Week 9: String Algorithms, Approximation

Strings

Strings

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A *string* is a sequence of characters.

An *alphabet* Σ is the set of possible characters in strings.

Examples of strings:

- C program
- HTML document
- DNA sequence
- Digitised image

Examples of alphabets:

- ASCII
- Unicode
- {0,1}
- {A,C,G,T}

... Strings

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Notation:

- *length(P)* ... #characters in *P*
- *λ* ... *empty* string   (*length(λ)* = 0)
- Σ^m ... set of all strings of length *m* over alphabet Σ
- Σ^* ... set of all strings over alphabet Σ

νω denotes the *concatenation* of strings ν and ω

Note: length(νω) = length(ν)+length(ω)   λω = ω = ωλ

... Strings

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Notation:

- *substring* of *P* ... any string *Q* such that *P* = *νQω*, for some $\nu,\omega \in \Sigma^*$
- *prefix* of *P* ... any string *Q* such that *P* = *Qω*, for some $\omega \in \Sigma^*$
- *suffix* of *P* ... any string *Q* such that *P* = *ωQ*, for some $\omega \in \Sigma^*$

Exercise #1: Strings

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The string a/a of length 3 over the ASCII alphabet has

- how many prefixes?
- how many suffixes?
- how many substrings?

- 4 prefixes: "" "a" "a/" "a/a"
- 4 suffixes: "a/a" "/a" "a" ""
- 6 substrings: "" "a" "/" "a/" "/a" "a/a"

Note:
"" means the same as λ (= empty string)

... Strings

ASCII (American Standard Code for Information Interchange)

- Specifies mapping of 128 characters to integers 0..127
- The characters encoded include:
  - upper and lower case English letters: A-Z and a-z
  - digits: 0-9
  - common punctuation symbols
  - special non-printing characters: e.g. *newline* and *space*

Ascii	Char	Ascii	Char	Ascii	Char	Ascii	Char
0	Null	32	Space	64	@	96	`
1	Start of heading	33	!	65	A	97	a
2	Start of text	34	"	66	B	98	b
3	End of text	35	#	67	C	99	c
4	End of transmit	36	$	68	D	100	d
5	Enquiry	37	%	69	E	101	e
6	Acknowledge	38	&	70	F	102	f
7	Audible bell	39	'	71	G	103	g
8	Backspace	40	(	72	H	104	h
9	Horizontal tab	41	)	73	I	105	i
10	Line feed	42	*	74	J	106	j
11	Vertical tab	43	+	75	K	107	k
12	Form feed	44	,	76	L	108	l
13	Carriage return	45	-	77	M	109	m
14	Shift in	46	.	78	N	110	n
15	Shift out	47	/	79	O	111	o
16	Data link escape	48	0	80	P	112	p
17	Device control 1	49	1	81	Q	113	q
18	Device control 2	50	2	82	R	114	r
19	Device control 3	51	3	83	S	115	s
20	Device control 4	52	4	84	T	116	t
21	Neg. acknowledge	53	5	85	U	117	u
22	Synchronous idle	54	6	86	V	118	v
23	End trans. block	55	7	87	W	119	w
24	Cancel	56	8	88	X	120	x
25	End of medium	57	9	89	Y	121	y
26	Substitution	58	:	90	Z	122	z
27	Escape	59	;	91	[	123	{
28	File separator	60	<	92	\	124	|
29	Group separator	61	=	93	]	125	}
30	Record separator	62	>	94	^	126	~
31	Unit separator	63	?	95	_	127	Forward del.

... Strings

UTF-8

- Most common Unicode standard
- Specifies mapping of around 150,000 characters
- ASCII-compatible

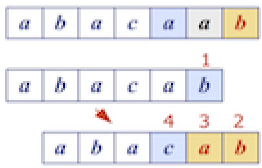
- `0b0xxxxxxx` ASCII letters 0-127
- Two, three and four byte long characters
  - `0b110xxxxxxxxxxxxx` most Latin-script alphabets
  - `0xEXXXXX` Chinese, Japanese, Korean characters
  - `0xFXXXXXXX` mathematical symbols, emojis

  unicode.org/emoji/charts/full-emoji-list.html

Pattern Matching

Pattern Matching

Example (pattern checked *backwards*):



- *Text* ... abacaab
- *Pattern* ... abacab

... Pattern Matching

Given two strings *T* (text) and *P* (pattern),
the *pattern matching problem* consists of finding a substring of *T* equal to *P*

Applications:

- Text editors
- Search engines
- Biological research

... Pattern Matching

Naive pattern matching algorithm

- checks for each possible shift of *P* relative to *T*
  - until a match is found, or
  - all placements of the pattern have been tried

```
NaiveMatching(T,P):
|  Input  text T of length n, pattern P of length m
|  Output starting index of a substring of T equal to P
|         -1 if no such substring exists
|
|  for all i=0..n-m do
|  |  j=0                          // check from left to right
|  |  while j<m and T[i+j]=P[j] do  // test ith shift of pattern
```

```
|   |   |   j=j+1
|   |   |   if j=m then
|   |   |      return i                    // entire pattern checked
|   |   |   end if
|   |   end while
|   end for
|   return -1                         // no match found
```

Analysis of Naive Pattern Matching

Naive pattern matching runs in O(n·m)

Examples of worst case (forward checking):

- *T* = aaa…ah
- *P* = aaah
- may occur in DNA sequences
- unlikely in English text

Exercise #2: Naive Matching

Suppose all characters in *P* are different.

