

Unimetry: A Phase-Space Reformulation of Special Relativity

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Abstract

We propose a compact reformulation of special relativity in which spacetime units (time and length) are treated as *phase velocities*—directional derivatives of a single underlying parameter, the phase $\vec{\chi} \in \mathbb{C}$. The observable Minkowski interval emerges as a conserved quantity under a change of parameter from the hidden phase coordinate χ to the observer’s proper time τ . In this *unimetry* formalism, familiar relativistic effects—time dilation, Lorentz factor, Doppler shift, and relativistic velocity composition—arise as elementary projections and rotations in a Euclidean phase plane. Hyperbolic features of Lorentz kinematics reappear after a reparameterization of time, yielding the standard relations without altering empirical content. We provide closed-form derivations of the longitudinal/transverse Doppler factors, identify a simple lemma equating the total phase speed to the conserved Minkowski norm, and outline connections to gauge phases, rapidity, and a cosmological time gauge.

Keywords: special relativity; phase; rapidity; Doppler shift; Lorentz factor; phase parameterization.

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1 Introduction

We usually take time and space as primitive. The *phase formalism* introduced here suggests a different viewpoint: time and space are *derived projections* of a single parameter $\vec{\chi} \in \mathbb{C}$ (“phase”). In this picture, relativistic effects such as time dilation and the Doppler shift are geometric consequences of phase-flow rotations.

The proposal does not modify physics; it reorganizes familiar relations in a simpler language. In spirit it is akin to Lagrangian/Hamiltonian re-descriptions of classical mechanics: same empirical content, different coordinates. Throughout, Greek θ will denote the *external* rotation angle associated with relative motion, while ζ denotes an *internal* angle associated with the object’s intrinsic state (mass/density heuristic). We emphasize that no modification of Einstein’s dynamics is proposed; all results are kinematical identities obtained by a change of parameter.

Notation. Tildes, dots and primes indicate derivatives with respect to the phase parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \quad \dot{X} := \frac{dX}{d\tau}, \quad X' := \frac{dX}{dl}.$$

We use c for the speed of light; $\beta := V/c$, $\gamma := 1/\sqrt{1-\beta^2}$, rapidity $\tanh \eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

2 Time and space as phase derivatives

Quaternionic phase (replacing the quaternionic ansatz). Henceforth the phase is *quaternionic*, not quaternionic. We write a unit rotor as

$$d(\hat{\mathbf{u}}, \psi) = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}, \quad (2.1)$$

with a unit pure quaternion $\hat{\mathbf{u}}$ (spatial direction) built from the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; the scalar part encodes the temporal projection, the vector part encodes the spatial projection. Any previous appearances of $[\cos(\psi) + \hat{\mathbf{u}} \sin(\psi)]$ or “quaternionic phase” are to be understood as their quaternionic counterparts via this rotor representation.

Let $\vec{\chi} \in \mathbb{C}$ be a variable whose change generates observable time-space effects. We treat the time and space units as directional derivatives (phase velocities) along the real and imaginary directions of a quaternionic basis (\hat{h}, \mathbf{l}) :

$$\hat{h} dx_0 = \frac{\partial \vec{\chi}}{\partial \chi_h} \frac{d\chi_h}{d\chi} d\chi = \tilde{H} d\chi, \quad \mathbf{l} dx_l = \frac{\partial \vec{\chi}}{\partial \chi_l} \frac{d\chi_l}{d\chi} d\chi = \tilde{L} d\chi, \quad l = 1, 2, 3. \quad (2.2)$$

Cheat sheet (actions & axes). *D-rotation (boost):* $d = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}$, $\mathbf{q} \mapsto d \mathbf{q} d$ (tilts the time axis to the 3-velocity).

R-rotation (spatial): $r = \cos \frac{\varphi}{2} + \hat{\mathbf{n}} \sin \frac{\varphi}{2}$, $\mathbf{q} \mapsto r \mathbf{q} r^{-1}$ (fixes the scalar/time axis).

