COMP9020

Foundations of Computer Science

Lecture 4: Set Theory

Outline

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Defining Sets

1. Explicitly list elements
2. Take a subset of an existing set by restricting the elements
3. Build up from existing sets using Set Operations

Set Operations

Definition

$A \cup B$ – **union** (a or b):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

$A \cap B$ – **intersection** (a and b):

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

A^c – **complement** (with respect to a universal set $\mathcal{U}$):

$$A^c = \{x : x \in \mathcal{U} \text{ and } x \notin A\}.$$

We say that A, B are **disjoint** if $A \cap B = \emptyset$

Set Operations

Other set operations

Definition

$A \setminus B$ – **set difference**, relative complement (a but not b):

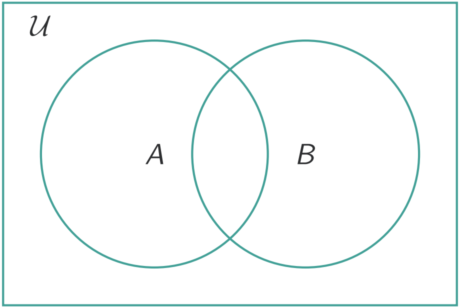
$$A \setminus B = A \cap B^c$$

$A \oplus B$ – **symmetric difference** (a and not b or b and not a; also known as a or b exclusively; a xor b):

$$A \oplus B = (A \setminus B) \cup (B \setminus A)$$

Venn Diagrams

A **Venn Diagram** is a simple graphical approach to visualize the basic set operations.



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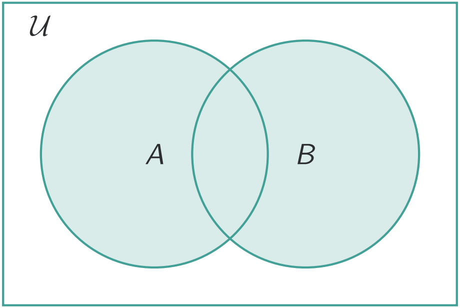

$A \cup B$

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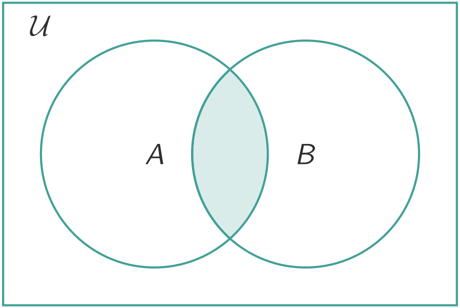

$A \cap B$

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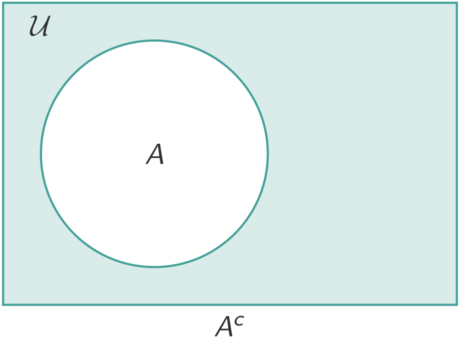

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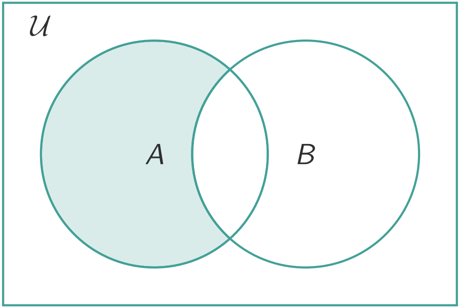

$A \setminus B$

Venn Diagrams

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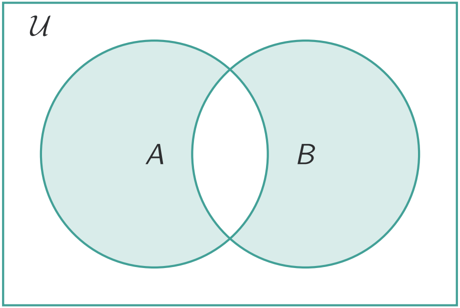


