Unimetry: A Phase-Space Reformulation of Special Relativity

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Abstract

We propose a compact reformulation of special relativity in which spacetime units (time and length) are treated as phase velocities - directional derivatives of a single underlying parameter, the phase $\vec{\chi} \in \mathbb{H}$. The observable Minkowski interval emerges as a conserved quantity under a change of parameter from the phase coordinate χ to the observer's proper time τ. In this formalism, we show that familiar relativistic effects - time dilation, Lorentz factor, Doppler shift, and relativistic velocity composition - arise as elementary projections and rotations in a Euclidean phase plane. Hyperbolic features of Lorentz kinematics reappear after a reparameterization of time, yielding the standard relations without altering empirical content. We provide closed-form derivations of the longitudinal/transverse Doppler factors, identify a simple lemma equating the total phase speed to the conserved Minkowski norm, and outline connections to gauge phases, rapidity, and a cosmological time gauge. Composition of non-collinear boosts (D-rotations strictly in $\mathbb{H}$) yields a Wigner rotation; in the continuous limit this gives Thomas precession; both arise here as purely kinematical consequences of the quaternionic phase formalism (see §4.7).

Keywords: special relativity; phase; rapidity; Doppler shift; Lorentz factor; Wigner rotation; Thomas precession; phase parameterization.

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1 Introduction

We usually take time and space as primitive. The unimetry formalism introduced here suggests a different viewpoint: time and space are derived projections of a single parameter $\vec{\chi} \in \mathbb{H}$ ("phase") which is suggested the intrinsic phase parameter of an object. In this picture, relativistic effects such as time dilation and the Doppler shift are geometric consequences of phase-vector rotations, when the quaternion algebra unites spatial rotations and boosts.

The proposal does not modify physics; it reorganizes familiar relations in a simpler language toward formalism of quantum mechanics demonstrating the physical equivalence between Lorentz transformations and Euclidean rotations under proper time parametrization. We will realize the phase kinematics with quaternionic rotors $d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\sin\frac{\psi}{2}$. Throughout, Greek θ will denote the *external* rotation angle associated with relative motion, while ζ denotes an *internal* angle associated with the object's intrinsic state (mass/density heuristic). We emphasize that no modification of Einstein's dynamics is proposed; all results are kinematical identities obtained by a change of parameter.

Notation. Tildes, dots and primes indicate derivatives with respect to the phase parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \qquad \dot{X} := \frac{dX}{d\tau}, \qquad X' := \frac{dX}{dl}.$$

Table 1: Notation used in the paper.

Symbol	Meaning
$\tilde{X}$	Normalized / phase–plane quantity (dimensionless), or a projection as defined at first occurrence.
$\dot{X}$	dX/dt (lab time) by default; derivatives with respect to proper time and phase are written $dX/d\tau$ and $dX/d\vartheta$.
X'	Value in a primed inertial frame or after one composition/transform; X'' after two compositions.

We use c for the speed of light; $\beta := v/\mathsf{c}$, $\gamma := 1/\sqrt{1-\beta^2}$, rapidity $\tanh\eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

2 Phase: operational definition and physical meaning

We will systematically replace hyperbolic functions and nested square roots by the circular trigonometry of a single *phase angle* ϑ, interpreting $\cos\vartheta$ as the temporal projection and $\sin\vartheta$ as the spatial one. This keeps all kinematic identities while avoiding hyperbolic parametrization.

We define the *kinematic angle* $\vartheta \in [0, \frac{\pi}{2})$ of a system with respect to a fixed observer as an **operationally measurable split** of a constant "flow budget" c between temporal and spatial projections:

$$\boxed{\cos\vartheta \equiv \frac{d\tau}{dt}, \qquad \sin\vartheta \equiv \frac{\|\mathbf{v}\|}{c} = \beta, \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}} \tag{2.1}$$

where t is the observer's time, τ is the proper time, and $\mathbf{v}$ is the 3-velocity. The identity $\cos^2\vartheta + \sin^2\vartheta = 1$ then restates the empirical invariance of the Minkowski interval.

Phase state and quaternionic slice. An (inertial) phase state is the pair $(\vartheta, \mathbf{u})$, with $\mathbf{u} \in S^2$. We associate to it the unit quaternion

$$q(\vartheta, \mathbf{u}) = \cos\vartheta + \mathbf{I}_{\mathbf{u}} \sin\vartheta, \qquad \mathbf{I}_{\mathbf{u}} := u_x\,\mathbf{i} + u_y\,\mathbf{j} + u_z\,\mathbf{k}, \quad \|\mathbf{u}\| = 1, \tag{2.2}$$

that is, a complex slice $\mathbb{C}_{\mathbf{u}} := \mathrm{Span}\{1, \mathbf{I}_{\mathbf{u}}\} \subset \mathbb{H}$ aligned with $\mathbf{u}$. This is the minimal structure that linearly encodes collinear compositions and naturally induces the Wigner–Thomas rotation for non-collinear boosts via quaternion multiplication.