Pullback to the observer's curvature: for observer rotor u , $\mathbf{q} \mapsto \bar{u} \mathbf{q} u$.

Introduce the phase speed of the SR interval $ds = \tilde{S} d\chi$. The interval conservation takes the form

$$\tilde{S}^2 = \frac{ds^2}{d\chi^2} = \frac{g_{ij} dx^i dx^j}{d\chi^2} = \tilde{H}^2 - \tilde{L}^2, \quad (2.3)$$

equivalently

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2. \quad (2.4)$$

Writing

$$\tilde{S} = \tilde{H} \cos \theta, \quad \tilde{L} = \tilde{H} \sin \theta, \quad (2.5)$$

where θ is the angle of the phase speed relative to the real axis. Algebraically, (??) is a Euclidean decomposition of a single speed into orthogonal projections; physically, we will see that under reparameterization the *projection* \tilde{S} , not the Euclidean norm \tilde{H} , is the conserved Minkowski quantity.

3 Phase space (*khōra*)

Let the phase vector space (“*khōra*”, after Plato) be \mathbb{C} with orthonormal basis (\hat{h}, \mathbf{l}) . For a phase vector $\vec{\chi} = R e^{\theta \mathbf{l}}$ with $\theta \in [-\pi, \pi]$,

$$\tilde{H} = R, \quad \tilde{S} = R \cos \theta, \quad \tilde{L} = R \sin \theta. \quad (3.1)$$

Choosing coordinates where the projectors onto (\hat{h}, \mathbf{l}) are unit, (??) simplifies to

$$\hat{h} dx_0 = \frac{d\chi_h}{d\chi} d\chi = \tilde{H} d\chi, \quad \mathbf{l} dx_l = \frac{d\chi_l}{d\chi} d\chi = \tilde{L} d\chi. \quad (3.2)$$

The map from phase to observables is an integral transform:

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}^i(u) du, \quad i = 0, 1, 2, 3, \quad (3.3)$$

where \tilde{X}^i are projections of $d\vec{\chi}/d\chi$ onto (\hat{h}, \mathbf{l}) and $x^i(\chi_0)$ fix initial conditions.

4 Objects

A *fundamental particle* is an *elementary object* with nonzero phase $\vec{\chi} \neq 0$. Composite *objects* are phase configurations; to represent them *in phase space* one may require additional dimensions, except for the *photon*, whose phase is always aligned with the imaginary axis:

$$\mathbf{p} = \frac{d\vec{\chi}}{d\chi_l} = p \mathbf{l} \in \Im. \quad (4.1)$$

Non-photonic phenomena are associated with nonzero real projection and nonzero mass. A quaternionic object can be identified with an *event* or worldline; the photon corresponds to a null-interval point encoding information about the event.

Any object's phase can be rotated to the *zero* (purely real) direction,

$$\vec{\chi}_0 = R \in \Re. \quad (4.2)$$

An object A moving with speed V relative to a rest observer has

$$\vec{\chi}_A = R e^{\theta_A \mathbf{l}}, \quad \sin \theta_A = \frac{V}{c} \equiv \beta. \quad (4.3)$$

4.1 Space as a symmetric phase pair

From (??), a naive zero-angle limit would remove the imaginary projection, contradicting observability. We enforce a nonvanishing spatial projection by pairing opposite-phase tilts:

$$\vec{\chi}^\pm = R e^{\pm \zeta \mathbf{l}}, \quad \vec{\chi}_l := \frac{\vec{\chi}^+ - \vec{\chi}^-}{2} = R \mathbf{l} \sin \zeta, \quad (4.4)$$

where ζ is an *internal angle* (intrinsic to the object; heuristically linked to mass/density). The local decomposition is

$$\vec{\chi}_0 = \vec{\chi}_\tau + \vec{\chi}_l = R \cos \zeta + R \mathbf{l} \sin \zeta, \quad (4.5)$$

with unit components (normalized by R): the real component is $\cos \zeta$ and the imaginary component is $\sin \zeta$.