Can you accelerate `NaiveMatching` to run in *O(n)* on an *n*-character text *T*?

When a mismatch occurs between *P[j]* and *T[i+j]*, shift the pattern all the way to align *P[0]* with *T[i+j]*

⇒ each character in *T* checked at most twice

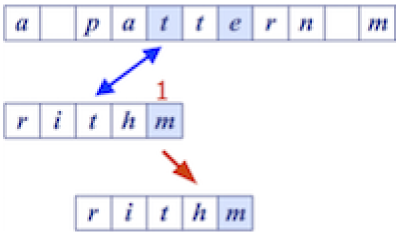
Example:

```
abcdabcdeabcc    abcdabcdeabcc
abcdexxxxxxxx    xxxxabcde
```

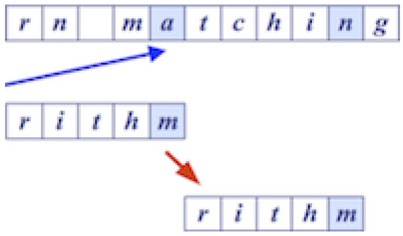
Boyer-Moore Algorithm

The *Boyer-Moore* pattern matching algorithm is based on two heuristics:

- *Looking-glass heuristic*: Compare *P* with subsequence of *T* moving *backwards*
- *Character-jump heuristic*: When a mismatch occurs at *T[i]*=c
  - if *P* contains c ⇒ shift *P* so as to align the last occurrence of c in *P* with *T[i]*



- otherwise ⇒ shift *P* so as to align *P[0]* with *T[i+1]* (a.k.a. "big jump")



... Boyer-Moore Algorithm

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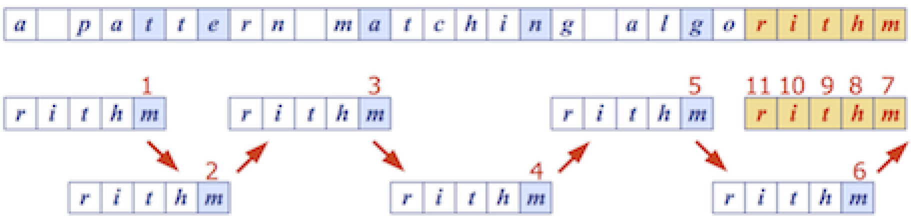
Example:



... Boyer-Moore Algorithm

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Boyer-Moore algorithm preprocesses pattern *P* and alphabet Σ to build

- *last-occurrence function L*
  - *L* maps Σ to integers such that *L(c)* is defined as
    - the largest index *i* such that *P[i]=c*, or
    - -1 if no such index exists

Example: Σ = {a, b, c, d}, *P* = acab

c	a	b	c	d
L(c)	2	3	1	-1

- *L* can be represented by an array indexed by the numeric codes of the characters
- *L* can be computed in *O(m+s)* time (*m* ... length of pattern, *s* ... size of Σ)

