Week 5: Graph Algorithms
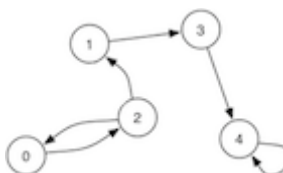
Directed Graphs

Directed Graphs (Digraphs)

In our previous discussion of graphs:

- an edge indicates a relationship between two vertices
- an edge indicates nothing more than a relationship

In many real-world applications of graphs:

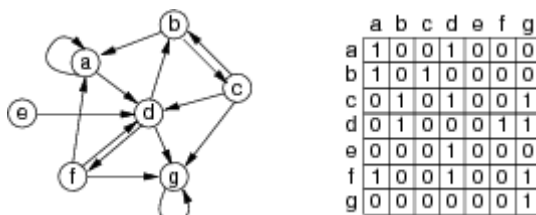- edges are directional $(v \rightarrow w \neq w \rightarrow v)$

- edges have a *weight*   (cost to go from $v \rightarrow w$)

... Directed Graphs (Digraphs)

Example digraph and adjacency matrix representation:

Undirectional ⇒ symmetric matrix
Directional ⇒ non-symmetric matrix

Maximum #edges in a digraph with V vertices: V^2

... Directed Graphs (Digraphs)

Terminology for digraphs ...

Directed path: sequence of $n \geq 2$ vertices $v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_n$

- where $(v_i, v_{i+1}) \in edges(G)$ for all v_i, v_{i+1} in sequence
- if $v_1 = v_n$, we have a *directed cycle*

Reachability: *w* is reachable from *v* if ∃ directed path *v,...,w*

Digraph Applications

Potential application areas:

Domain	Vertex	Edge
Web	web page	hyperlink
scheduling	task	precedence
chess	board position	legal move
science	journal article	citation
dynamic data	malloc'd object	pointer
programs	function	function call
`make`	file	dependency

... Digraph Applications

Problems to solve on digraphs:

- is there a directed path from *s* to *t*? (transitive closure)
- what is the shortest path from *s* to *t*? (shortest path)
- are all vertices mutually reachable? (strong connectivity)
- how to organise a set of tasks? (topological sort)
- which web pages are "important"? (PageRank)
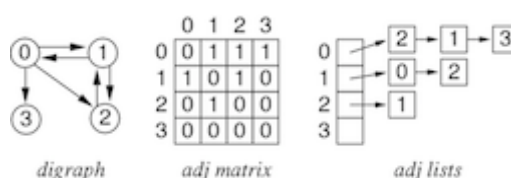- how to build a web crawler? (graph traversal)

Digraph Representation

Similar set of choices as for undirectional graphs:

- array of edges   (directed)
- vertex-indexed adjacency matrix   (non-symmetric)
- vertex-indexed adjacency lists

V vertices identified by *0 .. V-1*

digraph   *adj matrix*   *adj lists*

Reachability

Transitive Closure

Given a digraph *G* it is potentially useful to know

- is vertex *t* reachable from vertex *s*?

Example applications:

- can I complete a schedule from the current state?
- is a malloc'd object being referenced by any pointer?

How to compute transitive closure?

... Transitive Closure

One possibility:

- implement it via `hasPath(G, s, t)`   (itself implemented by DFS or BFS algorithm)
- feasible if *reachable(G,s,t)* is infrequent operation

What if we have an algorithm that frequently needs to check reachability?
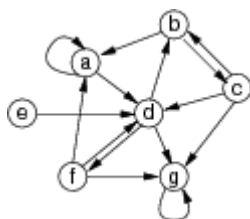
Would be very convenient/efficient to have:

```
reachable(G, s, t):
|   return G.tc[s][t]   // transitive closure matrix
```

Of course, if *V* is *very* large, then this is not feasible.
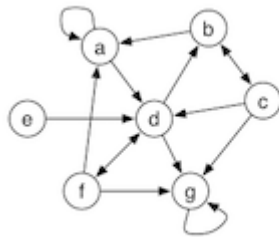

Exercise #1: Transitive Closure Matrix

Which reachable *s .. t* exist in the following graph?

	a	b	c	d	e	f	g
a	1	0	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	1	0	0	1
d	0	1	0	0	0	1	1
e	0	0	0	1	0	0	0
f	1	0	0	1	0	0	1
g	0	0	0	0	0	0	1

Transitive closure of example graph:

adjacency matrix   reachability matrix

... Transitive Closure

Goal: produce a matrix of reachability values

- if *tc[s][t]* is 1, then *t* is reachable from *s*
- if *tc[s][t]* is 0, then *t* is not reachable from *s*

So, how to create this matrix?

