Week 7: Search Tree Data Structures

Searching

An extremely common application in computing

- given a (large) collection of *items* and a *key* value
- find the item(s) in the collection containing that key
  - item = (key, val_1, val_2, ...) (i.e. a structured data type)
  - key = value used to distinguish items (e.g. student ID)

Applications: Google, databases, ...

... Searching

Since searching is a very important/frequent operation,
many approaches have been developed to do it

Linear structures: arrays, linked lists, files

Arrays = random access. Lists, files = sequential access.

Cost of searching:

	Array	List	File
Unsorted	O(n) (linear scan)	O(n) (linear scan)	O(n) (linear scan)
Sorted	O(log n) (binary search)	O(n) (linear scan)	O(log n) (*seek*, *seek*, ...)

- *O(n)* ... linear scan (search technique of last resort)
- *O(log n)* ... binary search, *search trees* (trees also have other uses)

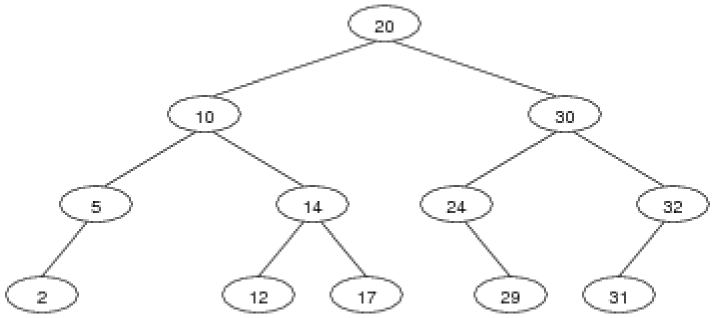
Also (cf. COMP9021): hash tables (*O(1)*, but only under optimal conditions)

... Searching

Maintaining the order in sorted arrays and files is a costly operation.

Search trees are as efficient to search but more efficient to maintain.

Example: the following tree corresponds to the sorted array $[2, 5, 10, 12, 14, 17, 20, 24, 29, 30, 31, 32]$:



Tree Data Structures

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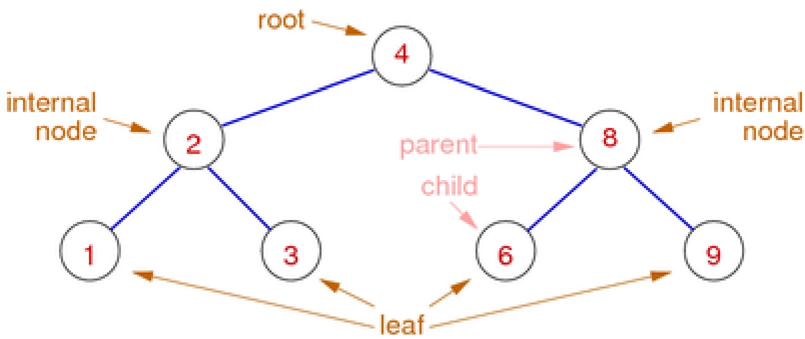
Trees are connected graphs

- consisting of nodes and edges (called *links*), with no cycles (no "up-links")
- each node contains a data value (or key+data)
- each node has links to ≤ *k* other child nodes (*k=2* below)



... Tree Data Structures

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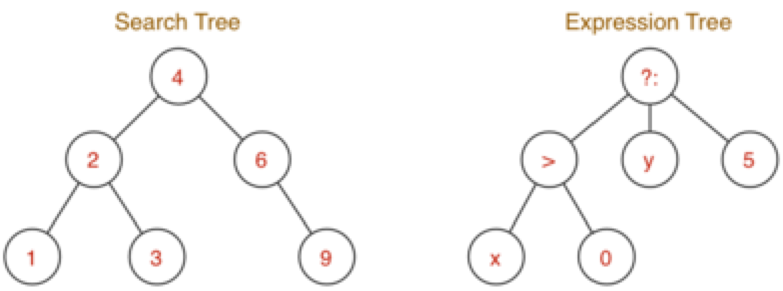
Trees are used in many contexts, e.g.

- representing hierarchical data structures (e.g. expressions)
- efficient searching (e.g. sets, symbol tables, ...)



... Tree Data Structures

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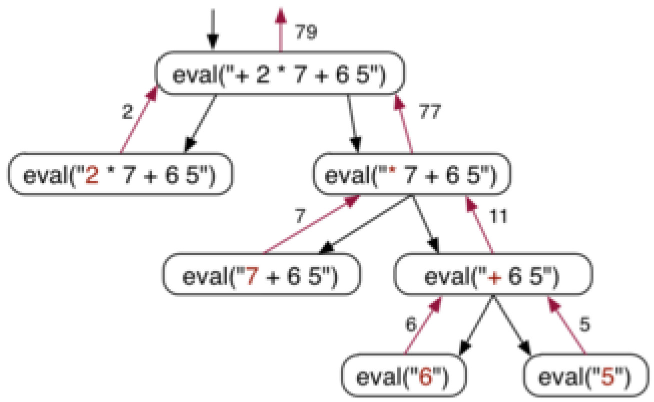
Trees can be used as a data structure, but also for *illustration*.