$A \oplus B$

Outline

Set Equality

Two sets are **equal** ($A = B$) if they contain the same elements

To show equality:

- Examine all the elements
- Show $A \subseteq B$ and $B \subseteq A$
- Use the Laws of Set Operations

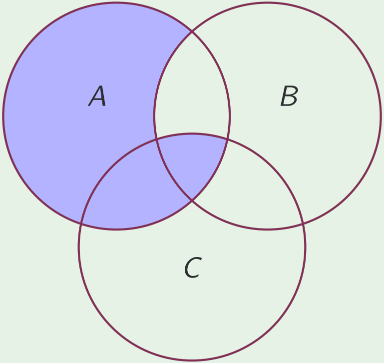
Important!

Venn diagrams can help visualize, but are **not** rigorous.

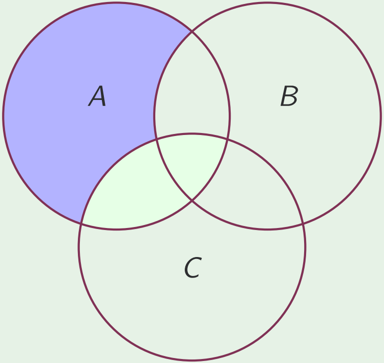
Example

Example

$$A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$$



$A \setminus (B \setminus C)$



$(A \setminus B) \setminus C$

Examples

Example

Show $\{3, 2, 1\} = (0, 4)$.

$(0, 4) = \{1, 2, 3\} = \{3, 2, 1\}$.

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}$

$$\begin{aligned}\{n : n \in \mathbb{Z} \text{ and } n^2 < 5\} &= \{-2, -1, 0, 1, 2\} \\ &= \{n : n \in \mathbb{Z} \text{ and } |n| \leq 2\}\end{aligned}$$

Example

Show $\{n : n \in \mathbb{Z} \text{ and } n^2 > 5\} = \{n : n \in \mathbb{Z} \text{ and } |n| > 2\}$

Show:

- For all $n \in \mathbb{Z}$, if $n^2 > 5$ then $|n| > 2$; and
- For all $n \in \mathbb{Z}$, if $|n| > 2$ then $n^2 > 5$.

That is, show:

$$\text{For all } n \in \mathbb{Z}\text{:} \quad n^2 > 5 \quad \text{if, and only if} \quad |n| > 2$$

Outline

Laws of Set Operations

For all sets A, B, C:

Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity	$A \cup \emptyset = A$
	$A \cap \mathcal{U} = A$
Complementation	$A \cup (A^c) = \mathcal{U}$
	$A \cap (A^c) = \emptyset$

Substitution

Because the laws hold for all sets, we can substitute complex expressions for each set symbol.

Example

Commutativity $$A \cup B = B \cup A$$

Therefore: $$(C \cap D) \cup (D \oplus E) = (D \oplus E) \cup (C \cap D)$$

Example

Example

Show that for all sets $A \cap (B \cap C) = C \cap (B \cap A)$:

$$\begin{aligned} A \cap (B \cap C) &= (A \cap B) \cap C && \text{[Associativity]} \\ &= C \cap (A \cap B) && \text{[Commutativity]} \\ &= C \cap (B \cap A) && \text{[Commutativity]} \end{aligned}$$

Important!

(Aim to) limit each step to a non-overlapping applications of a single rule

Outline

Other useful set laws

The following are all derivable from the previous 10 laws.

