# Unimetry: Proto-Space Reformulation of Special Relativity

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**Abstract**

We present a coordinate–free reformulation of inertial special–relativistic kinematics in a four–dimensional Euclidean proto–space $(\mathcal{E}, \delta)$. An observer is specified by a unit axis $N$, with time 1–form $\alpha = \delta(N, \cdot)$ and spatial projector $h = \delta - \alpha \otimes \alpha$. The Lorentzian interval is then the induced quadratic form $g = 2\,\alpha \otimes \alpha - \delta$, and proper time arises operationally via $c^2 d\tau^2 = g(dX, dX)$ as a reparametrization along worldlines. Introducing a $\delta$–calibrated proto–parameter $\chi$ yields a constant–norm protovelocity and identifies the instantaneous kinematic state space with the calibrated sphere $S^3_c \subset T_p\mathcal{E}$, with the physically admissible sector given by $g > 0$ and a null boundary at a finite Euclidean tilt. We distinguish the circular tilt $\xi$ from the conic ratio coordinate $\vartheta$ and relate both to rapidity $\eta$, providing a compact Euclidean description of boosts. Within $\mathcal{C}\ell_{4,0}$, observer axes and kinematics admit rotor and quaternionic representations that streamline projection–based derivations of standard SR optical effects.

**Keywords:** special relativity; phase; rapidity; Doppler shift; aberration; Lorentz factor; Wigner-Thomas rotation; phase parametrization.

**MSC (2020):** 83A05; 70A05.

## 1 Introduction

### 1.1 Motivation

Special relativity (SR) is usually presented as geometry in a Lorentzian spacetime. While conceptually economical, that viewpoint can hide two practical aspects that matter operationally:

(i) An observer does not measure an arbitrary curve parameter; physical clocks implement a specific phase (proper–time) parametrization along worldlines.

(ii) Many elementary relativistic effects (time dilation, the light cone, Doppler shift, aberration) can be traced to a single observer splitting: how a total displacement budget decomposes into a component along an observer time axis and a component orthogonal to it.

The present paper develops a Euclidean proto–space formulation that isolates this splitting as the primary geometric datum. We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$ and introduce a "time" axis $N$ (a unit vector field, $\delta(N, N) = 1$). The associated time 1–form $\alpha := N^\flat = \delta(N, \cdot)$ and spatial projector $h := \delta - \alpha \otimes \alpha$ encode the operational notions of "time reading" and "spatial distance" for that observer. The Lorentzian interval then appears as the induced quadratic form

$$g = \alpha \otimes \alpha - h = 2\,\alpha \otimes \alpha - \delta,$$

and proper time emerges as the operational reparametrization $c^2 d\tau^2 = g(dX, dX)$ along worldlines.

A second motivation is structural and computational. The Euclidean proto–space admits the strictly Euclidean Clifford algebra $\mathcal{C}\ell(4,0)$, so that orthogonal changes of observer gauge can be encoded by rotors acting by sandwiching. After fixing a fiducial unit vector $e_0$ (used to define a stable quaternionic coordinate subalgebra), an observer axis $N$ may be represented as a rotor image $N = qe_0\widetilde{q}$, while relative states between an observer axis and an object axis can be packaged as even elements built from the product of two unit vectors. In this language, calibrated proto–velocities form a compact state space: at each point $p \in \mathcal{E}$, the kinematically admissible calibrated proto–velocity states lie on the Euclidean sphere $S_c^3 \subset T_p\mathcal{E}$, and causal classification with respect to $g$ selects a physical sector of that sphere. The null boundary is reached at a finite Euclidean tilt to the time axis, corresponding to an equal split of the proto–space budget between the longitudinal and transverse components.

**Axiomatic shift (primitives and ontology).** The present reformulation does not modify the empirical content of inertial SR. It changes the choice of primitives and the direction of explanation. Instead of taking a Lorentzian spacetime metric as fundamental, we start from a Euclidean proto–space $(\mathcal{E}, \delta)$ (and its Euclidean geometric algebra $\mathcal{C}\ell(4,0)$) and treat "time" and "space" as observer–induced notions. The minimal observer input is a choice of a coherent unit axis $N$ (equivalently, its time 1–form $\alpha = \delta(N, \cdot)$), which induces a $1+3$ split and defines simultaneity as the $\delta$–orthogonal hyperplanes $\alpha(\Delta X) = 0$. The Lorentzian bilinear form used to encode causal classification and intervals is then constructed from this observer data by the pointwise projector correspondence of Reddy–Sharma–Sivaramakrishnan,

$$g \;=\; 2\,\alpha \otimes \alpha \;-\; \delta,$$

rather than postulated as a background structure. In this sense, Minkowski geometry enters as an operationally induced structure on top of the Euclidean proto–space.

**Status of SR postulates in the proto–space picture.** With this choice of primitives, several standard SR statements move from the axiomatic level to the level of derived consequences. Proper time is not taken as an a priori coordinate but is defined along worldlines by the constructed interval, $c^2 d\tau^2 = g(dX, dX)$. A convenient unobservable proto–parameter $\chi$ calibrates the total Euclidean budget through $\delta(dX/d\chi, dX/d\chi) = c^2$, while time dilation and the usual $\gamma$–factor emerge from the reparametrization $\chi \mapsto \tau$ (equivalently, from the projection of the calibrated proto–velocity onto the observer axis). The speed–of–light barrier corresponds to the null condition $g(dX, dX) = 0$ and is realized as a finite boundary in the Euclidean tilt geometry of the calibrated sphere. Inertial frame changes are implemented as changes of the observer axis, encoded by rotors in $\mathcal{C}\ell(4,0)$ after fixing the fiducial axis $e_0$. Textbook velocity composition, including transverse $1/\gamma$ suppression and Wigner–Thomas rotation, arises once one consistently works with proper–time normalized axes rather than with raw $\chi$–calibrated directions on $S_c^3$.

### 1.2 Relation to previous work

Attempts to express Lorentzian kinematics in Euclidean terms have a long history. Early geometric constructions already appear in Karapetoff [1], where relativistic transformations are visualized by Euclidean angle geometry. More recent works study various embeddings and correspondences between Euclidean and Lorentzian structures, including [2, 3].

A distinct line of literature aims at an explicitly Euclidean reformulation of special relativity. Euclidean SR can be obtained by changes of variables in which Lorentz transformations are represented as rotations in a Euclidean space (e.g. Gersten [9]). Other proposals postulate an absolute Euclidean background and reinterpret relativistic observables in terms of proper time (e.g. Montanus [10, 11]), or develop related "four–dimensional optics" frameworks in which $\tau$ plays a central operational role (e.g. Almeida [12, 13]). For broader context on Euclidean viewpoints beyond SR, see also Atkinson [14].

Our construction differs in emphasis. We do not identify the Euclidean norm with the Lorentz interval by a coordinate trick, and we do not postulate an absolute Euclidean time. Instead, we start from an observer splitting encoded by a coherent unit axis $N$ and its time 1–form $\alpha$, and interpret the Lorentzian interval as the operational quadratic form naturally associated with this splitting. Proper time arises as a reparametrization along worldlines, rather than being built in as a primary coordinate. This standpoint is aligned with relational viewpoints in generally covariant physics, where "time" is implemented by a chosen clock observable (cf. Rovelli's partial observables perspective [7, 8]).

A rigorous pointwise correspondence between a Riemannian metric and a Lorentzian one was established by Reddy, Sharma and Sivaramakrishnan [4]. Given a Riemannian manifold $(M, h)$ and a unit vector field $U$, they define a Lorentzian metric by $g = h - 2\,U^\flat \otimes U^\flat$. We adopt the sign–flipped variant adapted to the particle–physics convention $(+\,-\,-\,-)$,

$$g \;=\; 2\,N^\flat \otimes N^\flat \;-\; \delta,$$

and then make explicit how SR kinematics follows from phase reparametrization and Euclidean tilt geometry relative to $\alpha$.

**Inertial specialization (SR kinematics).** Although the construction applies to any smooth unit field $N$ on $(\mathcal{E}, \delta)$, the present paper focuses on inertial SR kinematics and therefore restricts to the case where $N$ is parallel with respect to the flat Euclidean connection:

$$\nabla^\delta N = 0. \tag{1.1}$$

Equivalently, in global Cartesian coordinates on $\mathcal{E} \simeq \mathbb{R}^4$, the components $N^A$ are constant and the induced form $g = 2\,N^\flat \otimes N^\flat - \delta$ is constant (globally Minkowskian). Relativistic effects in the present framework arise from the operational reparametrization and tilt geometry, rather than from curvature.

**Observer dependence and Lorentz invariance.** The axis field $N$ is a geometric datum (no dynamics is postulated for it in this paper), but it is not an absolute frame in the physical sense. In the inertial sector, choosing a constant $N$ is choosing an inertial observer. Any other constant unit axis $N'$ is related to $N$ by a Euclidean isometry $R \in O(4)$, $N' = R_* N$, and the induced form transforms tensorially, $g' = R^* g$. Physical statements are formulated in $g$–covariant (relational) quantities, so the theory does not privilege a particular representative $N$. Note that the quadratic form $g$ is invariant under $N \mapsto -N$, whereas the oriented time 1–form $\alpha = N^\flat$ changes sign; causal classification is insensitive to this sign, while a choice of "future" requires a consistent orientation.

**Geometric algebra viewpoint (role of $\mathcal{C}\ell(4,0)$).** Algebraically, our presentation is close in spirit to the geometric–algebra approach to relativity, in which boosts and rotations are handled by rotors (see, e.g., Hestenes' Space–Time Algebra [5, 6] and the discussion of spacetime algebra versus "imaginary time" in [15]). The difference is that we work throughout in the strictly Euclidean Clifford algebra $\mathcal{C}\ell(4,0)$: the observer split is encoded by the choice of $N$ (hence $\alpha$) and is observer–dependent, while Lorentzian boost kinematics is recovered by rotor transport of properly normalized axes and directions.

### 1.3 Contributions

The main contributions of the present paper are:

(C1) Observer splitting and operational origin of the Lorentz interval. Starting from a Euclidean proto–space $(\mathcal{E}, \delta)$ and a unit axis $N$, we define the time 1–form $\alpha := N^\flat = \delta(N, \cdot)$ and the spatial projector $h = \delta - \alpha \otimes \alpha$, and construct the induced bilinear form $g = \alpha \otimes \alpha - h = 2\,\alpha \otimes \alpha - \delta$. Proper time is then defined operationally by $c^2 d\tau^2 = g(dX, dX)$.

(C2) Euclidean Clifford algebra implementation of observer axes. We formulate the construction in $\mathcal{C}\ell(4,0)$, represent observer axes as rotor images $N = q e_0 \widetilde{q}$, and make explicit the associated gauge nonuniqueness and the $N \mapsto -N$ sign issue at the level of time orientation.

(C3) Reparameterization equivalence on the calibrated sphere. Proper–time normalization in the induced Lorentzian form is shown to be equivalent to a constant–speed Euclidean proto–velocity constrained to the calibrated sphere $S^3_c \subset T\mathcal{E}$.

(C4) Geometric characterization of the physical sector. The timelike condition $g(X, X) > 0$ selects an open subset of $S^3_c$ (two polar caps), with the null boundary reached at the finite Euclidean tilt $\xi = \pi/4$.

(C5) Circular versus conic/hyperbolic parametrizations of tilt. We distinguish the circular angle $\xi$ (normalized by $\|X\|_\delta$) from the conic ratio coordinate $\vartheta$ (normalized by $N \cdot X$), and relate both to the additive rapidity $\eta$ via $\tanh \eta = \tan \vartheta$.

(C6) Projection–based optics and Euclidean GA derivations. Frequency and direction of null rays are obtained as $g$–projections, yielding compact derivations of Doppler shift and aberration; boosts are encoded within the Euclidean rotor framework tied to the $N$–split.

### 1.4 Outline

The paper is organized as follows:

- Section 2 introduces the Euclidean proto–space $(\mathcal{E}, \delta)$, formulates $\mathcal{C}\ell(4,0)$ on $T\mathcal{E}$, and defines observer data ($e_0$ as a fiducial axis and $N$ as a coherent time axis). The induced Lorentzian form $g = 2\,N^\flat \otimes N^\flat - \delta$ is constructed pointwise.

- Section 3 develops basic properties of the induced Lorentzian structure: the orthogonal decomposition relative to $N$, norm identities, causal classification, and the geometry of the null cone in the $\xi$–tilt picture.

- Section 4 formulates the operational clock viewpoint: we introduce a $\delta$–calibrated proto–parameter $\chi$, define coordinate time and proper time via $\alpha$, and make explicit the phase reparametrization $\chi \mapsto \tau$.

