Week 4: Graph Data Structures

Graph Definitions

Graphs

Many applications require

- a collection of *items* (i.e. a set)
- *relationships*/connections between items

Examples:

- maps: items are cities, connections are roads
- web: items are pages, connections are hyperlinks

Collection types you're familiar with

- arrays and lists ... linear sequence of items

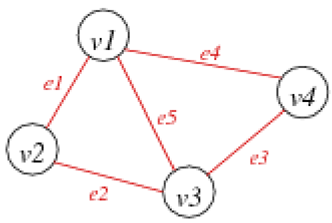
Graphs are more general ... allow arbitrary connections

... Graphs

A *graph G = (V,E)*

- *V* is a set of *vertices*
- *E* is a set of *edges* (subset of *V×V*)

Example:



$V = \{v1, v2, v3, v4\}$

$E = \{e1, e2, e3, e4, e5\}$

... Graphs

A real example: Australian road distances

Distance	Adelaide	Brisbane	Canberra	Darwin	Melbourne	Perth	Sydney
Adelaide	-	2055	-	3051	732	2716	-
Brisbane	2055	-	-	3429	1671	-	982

Canberra	-	-	-	-	658	-	309
Darwin	3051	3429	-	-	-	4049	-
Melbourne	732	1671	658	-	-	-	873
Perth	2716	-	-	4049	-	-	-
Sydney	-	982	309	-	873	-	-

Notes: vertices are cities, edges are distance between cities, symmetric

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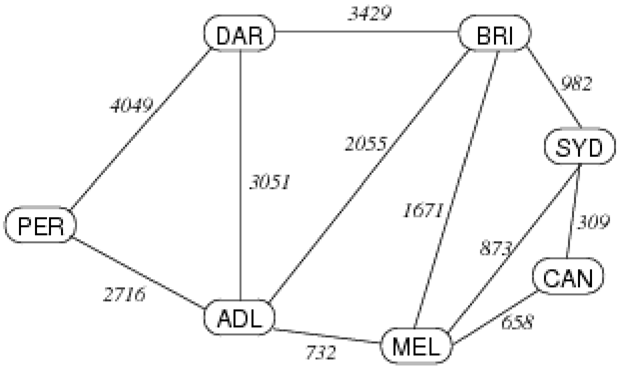
... Graphs

9/110

Alternative representation of above:



... Graphs

10/110

Questions we might ask about a graph:

- is there a way to get from item A to item B?
- what is the best way to get from A to B?
- which items are connected?

Graph algorithms are generally more complex than tree/list ones:

- no implicit order of items
- graphs may contain cycles
- concrete representation is less obvious
- algorithm complexity depends on connection complexity

Properties of Graphs

11/110

Terminology: *|V|* and *|E|* (cardinality) normally written just as *V* and *E*.

A graph with *V* vertices has at most *V(V-1)/2* edges.

The ratio *E:V* can vary considerably.

- if *E* is closer to V^2, the graph is *dense*

- if E is closer to V, the graph is *sparse*
    - Example: web pages and hyperlinks, intersections and roads on street map

Knowing whether a graph is sparse or dense is important

- may affect choice of data structures to represent graph
- may affect choice of algorithms to process graph

Exercise #1: Number of Edges

The edges in a graph represent pairs of connected vertices. A graph with V has V^2 such pairs.

Consider $V = \{1,2,3,4,5\}$ with all possible pairs:

$E = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (2,1), (2,2), ..., (4,5), (5,5) \}$

Why do we say that the maximum #edges is $V(V-1)/2$?

... because

- (v,w) and (w,v) denote the same edge (in an undirected graph)
- we do not consider loops (v,v) (in undirected graphs)

Graph Terminology

For an edge e that connects vertices v and w

- v and w are *adjacent* (neighbours)
- e is *incident* on both v and w

Degree of a vertex v

- number of edges incident on v

Synonyms:

- vertex = node, edge = arc = link (NB: some people use arc for *directed* edges)

... Graph Terminology

Path: a sequence of vertices where

- each vertex has an edge to its predecessor

Simple path: a path where

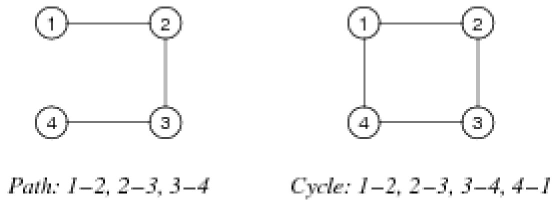
- all vertices and edges are different

Cycle: a path

- that is simple except last vertex = first vertex

Length of path or cycle:

- #edges



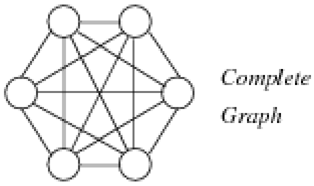
... Graph Terminology

Connected graph

- there is a *path* from each vertex to every other vertex
- if a graph is not connected, it has ≥2 *connected components*

Complete graph K_V

- there is an *edge* from each vertex to every other vertex
- in a complete graph, $E = V(V-1)/2$



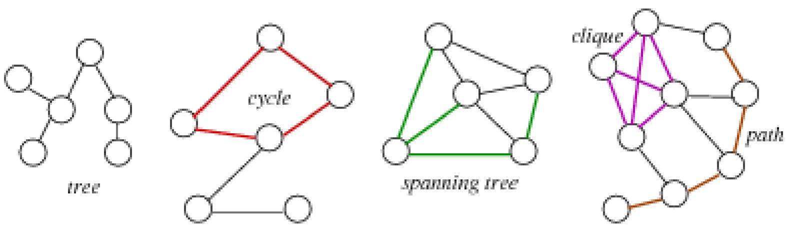
... Graph Terminology

Tree: connected (sub)graph with no cycles

Spanning tree: tree containing all vertices

Clique: complete subgraph

Consider the following single graph:



This graph has 26 vertices, 33 edges, and 4 connected components

Note: The entire graph has no spanning tree; what is shown in green is a spanning tree of the third connected component

