Unimetry: Proto-Space Reformulation of Special Relativity

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Abstract

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1 Introduction

1.1 Motivation

Special relativity (SR) is usually presented as geometry in a Lorentzian spacetime. While conceptually economical, this viewpoint can obscure two practical facts:

(i) An observer does not measure an abstract curve parameter; physical clocks implement a specific *phase* (proper–time) parametrization along worldlines.

(ii) Many elementary relativistic effects (time dilation, light cone, Doppler shift, aberration) can be viewed as consequences of a single *observer splitting*: how a total "flow budget" decomposes into a component along an observer's time axis and a component orthogonal to it.

The present paper develops a Euclidean proto–space formulation that isolates this observer splitting as the primary geometric datum. We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$ and introduce an *oriented coherent time axis* N (a unit vector field, $\delta(N, N) = 1$). The associated time 1–form $\alpha := N^\flat = \delta(N, \cdot)$ and spatial projector $h := \delta - \alpha \otimes \alpha$ encode the operational notions of "time reading" and "spatial distance" for that observer. The Lorentzian interval then appears as the induced quadratic form

$$g = \alpha \otimes \alpha - h = 2\,\alpha \otimes \alpha - \delta,$$

and proper time emerges as the operational reparametrization $c^2 d\tau^2 = g(dX, dX)$.

A second aim is algebraic: the Euclidean proto–space admits the strictly Euclidean Clifford algebra $\mathcal{C}\ell_{4,0}$. After fixing a reference unit vector e_0, any observer axis N can be represented as a rotor image $N = q e_0 \widetilde{q}$, and calibrated flow states naturally live on a Euclidean 3–sphere $S^3_c \subset T_p\mathcal{E}$, which can be encoded (after a gauge choice) by unit quaternions in a quaternionic subalgebra of $\mathcal{C}\ell_{4,0}$. This provides a compact computational language in which (i) Euclidean circular tilt geometry and (ii) Lorentzian boost kinematics (hyperbolic geometry) are handled uniformly via rotors, with the distinction controlled by the sign of the relevant bivector square.

1.2 Relation to previous work

Attempts to express Lorentzian kinematics in Euclidean terms have a long history. Early geometric constructions already appear in Karapetoff [1], where relativistic transformations are visualized by Euclidean angle geometry. More recent works study various embeddings and correspondences between Euclidean and Lorentzian structures, including [2, 3].

A distinct line of literature aims at an explicitly *Euclidean* reformulation of special relativity. In particular, Euclidean SR can be obtained by changes of variables in which Lorentz transformations are represented as rotations in a Euclidean space (e.g. Gersten [9]). Other proposals postulate an absolute Euclidean background and reinterpret relativistic observables in terms of proper time (e.g. Montanus [10, 11]), or develop related "four–dimensional optics" frameworks in which τ plays a central operational role (e.g. Almeida [12, 13]). For broader context on Euclidean viewpoints beyond SR, see also Atkinson [14].

Our construction differs in emphasis. We do *not* identify the Euclidean norm with the Lorentz interval by a coordinate trick, nor do we postulate an absolute Euclidean time. Instead, we start from an observer splitting encoded by a coherent unit axis N and its time 1–form $\alpha = N^\flat$, and we interpret the Lorentzian interval as the operational quadratic form naturally associated with this splitting. Proper time arises as a reparametrization along worldlines, rather than being built in as a primary coordinate. This standpoint is conceptually aligned with relational–time viewpoints in generally covariant physics, where "time" is implemented by a chosen clock observable (cf. Rovelli's partial observables perspective [7, 8]).

A rigorous pointwise correspondence between a Riemannian metric and a Lorentzian one was established by Reddy, Sharma and Sivaramakrishnan [4]. Given a Riemannian manifold (M, h) and a unit vector field U, they define a Lorentzian metric by $g = h - 2\,U^\flat \otimes U^\flat$. We adopt the sign–flipped variant adapted to the particle–physics convention $(+\,-\,-\,-)$, namely

$$g \;=\; 2\,N^\flat \otimes N^\flat \;-\; \delta,$$

and then make explicit how SR kinematics follows from phase reparametrization and tilt geometry relative to α.

Inertial specialization (SR kinematics). Although the construction applies to any smooth unit field N on $(\mathcal{E}, \delta)$, the present paper focuses on SR kinematics and therefore restricts to the inertial case, assuming that N is parallel with respect to the flat Euclidean connection:

$$\nabla^\delta N = 0. \tag{1.1}$$

Equivalently, in global Cartesian coordinates on $\mathcal{E} \simeq \mathbb{R}^4$, the components N^A are constant and the induced metric $g = 2\,N^\flat \otimes N^\flat - \delta$ is constant (globally Minkowskian). Relativistic effects in the present framework arise from the operational reparametrization and the tilt geometry, rather than from curvature.

Observer dependence and Lorentz invariance. The axis field N is a *geometric datum* (no dynamics is postulated for it in this paper), but it is not an "absolute frame" in the physical sense. In the inertial sector, choosing a constant N is choosing an inertial observer. Any other constant unit axis N' is related to N by a Euclidean isometry $R \in O(4)$, $N' = R_* N$, and the induced form transforms tensorially, $g' = R^* g$. Physical statements are formulated in g–covariant (relational) quantities, so the theory does not privilege a particular representative N. Note that the quadratic form g is invariant under $N \mapsto -N$, whereas the oriented time 1–form $\alpha = N^\flat$ changes sign; accordingly, causal structure is insensitive to this sign, while the choice of "future" requires a consistent orientation.

Geometric algebra viewpoint. Algebraically, our presentation is close in spirit to the geometric–algebra approach to relativity, in which boosts and rotations are treated uniformly as rotors (see, e.g., Hestenes' Space–Time Algebra [5, 6] and the discussion of spacetime algebra versus "imaginary time" in [15]). The difference is that we work throughout in the strictly Euclidean Clifford algebra $\mathcal{C}\ell_{4,0}$: the "space–time split" is encoded by the choice of N (hence α) and is therefore observer–dependent, while Lorentzian boosts are represented by Euclidean rotors after an appropriate change of bivector basis tied to the N–split.

1.3 Contributions

The main contributions of the present paper are:

(C1) **Observer splitting and operational origin of the Lorentz interval.** Starting from a Euclidean proto–space $(\mathcal{E}, \delta)$ and an oriented unit axis N, we introduce the time 1–form $\alpha := N^\flat = \delta(N, \cdot)$ and the spatial projector $h = \delta - \alpha \otimes \alpha$. We make explicit that the observer's interval is the induced quadratic form

$$g = \alpha \otimes \alpha - h = 2\,\alpha \otimes \alpha - \delta,$$

and that proper time arises as an operational reparametrization along worldlines via $c^2 d\tau^2 = g(dX, dX)$ (Sections 2–3 and §4).

(C2) **Euclidean Clifford algebra implementation of observer axes.** We formulate the construction in $\mathcal{C}\ell_{4,0}$ and represent observer axes as rotor images $N = q e_0 \widetilde{q}$, clarifying the associated gauge nonuniqueness and the $N \mapsto -N$ sign issue at the operational level (Section 2).

(C3) **Reparameterization equivalence on the calibrated sphere.** We show that proper–time normalization in the induced Lorentzian metric is equivalent to a constant–speed Euclidean flow constrained to the calibrated sphere $S^3_c \subset T\mathcal{E}$ (Theorem 6.6 and §6).

(C4) **Geometric characterization of the physical sector.** The timelike condition $g(X, X) > 0$ selects an open subset of S^3_c (two polar caps), with the lightlike boundary reached at the finite Euclidean tilt $\xi = \pi/4$ (cf. §**??** and Remark 6.8).

(C5) **Circular versus conic/hyperbolic parametrizations of tilt.** We distinguish the circular angle ξ (normalized by $\|X\|_\delta$) from the conic ratio coordinate ϑ (normalized by $N \cdot X$), and relate both to the additive rapidity η via $\tanh \eta = \tan \vartheta$, yielding a bounded Euclidean description of boost kinematics (Section 5).

(C6) **Projection-based optics and Euclidean GA derivations.** Frequency and direction of null rays are obtained as g–projections, yielding concise derivations of the Doppler shift and aberration. Boosts are encoded within the Euclidean geometric algebra framework as rotor actions tied to the N–split (Section 7).

1.4 Outline

The paper is organized as follows:

- **Section 2** introduces the Euclidean proto–space $(\mathcal{E}, \delta)$, formulates $\mathcal{C}\ell_{4,0}$ on $T\mathcal{E}$, and defines the observer axis N (including its rotor representation). The induced Lorentzian form $g = 2\, N^\flat \otimes N^\flat - \delta$ is then constructed.

- **Section 3** develops basic properties of the induced Lorentzian structure: the orthogonal decomposition relative to N, norm identities, causal classification, and the geometry of the null cone.

- **Section 4** formulates the operational clock viewpoint: we introduce a δ–calibrated proto–parameter χ, define coordinate time and proper time via α, and make explicit the phase reparametrization $\chi \mapsto \tau$.
- **Section 5** develops the tilt geometry and distinguishes circular (ξ) and conic/hyperbolic (ϑ, η) parametrizations, including the identity $\tanh \eta = \tan \vartheta$ and the $45°$ null boundary.
- **Section 6** reformulates 4–velocity normalization as a constraint on the Euclidean full flow, mapping calibrated states to the sphere $S^3_c \subset T\mathcal{E}$ and identifying the physical sector.
- **Section 7** illustrates the explanatory payoff: optical effects (Doppler shift, aberration) are derived as g–projections, and boosts are reformulated using Euclidean geometric algebra.

2 Lorentzian metric construction

2.1 Euclidean proto-space $(\mathcal{E}, \delta)$

We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$ equipped with the flat metric

$$\delta_{AB} = \mathrm{diag}(1, 1, 1, 1).$$

Indices are raised and lowered with δ:

$$X_A := \delta_{AB} X^B, \qquad X^A := \delta^{AB} X_B,$$

and we use the δ–inner product notation

$$X \cdot Y := \delta(X, Y) = \delta_{AB} X^A Y^B.$$

Remark 2.1 (Index conventions: δ vs. g). Throughout, δ is treated as the *background* Euclidean metric on $\mathcal{E}$ and is used for index gymnastics unless explicitly stated otherwise. The Lorentzian tensor g constructed later in §2.5 is regarded as a derived bilinear form on $T\mathcal{E}$ used to define interval–type scalars such as $g(X, X)$, rather than as the default device for raising/lowering.