- Section 5 develops the tilt geometry and distinguishes circular $(\xi)$ and conic/hyperbolic $(\vartheta, \eta)$ parametrizations, including the identity $\tanh \eta = \tan \vartheta$ and the $45^\circ$ null boundary.

- Section 6 reformulates 4–velocity normalization as a constraint on calibrated proto–velocities, mapping admissible states to the sphere $S^3_c \subset T\mathcal{E}$ and identifying the physical sector.

- Section 7 illustrates the computational payoff: optical effects are derived as $g$–projections and boosts are reformulated using Euclidean geometric algebra.

# 2 Lorentzian metric construction

## 2.1 Euclidean proto-space $(\mathcal{E}, \delta)$

We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$ equipped with the flat metric

$$\delta_{AB} = \mathrm{diag}(1, 1, 1, 1).$$

Indices are raised and lowered with $\delta$:

$$X_A := \delta_{AB} X^B, \qquad X^A := \delta^{AB} X_B,$$

and we use the $\delta$–inner product notation

$$X \cdot Y := \delta(X, Y) = \delta_{AB} X^A Y^B.$$

*Remark* 2.1 (Index conventions: $\delta$ vs. $g$). Throughout, $\delta$ is treated as the background Euclidean metric on $\mathcal{E}$ and is used for index gymnastics unless explicitly stated otherwise. The Lorentzian tensor $g$ constructed in §2.5 is regarded as a derived bilinear form on $T\mathcal{E}$ used to define interval–type scalars such as $g(X, X)$, rather than as the default device for raising/lowering.

In particular, we distinguish the $\delta$–raised components

$$g^{AB}_{(\delta)} := \delta^{AC} \delta^{BD} g_{CD}$$

from the inverse metric $(g^{-1})^{AB}$ defined by $(g^{-1})^{AC} g_{CB} = \delta^A{}_B$. For the special form $g_{AB} = 2N_A N_B - \delta_{AB}$ with $\delta(N, N) = 1$, one indeed has $(g^{-1})^{AB} = g^{AB}_{(\delta)} = 2N^A N^B - \delta^{AB}$, but the two notions remain conceptually distinct.

## 2.2 Clifford algebra $\mathcal{C}\ell(4, 0)$ on $\mathcal{E}$

At each point $p \in \mathcal{E}$ we equip the tangent space $T_p\mathcal{E} \simeq \mathbb{R}^4$ with its geometric (Clifford) algebra $\mathcal{C}\ell(4, 0)$ generated by vectors and the Euclidean metric $\delta$. Concretely, choose an oriented $\delta$–orthonormal basis $\{e_0, e_1, e_2, e_3\}$ of $T_p\mathcal{E}$ with

$$e_a^2 = +1, \qquad e_a e_b = -e_b e_a \quad (a \neq b).$$

The geometric product of vectors $a, b \in T_p\mathcal{E}$ decomposes as

$$ab \;=\; a \cdot b \;+\; a \wedge b,$$

where $a \cdot b = \delta(a, b)$ is the scalar (inner) product and $a \wedge b$ is the bivector (outer) product. We use standard grade projections $\langle \cdot \rangle_k$ $(k = 0, \ldots, 4)$ and the reversion (reverse) $\widetilde{(\cdot)}$, defined on basis blades by reversing the order of basis vectors (hence $\widetilde{ab} = \widetilde{b}\,\widetilde{a}$ and $\widetilde{a} = a$ for vectors).

**Rotors.** A rotor is an even multivector $r \in \mathcal{C}\ell^+(4, 0)$ satisfying

$$r\widetilde{r} = 1.$$

Rotors act on vectors by the sandwich map $v \mapsto rv\widetilde{r}$ and realize orthogonal transformations ($Spin(4) \to SO(4)$ is the usual double cover).

**Quaternionic coordinates (observer gauge).** Fixing a unit vector $e_0$ selects the even subspace

$$\mathbb{H}_{e_0} := \mathrm{span}\{1,\ e_{10},\ e_{20},\ e_{30}\} \subset \mathcal{C}\ell^+(4, 0), \qquad e_{i0} := e_i e_0,$$

which is isomorphic to the quaternion algebra once the orientation is fixed. This provides a convenient coordinate language for even elements (notably the unit 3–sphere $S^3$ of unit quaternions), but it does not introduce additional physics by itself.

### 2.3 Observer axis, inertial SR sector, and the role of $\nabla^\delta$

To speak about "time" and "space" within the Euclidean proto–space, one must choose a reference unit direction. In the special–relativistic (inertial) sector studied in this paper we encode the observer by a constant unit vector field $e_0$ on the region of interest:

$$\delta(e_0, e_0) = 1, \qquad \nabla^\delta e_0 = 0.$$

Here $\nabla^\delta$ denotes the Levi–Civita connection of $\delta$; in Cartesian coordinates on $\mathcal{E} \simeq \mathbb{R}^4$ one has $\Gamma(\delta) = 0$ and $\nabla^\delta$ reduces to the ordinary derivative, so $\nabla^\delta e_0 = 0$ simply means that the components of $e_0$ are constant.

*Remark* 2.2 (Physical axis $N$ versus fiducial axis $e_0$)*.* The Lorentzian form constructed below depends on a unit direction $N$ (the observer's coherent time axis). For a single inertial observer one may, and often will, identify $N \equiv e_0$ without loss of generality. We nevertheless keep the notation $e_0$ available as a fixed fiducial axis when working in $\mathcal{C}\ell(4, 0)$: it provides a stable quaternionic coordinate subalgebra $\mathbb{H}_{e_0}$ even in later extensions where a physical field $N$ may vary. In the present SR sector, no observable depends on this bookkeeping choice.

### 2.4 Two axes give a relative quaternion and a rotor

Let $u, v \in T_p\mathcal{E}$ be two unit vectors, $\delta(u, u) = \delta(v, v) = 1$. Think of $u$ as the observer time axis (typically $u = N_p$, and in the inertial SR sector $u = e_0$), and $v$ as the unit proto–velocity direction of an object. (Operationally, later we will identify $v = \tilde{X}/c$ for a calibrated proto–velocity $\tilde{X} = dX/d\chi$.)

**Relative quaternion from two vectors.** Define the associated even element

$$q(v, u) \;:=\; vu \;\in\; \mathcal{C}\ell^+(4, 0). \tag{2.1}$$

It is automatically unit:

$$q(v, u)\,\widetilde{q(v, u)} = (vu)(uv) = v(u^2)v = v^2 = 1.$$

Moreover,

$$q(v, u) = \langle q\rangle_0 + \langle q\rangle_2 = (v\cdot u) \;+\; (v\wedge u).$$

Let $\xi \in [0, \pi]$ be the Euclidean angle between $u$ and $v$, defined by

$$\cos\xi := u\cdot v. \tag{2.2}$$

Then $\|v \wedge u\|_\delta = \sin\xi$ and, for $\xi \notin \{0, \pi\}$,

$$q(v, u) = \cos\xi + \sin\xi\,\widehat{B}, \qquad \widehat{B} := \frac{v \wedge u}{\|v \wedge u\|_\delta}, \qquad \widehat{B}^2 = -1. \tag{2.3}$$

If $u = e_0$, then $q(v, e_0) = ve_0 \in \mathbb{H}_{e_0}$ and may be treated literally as a unit quaternion in the basis $\{1, e_{10}, e_{20}, e_{30}\}$.

**A canonical rotor mapping $u$ to $v$.** While $q(v, u) = vu$ compactly encodes the relative state, the rotor that acts by conjugation and maps $u$ to $v$ is the normalized "half–angle" element

$$r(v, u) \;:=\; \frac{1 + vu}{\sqrt{2(1 + u\cdot v)}} \;=\; \cos\!\Big(\frac{\xi}{2}\Big) + \sin\!\Big(\frac{\xi}{2}\Big)\,\widehat{B}, \qquad (\xi \neq \pi), \tag{2.4}$$

which satisfies

$$r(v, u)\,u\,\widetilde{r(v, u)} \;=\; v. \tag{2.5}$$

*Remark* 2.3 (Gauge nonuniqueness (stabilizer)). The rotor implementing $u \mapsto v$ is not unique: if $s$ is any rotor fixing $u$ (i.e. $su\widetilde{s} = u$), then $r' = rs$ yields the same image $r'u\widetilde{r'} = v$. Equivalently, $r$ is defined up to right multiplication by the stabilizer $\mathrm{Stab}(u) \cong Spin(3) \cong SU(2)$. For example, when $u = e_0$, $\mathrm{Stab}(e_0)$ consists of the purely spatial rotations in $e_0^{\perp_\delta}$; composing $r$ with such a spatial rotor changes the spatial gauge but does not change the image of the time axis.

### 2.5 Coherent time axis $N$ and the induced Lorentzian form

Let $N$ be a smooth unit vector field on an open set $U \subset \mathcal{E}$:

$$\delta(N, N) = 1 \qquad \text{on } U. \tag{2.6}$$

In the inertial SR sector we assume

$$\nabla^\delta N = 0 \qquad \text{on } U, \tag{2.7}$$

so that $N$ is constant (in Cartesian coordinates). For a single inertial observer we may set $N \equiv e_0$.

**Time 1–form and spatial projector.** Let

$$\alpha := N^\flat := \delta(N, \cdot) \qquad (\text{so } \alpha_A = \delta_{AB} N^B = N_A),$$

and define the spatial projector

$$h := \delta - \alpha \otimes \alpha \qquad (\text{i.e. } h_{AB} = \delta_{AB} - N_A N_B). \tag{2.8}$$

Then $h$ has rank 3, satisfies $h(\cdot, N) = 0$, and projects onto $N^{\perp_\delta}$.

**Reddy–Sharma–Sivaramakrishnan Lorentzization.** Define a symmetric $(0,2)$–tensor field $g$ on $U$ by

$$g \;:=\; 2\,\alpha \otimes \alpha \;-\; \delta, \qquad \text{i.e.} \qquad g_{AB} = 2N_A N_B - \delta_{AB}. \tag{2.9}$$

In an adapted $\delta$–orthonormal basis $\{N, e_1, e_2, e_3\}$ this has signature $(+\,-\,-\,-)$. The definition is insensitive to the sign flip $N \mapsto -N$ because $N \otimes N$ is unchanged.

*Remark* 2.4 (Relative quaternion viewpoint (observer–object)). Given an object proto–velocity direction $v$ (unit) and an observer axis $u = N_p$, the even element $q(v, u) = vu$ encodes the circular angle $\xi$ via $\langle q \rangle_0 = \cos\xi$ and $\|\langle q \rangle_2\|_\delta = \sin\xi$. This is the basic algebraic device used later to package the longitudinal/transverse proto–velocity budget split relative to $N$ into a single unit quaternion–like element.

## 3 Lorentzian metric properties

Throughout this section, $p \in \mathcal{E}$ is arbitrary and all statements are understood pointwise at $p$. We write $T_p\mathcal{E}$ for the tangent space, endowed with the Euclidean inner product $\delta$ and the associated Clifford algebra $\mathcal{C}\ell(4,0)$ from §2.2. The coherent time axis at $p$ is a $\delta$–unit vector $N \in T_p\mathcal{E}$, $\delta(N, N) = 1$, with associated Lorentzian bilinear form

$$g(X, Y) = 2\,(N \cdot X)\,(N \cdot Y) - \delta(X, Y) \qquad (\text{cf. } (2.9)).$$

When convenient, we also fix a fiducial axis $e_0$ and a rotor $q \in \mathcal{C}\ell^+(4,0)$ such that

$$N = q\, e_0\, \widetilde{q},$$

in which case $q$ may be regarded as a time–axis rotor (an "observer gauge") relative to the bookkeeping axis $e_0$.

### 3.1 Orthogonal decomposition of tangent vectors

For any $X \in T_p\mathcal{E}$ we define the $\delta$–longitudinal and $\delta$–transverse components relative to $N$ by

$$X_{\|} := (N \cdot X)\, N, \qquad X_{\perp} := h(X) = X - (N \cdot X)\, N, \tag{3.1}$$

where $h = \delta - N^{\flat} \otimes N^{\flat}$ is the $\delta$–orthogonal projector onto $N^{\perp_\delta}$ (cf. (2.8)).