Observation:

$\forall i,s,t \in$ vertices(G):
  (s,i) $\in$ edges(G) and *(i,t)* $\in$ edges(G) $\Rightarrow$ *tc[s][t]* = 1

`tc[s][t]=1` if there is a path from *s* to *t* of length 2   ($s \rightarrow i \rightarrow t$)

... Transitive Closure

If we implement the above as:

```
make tc[][] a copy of edges[][]
for all i∈vertices(G) do
   for all s∈vertices(G) do
      for all t∈vertices(G) do
         if tc[s][i]=1 and tc[i][t]=1 then
            tc[s][t]=1
         end if
      end for
   end for
end for
```

then we get an algorithm to convert `edges` into a *tc*

This is known as *Warshall's algorithm*

... Transitive Closure

How it works ...

After iteration 1, `tc[s][t]` is 1 if

- either $s \rightarrow t$ exists or $s \rightarrow 0 \rightarrow t$ exists

After iteration 2, `tc[s][t]` is 1 if any of the following exist

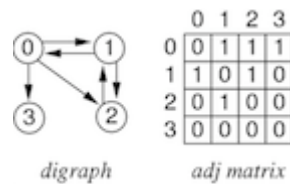- *s→t* or *s→0→t* or *s→1→t* or *s→0→1→t* or *s→1→0→t*

Etc. ... so after the Vth iteration, `tc[s][t]` is 1 if

- there is any directed path in the graph from *s* to *t*

Exercise #2: Transitive Closure

Trace Warshall's algorithm on the following graph:

digraph   adj matrix

1st iteration i=0:

tc	[0]	[1]	[2]	[3]
[0]	0	1	1	1
[1]	1	1	1	1
[2]	0	1	0	0
[3]	0	0	0	0

2nd iteration i=1:

tc	[0]	[1]	[2]	[3]
[0]	1	1	1	1
[1]	1	1	1	1
[2]	1	1	1	1
[3]	0	0	0	0

3rd iteration i=2: unchanged

4th iteration i=3: unchanged

... Transitive Closure

Cost analysis:

- storage: additional V^2 items (each item may be 1 bit)
- computation of transitive closure: $O(V^3)$
- computation of `reachable()`: *O(1)* after having generated `tc[][]`

Amortisation: would need many calls to `reachable()` to justify other costs

Alternative: use DFS in each call to `reachable()`
Cost analysis:

- storage: cost of queue and set during reachable
- computation of `reachable()`: cost of DFS = $O(V^2)$ (for adjacency matrix)

Digraph Traversal

Same algorithms as for undirected graphs:

`depthFirst(v):`

1. mark `v` as visited
2. for each $(v, w) \in edges$(G) do
    if `w` has not been visited then
        `depthFirst(w)`

`breadth-first(v):`

1. enqueue `v`
2. while queue not empty do
    dequeue `v`
    if `v` not already visited then
        mark `v` as visited
        enqueue each vertex `w` adjacent to `v`

Example: Web Crawling

Goal: visit every page on the web

Solution: breadth-first search with "implicit" graph

```
webCrawl(startingURL):
|   mark startingURL as alreadySeen
|   enqueue(Q, startingURL)
|   while Q is not empty do
|   |   nextPage=dequeue(Q)
|   |   visit nextPage
|   |   for each hyperLink on nextPage do
|   |   |   if hyperLink not alreadySeen then
|   |   |       mark hyperLink as alreadySeen
|   |   |       enqueue(Q, hyperLink)
|   |   |   end if
|   |   end for
|   end while
```

`visit` scans page and collects e.g. keywords and links

Weighted Graphs

Weighted Graphs

Graphs so far have considered

- edge = an association between two vertices/nodes
- may be a precedence in the association (directed)
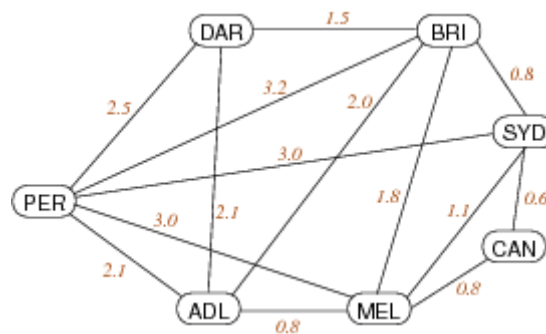
Some applications require us to consider

- a *cost* or *weight* of an association
- modelled by assigning values to edges (e.g. positive reals)

Weights can be used in both directed and undirected graphs.

... Weighted Graphs

Example: major airline flight routes in Australia

Representation: edge = direct flight; weight = approx flying time (hours)

... Weighted Graphs

Weights lead to minimisation-type questions, e.g.

1. Cheapest way to connect all vertices?

- a.k.a. *minimum spanning tree* problem
- assumes: edges are weighted and undirected

2. Cheapest way to get from *A* to *B*?

- a.k.a *shortest path* problem
- assumes: edge weights positive, directed or undirected

Exercise #3: Implementing a Route Finder

If we represent a street map as a graph

- what are the vertices?
- what are the edges?
- are edges directional?
- what are the weights?
- are the weights fixed?