In particular, we distinguish the δ–raised components

$$g^{AB}_{(\delta)} := \delta^{AC} \delta^{BD} g_{CD}$$

from the inverse metric $(g^{-1})^{AB}$ defined by $(g^{-1})^{AC} g_{CB} = \delta^A{}_B$. For the special form $g_{AB} = 2N_A N_B - \delta_{AB}$ with $\delta(N, N) = 1$, one indeed has $(g^{-1})^{AB} = g^{AB}_{(\delta)} = 2N^A N^B - \delta^{AB}$, but the two notions remain conceptually distinct.

2.2 Clifford algebra $\mathcal{C}\ell(4, 0)$ on $\mathcal{E}$

At each point $p \in \mathcal{E}$ we equip the tangent space $T_p\mathcal{E} \simeq \mathbb{R}^4$ with its geometric (Clifford) algebra $\mathcal{C}\ell(4, 0)$ generated by vectors and the Euclidean metric δ. Concretely, choose an oriented δ–orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $T_p\mathcal{E}$ with

$$e_a^2 = +1, \qquad e_a e_b = -e_b e_a \quad (a \neq b).$$

The geometric product of vectors $a, b \in T_p\mathcal{E}$ decomposes as

$$ab \;=\; a \cdot b \;+\; a \wedge b,$$

where $a \cdot b = \delta(a, b)$ is the scalar (inner) product and $a \wedge b$ is the bivector (outer) product. We use standard grade projections $\langle \cdot \rangle_k$ ($k = 0, \ldots, 4$) and the reversion (reverse) $\widetilde{(\cdot)}$, defined on basis blades by reversing the order of basis vectors (hence $\widetilde{ab} = \widetilde{b}\,\widetilde{a}$ and $\widetilde{a} = a$ for vectors).

The unit pseudoscalar (oriented volume element) is

$$I := e_0 e_1 e_2 e_3, \qquad I^2 = +1.$$

Rotors. A *rotor* is an even multivector $q \in \mathcal{C}\ell^{+}(4,0)$ satisfying

$$q\widetilde{q} = 1.$$

Rotors act on vectors by the sandwich map $v \mapsto qv\widetilde{q}$ and realize orthogonal transformations ($Spin(4) \to SO(4)$ is the usual double cover). We will use rotors as the algebraically clean way to encode changes of "time axis" (observer) and, later, local flow orientations.

Quaternionic subalgebras. Fixing a reference unit vector e_0 selects the even subspace

$$\mathbb{H}_{e_0} := \mathrm{span}\{1,\ e_{10},\ e_{20},\ e_{30}\} \subset \mathcal{C}\ell^{+}(4,0), \qquad e_{i0} := e_i e_0,$$

which is isomorphic to the quaternion algebra once the orientation is fixed. Elements of $\mathbb{H}_{e_0}$ have the form

$$q = a + b_1 e_{10} + b_2 e_{20} + b_3 e_{30},$$

and the unit condition $q\widetilde{q} = 1$ is simply $a^2 + b_1^2 + b_2^2 + b_3^2 = 1$, i.e. the unit 3–sphere S^3. In what follows we will call such unit elements *quaternionic rotors.*

2.3 Observer time axis: the reference vector e_0 and its rotations

To speak about "time" and "space" within the Euclidean proto–space, one must choose a reference unit direction. An inertial observer in our SR sector is encoded by a constant unit vector e_0 (constant on the region of interest):

$$\delta(e_0, e_0) = 1, \qquad \nabla^{\delta} e_0 = 0.$$

Its δ–orthogonal complement $e_0^{\perp_\delta}$ plays the role of the associated spatial hyperplane in that observer's description.

Space of possible time axes. At any point, the set of all possible unit time axes is the sphere

$$S^3 = \{u \in T_p\mathcal{E} :\ \delta(u,u) = 1\}.$$

Using the identification $T_p\mathcal{E} \simeq \mathbb{R}^4 \simeq \mathbb{H}$ (as a real vector space), this same S^3 may be regarded as the set of unit quaternions. However, the representation we will actually use is the rotor representation, because it makes changes of observer purely algebraic.

Changing the observer: rotating the zero vector. Given any other unit direction e_0' (another inertial observer axis), there exists a rotor r such that

$$e_0' \ =\ r\, e_0\, \widetilde{r}, \qquad r\widetilde{r} = 1. \tag{2.1}$$

Thus passing from one observer time axis to another is an orthogonal rotation of the reference vector e_0 implemented by a rotor.

Remark 2.2 (Nonuniqueness (stabilizer gauge)). The rotor r in (2.1) is not unique: if s is any rotor fixing e_0 (i.e. $se_0\widetilde{s} = e_0$), then $r' = rs$ yields the same e_0'. Equivalently, r is defined up to right multiplication by the stabilizer $\mathrm{Stab}(e_0) \cong Spin(3) \cong SU(2)$. This is a benign internal gauge: the induced axis e_0' is unambiguous.

2.4 Coherent time field N and flow quaternions

We now pass from a single inertial observer axis to a *coherent observer time field* on a region of $\mathcal{E}$.

Coherent observer time. Let N be a smooth δ–unit vector field on an open set $U \subset \mathcal{E}$:

$$\delta(N, N) = 1 \qquad \text{on } U. \tag{2.2}$$

In the inertial SR sector we assume N is parallel with respect to the flat Euclidean connection,

$$\nabla^{\delta} N = 0 \qquad \text{on } U, \tag{2.3}$$

so that one may choose an adapted global orthonormal frame on U and the induced Lorentzian geometry will be globally Minkowskian. Allowing $\nabla^{\delta} N \neq 0$ corresponds to non–inertial motion or to a dynamical/field interpretation of N, which we do not develop in the present SR–focused section.

Flow quaternion field. Fix once and for all a reference unit vector e_0 (as in §2.3). A *flow quaternion* (or *flow rotor*) is a unit even multivector field $q : U \to \mathcal{C}\ell^{+}(4,0)$ such that

$$q\widetilde{q} = 1, \qquad N = q\, e_0\, \widetilde{q}. \tag{2.4}$$

When q is chosen to lie in the quaternionic subalgebra $\mathbb{H}_{e_0}$ (§2.2), we may literally regard q as a unit quaternion. In any case, (2.4) provides an algebraically clean encoding of the coherent time axis N as a rotated copy of the reference axis e_0.

As in Remark 2.2, the field q is defined only up to right multiplication by stabilizer rotors fixing e_0; this gauge does not affect N.

Calibrated flow (proto–velocity) from q. Fix $c > 0$ and define the calibrated flow vector field

$$X := c\, N = c\, q\, e_0\, \widetilde{q}, \qquad \text{so that} \qquad \delta(X, X) = c^2. \tag{2.5}$$

In the intended reading, X is the local flow (or "flow gradient direction") in the proto–space, and q is the quaternionic rotor encoding its orientation relative to the observer reference e_0.

Remark 2.3 (Circular angle ξ and budget, extracted from q). Relative to the reference axis e_0, the temporal/spatial split of X is

$$H := (X \cdot e_0)\, e_0, \qquad L := X - H,$$

and for a calibrated flow one may define the *circular* angle $\xi \in [0, \pi]$ by

$$\cos\xi := \frac{X \cdot e_0}{c} = N \cdot e_0, \qquad \sin\xi := \frac{\|L\|_{\delta}}{c}.$$

For simple quaternionic rotors $q = a + \mathbf{b}$ in $\mathbb{H}_{e_0}$, $a = \langle q \rangle_0$ is the scalar part and ξ is recovered by $\xi = 2\arccos(a)$ on $0 \le \xi \le \pi$.

2.5 Reddy–Sharma–Sivaramakrishnan Lorentz metric construction

Having fixed a coherent time axis N (equivalently, flow quaternions q) we now construct the Lorentzian bilinear form associated with the observer.

Projector and 1–form. Let

$$\alpha := N^{\flat} := \delta(N, \cdot) \qquad (\text{so } \alpha_A = \delta_{AB} N^B = N_A),$$

and define the spatial projector

$$h := \delta - \alpha \otimes \alpha \qquad (\text{i.e. } h_{AB} = \delta_{AB} - N_A N_B). \tag{2.6}$$

Then h has rank 3, satisfies $h(\cdot, N) = 0$, and projects onto $N^{\perp_{\delta}}$.

Lorentzian form. Define a symmetric $(0,2)$–tensor field g on U by

$$g \;:=\; 2\,\alpha\otimes\alpha \;-\; \delta, \qquad \text{i.e.} \qquad g_{AB} = 2N_A N_B - \delta_{AB}. \tag{2.7}$$

This is precisely the RSS "Lorentzization" of the Euclidean metric along the unit direction N. The definition is insensitive to the sign flip $N \mapsto -N$ because $N\otimes N$ is unchanged.

(i) N is g–unit:
$$g(N,N) = 2(\delta(N,N))^2 - \delta(N,N) = 2\cdot 1 - 1 = 1.$$

(ii) N is g–orthogonal to $N^{\perp_\delta}$: if $X \in N^{\perp_\delta}$, i.e. $N\cdot X = 0$, then $g(N,X) = 0$.

(iii) On $N^{\perp_\delta}$ one has $g = -\delta$: if $X, Y \in N^{\perp_\delta}$, then $g(X,Y) = -\delta(X,Y)$.

Hence, at each $p \in U$,
$$T_p\mathcal{E} = \operatorname{span}\{N_p\} \oplus N_p^{\perp_\delta},$$
and in an adapted δ–orthonormal basis $\{e_0' = N, e_1', e_2', e_3'\}$ the bilinear form g_p has the Minkowski block form
$$g_p = (+1)\oplus(-1)\oplus(-1)\oplus(-1),$$
so g has Lorentzian signature $(+\,-\,-\,-)$.

Proposition 2.4 (Closed form and split identity)**.** *For every $X \in T_p\mathcal{E}$ one has*

$$g(X,X) = \alpha(X)^2 - h(X,X) = (N\cdot X)^2 - \delta(X_\perp, X_\perp) = 2(N\cdot X)^2 - \delta(X,X), \tag{2.8}$$

where $X_\perp := X - (N\cdot X)\,N$ is the δ–orthogonal projection onto $N^{\perp_\delta}$.