E.g. showing evaluation of a prefix arithmetic expression

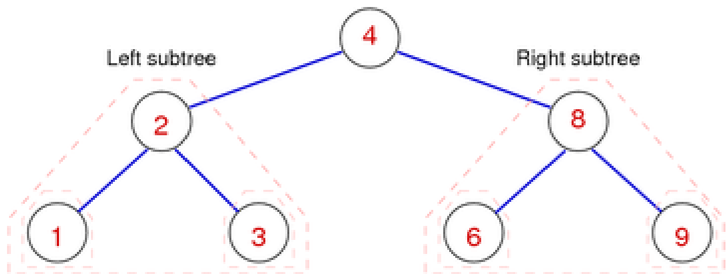


... Tree Data Structures

Binary trees (*k=2* children per node) can be defined recursively, as follows:

A *binary tree* is either

- empty (contains no nodes)
- consists of a *node*, with two *subtrees*
    - node contains a value
    - left and right subtrees are *binary trees*



... Tree Data Structures

Other special kinds of tree

- *m-ary tree*: each internal node has exactly *m* children
- *Ordered tree*: all left values < root, all right values > root
- *Balanced tree*: has ≅minimal height for a given number of nodes
- *Degenerate tree*: has ≅maximal height for a given number of nodes

---

Search Trees

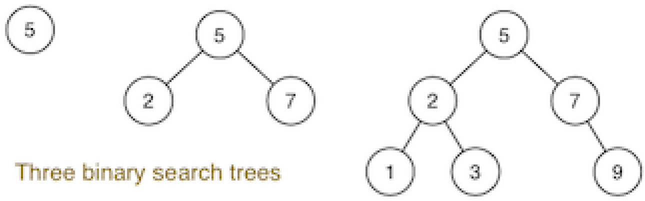
Binary Search Trees

Binary search trees (or *BSTs*) have the characteristic properties

- each node is the root of 0, 1 or 2 subtrees
- all values in any left subtree are less than root
- all values in any right subtree are greater than root
- these properties applies over all nodes in the tree

perfectly balanced trees have the properties

- #nodes in left subtree = #nodes in right subtree
- this property applies over all nodes in the tree



... Binary Search Trees

Operations on BSTs:

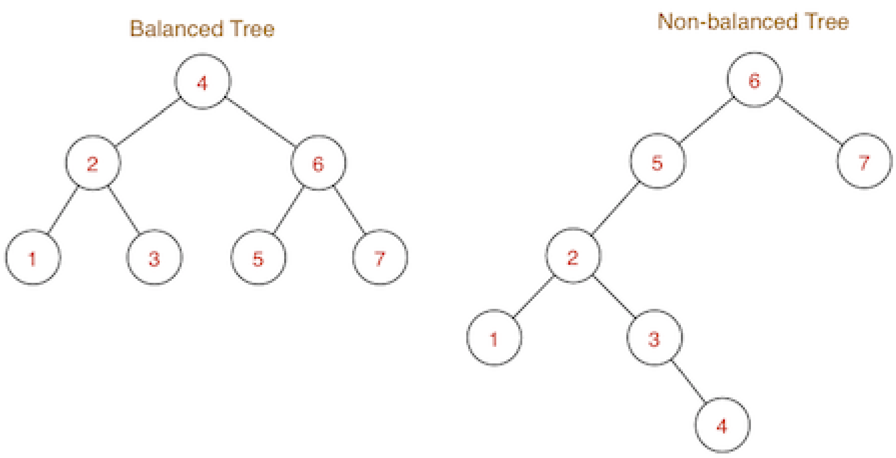
- *insert*(Tree,Item) ... add new item to tree via key
- *delete*(Tree,Key) ... remove item with specified key from tree
- *search*(Tree,Key) ... find item containing key in tree
- plus, "bookkeeping" ... new(), free(), show(), ...

Notes:

- in general, nodes contain `Item`s; we just show `Item.key`
- keys are unique (not technically necessary)

... Binary Search Trees

Examples of binary search trees:

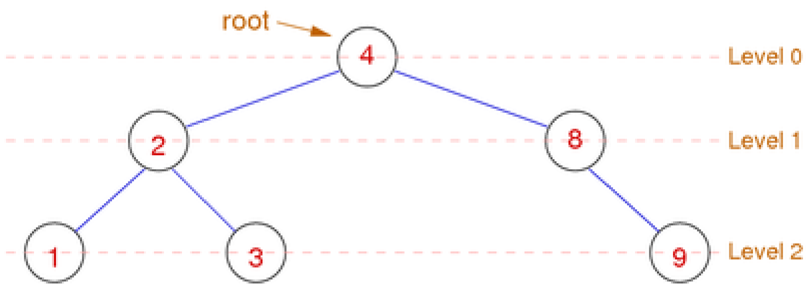


Shape of tree is determined by order of insertion.

