Thanks to (B.1, B.6), the third of (2.43) can be written as

$$p_\varphi = m\,(x\,\dot{y} - y\,\dot{x}) + e\,(x\,A_y - y\,A_x) = x\,(m\,\dot{y} + e\,A_y) - y\,(m\,\dot{x} + e\,A_x), \tag{2.44}$$

that coincides with the component of the angular momentum $\mathbf{M} = \mathbf{r} \wedge \mathbf{p} = \mathbf{r} \wedge (m\,\mathbf{u} + e\mathbf{A})$ along the z axis. This result shows that (2.41) holds also when the force acting onto the particle derives from an electromagnetic field.

2.9 Linear Motion

The *linear motion* is the motion of a system having only one degree of freedom. Using the Cartesian coordinate x, and assuming the case where the force acting onto the particle derives from a potential energy $V(x)$, gives the Hamiltonian function (1.32) the form $H = p^2/(2\,m) + V(x)$. As shown in Sect. 1.4, a Hamiltonian function of this type is a constant of motion whence, remembering that here it is $p = m\,\dot{x}$,

$$\frac{1}{2}\,m\,\dot{x}^2 + V(x) = E = \text{const.} \tag{2.45}$$

The constant E is called *total energy*. Its value is given by the initial conditions $x_0 = x(t = a)$, $\dot{x}_0 = \dot{x}(t = a)$. As the kinetic energy $m\,\dot{x}^2/2$ can not be negative, the motion is possible only in the intervals of the x axis such that $V(x) \le E$. In particular, the velocity $\dot{x}$ vanishes at the points where $V = E$. Instead, the intervals where $V > E$ can not be reached by the particle. Equation (2.45) is separable and provides a relation of the form $t = t(x)$,

$$t = a \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{\mathrm{d}\xi}{\sqrt{E - V(\xi)}}. \tag{2.46}$$

By way of example consider a situation like that shown in Fig. 2.1, where it is assumed that to the right of x_C the potential energy V keeps decreasing as $x \to \infty$. If the initial position of the particle is $x_0 = x_C$, there the velocity vanishes and the particle is subjected to a positive force $F = -\mathrm{d}V/\mathrm{d}x > 0$. As a consequence, the particle's motion will always be oriented to the right starting from x_C. Such a motion is called *unlimited*. If the initial position is $x_0 > x_C$ and the initial velocity is negative, the particle moves to the left until it reaches the position x_C, where it bounces back. The subsequent motion is the same as described above.

A different situation arises when the initial position of the particle belongs to an interval limited by two zeros of the function $E - V(x)$ like, e.g., x_A and x_B in Fig. 2.1. The motion is confined between x_A and x_B and, for this reason, is called *limited*. The particle bounces back and forth endlessly under the effect of a force that does not depend on time. As a consequence, the time necessary to complete a cycle $x_A \to x_B \to x_A$ is the same for each cycle. In other terms, the motion is periodic in time. Also, from (2.46) it is found by inspection that the time spent by the particle