Weighted Graph Representation
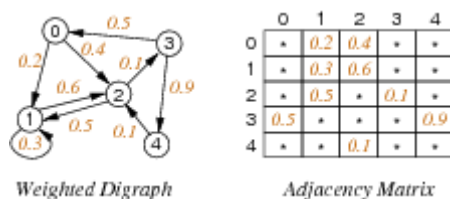
Weights can easily be added to:

- adjacency matrix representation   (0/1 → int or float)
- adjacency lists representation   (add int/float to list node)

Both representations work whether edges are directed or not.

... Weighted Graph Representation
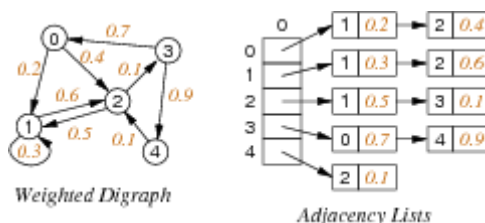
Adjacency matrix representation with weights:

Weighted Digraph   Adjacency Matrix

Note: need distinguished value to indicate "no edge".

... Weighted Graph Representation

Adjacency lists representation with weights:

Weighted Digraph   Adjacency Lists

Note: if undirected, each edge appears twice with same weight

... Weighted Graph Representation

Sample adjacency matrix implementation in C requires minimal changes to previous Graph ADT:

WGraph.h

```
// edges are pairs of vertices (end-points) plus positive weight
typedef struct Edge {
    Vertex v;
```

```
    Vertex w;
    int    weight;
} Edge;

// returns weight, or 0 if vertices not adjacent
int adjacent(Graph, Vertex, Vertex);
```

... Weighted Graph Representation

WGraph.c

```
typedef struct GraphRep {
    int **edges;  // adjacency matrix storing positive weights
                  // 0 if nodes not adjacent
    int   nV;     // #vertices
    int   nE;     // #edges
} GraphRep;


void insertEdge(Graph g, Edge e) {
    assert(g != NULL && validV(g,e.v) && validV(g,e.w));
    if (g->edges[e.v][e.w] == 0) {  // edge e not in graph
        g->edges[e.v][e.w] = e.weight;
        g->nE++;
    }
}

int adjacent(Graph g, Vertex v, Vertex w) {
    assert(g != NULL && validV(g,v) && validV(g,w));
    return g->edges[v][w];
}
```
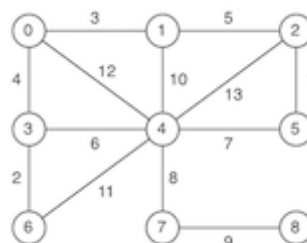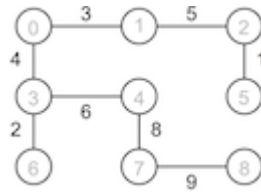
Minimum Spanning Trees

Exercise #4: Minimising Wires in Circuits

Electronic circuit designs often need to make the pins of several components electrically equivalent by wiring them together.

To interconnect a set of *n* pins we can use an arrangement of *n-1* wires each connecting two pins.

What kind of algorithm would ...

- help us find the arrangement with the least amount of wire?

Minimum Spanning Trees

Reminder: *Spanning tree ST* of graph *G=(V,E)*

- *spanning* = all vertices, *tree* = no cycles
  - *ST* is a subgraph of *G*   (*G'=(V,E')* where *E' ⊆ E*)
  - *ST* is *connected* and *acyclic*

Minimum spanning tree MST of graph *G*

- *MST* is a spanning tree of *G*
- sum of edge weights is no larger than any other ST

Applications: Computer networks, Electrical grids, Transportation networks ...

Problem: how to (efficiently) find MST for graph *G*?

NB: MST may not be unique   (e.g. all edges have same weight ⇒ every ST is MST)

... Minimum Spanning Trees

Example:

An MST ...

Brute force solution:

```
findMST(G):
|  Input  graph G
|  Output a minimum spanning tree of G
|
|  bestCost=∞
|  for all spanning trees t of G do
|  |   if cost(t)<bestCost then
|  |      bestTree=t
|  |      bestCost=cost(t)
|  |   end if
|  end for
|  return bestTree
```

Example of *generate-and-test* algorithm.

Not useful because #spanning trees is potentially large (e.g. n^{n-2} for a complete graph with n vertices)

Simplifying assumption:

- edges in *G* are not directed   (MST for digraphs is harder)

One approach to computing MST for graph *G* with *V* nodes:

1. start with empty MST
2. consider edges in increasing weight order
    - add edge if it does not form a cycle in MST
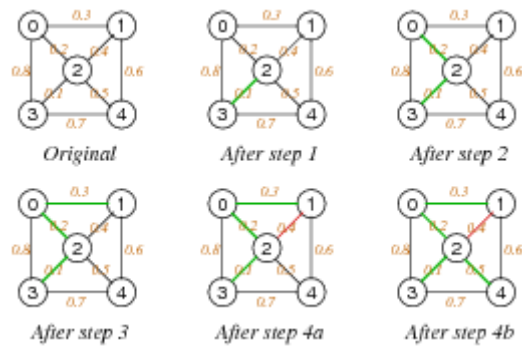3. repeat until *V-1* edges are added