**Lemma 3.1.** *For every $X \in T_p\mathcal{E}$,*

$$X = X_{\|} + X_{\perp},$$

*where $X_{\|} \in \operatorname{span}\{N\}$ and $X_{\perp} \in N^{\perp_\delta}$. The decomposition is unique.*

*Proof.* Since $h$ is a projector with $\ker(h) = \operatorname{span}\{N\}$ and $\operatorname{Im}(h) = N^{\perp_\delta}$, the splitting is the standard direct sum decomposition associated with complementary subspaces. □

*Remark* 3.2 (Clifford reflection viewpoint)*.* In $\mathcal{C}\ell(4,0)$, the sandwich map by a unit vector implements an orthogonal reflection. Define, for $Y \in T_p\mathcal{E}$,

$$Y^{\star} := -\, N\, Y\, N.$$

Then $Y^{\star}$ is the $\delta$–reflection of $Y$ across the hyperplane $N^{\perp_\delta}$ and satisfies

$$Y^{\star} = Y_{\|} - Y_{\perp}.$$

Consequently, the Lorentzian bilinear form can be written as a Euclidean pairing with a reflected argument,

$$g(X, Y) \;=\; \delta\big(X,\, Y^{\star}\big) \;=\; \delta\big(X,\, Y_{\|} - Y_{\perp}\big), \tag{3.2}$$

which is the Clifford–algebraic content of the RSS "Lorentzization".

### 3.2 Circular angle $\xi$ and norm identities

The decomposition (3.1) naturally defines a circular Euclidean angle between $X$ and the time axis $N$.

**Definition 3.3** (Circular angle $\xi$ relative to $N$)**.** Let $X \in T_p\mathcal{E}$ be nonzero. Define $\xi \in [0, \pi]$ by

$$\cos\xi := \frac{N \cdot X}{\|X\|_\delta}, \qquad \sin\xi := \frac{\|X_{\perp}\|_\delta}{\|X\|_\delta}, \tag{3.3}$$

where $\|X\|_\delta := \sqrt{\delta(X, X)}$. Equivalently, $\xi$ is the Euclidean angle between $X$ and $N$.

*Remark* 3.4 (Why $\xi$ is distinguished from $\vartheta$)*.* The angle $\xi$ is the circular (trigonometric) Euclidean angle defined by (7.1). Later we introduce a tangential parameter $\vartheta$ adapted to velocity ratios (e.g. $\beta = \tan\vartheta$ and rapidity). Keeping $\xi$ for the circular parametrization avoids conflating these two choices.

**Proposition 3.5.** *For any $X \in T_p\mathcal{E}$,*

$$g(X, X) = (N \cdot X)^2 - \delta(X_{\perp}, X_{\perp}). \tag{3.4}$$

*Equivalently, in terms of the circular angle $\xi$,*

$$g(X, X) = \|X\|_\delta^2 \, \cos(2\xi). \tag{3.5}$$

*Proof.* Insert (3.1) into $g(X, X)$ and use: $g(N, N) = 1$, $g(N, X_{\perp}) = 0$ (since $X_{\perp} \in N^{\perp_\delta}$), and $g(X_{\perp}, X_{\perp}) = -\delta(X_{\perp}, X_{\perp})$. This yields (3.4). Using $(N \cdot X) = \|X\|_\delta \cos\xi$ and $\|X_{\perp}\|_\delta = \|X\|_\delta \sin\xi$ from (7.1) gives (3.5). □

**Corollary 3.6.** *A vector* $X \neq 0$ *satisfies:*

- $g(X,X) > 0$ *iff* $\cos(2\xi) > 0$ *iff* $\xi \in (0, \frac{\pi}{4}) \cup (\frac{3\pi}{4}, \pi)$,
- $g(X,X) = 0$ *iff* $\cos(2\xi) = 0$ *iff* $\xi \in \{\frac{\pi}{4}, \frac{3\pi}{4}\}$,
- $g(X,X) < 0$ *iff* $\cos(2\xi) < 0$ *iff* $\xi \in (\frac{\pi}{4}, \frac{3\pi}{4})$.

*Equivalently, in split form:*

$$g(X,X) \gtrless 0 \quad \Longleftrightarrow \quad (N \cdot X)^2 \gtrless \delta(X_\perp, X_\perp).$$

Define the three disjoint subsets of $T_p\mathcal{E}$:

$$\mathcal{T}_p := \{X \in T_p\mathcal{E} : \; g(X,X) > 0\}, \qquad \mathcal{P}_p := \{X \in T_p\mathcal{E} : \; g(X,X) = 0\}, \qquad \mathcal{S}_p := \{X \in T_p\mathcal{E} : \; g(X,X) < 0\}.$$

We also single out the future time cone (relative to $N$):

$$\mathcal{T}_p^+ := \{X \in \mathcal{T}_p : \; N \cdot X > 0\}. \tag{3.6}$$

### 3.3 Geometry of the null cone and the $45^\circ$ Euclidean tilt

**Proposition 3.7.** *The set of* $g$*–null vectors at* $p$ *is the quadratic cone*

$$\mathcal{C}_p = \{\, X \in T_p\mathcal{E} : \; \delta(X_\perp, X_\perp) = (N \cdot X)^2 \,\}.$$

*Under the decomposition* $T_p\mathcal{E} = \operatorname{span}\{N\} \oplus N^{\perp_\delta}$*, it is a double cone given by*

$$N \cdot X = \pm \|X_\perp\|_\delta.$$

*Proof.* Immediate from Corollary 3.6 (or directly from (3.4)). □

*Remark* 3.8 (Why "light" corresponds to a $45^\circ$ Euclidean tilt)*.* Operationally, "light propagation" refers to a causal relation between an emission event and a detection event. During the signal's flight the observer advances along the time axis $N$, hence the proto–space separation between emission and detection is not purely spatial but decomposes as

$$X \;=\; X_\parallel + X_\perp, \qquad X_\parallel := (N \cdot X)\, N, \qquad X_\perp := X - (N \cdot X)\, N.$$

The defining kinematic content of "light" is that the induced Lorentzian interval vanishes,

$$g(X,X) = 0.$$

Using (3.4), this is equivalent to equality of the longitudinal and transverse $\delta$–magnitudes,

$$\|X_\parallel\|_\delta \;=\; \|X_\perp\|_\delta. \tag{3.7}$$

In terms of the circular Euclidean angle $\xi$ between $X$ and $N$ (Definition 3.3), the condition (3.7) reads $\cos\xi = \sin\xi$, hence

$$\xi = \frac{\pi}{4} \quad \text{or} \quad \xi = \frac{3\pi}{4}.$$

Thus the $g$–null directions are precisely those at a $45^\circ$ Euclidean tilt to the time axis: the proto–space displacement budget is split equally between $N$ and its orthogonal complement.

### 3.4 Proto–velocity viewpoint and the observer split

While the results above hold for arbitrary tangent vectors, the proto–space formulation is most transparent when applied to a calibrated proto–velocity. Let $X(\chi)$ be a regular curve and define its proto–velocity by

$$\tilde{X} := \frac{dX}{d\chi} \in T_p\mathcal{E}.$$

If $\chi$ is calibrated (cf. Definition 4.1), then

$$\delta(\tilde{X}, \tilde{X}) = c^2. \tag{3.8}$$

Thus, at each point, the set of admissible calibrated proto–velocities is the Euclidean 3–sphere $S^3_c(p) \subset T_p\mathcal{E}$.

Relative to the coherent time axis $N$, write the observer split

$$\tilde{X} = \tilde{H}\, N + \tilde{X}_\perp, \qquad \tilde{H} := N\cdot\tilde{X}, \qquad \tilde{X}_\perp := h(\tilde{X}) \in N^{\perp_\delta},$$

and set $\tilde{L} := \|\tilde{X}_\perp\|_\delta$. Then (3.8) implies the proto–velocity budget identity

$$\tilde{H}^{\,2} + \tilde{L}^{\,2} = c^2,$$

while the induced Lorentzian rate is

$$g(\tilde{X}, \tilde{X}) = \tilde{H}^{\,2} - \tilde{L}^{\,2} \;=\; c^2 \cos(2\xi), \tag{3.9}$$

where $\xi$ is the circular tilt angle of $\tilde{X}$ relative to $N$.

### 3.5 Spatial rotations preserving $\delta$ and $N$

Let $\mathrm{Aut}(\delta, N)$ denote the stabilizer of $N$ in the Euclidean orthogonal group:

$$\mathrm{Aut}(\delta, N) := \{\, L : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} :\ \delta(LX, LY) = \delta(X, Y),\ LN = N \,\}.$$

In an adapted $\delta$–orthonormal basis $\{e'_0 = N, e'_1, e'_2, e'_3\}$ one has

$$L = \mathrm{diag}(1, R), \qquad R \in O(3),$$

so $\mathrm{Aut}(\delta, N) \cong O(3)$ and contains no boost–like maps mixing $N$ with $N^{\perp_\delta}$.

*Remark* 3.9 (Rotor description of $\mathrm{Aut}(\delta, N)$). In the Clifford algebra, the connected component of $\mathrm{Aut}(\delta, N)$ is realized by spatial rotors $s \in \mathcal{C}\ell^+(4, 0)$ satisfying

$$s\widetilde{s} = 1, \qquad sN\widetilde{s} = N.$$

Equivalently, $s = \exp\!\left(-\frac{\phi}{2}B\right)$ with a bivector generator $B \in \Lambda^2(N^{\perp_\delta}) = \mathrm{span}\{e'_{12}, e'_{23}, e'_{31}\}$. This is the same stabilizer mechanism as in Remark 2.3: the time axis is held fixed while the spatial triad is rotated. Such stabilizer freedom is precisely the source of rotor nonuniqueness when one encodes only the axis $N$ (and not a full tetrad).

**Lemma 3.10.** *Every $L \in \mathrm{Aut}(\delta, N)$ preserves $g$:*

$$g(LX, LY) = g(X, Y) \qquad \text{for all } X, Y \in T_p\mathcal{E}.$$

*Proof.* Since $LN = N$ and $L$ is $\delta$–orthogonal,

$$g(LX, LY) = 2(N\cdot LX)(N\cdot LY) - \delta(LX, LY) = 2(N\cdot X)(N\cdot Y) - \delta(X, Y) = g(X, Y).$$

□

Thus $\mathrm{Aut}(\delta, N)$ is a spatial subgroup of $\mathrm{O}(g)$: it preserves $g$ and fixes $N$, but generates only Euclidean rotations on $N^{\perp_\delta}$.

*Remark* 3.11 (Time–axis rotor gauge, revisited)*.* If $N = qe_0\widetilde{q}$ for a rotor $q$, then $q$ is defined only up to right multiplication by a stabilizer rotor $s$ that fixes $e_0$ (cf. Remark 2.3), because $(qs)e_0\widetilde{(qs)} = qe_0\widetilde{q}$. Pointwise, this means that specifying the axis $N$ does not specify a unique orientation of the spatial complement: the latter may be rotated by an element of $Spin(3)$ without affecting $N$, hence without affecting $g_N$.

# 4 Phase reparametrization and the operational origin of the Lorentz interval

This section makes explicit the logical bridge between the Euclidean proto–metric $\delta$ on $\mathcal{E}$ and the Lorentzian form $g$ constructed in §2. The construction starts from an oriented coherent observer axis $N$ (and, if desired, a rotor gauge $q$ such that $N = qe_0\widetilde{q}$), and then defines operational time and space measurements as $\delta$–projections relative to $N$. The key point is operational: an observer does not have direct access to an arbitrary curve parameter. Instead, clocks and rulers implement specific projection rules; the Lorentzian interval is the quadratic form that governs the proper–time phase induced by those rules.

## 4.1 A $\delta$–calibrated proto–parameter $\chi$ and proto–velocity

Let $X : I \to \mathcal{E}$ be a $C^1$ curve. For a parameter $\lambda$ on $I$ we write

$$X'(\lambda) := \frac{dX}{d\lambda} \in T_{X(\lambda)}\mathcal{E}.$$

Because $(\mathcal{E}, \delta)$ is Euclidean, one may always reparametrize $X$ by a scaled $\delta$–arc length. We single out the following calibration, which fixes the total proto–space budget along the curve.

**Definition 4.1** ($\delta$–calibrated (proto–affine) parameter)**.** A parameter $\chi$ along $X$ is called $\delta$–calibrated (or *proto–affine*) if the associated proto–velocity

$$\tilde{X} := \frac{dX}{d\chi}$$

has constant $\delta$–norm equal to $c$, i.e.

$$\delta(\tilde{X}, \tilde{X}) = c^2. \tag{4.1}$$

*Remark* 4.2 (Interpretation of $\chi$)*.* In flat Euclidean geometry, (4.1) is simply a scaled arc–length parametrization:

$$d\chi = \frac{1}{c}\,\|dX\|_\delta.$$

The parameter $\chi$ is an auxiliary calibration: it fixes a convenient reference parameter in $\mathcal{E}$ relative to which the total Euclidean expenditure $\|\tilde{X}\|_\delta = c$ is constant. By itself, $\chi$ is not an operational time variable; operational time is introduced via the coherent axis $N$ below.