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... Graph Terminology

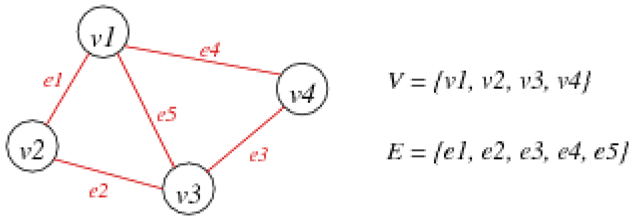
A *spanning tree* of connected graph *G = (V,E)*

- is a subgraph of *G* containing all of *V*
- and is a single tree (connected, no cycles)

A *spanning forest* of non-connected graph *G = (V,E)*

- is a subgraph of *G* containing all of *V*
- and is a set of trees (not connected, no cycles),
    - with one tree for each *connected component*

Exercise #2: Graph Terminology



1. How many edges need to be removed to obtain a spanning tree?
2. How many different spanning trees?

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1. 2
2. $\frac{5 \cdot 4}{2} - 2 = 8$ spanning trees (no spanning tree if we remove *{e1,e2}* or *{e3,e4}*)

... Graph Terminology

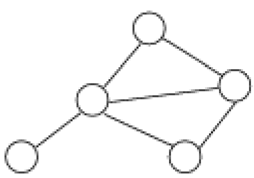
Undirected graph

- *edge(u,v) = edge(v,u),* no self-loops (i.e. no *edge(v,v)*)

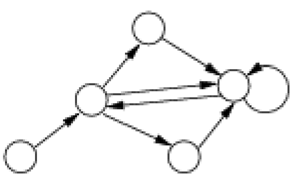
Directed graph

- *edge(u,v) ≠ edge(v,u),* can have self-loops (i.e. *edge(v,v)*)

Examples:



Undirected graph



Directed graph

... Graph Terminology

Other types of graphs ...

Weighted graph

- each edge has an associated value (weight)
- e.g. road map (weights on edges are distances between cities)

Multi-graph

- allow multiple edges between two vertices
- e.g. function call graph (`f()` calls `g()` in several places)

Graph Data Structures

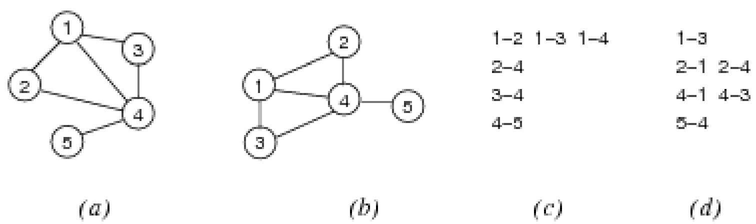
Graph Representations

24/110

Defining graphs:

- need some way of identifying vertices
- could give diagram showing edges and vertices
- could give a list of edges

E.g. four representations of the same graph:



... Graph Representations

25/110

We will discuss three different graph data structures:

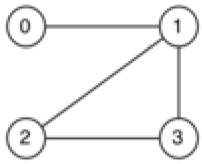
1. Array of edges
2. Adjacency matrix
3. Adjacency list

Array-of-edges Representation

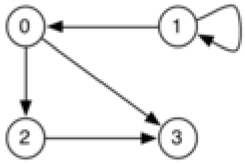
26/110

Edges are represented as an array of `Edge` values (= pairs of vertices)

- space efficient representation
- adding and deleting edges is slightly complex
- undirected: order of vertices in an `Edge` doesn't matter
- directed: order of vertices in an `Edge` encodes direction



[(0,1), (1,2), (1,3), (2,3)]



[(1,0), (1,1), (0,2), (0,3), (2,3)]

For simplicity, we always assume vertices to be numbered `0..V-1`

... Array-of-edges Representation

27/110

Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate enough memory for g.edges[]
|  return g
```

How much is enough? ... No more than *V(V-1)/2* ... Much less in practice (sparse graph)

... Array-of-edges Representation

28/110

Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)  // assumption: (v,w) not in g
|
|  g.edges[g.nE]=(v,w)
|  g.nE=g.nE+1
```

... Array-of-edges Representation

29/110

Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)  // assumption: (v,w) in g
|
|  i=0
|  while (v,w)≠g.edges[i] do
|     i=i+1
|  end while
|  g.edges[i]=g.edges[g.nE-1] // replace (v,w) by last edge in array
|  g.nE=g.nE-1
```

Cost Analysis

30/110

Storage cost: *O(E)*

Cost of operations:

- initialisation: *O(1)*
- insert edge: *O(1)* (assuming edge array has space)
- find/delete edge: *O(E)* (need to find edge in edge array)

If array is full on insert

- allocate space for a bigger array, copy edges across ⇒ *O(E)*

If we maintain edges in order

- use binary search to insert/find edge ⇒ *O(log E)*

Exercise #3: Array-of-edges Representation

Assuming an array-of-edges representation ...

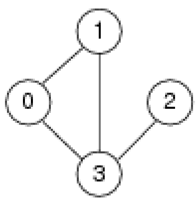
Write an algorithm to output all edges of the graph

```
show(g):
|  Input graph g
|
|  for all i=0 to g.nE-1 do
|     print g.edges[i]
|  end for
```

Time complexity: *O(E)*

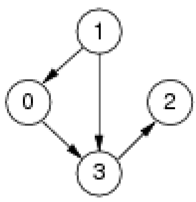
Adjacency Matrix Representation

Edges represented by a $V \times V$ matrix



Undirected graph

A	0	1	2	3
0	0	1	0	1
1	1	0	0	1
2	0	0	0	1
3	1	1	1	0



Directed graph

A	0	1	2	3
0	0	0	0	1
1	1	0	0	1
2	0	0	0	0
3	0	0	1	0

... Adjacency Matrix Representation

Advantages

* easily implemented as 2-dimensional array
* can represent graphs, digraphs and weighted graphs
    * graphs: symmetric boolean matrix
    * digraphs: non-symmetric boolean matrix
    * weighted: non-symmetric matrix of weight values

Disadvantages:

* if few edges (sparse) ⇒ memory-inefficient

... Adjacency Matrix Representation

Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate memory for g.edges[][]
|  for all i,j=0..V-1 do
|     g.edges[i][j]=0   // false
|  end for
|  return g
```