Idempotence	$A \cap A = A$
	$A \cup A = A$
Double complementation	$(A^c)^c = A$
Annihilation	$A \cap \emptyset = \emptyset$
	$A \cup \mathcal{U} = \mathcal{U}$
de Morgan's Laws	$(A \cap B)^c = A^c \cup B^c$
	$(A \cup B)^c = A^c \cap B^c$

Example (Idempotence of $\cup$)

$$
\begin{aligned}
A &= A \cup \emptyset && \text{(Identity)} \\
&= A \cup (A \cap A^c) && \text{(Complementation)} \\
&= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\
&= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\
&= (A \cup A) && \text{(Identity)}
\end{aligned}
$$

Outline

Two useful results

Definition

If A is a set defined using $\cap$, $\cup$, $\emptyset$ and $\mathcal{U}$, then $\mathrm{dual}(A)$ is the expression obtained by replacing $\cap$ with $\cup$ (and vice-versa) and $\emptyset$ with $\mathcal{U}$ (and vice-versa).

Theorem (Principle of Duality)

If you can prove $A_1 = A_2$ using the Laws of Set Operations then you can prove $dual(A_1) = dual(A_2)$

Example

Absorption law:	$A \cup (A \cap B) = A$
Dual:	$A \cap (A \cup B) = A$

Application (Idempotence of $\cap$)

Recall Idempotence of $\cup$:

$$\begin{aligned}
A &= A \cup \emptyset && \text{(Identity)} \\
&= A \cup (A \cap A^c) && \text{(Complementation)} \\
&= (A \cup A) \cap (A \cup A^c) && \text{(Distributivity)} \\
&= (A \cup A) \cap \mathcal{U} && \text{(Complementation)} \\
&= (A \cup A) && \text{(Identity)}
\end{aligned}$$

Application (Idempotence of $\cap$)

Invoke the dual laws!

$$
\begin{aligned}
A &= A \cap \mathcal{U} && \text{(Identity)} \\
&= A \cap (A \cup A^c) && \text{(Complementation)} \\
&= (A \cap A) \cup (A \cap A^c) && \text{(Distributivity)} \\
&= (A \cap A) \cup \emptyset && \text{(Complementation)} \\
&= (A \cap A) && \text{(Identity)}
\end{aligned}
$$

Two useful results

Theorem (Uniqueness of complement)

$A \cap B = \emptyset$ *and* $A \cup B = \mathcal{U}$ *if, and only if,* $B = A^c$*.*

Proof (Only if).

$$
\begin{aligned}
B &= B \cap \mathcal{U} && \text{(Identity)}\\
&= B \cap (A \cup A^c) && \text{(Complement)}\\
&= (B \cap A) \cup (B \cap A^c) && \text{(Distributivity)}\\
&= (A \cap B) \cup (A^c \cap B) && \text{(Commutativity)}\\
&= \emptyset \cup (A^c \cap B) && \text{(Given)}\\
&= (A \cap A^c) \cup (A^c \cap B) && \text{(Complement)}\\
&= (A^c \cap A) \cup (A^c \cap B) && \text{(Commutativity)}\\
&= A^c \cap (A \cup B) && \text{(Distributivity)}\\
&= A^c \cap \mathcal{U} && \text{(Given)}\\
&= A^c && \text{(Identity)}
\end{aligned}
$$

$\square$

Application (Double complement)

Take $A = X^c$ and $B = X$:

$$
\begin{aligned}
X^c \cap X &= X \cap X^c && \text{(Commutativity)} \\
&= \emptyset && \text{(Identity)} \\[1em]
X^c \cup X &= \mathcal{U} && \text{(Principle of duality)}
\end{aligned}
$$

By the uniqueness of complement, $(X^c)^c = X$.

Exercises

Exercises

Show the following for all sets A, B, C:

- $B \cup (A \cap \emptyset) = B$
- $(C \cup A) \cap (B \cup A) = A \cup (B \cap C)$
- $(A \cap B) \cup (A \cup B^c)^c = B$

Exercises

Give counterexamples to show the following do not hold for all sets:

- $A \setminus (B \setminus C) = (A \setminus B) \setminus C$
- $(A \cup B) \setminus C = A \cup (B \setminus C)$
- $(A \setminus B) \cup B = A$