3 Mapping to standard SR variables

The phase angle ϑ is *not* the rapidity η; they are related by a Gudermann-type [1] bridge

$$\boxed{\beta = \tanh\eta = \sin\vartheta, \qquad \gamma = \cosh\eta = \sec\vartheta, \qquad \tan\frac{\vartheta}{2} = \tanh\frac{\eta}{2}}\,. \tag{3.1}$$

Consequently, all standard kinematic relations follow from circular trigonometry in ϑ:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sec\vartheta, \qquad k_{\|} = e^{\pm\eta} = \frac{1+\tan(\vartheta/2)}{1-\tan(\vartheta/2)}. \tag{3.2}$$

All collinear compositions reduce to angle addition inside the slice $\mathbb{C}_{\mathbf{u}}$.

Collinear composition. For $\mathbf{u}$ fixed,

$$q(\vartheta_2, \mathbf{u})\, q(\vartheta_1, \mathbf{u}) = q(\vartheta_1 \oplus \vartheta_2, \mathbf{u}), \qquad \cos(\vartheta_1 \oplus \vartheta_2) = \cos\vartheta_1 \cos\vartheta_2 - \sin\vartheta_1 \sin\vartheta_2, \tag{3.3}$$

which implies Einstein's velocity addition

$$\beta_{12} = \sin(\vartheta_1 \oplus \vartheta_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}. \tag{3.4}$$

Non-collinear composition and Wigner–Thomas rotation. For $\mathbf{u}_1 \neq \mathbf{u}_2$ one has the factorization

$$q(\vartheta_2, \mathbf{u}_2)\, q(\vartheta_1, \mathbf{u}_1) = R_W\; q(\vartheta_{12}, \mathbf{u}_{12}), \tag{3.5}$$

where $R_W \in \mathrm{SO}(3)$ is the Wigner–Thomas rotation (a genuine 3D rotation), while $q(\vartheta_{12}, \mathbf{u}_{12})$ is the effective boost in the slice $\mathbb{C}_{\mathbf{u}_{12}}$. The rotation angle and axis can be extracted from the vector part of the quaternion product; a closed expression equivalent to the standard formulas is provided in Appendix A.

4 SR–phase correspondences

Below is a minimal dictionary of correspondences between the hyperbolic SR picture and the circular phase picture.

Quantity	Standard SR (hyperbolic)	Phase picture (circular)
Rapidity	$\eta = \operatorname{artanh}\beta$	$\tanh\eta = \sin\vartheta$
Lorentz factor	$\gamma = \cosh\eta$	$\gamma = \sec\vartheta$
Speed	$\beta = \tanh\eta$	$\beta = \sin\vartheta$
Doppler (longitudinal)	$k = e^{\pm\eta}$	$k = \dfrac{1+\tan(\vartheta/2)}{1-\tan(\vartheta/2)}$
Temporal projection	$\operatorname{sech}\eta$	$\cos\vartheta$

Table 2: One-to-one correspondences between hyperbolic (rapidity) and circular (phase) parametrizations.

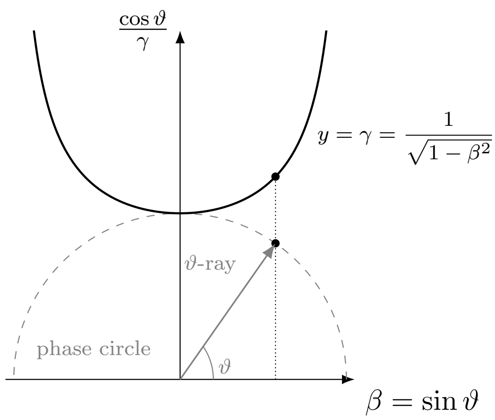

Figure 1: Phase circle vs. Lorentz hyperbola at common $\beta = \sin\vartheta$. Vertical mapping at fixed β illustrates the Gudermann bridge: $\gamma = \sec\vartheta = \cosh\eta$.

