To calculate the extremum functions of (2.30) it is convenient to parametrize the coordinates in the form $x_i = x_i(\xi)$, where the parameter ξ takes the same limiting values, say, $\xi = a$ at A and $\xi = b$ at B, for the natural and all the virtual trajectories. Letting $\dot{x}_i = \mathrm{d}x_i/\mathrm{d}\xi$ one finds $(\mathrm{d}s/\mathrm{d}\xi)^2 = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2$ which, remembering (1.7), yields the extremum condition for (2.30):

$$\delta \int_a^b \theta \, \mathrm{d}\xi = 0, \qquad \theta = \sqrt{E-V}\,\sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2}, \qquad \frac{\mathrm{d}}{\mathrm{d}\xi}\frac{\partial \theta}{\partial \dot{x}_i} = \frac{\partial \theta}{\partial x_i}. \tag{2.31}$$

The following relations are useful to work out the last of (2.31): $\mathrm{d}s/\mathrm{d}t = u = \sqrt{(2/m)(E-V)}$, $\mathrm{d}x_i = \dot{x}_i \, \mathrm{d}\xi$, $\mathrm{d}x_i/\mathrm{d}t = u_i$. One finds

$$\frac{\partial \theta}{\partial \dot{x}_i} = \sqrt{E-V}\,\frac{\dot{x}_i}{\mathrm{d}s/\mathrm{d}\xi} = \sqrt{\frac{m}{2}}\, u \,\frac{\dot{x}_i \, \mathrm{d}\xi}{\mathrm{d}s} = \sqrt{\frac{m}{2}}\,\frac{\mathrm{d}x_i}{\mathrm{d}t} = \sqrt{\frac{m}{2}}\, u_i, \tag{2.32}$$

$$\frac{\partial \theta}{\partial x_i} = \frac{\mathrm{d}s}{\mathrm{d}\xi}\,\frac{-\partial V/\partial x_i}{2\sqrt{E-V}} = \frac{\mathrm{d}s}{\mathrm{d}\xi}\,\frac{F_i}{2\sqrt{m/2}u} = \frac{\mathrm{d}t}{\mathrm{d}\xi}\,\frac{F_i}{\sqrt{2m}}, \tag{2.33}$$

with F_i the ith component of the force. The last of (2.31) then yields

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\sqrt{\frac{m}{2}}\, u_i\right) = \frac{\mathrm{d}t}{\mathrm{d}\xi}\,\frac{F_i}{\sqrt{2m}}, \qquad F_i = m\,\frac{\mathrm{d}u_i}{\mathrm{d}\xi}\,\frac{\mathrm{d}\xi}{\mathrm{d}t} = m\,\frac{\mathrm{d}u_i}{\mathrm{d}t}. \tag{2.34}$$

In conclusion, the equation that provides the extremum condition for functional G is equivalent to Newton's second law $\mathbf{F} = m\mathbf{a}$.

2.8 Spherical Coordinates—Angular Momentum

Consider a single particle of mass m and use the transformation from the Cartesian (x, y, z) to the spherical (r, ϑ, φ) coordinates shown in Sect. B.1. The kinetic energy is given by (B.7), namely,

$$T = \frac{m}{2}\,(\dot{r}^2 + r^2\dot{\vartheta}^2 + r^2\dot{\varphi}^2 \sin^2\vartheta). \tag{2.35}$$

If the force acting onto the particle is derivable from a potential energy $V = V(x, y, z, t)$, the Lagrangian function in the spherical reference is $L = T - V(r, \vartheta, \varphi, t)$, where T is given by (2.35). The momenta conjugate to the spherical coordinates are

$$\begin{cases} p_r = \partial L/\partial \dot{r} = m\dot{r} \\ p_\vartheta = \partial L/\partial \dot{\vartheta} = mr^2\,\dot{\vartheta} \\ p_\varphi = \partial L/\partial \dot{\varphi} = mr^2\,\dot{\varphi}\,\sin^2\vartheta \end{cases} \tag{2.36}$$

Using (2.36), the kinetic energy is recast as