... Boyer-Moore Algorithm

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```
BoyerMooreMatch(T,P,Σ):
|  Input  text T of length n, pattern P of length m, alphabet Σ
|  Output starting index of a substring of T equal to P
|         -1 if no such substring exists
```

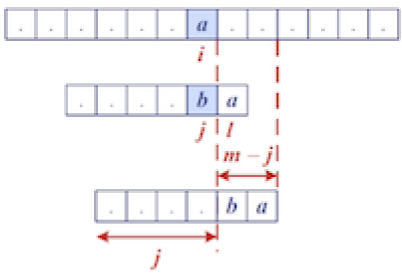
```
|
|  L=lastOccurenceFunction(P, Σ)
|  i=m-1,  j=m-1                    // start at end of pattern
|  repeat
|  |  if T[i]=P[j] then
|  |  |  if j=0 then
|  |  |     return i                  // match found at i
|  |  |  else
|  |  |     i=i-1,  j=j-1            // keep comparing
|  |  |  end if
|  |  else                           // character-jump
|  |  |  i=i+m-min(j,1+L[T[i]])
|  |  |  j=m-1
|  |  end if
|  until i⩾n
|  return -1                         // no match
```

- Biggest jump (m characters ahead) occurs when *L[T[i]]* = -1

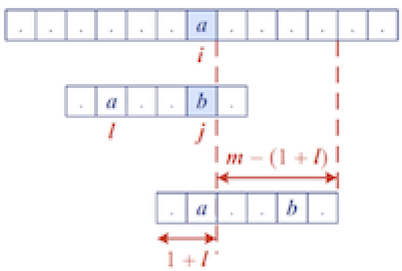
... Boyer-Moore Algorithm

l ... last occurrence L[T[i]] of character T[i]

- Case 1: *j ≤ 1+l*   ⇒ *i = i+m-j*



- Case 2: *1+l < j*   ⇒ *i = i+m-(1+l)*



Exercise #3: Boyer-Moore algorithm

For the alphabet Σ = {a, b, c, d}

1. compute last-occurrence function *L* for pattern *P* = abacab
2. trace Boyer-More on *P* and text *T* = abacaabadcabacabaabb
   - how many comparisons are needed?

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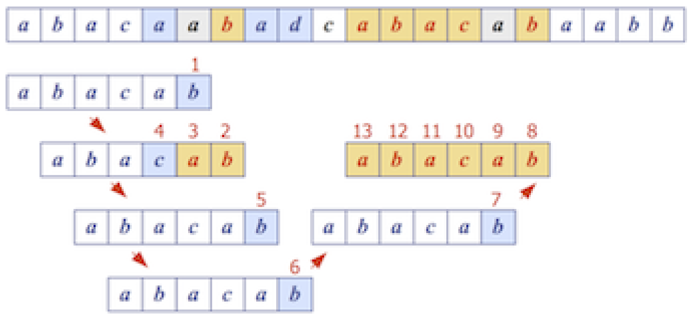
c	a	b	c	d
$L(c)$	4	5	3	-1



13 comparisons in total

... Boyer-Moore Algorithm

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Analysis of Boyer-Moore algorithm:

- Runs in *O(nm+s)* time
  - *m* ... length of pattern *n* ... length of text *s* ... size of alphabet
- Example of worst case:
  - *T* = aaa ⋯ a
  - *P* = baaa
- Worst case may occur in images and DNA sequences but unlikely in English texts
  ⇒ Boyer-Moore significantly faster than naive matching on English text

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Knuth-Morris-Pratt Algorithm

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The *Knuth-Morris-Pratt* algorithm ...

- compares the pattern to the text *left-to-right*
- but shifts the pattern more intelligently than the naive algorithm

Reminder:

- *Q* is a *prefix* of *P* ... *P = Qω*, for some $\omega \in \Sigma^*$
- *Q* is a *suffix* of *P* ... *P = ωQ*, for some $\omega \in \Sigma^*$

... Knuth-Morris-Pratt Algorithm

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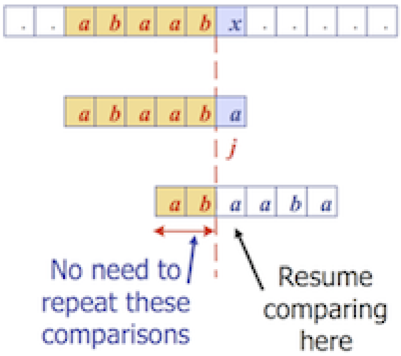
When a mismatch occurs ...

- what is the most we can shift the pattern to avoid redundant comparisons?
- Answer: the largest *prefix* of *P[0..j]* that is a *suffix* of *P[1..j]*



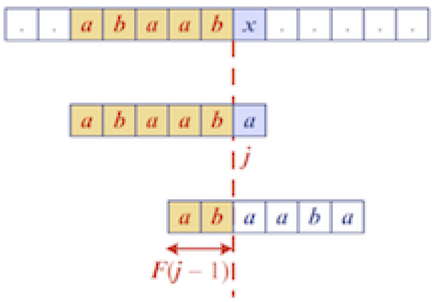
... Knuth-Morris-Pratt Algorithm

KMP preprocesses the pattern *P[0..m-1]* to find matches of its prefixes with itself

- *Failure function F(j)* defined as
  - the size of the *largest prefix* of *P[0..j]* that is also a *suffix* of *P[1..j]*
    - for each position *j*=0..*m-1*
- if mismatch occurs at P_j ⇒ advance *j* to *F(j-1)*

Example: *P* = abaaba

j	0	1	2	3	4	5
P_j	a	b	a	a	b	a
F(j)	0	0	1	1	2	3



... Knuth-Morris-Pratt Algorithm


```
KMPMatch(T,P):
|  Input  text T of length n, pattern P of length m
|  Output starting index of a substring of T equal to P
|         -1 if no such substring exists
|
|  F=failureFunction(P)
|  i=0, j=0                 // start from left
|  while i<n do
|  |  if T[i]=P[j] then
|  |  |  if j=m-1 then
|  |  |     return i-j       // match found at i-j
|  |  |  else
|  |  |     i=i+1, j=j+1    // keep comparing
|  |  |  end if
```