... Adjacency Matrix Representation

Edge insertion

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]=0 then  // (v,w) not in graph
|     g.edges[v][w]=1       // set to true
|     g.edges[w][v]=1
|     g.nE=g.nE+1
|  end if
```

... Adjacency Matrix Representation

Edge removal

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  if g.edges[v][w]≠0 then  // (v,w) in graph
|     g.edges[v][w]=0       // set to false
|     g.edges[w][v]=0
|     g.nE=g.nE-1
|  end if
```

Exercise #4: Show Graph

Assuming an adjacency matrix representation ...

Write an algorithm to output all edges of the graph (no duplicates!)

... Adjacency Matrix Representation


```
show(g):
|  Input graph g
|
|  for all i=0 to g.nV-2 do
|  |  for all j=i+1 to g.nV-1 do
|  |     if g.edges[i][j] then
|  |        print i"—"j
|  |     end if
|  |  end for
|  end for
```

Time complexity: $O(V^2)$

Exercise #5:

Analyse storage cost and time complexity of adjacency matrix representation

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Storage cost: $O(V^2)$

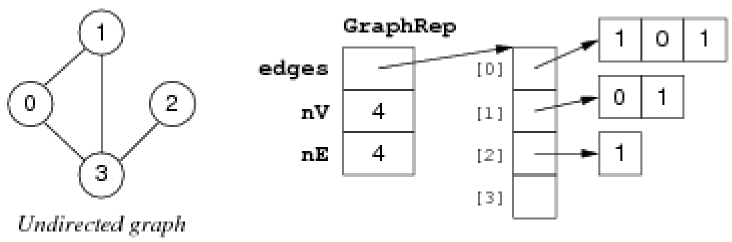
If the graph is sparse, most storage is wasted.

Cost of operations:

- initialisation: $O(V^2)$ (initialise $V \times V$ matrix)
- insert edge: $O(1)$ (set two cells in matrix)
- delete edge: $O(1)$ (unset two cells in matrix)

... Adjacency Matrix Representation

A storage optimisation: store only top-right part of matrix.



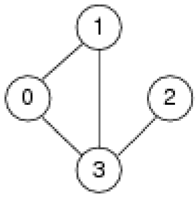
New storage cost: $V-1$ int ptrs + $V(V-1)/2$ ints (but still $O(V^2)$)

Requires us to always use edges (v,w) such that $v < w$.

Adjacency List Representation

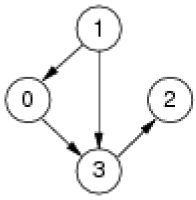
43/110

For each vertex, store linked list of adjacent vertices:



Undirected graph

A[0] = <1, 3>
A[1] = <0, 3>
A[2] = <3>
A[3] = <0, 1, 2>



Directed graph

A[0] = <3>
A[1] = <0, 3>
A[2] = < >
A[3] = <2>

... Adjacency List Representation

44/110

Advantages

- relatively easy to implement in languages like C
- can represent graphs and digraphs
- memory efficient if *E:V* relatively small

Disadvantages:

- one graph has many possible representations
  (unless lists are ordered by same criterion e.g. ascending)

... Adjacency List Representation

45/110

Graph initialisation

```
newGraph(V):
|  Input  number of nodes V
|  Output new empty graph
|
|  g.nV = V   // #vertices (numbered 0..V-1)
|  g.nE = 0   // #edges
|  allocate memory for g.edges[]
|  for all i=0..V-1 do
|     g.edges[i]=NULL   // empty list
|  end for
|  return g
```

... Adjacency List Representation

46/110

Edge insertion:

```
insertEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  insertLL(g.edges[v],w)
|  insertLL(g.edges[w],v)
|  g.nE=g.nE+1
```

... Adjacency List Representation

Edge removal:

```
removeEdge(g,(v,w)):
|  Input graph g, edge (v,w)
|
|  deleteLL(g.edges[v],w)
|  deleteLL(g.edges[w],v)
|  g.nE=g.nE-1
```

Exercise #6:

Analyse storage cost and time complexity of adjacency list representation

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Storage cost: *O(V+E)* (*V* list pointers, total of 2·*E* list elements)

- the larger of *V*,*E* determines the complexity

Cost of operations:

- initialisation: *O(V)* (initialise *V* lists)
- insert edge: *O(1)* (insert one vertex into list)
  - if you don't check for duplicates
- find/delete edge: *O(V)* (need to find vertex in list)

If vertex lists are sorted

- insert requires search of list ⇒ *O(V)*
- delete always requires a search, regardless of list order

Comparison of Graph Representations

	array of edges	adjacency matrix	adjacency list
space usage	*E*	V^2	*V+E*
initialise	*1*	V^2	*V*
insert edge	*1*	*1*	*1*
find/delete edge	*E*	*1*	*V*

Other operations:

	array of edges	adjacency matrix	adjacency list
disconnected(v)?	*E*	*V*	*1*
isPath(x,y)?	$E \cdot \log V$	V^2	$V+E$
copy graph	*E*	V^2	*E*
destroy graph	*1*	*V*	*E*

Graph Abstract Data Type

Graph ADT

Data:

- set of edges, set of vertices

Operations:

- building: create graph, add edge
- deleting: remove edge, drop whole graph
- scanning: check if graph contains a given edge

Things to note:

- set of vertices is fixed when graph initialised
- we treat vertices as `int`s, but could be arbitrary `Item`s