4.2 Absolute, local, and observed time

Define *absolute* time $t = t(\tilde{H})$ at the zero phase direction; it is the fastest clock and useful for normalization between different phase speeds. Along the local real direction,

$$dx_0 = \frac{d}{d\chi} \Re(\vec{\chi}) d\chi = \frac{\vec{\chi}^+ + \vec{\chi}^-}{2} d\chi = \cos \zeta d\chi =: d\tau. \quad (4.6)$$

Here $d\chi_0 := \cos \zeta d\chi$ is the projection of $d\chi$ onto the local real axis; in Sec. ?? we calibrate $d\tau = (1/\nu_0) d\chi_0$. The observed proper time of A relative to the rest observer is

$$\tilde{H}_A = \Re\left(\frac{d\vec{\chi}_A}{d\vec{\chi}_0}\right) = \cos \theta_A = \sqrt{1 - \sin^2 \theta_A} = \sqrt{1 - \frac{V^2}{c^2}} = \frac{1}{\gamma}. \quad (4.7)$$

4.3 Normalization

Let local time be parameterized by *phase*; introduce a reference frequency ν_0 and set

$$d\tau = \frac{1}{\nu_0} d\chi_0. \quad (4.8)$$

By the chain rule,

$$dx_0 = \tilde{H} d\chi = \frac{dx_0}{d\chi_0} \frac{d\chi_0}{d\tau} d\tau = \tilde{H} \dot{\chi} d\tau =: \dot{H} d\tau, \quad (4.9)$$

where $\nu := d\chi/d\tau$, $\dot{\chi} := \nu/\nu_0$, and $\dot{H} := \tilde{H} \dot{\chi}$. Choosing the calibration $\dot{H} \equiv c$ gives $dx_0 = c d\tau$. Similarly for space,

$$dx_l = \tilde{L} d\chi = \frac{dx_l}{d\chi_0} \frac{d\chi_0}{dl} dl = \tilde{L} \chi' dl =: L' dl, \quad \chi' := \frac{d\chi}{dl}. \quad (4.10)$$

From $dx_0 = dx_l$ for light one gets

$$c = \tilde{L}' \frac{dl}{d\tau}, \quad (4.11)$$

hence with temporal calibration to c the spatial scale becomes unit: $\tilde{L}' = 1$.

4.4 Light and c as a calibration constant

From the normalized forms,

$$\frac{c}{\dot{\chi}} d\chi = \frac{1}{\chi'} d\chi \quad \Rightarrow \quad c = \frac{\dot{\chi}}{\chi'} = \frac{dl}{d\tau}, \quad (4.12)$$

i.e. c is a *calibration constant* tying temporal and spatial measures, independent of local phase variation. Equation (??) also reads

$$c = \left(\frac{d\chi}{d\tau} \right) \left[\frac{dl}{d\chi} \right] \sim (\nu) [\lambda], \quad (4.13)$$

matching frequency and wavelength of a photon, with χ as its phase. For a lightlike trajectory,

$$ds^2 = c^2 \left(\frac{d\chi^2}{\dot{\chi}^2} - \frac{d\chi^2}{\dot{\chi}^2} \right) = 0. \quad (4.14)$$

At unit frequency, $\tau = \chi$: the photon's “proper time” is its phase, and the length of its phase-speed vector equals its wavelength, $\tilde{H}_p = \lambda$. Finally, the kinematic slope in phase coordinates is

$$\frac{dx_l}{dx_0} = \frac{\tilde{L} d\chi}{\tilde{H} d\chi} = \sin \theta = \frac{V}{c} \equiv \beta, \quad (4.15)$$

so $\theta = \pi/2$ implies $V = c$.