Critical operations:

- iterating over edges in weight order
- checking for cycles in a graph
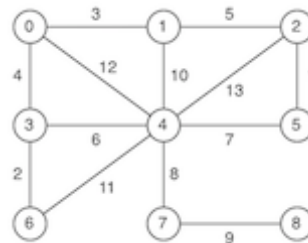
Execution trace of Kruskal's algorithm:

Original After step 1 After step 2

After step 3 After step 4a After step 4b
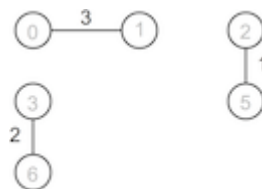

Exercise #5: Kruskal's Algorithm
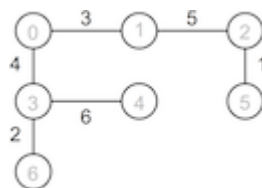
Show how Kruskal's algorithm produces an MST on:

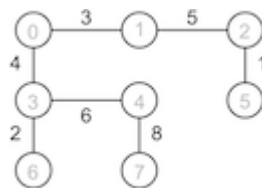## After 3rd iteration:

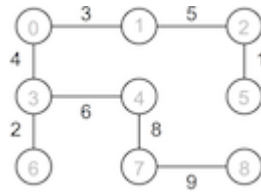## After 6th iteration:

After 7th iteration:

After 8th iteration (V-1=8 edges added):

... Kruskal's Algorithm

Pseudocode:

```
KruskalMST(G):
│   Input  graph G with n nodes
│   Output a minimum spanning tree of G
│
│   MST=empty graph
│   sort edges(G) by weight
│   for each e∈sortedEdgeList do
│   │   MST = MST ∪ {e}
│   │   if MST has a cyle then
│   │      MST = MST \ {e}
│   │   end if
│   │   if MST has n-1 edges then
│   │      return MST
│   │   end if
│   end for
```

... Kruskal's Algorithm

Time complexity analysis ...

- sorting edge list is $O(E \cdot log\ E)$
- min V, max E iterations over sorted edges
- on each iteration ...
  - getting next lowest cost edge is $O(1)$
  - checking whether adding it forms a cycle: cost = ??
    - use DFS ... too expensive?
    - could use *Union-Find data structure* (see Sedgewick Ch.1) to maintain sets of connected components
      $\Rightarrow$ loop is $O(E \cdot log\ V)$
- overall complexity $O(E \cdot log\ E) = O(E \cdot log\ V)$

Exercise #6: Kruskal's Algorithm

Why is $O(E \cdot log\ E) = O(E \cdot log\ V)$ in this case?

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1. at most $E = V^2$ edges $\Rightarrow$ $log\ E = 2 \cdot log\ V = O(log\ V)$
2. if $V > E+1 \Rightarrow$ can ignore all unconnected vertices

---

Prim's Algorithm

Another approach to computing MST for graph *G=(V,E)*:

1. start from any vertex *v* and empty MST
2. choose edge not already in MST to add to MST
   - must be incident on a vertex *s* already connected to *v* in MST
   - must be incident on a vertex *t* not already connected to *v* in MST
   - must have minimal weight of all such edges
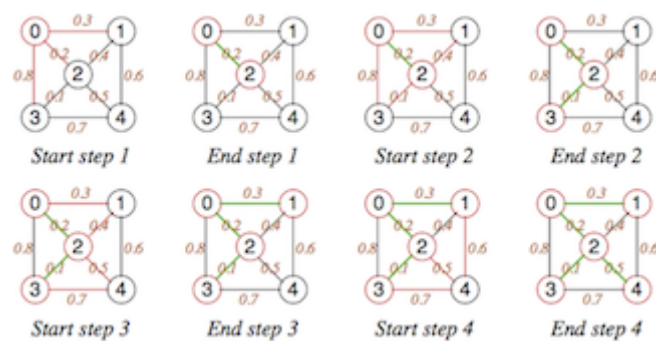3. repeat until MST covers all vertices

Critical operations:

- checking for vertex being connected in a graph
- finding min weight edge in a set of edges

... Prim's Algorithm

Execution trace of Prim's algorithm (starting at *s=0*):

Start step 1   *End step 1*   *Start step 2*   *End step 2*

Start step 3   *End step 3*   *Start step 4*   *End step 4*

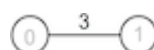## Exercise #7: Prim's Algorithm

Show how Prim's algorithm produces an MST on:
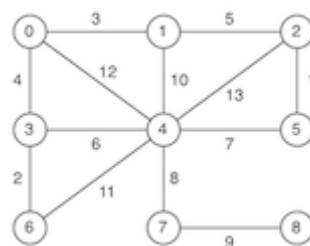
Start from vertex 0