... Graph ADT

Graph ADT interface `graph.h`

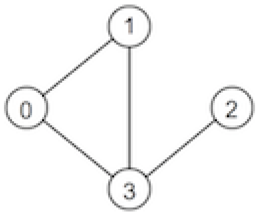
```
// graph representation is hidden
typedef struct GraphRep *Graph;

// vertices denoted by integers 0..N-1
typedef int Vertex;

// edges are pairs of vertices (end-points)
typedef struct Edge { Vertex v; Vertex w; } Edge;

// operations on graphs
Graph newGraph(int V);                 // new graph with V vertices
int   numOfVertices(Graph);            // get number of vertices in a graph
void  insertEdge(Graph, Edge);
void  removeEdge(Graph, Edge);
bool  adjacent(Graph, Vertex, Vertex); /* is there an edge
```

```
                                         between two vertices */
void  showGraph(Graph);               // print all edges in a graph
void  freeGraph(Graph);
```

Exercise #7: Graph ADT Client

Write a program that uses the graph ADT to

- build the graph depicted below
- print all the nodes that are incident to vertex `1` in ascending order

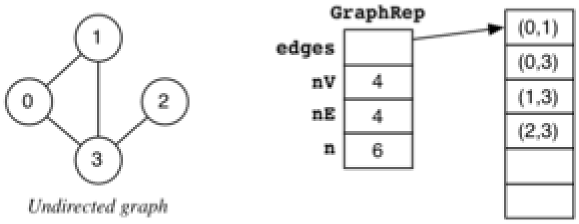



```
#include <stdio.h>
#include "Graph.h"

#define NODES 4
#define NODE_OF_INTEREST 1

int main(void) {
   Graph g = newGraph(NODES);

   Edge e;
   e.v = 0; e.w = 1; insertEdge(g,e);
   e.v = 0; e.w = 3; insertEdge(g,e);
   e.v = 1; e.w = 3; insertEdge(g,e);
   e.v = 3; e.w = 2; insertEdge(g,e);

   int v;
   for (v = 0; v < NODES; v++) {
      if (adjacent(g, v, NODE_OF_INTEREST))
         printf("%d\n", v);
   }

   freeGraph(g);
   return 0;
}
```

Graph ADT (Array of Edges)

Implementation of `GraphRep` (array-of-edges representation)

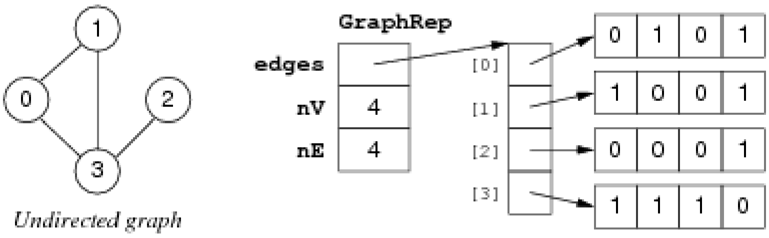
```
typedef struct GraphRep {
   Edge *edges; // array of edges
   int   nV;    // #vertices (numbered 0..nV-1)
   int   nE;    // #edges
   int   n;     // size of edge array
} GraphRep;
```



Graph ADT (Adjacency Matrix)

Implementation of `GraphRep` (adjacency-matrix representation)

```
typedef struct GraphRep {
   int  **edges; // adjacency matrix
   int    nV;    // #vertices
   int    nE;    // #edges
} GraphRep;
```



... Graph ADT (Adjacency Matrix)

Implementation of graph initialisation (adjacency-matrix representation)

```
Graph newGraph(int V) {
   assert(V >= 0);
   int i;

   Graph g = malloc(sizeof(GraphRep));      assert(g != NULL);
   g->nV = V;  g->nE = 0;

   // allocate memory for each row
   g->edges = malloc(V * sizeof(int *));    assert(g->edges != NULL);
   // allocate memory for each column and initialise with 0
   for (i = 0; i < V; i++) {
      g->edges[i] = calloc(V, sizeof(int)); assert(g->edges[i] != NULL);
   }
   return g;
}
```

standard library function `calloc(size_t nelems, size_t nbytes)`

- allocates a memory block of size `nelems*nbytes`
- and sets all bytes in that block to *zero*

... Graph ADT (Adjacency Matrix)

Implementation of edge insertion/removal (adjacency-matrix representation)

```
// check if vertex is valid in a graph
bool validV(Graph g, Vertex v) {
   return (g != NULL && v >= 0 && v < g->nV);
}

void insertEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (!g->edges[e.v][e.w]) {  // edge e not in graph
      g->edges[e.v][e.w] = 1;
      g->edges[e.w][e.v] = 1;
      g->nE++;
   }
}

void removeEdge(Graph g, Edge e) {
   assert(g != NULL && validV(g,e.v) && validV(g,e.w));

   if (g->edges[e.v][e.w]) {   // edge e in graph
      g->edges[e.v][e.w] = 0;
      g->edges[e.w][e.v] = 0;
      g->nE--;
   }
}
```

Exercise #8: Checking Neighbours (i)

Assuming an adjacency-matrix representation ...

Implement a function to check whether two vertices are directly connected by an edge

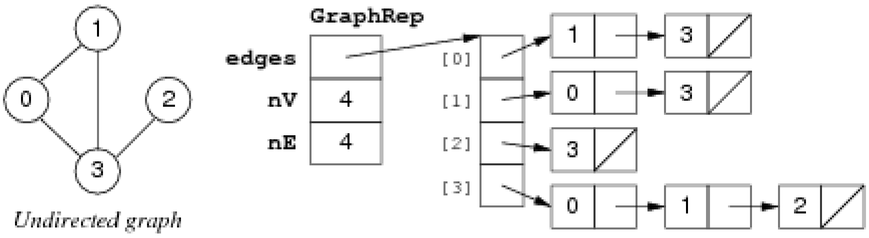
```
bool adjacent(Graph g, Vertex x, Vertex y) { … }
```

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x) && validV(g,y));

   return (g->edges[x][y] != 0);
}
```

Graph ADT (Adjacency List)

Implementation of `GraphRep` (adjacency-list representation)