... Binary Search Trees

Level of node = path length from root to node

Height (or: *depth*) of tree = max path length from root to leaf



Height-balanced tree: ∀ nodes: height(left subtree) = height(right subtree) ± 1

Time complexity of tree algorithms is typically *O(height)*

Exercise #1: Insertion into BSTs

For each of the sequences below

- start from an initially empty binary search tree
- show tree resulting from inserting values in order given

(a) 4 2 6 5 1 7 3

(b) 6 5 2 3 4 7 1

(c) 1 2 3 4 5 6 7

Assume new values are always inserted as new leaf nodes

(a) the balanced tree from 3 slides ago (height = 2)

(b) the non-balanced tree from 3 slides ago (height = 4)

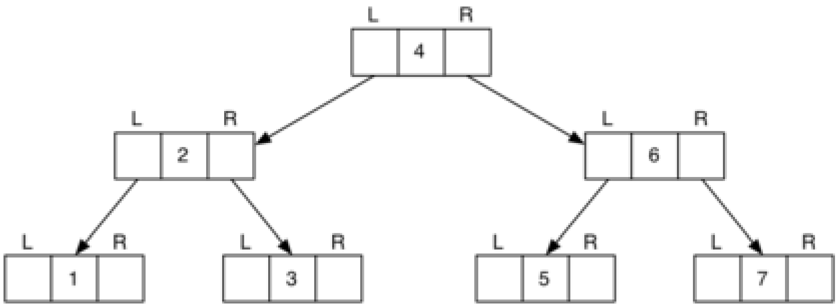
(c) a fully degenerate tree of height 6

Representing BSTs

Binary trees are typically represented by node structures

- containing a value, and pointers to child nodes

Most tree algorithms move *down* the tree.
If upward movement needed, add a pointer to parent.



... Representing BSTs

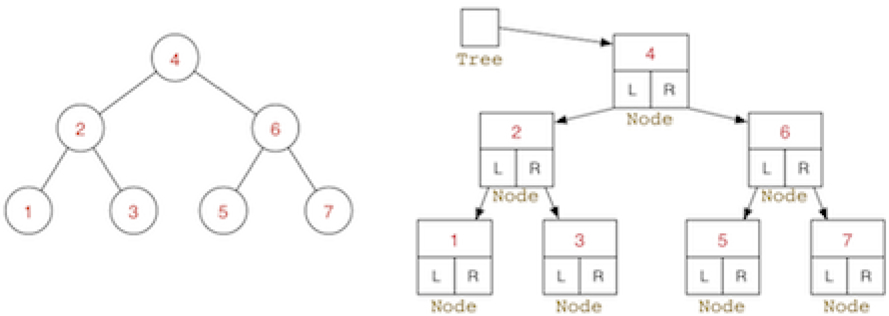
Typical data structures for trees ...

```
// a Tree is represented by a pointer to its root node
typedef struct Node *Tree;

// a Node contains its data, plus left and right subtrees
typedef struct Node {
   Item data;          // We will only use an int for the value of a node
   Tree left, right;
} Node;

// some macros that we will use frequently
#define data(tree)  ((tree)->data)
#define left(tree)  ((tree)->left)
#define right(tree) ((tree)->right)
```

We ignore items ⇒ `data` in `Node` is just a key

... Representing BSTs

Abstract data vs concrete data ...



Tree Algorithms

Searching in BSTs

Most tree algorithms are best described recursively

```
TreeSearch(tree,item):
|  Input  tree, item
|  Output true if item found in tree, false otherwise
|
|  if tree is empty then
|     return false
|  else if item < data(tree) then
|     return TreeSearch(left(tree),item)
|  else if item > data(tree) then
|     return TreeSearch(right(tree),item)
|  else         // found
|    return true
|  end if
```

Insertion into BSTs

Insert an item into appropriate subtree

```
insertAtLeaf(tree,item):
|  Input  tree, item
|  Output tree with item inserted
|
|  if tree is empty then
|     return new node containing item
|  else if item < data(tree) then
|     return insertAtLeaf(left(tree),item)
|  else if item > data(tree) then
|     return insertAtLeaf(right(tree),item)
|  else
|     return tree    // avoid duplicates
|  end if
```

Tree Traversal

Iteration (traversal) on ...

- `List`s ... visit each value, from first to last
- `Graph`s ... visit each vertex, order determined by DFS/BFS/...

For binary `Tree`s, several well-defined visiting orders exist:

- *preorder* (NLR) ... visit root, then left subtree, then right subtree
- *inorder* (LNR) ... visit left subtree, then root, then right subtree
- *postorder* (LRN) ... visit left subtree, then right subtree, then root
- *level-order* ... visit root, then all its children, then all their children

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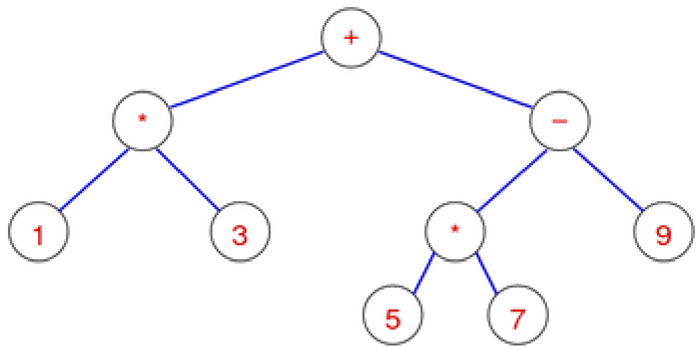
... Tree Traversal

Consider "visiting" an expression tree like:



NLR: + * 1 3 - * 5 7 9 (prefix-order: useful for building tree)
LNR: 1 * 3 + 5 * 7 - 9 (infix-order: "natural" order)
LRN: 1 3 * 5 7 * 9 - + (postfix-order: useful for evaluation)
Level: + * - 1 3 * 9 5 7 (level-order: useful for printing tree)

Exercise #2: Tree Traversal

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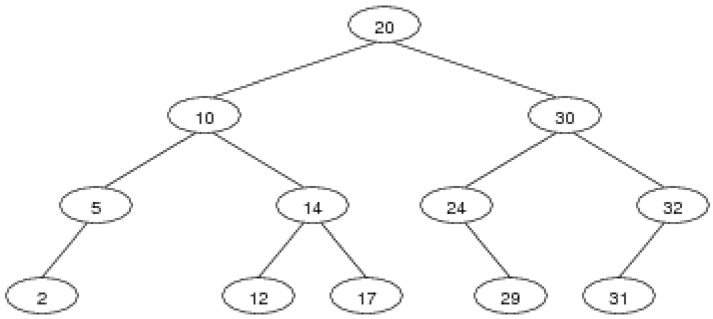
Show NLR, LNR, LRN traversals for the tree



NLR (preorder): 20 10 5 2 14 12 17 30 24 29 32 31

LNR (inorder): 2 5 10 12 14 17 20 24 29 30 31 32

LRN (postorder): 2 5 12 17 14 10 29 24 31 32 30 20

Exercise #3: Non-recursive traversals

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Write a non-recursive *preorder* traversal algorithm.

Assume that you have a stack ADT available.

```
showBSTreePreorder(t):
|  Input tree t
|
|  push t onto new stack S
|  while stack is not empty do
|  |  t=pop(S)
|  |  print data(t)
|  |  if right(t) is not empty then
|  |     push right(t) onto S
|  |  end if
|  |  if left(t) is not empty then
```

```
|   |     push left(t) onto S
|   |   end if
|   end while
```

Joining Two Trees

An auxiliary tree operation ...

Tree operations so far have involved just one tree.

An operation on two trees: `t = joinTrees(`t_1`, `t_2`)`

- Pre-conditions:
  - takes two BSTs; returns a single BST
  - max(key(t_1)) < min(key(t_2))
- Post-conditions:
  - result is a BST (i.e. fully ordered)
  - containing all items from t_1 and t_2

... Joining Two Trees

Method for performing tree-join:

- find the min node in the right subtree (t_2)
- replace min node by its right subtree
- elevate min node to be new root of both trees

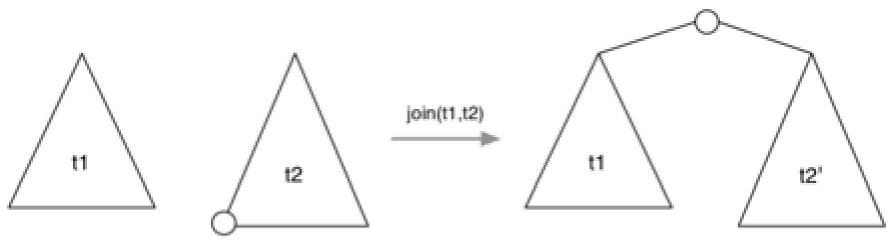
Advantage: doesn't increase height of tree significantly

x ≤ height(t) ≤ x+1, where x = max(height(t_1),height(t_2))

Variation: choose deeper subtree; take root from there.

... Joining Two Trees

Joining two trees:



Note: t2' may be less deep than t2

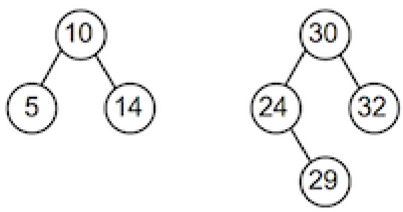
... Joining Two Trees

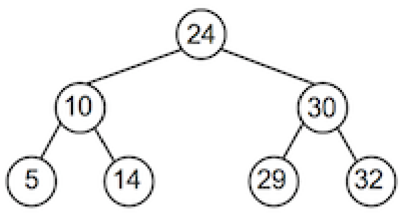
Implementation of tree-join

```
joinTrees(t1,t2):
|  Input  trees t1,t2
|  Output t1 and t2 joined together
|
|  if t1 is empty then return t2
|  else if t2 is empty then return t1
|  else
|  |  curr=t2, parent=NULL
|  |  while left(curr) is not empty do     // find min element in t2
|  |     parent=curr
|  |     curr=left(curr)
|  |  end while
|  |  if parent≠NULL then
|  |     left(parent)=right(curr)   // unlink min element from parent
|  |     right(curr)=t2
|  |  end if
|  |  left(curr)=t1
|  |  return curr                   // curr is new root
|  end if
```

Exercise #4: Joining Two Trees

Join the trees





Deletion from BSTs

Insertion into a binary search tree is easy.

Deletion from a binary search tree is harder.

Four cases to consider ...