After 1st iteration:

After 2nd iteration:

After 3rd iteration:
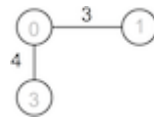
After 3rd iteration:
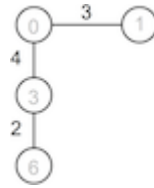
After 4th iteration:
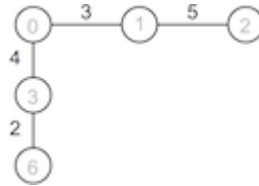
After 8th iteration (all vertices covered):
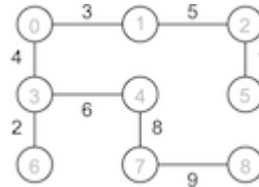
... Prim's Algorithm

Pseudocode:

```
PrimMST(G):
|  Input  graph G with n nodes
|  Output a minimum spanning tree of G
|
|  MST=empty graph
|  usedV={0}
|  unusedE=edges(g)
|  while |usedV|<n do
|  |  find e=(s,t,w) ∈unusedE such that {
|  |     s∈usedV, t∉usedV and w is min weight of all such edges
|  |  }
|  |  MST = MST ∪ {e}
|  |  usedV = usedV ∪ {t}
|  |  unusedE = unusedE \ {e}
|  end while
|  return MST
```

Critical operation: finding best edge

... Prim's Algorithm

Rough time complexity analysis ...

- *V* iterations of outer loop
- in each iteration ...
    - find min edge with set of edges is $O(E) \Rightarrow O(V \cdot E)$ overall
    - find min edge with *priority queue* is $O(\log E) \Rightarrow O(V \cdot \log E)$ overall

---

Sidetrack: Priority Queues

Some applications of queues require

- items processed in order of "priority"
- rather than in order of entry (FIFO — first in, first out)

Priority Queues (*PQueues*) provide this via:

- `join`: insert item into PQueue with an associated priority (replacing `enqueue`)
- `leave`: remove item with highest priority (replacing `dequeue`)

Time complexity for naive implementation of a PQueue containing *N* items ...

- *O(1)* for `join`   *O(N)* for `leave`

Most efficient implementation ("heap") ...

- *O(log N)* for `join`, `leave`

Other MST Algorithms

Boruvka's algorithm ... complexity *O(E·log V)*

- the oldest MST algorithm
- start with *V* separate components
- join components using min cost links
- continue until only a single component

Karger, Klein, and Tarjan ... complexity *O(E)*

- based on Boruvka, but non-deterministic
- randomly selects subset of edges to consider
- for the keen, here's the paper describing the algorithm

Shortest Path

Shortest Path

Path = sequence of edges in graph G $p = (v_0, v_1), (v_1, v_2), ..., (v_{m-1}, v_m)$

cost(path) = sum of edge weights along path

Shortest path between vertices *s* and *t*

- a simple path *p(s,t)* where *s = first(p)*, *t = last(p)*
- no other simple path *q(s,t)* has *cost(q) < cost(p)*

Assumptions: weighted digraph, no negative weights.

Finding shortest path between two given nodes known as *source-target* SP problem

Variations: *single-source* SP, *all-pairs* SP

Applications: navigation, routing in data networks, ...

Single-source Shortest Path (SSSP)

Given: weighted digraph *G*, source vertex *s*

Result: shortest paths from *s* to all other vertices

- `dist[]` *V*-indexed array of cost of shortest path from *s*
- `pred[]` *V*-indexed array of predecessor in shortest path from *s*

Example:

Edge Relaxation

Assume: `dist[]` and `pred[]` as above   (but containing data for shortest paths *discovered so far*)

`dist[v]` is length of shortest known path from *s* to *v*
`dist[w]` is length of shortest known path from *s* to *w*

Relaxation updates data for *w* if we find a shorter path from *s* to *w*:


```
dist[v]=8, dist[w]=12          dist[v]=8, dist[w]=11
pred[v]=?, pred[w]=?           pred[v]=?, pred[w]=v
```
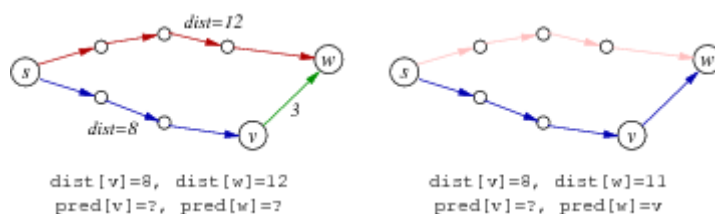
Relaxation along edge *e=*(`v`,`w`,`weight`):

- if `dist[v]+weight < dist[w]` then
  update `dist[w]:=dist[v]+weight` and `pred[w]:=v`

Dijkstra's Algorithm

One approach to solving single-source shortest path problem ...