```
typedef struct GraphRep {
   Node **edges;  // array of lists
   int    nV;     // #vertices
   int    nE;     // #edges
} GraphRep;

typedef struct Node {
   Vertex       v;
   struct Node *next;
} Node;
```



Exercise #9: Checking Neighbours (ii)

Assuming an adjacency list representation ...

Implement a function to check whether two vertices are directly connected by an edge

```
bool adjacent(Graph g, Vertex x, Vertex y) { … }
```

```
bool adjacent(Graph g, Vertex x, Vertex y) {
   assert(g != NULL && validV(g,x));

   return inLL(g->edges[x], y);
}
```

`inLL()` checks if linked list contains an element

Problems on Graphs

Problems on Graphs

What kind of problems do we want to solve on/via graphs?

- is the graph fully-connected?
- can we remove an edge and keep it fully-connected?
- which vertices are reachable from *v*? (transitive closure)
- is there a cycle that passes through all vertices? (circuit)
- what is the cheapest cost path from *v* to *w*?
- is there a tree that links all vertices? (spanning tree)
- what is the minimum spanning tree?
- what is the maximal flow through a graph?
- ...
- can a graph be drawn in a plane with no crossing edges? (planar graphs)
- are two graphs "equivalent"? (isomorphism)
- ...

Graph Algorithms

In this course we examine algorithms for

- graph traversal (simple graphs)
- reachability (directed graphs)
- minimum spanning trees (weighted graphs)
- shortest path (weighted graphs)

- maximum flow (weighted graphs)

Graph Traversal

Finding a Path

69/110

Questions on paths:

- is there a path between two given vertices (*src*,*dest*)?
- what is the sequence of vertices from *src* to *dest*?

Approach to solving problem:

- examine vertices adjacent to *src*
- if any of them is *dest*, then done
- otherwise try vertices two edges from *src*
- repeat looking further and further from *src*

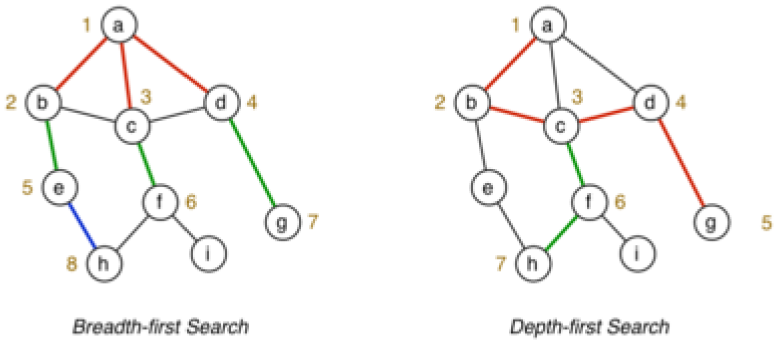
Two strategies for graph traversal/search: *depth-first*, *breadth-first*

- DFS follows one path to completion before considering others
- BFS "fans-out" from the starting vertex ("spreading" subgraph)

... Finding a Path

70/110

Comparison of BFS/DFS search for checking if there is a path from *a* to *h* ...



Both approaches ignore some edges by remembering previously visited vertices.

Depth-first Search

71/110

Depth-first search can be described recursively as

```
depthFirst(G,v):
```

1. mark `v` as visited
2. for each `(v,w)` ∈ *edges*(G) do
   if `w` has not been visited then

```
        depthFirst(w)
```

The recursion induces *backtracking*

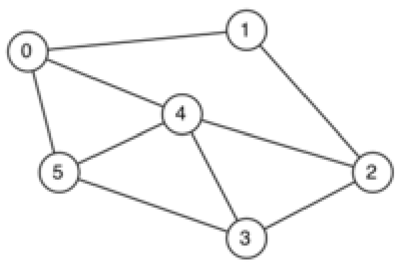
... Depth-first Search

Recursive DFS path checking

```
hasPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if there is a path from src to dest in G,
|         false otherwise
|
|  mark all vertices in G as unvisited
|  return dfsPathCheck(G,src,dest)

dfsPathCheck(G,v,dest):
|  mark v as visited
|  if v=dest then       // found dest
|     return true
|  else
|  |  for all (v,w)∈edges(G) do
|  |  |  if w has not been visited then
|  |  |     return dfsPathCheck(G,w,dest) // found path via w to dest
|  |  |  end if
|  |  end for
|  end if
|  return false         // no path from v to dest
```

Exercise #10: Depth-first Traversal (i)

Trace the execution of `dfsPathCheck(G,0,5)` on:



Consider neighbours in ascending order

Answer:

0 - 1 - 2 - 3 - 4 - 5

... Depth-first Search

Cost analysis:

- all vertices marked as unvisited, each vertex visited at most once ⇒ cost = *O(V)*
- visit all edges incident on visited vertices ⇒ cost = *O(E)*
  - assuming an adjacency list representation

Time complexity of DFS: O(V+E) (adjacency list representation)

- the *larger* of *V,E* determines the complexity

For *dense graphs* … $E \cong V^2 \Rightarrow O(V+E) = O(V^2)$
For *sparse graphs* … $E \cong V \Rightarrow O(V+E) = O(V)$

... Depth-first Search

76/110

Note how different graph data structures affect cost:

- array-of-edges representation
  - visit all edges incident on visited vertices ⇒ cost = $O(V \cdot E)$
  - cost of DFS: $O(V \cdot E)$
- adjacency-matrix representation
  - visit all edges incident on visited vertices ⇒ cost = $O(V^2)$
  - cost of DFS: $O(V^2)$

... Depth-first Search

77/110

Knowing whether a path exists can be useful

Knowing what the path is even more useful
⇒ record the previously visited node as we search through the graph (so that we can then trace path through graph)

Make use of global variable:

- `visited[]` … array to store previously visited node, for each node being visited

---

... Depth-first Search

78/110

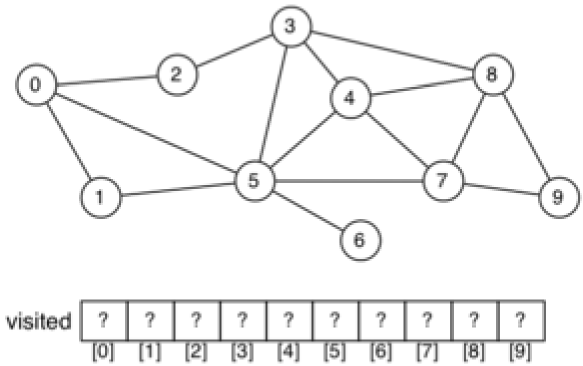
```
visited[] // store previously visited node, for each vertex 0..nV-1