```
|   |  else if j>0 then        // mismatch and j>0?
|   |  |  j=F[j-1]            //  → shift pattern to i-F[j-1]
|   |  else                    // mismtach and j still 0?
|   |  |  i=i+1               //  → begin at next text character
|   |  end if
|  end while
|  return -1                   // no match
```

Exercise #4: KMP-Algorithm

1. compute failure function *F* for pattern *P* = `abacab`
2. trace Knuth-Morris-Pratt on *P* and text *T* = abacaabaccabacabaabb
    - how many comparisons are needed?

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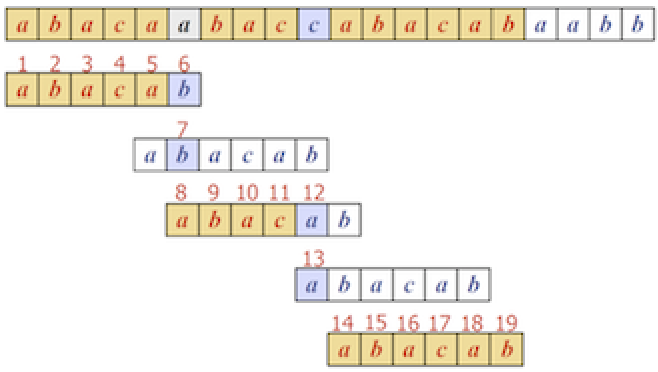
j	0	1	2	3	4	5
P_j	a	b	a	c	a	b
F(j)	0	0	1	0	1	2



19 comparisons in total

... Knuth-Morris-Pratt Algorithm

Analysis of Knuth-Morris-Pratt algorithm:

- Failure function can be computed in *O(m)* time   (→ next slide)
- At each iteration of the while-loop, either
    - *i* increases by one, or
    - the pattern is shifted ≥1 to the right   ("shift amount" *i-j* increases since always F(j-1)<j)

  ⇒ *i* can be incremented at most *n* times, pattern can be shifted at most *n* times

  ⇒ there are no more than 2·*n* iterations of the while-loop

⇒ KMP's algorithm runs in *optimal time O(m+n)*

... Knuth-Morris-Pratt Algorithm

Construction of the failure function matches pattern against *itself*:

```
failureFunction(P):
|  Input  pattern P of length m
|  Output failure function for P
|
|  F[0]=0                  // F[0] is always 0
|  j=1, len=0
|  while j<m do
|  |  if P[j]=P[len] then
|  |     len=len+1         // we have matched len+1 characters
|  |     F[j]=len          // P[0..len-1] = P[len-1..j]
|  |     j=j+1
|  |  else if len>0 then    // mismatch and len>0?
|  |     len=F[len-1]      //  → use already computed F[len] for new len
|  |  else                  // mismatch and len still 0?
|  |     F[j]=0            //  → no prefix of P[0..j] is also suffix of P[1..j]
|  |     j=j+1             //  → continue with next pattern character
|  |  end if
|  end while
|  return F
```

Exercise #5:

Trace the `failureFunction` algorithm for pattern P = `abaaba`