Proof. By (2.7) and (2.6) we have $g = \alpha\otimes\alpha - h$. Evaluating on X yields $g(X,X) = \alpha(X)^2 - h(X,X)$. Writing $X = (N\cdot X)N + X_\perp$ gives $\delta(X,X) = (N\cdot X)^2 + \delta(X_\perp, X_\perp)$, hence the last equality in (2.8). □

Remark 2.5 (Inertial SR scope and equivariance). In the SR sector we restrict to local inertial patches where $\nabla^\delta N = 0$ (and therefore g is globally Minkowskian on the patch). Changing inertial observers corresponds to replacing N by another constant unit axis $N' = R_*N$ for some $R \in O(4)$, equivalently rotating the reference vector e_0 by a rotor. The construction is $O(4)$–equivariant: $g_{N'} = R^*g_N$. Observable statements are formulated in terms of g–covariant (relational) quantities, so no preferred inertial frame is implied by the use of a reference axis.

3 Lorentzian metric properties

Throughout this section, $p \in \mathcal{E}$ is arbitrary and all statements are understood pointwise at p. We write $T_p\mathcal{E}$ for the tangent space, endowed with the Euclidean inner product δ and the associated Clifford algebra $\mathcal{C}\ell(4,0)$ from §2.2. The coherent time axis at p is a δ–unit vector $N \in T_p\mathcal{E}$, $\delta(N,N) = 1$, with associated Lorentzian form

$$g(\cdot,\cdot) = 2\,(N\cdot\cdot)\,(N\cdot\cdot) - \delta(\cdot,\cdot) \qquad \text{(cf. (2.7))}.$$

When convenient, we also fix a reference axis e_0 and a flow quaternion (rotor) q at p such that $N = qe_0\widetilde{q}$ (cf. (2.4)).

3.1 Orthogonal decomposition of tangent vectors

For any $X \in T_p\mathcal{E}$ we define the δ–longitudinal and δ–transverse components relative to N by

$$X_\| := (N \cdot X)\, N, \qquad X_\perp := h(X) = X - (N \cdot X)\, N, \tag{3.1}$$

where $h = \delta - N^\flat \otimes N^\flat$ is the δ–orthogonal projector onto $N^{\perp_\delta}$ (cf. (2.6)).

Lemma 3.1. *For every $X \in T_p\mathcal{E}$,*

$$X = X_\| + X_\perp,$$

where $X_\| \in \operatorname{span}\{N\}$ and $X_\perp \in N^{\perp_\delta}$. The decomposition is unique.

Proof. Since h is a projector with $\ker(h) = \operatorname{span}\{N\}$ and $\operatorname{Im}(h) = N^{\perp_\delta}$, the splitting is the standard direct sum decomposition associated with complementary subspaces. □

Remark 3.2 (Clifford reflection viewpoint)*.* In $\mathcal{C}\ell(4,0)$, the sandwich map by a unit vector implements an orthogonal reflection. Define, for $Y \in T_p\mathcal{E}$,

$$Y^\star := -\, N\, Y\, N.$$

Then $Y^\star$ is the δ–reflection of Y across the hyperplane $N^{\perp_\delta}$ and satisfies

$$Y^\star = Y_\| - Y_\perp.$$

Consequently, the Lorentzian bilinear form can be written as a Euclidean pairing with a reflected argument,

$$g(X,Y) \;=\; \delta\big(X,\, Y^\star\big) \;=\; \delta\big(X,\, Y_\| - Y_\perp\big), \tag{3.2}$$

which is the Clifford–algebraic content of the RSS "Lorentzization".

3.2 Circular angle ξ and norm identities

The decomposition (3.1) naturally defines a *circular* Euclidean angle between X and the time axis N.

Definition 3.3 (Circular angle ξ relative to N)**.** Let $X \in T_p\mathcal{E}$ be nonzero. Define $\xi \in [0,\pi]$ by

$$\cos\xi := \frac{N \cdot X}{\|X\|_\delta}, \qquad \sin\xi := \frac{\|X_\perp\|_\delta}{\|X\|_\delta}, \tag{3.3}$$

where $\|X\|_\delta := \sqrt{\delta(X,X)}$. Equivalently, ξ is the Euclidean angle between X and N.

Remark 3.4 (Why ξ is distinguished from ϑ)*.* The angle ξ is the *circular* (trigonometric) Euclidean angle defined by (3.3). Later we will introduce a *tangential* parameter ϑ adapted to velocity ratios (e.g. $\beta = \tan\vartheta$). Keeping ξ for the circular parametrization avoids conflating these two parameter choices.

Proposition 3.5. *For any $X \in T_p\mathcal{E}$,*

$$g(X,X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp). \tag{3.4}$$

Equivalently, in terms of the circular angle ξ,

$$g(X,X) = \|X\|_\delta^2\, \cos(2\xi). \tag{3.5}$$

Proof. Insert (3.1) into $g(X,X)$ and use: $g(N,N) = 1$, $g(N,X_\perp) = 0$ (since $X_\perp \in N^{\perp_\delta}$), and $g(X_\perp,X_\perp) = -\delta(X_\perp,X_\perp)$. This yields (3.4). Using $(N \cdot X) = \|X\|_\delta \cos\xi$ and $\|X_\perp\|_\delta = \|X\|_\delta \sin\xi$ from (3.3) gives (3.5). □

Corollary 3.6. *A vector $X \neq 0$ satisfies:*

- $g(X,X) > 0$ *iff* $\cos(2\xi) > 0$ *iff* $\xi \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$,
- $g(X,X) = 0$ *iff* $\cos(2\xi) = 0$ *iff* $\xi \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$,
- $g(X,X) < 0$ *iff* $\cos(2\xi) < 0$ *iff* $\xi \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

Equivalently, in split form:

$$g(X,X) \gtrless 0 \quad \Longleftrightarrow \quad (N \cdot X)^2 \gtrless \delta(X_\perp, X_\perp).$$

Define the three disjoint subsets of $T_p\mathcal{E}$:

$$\mathcal{T}_p := \{X \in T_p\mathcal{E} : \; g(X,X) > 0\}, \qquad \mathcal{P}_p := \{X \in T_p\mathcal{E} : \; g(X,X) = 0\}, \qquad \mathcal{S}_p := \{X \in T_p\mathcal{E} : \; g(X,X) < 0\}.$$

We also single out the *future* time cone (relative to N):

$$\mathcal{T}_p^+ := \{X \in \mathcal{T}_p : \; N \cdot X > 0\}. \tag{3.6}$$

3.3 Geometry of the null cone and the $45°$ Euclidean tilt

Proposition 3.7. *The set of g–null vectors at p is the quadratic cone*

$$\mathcal{C}_p = \{\, X \in T_p\mathcal{E} : \; \delta(X_\perp, X_\perp) = (N \cdot X)^2 \,\}.$$

Under the decomposition $T_p\mathcal{E} = \operatorname{span}\{N\} \oplus N^{\perp_\delta}$, it is a double cone given by

$$N \cdot X = \pm \|X_\perp\|_\delta.$$

Proof. Immediate from Corollary 3.6 (or directly from (3.4)). □

Remark 3.8 (Why "light" corresponds to a $45°$ Euclidean tilt). Operationally, "light propagation" refers to a causal relation between an emission event and a detection event. During the signal's flight the observer advances along the time axis N, hence the proto–space separation between emission and detection is not purely spatial but decomposes as

$$X \;=\; X_\parallel + X_\perp, \qquad X_\parallel := (N \cdot X)\, N, \qquad X_\perp := X - (N \cdot X)\, N.$$

The defining kinematic content of "light" is that the induced Lorentzian interval vanishes,

$$g(X,X) = 0.$$

Using (3.4), this is equivalent to equality of the longitudinal and transverse δ–magnitudes,

$$\|X_\parallel\|_\delta \;=\; \|X_\perp\|_\delta. \tag{3.7}$$

In terms of the circular Euclidean angle ξ between X and N (Definition 3.3), the condition (3.7) reads $\cos\xi = \sin\xi$, hence

$$\xi = \frac{\pi}{4} \quad \text{or} \quad \xi = \frac{3\pi}{4}.$$

Thus the g–null directions are precisely those at a $45°$ Euclidean tilt to the time axis: the proto–space displacement budget is split equally between N and its orthogonal complement.

3.4 Spatial rotations preserving δ and N

Let $\mathrm{Aut}(\delta, N)$ denote the stabilizer of N in the Euclidean orthogonal group:

$$\mathrm{Aut}(\delta, N) := \{\, L : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \ \delta(LX, LY) = \delta(X, Y), \ LN = N \,\}.$$

In an adapted δ–orthonormal basis $\{e'_0 = N, e'_1, e'_2, e'_3\}$ one has

$$L = \mathrm{diag}(1, R), \qquad R \in O(3),$$

so $\mathrm{Aut}(\delta, N) \cong O(3)$ and contains no boost–like maps mixing N with $N^{\perp_\delta}$.

Remark 3.9 (Rotor description of $\mathrm{Aut}(\delta, N)$). In the Clifford algebra, the connected component of $\mathrm{Aut}(\delta, N)$ is realized by *spatial rotors* $s \in \mathcal{C}\ell^+(4, 0)$ satisfying

$$s\widetilde{s} = 1, \qquad sN\widetilde{s} = N.$$

Equivalently, $s = \exp\left(-\frac{\phi}{2}B\right)$ with a bivector generator $B \in \Lambda^2(N^{\perp_\delta}) = \mathrm{span}\{e'_{12}, e'_{23}, e'_{31}\}$. This is the same "stabilizer" mechanism as in Remark 2.2 for the reference axis e_0: the time axis is held fixed while the spatial triad is rotated. Such stabilizer freedom is precisely the source of rotor nonuniqueness when one encodes only the axis N (and not a full tetrad).