5 Time and space as phase derivatives

Why a complex slice of a quaternion? For local kinematics any unit direction $\hat{\mathbf{u}}$ singles out the two–dimensional subalgebra $\mathrm{Span}\{1, \hat{\mathbf{u}}\} \cong \mathbb{C} \subset \mathbb{H}$. Working in this complex *slice* preserves all boost/rotation algebra along $\hat{\mathbf{u}}$, but keeps formulas elementary. When the direction changes, one updates the slice; the full quaternionic structure is retained.

Let $\vec{\chi} \in \mathbb{C}$ be a variable whose change generates observable time-space effects. We treat the time and space units as directional derivatives (phase velocities) along the real and imaginary directions of a complex basis $(\hat{h}, \mathbf{l})$:

$$\hat{h}\, dx_0 = \frac{\partial \vec{\chi}}{\partial \chi_h} \frac{d\chi_h}{d\chi}\, d\chi = \tilde{H}\, d\chi, \qquad \mathbf{l}\, dx_l = \frac{\partial \vec{\chi}}{\partial \chi_l} \frac{d\chi_l}{d\chi}\, d\chi = \tilde{L}\, d\chi, \quad l = 1, 2, 3. \tag{5.1}$$

Introduce the phase speed of the SR interval $ds = \tilde{S}\, d\chi$. The interval conservation takes the form

$$\tilde{S}^2 = \frac{ds^2}{d\chi^2} = \frac{g_{ij}\, dx^i dx^j}{d\chi^2} = \left(\frac{\mathsf{c}^2 dt^2}{d\chi^2}\right) - \left[\frac{\mathbf{dx}^2}{d\chi^2}\right] = \left(\tilde{H}^2\right) - \left[\tilde{L}^2\right], \tag{5.2}$$

equivalently

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2. \tag{5.3}$$

Writing

$$\tilde{S} = \tilde{H} \cos\theta, \qquad \tilde{L} = \tilde{H} \sin\theta, \tag{5.4}$$

where θ is the angle of the phase speed relative to the real axis. Algebraically, (5.3) is a Euclidean decomposition of a single speed into orthogonal projections; physically, we will see that under reparameterization the *projection* $\tilde{S}$, not the Euclidean norm $\tilde{H}$, is the conserved Minkowski quantity.

Flow and phase 1-form. Let $\Phi : \mathcal{E} \to \mathbb{R}$ be a scalar *phase potential* on a (possibly infinite-dimensional) Euclidean/Hilbert proto-space $(\mathcal{E}, \langle \cdot, \cdot \rangle)$. Define the phase 1-form $\alpha := d\Phi$ and the associated *flow vector* $\boldsymbol{\chi} := \nabla\Phi$, where the gradient is taken with respect to $\langle \cdot, \cdot \rangle$.

Fix an observer's orthonormal spatial triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathcal{E}$ and let $S = \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with orthogonal projectors P_S and $P_{S^\perp}$. Decompose

$$\boldsymbol{\chi} = \boldsymbol{\chi}_S + \boldsymbol{\chi}_\perp, \qquad \boldsymbol{\chi}_S := P_S \boldsymbol{\chi}, \quad \boldsymbol{\chi}_\perp := P_{S^\perp} \boldsymbol{\chi}.$$

Define observable spatial components and the orthogonal magnitude

$$\ell_i := \langle \boldsymbol{\chi}, \mathbf{e}_i \rangle, \qquad \mathbf{l} := \sum_{i=1}^{3} \ell_i\, \mathbf{e}_i, \qquad t := \|\boldsymbol{\chi}_\perp\| = \sqrt{\|\boldsymbol{\chi}\|^2 - \|\boldsymbol{\chi}_S\|^2},$$

and, for orientation when $t > 0$, the unit direction $\mathbf{e}_t := \boldsymbol{\chi}_\perp / \|\boldsymbol{\chi}_\perp\|$. Then the phase angle ϑ and the direction $\mathbf{u}$ used throughout this paper are recovered as

$$\cos\vartheta = \frac{t}{\|\boldsymbol{\chi}\|}, \qquad \sin\vartheta = \frac{\|\mathbf{l}\|}{\|\boldsymbol{\chi}\|}, \qquad \mathbf{u} = \frac{\mathbf{l}}{\|\mathbf{l}\|} \quad (\|\mathbf{l}\| > 0).$$

This complements the operational definition (2.1) and ties the phase picture to a differential-form language.