## 4.2 A coherent axis $N$: time 1–form and spatial projector

Throughout, $N$ denotes the coherent $\delta$–unit axis field of §2,

$$\delta(N, N) = 1.$$

When working in $\mathcal{C}\ell(4, 0)$ one may (optionally) encode $N$ by a rotor $q$ relative to a fixed fiducial axis $e_0$ via

$$N = q\, e_0\, \widetilde{q},$$

but the operational content depends only on $N$ (and its chosen orientation), not on a particular representative $q$.

The $\delta$–dual 1–form (the time form) of $N$ is

$$\alpha := N^\flat := \delta(N, \cdot), \qquad \text{i.e.} \qquad \alpha_A = \delta_{AB} N^B = N_A. \tag{4.2}$$

The induced spatial projector is

$$h := \delta - \alpha \otimes \alpha, \qquad \text{i.e.} \qquad h_{AB} = \delta_{AB} - N_A N_B, \tag{4.3}$$

so that $h(\,\cdot\,, N) = 0$ and $\mathrm{Im}(h) = N^{\perp_\delta}$.

For any $V \in T_p\mathcal{E}$ one has the orthogonal $\delta$–split

$$\delta(V, V) = \alpha(V)^2 + h(V, V), \tag{4.4}$$

i.e. "total proto–budget = longitudinal$^2$ + transverse$^2$" relative to $N$.

*Remark* 4.3 (Sign of $N$ and what is (in)sensitive to it). The RSS Lorentzization $g = 2\,N^\flat \otimes N^\flat - \delta$ depends on $N$ only through $N \otimes N$ and is therefore invariant under the sign flip $N \mapsto -N$. Hence the light cone and causal classification determined by $g$ are insensitive to the orientation of $N$.

However, the time 1–form $\alpha = N^\flat$ changes sign under $N \mapsto -N$, and so do the projected time increments defined below. Thus, to speak about an oriented "future" direction and about a monotone time coordinate, one must choose an orientation of $N$ (a future–pointing branch) and keep it coherent on the region under consideration.

*Remark* 4.4 (Rotor gauge does not affect $\alpha$, $h$, or $g$). If $N = qe_0\widetilde{q}$, then replacing $q$ by $qs$ with any stabilizer rotor $s$ satisfying $se_0\widetilde{s} = e_0$ leaves $N$ unchanged, hence also leaves $\alpha$, $h$, and the induced $g$ unchanged. Operational quantities depend on $N$ (and its chosen orientation), not on a particular representative $q$.

### 4.3 Coordinate time $t$ and spatial distance $\ell$ as projections

Given an oriented coherent axis $N$, the time form $\alpha$ defines the observer–adapted coordinate time $t$ along a worldline by the operational projection rule

$$dt := \frac{1}{c}\,\alpha(dX) \qquad \Longleftrightarrow \qquad \frac{dt}{d\lambda} = \frac{1}{c}\,\alpha\big(X'(\lambda)\big). \tag{4.5}$$

In general, (4.5) defines $t$ only along the given curve; a global time function $t$ on an open set requires an integrability condition on $\alpha$ (e.g. $\alpha$ closed/exact, equivalently $N$ hypersurface–orthogonal).

Likewise, the induced spatial line element along the curve is defined by

$$d\ell^2 := h(dX, dX) \qquad \Longleftrightarrow \qquad \Big(\frac{d\ell}{d\lambda}\Big)^2 = h\big(X'(\lambda), X'(\lambda)\big). \tag{4.6}$$

**Proposition 4.5** (Operational form of the Lorentz interval)**.** *For every curve $X$ and every parameter $\lambda$ one has the identity*

$$g(dX, dX) = c^2\, dt^2 - d\ell^2, \tag{4.7}$$

*where $dt$ and $d\ell$ are given by* (4.5)*–*(4.6)*.*

*Proof.* By the RSS definition $g = \alpha \otimes \alpha - h$ (equivalently $g = 2\alpha \otimes \alpha - \delta$), for any $V$ we have $g(V, V) = \alpha(V)^2 - h(V, V)$. Apply this to $V = dX$ and substitute (4.5) and (4.6). □

### 4.4 Proper time $\tau$ and the $\chi \mapsto \tau$ phase reparametrization

We define the proper time $\tau$ along a $g$–timelike curve as the $g$–arc length parameter.

**Definition 4.6** (Proper time)**.** Along a $g$–timelike curve (i.e. $g(dX, dX) > 0$) the proper time is defined by

$$c^2\, d\tau^2 := g(dX, dX). \tag{4.8}$$

Equivalently, combining (4.8) with (4.7) yields

$$d\tau^2 = dt^2 - \frac{1}{c^2}\, d\ell^2. \tag{4.9}$$

Assume henceforth that $\chi$ is $\delta$–calibrated in the sense of Definition 4.1, and write $\tilde{X} = dX/d\chi$. Define the longitudinal and transverse rates (per unit $\chi$) by

$$S(\chi) := \alpha(\tilde{X}) = N \cdot \tilde{X}, \qquad L(\chi)^2 := h(\tilde{X}, \tilde{X}) = \delta(\tilde{X}_\perp, \tilde{X}_\perp). \tag{4.10}$$

Then (4.4) becomes the exact budget identity

$$\underbrace{\delta(\tilde{X}, \tilde{X})}_{c^2} = \underbrace{S^2}_{\text{longitudinal (time-axis) rate}} + \underbrace{L^2}_{\text{transverse (spatial) rate}}. \tag{4.11}$$

**Theorem 4.7** (Phase reparametrization $\chi \mapsto \tau$)**.** *Let $X(\chi)$ be $\delta$–calibrated and $g$–timelike along the curve, i.e. $g(\tilde{X}, \tilde{X}) > 0$. Then*

$$\frac{d\tau}{d\chi} = \frac{1}{c}\sqrt{g(\tilde{X}, \tilde{X})} = \frac{1}{c}\sqrt{S^2 - L^2}. \tag{4.12}$$

*Equivalently, reparametrizing the same geometric curve by $\tau$ (i.e. setting $\dot{X} := dX/d\tau$) yields the unit–speed condition*

$$g(\dot{X}, \dot{X}) = c^2. \tag{4.13}$$

*Proof.* By Definition 4.6, $c^2(d\tau/d\chi)^2 = g(\tilde{X}, \tilde{X})$, giving (4.12). Then $\dot{X} = (d\chi/d\tau)\tilde{X}$ implies

$$g(\dot{X}, \dot{X}) = \left(\frac{d\chi}{d\tau}\right)^2 g(\tilde{X}, \tilde{X}) = \frac{c^2}{g(\tilde{X}, \tilde{X})}\, g(\tilde{X}, \tilde{X}) = c^2,$$

which is (4.13). □

*Remark* 4.8 (Roles of $\chi$, $t$, and $\tau$). The three parameters used in this paper have distinct status:

- $\chi$ is an auxiliary $\delta$–calibration fixing the total proto–space budget $\|\tilde{X}\|_\delta = c$;
- $t$ is the observer–adapted coordinate time obtained by projecting $dX$ onto the coherent axis $N$ via (4.5);
- $\tau$ is the proper time of an ideal comoving clock, defined invariantly as the $g$–arc length by (4.8).

Operationally, the observer has access to $t$ (relative to its chosen coherent axis $N$) and to $\tau$ (along its own worldline); $\chi$ is a convenient proto–space gauge used to expose the fixed Euclidean budget underlying the reparametrization equivalence.

*Remark* 4.9 (Circular tilt $\xi$ and the explicit $\chi$–$\tau$ factor). Under the $\chi$–calibration (4.1), the proto–velocity $\tilde{X}$ lies on the sphere $S^3_c(p) \subset T_p\mathcal{E}$. Let $\xi$ denote the circular Euclidean angle between $\tilde{X}$ and $N$,

$$\cos\xi = \frac{N \cdot \tilde{X}}{\|\tilde{X}\|_\delta}, \qquad \sin\xi = \frac{\|\tilde{X}_\perp\|_\delta}{\|\tilde{X}\|_\delta}.$$

Then $S = c\cos\xi$ and $L = c\sin\xi$, so

$$g(\tilde{X}, \tilde{X}) = S^2 - L^2 = c^2\cos(2\xi),$$

and Theorem 4.7 becomes

$$\frac{d\tau}{d\chi} = \sqrt{\cos(2\xi)}.$$

Thus the proper–time rate is controlled by the double–angle factor $\cos(2\xi)$ already at the circular level; the tangential (hyperbolic) parametrization introduced later is a different coordinate choice adapted to velocity composition, not a different source of the $\cos(2\,\cdot)$ structure.

*Remark* 4.10 (Why $g$ is observed rather than $\delta$). The proto–metric $\delta$ measures the total Euclidean budget with respect to the auxiliary calibration $\chi$. Physical clocks, however, realize the proper–time parametrization $\tau$ defined by (4.8), hence by the quadratic form $g$. In this operational sense the observer "lives by its own phase": the time variable implemented by ideal clocks is $\tau$, and the interval controlling it is $g$, not $\delta$.

# 5 Tilt geometry: circular ($\xi$) versus conic/hyperbolic ($\vartheta, \eta$) parametrizations

Throughout this section, $p \in \mathcal{E}$ is fixed and all statements are understood pointwise at $p$. We work with the $\delta$–orthogonal splitting relative to the coherent time axis $N$ (cf. §3):

$$X = X_\parallel + X_\perp, \qquad X_\parallel := (N\cdot X)\,N, \qquad X_\perp := X - (N\cdot X)\,N.$$

Recall that the induced Lorentzian form is $g = 2N^\flat \otimes N^\flat - \delta$, so that

$$g(X,X) = (N\cdot X)^2 - \delta(X_\perp, X_\perp) \qquad \text{and} \qquad \delta(X,X) = (N\cdot X)^2 + \delta(X_\perp, X_\perp).$$

## 5.1 Circular parametrization: the Euclidean tilt angle $\xi$

For any nonzero $X \in T_p\mathcal{E}$ define its circular (Euclidean) tilt angle $\xi \in [0, \pi]$ relative to the axis $N$ by

$$\cos\xi := \frac{N \cdot X}{\|X\|_\delta}, \qquad \sin\xi := \frac{\|X_\perp\|_\delta}{\|X\|_\delta}, \qquad \|X\|_\delta := \sqrt{\delta(X,X)}. \tag{5.1}$$

Whenever $X_\perp \neq 0$, define the transverse unit direction

$$E := \frac{X_\perp}{\|X_\perp\|_\delta} \in N^{\perp_\delta}, \qquad \delta(E,E) = 1, \qquad N \cdot E = 0.$$

Then $X$ admits the explicit circular decomposition

$$X \;=\; \|X\|_\delta\big(\cos\xi\, N + \sin\xi\, E\big). \tag{5.2}$$

**Lemma 5.1** (Circular budget identities)**.** *For any nonzero $X \in T_p\mathcal{E}$,*

$$\|X\|^2_\delta = (N \cdot X)^2 + \delta(X_\perp, X_\perp), \qquad \cos^2\xi + \sin^2\xi = 1.$$

*Proof.* Immediate from $\delta(X_\parallel, X_\perp) = 0$ and Definition (5.1). $\square$

**Proposition 5.2** (Lorentzian norm in circular form). *For any nonzero $X \in T_p\mathcal{E}$,*

$$g(X,X) = \|X\|_\delta^2(\cos^2\xi - \sin^2\xi) = \|X\|_\delta^2 \cos(2\xi). \tag{5.3}$$

*Proof.* Substitute $(N\cdot X) = \|X\|_\delta \cos\xi$ and $\|X_\perp\|_\delta = \|X\|_\delta \sin\xi$ into $g(X,X) = (N\cdot X)^2 - \|X_\perp\|_\delta^2$. □

*Remark* 5.3 (Timelike domain in circular coordinates). In the future timelike cone $\mathcal{T}_p^+ = \{X : g(X,X) > 0,\ N\cdot X > 0\}$ one has $\cos(2\xi) > 0$ and $\cos\xi > 0$, hence

$$X \in \mathcal{T}_p^+ \iff \xi \in \Big[0, \frac{\pi}{4}\Big).$$

The boundary $\xi = \pi/4$ corresponds to $g(X,X) = 0$ (null directions).