Lemma 3.10. *Every* $L \in \mathrm{Aut}(\delta, N)$ *preserves* g*:*

$$g(LX, LY) = g(X, Y) \qquad \textit{for all } X, Y \in T_p\mathcal{E}.$$

Proof. Since $LN = N$ and L is δ–orthogonal,

$$g(LX, LY) = 2(N \cdot LX)(N \cdot LY) - \delta(LX, LY) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = g(X, Y).$$

□

Thus $\mathrm{Aut}(\delta, N)$ is a spatial subgroup of $\mathrm{O}(g)$: it preserves g and fixes N, but generates only Euclidean rotations on $N^{\perp_\delta}$.

Remark 3.11 (Flow quaternion gauge, revisited). If $N = qe_0\widetilde{q}$ for a flow quaternion (rotor) q, then q is defined only up to right multiplication by a stabilizer rotor s that fixes e_0 (cf. Remark 2.2), because $(qs)e_0\widetilde{(qs)} = qe_0\widetilde{q}$. Pointwise, this means that specifying the axis N does not specify a unique orientation of the spatial complement: the latter may be rotated by an element of $Spin(3)$ without affecting N, hence without affecting g_N.

4 Phase reparametrization and the operational origin of the Lorentz interval

This section makes explicit the logical bridge between the Euclidean proto–metric δ on $\mathcal{E}$ and the Lorentzian form g constructed in §2. The construction starts from a *coherent observer time axis* N (or, equivalently, a field of flow rotors q with $N = qe_0\widetilde{q}$) and then defines operational time and space measurements as projections relative to N. The key point is operational: an observer does *not* have direct access to an arbitrary curve parameter. Instead, clocks and rulers implement specific projection rules, from which the Lorentzian interval emerges as the quadratic form governing the proper–time phase along worldlines.

4.1 A δ–affine proto–parameter χ

Let $X : I \to \mathcal{E}$ be a C^1 curve. For a parameter λ on I we write

$$X'(\lambda) := \frac{dX}{d\lambda} \in T_{X(\lambda)}\mathcal{E}.$$

Because $(\mathcal{E}, \delta)$ is Euclidean, one may always reparametrize X by a scaled δ–arc length. We single out the following calibration, which fixes the *total* proto–space budget along the curve.

Definition 4.1 (δ–calibrated (proto–affine) parameter)**.** A parameter χ along X is called *δ–calibrated* (or *proto–affine*) if the δ–speed is constant and equal to c:

$$\delta(\tilde{X}, \tilde{X}) = c^2, \qquad \tilde{X} := \frac{dX}{d\chi}. \tag{4.1}$$

Remark 4.2 (Interpretation of χ)*.* In flat Euclidean geometry, (4.1) is simply a scaled arc–length parametrization:

$$d\chi = \frac{1}{c}\,\|dX\|_\delta.$$

The parameter χ is an *auxiliary* calibration: it fixes a convenient reference clock in $\mathcal{E}$ relative to which the total Euclidean expenditure $\|\tilde{X}\|_\delta = c$ is constant. By itself, χ is not an operational time variable of an observer; operational time is introduced via the coherent axis N below.

4.2 A coherent axis N: time 1–form and spatial projector

Throughout, N denotes the coherent δ–unit axis field of §2,

$$\delta(N, N) = 1,$$

encoded (when desired) by a flow rotor q via $N = qe_0\widetilde{q}$ (cf. (2.4)). The δ–dual 1–form (the *time form*) of the field is

$$\alpha := N^\flat := \delta(N, \cdot), \qquad \text{i.e.} \qquad \alpha_A = \delta_{AB}N^B = N_A. \tag{4.2}$$

The induced spatial projector is

$$h := \delta - \alpha \otimes \alpha, \qquad \text{i.e.} \qquad h_{AB} = \delta_{AB} - N_A N_B, \tag{4.3}$$

so that $h(\,\cdot\,, N) = 0$ and $\mathrm{Im}(h) = N^{\perp_\delta}$.

For any $V \in T_p\mathcal{E}$ one has the orthogonal δ–split

$$\delta(V, V) = \alpha(V)^2 + h(V, V), \tag{4.4}$$

which is the Pythagorean identity "total budget = longitudinal^2 + spatial^2" relative to the axis N.

Remark 4.3 (Sign of N and what is (in)sensitive to it)*.* The RSS Lorentzization $g = 2\,N^\flat \otimes N^\flat - \delta$ depends on N only through $N \otimes N$ and is therefore invariant under the sign flip $N \mapsto -N$. Hence the light cone and causal classification determined by g are insensitive to the orientation of N.

However, the 1–form $\alpha = N^\flat$ changes sign under $N \mapsto -N$, and so do the projected time increments defined below. Thus, to speak about an oriented "future" direction (e.g. $N \cdot X > 0$) and about a monotone time coordinate, one must choose an orientation of N (future–pointing branch) and keep it coherent on the region under consideration.

Remark 4.4 (Rotor gauge does not affect α, h, or g)*.* If $N = qe_0\widetilde{q}$, then replacing q by qs with any stabilizer rotor s satisfying $se_0\widetilde{s} = e_0$ leaves N unchanged, hence also leaves α, h, and the induced g unchanged. Operational quantities depend on N (and its chosen orientation), not on a particular representative q.

4.3 Coordinate time t versus proper time τ

Given an oriented coherent axis N, the time form α defines the observer–adapted coordinate time t *along a worldline* by the operational projection rule

$$dt := \frac{1}{c}\,\alpha(dX) \qquad \Longleftrightarrow \qquad \frac{dt}{d\lambda} = \frac{1}{c}\,\alpha\big(X'(\lambda)\big). \tag{4.5}$$

In general, (4.5) defines t only along the given curve; a global time function t on an open set requires an integrability condition on α (e.g. α closed/exact, equivalently N hypersurface–orthogonal).

Likewise, the induced spatial line element along the curve is defined by

$$d\ell^2 := h(dX, dX) \qquad \Longleftrightarrow \qquad \Big(\frac{d\ell}{d\lambda}\Big)^2 = h\big(X'(\lambda), X'(\lambda)\big). \tag{4.6}$$

Proposition 4.5 (Operational form of the Lorentz interval)**.** *For every curve X and every parameter λ one has the identity*

$$g(dX, dX) = c^2\,dt^2 - d\ell^2, \tag{4.7}$$

where dt and $d\ell$ are given by (4.5)–(4.6)*.*

Proof. By the RSS definition $g = \alpha \otimes \alpha - h$ (equivalently $g = 2\alpha \otimes \alpha - \delta$), for any V we have $g(V, V) = \alpha(V)^2 - h(V, V)$. Apply this to $V = dX$ and substitute (4.5) and (4.6). $\square$

We define the *proper time* τ along a g–timelike curve as the g–arc length parameter.

Definition 4.6 (Proper time)**.** Along a g–timelike curve (i.e. $g(dX, dX) > 0$) the proper time is defined by

$$c^2\,d\tau^2 := g(dX, dX). \tag{4.8}$$

Equivalently, combining (4.8) with (4.7) yields

$$d\tau^2 = dt^2 - \frac{1}{c^2}\,d\ell^2. \tag{4.9}$$

Remark 4.7 (Roles of χ, t, and τ)*.* The three parameters used in this paper have distinct operational status:

- χ is an auxiliary δ–calibration fixing the total proto–space budget $\|\tilde{X}\|_\delta = c$;
- t is the observer–adapted coordinate time obtained by projecting dX onto the coherent axis N via (4.5);
- τ is the proper time measured by an ideal comoving clock, defined invariantly as the g–arc length by (4.8).

The Lorentzian form g is the unique quadratic form in this construction for which (4.7) holds identically once N (hence α and h) has been fixed.

4.4 Equivalence of δ–budget and the proper–time interval

Assume henceforth that χ is δ–calibrated in the sense of Definition 4.1, and write $\tilde{X} = dX/d\chi$. Define the longitudinal and transverse flow rates (per unit χ) by

$$S(\chi) := \alpha(\tilde{X}) = N\cdot\tilde{X}, \qquad L(\chi)^2 := h(\tilde{X}, \tilde{X}) = \delta(\tilde{X}_\perp, \tilde{X}_\perp). \tag{4.10}$$

Then (4.4) becomes the exact budget identity

$$\underbrace{\delta(\tilde{X}, \tilde{X})}_{c^2} = \underbrace{S^2}_{\text{longitudinal (time-axis) rate}} + \underbrace{L^2}_{\text{transverse (spatial) rate}}. \tag{4.11}$$

Theorem 4.8 (Phase reparametrization $\chi \mapsto \tau$)**.** *Let $X(\chi)$ be δ–calibrated and g–timelike along the curve, i.e. $g(\tilde{X}, \tilde{X}) > 0$. Then*

$$\frac{d\tau}{d\chi} \;=\; \frac{1}{c}\sqrt{g(\tilde{X}, \tilde{X})} \;=\; \frac{1}{c}\sqrt{S^2 - L^2}. \tag{4.12}$$

Equivalently, reparametrizing the same geometric curve by τ (i.e. setting $\dot{X} := dX/d\tau$) yields the unit–speed condition

$$g(\dot{X}, \dot{X}) = c^2. \tag{4.13}$$

Proof. By Definition 4.6, $c^2(d\tau/d\chi)^2 = g(\tilde{X}, \tilde{X})$, giving (4.12). Then $\dot{X} = (d\chi/d\tau)\tilde{X}$ implies

$$g(\dot{X}, \dot{X}) = \Big(\frac{d\chi}{d\tau}\Big)^2 g(\tilde{X}, \tilde{X}) = \frac{c^2}{g(\tilde{X}, \tilde{X})}\, g(\tilde{X}, \tilde{X}) = c^2,$$

which is (4.13). □

Remark 4.9 (Why g is observed rather than δ). The proto–metric δ measures the total Euclidean budget with respect to the auxiliary calibration χ. Physical clocks, however, realize the proper–time parametrization τ defined by (4.8), hence by the quadratic form g. In this operational sense the observer “lives by its own phase”: the time variable implemented by ideal clocks is τ, and the interval controlling it is g, not δ.