6 Phase space

Let the phase vector space be $\mathbb{H}$ with orthonormal basis $(\hat{h}, \mathbf{l})$. For a phase vector $\vec{\chi} = R\, e^{\theta \mathbf{l}}$ with $\theta \in [-\pi, \pi]$,

$$\tilde{H} = R, \qquad \tilde{S} = R\cos\theta, \qquad \tilde{L} = R\,\mathbf{l}\sin\theta. \tag{6.1}$$

Choosing coordinates where the projectors onto $(\hat{h}, \mathbf{l})$ are unit, (5.1) simplifies to

$$\hat{h}\, dx_0 = \frac{d\chi_h}{d\chi}\, d\chi = \tilde{H}\, d\chi, \qquad \mathbf{l}\, dx_l = \frac{d\chi_l}{d\chi}\, d\chi = \tilde{L}\, d\chi. \tag{6.2}$$

The map from phase to observables is an integral transform:

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}^i(u)\, du, \qquad i = 0, 1, 2, 3, \tag{6.3}$$

where $\tilde{X}^i$ are projections of $d\vec{\chi}/d\chi$ onto $(\hat{h}, \mathbf{l})$ and $x^i(\chi_0)$ fix initial conditions.

7 Objects

Roadmap. The next formulas fix notation and the geometric carriers we use throughout. In particular, the phase state $(\vartheta, \mathbf{u})$ selects a complex slice $\mathbb{C}_{\mathbf{u}} \subset \mathbb{H}$; collinear compositions become ordinary circular sums on this slice, while non-collinear compositions generate a genuine 3D rotation (Wigner–Thomas) via quaternion multiplication. This explains why we keep both ϑ and $\mathbf{u}$ as primary objects.

A fundamental particle is an elementary object with nonzero phase $\vec{\chi} \neq 0$. Composite objects are phase configurations; to represent them in phase space one may require additional dimensions, except for the photon, whose phase is always aligned with the imaginary axis:

$$\mathbf{p} = \frac{d\vec{\chi}}{d\chi_l} = p\,\mathbf{l} \in \Im. \tag{7.1}$$

Non-photonic phenomena are associated with nonzero real projection and nonzero mass. A complex object can be identified with an event or worldline; the photon corresponds to a null-interval point encoding information about the event.

Any object's phase can be rotated to the *zero* (purely real) direction,

$$\vec{\chi}_0 = R \in \Re. \tag{7.2}$$

An object A moving with speed v relative to a rest observer has

$$\vec{\chi}_A = R\, e^{\theta_A \mathbf{l}}, \qquad \sin\theta_A = \frac{v}{\mathsf{c}} \equiv \beta. \tag{7.3}$$

From unit norm to the interval. We will repeatedly use that $cd\tau = c\,dt\cos\vartheta$ and $d\mathbf{x} = c\,dt\,\sin\vartheta\,\mathbf{u}$. Thus the identity $\cos^2\vartheta + \sin^2\vartheta = 1$ is exactly the Minkowski metric statement $(cd\tau)^2 = (cdt)^2 - d\mathbf{x}^2$; from now on, square-root expressions are traded for circular trigonometry in ϑ.

7.1 Space as a symmetric phase pair

From (5.4), a naive zero-angle limit would remove the imaginary projection, contradicting observability. We enforce a nonvanishing spatial projection by pairing opposite-phase tilts:

$$\vec{\chi}^{\pm} = R\, e^{\pm\zeta\mathbf{l}}, \qquad \vec{\chi}_l := \frac{\vec{\chi}^{+} - \vec{\chi}^{-}}{2} = R\,\mathbf{l}\sin\zeta, \tag{7.4}$$

where ζ is an *internal angle* (intrinsic to the object; heuristically linked to mass/density). The local decomposition is

$$\vec{\chi}_0 = \vec{\chi}_\tau + \vec{\chi}_l = R\cos\zeta + R\,\mathbf{l}\sin\zeta, \tag{7.5}$$

with unit components (normalized by R): the real component is $\cos\zeta$ and the imaginary component is $\sin\zeta$.

7.2 Absolute, local, and observed time

Define *absolute* time $t = t(\tilde{H})$ at the zero phase direction; it is the fastest clock and useful for normalization between different phase speeds. Along the local real direction,

$$dx_0 = \frac{d}{d\chi}\Re(\vec{\chi})\,d\chi = \frac{\vec{\chi}^+ + \vec{\chi}^-}{2}\,d\chi = \cos\zeta\,d\chi =: d\tau. \tag{7.6}$$

Here $d\chi_0 := \cos\zeta\,d\chi$ is the projection of $d\chi$ onto the local real axis; in Sec. 7.3 we calibrate $d\tau = (1/\nu_0)\,d\chi_0$. The observed proper time of A relative to the rest observer is

$$\tilde{H}_A = \Re\!\left(\frac{d\vec{\chi}_A}{d\vec{\chi}_0}\right) = \cos\theta_A = \sqrt{1-\sin^2\theta_A} = \sqrt{1-\frac{v^2}{\mathsf{c}^2}} = \frac{1}{\gamma}. \tag{7.7}$$

7.3 Normalization

Calibration. We fix the calibration by the observer's clock and speed budget: $\cos\vartheta \equiv d\tau/dt$ and $\sin\vartheta \equiv \beta$. This choice does not restrict generality: any overall rescaling of the underlying flow is absorbed into the definitions of t and c, leaving all dimensionless observables unchanged.