- empty tree ... new tree is also empty

- zero subtrees ... unlink node from parent
- one subtree ... replace by child
- two subtrees ... replace by successor, join two subtrees

... Deletion from BSTs

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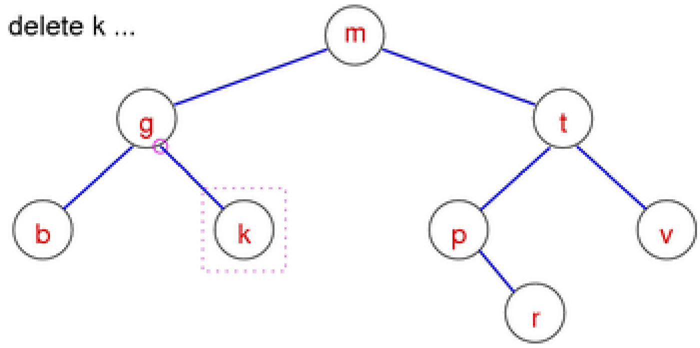
Case 2: item to be deleted is a leaf (zero subtrees)



Just delete the item

... Deletion from BSTs

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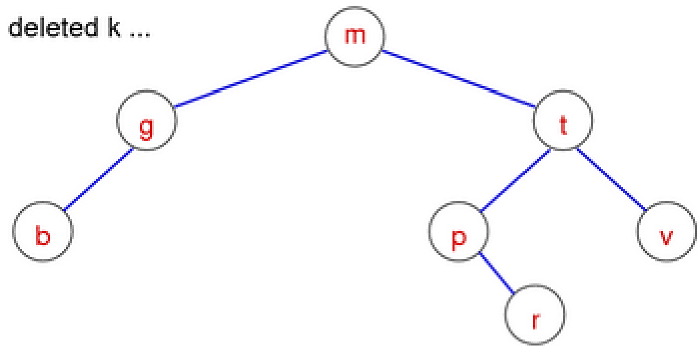
Case 2: item to be deleted is a leaf (zero subtrees)



... Deletion from BSTs

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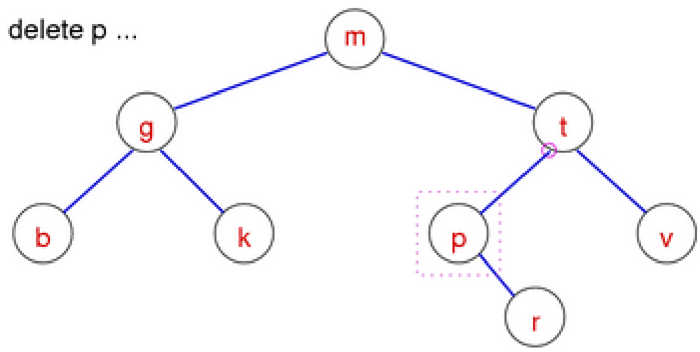
Case 3: item to be deleted has one subtree



Replace the item by its only subtree

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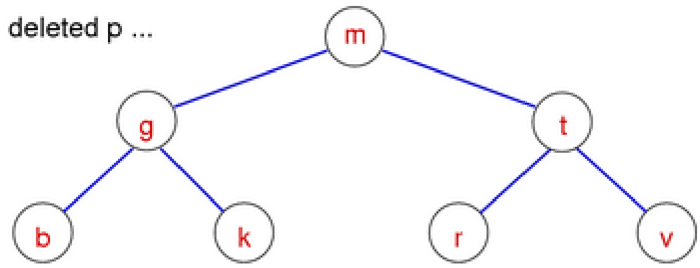
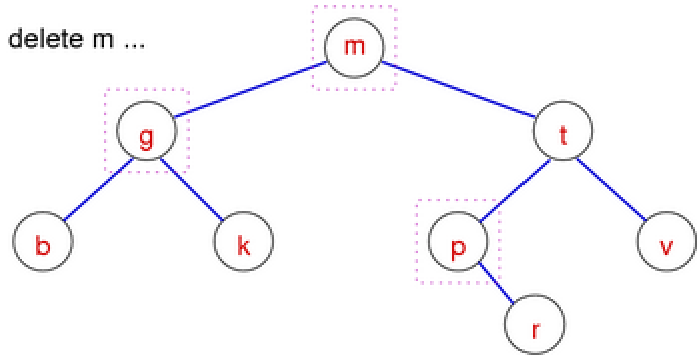
... Deletion from BSTs

Case 3: item to be deleted has one subtree



... Deletion from BSTs

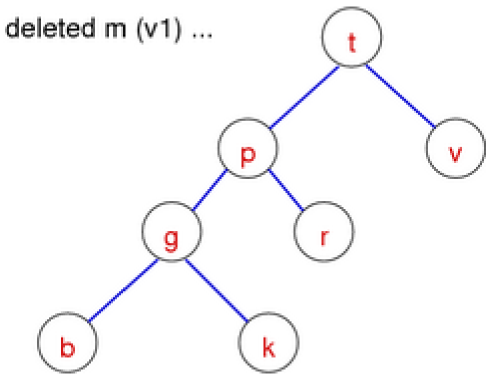
Case 4: item to be deleted has two subtrees



Version 1: right child becomes new root, attach left subtree to min element of right subtree