4.5 Circular angle ξ and the explicit χ–τ link

For a δ–calibrated tangent $\tilde{X}$ we define the *circular* Euclidean angle $\xi \in [0, \pi]$ between $\tilde{X}$ and the axis N by

$$\cos\xi := \frac{N \cdot \tilde{X}}{c} = \frac{S}{c}, \qquad \sin\xi := \frac{\|\tilde{X}_\perp\|_\delta}{c} = \frac{L}{c}. \tag{4.14}$$

Then (4.11) becomes $\cos^2\xi + \sin^2\xi = 1$ and the RSS split gives the explicit double–angle form

$$g(\tilde{X}, \tilde{X}) = S^2 - L^2 = c^2(\cos^2\xi - \sin^2\xi) = c^2\cos(2\xi). \tag{4.15}$$

Substituting into (4.12) yields the explicit phase–rate relation

$$\frac{d\tau}{d\chi} = \sqrt{\cos(2\xi)}. \tag{4.16}$$

Remark 4.10 (Orientation, future sector, and admissible domain). Since g is insensitive to $N \mapsto -N$ but t is not, one typically fixes a future–pointing branch of N by requiring $S = N\cdot\tilde{X} > 0$ along the worldline (equivalently $dt/d\chi > 0$). In the future timelike sector, $g(\tilde{X}, \tilde{X}) > 0$ implies $\cos(2\xi) > 0$, hence $\xi \in [0, \pi/4)$. The boundary $\xi = \pi/4$ corresponds to $g(\tilde{X}, \tilde{X}) = 0$ (null case), for which $d\tau/d\chi = 0$.

Remark 4.11 (Relation to the tangential parametrization). On the future sector ($S > 0$) one may also use the ratio parameter

$$\beta := \frac{L}{S} = \tan\xi,$$

which provides a convenient “tangential” description of the same tilt. If one prefers to reserve a separate symbol ϑ for the tangential parameterization used later, one may set $\beta = \tan\vartheta$ and regard ϑ as a re–labeling emphasizing the ratio coordinate. In terms of β,

$$\cos(2\xi) = \frac{1-\beta^2}{1+\beta^2},$$

so the same double–angle structure is present; it is merely expressed in a different parameter.

5 Tilt geometry: circular (ξ) versus conic/hyperbolic (ϑ, η) parametrizations

Throughout this section, $p \in \mathcal{E}$ is fixed and all statements are understood pointwise at p. We work with the δ–orthogonal splitting relative to the coherent time axis N (cf. §3):

$$X = X_{\parallel} + X_{\perp}, \qquad X_{\parallel} := (N\cdot X)\,N, \qquad X_{\perp} := X - (N\cdot X)\,N.$$

Recall that the induced Lorentzian form is $g = 2N^{\flat}\otimes N^{\flat} - \delta$, so that

$$g(X,X) = (N\cdot X)^2 - \delta(X_{\perp}, X_{\perp}) \qquad \text{and} \qquad \delta(X,X) = (N\cdot X)^2 + \delta(X_{\perp}, X_{\perp}).$$

5.1 Circular parametrization: the Euclidean tilt angle ξ

For any nonzero $X \in T_p\mathcal{E}$ define its *circular* (Euclidean) tilt angle $\xi \in [0,\pi]$ relative to the axis N by

$$\cos\xi := \frac{N\cdot X}{\|X\|_{\delta}}, \qquad \sin\xi := \frac{\|X_{\perp}\|_{\delta}}{\|X\|_{\delta}}, \qquad \|X\|_{\delta} := \sqrt{\delta(X,X)}. \tag{5.1}$$

Whenever $X_{\perp} \neq 0$, define the transverse unit direction

$$E := \frac{X_{\perp}}{\|X_{\perp}\|_{\delta}} \in N^{\perp_{\delta}}, \qquad \delta(E,E) = 1, \qquad N\cdot E = 0.$$

Then X admits the explicit circular decomposition

$$X \;=\; \|X\|_{\delta}\big(\cos\xi\, N + \sin\xi\, E\big). \tag{5.2}$$

Lemma 5.1 (Circular budget identities)**.** *For any nonzero $X \in T_p\mathcal{E}$,*

$$\|X\|_{\delta}^2 = (N\cdot X)^2 + \delta(X_{\perp}, X_{\perp}), \qquad \cos^2\xi + \sin^2\xi = 1.$$

Proof. Immediate from $\delta(X_{\parallel}, X_{\perp}) = 0$ and Definition (5.1). □

Proposition 5.2 (Lorentzian norm in circular form)**.** *For any nonzero $X \in T_p\mathcal{E}$,*

$$g(X,X) = \|X\|_{\delta}^2(\cos^2\xi - \sin^2\xi) = \|X\|_{\delta}^2\cos(2\xi). \tag{5.3}$$

Proof. Substitute $(N\cdot X) = \|X\|_{\delta}\cos\xi$ and $\|X_{\perp}\|_{\delta} = \|X\|_{\delta}\sin\xi$ into $g(X,X) = (N\cdot X)^2 - \|X_{\perp}\|_{\delta}^2$. □

Remark 5.3 (Timelike domain in circular coordinates)*.* In the future timelike cone $\mathcal{T}_p^+ = \{X : g(X,X) > 0,\ N\cdot X > 0\}$ one has $\cos(2\xi) > 0$ and $\cos\xi > 0$, hence

$$X \in \mathcal{T}_p^+ \quad\Longleftrightarrow\quad \xi \in \Big[0, \frac{\pi}{4}\Big).$$

The boundary $\xi = \pi/4$ corresponds to $g(X,X) = 0$ (null directions).

5.2 Conic (tangential) parametrization: the ratio angle ϑ

While ξ is defined by normalizing with the *total* Euclidean magnitude $\|X\|_{\delta}$, the physically relevant kinematic ratio on the future timelike cone is the transverse–to–longitudinal budget ratio

$$\beta := \frac{\|X_{\perp}\|_{\delta}}{N\cdot X}, \qquad (X \in \mathcal{T}_p^+).$$

This suggests a *conic* (tangential) coordinate ϑ defined by

$$\tan\vartheta := \frac{\|X_{\perp}\|_{\delta}}{N\cdot X}, \qquad (X \in \mathcal{T}_p^+), \tag{5.4}$$

so that $\beta = \tan\vartheta \in [0,1)$ on $\mathcal{T}_p^+$. Null vectors satisfy $\tan\vartheta = 1$ (equivalently $\vartheta = \pi/4$), and g–spacelike vectors satisfy $\tan\vartheta > 1$.

Remark 5.4 (Relation between ξ and ϑ). On the future sector, $\cos\xi > 0$, hence the ratio definition gives

$$\tan\vartheta = \frac{\|X_\perp\|_\delta}{N\cdot X} = \frac{\|X\|_\delta \sin\xi}{\|X\|_\delta \cos\xi} = \tan\xi,$$

so numerically $\vartheta = \xi$ on $\mathcal{T}_p^+$. The distinction is conceptual: ξ is a *circular* normalization by $\|X\|_\delta$, whereas ϑ is a *conic* normalization by the longitudinal component $N\cdot X$. This is exactly the choice that makes $\beta = \tan\vartheta$ the natural speed–like parameter saturating at $\beta \to 1$ in the null limit.

Proposition 5.5 (Lorentzian norm in conic form). *For $X \in \mathcal{T}_p^+$,*

$$g(X,X) = (N\cdot X)^2\Big(1-\tan^2\vartheta\Big) = \frac{(N\cdot X)^2}{1+\beta^2}\,(1-\beta^2), \qquad \beta = \tan\vartheta. \tag{5.5}$$

Equivalently,

$$\cos(2\xi) = \frac{1-\beta^2}{1+\beta^2}. \tag{5.6}$$

Proof. From $g(X,X) = (N\cdot X)^2 - \|X_\perp\|_\delta^2$ and $\|X_\perp\|_\delta = (N\cdot X)\tan\vartheta$ one gets (5.5). Using $\beta = \tan\xi$ and $\cos(2\xi) = (1-\tan^2\xi)/(1+\tan^2\xi)$ yields (5.6). □

5.3 Hyperbolic parameter (rapidity) η

A genuine group parameter for boosts is the *rapidity* η. For $X \in \mathcal{T}_p^+$ define $\eta \ge 0$ by

$$\tanh\eta := \tan\vartheta = \beta. \tag{5.7}$$

Equivalently, η can be defined invariantly by the pair of relations

$$\cosh\eta := \frac{N\cdot X}{\sqrt{g(X,X)}}, \qquad \sinh\eta := \frac{\|X_\perp\|_\delta}{\sqrt{g(X,X)}}, \qquad (X \in \mathcal{T}_p^+), \tag{5.8}$$

which immediately implies $\tanh\eta = \|X_\perp\|_\delta/(N\cdot X) = \tan\vartheta$.

Proposition 5.6 (Hyperbolic decomposition of timelike vectors). *For $X \in \mathcal{T}_p^+$ and $E = X_\perp/\|X_\perp\|_\delta$ one has*

$$X \;=\; \sqrt{g(X,X)}\big(\cosh\eta\, N + \sinh\eta\, E\big), \tag{5.9}$$

and the identities (5.8) *hold.*

Proof. Write $X = (N\cdot X)N + \|X_\perp\|_\delta E$ and factor out $\sqrt{g(X,X)}$ using (5.8). □

Lemma 5.7. *For $X \in \mathcal{T}_p^+$,*

$$\cosh\eta = \frac{1}{\sqrt{1-\tan^2\vartheta}} = \frac{\cos\vartheta}{\sqrt{\cos(2\vartheta)}}, \qquad \sinh\eta = \frac{\tan\vartheta}{\sqrt{1-\tan^2\vartheta}} = \frac{\sin\vartheta}{\sqrt{\cos(2\vartheta)}}.$$

Proof. From $\tanh\eta = \tan\vartheta$ we have

$$\cosh^2\eta = \frac{1}{1-\tanh^2\eta} = \frac{1}{1-\tan^2\vartheta}.$$

Taking the positive square root (since $\eta \ge 0$ and $\vartheta \in [0,\pi/4)$) gives $\cosh\eta$. Then $\sinh\eta = \tanh\eta\,\cosh\eta = \tan\vartheta\,\cosh\eta$. Finally, $\cos(2\vartheta) = \cos^2\vartheta - \sin^2\vartheta = \cos^2\vartheta(1-\tan^2\vartheta)$ yields the alternative expressions. □