Data: *G*, *s*, `dist[]`, `pred[]` and

- *vSet*: set of vertices whose shortest path from *s* is unknown

Algorithm:

```
dist[]  // array of cost of shortest path from s
pred[]  // array of predecessor in shortest path from s

dijkstraSSSP(G, source):
|  Input graph G, source node
|
|  initialise dist[] to all ∞, except dist[source]=0
|  initialise pred[] to all -1
|  vSet=all vertices of G
|  while vSet≠∅ do
|  |    find s∈vSet with minimum dist[s]
|  |    for each (s,t,w)∈edges(G) do
|  |       relax along (s,t,w)
|  |    end for
|  |    vSet=vSet\{s}
|  end while
```

Exercise #8: Dijkstra's Algorithm

Show how Dijkstra's algorithm runs on (source node = 0):
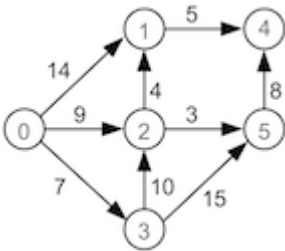
	[0]	[1]	[2]	[3]	[4]	[5]
dist	0	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

---

	[0]	[1]	[2]	[3]	[4]	[5]
dist	0	∞	∞	∞	∞	∞
pred	–	–	–	–	–	–

dist	0	14	9	7	∞	∞

pred	–	0	0	0	–	–

dist	0	14	9	7	∞	22
pred	–	0	0	0	–	3

dist	0	13	9	7	∞	12
pred	–	2	0	0	–	2

dist	0	13	9	7	20	12
pred	–	2	0	0	5	2

dist	0	13	9	7	18	12
pred	–	2	0	0	1	2

... Dijkstra's Algorithm

Why Dijkstra's algorithm is correct:

Hypothesis.
(a) For visited *s* ... *dist[s]* is shortest distance from source
(b) For unvisited *t* ... *dist[t]* is shortest distance from source *via visited nodes*

Proof.
Base case: no visited nodes, *dist[source]=0*, *dist[s]=∞* for all other nodes

Induction step:

1. If *s* is unvisited node with minimum *dist[s]*, then *dist[s]* is shortest distance from source to *s*:
   - if ∃ shorter path via only visited nodes, then *dist[s]* would have been updated when processing the predecessor of *s* on this path
   - if ∃ shorter path via an unvisited node *u*, then *dist[u]<dist[s]*, which is impossible if *s* has min distance of all unvisited nodes
2. This implies that (a) holds for *s* after processing *s*
3. (b) still holds for all unvisited nodes *t* after processing *s*:
   - if ∃ shorter path via *s* we would have just updated *dist[t]*
   - if ∃ shorter path without *s* we would have found it previously

---

... Dijkstra's Algorithm

Time complexity analysis ...

Each edge needs to be considered once ⇒ *O(E)*.

Outer loop has *O(V)* iterations.

Implementing "`find s∈vSet with minimum dist[s]`"

1. try all $s \in \text{vSet} \Rightarrow$ cost = $O(V) \Rightarrow$ overall cost = $O(E + V^2)$ = $O(V^2)$
2. using a PQueue to implement extracting minimum
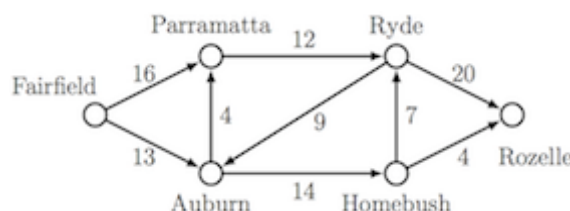   - can improve overall cost to $O(E + V \cdot log\ V)$ (for best-known implementation)

Network Flow

Exercise #9: Merchandise Distribution

Lucky Cricket Company ...

- produces cricket balls in Fairfield
- has a warehouse in Rozelle that stocks them
- ships them from factory to warehouse by leasing space on trucks with limited capacity:

What kind of algorithm would ...

- help us find the maximum number of crates that can be shipped from Fairfield to Rozelle per day?

Flow Networks

Flow network ...

- weighted graph *G=(V,E)*
- distinct nodes *s∈V* (*source*), *t∈V* (*sink*)

Edge weights denote *capacities* Applications:

- Distribution networks, e.g.
    - source: oil field
    - sink: refinery
    - edges: pipes
- Traffic flow

... Flow Networks

Flow in a network *G=(V,E)* ... nonnegative *f(v,w)* for all vertices *v,w∈V* such that

- *f(v,w)≤capacity* for each edge *e=(v,w,capacity)* ∈ *E*
- *f(v,w)=0* if no edge between *v* and *w*

- total flow *into* a vertex = total flow *out of* a vertex:

$$\sum_{x \in V} f(x, v) = \sum_{y \in V} f(v, y) \quad \text{for all } v \in V \setminus \{s, t\}$$

Maximum flow ... no other flow from *s* to *t* has larger value

... Flow Networks
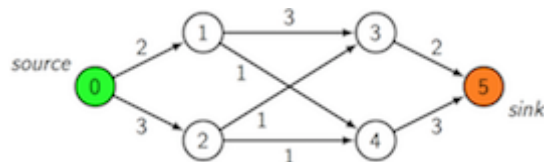
Example:

A (maximum) flow ...
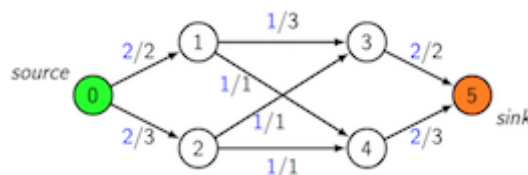
Augmenting Paths

Assume ...   *f(v,w)* contains current flow

Augmenting path: any path from source *s* to sink *t* that can currently take more flow

Example:
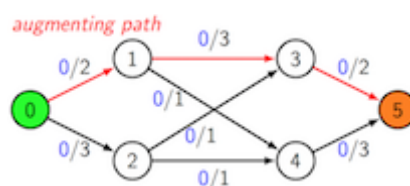
Residual Network

Assume ...   flow network *G=(V,E)* and flow *f(v,w)*

Residual network (V,E'):

- same vertex set *V*
- for each edge *v* →*c* *w* ∈ *E* ...
  - *f(v,w) < c*   ⇒ add edge *(v* →*c-f(v,w)* *w)* to *E'*
  - *f(v,w) > 0*   ⇒ add edge *(v* ←*f(v,w)* *w)* to *E'*

Example:
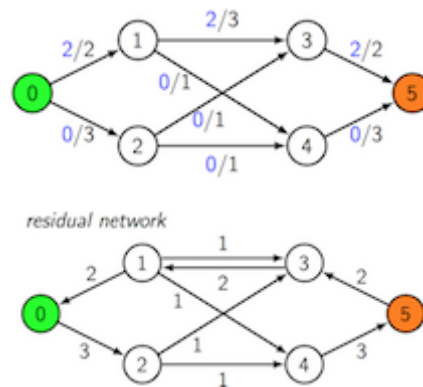
residual network

Exercise #10: Augmenting Paths and Residual Networks

Find an augmenting path in:
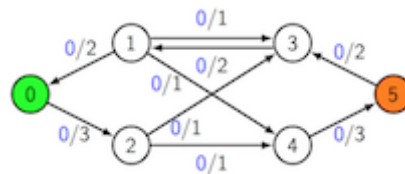
and show the residual network after augmenting the flow

---

1. Augmenting path:
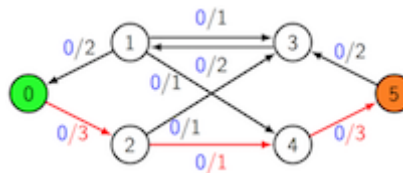
maximum additional flow = 1

2. Residual network:
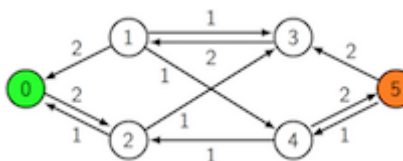
Can you find a further augmenting path in the new residual network?

Edmonds-Karp Algorithm

One approach to solving maximum flow problem ...

`maxflow(G):`

1. Find a shortest augmenting path
2. Update `flow[][]` so as to represent residual graph
3. Repeat until no augmenting path can be found

Algorithm:

```
flow[][]  // V×V array of current flow
visited[] /* array of predecessor nodes on shortest path
            from source to sink in residual network */

maxflow(G):
│  Input  flow network G with source s and sink t
│  Output maximum flow value
│
│  initialise flow[v][w]=0 for all vertices v, w
│  maxflow=0
│  while ∃shortest augmenting path visited[] from s to t do
│  │   df = maximum additional flow via visited[]
│  │   // adjust flow so as to represent residual graph
│  │   v=t
│  │   while v≠s do
│  │   │   flow[visited[v]][v] = flow[visited[v]][v] + df;
│  │   │   flow[v][visited[v]] = flow[v][visited[v]] - df;
│  │   │   v=visited[v]
│  │   end while
│  │   maxflow=maxflow+df
│  end while
│  return maxflow
```

Shortest augmenting path can be found by standard BFS

Time complexity analysis ...

- *Theorem.* The number of augmenting paths needed is at most $V{\cdot}E/2$.
  $\Rightarrow$ Outer loop has $O(V{\cdot}E)$ iterations.
- Finding augmenting path $\Rightarrow O(E)$ (consider only vertices connected to source and sink $\Rightarrow$ $O(V+E)=O(E)$)

Overall cost of Edmonds-Karp algorithm: $O(V{\cdot}E^2)$

Note: Edmonds-Karp algorithm is an implementation of general *Ford-Fulkerson method*

Show how Edmonds-Karp algorithm runs on:

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]						
[1]						
[2]						
[3]						
[4]						
[5]						

c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]		2	3			
[1]				3	1	
[2]				1	1	
[3]						2
[4]						3
[5]						