4.5 Lorentz factor via reparameterization

A change of direction of the phase speed transforms

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2 \mapsto \dot{H}^2 = \dot{S}^2 + \dot{L}^2. \quad (4.16)$$

Lemma (parameter-change identity). The transition $\tilde{H} \rightarrow \dot{S}$ is the manifestation of evolving phase speed under the parameter change $\chi \mapsto \tau(\chi)$, with local Jacobian

$$\frac{d\tau}{d\chi} = \cos \zeta(\chi) \cos \theta(\chi) \quad \Rightarrow \quad \mathcal{J}(\zeta, \theta) := \frac{d\chi}{d\tau} = \frac{1}{\cos \zeta \cos \theta}. \quad (4.17)$$

Then

$$\dot{H} = \tilde{H} \mathcal{J}, \quad \dot{L} = \tilde{L} \mathcal{J}. \quad (4.18)$$

In differential form,

$$d \ln \dot{H} = d \ln \mathcal{J} = \tan \zeta d\zeta + \tan \theta d\theta. \quad (4.19)$$

For a *pure boost* ($d\zeta = 0$) one has $d\dot{H} = \dot{H} \tan \theta d\theta$. Absorbing a constant $\cos \zeta$ into the calibration (set $\zeta = 0$ henceforth), we obtain

$$\tilde{H}^2 = \dot{H}^2 - \dot{L}^2 = \sec^2 \theta (\tilde{H}^2 - \tilde{L}^2) = \gamma^2 (\tilde{H}^2 - \tilde{L}^2). \quad (4.20)$$

Corollary. In phase space the Euclidean norm \tilde{H} is conserved; in observed time the Minkowski norm \dot{S} is conserved; they are identical as quantities:

$$\boxed{\tilde{H} = \dot{S}}. \quad (4.21)$$

4.6 Rapidity and the phase angle

By definition,

$$\beta = \frac{V}{c} = \sin \theta, \quad \tanh \eta = \beta, \quad d\eta = \frac{d\beta}{1 - \beta^2}. \quad (4.22)$$

With $d\beta = \cos \theta d\theta$ and $1 - \beta^2 = \cos^2 \theta$,

$$d\eta = \sec \theta d\theta, \quad \eta(\theta) = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| = \frac{1}{2} \ln \frac{|1 + \sin \theta|}{|1 - \sin \theta|}. \quad (4.23)$$

Fixing $\eta(0) = 0$,

$$e^{\eta(\theta)} = \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \sec \theta = \cosh \eta. \quad (4.24)$$

Remark (groups). Observables satisfy $\beta = \sin \theta = \tanh \eta$ and $\gamma = \sec \theta = \cosh \eta$. Thus Euclidean rotations in the phase circle ($U(1)$ with angle θ) reproduce the numerical factors of hyperbolic boosts in $SO^+(1, 1)$ (rapidity η) *after* reparameterizing time. We do not claim an isomorphism $U(1) \cong SO(1, 1)$; only the equality of observable combinations under the change of parameter.

4.7 Velocity addition

Notation. In the unimetry formalism, an inertial boost is a *D-rotation*

$$\mathcal{B}(\hat{\mathbf{u}}, \psi) : \quad \mathbf{q} \mapsto d \mathbf{q} d, \quad d = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}, \quad (4.25)$$

and a spatial rotation is an *R-rotation*

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) : \quad \mathbf{q} \mapsto r \mathbf{q} r^{-1}, \quad r = \cos \frac{\varphi}{2} + \hat{\mathbf{n}} \sin \frac{\varphi}{2}. \quad (4.26)$$

We parametrize kinematics by the unimetry angle ψ via

$$\beta \equiv v/c = \sin \psi, \quad \gamma = \frac{1}{\cos \psi}, \quad \tan \frac{\psi}{2} = \frac{\sin \psi}{1 + \cos \psi} = \frac{\gamma \beta}{\gamma + 1}. \quad (4.27)$$

(Recovering the standard SR formulas is achieved by the replacement $\sin \leftrightarrow \sinh$, $\cos \leftrightarrow \cosh$.)