5.4 Differential relation between η and the conic angle ϑ

Proposition 5.8. *For $X \in \mathcal{T}_p^+$, the parameters η and ϑ satisfy*

$$\frac{d\eta}{d\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

Proof. Differentiate $\tanh\eta = \tan\vartheta$:

$$\operatorname{sech}^2\eta\, d\eta = \sec^2\vartheta\, d\vartheta.$$

Using $\operatorname{sech}^2\eta = 1 - \tanh^2\eta = 1 - \tan^2\vartheta = \cos(2\vartheta)/\cos^2\vartheta$ gives

$$\frac{d\eta}{d\vartheta} = \frac{\sec^2\vartheta}{\operatorname{sech}^2\eta} = \frac{1/\cos^2\vartheta}{\cos(2\vartheta)/\cos^2\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

□

5.5 Boost subgroup and additivity of the hyperbolic parameter

Let $\mathrm{O}(g)$ denote the Lorentz group of $(T_p\mathcal{E}, g)$:

$$\mathrm{O}(g) := \{\, \Lambda : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \; g(\Lambda X, \Lambda Y) = g(X, Y)\,\}.$$

Fix a δ–unit transverse direction $E \in N^{\perp_\delta}$, $\delta(E,E) = 1$. The *boost* in the 2–plane $\operatorname{span}\{N, E\}$ with rapidity η is the unique $\Lambda(\eta) \in \mathrm{O}(g)$ acting as a hyperbolic rotation on $\operatorname{span}\{N, E\}$ and as the identity on its g–orthogonal complement:

$$\Lambda(\eta)N = (\cosh\eta)\, N + (\sinh\eta)\, E, \qquad \Lambda(\eta)E = (\sinh\eta)\, N + (\cosh\eta)\, E,$$

$$\Lambda(\eta)X = X \quad \text{for } X \perp_g \operatorname{span}\{N, E\}.$$

Such boosts preserve g but, in general, do not preserve δ and do not fix N.

Theorem 5.9 (Additivity of rapidity)**.** *For boosts $\Lambda(\eta_1)$ and $\Lambda(\eta_2)$ in the same (N, E)–plane, their composition is a boost with parameter $\eta_1 + \eta_2$:*

$$\Lambda(\eta_1) \circ \Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2).$$

Proof. On $\operatorname{span}\{N, E\}$ the boosts are represented (in the basis $\{N, E\}$) by

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix},$$

whose multiplication adds rapidities. On the g–orthogonal complement the action is the identity, hence the statement holds on all of $T_p\mathcal{E}$. □

5.6 Comparison with classical angle conventions

The *geometric* Euclidean tilt is captured by the circular angle ξ (cf. (5.1)), but different reformulations choose different dimensionless parameters derived from it.

A common choice is the sine–based parameter

$$\beta_{\sin} := \sin\xi = \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

whereas in the present work the kinematically natural parameter is the ratio (conic) parameter

$$\beta := \frac{\|X_\perp\|_\delta}{N \cdot X} = \tan\vartheta = \tan\xi, \qquad (X \in \mathcal{T}_p^+),$$

which satisfies $\beta \in [0, 1)$ on the future timelike cone and reaches the null limit at $\beta \to 1$.

Remark 5.10 (Photon limit and the 45° Euclidean tilt). The null cone is characterized by $g(X,X) = 0$, equivalently $\cos(2\xi) = 0$, hence the lightlike limit corresponds to $\xi \to \pi/4$ in the Euclidean picture. In this limit one has

$$\beta = \tan\xi \to 1, \qquad \beta_{\sin} = \sin\xi \to \frac{1}{\sqrt{2}}.$$

Thus a light ray is reached at a *finite* Euclidean tilt of 45° relative to N (not at 90°). The ratio parameter $\beta = \tan\xi$ is therefore better adapted to the speed–of–light barrier than the sine parameter.

Remark 5.11 (Nonadditivity of ξ and ϑ). Neither the circular angle ξ nor the conic coordinate ϑ is a group parameter for boosts. Even in the collinear case, where rapidities add, $\eta_{12} = \eta_1 + \eta_2$, the corresponding tilt coordinates do not add:

$$\xi_{12} \neq \xi_1 + \xi_2, \qquad \vartheta_{12} \neq \vartheta_1 + \vartheta_2.$$

Indeed, the collinear velocity–composition law

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$$

translates into

$$\tan\vartheta_{12} = \frac{\tan\vartheta_1 + \tan\vartheta_2}{1 + \tan\vartheta_1 \tan\vartheta_2},$$

showing explicitly that ϑ is a nonlinear reparameterization of the additive rapidity.

Remark 5.12 (Why we keep both parametrizations). The circular angle ξ is geometrically immediate and makes the double–angle structure $g(X,X) = \|X\|_\delta^2 \cos(2\xi)$ transparent. The conic/hyperbolic parametrization (ϑ, η) is operationally adapted to velocity ratios and to the boost subgroup: $\beta = \tan\vartheta$ saturates at $\beta \to 1$ in the null limit and η is additive under boosts. The two descriptions therefore complement each other.

6 Flow invariants and the emergence of S^3 in the Euclidean proto-space

This section makes precise a central equivalence of the proto–space approach. Once a coherent observer axis N (hence g) is fixed, the standard SR normalization of the 4–velocity in proper time,

$$g(\dot{X}, \dot{X}) = c^2,$$

is equivalent (up to reparameterization) to a constant Euclidean flow budget in proto–space,

$$\delta(\tilde{X}, \tilde{X}) = c^2,$$

with respect to a calibrated proto–parameter χ. The latter condition forces the instantaneous flow states to lie on the Euclidean sphere $S_c^3 \subset T_p\mathcal{E}$.

6.1 Worldlines, proto–parameters, and the full flow vector

Let $X : I \to \mathcal{E}$ be a smooth regular worldline (geometric curve). A *proto–parameter* along X is any smooth parameter χ with nowhere–vanishing derivative. The associated *full flow vector* (proto–space tangent) is

$$\tilde{X} \;:=\; \frac{dX}{d\chi} \;\in T_{X(\chi)}\mathcal{E}. \tag{6.1}$$

Definition 6.1 (Calibrated proto–parameter)**.** A proto–parameter χ is called *δ–calibrated* (with scale c) if

$$\delta(\tilde{X}, \tilde{X}) \;=\; c^2 \qquad \text{along } X. \tag{6.2}$$

Remark 6.2 (Existence and gauge nature). For any regular curve X with $\dot{X} \neq 0$, a δ–calibrated parameter always exists: one may define χ (up to an additive constant) by

$$d\chi := \frac{1}{c}\, \|dX\|_\delta.$$

Fixing a time orientation ($d\chi > 0$ along the chosen direction of traversal), the calibrated proto–parameter is unique up to translation. This calibration is auxiliary: it fixes the *total* Euclidean flow budget in $\mathcal{E}$ but is not, by itself, an observer's operational time variable.

In words: in a calibrated proto–parameter, the full proto–space flow vector $\tilde{X}$ has fixed Euclidean norm. This is the proto–space counterpart of the standard SR statement that the 4–velocity has fixed Minkowski norm in proper time.

6.2 Observer splitting and the interval–rate identity

Fix a coherent δ–unit axis field N as in §2–§3 (with a chosen time orientation). Pointwise along X, decompose $\tilde{X}$ into δ–longitudinal and δ–transverse parts relative to N:

$$\tilde{X} \;=\; S\,N + \tilde{X}_\perp, \qquad S := N \cdot \tilde{X}, \qquad \tilde{X}_\perp := h(\tilde{X}) \in N^{\perp_\delta}. \tag{6.3}$$

Let $L := \|\tilde{X}_\perp\|_\delta$ and, when $L \neq 0$, $E := \tilde{X}_\perp / L \in N^{\perp_\delta}$ so that $\tilde{X} = S\,N + L\,E$ with $\delta(E, E) = 1$.

Lemma 6.3 (Euclidean and Lorentzian norms of the flow)**.** *Along X one has*

$$\delta(\tilde{X}, \tilde{X}) = S^2 + L^2, \qquad g(\tilde{X}, \tilde{X}) = S^2 - L^2. \tag{6.4}$$

Proof. Since $\tilde{X}_\perp \in N^{\perp_\delta}$, one has $\delta(N, \tilde{X}_\perp) = 0$, hence $\delta(\tilde{X}, \tilde{X}) = S^2 + \delta(\tilde{X}_\perp, \tilde{X}_\perp) = S^2 + L^2$. For g, use $g(N, N) = 1$, $g(N, \tilde{X}_\perp) = 0$, and $g(\tilde{X}_\perp, \tilde{X}_\perp) = -\delta(\tilde{X}_\perp, \tilde{X}_\perp) = -L^2$. □

Motivated by the phase–formalism viewpoint, we define the *interval rate* with respect to χ by

$$\tilde{s} \;:=\; \sqrt{g(\tilde{X}, \tilde{X})} \;=\; \sqrt{S^2 - L^2} \qquad (g\text{–timelike case}). \tag{6.5}$$

Equivalently, along a timelike segment one may write $g(dX, dX) = \tilde{s}^{\,2}\, d\chi^2$.

Remark 6.4 (Circular angle ξ on calibrated flows). If χ is δ–calibrated, $\delta(\tilde{X}, \tilde{X}) = c^2$, then (6.4) implies $S^2 + L^2 = c^2$ and one may set

$$\cos\xi := \frac{S}{c}, \qquad \sin\xi := \frac{L}{c}.$$

Then

$$g(\tilde{X}, \tilde{X}) = c^2(\cos^2\xi - \sin^2\xi) = c^2\cos(2\xi), \qquad \tilde{s} = c\sqrt{\cos(2\xi)}.$$

This is the same double–angle structure that governs the phase rate $d\tau/d\chi$ in §4.

6.3 Equivalence: proper–time normalization $\Longleftrightarrow$ calibrated full flow

Let τ denote the proper time along a g–timelike worldline X, i.e. a parameter such that $\dot{X} := dX/d\tau$ satisfies

$$g(\dot{X}, \dot{X}) = c^2. \tag{6.6}$$

Equivalently, $g(dX, dX) = c^2\, d\tau^2$ along X.

Remark 6.5 (Units and normalized 4–velocity). The proper–time tangent $\dot{X}$ has the physical dimension of a speed and is normalized by (6.6). It is often convenient to introduce the dimensionless unit 4–velocity

$$U := \frac{1}{c}\dot{X}, \qquad \text{so that} \qquad g(U,U) = 1.$$

Whenever U arises from a worldline it is understood as the normalized proper–time tangent.