findPath(G,src,dest):
|  Input graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  visited[src]=src                  // starting node of the path
|  if dfsPathCheck(G,src,dest) then  // show path in dest..src order
|  |  v=dest
|  |  while v≠src do
|  |     print v"-"
|  |     v=visited[v]
```

```
|  |  end while
|  |  print src
|  end if
```

```
dfsPathCheck(G,v,dest):
|  if v=dest then                // found edge from v to dest
|     return true
|  else
|  |  for all (v,w)∈edges(G) do
|  |  |  if visited[w]=-1 then
|  |  |  |  visited[w]=v
|  |  |  |  if dfsPathCheck(G,w,dest) then
|  |  |  |     return true        // found path via w to dest
|  |  |  |  end if
|  |  |  end if
|  |  end for
|  end if
|  return false                  // no path from v to dest
```

Exercise #11: Depth-first Traversal (ii)

Show the DFS order in which we visit vertices in this graph when searching for a path from 0 to 6:



Consider neighbours in ascending order

0	0	3	5	3	1	5	4	7	8
[0]	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**

Path: 6-5-1-0

... Depth-first Search

DFS can also be described non-recursively (via a *stack*):

```
hasPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if there is a path from src to dest in G,
|         false otherwise
|
```

```
|  mark all vertices in G as unvisited
|  push src onto new stack s
|  found=false
|  while not found and s is not empty do
|  |  pop v from s
|  |  mark v as visited
|  |  if v=dest then
|  |     found=true
|  |  else
|  |  |  for each (v,w)∈edges(G) such that w has not been visited
|  |  |     push w onto s
|  |  |  end for
|  |  end if
|  end while
|  return found
```

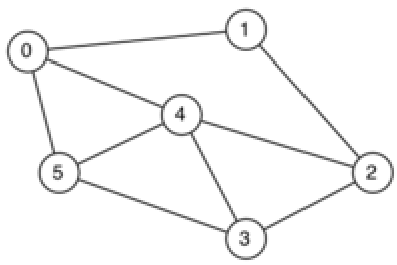
Uses standard stack operations (push, pop, check if empty)

Time complexity is the same: *O(V+E)*

Exercise #12: Depth-first Traversal (iii)

82/110

Show how the stack evolves when executing `findPathDFS(g,0,5)` on:



Push neighbours in *descending* order ... so they get popped in ascending order

---

										4		5		
								3		5		5		5
				1		2		4		4		4		4
				4		4		4		4		4		4
(empty)	→	0	→	5	→	5	→	5	→	5	→	5	→	5

Breadth-first Search

84/110

Basic approach to breadth-first search (BFS):

- visit and mark current vertex
- visit all neighbours of current vertex
- then consider neighbours of neighbours

Notes:

- tricky to describe recursively
- a minor variation on non-recursive DFS search works
  ⇒ switch the *stack* for a *queue*

... Breadth-first Search

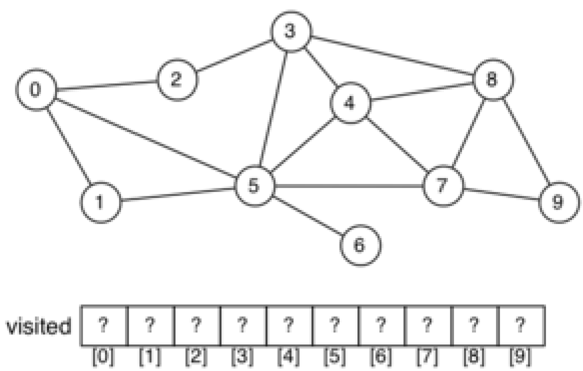
BFS algorithm (records visiting order, marks vertices as visited when put *on* queue):

```
visited[] // array of visiting orders, indexed by vertex 0..nV-1

findPathBFS(G,src,dest):
|  Input  graph G, vertices src,dest
|
|  for all vertices v∈G do
|     visited[v]=-1
|  end for
|  enqueue src into new queue q
|  visited[src]=src
|  found=false
|  while not found and q is not empty do
|  |  dequeue v from q
|  |  if v=dest then
|  |     found=true
|  |  else
|  |  |  for each (v,w)∈edges(G) such that visited[w]=-1 do
|  |  |     enqueue w into q
|  |  |     visited[w]=v
|  |  |  end for
|  |  end if
|  end while
|  if found then
|     display path in dest..src order
|  end if
```

Uses standard queue operations (enqueue, dequeue, check if empty)

Exercise #13: Breadth-first Traversal

Show the BFS order in which we visit vertices in this graph when searching for a path from 0 to 6:



Consider neighbours in ascending order

0	**0**	**0**	**2**	**5**	**0**	**5**	**5**	**3**	**-1**
[0]	**[1]**	**[2]**	**[3]**	**[4]**	**[5]**	**[6]**	**[7]**	**[8]**	**[9]**

Path: 6-5-0

... Breadth-first Search

Time complexity of BFS: O(V+E) (adjacency list representation, same as DFS)

BFS finds a "shortest" path

- based on minimum # edges between *src* and *dest*.
- stops with first-found path, if there are multiple ones

In many applications, edges are weighted and we want path

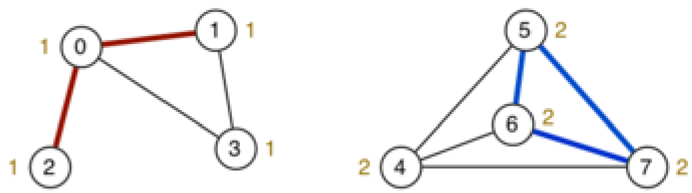
- based on minimum sum-of-weights along path *src* .. *dest*

We discuss weighted/directed graphs later.

Other DFS Examples

Other problems to solve via DFS graph search

- checking for the existence of a cycle
- determining which connected component each vertex is in



Graph with two connected components, a path and a cycle

Exercise #14: Buggy Cycle Check

A graph has a *cycle* if

- it has a path of length > 1
- with start vertex *src* = end vertex *dest*
- and without using any edge more than once

We are not required to give the path, just indicate its presence.

The following DFS cycle check has two bugs. Find them.