4.7.1 Wigner rotation

Statement (boost composition). The composition of two non-collinear unimetry boosts factorizes into a net boost followed by a spatial rotation (the Wigner rotation):

$$\boxed{\mathcal{B}(\hat{\mathbf{u}}_2, \psi_2) \mathcal{B}(\hat{\mathbf{u}}_1, \psi_1) = \mathcal{R}(\hat{\mathbf{u}}_W, \phi_U) \mathcal{B}(\hat{\mathbf{u}}_{12}, \psi_{12})} \quad (4.28)$$

Quaternionic pullback (observer's curvature). Let u denote the observer's D -rotor (so that passing to the observer's curvature amounts to the sandwich $X \mapsto \bar{u} X u$). Consider two boosts with rotors $d_i = \cos \frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin \frac{\psi_i}{2}$ and let

$$L \equiv d_2 d_1, \quad L_u \equiv \bar{u} L u = \bar{u} d_2 d_1 u, \quad (4.29)$$

be the composed transformation as *seen by the observer*. Let $d_{12}^{(u)}$ be the unique unimetry boost obtained from the velocity-addition rule in the observer frame, i.e.

$$d_{12}^{(u)} = \cos \frac{\psi_{12}}{2} + \hat{\mathbf{u}}_{12} \sin \frac{\psi_{12}}{2}, \quad (4.30)$$

with $(\hat{\mathbf{u}}_{12}, \psi_{12})$ computed from $(\hat{\mathbf{u}}_1, \psi_1)$, $(\hat{\mathbf{u}}_2, \psi_2)$ in §?? but expressed in the observer's frame. Then the *observed* Wigner rotation is simply the residual rotor in the polar factorization of L_u :

$$\boxed{r_W^{(u)} = L_u (d_{12}^{(u)})^{-1} = \bar{u} d_2 d_1 u \left(\cos \frac{\psi_{12}}{2} - \hat{\mathbf{u}}_{12} \sin \frac{\psi_{12}}{2} \right)} \quad (4.31)$$

(a unit quaternion). It is purely spatial in the observer frame, i.e. it fixes the scalar subspace: $r_W^{(u)} \lambda (r_W^{(u)})^{-1} = \lambda$ for all scalars λ . Writing $r_W^{(u)} = \cos \frac{\phi_U}{2} + \hat{\mathbf{n}}_W \sin \frac{\phi_U}{2}$ identifies the observed axis $\hat{\mathbf{n}}_W^{(u)}$ and angle ϕ_U .

Proof sketch. In the observer frame, $d_{12}^{(u)}$ is the unique D -rotor that maps the time axis to the composed 3-velocity (Sec. ??). Therefore $L_u (d_{12}^{(u)})^{-1}$ must leave the time axis invariant and hence is a pure R -rotation—the Wigner rotation.

with the Wigner axis along the cross product of the boost directions:

$$\hat{\mathbf{n}}_W = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1}{\sin \theta}, \quad \cos \theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1. \quad (4.32)$$

Angle (unimetry half-angles). In terms of unimetry half-angles, the Wigner angle is

$$\boxed{\tan \frac{\phi_U}{2} = \frac{\tan \frac{\psi_1}{2} \tan \frac{\psi_2}{2} \sin \theta}{1 + \tan \frac{\psi_1}{2} \tan \frac{\psi_2}{2} \cos \theta}} \quad (4.33)$$

equivalently,

$$\tan \frac{\phi_U}{2} = \frac{\sin \frac{\psi_1}{2} \sin \frac{\psi_2}{2} \sin \theta}{\cos \frac{\psi_1}{2} \cos \frac{\psi_2}{2} + \sin \frac{\psi_1}{2} \sin \frac{\psi_2}{2} \cos \theta}. \quad (4.34)$$

Limits: for collinear boosts $\theta = 0$ one has $\phi_U = 0$; for $\beta \ll 1$,

$$\phi_U \approx \frac{1}{2} |\boldsymbol{\beta}_2 \times \boldsymbol{\beta}_1|, \quad \boldsymbol{\beta}_i \equiv \beta_i \hat{\mathbf{u}}_i. \quad (4.35)$$