Theorem 6.6 (Reparameterization equivalence)**.** *Let X be a regular g–timelike worldline (geometric curve). Then the following statements are equivalent up to reparameterization:*

(A) X is parameterized by proper time τ so that $g(\dot{X},\dot{X}) = c^2$.

(B) X is parameterized by a δ–calibrated proto–parameter χ so that $\delta(\tilde{X},\tilde{X}) = c^2$.

Moreover, when both parameters are used on the same curve, they satisfy

$$\frac{d\tau}{d\chi} = \frac{\sqrt{g(\tilde{X},\tilde{X})}}{c} = \frac{\tilde{s}}{c}, \qquad \frac{d\chi}{d\tau} = \frac{\|\dot{X}\|_\delta}{c}. \tag{6.7}$$

Proof. Assume (A). Define χ (up to an additive constant) by

$$\frac{d\chi}{d\tau} := \frac{\|\dot{X}\|_\delta}{c},$$

which is smooth and positive since $\dot{X} \neq 0$. Then $\tilde{X} = dX/d\chi = (d\tau/d\chi)\dot{X}$, so

$$\delta(\tilde{X},\tilde{X}) = \left(\frac{d\tau}{d\chi}\right)^2 \delta(\dot{X},\dot{X}) = \frac{c^2}{\|\dot{X}\|_\delta^2}\,\|\dot{X}\|_\delta^2 = c^2,$$

which is (B).

Conversely, assume (B). Define τ (up to an additive constant) by

$$\frac{d\tau}{d\chi} := \frac{\sqrt{g(\tilde{X},\tilde{X})}}{c},$$

which is well–defined and positive for timelike $\tilde{X}$ since $g(\tilde{X},\tilde{X}) > 0$. Then $\dot{X} = dX/d\tau = (d\chi/d\tau)\tilde{X}$, hence

$$g(\dot{X},\dot{X}) = \left(\frac{d\chi}{d\tau}\right)^2 g(\tilde{X},\tilde{X}) = \frac{c^2}{g(\tilde{X},\tilde{X})}\, g(\tilde{X},\tilde{X}) = c^2,$$

which is (A). The relations (6.7) are exactly the two defining ODEs. □

Operational meaning. Statement (A) is the standard SR normalization of the 4–velocity in proper time. Statement (B) is the corresponding calibration of the proto–space flow. Theorem 6.6 shows that these two "invariants" are equivalent and amount to a change of parameter: constant norm in one metric corresponds to constant norm in the other once the operational projection structure (N, hence g) has been fixed.

6.4 The S^3 of admissible flow states

Fix $p \in \mathcal{E}$. The set of all δ–calibrated flow vectors at p is the Euclidean 3–sphere of radius c inside $T_p\mathcal{E}$:

$$S^3_c(p) \;:=\; \{\, V \in T_p\mathcal{E} :\; \delta(V,V) = c^2 \,\} \;\cong\; S^3. \tag{6.8}$$

Thus, once the calibration (6.2) is imposed, every instantaneous proto–kinematic state is a point on $S^3_c(p)$.

Relative to the coherent axis N_p, each $V \in S^3_c(p)$ admits the decomposition

$$V = c\big(\cos\xi\, N + \sin\xi\, E\big), \qquad \xi \in [0,\pi], \quad E \in N^{\perp_\delta},\ \delta(E,E) = 1, \tag{6.9}$$

where ξ is the circular tilt angle and E is the transverse unit direction. When $\sin\xi = 0$ (the poles), the direction E is irrelevant and the state reduces to $V = \pm c\, N_p$.

Remark 6.7 (Quaternionic encoding of S^3 (optional viewpoint)). Fix a reference unit vector e_0 and the associated quaternionic subalgebra $\mathbb{H}_{e_0} \subset \mathcal{C}\ell^+(4,0)$ generated by $I_i := e_i e_0$ $(i = 1,2,3)$, so that $I_i^2 = -1$ and $I_i I_j = -I_j I_i$ for $i \neq j$. Then any vector $V = V^0 e_0 + V^1 e_1 + V^2 e_2 + V^3 e_3 \in T_p\mathcal{E}$ can be encoded as

$$Q(V) := V^0 + V^1 I_1 + V^2 I_2 + V^3 I_3 \in \mathbb{H}_{e_0},$$

and one has $\delta(V,V) = |Q(V)|^2$ (quaternionic norm). Hence the calibrated sphere $S^3_c(p)$ corresponds to the set of quaternions of norm c, and the unit sphere corresponds to unit quaternions. This identification depends on the chosen reference axis (a gauge choice); the underlying manifold structure is canonical.

Remark 6.8 (The physical sector of S^3 and time orientation). While the calibrated flow states form the full sphere $S^3_c(p)$, the requirement that a worldline be timelike with respect to g,

$$g(V,V) > 0,$$

restricts admissible states to

$$|S| > |L| \qquad \text{where} \qquad S := N \cdot V, \quad L := \|h(V)\|_\delta,$$

i.e. to the union of two polar caps on $S^3_c(p)$ centered at $\pm cN$. The boundary $|S| = |L|$ is the null locus.

Because g is insensitive to $N \mapsto -N$ but the oriented time projection $\alpha = N^\flat$ is not, an observer fixes a *future* branch by imposing $S = N \cdot V > 0$. This selects a single polar cap (the future timelike sector). Along any sequence approaching the null boundary within that cap one has $d\tau/d\chi = \sqrt{g(V,V)}/c \to 0$, consistently with (6.7).

7 Why the Euclidean proto–space viewpoint

The induced Lorentzian form g constructed from (δ, N) equips each tangent space $(T_p\mathcal{E}, g)$ with the standard causal classification and light cone. The additional advantage of the Euclidean proto–space viewpoint is that it keeps, in a single geometric package, both (i) *directional* data in the Euclidean spatial complement $\mathrm{Im}(h_p) = N_p^{\perp_\delta}$ and (ii) *clock calibration* data encoded by g–projections onto oriented time axes. In particular, "light" does not require an extra postulate: a null direction is a g–null ray, and choosing a convenient δ–normalization picks a canonical cross–section of that ray on the calibrated Euclidean sphere.

The material in this section provides the operational dictionary needed for later frequency and angle computations. It uses only the observer splitting determined by a chosen oriented coherent axis N.

7.1 Light rays: canonical frequency and direction from the N–split

Fix $p \in \mathcal{E}$ and an oriented coherent time axis N_p (future branch chosen). A *light ray direction* at p is a projective class $[K]$ of nonzero vectors $K \in T_p\mathcal{E}$ satisfying

$$g(K, K) = 0, \qquad N \cdot K > 0,$$

where $K \sim \lambda K$ for $\lambda > 0$ represents the same ray.

A convenient representative is obtained by encoding the scale of K as the frequency measured by N.

Definition 7.1 (Proto–frequency of a null ray (observer N))**.** Let $K \neq 0$ be future–directed and g–null. The frequency measured by N is

$$\omega := g(K, N). \tag{7.1}$$

Lemma 7.2 (Euclidean and Lorentzian projections coincide on N)**.** *For any $K \in T_p\mathcal{E}$ one has*

$$g(K, N) = N \cdot K. \tag{7.2}$$

In particular, a future direction satisfies $\omega = N \cdot K > 0$.

Proof. Using $g = 2\, N^\flat \otimes N^\flat - \delta$ and $\delta(N, N) = 1$,

$$g(K, N) = 2(N \cdot K)(N \cdot N) - \delta(K, N) = 2(N \cdot K) - (N \cdot K) = N \cdot K.$$

□

Decompose K into longitudinal and transverse parts relative to N:

$$K = (N \cdot K)\, N + K_\perp, \qquad K_\perp \in N^{\perp_\delta}.$$

The null condition fixes the transverse magnitude.

Lemma 7.3 (Canonical null decomposition)**.** *Every future–directed null vector $K \neq 0$ admits a unique decomposition*

$$K = \omega\, (N + E), \tag{7.3}$$

where $\omega = g(K, N) > 0$ and $E \in N^{\perp_\delta}$ is uniquely determined by

$$\delta(E, E) = 1, \qquad N \cdot E = 0.$$

Proof. Write $K = \omega N + K_\perp$ with $\omega = N \cdot K > 0$ and $K_\perp \in N^{\perp_\delta}$. Then

$$0 = g(K, K) = g(\omega N, \omega N) + 2g(\omega N, K_\perp) + g(K_\perp, K_\perp) = \omega^2 - \delta(K_\perp, K_\perp),$$

so $\|K_\perp\|_\delta = \omega$. Set $E := K_\perp/\omega$ to obtain (7.3). Uniqueness follows from uniqueness of the N–split and the positivity of ω. □

Remark 7.4 (Null rays as a calibrated slice of the null cone)*.* The scale in (7.3) is encoded by ω; correspondingly,

$$\delta(K, K) = \delta(\omega(N + E), \omega(N + E)) = 2\omega^2.$$

If one prefers a *δ–calibrated* representative on the Euclidean sphere, define

$$\widehat{K} \;:=\; \frac{c}{\sqrt{2}\,\omega}\, K \;=\; \frac{c}{\sqrt{2}}\, (N + E).$$

Then $\delta(\widehat{K}, \widehat{K}) = c^2$ and $g(\widehat{K}, \widehat{K}) = 0$. Thus the set of (future) null directions corresponds to the S^2 cross–section

$$S^3_c(p) \cap \{V :\ g(V, V) = 0,\ N \cdot V > 0\} = \Big\{\frac{c}{\sqrt{2}}(N + E) :\ E \in S^2 \subset N^{\perp_\delta}\Big\}.$$

In the circular tilt language of §4–§5, this is the boundary $\xi = \pi/4$ of the future timelike cap on $S^3_c(p)$.

7.2 Observers as unit timelike states; frequency as a g–projection

A local observer at p is represented by a future unit timelike vector (dimensionless)

$$U \in T_p\mathcal{E}, \qquad g(U,U) = 1, \qquad g(U,N) > 0.$$

All measurable scalars are obtained by taking g–contractions.