Let local time be parameterized by *phase*; introduce a reference frequency ν_0 and set

$$d\tau = \frac{1}{\nu_0}\,d\chi_0. \tag{7.8}$$

By the chain rule,

$$dx_0 = \tilde{H}\,d\chi = \frac{dx_0}{d\chi_0}\frac{d\chi_0}{d\tau}\,d\tau = \tilde{H}\,\dot{\chi}\,d\tau =: \dot{H}\,d\tau, \tag{7.9}$$

where $\nu := d\chi/d\tau$, $\dot{\chi} := \nu/\nu_0$, and $\dot{H} := \tilde{H}\,\dot{\chi}$. Choosing the calibration $\dot{H} \equiv \mathsf{c}$ gives $dx_0 = \mathsf{c}\,d\tau$. Similarly for space,

$$dx_l = \tilde{L}\,d\chi = \frac{dx_l}{d\chi_0}\frac{d\chi_0}{dl}\,dl = \tilde{L}\,\chi'\,dl =: L'\,dl, \qquad \chi' := \frac{d\chi}{dl}. \tag{7.10}$$

From $dx_0 = dx_l$ for light one gets

$$\mathsf{c} = \tilde{L}'\,\frac{dl}{d\tau}, \tag{7.11}$$

hence with temporal calibration to c the spatial scale becomes unit: $\tilde{L}' = 1$.

7.4 Light and c as a calibration constant

From the normalized forms,

$$\frac{\mathsf{c}}{\dot{\chi}}\,d\chi = \frac{1}{\chi'}\,d\chi \quad\Rightarrow\quad \mathsf{c} = \frac{\dot{\chi}}{\chi'} = \frac{dl}{d\tau}, \tag{7.12}$$

i.e. c is a *calibration constant* tying temporal and spatial measures, independent of local phase variation. Equation (7.12) also reads

$$\mathsf{c} = \left(\frac{d\chi}{d\tau}\right)\left[\frac{dl}{d\chi}\right] \sim (\nu)\,[\lambda], \tag{7.13}$$

matching frequency and wavelength of a photon, with χ as its phase. For a lightlike trajectory,

$$ds^2 = \mathsf{c}^2\left(\frac{d\chi^2}{\dot{\chi}^2} - \frac{d\chi^2}{\dot{\chi}^2}\right) = 0. \tag{7.14}$$

At unit frequency, $\tau = \chi$: the photon's "proper time" is its phase, and the length of its phase-speed vector equals its wavelength, $\tilde{H}_p = \lambda$. Finally, the kinematic slope in phase coordinates is

$$\frac{dx_l}{dx_0} = \frac{\tilde{L}\,d\chi}{\tilde{H}\,d\chi} = \sin\theta = \frac{v}{\mathsf{c}} \equiv \beta, \tag{7.15}$$

so $\theta = \pi/2$ implies $v = \mathsf{c}$.

7.5 Lorentz factor via reparameterization

A change of direction of the phase speed transforms

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2 \longmapsto \dot{H}^2 = \dot{S}^2 + \dot{L}^2. \tag{7.16}$$

Lemma (parameter-change identity). The transition $\tilde{H} \to \dot{S}$ is the manifestation of evolving phase speed under the parameter change $\chi \mapsto \tau(\chi)$, with local Jacobian

$$\frac{d\tau}{d\chi} = \cos\zeta(\chi)\cos\theta(\chi) \quad \Rightarrow \quad \mathcal{J}(\zeta,\theta) := \frac{d\chi}{d\tau} = \frac{1}{\cos\zeta\,\cos\theta}. \tag{7.17}$$

Then

$$\dot{H} = \tilde{H}\,\mathcal{J}, \qquad \dot{L} = \tilde{L}\,\mathcal{J}. \tag{7.18}$$

In differential form,

$$d\ln\dot{H} = d\ln\mathcal{J} = \tan\zeta\,d\zeta + \tan\theta\,d\theta. \tag{7.19}$$