Operator form. If $d_i = \cos \frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin \frac{\psi_i}{2}$, then

$$d_2 d_1 = r_W d_{12}, \quad r_W = \cos \frac{\phi_U}{2} + \hat{\mathbf{n}}_W \sin \frac{\phi_U}{2}, \quad (4.36)$$

and the action on any unimetry 4-object is

$$\mathbf{q}' = r_W (d_{12} \mathbf{q} d_{12}) r_W^{-1}.$$

Quaternionic pullback as angle compensation. Let d_1 and d_2 be the D -rotors of two successive boosts. Their raw action on any unimetry 4-object is

$$\mathbf{q}' = d_2 d_1 \mathbf{q} d_1 d_2. \quad (4.38)$$

Let d_{12} denote the unique D -rotor that reproduces the combined space–time tilt (the change of the spatio–temporal angle) of $d_2 d_1$, i.e. it maps the time axis to the composite 3-velocity given by the velocity-addition law in §??. Pulling back by the conjugate rotor \bar{d}_{12} on both sides removes this tilt:

$$\mathbf{q}^{(u)} = \bar{d}_{12} \mathbf{q}' \bar{d}_{12} = (\bar{d}_{12} d_2 d_1) \mathbf{q} (d_1 d_2 \bar{d}_{12}). \quad (4.39)$$

Define the residual rotor

$$\boxed{r_W \equiv \bar{d}_{12} d_2 d_1} \quad \Rightarrow \quad \mathbf{q}^{(u)} = r_W \mathbf{q} r_W^{-1}, \quad (4.40)$$

Uniqueness of d_{12} and factorization lemma. Let \mathbf{e}_t denote the unit temporal basis (observer’s time axis). The composite D -rotor d_{12} is uniquely fixed by requiring it to map the time axis to that of the product $d_2 d_1$,

$$d_{12} \mathbf{e}_t d_{12} = d_2 d_1 \mathbf{e}_t d_1 d_2, \quad \text{with } \Re(d_{12}) \geq 0, \quad (4.41)$$

(the sign choice $\Re(d_{12}) \geq 0$ removes the trivial two-fold ambiguity $d \rightarrow -d$).

Lemma (D–R factorization). For any product of D -rotors $L = d_2 d_1$ there exist unique rotors d_{12} (of type D) and r_W (of type R) such that

$$\boxed{L = d_{12} r_W}, \quad r_W = \bar{d}_{12} L = \bar{d}_{12} d_2 d_1. \quad (4.42)$$

Order dependence (noncommutativity). Write

$$L_{12} \equiv d_2 d_1 = d_{12} r_W, \quad L_{21} \equiv d_1 d_2 = d_{21} r_W^{-1}, \quad (4.43)$$

where d_{12} and d_{21} are the unique D -rotors specified by (??) for L_{12} and L_{21} , respectively, and r_W is the Wigner rotor defined in (??). In general,

$$d_{12} \neq d_{21}, \quad r_W^{-1} \neq r_W, \quad (4.44)$$

except for the collinear (or trivial) case. Swapping the order flips the Wigner rotation ($r_W \mapsto r_W^{-1}$) and changes the net 3-velocity, reflecting the noncommutativity of velocity addition ($\mathbf{v}_2 \oplus \mathbf{v}_1 \neq \mathbf{v}_1 \oplus \mathbf{v}_2$). The quaternionic compensation picture remains the same: for the $(1 \rightarrow 2)$ order the observed map is $r_W \mathbf{q} r_W^{-1}$, for $(2 \rightarrow 1)$ it is $r_W^{-1} \mathbf{q} r_W$.

One-line algorithm for r_W . Compute $L = d_2 d_1$, find the unique d_{12} by $d_{12} \mathbf{e}_t d_{12} = L \mathbf{e}_t L$ with $\Re(d_{12}) \geq 0$, then $r_W = \bar{d}_{12} L$ and the observed action is $\mathbf{q} \mapsto r_W \mathbf{q} r_W^{-1}$.