Definition 7.5 (Frequency measured by an observer)**.** For a null ray represented by $K \neq 0$, the frequency measured by the observer U is

$$\omega_U := g(U,K). \tag{7.4}$$

Remark 7.6 (Ray vector vs. wave covector)*.* We represent a light ray by a future–directed null vector $K \in T_p\mathcal{E}$. Its metric dual 1–form

$$k \;:=\; g(K,\cdot) \;\in T_p^*\mathcal{E}$$

is the standard wave covector of geometric optics. With this notation, $\omega_U = k(U) = g(U,K)$.

7.3 Doppler shift as a one–line contraction

Let $E_v \in N^{\perp_\delta}$ be a δ–unit direction, $\delta(E_v,E_v) = 1$. Let U be obtained from N by a boost of rapidity $\eta \geq 0$ in the (N,E_v)–plane:

$$U := (\cosh\eta)\,N + (\sinh\eta)\,E_v. \tag{7.5}$$

Let the null ray be written in canonical form (7.3),

$$K = \omega\,(N+E), \qquad \delta(E,E) = 1, \quad E \in N^{\perp_\delta}.$$

Using bilinearity and $g(N,N) = 1$, $g(N,E) = 0$, $g(E_v,N) = 0$, $g(E_v,E) = -\delta(E_v,E)$, we obtain

$$\begin{aligned}
\omega_U &= g(U,K)\\
&= \omega\, g\big((\cosh\eta)N + (\sinh\eta)E_v,\; N+E\big)\\
&= \omega\Big(\cosh\eta - \sinh\eta\,\delta(E_v,E)\Big).
\end{aligned} \tag{7.6}$$

Introduce the standard parameters

$$\beta := \tanh\eta, \qquad \gamma := \cosh\eta,$$

and define the Euclidean angle $\psi \in [0,\pi]$ between the velocity axis E_v and the ray direction E inside the observer space $N^{\perp_\delta}$ by

$$\cos\psi := \delta(E_v,E). \tag{7.7}$$

Then (7.6) becomes the standard relativistic Doppler law

$$\frac{\omega_U}{\omega} = \gamma\big(1 - \beta\cos\psi\big). \tag{7.8}$$

Interpretation. In the proto–space picture, (7.8) is literally a projection: the measured frequency is the scalar g–projection of a null direction K onto an observer state U.

7.4 Aberration as projection plus normalization

The direction of the ray measured by U is the normalized U–spatial part of K (in the g–orthogonal complement of U). Define the g–spatial component

$$K_{\perp U} := K - (g(U,K))\,U. \tag{7.9}$$

Then $g(U, K_{\perp U}) = 0$, hence $K_{\perp U} \in U^{\perp_g}$, and

$$g(K_{\perp U}, K_{\perp U}) = -\,\omega_U^2 \qquad (\text{since } g(K,K) = 0,\ g(U,U) = 1).$$

Thus a unit spatial direction of the ray in the U–frame can be taken as

$$E_U := \frac{1}{\omega_U}\,K_{\perp U} = \frac{1}{g(U,K)}\Big(K - (g(U,K))U\Big), \qquad g(E_U, E_U) = -1, \quad g(U, E_U) = 0. \tag{7.10}$$

Specialize to the same kinematics as in §7.3. Let ψ' denote the angle between the ray direction in the U–rest space and the U–spatial image of the boost axis. A direct computation yields the standard aberration law

$$\cos\psi' = \frac{\cos\psi - \beta}{1 - \beta\cos\psi}, \tag{7.11}$$

with $\cos\psi = \delta(E_v, E)$ as in (7.7).

Interpretation. Doppler and aberration are the same operation in two steps:

- Doppler: take the g–projection $g(U,K)$ (a scalar).
- Aberration: subtract the time component $(g(U,K))U$ and normalize the remaining U–spatial part.

Both effects are thus immediate consequences of the null cone geometry together with the observer–dependent splitting induced by U.

7.5 Geometric Algebra viewpoint: circular versus hyperbolic rotors

The Euclidean proto–space admits the Clifford algebra $\mathcal{C}\ell(T_p\mathcal{E}, \delta) \cong \mathcal{C}\ell_{4,0}$. For vectors $u, v \in T_p\mathcal{E}$ the geometric product is defined by

$$uv = u \cdot v + u \wedge v,$$

where $u \cdot v = \delta(u,v)$ is the Euclidean inner product. The induced Lorentzian form g can be encoded by the metric extensor $G := 2\,N \otimes N - \mathrm{Id}$, satisfying

$$g(u,v) = \delta(Gu, v),$$

which makes explicit how the g–geometry is obtained from Euclidean data once N is fixed.

Observer split inside $\mathcal{C}\ell_{4,0}$. Multiplying a vector X by N decomposes it into a scalar (longitudinal) part and a bivector (transverse) part relative to N:

$$XN = \underbrace{X \cdot N}_{\text{scalar}} + \underbrace{X \wedge N}_{\text{bivector}}. \tag{7.12}$$

The bivectors of the form $E \wedge N$ (with $E \in N^{\perp_\delta}$) constitute the observer's Euclidean spatial algebra. The even subalgebra $\mathcal{C}\ell_{4,0}^{+}$ is isomorphic to the quaternion algebra; this is the algebraic origin of the "unit quaternion" representation of calibrated flow states on S^3.

Circular rotor in Euclidean signature. Let $E \in N^{\perp_\delta}$ be δ–unit. In $\mathcal{C}\ell_{4,0}$ one has

$$N^2 = 1, \qquad E^2 = 1,$$

and the bivector $\mathbf{I} := NE$ satisfies

$$\mathbf{I}^2 = (NE)(NE) = -\,N^2E^2 = -1.$$

Hence $\exp(-\mathbf{I}\,\xi/2)$ generates an ordinary *circular* rotation in the (N,E)–plane. This is the natural algebraic encoding of the circular tilt coordinate ξ used in §4 and §5.

Hyperbolic rotor in Lorentzian signature. In the induced Lorentzian geometry, N remains unit timelike while a spatial unit direction $\mathbf{e}$ satisfies $\mathbf{e}^2 \overset{g}{=} -1$. Accordingly, the boost bivector $\mathbf{K} := N\mathbf{e}$ satisfies

$$\mathbf{K}^2 = (N\mathbf{e})(N\mathbf{e}) = -\,N^2\mathbf{e}^2 = +1,$$

so $\exp(-\mathbf{K}\,\eta/2)$ generates a *hyperbolic* rotor (a boost) with rapidity η. Applying such a rotor to N yields the standard boosted observer state

$$U = \cosh\eta\,N + \sinh\eta\,\mathbf{e},$$

and η is additive under collinear boost composition. In the conic parametrization of §5 one has $\tanh\eta = \beta = \tan\vartheta$.

Remark 7.7 (What "emerges" from the proto–space viewpoint)*.* No analytic continuation is invoked: the distinction between circular and hyperbolic behavior is controlled by the sign of the square of the generator bivector (-1 versus $+1$), which in turn is fixed by whether the spatial basis squares to $+1$ (Euclidean δ) or to -1 (Lorentzian g). The proto–space framework makes this dependence explicit by separating the Euclidean algebra $\mathcal{C}\ell_{4,0}$ from the induced Lorentzian structure determined by N.

7.6 Scope and limitations

The present work is purely kinematical. We do not introduce dynamical equations for the field N, nor do we model localized sources, defects, or global topology. Such extensions require additional structure (e.g. a field theory for N and/or a source model) and are deferred to separate work.

8 Discussion

8.1 What this reformulation does (and does not) claim

The primary contribution of this paper is a coordinate–free geometric reformulation of special–relativistic kinematics in a Euclidean phase space $(\mathcal{E},\delta)$ equipped with an induced Lorentzian metric $g = 2\,\alpha\otimes\alpha - \delta$ generated by a unit field N (or $\alpha := \delta(N,\cdot)$). The reformulation is designed to make the kinematic content of SR visible as simple Euclidean geometry on calibrated spheres S^3_c and in the orthogonal decomposition $X = X_\parallel + X_\perp$. At the same time, we emphasize that this construction is not a "Euclideanization" of spacetime physics: physical statements are made with respect to g, and δ serves as an underlying bookkeeping geometry for calibration and decomposition.

8.2 On the proto–parameter χ, proper time τ, and the locality of t

A central technical device is the calibrated proto–parameter χ, defined by $\delta(\tilde{X}, \tilde{X}) = c^2$, which provides a uniform Euclidean arclength normalization in $\mathcal{E}$. Proper time τ remains the g–invariant parameter along timelike worldlines, related to χ by $d\tau/d\chi = \sqrt{g(\tilde{X}, \tilde{X})}/c$ (Theorem 4.8). The quantity t introduced via $dt = \alpha(dX)/c$ should be understood as a line integral along a given worldline; a global time coordinate exists only under additional integrability conditions on α (e.g. hypersurface orthogonality).

8.3 Angle conventions and comparison with standard SR

The reformulation naturally introduces a Euclidean tilt angle ϑ via $\tan\vartheta = \|X_\perp\|_\delta/(N\cdot X)$, so that the physical speed parameter is $\beta = v/c = \tan\vartheta$, while alternative bounded parameters such as $\sin\vartheta$ may be used as auxiliary reparameterizations. This choice is adapted to the orthogonal decomposition and yields compact identities such as $g(X, X) = \|X\|_\delta^2 \cos(2\vartheta)$, making the timelike/null/spacelike sectors transparent in the calibrated geometry.

8.4 Advantages, limitations, and potential points of confusion

The main advantage is conceptual: several standard SR relations (Doppler, aberration) reduce to Euclidean projection identities once g is induced by (δ, N) and the calibration is fixed. A limitation is that, for general (nonintegrable) N, the "coordinate time" t is not globally defined, and the formalism should be interpreted locally or along worldlines. Another limitation is that the present paper focuses on inertial kinematics; extensions to accelerated motion and to dynamical fields require additional structure (e.g. evolution laws for N and/or α).

8.5 Relation to other constructions and outlook

The induced–metric mechanism is compatible with known recipes for obtaining Lorentzian metrics from Riemannian data (cf. [4]), but the present emphasis is different: we treat S_c^3 calibration and the tilt geometry as the primary kinematic arena. Natural next steps include: (i) a systematic treatment of accelerated observers in the calibrated picture and the emergence of Wigner–Thomas rotation; (ii) coupling to a field sector where N (or α) becomes dynamical; and (iii) identifying observationally accessible invariants that remain compact in the calibrated geometry while reproducing the standard SR predictions.

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