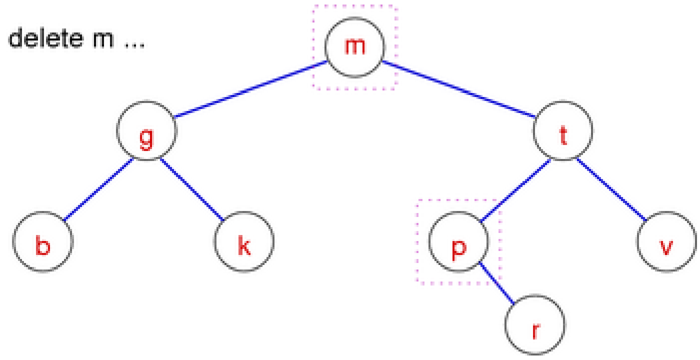
... Deletion from BSTs

Case 4: item to be deleted has two subtrees



... Deletion from BSTs

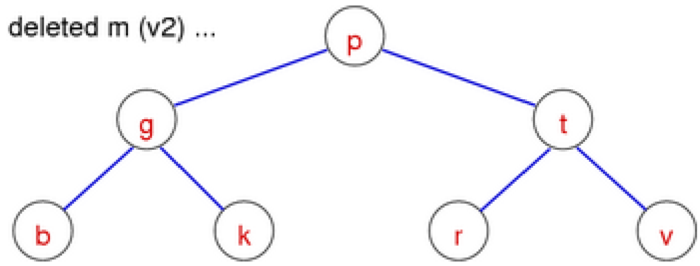
Case 4: item to be deleted has two subtrees



Version 2: *join* left and right subtree

... Deletion from BSTs

Case 4: item to be deleted has two subtrees



... Deletion from BSTs

Pseudocode (version 2 for case 4)

```
TreeDelete(t,item):
|  Input  tree t, item
|  Output t with item deleted
|
|  if t is not empty then          // nothing to do if tree is empty
|  |  if item < data(t) then       // delete item in left subtree
|  |     left(t)=TreeDelete(left(t),item)
|  |  else if item > data(t) then  // delete item in right subtree
|  |     right(t)=TreeDelete(right(t),item)
|  |  else                         // node 't' must be deleted
|  |  |  if left(t) and right(t) are empty then
|  |  |     new=empty tree                    // 0 children
|  |  |  else if left(t) is empty then
|  |  |     new=right(t)                      // 1 child
|  |  |  else if right(t) is empty then
|  |  |     new=left(t)                       // 1 child
|  |  |  else
|  |  |     new=joinTrees(left(t),right(t))  // 2 children
|  |  |  end if
|  |  |  free memory allocated for current node t
|  |  |  t=new
|  |  end if
|  end if
|  return t
```

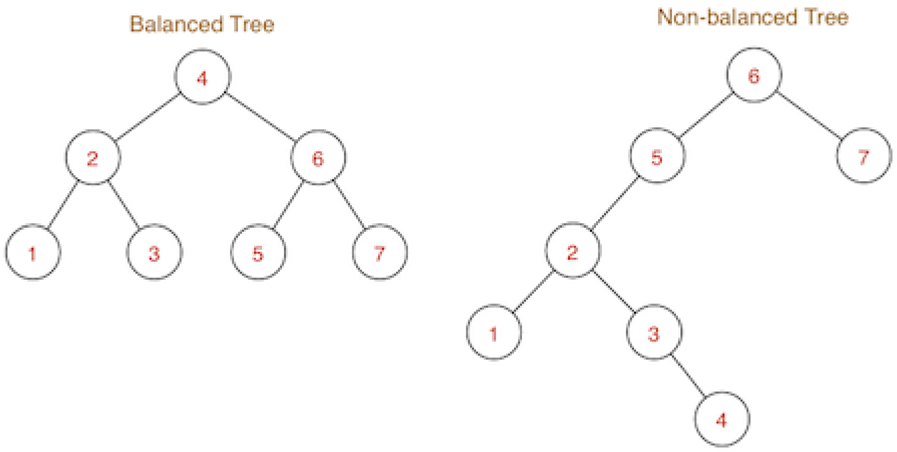
Balanced Binary Search Trees

Goal: build binary search trees which have

- minimum height ⇒ minimum worst case search cost

Best balance you can achieve for tree with *N* nodes:

- abs(#nodes(LeftSubtree) - #nodes(RightSubtree)) ≤ 1, for every node
- height of $log_2 N$ ⇒ worst case search *O(log N)*



Operations for Rebalancing

To assist with rebalancing, we consider new operations:

Left rotation

- move right child to root; rearrange links to retain order

Right rotation

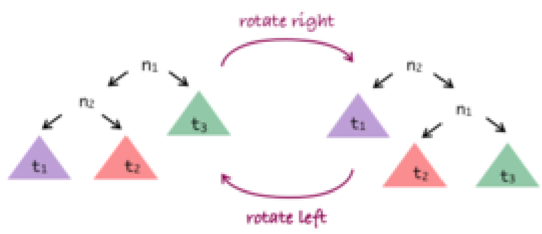
- move left child to root; rearrange links to retain order

Insertion at root

- each new item is added as the new root node

Tree Rotation

In tree below: $t_1 < n_2 < t_2 < n_1 < t_3$



... Tree Rotation

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Method for rotating tree T right:

- N_1 is current root; N_2 is root of N_1's left subtree
- N_1 gets new left subtree, which is N_2's right subtree
- N_1 becomes root of N_2's new right subtree
- N_2 becomes new root

Left rotation: swap left/right in the above.