## 5.2 Conic (tangential) parametrization: the ratio angle $\vartheta$

While $\xi$ is defined by normalizing with the total Euclidean magnitude $\|X\|_\delta$, the physically relevant kinematic ratio on the future timelike cone is the transverse–to–longitudinal budget ratio

$$\beta := \frac{\|X_\perp\|_\delta}{N\cdot X}, \qquad (X \in \mathcal{T}_p^+).$$

This suggests a conic (tangential) coordinate $\vartheta$ defined by

$$\tan\vartheta := \frac{\|X_\perp\|_\delta}{N\cdot X}, \qquad (X \in \mathcal{T}_p^+), \tag{5.4}$$

so that $\beta = \tan\vartheta \in [0,1)$ on $\mathcal{T}_p^+$. Null vectors satisfy $\tan\vartheta = 1$ (equivalently $\vartheta = \pi/4$), and $g$–spacelike vectors satisfy $\tan\vartheta > 1$.

*Remark* 5.4 (Relation between $\xi$ and $\vartheta$). On the future sector, $\cos\xi > 0$, hence the ratio definition gives

$$\tan\vartheta = \frac{\|X_\perp\|_\delta}{N\cdot X} = \frac{\|X\|_\delta \sin\xi}{\|X\|_\delta \cos\xi} = \tan\xi,$$

so numerically $\vartheta = \xi$ on $\mathcal{T}_p^+$. The distinction is conceptual: $\xi$ is a circular normalization by $\|X\|_\delta$, whereas $\vartheta$ is a conic normalization by the longitudinal component $N\cdot X$. This is exactly the choice that makes $\beta = \tan\vartheta$ the natural speed–like parameter saturating at $\beta \to 1$ in the null limit.

**Proposition 5.5** (Lorentzian norm in conic form). *For $X \in \mathcal{T}_p^+$,*

$$g(X,X) = (N\cdot X)^2\Big(1 - \tan^2\vartheta\Big) = (N\cdot X)^2(1-\beta^2) = \delta(X,X)\,\frac{1-\beta^2}{1+\beta^2}, \qquad \beta = \tan\vartheta. \tag{5.5}$$

*Equivalently,*

$$\cos(2\xi) = \frac{1-\beta^2}{1+\beta^2}. \tag{5.6}$$

*Proof.* From $g(X,X) = (N\cdot X)^2 - \|X_\perp\|_\delta^2$ and $\|X_\perp\|_\delta = (N\cdot X)\tan\vartheta$ one gets $g(X,X) = (N\cdot X)^2(1-\tan^2\vartheta) = (N\cdot X)^2(1-\beta^2)$. Moreover,

$$\delta(X,X) = (N\cdot X)^2 + \|X_\perp\|_\delta^2 = (N\cdot X)^2(1+\beta^2),$$

hence $g(X,X) = \delta(X,X)\,(1-\beta^2)/(1+\beta^2)$. Using $\beta = \tan\xi$ and $\cos(2\xi) = (1-\tan^2\xi)/(1+\tan^2\xi)$ yields (5.6). □

### 5.3 Hyperbolic parameter (rapidity) $\eta$

A genuine group parameter for boosts is the rapidity $\eta$. For $X \in \mathcal{T}_p^+$ define $\eta \geq 0$ by

$$\tanh \eta := \tan \vartheta = \beta. \tag{5.7}$$

Equivalently, $\eta$ can be defined invariantly by the pair of relations

$$\cosh \eta := \frac{N \cdot X}{\sqrt{g(X,X)}}, \qquad \sinh \eta := \frac{\|X_\perp\|_\delta}{\sqrt{g(X,X)}}, \qquad (X \in \mathcal{T}_p^+), \tag{5.8}$$

which immediately implies $\tanh \eta = \|X_\perp\|_\delta/(N \cdot X) = \tan \vartheta$.

**Proposition 5.6** (Hyperbolic decomposition of timelike vectors)**.** *For $X \in \mathcal{T}_p^+$ and $E = X_\perp/\|X_\perp\|_\delta$ one has*

$$X \;=\; \sqrt{g(X,X)}\big(\cosh \eta\, N + \sinh \eta\, E\big), \tag{5.9}$$

*and the identities* (5.8) *hold.*

*Proof.* Write $X = (N \cdot X)N + \|X_\perp\|_\delta E$ and factor out $\sqrt{g(X,X)}$ using (5.8). □

**Lemma 5.7.** *For $X \in \mathcal{T}_p^+$,*

$$\cosh \eta = \frac{1}{\sqrt{1-\tan^2 \vartheta}} = \frac{\cos \vartheta}{\sqrt{\cos(2\vartheta)}}, \qquad \sinh \eta = \frac{\tan \vartheta}{\sqrt{1-\tan^2 \vartheta}} = \frac{\sin \vartheta}{\sqrt{\cos(2\vartheta)}}.$$

*Proof.* From $\tanh \eta = \tan \vartheta$ we have

$$\cosh^2 \eta = \frac{1}{1-\tanh^2 \eta} = \frac{1}{1-\tan^2 \vartheta}.$$

Taking the positive square root (since $\eta \geq 0$ and $\vartheta \in [0, \pi/4)$) gives $\cosh \eta$. Then $\sinh \eta = \tanh \eta \, \cosh \eta = \tan \vartheta \, \cosh \eta$. Finally, $\cos(2\vartheta) = \cos^2 \vartheta - \sin^2 \vartheta = \cos^2 \vartheta (1 - \tan^2 \vartheta)$ yields the alternative expressions. □

### 5.4 Differential relation between $\eta$ and the conic angle $\vartheta$

**Proposition 5.8.** *For $X \in \mathcal{T}_p^+$, the parameters $\eta$ and $\vartheta$ satisfy*

$$\frac{d\eta}{d\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

*Proof.* Differentiate $\tanh \eta = \tan \vartheta$:

$$\operatorname{sech}^2 \eta \, d\eta = \sec^2 \vartheta \, d\vartheta.$$

Using $\operatorname{sech}^2 \eta = 1 - \tanh^2 \eta = 1 - \tan^2 \vartheta = \cos(2\vartheta)/\cos^2 \vartheta$ gives

$$\frac{d\eta}{d\vartheta} = \frac{\sec^2 \vartheta}{\operatorname{sech}^2 \eta} = \frac{1/\cos^2 \vartheta}{\cos(2\vartheta)/\cos^2 \vartheta} = \frac{1}{\cos(2\vartheta)}.$$

□

## 5.5 Boost subgroup and additivity of the hyperbolic parameter

Let $\mathrm{O}(g)$ denote the Lorentz group of $(T_p\mathcal{E}, g)$:

$$\mathrm{O}(g) := \{\, \Lambda : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \; g(\Lambda X, \Lambda Y) = g(X, Y) \,\}.$$

Fix a $\delta$–unit transverse direction $E \in N^{\perp_\delta}$, $\delta(E, E) = 1$. The boost in the 2–plane $\operatorname{span}\{N, E\}$ with rapidity $\eta$ is the unique $\Lambda(\eta) \in \mathrm{O}(g)$ acting as a hyperbolic rotation on $\operatorname{span}\{N, E\}$ and as the identity on its $g$–orthogonal complement:

$$\Lambda(\eta)N = (\cosh\eta)\, N + (\sinh\eta)\, E, \qquad \Lambda(\eta)E = (\sinh\eta)\, N + (\cosh\eta)\, E,$$

$$\Lambda(\eta)X = X \quad \text{for } X \perp_g \operatorname{span}\{N, E\}.$$

Such boosts preserve $g$ but, in general, do not preserve $\delta$ and do not fix $N$.

**Theorem 5.9** (Additivity of rapidity)**.** *For boosts $\Lambda(\eta_1)$ and $\Lambda(\eta_2)$ in the same $(N, E)$–plane, their composition is a boost with parameter $\eta_1 + \eta_2$:*

$$\Lambda(\eta_1) \circ \Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2).$$

*Proof.* On $\operatorname{span}\{N, E\}$ the boosts are represented (in the basis $\{N, E\}$) by

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix},$$

whose multiplication adds rapidities. On the $g$–orthogonal complement the action is the identity, hence the statement holds on all of $T_p\mathcal{E}$. □

## 5.6 Comparison with classical angle conventions

The geometric Euclidean tilt is captured by the circular angle $\xi$ (cf. (5.1)), but different reformulations choose different dimensionless parameters derived from it.

A common choice is the sine–based parameter

$$\beta_{\sin} := \sin\xi = \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

whereas in the present work the kinematically natural parameter is the ratio (conic) parameter

$$\beta := \frac{\|X_\perp\|_\delta}{N \cdot X} = \tan\vartheta = \tan\xi, \qquad (X \in \mathcal{T}_p^+),$$

which satisfies $\beta \in [0, 1)$ on the future timelike cone and reaches the null limit at $\beta \to 1$.

*Remark* 5.10 (Photon limit and the 45° Euclidean tilt)*.* The null cone is characterized by $g(X, X) = 0$, equivalently $\cos(2\xi) = 0$, hence the lightlike limit corresponds to $\xi \to \pi/4$ in the Euclidean picture. In this limit one has

$$\beta = \tan\xi \to 1, \qquad \beta_{\sin} = \sin\xi \to \frac{1}{\sqrt{2}}.$$

Thus a light ray is reached at a finite Euclidean tilt of 45° relative to $N$ (not at 90°). The ratio parameter $\beta = \tan\xi$ is therefore better adapted to the speed–of–light barrier than the sine parameter.

*Remark* 5.11 (Nonadditivity of $\xi$ and $\vartheta$). Neither the circular angle $\xi$ nor the conic coordinate $\vartheta$ is a group parameter for boosts. Even in the collinear case, where rapidities add, $\eta_{12} = \eta_1 + \eta_2$, the corresponding tilt coordinates do not add:

$$\xi_{12} \neq \xi_1 + \xi_2, \qquad \vartheta_{12} \neq \vartheta_1 + \vartheta_2.$$

Indeed, the collinear velocity–composition law

$$\beta_{12} = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}$$

translates into

$$\tan\vartheta_{12} = \frac{\tan\vartheta_1 + \tan\vartheta_2}{1 + \tan\vartheta_1 \tan\vartheta_2},$$

showing explicitly that $\vartheta$ is a nonlinear reparameterization of the additive rapidity.

*Remark* 5.12 (Why we keep both parametrizations). The circular angle $\xi$ is geometrically immediate and makes the double–angle structure $g(X, X) = \|X\|_\delta^2 \cos(2\xi)$ transparent. The conic/hyperbolic parametrization $(\vartheta, \eta)$ is operationally adapted to velocity ratios and to the boost subgroup: $\beta = \tan\vartheta$ saturates at $\beta \to 1$ in the null limit and $\eta$ is additive under boosts. The two descriptions therefore complement each other.

# 6 Protovelocity invariants and the emergence of $S^3$ in the Euclidean proto-space

This section formalizes a central equivalence of the proto–space approach. Once a coherent observer axis $N$ (hence the Lorentzian form $g$) is fixed, the standard SR normalization of the 4–velocity in proper time,

$$g(\dot{X}, \dot{X}) = c^2,$$

is equivalent (up to reparameterization) to a constant Euclidean protovelocity budget in proto–space,

$$\delta(\widetilde{X}, \widetilde{X}) = c^2,$$

with respect to a calibrated proto–parameter $\chi$. The latter condition forces the instantaneous protovelocity states to lie on the Euclidean sphere $S_c^3 \subset T_p\mathcal{E}$.

## 6.1 Worldlines, proto–parameters, and the protovelocity

Let $X : I \to \mathcal{E}$ be a smooth regular worldline (geometric curve). A *proto–parameter* along $X$ is any smooth parameter $\chi$ with nowhere–vanishing derivative. The associated *protovelocity* is

$$\widetilde{X} \;:=\; \frac{dX}{d\chi} \;\in T_{X(\chi)}\mathcal{E}. \tag{6.1}$$

**Definition 6.1** ($\delta$–calibrated proto–parameter)**.** A proto–parameter $\chi$ is called *$\delta$–calibrated* (with scale $c$) if

$$\delta(\widetilde{X}, \widetilde{X}) \;=\; c^2 \qquad \text{along } X. \tag{6.2}$$

*Remark* 6.2 (Existence and gauge nature). For any regular curve $X$ with $dX \neq 0$, a $\delta$–calibrated parameter always exists: one may define $\chi$ (up to an additive constant) by

$$d\chi := \frac{1}{c}\,\|dX\|_\delta.$$

Fixing an orientation ($d\chi > 0$ along the chosen direction of traversal), the calibrated proto–parameter is unique up to translation. This calibration is auxiliary: it fixes the *total* Euclidean protovelocity budget in $\mathcal{E}$ but is not, by itself, the observer's operational time variable.