```
                        ⇒          F[0]=0
j=1, len=0, P[1]≠P[0] ⇒          F[1]=0
j=2, len=0, P[2]=P[0] ⇒ len=1, F[2]=1
j=3, len=1, P[3]≠P[1] ⇒ len=F[0]=0
j=3, len=0, P[3]=P[0] ⇒ len=1, F[3]=1
j=4, len=1, P[4]=P[1] ⇒ len=2, F[4]=2
j=5, len=2, P[5]=P[2] ⇒ len=3, F[5]=3
```

... Knuth-Morris-Pratt Algorithm

Analysis of failure function computation:

- At each iteration of the while-loop, either
  - i increases by one, or
  - the "shift amount" i-j increases by at least one (remember that always F(j-1)<j)
- Hence, there are no more than 2·m iterations of the while-loop

⇒ failure function can be computed in *O(m)* time

Boyer-Moore vs KMP

Boyer-Moore algorithm

- decides how far to jump ahead based on the mismatched character in the text
- works best on large alphabets and natural language texts (e.g. English)

Knuth-Morris-Pratt algorithm

- uses information embodied in the pattern to determine where the next match could begin
- works best on small alphabets (e.g. `A`, `C`, `G`, `T`)

For the keen: The article "Average running time of the Boyer-Moore-Horspool algorithm" shows that the time is inversely proportional to size of alphabet

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Word Matching With Tries

Preprocessing Strings

Preprocessing the *pattern* speeds up pattern matching queries

- After preprocessing *P*, KMP algorithm performs pattern matching in time proportional to the text length

If the text is large, immutable and searched for often (e.g., works by Shakespeare)

- we can preprocess the *text* instead of the pattern

... Preprocessing Strings

Reminder: A *trie* ...

- is a compact data structure for representing a set of strings
  - e.g. all the words in a text, a dictionary etc.

Tries support pattern matching queries in time *proportional* to the *pattern* size

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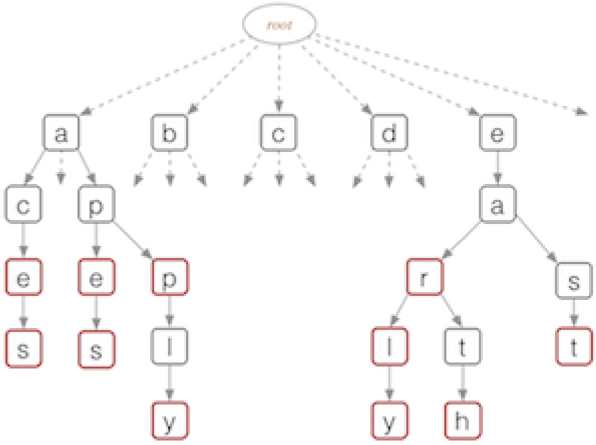
... Preprocessing Strings

Tries are trees organised using parts of keys (rather than whole keys)



... Preprocessing Strings

Complexity of standard tries:

- *O(n)* space
- insertion and search in time *O(m)*
  - *n* ... total size of text (e.g. sum of lengths of all strings in a given dictionary)
  - *m* ... size of the string parameter of the operation (the "key")

Word Matching With Tries

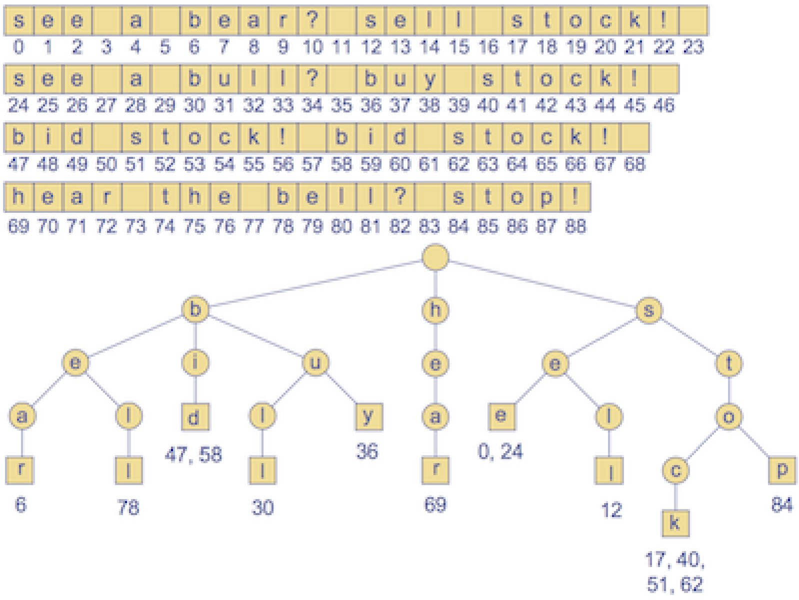
Word Matching with Tries

Preprocessing the text:

1. Insert all searchable words of a text into a trie
2. Each finishing node stores the occurrence(s) of the associated word in the text

... Word Matching with Tries

Example text and corresponding trie of searchable words:

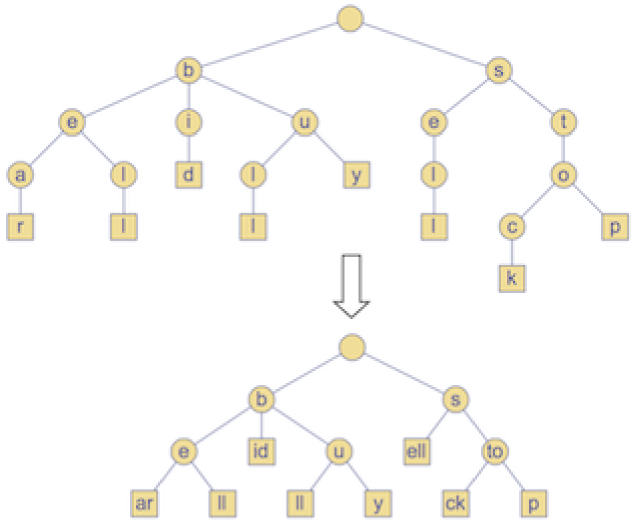


Compressed Tries

Compressed tries ...

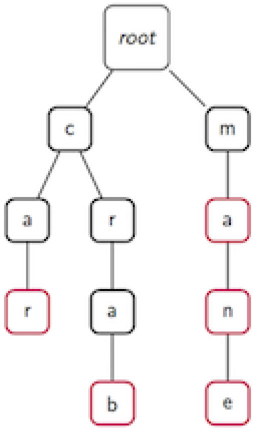
- have internal nodes of degree ≥ 2
- are obtained from standard tries by compressing "redundant" chains of nodes

Example:



Exercise #6: Compressed Tries

Consider this uncompressed trie:



How many nodes (including the root) are needed for the compressed trie?

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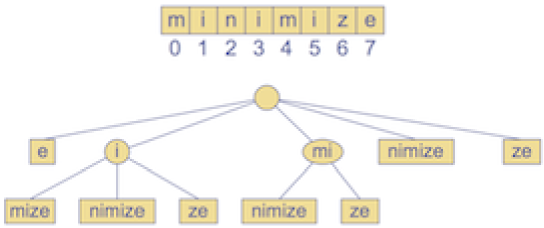
Pattern Matching With Suffix Tries

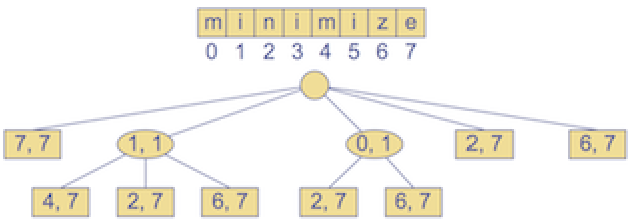
The *suffix trie* of a text *T* is the compressed trie of all the suffixes of *T*

Example:



... Pattern Matching With Suffix Tries

Compact representation:



... Pattern Matching With Suffix Tries

Input:

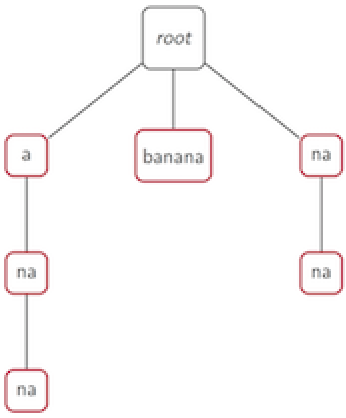
- compact suffix trie for text *T*
- pattern *P*

Goal:

- find starting index of a substring of T equal to P

Exercise #7: Suffix Tries

Construct the compressed suffix trie for T = `banana`



... Pattern Matching With Suffix Tries