Passive vs active actions (disambiguation).

| Name | Map on \mathbf{q} | Meaning | Fixes time? |
|------------------|------------------------------|-----------------------------------------------|-------------|
| Passive pullback | $\bar{u} \mathbf{q} u$ | change of frame / curvature (observer’s view) | yes |
| Active D-cancel | $\bar{u} \mathbf{q} \bar{u}$ | undo D-tilt (acts on the object) | no |
| R-rotation | $r \mathbf{q} r^{-1}$ | pure spatial rotation | yes |
| D-rotation | $d \mathbf{q} d$ | boost (tilt of time axis) | no |

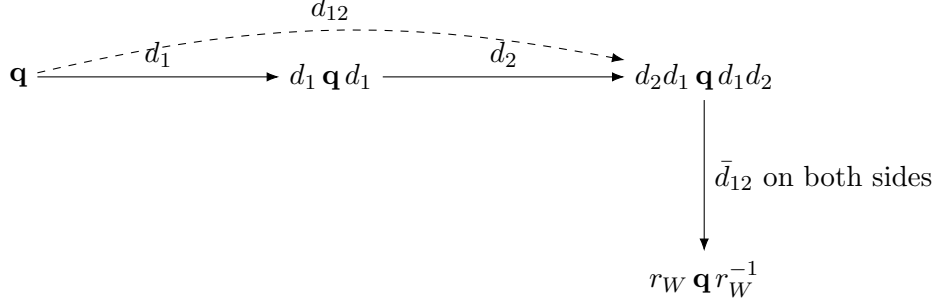


Figure 1: Two successive D-rotations (boosts) and compensation of the net spatio-temporal angle by the conjugate of d_{12} , leaving a pure R-rotation r_W .

With this choice, r_W fixes \mathbf{e}_t and acts as a pure spatial rotation in the observer frame, reproducing the Wigner rotation with axis/angle given by (??)–(??).

so the *observed* transformation is a pure spatial rotation. Writing $r_W = \cos \frac{\phi_U}{2} + \hat{\mathbf{n}}_W \sin \frac{\phi_U}{2}$ recovers the axis and angle formulas (??)–(??). This exhibits the Wigner rotation as the quaternionic *compensation* of the net spatio-temporal angle.

4.7.2 Thomas precession

Definition (continuous limit of Wigner rotation). For a worldline with time-dependent velocity direction $\hat{\mathbf{u}}(t)$, Thomas precession is the instantaneous angular velocity of the Wigner rotation accumulated by the sequence of infinitesimal boosts:

$$\boxed{\boldsymbol{\omega}_T = (\gamma - 1) (\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}) = \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{a} \times \mathbf{v}}{c^2}} \quad (4.45)$$

where $\mathbf{v} = v \hat{\mathbf{u}}$, $\mathbf{a} = \dot{\mathbf{v}}$, and $\gamma = 1/\cos \psi$. Equivalently, in pure unimetry variables,

$$\boldsymbol{\omega}_T = \frac{1 - \cos \psi}{\cos \psi} (\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}). \quad (4.46)$$

Special cases and limits. For uniform circular motion ($|\mathbf{v}| = \text{const}$) with orbital angular velocity $\boldsymbol{\Omega}$ (defined by $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega} \times \hat{\mathbf{u}}$),

$$\boxed{|\boldsymbol{\omega}_T| = (\gamma - 1) \Omega} \quad (\text{axis opposite to } \boldsymbol{\Omega} \text{ in the standard convention}). \quad (4.47)$$

In the nonrelativistic limit $\beta \ll 1$,

$$\boldsymbol{\omega}_T \approx \frac{1}{2} \frac{\mathbf{a} \times \mathbf{v}}{c^2}. \quad (4.48)$$

Remark on placement. Since Thomas precession is the differential (continuous) limit of the Wigner rotation for a sequence of infinitesimal non-collinear boosts, it is natural to present §?? first and then §??.