In words: in a $\delta$–calibrated proto–parameter, the protovelocity $\widetilde{X}$ has fixed Euclidean norm. This is the proto–space counterpart of the standard SR statement that the 4–velocity has fixed Minkowski norm in proper time.

## 6.2 Observer splitting and the interval–rate identity

Fix a coherent $\delta$–unit axis field $N$ as in §2–§3 (with a chosen time orientation). Pointwise along $X$, decompose the protovelocity into $\delta$–longitudinal and $\delta$–transverse parts relative to $N$:

$$\widetilde{X} \;=\; S\,N + \widetilde{X}_\perp, \qquad S := N\cdot\widetilde{X}, \qquad \widetilde{X}_\perp := h(\widetilde{X}) \in N^{\perp_\delta}. \tag{6.3}$$

Let $L := \|\widetilde{X}_\perp\|_\delta$ and, when $L \neq 0$, $E := \widetilde{X}_\perp/L \in N^{\perp_\delta}$ so that $\widetilde{X} = S\,N + L\,E$ with $\delta(E,E) = 1$.

**Lemma 6.3** (Euclidean and Lorentzian norms of the protovelocity)**.** *Along $X$ one has*

$$\delta(\widetilde{X},\widetilde{X}) = S^2 + L^2, \qquad g(\widetilde{X},\widetilde{X}) = S^2 - L^2. \tag{6.4}$$

*Proof.* Since $\widetilde{X}_\perp \in N^{\perp_\delta}$, one has $\delta(N,\widetilde{X}_\perp) = 0$, hence $\delta(\widetilde{X},\widetilde{X}) = S^2 + \delta(\widetilde{X}_\perp,\widetilde{X}_\perp) = S^2 + L^2$. For $g$, use $g(N,N) = 1$, $g(N,\widetilde{X}_\perp) = 0$, and $g(\widetilde{X}_\perp,\widetilde{X}_\perp) = -\delta(\widetilde{X}_\perp,\widetilde{X}_\perp) = -L^2$. □

Motivated by the phase/clock viewpoint, define the *interval rate* with respect to $\chi$ by

$$\widetilde{s} \;:=\; \sqrt{g(\widetilde{X},\widetilde{X})} \;=\; \sqrt{S^2 - L^2} \qquad (g\text{–timelike case}). \tag{6.5}$$

Equivalently, along a timelike segment one may write $g(dX,dX) = \widetilde{s}^{\,2}\,d\chi^2$.

*Remark* 6.4 (Circular angle $\xi$ on calibrated protovelocities)*.* If $\chi$ is $\delta$–calibrated, $\delta(\widetilde{X},\widetilde{X}) = c^2$, then (6.4) implies $S^2 + L^2 = c^2$ and one may set

$$\cos\xi := \frac{S}{c}, \qquad \sin\xi := \frac{L}{c}.$$

Then

$$g(\widetilde{X},\widetilde{X}) = c^2(\cos^2\xi - \sin^2\xi) = c^2\cos(2\xi), \qquad \widetilde{s} = c\sqrt{\cos(2\xi)}.$$

This is the same double–angle structure that governs the phase rate $d\tau/d\chi$ in §4.

## 6.3 Equivalence: proper–time normalization $\Longleftrightarrow$ calibrated protovelocity

Let $\tau$ denote the proper time along a $g$–timelike worldline $X$, i.e. a parameter such that $\dot{X} := dX/d\tau$ satisfies

$$g(\dot{X},\dot{X}) = c^2. \tag{6.6}$$

Equivalently, $g(dX,dX) = c^2\,d\tau^2$ along $X$.

*Remark* 6.5 (Units and normalized 4–velocity)*.* The proper–time tangent $\dot{X}$ has the physical dimension of a speed and is normalized by (6.6). It is often convenient to introduce the dimensionless unit 4–velocity

$$U \;:=\; \frac{1}{c}\,\dot{X}, \qquad \text{so that} \qquad g(U,U) = 1.$$

Whenever $U$ arises from a worldline it is understood as the normalized proper–time tangent.

**Theorem 6.6** (Reparameterization equivalence)**.** *Let $X$ be a regular $g$–timelike worldline (geometric curve). Then the following statements are equivalent up to reparameterization:*

*(A) $X$ is parameterized by proper time $\tau$ so that $g(\dot{X},\dot{X}) = c^2$.*

*(B) $X$ is parameterized by a $\delta$–calibrated proto–parameter $\chi$ so that $\delta(\widetilde{X}, \widetilde{X}) = c^2$.*

*Moreover, when both parameters are used on the same curve, they satisfy*

$$\frac{d\tau}{d\chi} = \frac{\sqrt{g(\widetilde{X}, \widetilde{X})}}{c} = \frac{\widetilde{s}}{c}, \qquad \frac{d\chi}{d\tau} = \frac{\|\dot{X}\|_\delta}{c}. \tag{6.7}$$

*Proof.* Assume (A). Define $\chi$ (up to an additive constant) by

$$\frac{d\chi}{d\tau} := \frac{\|\dot{X}\|_\delta}{c},$$

which is smooth and positive since $\dot{X} \neq 0$. Then $\widetilde{X} = dX/d\chi = (d\tau/d\chi)\dot{X}$, so

$$\delta(\widetilde{X}, \widetilde{X}) = \Big(\frac{d\tau}{d\chi}\Big)^2 \delta(\dot{X}, \dot{X}) = \frac{c^2}{\|\dot{X}\|_\delta^2} \, \|\dot{X}\|_\delta^2 = c^2,$$

which is (B).

Conversely, assume (B). Define $\tau$ (up to an additive constant) by

$$\frac{d\tau}{d\chi} := \frac{\sqrt{g(\widetilde{X}, \widetilde{X})}}{c},$$

which is well–defined and positive for timelike $\widetilde{X}$ since $g(\widetilde{X}, \widetilde{X}) > 0$. Then $\dot{X} = dX/d\tau = (d\chi/d\tau)\widetilde{X}$, hence

$$g(\dot{X}, \dot{X}) = \Big(\frac{d\chi}{d\tau}\Big)^2 g(\widetilde{X}, \widetilde{X}) = \frac{c^2}{g(\widetilde{X}, \widetilde{X})} \, g(\widetilde{X}, \widetilde{X}) = c^2,$$

which is (A). The relations (6.7) are exactly the two defining ODEs. □

**Operational meaning.** Statement (A) is the standard SR normalization of the 4–velocity in proper time. Statement (B) is the corresponding calibration of the proto–space protovelocity. Theorem 6.6 shows that these two "invariants" are equivalent and amount to a change of parameter: constant norm in one metric corresponds to constant norm in the other once the operational projection structure ($N$, hence $g$) has been fixed.

### 6.4 The $S^3$ of admissible protovelocity states

Fix $p \in \mathcal{E}$. The set of all $\delta$–calibrated protovelocities at $p$ is the Euclidean 3–sphere of radius $c$ inside $T_p\mathcal{E}$:

$$S_c^3(p) \;:=\; \big\{\, V \in T_p\mathcal{E} : \; \delta(V, V) = c^2 \,\big\} \;\cong\; S^3. \tag{6.8}$$

Thus, once the calibration (6.2) is imposed, every instantaneous proto–kinematic state is a point on $S_c^3(p)$.

Relative to the coherent axis $N_p$, each $V \in S_c^3(p)$ admits the decomposition

$$V = c\big(\cos\xi\, N + \sin\xi\, E\big), \qquad \xi \in [0, \pi], \quad E \in N^{\perp_\delta}, \; \delta(E, E) = 1, \tag{6.9}$$

where $\xi$ is the circular tilt angle and $E$ is the transverse unit direction. When $\sin\xi = 0$ (the poles), the direction $E$ is irrelevant and the state reduces to $V = \pm c\, N_p$.

*Remark* 6.7 (Quaternionic encoding of $S^3$ (optional viewpoint)). Fix a reference unit vector $e_0$ and the associated quaternionic subalgebra $\mathbb{H}_{e_0} \subset \mathcal{C}\ell^+(4,0)$ generated by $I_i := e_i e_0$ $(i = 1, 2, 3)$, so that $I_i^2 = -1$ and $I_i I_j = -I_j I_i$ for $i \neq j$. Then any vector $V = V^0 e_0 + V^1 e_1 + V^2 e_2 + V^3 e_3 \in T_p\mathcal{E}$ can be encoded as

$$Q(V) := V^0 + V^1 I_1 + V^2 I_2 + V^3 I_3 \in \mathbb{H}_{e_0},$$

and one has $\delta(V, V) = |Q(V)|^2$ (quaternionic norm). Hence the calibrated sphere $S_c^3(p)$ corresponds to the set of quaternions of norm $c$, and the unit sphere corresponds to unit quaternions. This identification depends on the chosen reference axis (a gauge choice); the underlying manifold structure is canonical.

*Remark* 6.8 (The physical sector of $S^3$ and time orientation). While the calibrated protovelocity states form the full sphere $S_c^3(p)$, the requirement that a worldline be timelike with respect to $g$,

$$g(V, V) > 0,$$

restricts admissible states to

$$|S| > |L| \qquad \text{where} \qquad S := N \cdot V, \quad L := \|h(V)\|_\delta,$$

i.e. to the union of two polar caps on $S_c^3(p)$ centered at $\pm cN$. The boundary $|S| = |L|$ is the null locus.

Because $g$ is insensitive to $N \mapsto -N$ but the oriented time projection $\alpha = N^\flat$ is not, an observer fixes a *future* branch by imposing $S = N \cdot V > 0$. This selects a single polar cap (the future timelike sector). Along any sequence approaching the null boundary within that cap one has $d\tau/d\chi = \sqrt{g(V, V)}/c \to 0$, consistently with (6.7).

# 7 Why the Euclidean proto–space viewpoint

The induced Lorentzian form $g$ constructed from $(\delta, N)$ equips each tangent space $(T_p\mathcal{E}, g)$ with the standard causal classification and light cone. The additional advantage of the Euclidean proto–space viewpoint is that it keeps, in a single geometric package, both (i) directional data in the Euclidean spatial complement $\mathrm{Im}(h_p) = N_p^{\perp_\delta}$ and (ii) clock calibration data encoded by $g$–projections onto oriented time axes. In particular, "light" does not require an extra postulate: a null direction is a $g$–null ray, and choosing a convenient $\delta$–normalization picks a canonical cross–section of that ray on the calibrated Euclidean sphere.

The material in this section provides the operational dictionary needed for later frequency and angle computations. It uses only the observer splitting determined by a chosen oriented coherent axis $N$.

## 7.1 $\xi$–formulation of inertial composition

Fix an inertial observer axis $e_0$ (constant on the region of interest), $\delta(e_0, e_0) = 1$, $\nabla^\delta e_0 = 0$, and set $N \equiv e_0$ in the SR sector. Let $g = 2\, e_0^\flat \otimes e_0^\flat - \delta$ be the induced Lorentzian form. All statements below are pointwise at $p \in \mathcal{E}$.

**Calibrated proto–flows and circular angle data.** A $\chi$–calibrated proto–flow state is a vector $\tilde{X} \in T_p\mathcal{E}$ satisfying $\delta(\tilde{X}, \tilde{X}) = c^2$. Relative to $e_0$ write

$$\tilde{X} = \tilde{H}\, e_0 + \tilde{L}\, E, \qquad E \in e_0^{\perp_\delta},\ \delta(E, E) = 1, \qquad \tilde{H}^2 + \tilde{L}^2 = c^2,$$

and define the circular tilt $\xi \in [0, \pi]$ by

$$\cos\xi := \frac{\tilde{H}}{c}, \qquad \sin\xi := \frac{\tilde{L}}{c}. \tag{7.1}$$

The observer speed parameter (measured in the observer coordinate time $t$) is

$$\beta := \frac{v}{c} = \frac{\tilde{L}}{\tilde{H}} = \tan\xi. \tag{7.2}$$

**Operational time and the $\chi \to \tau$ rescaling.** Along a timelike worldline, proper time satisfies

$$\frac{d\tau}{d\chi} = \sqrt{\cos(2\xi)}, \qquad \cos(2\xi) = \frac{g(\tilde{X},\tilde{X})}{\delta(\tilde{X},\tilde{X})}. \tag{7.3}$$

Hence the proper–time tangent and the dimensionless unit $g$–velocity are

$$\dot{X} = \frac{dX}{d\tau} = \frac{1}{\sqrt{\cos(2\xi)}}\,\tilde{X}, \qquad U := \frac{1}{c}\dot{X} = \frac{1}{c\sqrt{\cos(2\xi)}}\,\tilde{X}, \tag{7.4}$$

so that $g(U,U) = 1$.