```
suffixTrieMatch(trie,P):
|  Input  compact suffix trie for text T, pattern P of length m
|  Output starting index of a substring of T equal to P
|         -1 if no such substring exists
|
|  j=0, v=root of trie
|  repeat
|  |  // we have matched j characters
|  |  if ∃w∈children(v) such that P[j]=T[start(w)] then
|  |  |  i=start(w)            // start(w) is the start index of w
|  |  |  x=end(w)-i+1         // end(w) is the end index of w
|  |  |  if m≤x then   // length of suffix ≤ length of the node label?
|  |  |     if P[j..j+m-1]=T[i..i+m-1] then
|  |  |        return i-j      // match at i-j
|  |  |     else
|  |  |        return -1       // no match
|  |  |  else if P[j..j+x-1]=T[i..i+x-1] then
|  |  |     j=j+x, m=m-x       // update suffix start index and length
|  |  |     v=w                // move down one level
|  |  |  else return -1         // no match
|  |  |  end if
|  |  else
|  |     return -1
|  |  end if
|  until v is leaf node
|  return -1                   // no match
```

... Pattern Matching With Suffix Tries

Analysis of pattern matching using suffix tries:

Suffix trie for a text of size n ...

- can be constructed in *O(n)* time
- uses *O(n)* space
- supports pattern matching queries in *O(m)* time
  - *m* ... length of the pattern

Text Compression

Text Compression

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Problem: Efficiently encode a given string *X* by a smaller string *Y*

Applications:

- Save memory and/or bandwidth

Huffman's algorithm

- computes frequency *f(c)* for each character *c*
- encodes high-frequency characters with short code
- no code word is a prefix of another code word
- uses optimal *encoding tree* to determine the code words

... Text Compression

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Code ... mapping of each character to a binary code word

Prefix code ... binary code such that no code word is prefix of another code word

Encoding tree ...

- represents a prefix code
- each leaf stores a character
- code word given by the path from the root to the leaf (0 for left child, 1 for right child)

... Text Compression

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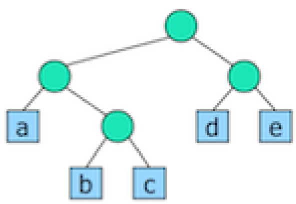
Example:

00	010	011	10	11
a	b	c	d	e



... Text Compression

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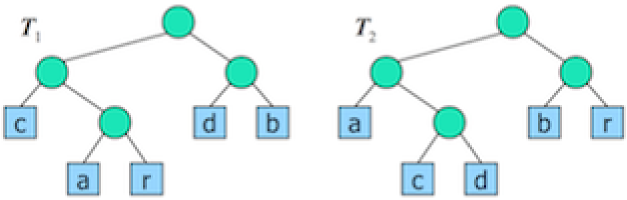
Text compression problem

Given a text T, find a prefix code that yields the shortest encoding of T

- short codewords for frequent characters
- long code words for rare characters

Exercise #8:

Two different prefix codes:



Which code is more efficient for T = `abracadabra`?

T_1 requires 29 bits to encode text T,
T_2 requires 24 bits.

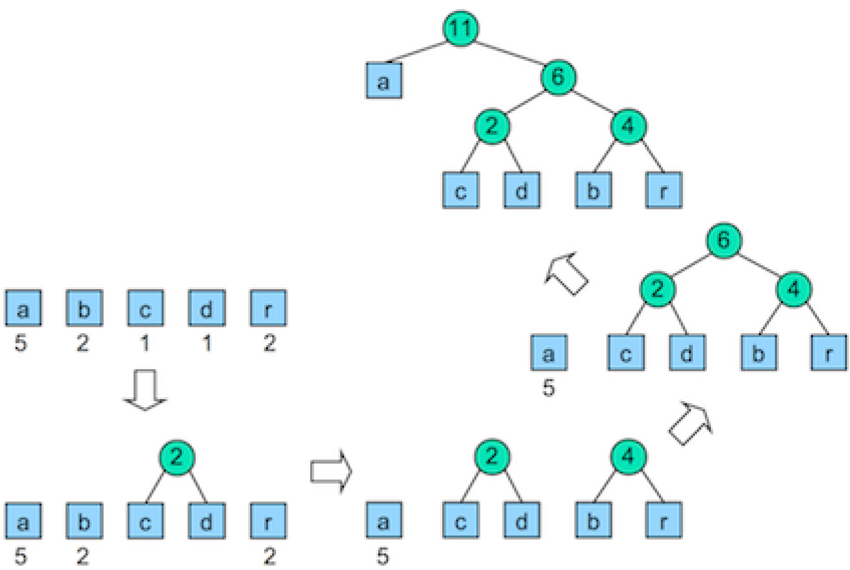
01011011010000101001011011010 vs 001011000100001100101100

Huffman Code

Huffman's algorithm

- computes frequency *f(c)* for each character
- successively combines pairs of lowest-frequency characters to build encoding tree "bottom-up"

Example: `abracadabra`



... Huffman Code

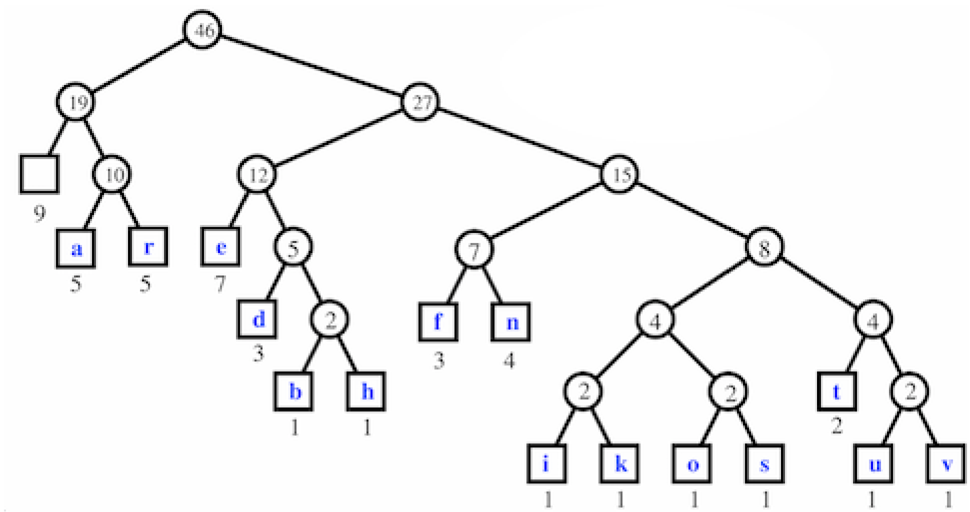
Huffman's algorithm using priority queue:

