

Unit 2: Probability and distributions

3. Normal distribution

Sta 101 - Spring 2015

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Slides posted at <http://bitly.com/sta101sp15>

- ▶ Peer evaluation 1 by Friday 11:59pm
- ▶ Office hours?

1

1. Two types of probability distributions: discrete and continuous

- ▶ A *discrete probability distribution* lists all possible events and the probabilities with which they occur
 - The events listed must be disjoint
 - Each probability must be between 0 and 1
 - The probabilities must total 1
- ▶ A *continuous probability distribution* differs from a discrete probability distribution in several ways:
 - The probability that a continuous random variable will equal to any specific value is zero.
 - As such, they cannot be expressed in tabular form.
 - Instead, we use an equation or a formula to describe its distribution via a probability density function (pdf).
 - We can calculate the probability for ranges of values the random variable takes (area under the curve).

2

Examples

Discrete:

In a card game if you draw an ace from a well-shuffled full deck you win \$10. If you draw a red card, you lose \$2.

Outcome	X	P(X)
Win \$10 (black aces)	10	$\frac{2}{52}$
Win \$8 (red aces: 10 - 2)	8	$\frac{2}{52}$
Lose \$2 (non-ace reds)	-2	$\frac{24}{52}$
No win / loss	0	$\frac{24}{52}$
		$\frac{52}{52} = 1$

Continuous:

Distribution of weekly expenditures of entertainment for a family is right skewed with median of \$70.

3

2. Normal distribution is unimodal, symmetric, and follows the 68-95-99.7 rule

$$N(\mu, \sigma)$$

- ▶ Unimodal and symmetric (bell shaped) that follows very strict guidelines about how variably the data are distributed around the mean
- ▶ **68-95-99.7 Rule:**
 - about 68% of the distribution falls within 1 SD of the mean
 - about 95% falls within 2 SD of the mean
 - about 99.7% falls within 3 SD of the mean
 - it is possible for observations to fall 4, 5, or more standard deviations away from the mean, but this is very rare if the data are nearly normal
- ▶ While most variables are nearly normal, but none are exactly normal

4

Clicker question

Speeds of cars on a highway are normally distributed with mean 65 miles / hour. The minimum speed recorded is 48 miles / hour and the maximum speed recorded is 83 miles / hour. Which of the following is most likely to be the standard deviation of the distribution?

- (a) -5
- (b) 5
- (c) 10
- (d) 15
- (e) 30

5

3. Z scores serve as a ruler for any distribution

$$Z = \frac{\text{obs} - \text{mean}}{\text{SD}}$$

- ▶ Z score: number of standard deviations it falls above or below the mean
- ▶ Defined for distributions of any shape, but only when the distribution is normal can we use Z scores to calculate percentiles
- ▶ Observations with $|Z| > 2$ are usually considered *unusual*

6

4. Z distribution is normal with $\mu = 0$ and $\sigma = 1$

- ▶ Linear transformations of normally distributed random variable will also be normally distributed.
- ▶ Hence, if

$$Z = \frac{X - \mu}{\sigma}, \text{ where } X \sim N(\mu, \sigma),$$

then

$$Z \sim N(0, 1)$$

- ▶ Z distribution is a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$ (unit normal distribution)

7

Clicker question

Scores on a standardized test are normally distributed with a mean of 100 and a standard deviation of 20. If these scores are converted to standard normal Z scores, which of the following statements will be correct?

- (a) The mean will equal 0, but the median cannot be determined.
- (b) The mean of the standardized Z-scores will equal 100.
- (c) The mean of the standardized Z-scores will equal 5.
- (d) Both the mean and median score will equal 0.
- (e) A score of 70 is considered unusually low on this test.

8

Clicker question

Which of the following is false?

- (a) Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.
- (b) Majority of Z scores in a right skewed distribution are negative.
- (c) In a normal distribution, Q1 and Q3 are more than one SD away from the mean.
- (d) Regardless of the shape of the distribution (symmetric vs. skewed) the Z score of the mean is always 0.

10

Application exercise: 2.3 Normal distribution

See the course website for instructions.