**Observer split in $\xi$–variables.** Define the observer Lorentz factor and the spatial speed vector by

$$\gamma := e_0 \cdot_g U, \qquad \boldsymbol{\beta} := \frac{1}{\gamma}\,h_g(U),$$

where $h_g$ denotes the $g$–orthogonal projector onto $e_0^{\perp_g}$. From (7.4) and (7.1) one obtains purely in terms of $\xi$

$$\gamma = \frac{\cos\xi}{\sqrt{\cos(2\xi)}}, \qquad \gamma\,\|\boldsymbol{\beta}\| = \frac{\sin\xi}{\sqrt{\cos(2\xi)}}, \qquad \|\boldsymbol{\beta}\| = \tan\xi. \tag{7.5}$$

Thus the proper–time normalization $g(U,U) = 1$ is exactly the $\chi$–calibrated state rescaled by $(\cos 2\xi)^{-1/2}$.

## 7.2 Light rays: canonical frequency and direction from the $N$–split

Fix $p \in \mathcal{E}$ and an oriented coherent time axis $N_p$ (future branch chosen). A light ray direction at $p$ is a projective class $[K]$ of nonzero vectors $K \in T_p\mathcal{E}$ satisfying

$$g(K,K) = 0, \qquad N \cdot K > 0,$$

where $K \sim \lambda K$ for $\lambda > 0$ represents the same ray.

A convenient representative is obtained by encoding the scale of $K$ as the frequency measured by $N$.

**Definition 7.1** (Proto–frequency of a null ray (observer $N$))**.** Let $K \neq 0$ be future–directed and $g$–null. The frequency measured by $N$ is

$$\omega := g(K,N). \tag{7.6}$$

**Lemma 7.2** (Euclidean and Lorentzian projections coincide on $N$)**.** *For any $K \in T_p\mathcal{E}$ one has*

$$g(K,N) = N \cdot K. \tag{7.7}$$

*In particular, a future direction satisfies $\omega = N \cdot K > 0$.*

*Proof.* Using $g = 2\,N^\flat \otimes N^\flat - \delta$ and $\delta(N,N) = 1$,

$$g(K,N) = 2(N\cdot K)(N\cdot N) - \delta(K,N) = 2(N\cdot K) - (N\cdot K) = N\cdot K.$$

□

Decompose $K$ into longitudinal and transverse parts relative to $N$:

$$K = (N \cdot K)\, N + K_\perp, \qquad K_\perp \in N^{\perp_\delta}.$$

The null condition fixes the transverse magnitude.

**Lemma 7.3** (Canonical null decomposition)**.** *Every future–directed null vector $K \neq 0$ admits a unique decomposition*

$$K = \omega\,(N + E), \tag{7.8}$$

*where $\omega = g(K, N) > 0$ and $E \in N^{\perp_\delta}$ is uniquely determined by*

$$\delta(E, E) = 1, \qquad N \cdot E = 0.$$

*Proof.* Write $K = \omega N + K_\perp$ with $\omega = N \cdot K > 0$ and $K_\perp \in N^{\perp_\delta}$. Then

$$0 = g(K, K) = g(\omega N, \omega N) + 2g(\omega N, K_\perp) + g(K_\perp, K_\perp) = \omega^2 - \delta(K_\perp, K_\perp),$$

so $\|K_\perp\|_\delta = \omega$. Set $E := K_\perp/\omega$ to obtain (7.8). Uniqueness follows from uniqueness of the $N$–split and the positivity of $\omega$. □

*Remark* 7.4 (Null rays as a calibrated slice of the null cone)*.* The scale in (7.8) is encoded by $\omega$; correspondingly,

$$\delta(K, K) = \delta(\omega(N + E), \omega(N + E)) = 2\omega^2.$$

If one prefers a $\delta$–calibrated representative on the Euclidean sphere, define

$$\widehat{K} \;:=\; \frac{c}{\sqrt{2}\,\omega}\, K \;=\; \frac{c}{\sqrt{2}}\,(N + E).$$

Then $\delta(\widehat{K}, \widehat{K}) = c^2$ and $g(\widehat{K}, \widehat{K}) = 0$. Thus the set of (future) null directions corresponds to the $S^2$ cross–section

$$S^3_c(p) \cap \{V :\; g(V, V) = 0,\; N \cdot V > 0\} = \Big\{ \frac{c}{\sqrt{2}}(N + E) :\; E \in S^2 \subset N^{\perp_\delta} \Big\}.$$

In the circular tilt language of §4–§5, this is the boundary $\xi = \pi/4$ of the future timelike cap on $S^3_c(p)$.

### 7.3 Observers as unit timelike states; frequency as a $g$–projection

A local observer at $p$ is represented by a future unit timelike vector (dimensionless)

$$U \in T_p\mathcal{E}, \qquad g(U, U) = 1, \qquad g(U, N) > 0.$$

All measurable scalars are obtained by taking $g$–contractions.

**Definition 7.5** (Frequency measured by an observer)**.** For a null ray represented by $K \neq 0$, the frequency measured by the observer $U$ is

$$\omega_U := g(U, K). \tag{7.9}$$

*Remark* 7.6 (Ray vector vs. wave covector)*.* We represent a light ray by a future–directed null vector $K \in T_p\mathcal{E}$. Its metric dual 1–form

$$k \;:=\; g(K, \cdot) \;\in T^*_p\mathcal{E}$$

is the standard wave covector of geometric optics. With this notation, $\omega_U = k(U) = g(U, K)$.

### 7.4 Doppler shift as a one–line contraction

Let $E_v \in N^{\perp_\delta}$ be a $\delta$–unit direction, $\delta(E_v, E_v) = 1$. Let $U$ be obtained from $N$ by a boost of rapidity $\eta \geq 0$ in the $(N, E_v)$–plane:

$$U := (\cosh \eta)\, N + (\sinh \eta)\, E_v. \tag{7.10}$$

Let the null ray be written in canonical form (7.8),

$$K = \omega\,(N + E), \qquad \delta(E, E) = 1, \quad E \in N^{\perp_\delta}.$$

Using bilinearity and $g(N, N) = 1$, $g(N, E) = 0$, $g(E_v, N) = 0$, $g(E_v, E) = -\delta(E_v, E)$, we obtain

$$\begin{aligned} \omega_U &= g(U, K) \\ &= \omega\, g\big((\cosh \eta)N + (\sinh \eta)E_v,\ N + E\big) \\ &= \omega\Big(\cosh \eta - \sinh \eta\, \delta(E_v, E)\Big). \end{aligned} \tag{7.11}$$

Introduce the standard parameters

$$\beta := \tanh \eta, \qquad \gamma := \cosh \eta,$$

and define the Euclidean angle $\psi \in [0, \pi]$ between the velocity axis $E_v$ and the ray direction $E$ inside the observer space $N^{\perp_\delta}$ by

$$\cos \psi := \delta(E_v, E). \tag{7.12}$$

Then (7.11) becomes the standard relativistic Doppler law

$$\frac{\omega_U}{\omega} = \gamma\big(1 - \beta \cos \psi\big). \tag{7.13}$$

**Interpretation.** In the proto–space picture, (7.13) is literally a projection: the measured frequency is the scalar $g$–projection of a null direction $K$ onto an observer state $U$.

### 7.5 Aberration as projection plus normalization

The direction of the ray measured by $U$ is the normalized $U$–spatial part of $K$ (in the $g$–orthogonal complement of $U$). Define the $g$–spatial component

$$K_{\perp U} := K - (g(U, K))\, U. \tag{7.14}$$

Then $g(U, K_{\perp U}) = 0$, hence $K_{\perp U} \in U^{\perp_g}$, and

$$g(K_{\perp U}, K_{\perp U}) = -\,\omega_U^2 \qquad (\text{since } g(K, K) = 0,\ g(U, U) = 1).$$

Thus a unit spatial direction of the ray in the $U$–frame can be taken as

$$E_U := \frac{1}{\omega_U}\, K_{\perp U} = \frac{1}{g(U, K)}\Big(K - (g(U, K))U\Big), \qquad g(E_U, E_U) = -1, \quad g(U, E_U) = 0. \tag{7.15}$$

Specialize to the same kinematics as in §7.4. Let $\psi'$ denote the angle between the ray direction in the $U$–rest space and the $U$–spatial image of the boost axis. A direct computation yields the standard aberration law

$$\cos \psi' = \frac{\cos \psi - \beta}{1 - \beta \cos \psi}, \tag{7.16}$$

with $\cos \psi = \delta(E_v, E)$ as in (7.12).

**Interpretation.** Doppler and aberration are the same operation in two steps:

- Doppler: take the $g$–projection $g(U, K)$ (a scalar).
- Aberration: subtract the time component $(g(U, K))U$ and normalize the remaining $U$–spatial part.

Both effects are thus immediate consequences of the null cone geometry together with the observer–dependent splitting induced by $U$.

**A $\xi$–rotor that maps $e_0$ to the unit axis $U(\xi, E)$.** Given circular data $(\xi, E)$ with $E \in e_0^{\perp_\delta}$, $\delta(E, E) = 1$, define

$$U(\xi, E) := \frac{1}{\sqrt{\cos(2\xi)}}\big(\cos\xi\, e_0 + \sin\xi\, E\big), \tag{7.17}$$

and define the associated (minimal) boost rotor by the standard axis–to–axis formula

$$R(\xi, E) := \frac{1 + U(\xi, E)\, e_0}{\sqrt{2\big(1 + e_0 \cdot_g U(\xi, E)\big)}} \quad \in Spin(1, 3) \subset \mathcal{C}\ell^+(4, 0), \tag{7.18}$$

so that

$$R(\xi, E)\, e_0\, \widetilde{R(\xi, E)} = U(\xi, E), \qquad R(\xi, E)\widetilde{R(\xi, E)} = 1.$$

No rapidity parameter is used: $R$ is defined directly from $\xi$ via the rescaled axis $U(\xi, E)$.

*Remark* 7.7 (Why the rescaling is essential). If one replaces $U(\xi, E)$ in (7.18) by the unrescaled proto–flow direction $\tilde{X}/c = \cos\xi\, e_0 + \sin\xi\, E$ (which lives on the calibrated sphere $S^3$), the resulting sandwich map is a Euclidean rotation in the $(e_0, E)$–plane and does not reproduce SR velocity composition. The factor $(\cos 2\xi)^{-1/2}$ in (7.17) is precisely the operational correction that converts $\chi$–calibrated geometry into proper–time (hence Lorentz) normalization.

**Proposition 7.8** (Two–step inertial composition in $\xi$–variables)**.** *Let observer* 0 *be encoded by* $e_0$*. Assume object* 1 *is given in the observer algebra by circular data* $(\xi_1, E_1)$*, where* $E_1 \in e_0^{\perp_\delta}$ *and* $\delta(E_1, E_1) = 1$*. Define its unit axis and rotor*

$$U_1 := U(\xi_1, E_1), \qquad R_{10} := R(\xi_1, E_1).$$

*Assume object* 2 *is given in the comoving algebra of object* 1 *by circular data* $(\xi_2', E_2')$*, where* $E_2' \in (U_1)^{\perp_\delta}$ *is unit in the spatial complement of* $U_1$ *(the "space" of object* 1*). Transport this direction into the observer algebra by*

$$E_2 := R_{10}\, E_2'\, \widetilde{R}_{10}. \tag{7.19}$$

*Define the unit axis of object* 2 *relative to* $U_1$ *by the same* $\xi$*–recipe:*

$$U_2 := \frac{1}{\sqrt{\cos(2\xi_2')}}\big(\cos\xi_2'\, U_1 + \sin\xi_2'\, E_2\big). \tag{7.20}$$

*Then the corresponding rotor* $R_{21} := R(U_2, U_1)$ *and the composite rotor*

$$R_{20} := R_{21}\, R_{10}$$

*map the observer axis to* $U_2$*:*

$$R_{20}\, e_0\, \widetilde{R}_{20} = U_2.$$

*Moreover, the 3–velocity of object* 2 *measured by observer* 0 *obtained from the observer split of* $U_2$ *coincides with the textbook SR velocity addition rule.*

*Proof sketch in $\xi$–language.* The only nontrivial step is operational normalization. Each circular datum $(\xi, E)$ defines a proto–flow direction $\cos\xi\,(\text{time}) + \sin\xi\,(\text{space})$ on the calibrated sphere, but physical composition of inertial frames acts on $g$–unit timelike axes. The proper–time rescaling (7.4) converts the circular datum into the $g$–unit axis $U(\xi, E)$, and thereby fixes the correct $\gamma$–factor $\gamma = \cos\xi/\sqrt{\cos(2\xi)}$ as in (7.5). Once $U_1$ is so normalized, the transport (7.19) is exactly the change of spatial basis from the comoving algebra of object 1 into the observer algebra by rotor conjugation. Finally, the definition (7.20) constructs the second axis from the same circular rule, but now in the $1+3$ split of $U_1$. Projecting the resulting $U_2$ onto the observer split recovers the standard longitudinal and transverse SR composition laws, with the characteristic $1/\gamma_1$ suppression of transverse components. □

**Worked example (orthogonal data; no rapidity).** Let object 1 have speed $\beta_1 = 0.6$ along $x$ in the observer algebra, hence $\xi_1 = \arctan(0.6)$ and $E_1 = e_x$. Let object 2 have speed $\beta_2' = 0.6$ along $y'$ in the comoving algebra of object 1, hence $\xi_2' = \arctan(0.6)$ and $E_2' = e_{y'}$.