```
HuffmanCode(T):
|  Input  string T of size n
|  Output optimal encoding tree for T
|
|  compute frequency array
|  Q=new priority queue
|  for all characters c do
|  |  T=new single-node tree storing c
|  |  join(Q,T) with frequency(c) as key
|  end for
|  while |Q|≥2 do
|  |  f1=Q.minKey(), T1=leave(Q)
|  |  f2=Q.minKey(), T2=leave(Q)
|  |  T=new tree node with subtrees T1 and T2
|  |  join(Q,T) with f1+f2 as key
|  end while
|  return leave(Q)
```

Exercise #9: Huffman Code

Construct a Huffman tree for: a fast runner need never be afraid of the dark

Character		a	b	d	e	f	h	i	k	n	o	r	s	t	u	v
Frequency	9	5	1	3	7	3	1	1	1	4	1	5	1	2	1	1



... Huffman Code

Analysis of Huffman's algorithm:

- *O(n+d·log d)* time
  - *n* ... length of the input text *T*
  - *d* ... number of distinct characters in *T*

Approximation

Approximation for Numerical Problems

Approximation is often used to solve numerical problems by

- solving a simpler, but much more easily solved, problem
- where this new problem gives an approximate solution
- and refine the method until it is "accurate enough"

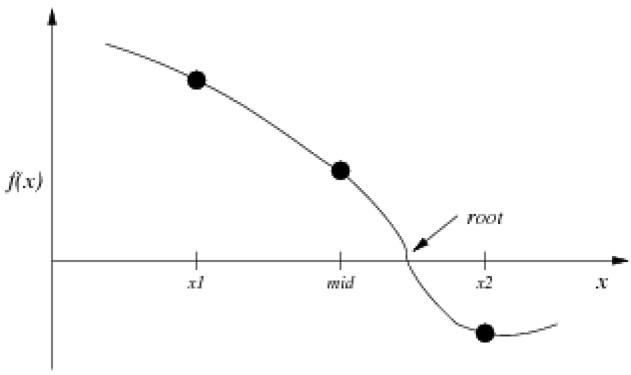
Examples:

- roots of a function *f*
- length of a curve determined by a function *f*
- ... and many more

... Approximation for Numerical Problems

Example: Finding Roots

Find where a function crosses the x-axis:



Generate and test: move x_1 and x_2 together until "close enough"

... Approximation for Numerical Problems

A simple approximation algorithm for finding a root in a given interval:

```
bisection(f,x₁,x₂):
|  Input  function f, interval [x₁,x₂]
|  Output x∈[x₁,x₂] with f(x)≅0
|
|  repeat
|  |  mid=(x₁+x₂)/2
|  |  if f(x₁)*f(mid)<0 then
|  |     x₂=mid      // root to the left of mid
|  |  else
```

```
|  |     x1=mid      // root to the right of mid
|  |  end if
|  until f(mid)=0 or x2-x1<ε    // ε : accuracy
|  end while
|  return mid
```

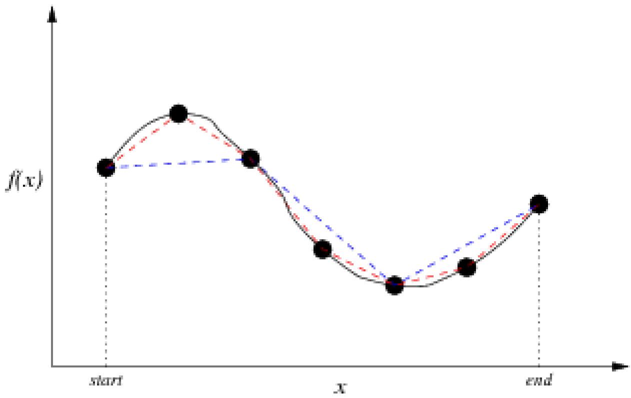
`bisection` guaranteed to converge to a root if *f* continuous on [x_1,x_2] and $f(x_1)$ and $f(x_2)$ have opposite signs

---

... Approximation for Numerical Problems

Example: Length of a Curve

Estimate length: approximate curve as sequence of straight lines.