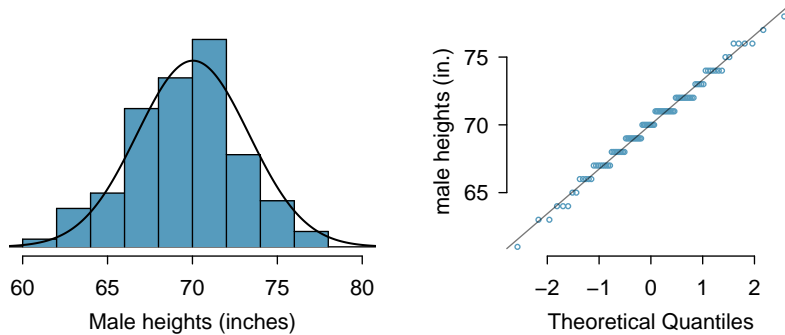
9

Anatomy of a normal probability plot

- ▶ Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis
- ▶ If there is a one-to-one relationship between the data and the theoretical quantiles, then the data follow a nearly normal distribution
- ▶ Since a one-to-one relationship would appear as a straight line on a scatter plot, the closer the points are to a perfect straight line, the more confident we can be that the data follow the normal model
- ▶ Constructing a normal probability plot requires calculating percentiles and corresponding Z-scores for each observation, which is tedious. Therefore we generally rely on software when making these plots

11

A histogram and *normal probability plot* of a sample of 100 male heights.



Why do the points on the normal probability have jumps?

12

We construct a normal probability plot for the heights of a sample of 100 men as follows:

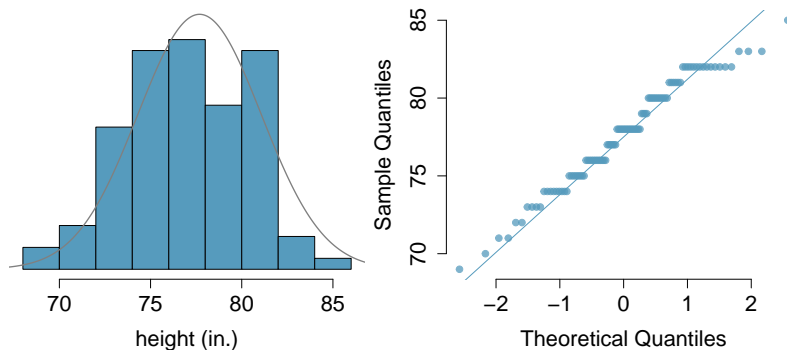
1. Order the observations.
2. Determine the percentile of each observation in the ordered data set.
3. Identify the Z score corresponding to each percentile.
4. Create a scatterplot of the observations (vertical) against the Z scores (horizontal)

Observation i	1	2	3	...	100
x_i	61	63	63	...	78
Percentile, $i/(n+1)$	0.99%	1.98%	2.97%	...	99.01%
z_i	-2.33	-2.06	-1.89	...	2.33

How are the Z scores corresponding to each percentile determined?

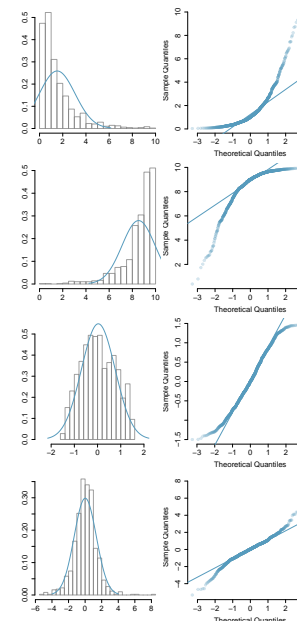
13

Below is a histogram and normal probability plot for the heights of Duke men's basketball players (from 1990s and 2000s). Do these data appear to follow a normal distribution?



Source: GoDuke.com

14



Right Skew - Points bend up and to the left

Left Skew - Points bend down and to the right

Skinny Tails - S shaped-curve indicating shorter than normal tails (narrower, less variable, than expected)

Fat Tails - Curve starting below the normal line, bends to follow it, and ends above it (wider, more variable, than expected)

15

1. Two types of probability distributions: discrete and continuous
2. Normal distribution is unimodal, symmetric, and follows the 69-95-99.7 rule
3. Z scores serve as a ruler for any distribution
4. Z distribution is normal with $\mu = 0$ and $\sigma = 1$
5. Normally distributed data plot as a straight line on the normal probability plot

At a pharmaceutical factory the amount of the active ingredient which is added to each pill is supposed to be 36 mg. The amount of the active ingredient added follows a nearly normal distribution with a standard deviation of 0.11 mg. Once every 30 minutes a pill is selected from the production line, and its composition is measured precisely. We know that the failure rate of the quality control is 3% at this factory. What are the bounds of the acceptable amount of the active ingredient?