For a *pure boost* ($d\zeta = 0$) one has $d\dot{H} = \dot{H}\tan\theta\,d\theta$. Absorbing a constant $\cos\zeta$ into the calibration (set $\zeta = 0$ henceforth), we obtain

$$\tilde{H}^2 = \dot{H}^2 - \dot{L}^2 = \sec^2\theta\,(\tilde{H}^2 - \tilde{L}^2) = \gamma^2(\tilde{H}^2 - \tilde{L}^2). \tag{7.20}$$

Corollary. In phase space the Euclidean norm $\tilde{H}$ is conserved; in observed time the Minkowski norm $\dot{S}$ is conserved; they are identical as quantities:

$$\boxed{\tilde{H} = \dot{S}}. \tag{7.21}$$

7.6 Rapidity and the phase angle

By definition,

$$\beta = \frac{v}{\mathsf{c}} = \sin\theta, \qquad \tanh\eta = \beta, \qquad d\eta = \frac{d\beta}{1-\beta^2}. \tag{7.22}$$

With $d\beta = \cos\theta\,d\theta$ and $1 - \beta^2 = \cos^2\theta$,

$$d\eta = \sec\theta\,d\theta, \qquad \eta(\theta) = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \tfrac{1}{2}\ln\frac{|1+\sin\theta|}{|1-\sin\theta|}. \tag{7.23}$$

Fixing $\eta(0) = 0$,

$$e^{\eta(\theta)} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \sec\theta = \cosh\eta. \tag{7.24}$$

Remark (groups). Observables satisfy $\beta = \sin\theta = \tanh\eta$ and $\gamma = \sec\theta = \cosh\eta$. Thus Euclidean rotations in the phase circle ($U(1)$ with angle θ) reproduce the numerical factors of hyperbolic boosts in $SO^+(1,1)$ (rapidity η) *after* reparameterizing time. We do not claim an isomorphism $U(1) \cong SO(1,1)$; only the equality of observable combinations under the change of parameter.

7.7 Velocity addition

Notation. In unimetry, an inertial boost is a *D-rotation*

$$\mathcal{B}(\hat{\mathbf{u}}, \psi): \quad \mathbf{q} \mapsto d\,\mathbf{q}\,d, \qquad d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\sin\frac{\psi}{2}, \tag{7.25}$$

and a spatial rotation is an *R-rotation*

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi): \quad \mathbf{q} \mapsto r\,\mathbf{q}\,r^{-1}, \qquad r = \cos\frac{\varphi}{2} + \hat{\mathbf{n}}\sin\frac{\varphi}{2}. \tag{7.26}$$

Kinematic mapping: $\beta \equiv v/c = \sin\psi$, $\gamma = 1/\cos\psi$, $\tan\dfrac{\psi}{2} = \dfrac{\gamma\beta}{\gamma+1}$. For quaternionic/GA accounts of rotors and Lorentz boosts see [5, 6, 7].

7.7.1 Wigner rotation

Let d_1, d_2 be D-rotors of two successive boosts. The raw action on any unimetry 4-object is

$$\mathbf{q}' = d_2 d_1\,\mathbf{q}\,d_1 d_2 \equiv L_{12}\,\mathbf{q}\,L_{21}, \qquad L_{12} = d_2 d_1, \quad L_{21} = d_1 d_2. \tag{7.27}$$

Define d_{12} to be the unique D-rotor reproducing the combined spatio–temporal tilt of L_{12}:

$$\boxed{d_{12}\,\mathbf{e}_t\,d_{12} \;=\; L_{12}\,\mathbf{e}_t\,L_{21}, \qquad \Re(d_{12}) \geq 0} \tag{7.28}$$

(the sign choice removes the trivial two-fold ambiguity). Then the *Wigner rotor* is the residual R-rotation in the symmetric D–R factorization:

$$\boxed{L_{12} = d_{12}\,r_W, \qquad L_{21} = r_W^{-1}\,d_{12}} \tag{7.29}$$

equivalently,

$$\boxed{r_W = \bar{d}_{12}\,L_{12} = L_{21}\,\bar{d}_{12}}. \tag{7.30}$$

Hence the observed map after compensating the tilt is $\bar{d}_{12}\,\mathbf{q}'\,\bar{d}_{12} = r_W\,\mathbf{q}\,r_W^{-1}$.

$$\begin{tikzcd}
\mathbf{q} \arrow[r, "d_1"] \arrow[rr, dashed, bend left, "d_{12}"] & d_1\,\mathbf{q}\,d_1 \arrow[r, "d_2"] & d_2 d_1\,\mathbf{q}\,d_1 d_2 \arrow[d, "\bar{d}_{12}\text{ on both sides}"] \\
 & & r_W\,\mathbf{q}\,r_W^{-1}
\end{tikzcd}$$

Figure 2: Two successive D-rotations (boosts) and compensation of the net spatio–temporal angle by the conjugate of d_{12}, leaving a pure R-rotation r_W.

