left

Left Skew - Points bend down and to the right

Skinny Tails - S shaped-curve indicating shorter than normal tails (narrower, less variable, than expected)

Fat Tails - Curve starting below the normal line, bends to follow it, and ends above it (wider, more variable, than expected)

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At a pharmaceutical factory the amount of the active ingredient which is added to each pill is supposed to be 36 mg. The amount of the active ingredient added follows a nearly normal distribution with a standard deviation of 0.11 mg. Once every 30 minutes a pill is selected from the production line, and its composition is measured precisely. We know that the failure rate of the quality control is 3% at this factory. What are the bounds of the acceptable amount of the active ingredient?