---

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	0	0	0	0	0
[1]	0	0	0	0	0	0
[2]	0	0	0	0	0	0
[3]	0	0	0	0	0	0
[4]	0	0	0	0	0	0
[5]	0	0	0	0	0	0

c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	–	2	3	–	–	–
[1]	–	–	–	3	1	–
[2]	–	–	–	1	1	–
[3]	–	–	–	–	–	2
[4]	–	–	–	–	–	3
[5]	–	–	–	–	–	–

augmenting path: 0-1-3-5, df: 2

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	0	0	0	0
[1]	-2	0	0	2	0	0
[2]	0	0	0	0	0	0
[3]	0	-2	0	0	0	2
[4]	0	0	0	0	0	0
[5]	0	0	0	-2	0	0

c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	–	0	3	–	–	–
[1]	2	–	–	1	1	–
[2]	–	–	–	1	1	–
[3]	–	2	–	–	–	0
[4]	–	–	–	–	–	3
[5]	–	–	–	2	–	–

augmenting path: 0-2-4-5, df: 1

flow	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	1	0	0	0
[1]	-2	0	0	2	0	0
[2]	-1	0	0	0	1	0
[3]	0	-2	0	0	0	2
[4]	0	0	-1	0	0	1

c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	–	0	2	–	–	–
[1]	2	–	–	1	1	–
[2]	1	–	–	1	0	–
[3]	–	2	–	–	–	0
[4]	–	–	1	–	–	2

[5]	0	0	0	-2	-1	0	[5]	–	–	–	2	1	–

augmenting path: 0-2-3-1-4-5, `df`: 1

flow	[0]	[1]	[2]	[3]	[4]	[5]	c-f	[0]	[1]	[2]	[3]	[4]	[5]
[0]	0	2	2	0	0	0	[0]	–	0	1	–	–	–
[1]	-2	0	0	1	1	0	[1]	2	–	–	2	0	–
[2]	-2	0	0	1	1	0	[2]	2	–	–	0	0	–
[3]	0	-1	-1	0	0	2	[3]	–	1	1	–	–	0
[4]	0	-1	-1	0	0	2	[4]	–	1	1	–	–	1
[5]	0	0	0	-2	-2	0	[5]	–	–	–	2	2	–

Digraph Applications

PageRank

Goal: determine which Web pages are "important"

Approach: ignore page contents; focus on hyperlinks

- treat Web as graph: page = vertex, hyperlink = directed edge
- pages with many incoming hyperlinks are important
- need to compute "incoming degree" for vertices

Problem: the Web is a *very* large graph

- approx. 10^{14} pages, 10^{15} hyperlinks

Assume for the moment that we could build a graph ...

Most frequent operation in algorithm "Does edge (v,w) exist?"

... PageRank

Simple PageRank algorithm:

```
PageRank(myPage):
|   rank=0
|   for each page in the Web do
|   |   if linkExists(page,myPage) then
|   |       rank=rank+1
|   |   end if
|   end for
```

Note: requires *inbound* link check

... PageRank

V = # pages in Web, E = # hyperlinks in Web

Costs for computing PageRank for each representation:

Representation	linkExists(v,w)	Cost
Adjacency *matrix*	`edge[v][w]`	*1*
Adjacency *lists*	`inLL(list[v],w)`	$\cong$ *E/V*

Not feasible ...

- adjacency matrix ... $V \cong 10^{14} \Rightarrow$ matrix has 10^{28} cells
- adjacency list ... V lists, each with $\cong$10 hyperlinks $\Rightarrow 10^{15}$ list nodes

So how to really do it?

... PageRank

Approach: the random web surfer

- if we randomly follow links in the web ...
- ... more likely to re-discover pages with many inbound links

```
curr=random page, prev=null
for a long time do
|  if curr not in array ranked[] then
|     rank[curr]=0
|  end if
|  rank[curr]=rank[curr]+1
|  if random(0,100)<85 then            // with 85% chance ...
|     prev=curr
|     curr=choose hyperlink from curr  // ... crawl on
|  else
|     curr=random page                 // avoid getting stuck
|     prev=null
|  end if
end for
```

Could be accomplished while we crawl web to build search index

Exercise #12: Implementing Facebook

Facebook could be considered as a giant "social graph"

- what are the vertices?
- what are the edges?
- are edges directional?

What kind of algorithm would ...

- help us find people that you might like to "befriend"?

Summary

- Digraphs, weighted graphs: representations, applications
- Reachability
    - Warshall
- Minimum Spanning Tree (MST)
    - Kruskal, Prim
- Shortest path problems
    - Dijkstra (single source SPP)
- Flow networks
    - Edmonds-Karp (maximum flow)

- Suggested reading (Sedgewick):
    - digraphs ... Ch. 19.1-19.3
    - weighted graphs ... Ch. 20-20.1
    - MST ... Ch. 20.2-20.4
    - SSP ... Ch. 21-21.3
    - network flows ... Ch. 22.1-22.2

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