```
length=0,  δ=(end-start)/StepSize
for each x∈[start+δ,start+2δ,..,end] do
   length = length + sqrt(δ² + (f(x)-f(x-δ))²)
end for
```

Approximation for NP-hard Problems

Approximation is often used for NP-hard problems ...

- computing a near-optimal solution
- in polynomial time

Examples:

- vertex cover of a graph
- subset-sum problem

Vertex Cover

Reminder: Graph *G = (V,E)*

- set of vertices *V*
- set of edges *E*

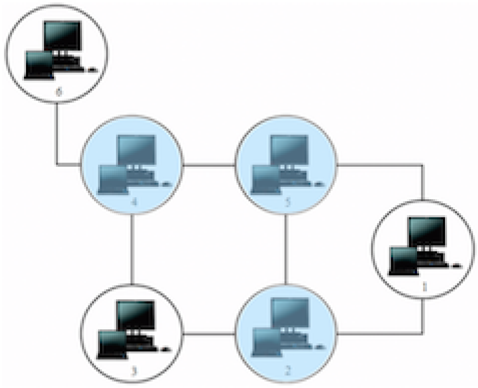
Vertex cover C of *G* ...

- $C \subseteq V$
- for all edges $(u,v) \in E$ either $v \in C$ or $u \in C$ (or both)

⇒ All edges of the graph are "covered" by vertices in C

... Vertex Cover

Example (6 nodes, 7 edges, 3-vertex cover):



Applications:

- Computer Network Security
  - compute minimal set of routers to cover all connections
- Biochemistry

... Vertex Cover

size of vertex cover C ... $|C|$ (number of elements in C)

optimal vertex cover ... a vertex cover of minimum size

Theorem.
Determining whether a graph has a vertex cover of a given size k is an NP-complete problem.

... Vertex Cover

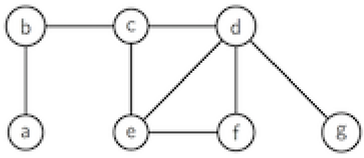
An approximation algorithm for vertex cover:

```
approxVertexCover(G):
|  Input  undirected graph G=(V,E)
|  Output vertex cover of G
|
|  C=∅
|  unusedE=E
|  while unusedE≠∅
|  |  choose any (v,w)∈unusedE
|  |  C = C∪{v,w}
|  |  unusedE = unusedE\{all edges incident on v or w}
```

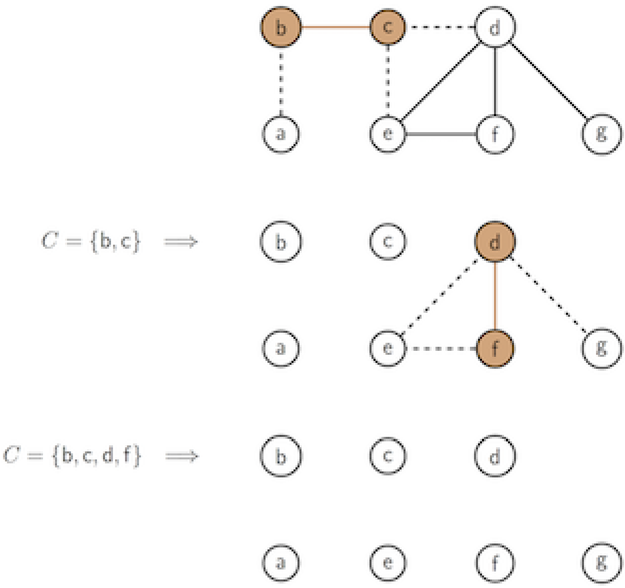
```
|   end while
|   return C
```

Exercise #10: Vertex Cover

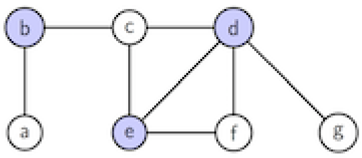
Show how the approximation algorithm produces a vertex cover on:



Possible result:



What would be an optimal vertex cover?



... Vertex Cover

Theorem.
The approximation algorithm returns a vertex cover *at most twice the size* of an optimal cover.

Proof. Any (optimal) cover must include at least one endpoint of each chosen edge.

Cost analysis ...

- repeatedly select an edge from E
  - add endpoints to C
  - delete all edges in E covered by endpoints

Time complexity: O(V+E) (adjacency list representation)

Summary

- Alphabets and words
- Pattern matching
  - Boyer-Moore, Knuth-Morris-Pratt
  - Tries
- Text compression
  - Huffman code
- Approximation
  - numerical problems
  - vertex cover

- Suggested reading:
  - approximation ... Moffat, Ch. 9.4

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