7.7.2 Thomas precession

The continuous limit of Wigner rotation for a time-dependent velocity direction $\hat{\mathbf{u}}(t)$ yields

$$\boldsymbol{\omega}_T = (\gamma - 1)\left(\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}\right) = \frac{\gamma^2}{\gamma + 1}\,\frac{\mathbf{a} \times \mathbf{v}}{c^2}, \qquad \gamma = \frac{1}{\cos\psi}. \tag{7.31}$$

For uniform circular motion ($|\mathbf{v}| = \text{const}$) with $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega} \times \hat{\mathbf{u}}$ one has $|\boldsymbol{\omega}_T| = (\gamma - 1)\,\Omega$.

7.8 Doppler shift

Define the observed frequency as the phase growth rate in the observer's proper time:

$$\nu := \frac{d\chi}{d\tau}. \tag{7.32}$$

For two successive wavefronts the phase increment is identical, hence

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{d\chi/d\tau_{\text{obs}}}{d\chi/d\tau_{\text{src}}} = \frac{d\tau_{\text{src}}}{d\tau_{\text{obs}}}. \tag{7.33}$$

Longitudinal case: during $\gamma\, d\tau_{\text{src}}$ in the observer frame the source displaces by $\pm v\,\gamma\, d\tau_{\text{src}}$ ("+" receding, "−" approaching). Then

$$d\tau_{\text{obs}} = \gamma\, d\tau_{\text{src}}(1 \pm \beta), \qquad \Rightarrow \qquad \boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 \pm \beta)}}. \tag{7.34}$$

Equivalent forms (with $\beta = \sin\theta$, $\gamma = \sec\theta$ and rapidity η):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec\theta\,(1 \mp \sin\theta) = e^{\mp\eta}. \tag{7.35}$$

Transverse Doppler ($\varphi = 90^\circ$ in the observer's frame):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma} = \cos\theta. \tag{7.36}$$

General line-of-sight (LOS) angle φ in the observer's frame:

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \gamma\,(1 - \beta\cos\varphi)}. \tag{7.37}$$

Wavelength ratios are inverse to frequency ratios.

8 Gravity as a phase rotation: local tetrads and clock angle

On a curved background $(\mathcal{M}, g)$ we work with orthonormal tetrads $e_a{}^\mu$ such that $g_{\mu\nu}e_a{}^\mu e_b{}^\nu = \eta_{ab}$ and take the observer's time leg to be $e_0{}^\mu$. We introduce the *gravitational (clock) angle* φ by

$$\cos\varphi \;:=\; e^0{}_\mu\, u^\mu \;=\; \frac{d\tau_{\text{stat}}}{dt} \;=\; \sqrt{-g_{00}} \qquad \text{(stationary case)}. \tag{8.1}$$

Kinematics remains encoded by the phase angle ϑ in the slice $\mathbb{C}_{\mathbf{u}}$, with $\cos\vartheta = d\tau/d\tau_{\text{stat}}$. Therefore the proper time factorizes as

$$d\tau \;=\; dt\,\cos\varphi\,\cos\vartheta, \qquad d\mathbf{x} \;=\; c\,dt\,\sin\vartheta\,\mathbf{u}, \tag{8.2}$$

and the total redshift factorizes into kinematic and gravitational parts:

$$1 + z_{\text{tot}} \;=\; \frac{\cos\vartheta_{\text{em}}}{\cos\vartheta_{\text{obs}}} \times \frac{\cos\varphi(x_{\text{em}})}{\cos\varphi(x_{\text{obs}})}. \tag{8.3}$$

For static emitter/observer ($\vartheta_{\text{em}} = \vartheta_{\text{obs}} = 0$) one recovers the standard gravitational redshift $1 + z_g = \sqrt{\frac{-g_{00}(x_{\text{obs}})}{-g_{00}(x_{\text{em}})}}$. In the weak-field limit $g_{00} \simeq -(1 + 2\Phi/c^2)$ this gives $z_g \simeq (\Phi_{\text{obs}} - \Phi_{\text{em}})/c^2$.