Cost of tree rotation: *O(1)*

... Tree Rotation

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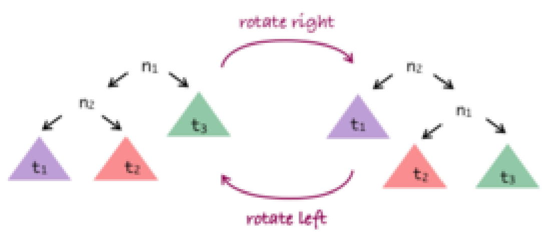
Algorithm for right rotation:

```
rotateRight(n1):
|  Input  tree n1
|  Output n1 rotated to the right
|
|  if n1 is empty or left(n1) is empty then
|     return n1
|  end if
|  n2=left(n1)
|  left(n1)=right(n2)
|  right(n2)=n1
|  return n2
```



Exercise #5: Tree Rotation

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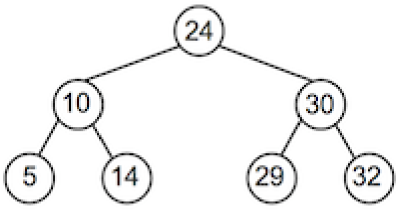
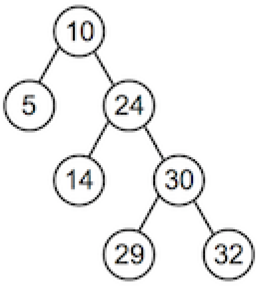
Consider the tree `t`:



Show the result of `rotateRight(t)`



Exercise #6: Tree Rotation

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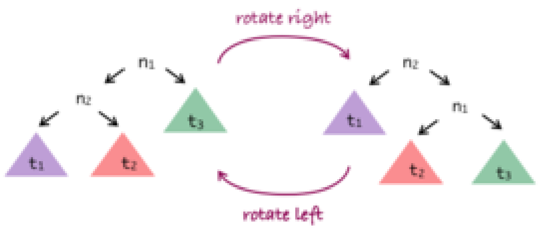
Write the algorithm for left rotation



```
rotateLeft(n2):
|  Input  tree n2
|  Output n2 rotated to the left
|
|  if n2 is empty or right(n2) is empty then
|     return n2
|  end if
|  n1=right(n2)
|  right(n2)=left(n1)
|  left(n1)=n2
|  return n1
```

Insertion at Root

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Previous description of BSTs inserted at leaves.

Different approach: insert new item at root.

Potential disadvantages:

- large-scale rearrangement of tree for each insert

Potential advantages:

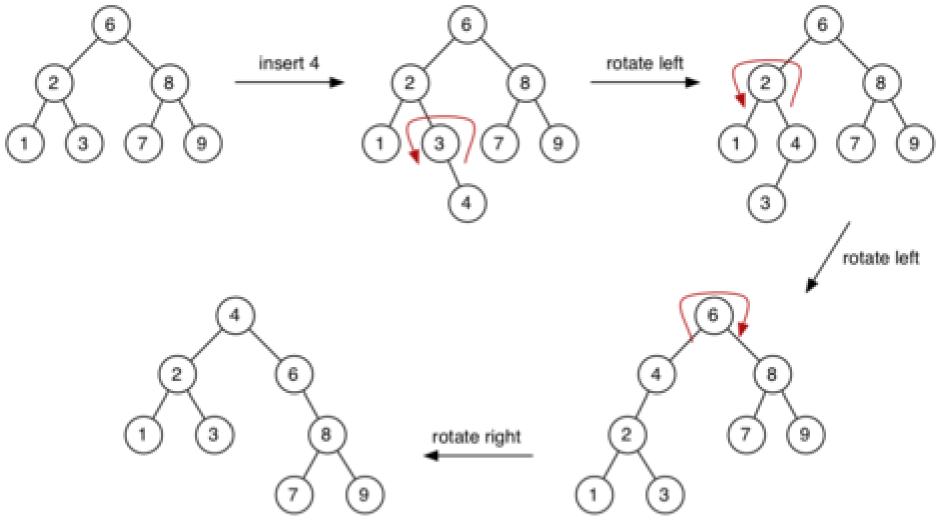
- recently-inserted items are close to root
- low cost if recent items more likely to be searched

... Insertion at Root

Method for inserting at root:

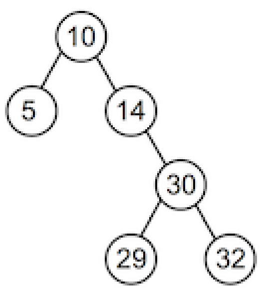
- base case:
  - tree is empty; make new node and make it root
- recursive case:
  - insert new node as root of appropriate subtree
  - lift new node to root by rotation