From $\beta = \tan\xi$ one has $\cos(2\xi) = \frac{1-\beta^2}{1+\beta^2}$, and therefore

$$\gamma_1 = \frac{\cos\xi_1}{\sqrt{\cos(2\xi_1)}} = \frac{1}{\sqrt{1-\beta_1^2}} = 1.25.$$

Applying Proposition 7.8 and projecting $U_2$ onto the observer split yields

$$\beta_x^{(20)} = \beta_1 = 0.6, \qquad \beta_y^{(20)} = \frac{\beta_2'}{\gamma_1} = \frac{0.6}{1.25} = 0.48,$$

which coincides exactly with the standard SR result for orthogonal velocity composition.

**Wigner–Thomas rotation (optional; no rapidity).** Let $R_{20}^{\min} := R(U_2, e_0)$ denote the minimal rotor mapping $e_0$ directly to $U_2$. Then

$$W := \widetilde{R}_{20}^{\min}\, R_{21}\, R_{10}$$

fixes $e_0$ and hence belongs to the observer's spatial stabilizer $Spin(3)$; this is the Wigner–Thomas rotation induced by the noncollinear two–step composition.

### 7.6 Geometric Algebra viewpoint: circular versus hyperbolic rotors

The Euclidean proto–space admits the Clifford algebra $\mathcal{C}\ell(T_p\mathcal{E}, \delta) \cong \mathcal{C}\ell_{4,0}$. For vectors $u, v \in T_p\mathcal{E}$ the geometric product is defined by

$$uv = u\cdot v + u\wedge v,$$

where $u\cdot v = \delta(u, v)$ is the Euclidean inner product. The induced Lorentzian form $g$ can be encoded by the metric extensor $G := 2\,N\otimes N - \mathrm{Id}$, satisfying

$$g(u, v) = \delta(Gu, v),$$

which makes explicit how the $g$–geometry is obtained from Euclidean data once $N$ is fixed.

**Observer split inside $\mathcal{C}\ell_{4,0}$.** Multiplying a vector $X$ by $N$ decomposes it into a scalar (longitudinal) part and a bivector (transverse) part relative to $N$:

$$XN = \underbrace{X\cdot N}_{\text{scalar}} + \underbrace{X\wedge N}_{\text{bivector}}. \tag{7.21}$$

The bivectors of the form $E\wedge N$ (with $E \in N^{\perp_\delta}$) constitute the observer's Euclidean spatial algebra. The even subalgebra $\mathcal{C}\ell_{4,0}^{+}$ is isomorphic to the quaternion algebra; this is the algebraic origin of the "unit quaternion" representation of calibrated flow states on $S^3$.

**Circular rotor in Euclidean signature.** Let $E \in N^{\perp_\delta}$ be $\delta$–unit. In $\mathcal{C}\ell_{4,0}$ one has

$$N^2 = 1, \qquad E^2 = 1,$$

and the bivector $\mathbf{I} := NE$ satisfies

$$\mathbf{I}^2 = (NE)(NE) = -\,N^2E^2 = -1.$$

Hence $\exp(-\mathbf{I}\,\xi/2)$ generates an ordinary circular rotation in the $(N, E)$–plane. This is the natural algebraic encoding of the circular tilt coordinate $\xi$ used in §4 and §5.

**Hyperbolic rotor in Lorentzian signature.** In the induced Lorentzian geometry, $N$ remains unit timelike while a spatial unit direction $\mathbf{e}$ satisfies $\mathbf{e}^2 \overset{g}{=} -1$. Accordingly, the boost bivector $\mathbf{K} := N\mathbf{e}$ satisfies

$$\mathbf{K}^2 = (N\mathbf{e})(N\mathbf{e}) = -\,N^2\mathbf{e}^2 = +1,$$

so $\exp(-\mathbf{K}\,\eta/2)$ generates a hyperbolic rotor (a boost) with rapidity $\eta$. Applying such a rotor to $N$ yields the standard boosted observer state

$$U = \cosh\eta\, N + \sinh\eta\,\mathbf{e},$$

and $\eta$ is additive under collinear boost composition. In the conic parametrization of §5 one has $\tanh\eta = \beta = \tan\vartheta$.

*Remark* 7.9 (What "emerges" from the proto–space viewpoint). No analytic continuation is invoked: the distinction between circular and hyperbolic behavior is controlled by the sign of the square of the generator bivector ($-1$ versus $+1$), which in turn is fixed by whether the spatial basis squares to $+1$ (Euclidean $\delta$) or to $-1$ (Lorentzian $g$). The proto–space framework makes this dependence explicit by separating the Euclidean algebra $\mathcal{C}\ell_{4,0}$ from the induced Lorentzian structure determined by $N$.

### 7.7 Scope and limitations

The present work is purely kinematical. We do not introduce dynamical equations for the field $N$, nor do we model localized sources, defects, or global topology. Such extensions require additional structure (e.g. a field theory for $N$ and/or a source model) and are deferred to separate work.

## 8 Discussion

The primary contribution of this paper is a coordinate–free geometric reformulation of special–relativistic kinematics in a Euclidean phase space $(\mathcal{E}, \delta)$ equipped with an induced Lorentzian metric $g = 2\,\alpha\otimes\alpha - \delta$ generated by a unit field $N$ (or $\alpha := \delta(N, \cdot)$). A natural question is why it is useful to represent SR in a strictly Euclidean 4D setting without introducing a distinguished time coordinate at the outset.

First, the Euclidean proto–space separates two logically different ingredients that are often conflated in the standard Minkowski presentation: the kinematic state of motion and the operational choice of clock. In our framework, $\delta$ provides a positive–definite reference geometry for proto–velocities and for the calibrated budget constraint $\delta(\tilde{X}, \tilde{X}) = c^2$, while the Lorentzian interval arises only after an observer chooses a coherent time axis $N$ and hence a time 1–form $\alpha$. Proper time is then obtained by reparametrization along worldlines, $c^2 d\tau^2 = g(dX, dX)$. This makes explicit that SR kinematics can be organized around a choice of axis (and a clock parametrization) rather than around a fundamental time coordinate built into the background manifold.

Second, the calibrated Euclidean picture provides a compact and geometrically transparent state space. At each point, the set of calibrated proto–velocities is the sphere $S^3_c \subset T_p\mathcal{E}$, and the timelike and null conditions with respect to $g$ select, respectively, an open sector (two polar caps) and its boundary. In this representation the speed–of–light barrier is realized as a finite geometric boundary on $S^3_c$ (a 45° Euclidean tilt relative to $N$), so the causal classification and its limiting behavior are read off directly from Euclidean angle geometry and the identity $g(X,X) = \|X\|^2_\delta \cos(2\xi)$.

Third, working in $\mathcal{C}\ell(4,0)$ keeps all kinematical transformations within a strictly Euclidean rotor calculus. Once a reference axis is fixed, relative states and frame changes can be encoded by even elements (quaternionic coordinates) and rotors acting by sandwiching, while the Lorentzian bilinear form is implemented by the RSS reflection/projection relative to $N$. This gives a uniform algebraic mechanism for transporting directions and proto–velocities between observer gauges, and it isolates the genuinely operational step: the reparametrization from the calibrated proto–parameter to proper time.

Finally, although the present paper is restricted to the inertial SR sector (where $N$ is taken constant), the formulation clarifies what must change beyond it: departures from inertial kinematics correspond to variations of the coherent time axis and/or its associated 1–form. In this sense the Euclidean proto–space is not introduced as an alternative time coordinate, but as a neutral geometric carrier in which the observer's time structure is a field–like datum that can later be allowed to vary without changing the basic kinematic bookkeeping.

### 8.1 Outlook: towards fields via a tilted field vector

The special–relativistic sector treated in this paper assumes an inertial (coherently constant) "time" axis represented by a field vector $N$ with $\nabla^\delta N = 0$, so that the induced Lorentzian form $g_N = 2\,N^\flat \otimes N^\flat - \delta$ is globally Minkowskian. A natural extension is to allow $N$ to vary smoothly, while keeping the same RSS "Lorentzization" rule for $g_N$. In this view, what is usually called a "gravitational field" is encoded kinematically by the spatial variation of the coherent time direction, rather than by introducing a Lorentzian metric as a primitive object.

To illustrate the mechanism in the simplest setting, consider a static, spherically symmetric configuration on the Euclidean proto–space $(\mathcal{E}, \delta)$ and write the flat background in spherical coordinates as

$$\delta = c^2\, dt^2 + dr^2 + r^2 d\Omega^2, \qquad d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2.$$

Let $e_0$ denote the constant fiducial time axis and $e_r$ the outward unit radial vector with respect to $\delta$. Introduce a radial tilt field $\alpha = \alpha(r)$ and set

$$N(r) \;:=\; \cos\alpha(r)\, e_0 \;+\; \sin\alpha(r)\, e_r, \qquad \delta(N,N) = 1. \tag{8.1}$$

The induced Lorentzian form $g_N = 2\,N^\flat \otimes N^\flat - \delta$ then acquires, in the $(t,r)$ block, the components

$$g_{tt} = \cos\big(2\alpha(r)\big), \qquad g_{tr} = \sin\big(2\alpha(r)\big), \qquad g_{rr} = -\cos\big(2\alpha(r)\big), \tag{8.2}$$

while $g_{\theta\theta} = -r^2$ and $g_{\phi\phi} = -r^2 \sin^2\theta$. Hence

$$ds^2 \;=\; \cos(2\alpha)\, c^2 dt^2 \;+\; 2\sin(2\alpha)\, c\, dt\, dr \;-\; \cos(2\alpha)\, dr^2 \;-\; r^2 d\Omega^2. \tag{8.3}$$

The off–diagonal term is removed by the shifted time coordinate

$$c\, dt_S \;:=\; c\, dt \;+\; \tan\big(2\alpha(r)\big)\, dr, \tag{8.4}$$

which yields the diagonal form

$$ds^2 \;=\; A(r)\, c^2 dt_S^2 \;-\; \frac{1}{A(r)}\, dr^2 \;-\; r^2 d\Omega^2, \qquad A(r) := \cos\big(2\alpha(r)\big). \tag{8.5}$$

Thus a radial variation of the coherent time axis produces the characteristic reciprocal pair of lapse–type factors $A$ and $A^{-1}$ by a purely geometric identity. Choosing, for $r > r_s$, the profile

$$A(r) \;=\; 1 - \frac{r_s}{r}, \qquad r_s := \frac{2GM}{c^2}, \tag{8.6}$$

recovers the Schwarzschild form (in the $(+ - --)$ convention) as a special case, with the "tilt angle" determined by

$$\cos\bigl(2\alpha(r)\bigr) = 1 - \frac{r_s}{r}. \tag{8.7}$$

In this parametrization the horizon corresponds to $A \to 0^+$, i.e. to the finite Euclidean tilt $2\alpha \to \pi/2$ (so $\alpha \to \pi/4$), which is the same distinguished $45^\circ$ value that appears in the null limit of the SR tilt geometry.

The point of the present paragraph is not to derive general relativity within the kinematical scope of this paper, but to clarify why the Euclidean proto–space formulation is structurally useful: the same data $(\delta, N)$ that encode inertial SR kinematics extend to a field setting by relaxing the constancy of $N$. The remaining dynamical question is then shifted from postulating a Lorentzian metric to specifying an evolution law for the tilt field (or for $N$), for instance via a variational principle on $(\mathcal{E}, \delta)$ or a coarse–grained source model. Developing such field equations and their relation to the Einstein vacuum equation is left for future work.

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