Beyond static fields. In a 3+1 split $ds^2 = -N^2c^2dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$ one may identify $\cos\varphi := N$ in the observer's tetrad, which keeps (**??**)–(**??**) coordinate-agnostic. Uniform acceleration (Rindler) accumulates phase according to $d\vartheta = \kappa\, dt$ inside the local slice, consistently reproducing accelerated-frame kinematics when combined with the clock angle φ.

9 Discussion: links to known structures

Gauge phases. A global shift $\chi \mapsto \chi + \chi_0$ is unobservable. Allowing local reparameterizations $\chi \mapsto \chi + \alpha(x)$ induces a connection when comparing phases at different points. On wavefunctions $\psi \sim e^{i\chi}$ this is the familiar $U(1)$ gauge freedom $\psi \to e^{i\alpha(x)}\psi$ with $D_\mu = \partial_\mu - iA_\mu$ as the *phase-transport connection.*

Mass and the internal angle. With the decomposition by ζ, mass heuristically correlates with an irreducible real projection: massless objects have $\zeta = \pm\pi/2$ (no proper time; photon subspace), while massive objects have $|\zeta| < \pi/2$ (proper time exists). In the present paper we set $\zeta = 0$ in boost kinematics by calibration; a detailed mass-generation mechanism is left for future work.

Cosmological gauge. A natural global calibration of "absolute" time is the comoving frame with vanishing CMB dipole. This fixes a cosmological time t (FLRW) as a gauge, without affecting local Lorentz invariance; Doppler factors are then operationally referenced to that frame.

10 Conclusion

In unimetry, time and space are integrals of phase velocities; the Minkowski interval appears as a conserved quantity under parameter change. The core relations of SR—γ, rapidity, velocity addition, and Doppler factors—follow from elementary phase-plane geometry with a single rotation angle θ, while hyperbolic structure re-emerges upon reparameterizing time. The formalism is empirically equivalent to standard SR but can clarify causality and composition by treating all effects as projections of a single flow.

Outlook. Future directions include (i) a more explicit group-theoretic embedding, (ii) a rigorous treatment of the internal angle ζ and its relation to mass, and (iii) exploration of curved metrics as spatially varying Jacobians $\mathcal{J}(x)$ in the phase-to-observable map.

References

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A Equivalence to the classical Wigner rotation

We sketch an intrinsic quaternionic proof that the unimetry expression for the Wigner rotation coincides with the standard special-relativistic formula.

Step 1: product of two D-rotors. For $d_i = \cos\frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin\frac{\psi_i}{2}$,

$$d_2 d_1 = (c_2 c_1 - s_2 s_1 \cos\theta) + \Big(c_2 s_1 \hat{\mathbf{u}}_1 + s_2 c_1 \hat{\mathbf{u}}_2 + s_2 s_1\, \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1\Big), \tag{A.1}$$

with $c_i = \cos(\psi_i/2)$, $s_i = \sin(\psi_i/2)$ and $\cos\theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1$.

Step 2: symmetric D–R factorization. Define d_{12} by $d_{12}\, \mathbf{e}_t\, d_{12} = L_{12}\, \mathbf{e}_t\, L_{21}$ and set $r_W = \bar{d}_{12}\, L_{12} = L_{21}\, \bar{d}_{12}$. Then r_W fixes $\mathbf{e}_t$ and is a pure spatial rotor, so $r_W = \cos\frac{\phi}{2} + \hat{\mathbf{n}} \sin\frac{\phi}{2}$ with $\hat{\mathbf{n}} \parallel \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. Matching scalar and bivector parts gives

$$\tan\frac{\phi}{2} = \frac{s_1 s_2\, \sin\theta}{c_1 c_2 + s_1 s_2\, \cos\theta}. \tag{A.2}$$

Step 3: map to rapidities. With the substitutions $\sin(\psi/2) \mapsto \sinh(\eta/2)$, $\cos(\psi/2) \mapsto \cosh(\eta/2)$, $\tan(\psi/2) \mapsto \tanh(\eta/2)$ (where $\tanh\eta = \beta$, $\cosh\eta = \gamma$), (A.2) becomes the textbook Wigner angle:

$$\tan\frac{\phi}{2} = \frac{\sinh\frac{\eta_1}{2}\, \sinh\frac{\eta_2}{2}\, \sin\theta}{\cosh\frac{\eta_1}{2}\, \cosh\frac{\eta_2}{2} + \sinh\frac{\eta_1}{2}\, \sinh\frac{\eta_2}{2}\, \cos\theta}, \tag{A.3}$$

with axis along $\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. This circular–hyperbolic correspondence is classical; cf. Gudermann [1].