4.8 Doppler shift

Define the observed frequency as the phase growth rate in the observer's proper time:

$$\nu := \frac{d\chi}{d\tau}. \quad (4.49)$$

For two successive wavefronts the phase increment is identical, hence

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{d\chi/d\tau_{\text{obs}}}{d\chi/d\tau_{\text{src}}} = \frac{d\tau_{\text{src}}}{d\tau_{\text{obs}}}. \quad (4.50)$$

Longitudinal case: during $\gamma d\tau_{\text{src}}$ in the observer frame the source displaces by $\pm V \gamma d\tau_{\text{src}}$ (“+” receding, “−” approaching). Then

$$d\tau_{\text{obs}} = \gamma d\tau_{\text{src}}(1 \pm \beta), \quad \Rightarrow \quad \boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 \pm \beta)}}. \quad (4.51)$$

Equivalent forms (with $\beta = \sin \theta$, $\gamma = \sec \theta$ and rapidity η):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec \theta (1 \mp \sin \theta) = e^{\mp \eta}. \quad (4.52)$$

Transverse Doppler ($\varphi = 90^\circ$ in the observer’s frame):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma} = \cos \theta. \quad (4.53)$$

General line-of-sight (LOS) angle φ in the observer’s frame:

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \gamma (1 - \beta \cos \varphi)}. \quad (4.54)$$

Wavelength ratios are inverse to frequency ratios.

Differential pullback and Thomas precession. Let $d(t)$ be the instantaneous boost rotor and fix the observer’s $u(t)$ as instantaneously comoving. Over a small interval Δt the observed residual rotation is

$$r_W^{(u)}(\Delta t) = \bar{u}(t) d(t + \Delta t) d(t) u(t) \left(d_{12}^{(u)}(t, \Delta t) \right)^{-1}, \quad (4.55)$$

whose first-order expansion is $r_W^{(u)}(\Delta t) \simeq 1 - \frac{1}{2} (\omega_T \Delta t) \cdot \hat{\mathbf{N}}$, yielding (??)–(??) with $\omega_T = (\gamma - 1) \hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}$. This makes explicit that the Thomas precession is the differential limit of the quaternionic pullback residual rotation.

5 Discussion: links to known structures

Gauge phases. A global shift $\chi \mapsto \chi + \chi_0$ is unobservable. Allowing local reparameterizations $\chi \mapsto \chi + \alpha(x)$ induces a connection when comparing phases at different points. On wavefunctions $\psi \sim [\cos(\chi) + \hat{\mathbf{u}} \sin(\chi)]$ this is the familiar $U(1)$ gauge freedom $\psi \rightarrow [\cos(\alpha(x)) + \hat{\mathbf{u}} \sin(\alpha(x))] \psi$ with $D_\mu = \partial_\mu - iA_\mu$ as the *phase-transport connection*.

Mass and the internal angle. With the decomposition by ζ , mass heuristically correlates with an irreducible real projection: massless objects have $\zeta = \pm\pi/2$ (no proper time; photon subspace), while massive objects have $|\zeta| < \pi/2$ (proper time exists). In the present paper we set $\zeta = 0$ in boost kinematics by calibration; a detailed mass-generation mechanism is left for future work.

Cosmological gauge. A natural global calibration of “absolute” time is the comoving frame with vanishing CMB dipole. This fixes a cosmological time t (FLRW) as a gauge, without affecting local Lorentz invariance; Doppler factors are then operationally referenced to that frame.

6 Conclusion

In unimetry, time and space are integrals of phase velocities; the Minkowski interval appears as a conserved quantity under parameter change. The core relations of SR— γ , rapidity, velocity addition, and Doppler factors—follow from elementary phase-plane geometry with a single rotation angle θ , while hyperbolic structure re-emerges upon reparameterizing time. The formalism is empirically equivalent to standard SR but can clarify causality and composition by treating all effects as projections of a single flow.

Outlook. Future directions include (i) a more explicit group-theoretic embedding, (ii) a rigorous treatment of the internal angle ζ and its relation to mass, and (iii) exploration of curved metrics as spatially varying Jacobians $\mathcal{J}(x)$ in the phase-to-observable map.

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