... Insertion at Root

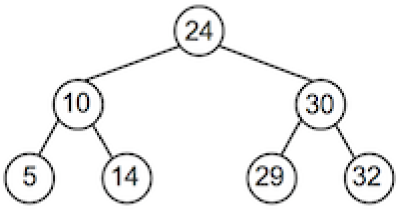


Exercise #7: Insertion at Root

Consider the tree `t`:



Show the result of `insertAtRoot(t, 24)`



... Insertion at Root

Analysis of insertion-at-root:

- same complexity as for insertion-at-leaf: *O(height)*
- tendency to be balanced, but no balance guarantee
- benefit comes in searching
  - for some applications, search favours recently-added items
  - insertion-at-root ensures these are close to root
- could even consider "move to root when found"
  - effectively provides "self-tuning" search tree

  ⇒ more on this in week 8 (real balanced trees)

Randomised BST Insertion

Effects of order of insertion on BST shape:

- best case (for at-leaf insertion): keys inserted in pre-order
  (median key first, then median of lower half, median of upper half, etc.)
- worst case: keys inserted in ascending/descending order
- average case: keys inserted in *random* order ⇒ $O(\log_2 n)$

Tree ADT has no control over order that keys are supplied.

Can the algorithm itself introduce some *randomness*?

In the hope that this randomness helps to balance the tree ...

... Randomised BST Insertion

How can a computer pick a number at random?

- it cannot

Software can only produce *pseudo random numbers*.

- a pseudo random number may appear unpredictable
  - but is actually predictable
- ⇒ implementation may deviate from expected theoretical behaviour
  - more on this in week 10

... Randomised BST Insertion

- Pseudo random numbers in C:

  ```
  rand() // generates random numbers in the range 0 .. RAND_MAX
  ```

  where the constant `RAND_MAX` is defined in `stdlib.h`
  (depends on the computer: on the CSE network, RAND_MAX = 2147483647)

To convert the return value of `rand()` to a number between 0 .. RANGE

- compute the remainder after division by RANGE+1

... Randomised BST Insertion

Approach: normally do leaf insert, randomly do root insert.

```
insertRandom(tree,item)
|  Input  tree, item
|  Output tree with item randomly inserted
|
|  if tree is empty then
|     return new node containing item
|  end if
|  // p/q chance of doing root insert
|  if random number mod q < p then
|     return insertAtRoot(tree,item)
|  else
|     return insertAtLeaf(tree,item)
|  end if
```

E.g. 30% chance ⇒ choose *p=3, q=10*

... Randomised BST Insertion

Cost analysis:

- similar to cost for inserting keys in random order: *O(log n)*
- does not rely on keys being supplied in random order

Approach can also be applied to deletion:

- standard method promotes inorder successor to root
- for the randomised method ...
  - promote inorder successor from right subtree, OR
  - promote inorder predecessor from left subtree

Application of BSTs: Sets

Trees provide efficient search.

Sets require efficient search

- to find where to insert/delete
- to test for set membership

Logical to implement a set ADT via `BSTree`

... Application of BSTs: Sets

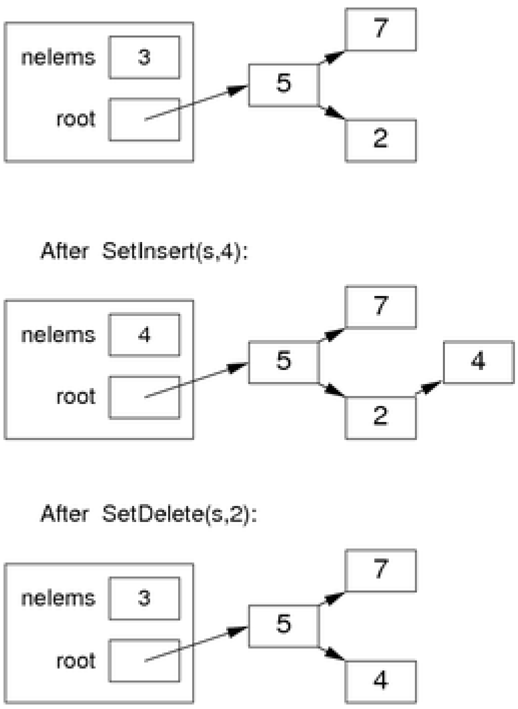
Assuming we have `Tree` implementation

- which precludes duplicate key values
- which implements insertion, search, deletion

then `Set` implementation is

- `SetInsert(Set,Item)` ≡ `TreeInsert(Tree,Item)`
- `SetDelete(Set,Item)` ≡ `TreeDelete(Tree,Item.Key)`
- `SetMember(Set,Item)` ≡ `TreeSearch(Tree,Item.Key)`

... Application of BSTs: Sets



... Application of BSTs: Sets

Concrete representation:

```
#include "BSTree.h"

typedef struct SetRep {
   int    nelems;
   Tree   root;
} SetRep;
```

```
typedef SetRep *Set;

Set newSet() {
   Set S = malloc(sizeof(SetRep));
   assert(S != NULL);
   S->nelems = 0;
   S->root = newTree();
   return S;
}
```

Summary

- Binary search tree (BST) data structure
- Tree traversal
- Basic BST operation: insertion, join, deletion, rotation
- Randomised at-leaf/at-root insertion

- Suggested reading:
  - Sedgewick, Ch. 12.5-12.6, 12.8

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