```
hasCycle(G):
|  Input  graph G
|  Output true if G has a cycle, false otherwise
|
|  choose any vertex v∈G
|  return dfsCycleCheck(G,v)

dfsCycleCheck(G,v):
|  mark v as visited
|  for each (v,w)∈edges(G) do
|  |  if w has been visited then   // found cycle
|  |     return true
|  |  else if dfsCycleCheck(G,w) then
|  |     return true
|  end for
|  return false                    // no cycle at v
```

---

1. Only one connected component is checked.
2. The loop

   ```
   for each (v,w)∈edges(G) do
   ```

   should exclude the neighbour of v from which you just came, so as to prevent a single edge *w-v* from being classified as a cycle.

Computing Connected Components

Problems:

- how many connected subgraphs are there?
- are two vertices in the same connected subgraph?

Both of the above can be solved if we can

- build an array, one element for each vertex V
- indicating which connected component V is in

- `componentOf[]` ... array [0..nV-1] of component IDs

... Computing Connected Components

Algorithm to assign vertices to connected components:

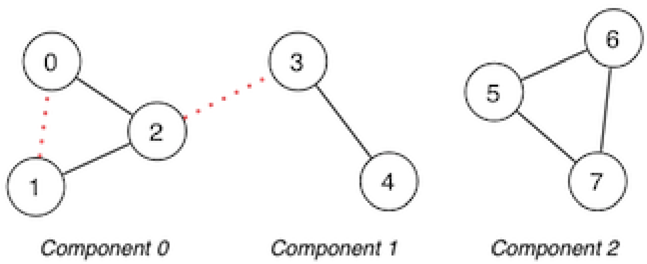
```
components(G):
|  Input graph G
|
|  for all vertices v∈G do
|     componentOf[v]=-1
|  end for
|  compID=0
|  for all vertices v∈G do
|  |  if componentOf[v]=-1 then
```

```
|   |     dfsComponents(G,v,compID)
|   |     compID=compID+1
|   |   end if
|   end for

dfsComponents(G,v,id):
|   componentOf[v]=id
|   for all vertices w adjacent to v do
|      if componentOf[w]=-1 then
|         dfsComponents(G,w,id)
|      end if
|   end for
```

Exercise #15: Connected components

Trace the execution of the algorithm

1. on the graph shown below
2. on the same graph but with the dotted edges added



Consider neighbours in ascending order

1.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1
0	-1	0	-1	-1	-1	-1	-1
0	0	0	-1	-1	-1	-1	-1
0	0	0	1	-1	-1	-1	-1
...							
0	0	0	1	1	2	2	2

2.

[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]
-1	-1	-1	-1	-1	-1	-1	-1
0	-1	-1	-1	-1	-1	-1	-1
0	0	-1	-1	-1	-1	-1	-1
0	0	0	-1	-1	-1	-1	-1
...							

0	0	0	0	0	1	1	1

Hamiltonian and Euler Paths

Hamiltonian Path and Circuit

97/110

Hamiltonian path problem:

- find a path connecting two vertices *v,w* in graph *G*
- such that the path includes each *vertex* exactly once

If *v* = *w*, then we have a *Hamiltonian circuit*

Simple to state, but difficult to solve (*NP*-complete)

Many real-world applications require you to visit all vertices of a graph:

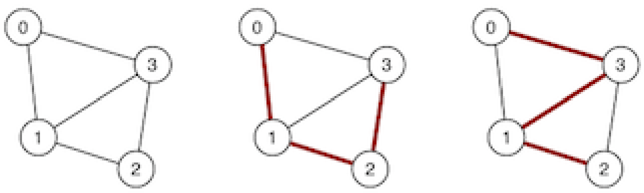
- Travelling salesman
- Bus routes
- ...

Problem named after Irish mathematician, physicist and astronomer Sir William Rowan Hamilton (1805 — 1865)

... Hamiltonian Path and Circuit

98/110

Graph and two possible Hamiltonian paths:



... Hamiltonian Path and Circuit

99/110

Approach:

- generate all possible simple paths (using e.g. DFS)
- keep a counter of vertices visited in current path
- stop when find a path containing *V* vertices

Can be expressed via a recursive DFS algorithm

- similar to simple path finding approach, except
  - keeps track of path length; succeeds if length = *v*-1 (length = *v* for circuit)
  - resets "visited" marker after unsuccessful path

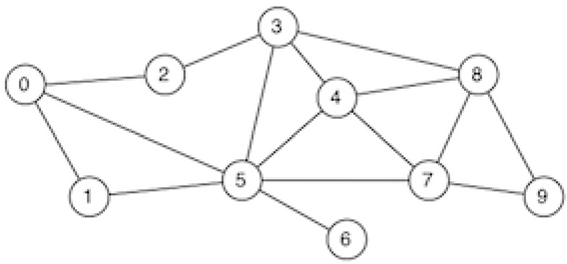
... Hamiltonian Path and Circuit

Algorithm for finding Hamiltonian path:

```
visited[] // array [0..nV-1] to keep track of visited vertices

hasHamiltonianPath(G,src,dest):
|  for all vertices v∈G do
|     visited[v]=false
|  end for
|  return hamiltonR(G,src,dest,#vertices(G)-1)

hamiltonR(G,v,dest,d):
|  Input G    graph
|        v    current vertex considered
|        dest destination vertex
|        d    distance "remaining" until path found
|
|  if v=dest then
|     if d=0 then return true else return false
|  else
|  |  mark v as visited
|  |  for each unvisited neighbour w of v in Gdo
|  |     if hamiltonR(G,w,dest,d-1) then
|  |        return true
|  |     end if
|  |  end for
|  end if
|  mark v as unvisited            // reset visited mark
|  return false
```

Exercise #16: Hamiltonian Path

Trace the execution of the algorithm when searching for a Hamiltonian path from 1 to 6:



Consider neighbours in ascending order

1-0-2-3-4-5-6	d≠0
1-0-2-3-4-5-7-8-9	no unvisited neighbour
1-0-2-3-4-5-7-9-8	no unvisited neighbour
1-0-2-3-4-7-5-6	d≠0
1-0-2-3-4-7-8-9	no unvisited neighbour
1-0-2-3-4-7-9-8	no unvisited neighbour

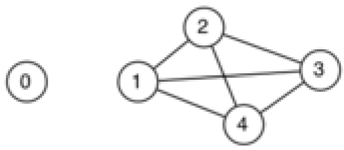
1-0-2-3-4-8-7-5-6	d≠0
1-0-2-3-4-8-7-9	no unvisited neighbour
1-0-2-3-4-8-9-7-5-6	✓

Repeat on your own with `src=0` and `dest=6`

... Hamiltonian Path and Circuit

Analysis: worst case requires *(V-1)!* paths to be examined

Consider a graph with isolated vertex and the rest fully-connected



Checking `hasHamiltonianPath(g,` *x*`, 0)` for any *x*

- requires us to consider every possible path
- e.g 1-2-3-4, 1-2-4-3, 1-3-2-4, 1-3-4-2, 1-4-2-3, ...
- starting from any *x*, there are 3! paths ⇒ 4! total paths
- there is no path of length 5 in these *(V-1)!* possibilities

There is no known simpler algorithm for this task ⇒ *NP*-hard.

Note, however, that the above case could be solved in constant time if we had a fast check for 0 and *x* being in the same connected component

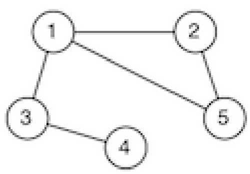
---

Euler Path and Circuit

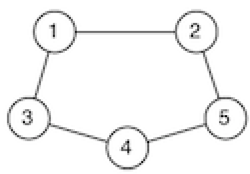
Euler path problem:

- find a path connecting two vertices *v,w* in graph *G*
- such that the path includes each *edge* exactly once
  (note: the path does not have to be simple ⇒ can visit vertices more than once)

If *v = w*, the we have an *Euler circuit*



Euler Path: 4-3-1-5-2-1



Euler Circuit: 1-2-5-4-3-1

Many real-world applications require you to visit all edges of a graph:

- Postman
- Garbage pickup
- ...

Problem named after Swiss mathematician, physicist, astronomer, logician and engineer Leonhard Euler (1707 — 1783)

... Euler Path and Circuit

One possible "brute-force" approach:

- check for each path if it's an Euler path
- would result in factorial time performance

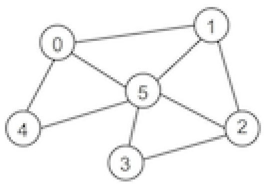
Can develop a better algorithm by exploiting:

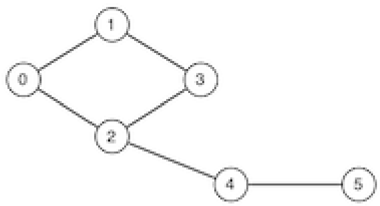
Theorem. A graph has an Euler circuit if and only if
it is connected and all vertices have even degree

Theorem. A graph has a non-circuitous Euler path if and only if
it is connected and exactly two vertices have odd degree

Exercise #17: Euler Paths and Circuits

Which of these two graphs have an Euler path? an Euler circuit?





No Euler circuit

Only the second graph has an Euler path, e.g. 2-0-1-3-2-4-5

... Euler Path and Circuit

Assume the existence of `degree(g,v)` (degree of a vertex, cf. homework exercise 2 this week)

Algorithm to check whether a graph has an Euler path:

```
hasEulerPath(G,src,dest):
|  Input  graph G, vertices src,dest
|  Output true if G has Euler path from src to dest
|         false otherwise
|
```

```
| if src≠dest then       // non~circuitous path
|    if degree(G,src) or degree(G,dest) is even then
|       return false
|    end if
| else if degree(G,src) is odd then  // circuit
|    return false
| end if
| for all vertices v∈G do
|    if v≠src and v≠dest and degree(G,v) is odd then
|       return false
|    end if
| end for
| return true
```

... Euler Path and Circuit

Analysis of `hasEulerPath` algorithm:

- assume that connectivity is already checked
- assume that degree is available via *O(1)* lookup
- single loop over all vertices ⇒ *O(V)*

If degree requires iteration over vertices

- cost to compute degree of a single vertex is *O(V)*
- overall cost is $O(V^2)$

⇒ problem tractable, even for large graphs (unlike Hamiltonian path problem)

For the keen: Linear-time (in the number of edges, *E*) algorithm to compute an Euler path described in [Sedgewick] Ch.17.7

Summary

- Graph terminology
  - vertices, edges, vertex degree, connected graph, tree
  - path, cycle, clique, spanning tree, spanning forest
- Graph representations
  - array of edges
  - adjacency matrix
  - adjacency lists
- Graph traversal
  - depth-first search (DFS)
  - breadth-first search (BFS)
  - cycle check, connected components
  - Hamiltonian paths/circuits, Euler paths/circuits

- Suggested reading (Sedgewick):
  - graph representations ... Ch. 17.1-17.5

- Hamiltonian/Euler paths ... Ch. 17.7
- graph search ... Ch. 18.1-18.3, 18.7

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