Unimetry: A Quaternionic Gravito–Electromagnetic Formulation

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Abstract

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2.1 Context and motivation

Unimetry is a proposed phase–geometric framework in which physical systems are described in terms of stationary flows in an underlying Euclidean proto-space $\mathcal{E}$. Rather than postulating space–time as a primary arena, unimetry treats the observed space–time geometry, relativistic kinematics, and field interactions as effective structures derived from the orientation and coupling of such flows. A dimensionless scalar phase potential $\Phi : \mathcal{E} \to \mathbb{R}$ and its gradient define a normalized flow direction; the familiar Minkowski metric and Lorentzian phenomena then appear as particular projections of this underlying flow geometry.

In this sense, special relativity (SR) is not the endpoint, but the first benchmark for the framework: unimetry aims at a unified phase-based description of kinematics, gravity and gauge interactions, with SR recovered as a specific limit of the general construction. The present paper develops one important sector of this programme, namely a quaternionic gravito–electromagnetic (GEM) formulation built on top of the unimetrical flow picture.

At the classical level, gravito–electromagnetic analogies are well known: in the weak-field, slow-motion limit of general relativity, the Einstein equations can be cast into a Maxwell-like form, and moving masses generate a "gravitomagnetic" field. Quaternions and related algebras have also long been used to encode rotations and the Maxwell equations in a compact way. What unimetry adds to this landscape is a concrete phase-geometric interpretation: a single quaternionic object encodes both the temporal and spatial parts of a flow, and bilinear forms of such objects naturally split into scalar, symmetric vector, and axial (vorticity-like) channels. This suggests that gravity and electromagnetism might be viewed as different faces of the same bilinear structure acting on suitably dressed flow quaternions.

Our goal here is to make this statement precise. We construct a quaternionic GEM formalism in which gravitational and electromagnetic interactions originate from the *same* bilinear machinery applied to metrically dressed "body quaternions". In particular, we show that Newton and Coulomb potentials arise as two branches of a single scalar form, while the magnetic and gravitomagnetic sectors are associated with a vortical bilinear form whose physical calibration reveals a natural role for the constants ε_0, μ_0, G and c. The resulting description remains Euclidean at the level of the proto-space, yet reproduces relativistic kinematics and GEM fields in the observable three-space.

2.2 Relation to the base unimetry paper

This work is a direct sequel to the base unimetry paper, *"Unimetry: A Phase-Space Reformulation of Special Relativity"* (henceforth "Paper I"). Paper I develops the core phase/flow structure: the phase potential Φ, the phase 1-form $\alpha = d\Phi$, the normalized flow $\widehat{\boldsymbol{\chi}}$, and the calibration $\boldsymbol{\chi} = c\,\widehat{\boldsymbol{\chi}}$, together with the derivation of the Minkowski interval and standard SR effects from a Euclidean proto-space. It also introduces the unimetrical D-rotation, which encodes Lorentz boosts as Euclidean rotations in a suitable plane of $\mathcal{E}$.

From the unimetry viewpoint, however, these SR results are only the first consistency test of a broader phase-based paradigm. The present paper assumes familiarity with the conceptual setting of Paper I, but is written to be as self-contained as reasonably possible. We briefly recall the key definitions of the phase proto-space, the flow vector, and the two calibrations of the flow that lead to kinematic and energetic interpretations. All constructions that are essential for the GEM sector are reproduced or adapted here; more detailed discussions of SR and cosmological applications remain in Paper I and are only referenced when needed.

2.3 Main results

The main technical contributions of this paper can be summarized as follows.

- We introduce *metrically dressed body quaternions* $\widetilde{\mathsf{q}}_i = L_{E,i}\,\hat{h} + L_{G,i}\,\widehat{\boldsymbol{n}}_i$, whose components have the dimension of length. The "electric" and "gravitational" lengths

$$L_{E,i} = \sqrt{\frac{G}{4\pi\varepsilon_0 c^4}}\,Q_i, \qquad L_{G,i} = \frac{G}{c^2}\,m_i$$

encode the charge Q_i and mass m_i of the body in a unified geometric fashion. The unit vector $\widehat{\boldsymbol{n}}_i$ represents the spatial flow direction associated with the body.

- We show that the scalar bilinear form

$$A(\widetilde{\mathsf{q}}_1, \widetilde{\mathsf{q}}_2) = L_{E,1}L_{E,2} - \boldsymbol{S}_1 \cdot \boldsymbol{S}_2$$

  (with $\boldsymbol{S}_i = L_{G,i}\widehat{\boldsymbol{n}}_i$) yields, after a single global calibration by c^4/G and a geometric $1/r$ factor, the combined Newton–Coulomb potential:

$$U(r) = \frac{c^4}{G}\,\frac{A}{r} = \frac{1}{4\pi\varepsilon_0}\,\frac{Q_1Q_2}{r} - G\,\frac{m_1m_2}{r}.$$

  Thus gravity and electrostatics arise as two channels of a single invariant scalar form.

- We identify two vector-valued bilinear forms, $\boldsymbol{B}(\widetilde{\mathsf{q}}_1, \widetilde{\mathsf{q}}_2)$ and $\boldsymbol{C}(\widetilde{\mathsf{q}}_1, \widetilde{\mathsf{q}}_2)$, corresponding to the symmetric and axial parts of the quaternion product. In the dressed setting these naturally describe current-like and vortical channels. In particular, the vortical form $\boldsymbol{C}$ reproduces the geometry of magnetic and gravitomagnetic fields generated by moving charges and masses.

- We construct a quaternionic GEM field $\mathcal{F}_{\mathrm{GEM}}(\boldsymbol{x})$ over the observable three-space by combining dressed source quaternions with purely imaginary distance quaternions. Its scalar channel reproduces the gravitational and electrostatic potentials, while its vortical channel yields a physically natural "phase-vortical" field C_{phys} with the same dimension as $\boldsymbol{E}$. The standard magnetic field $\boldsymbol{B}$ in SI units then appears as

$$\boldsymbol{B} = \frac{1}{c}\,C_{\mathrm{phys}},$$

  so that the familiar μ_0 and ε_0 can be interpreted in terms of linear and areal stiffness of the vacuum, combined into an effective volumetric stiffness proportional to $1/(\varepsilon_0 c^3)$.

- We analyze the action of unimetrical D-rotations and ordinary spatial rotors on dressed quaternions. Pure spatial rotations act in the usual way on the vector channels and leave the scalar form A invariant, while D-rotations mix the scalar channel and the longitudinal component of $\boldsymbol{B}$ in a two-dimensional "energy–current" plane. This provides a quaternionic encoding of relativistic kinematics in the GEM setting, with Lorentz-consistent transformation properties of the fields.

- Finally, we outline a Hamiltonian and Lagrangian formulation of the quaternionic GEM theory in terms of the self-form A and the norm-squares of $\boldsymbol{B}$ and $\boldsymbol{C}$, and discuss how the standard Maxwell Lagrangian and linearized GEM equations arise in appropriate limits.

2.4 Structure of the paper

The paper is organized as follows. In Section 3 we recall the basic quaternion algebra and introduce the bilinear forms A, $\boldsymbol{B}$, and $\boldsymbol{C}$ that arise from the quaternion product, together with their matrix representation and geometric interpretation. Section 4 provides a brief overview of the unimetrical phase proto-space, the phase potential, the flow vector, and the two calibrations of the flow that lead to kinematic and energetic interpretations.

In **??** we introduce metrically dressed body quaternions and define the electric and gravitational lengths L_E and L_G. **??** shows how the scalar form A for dressed quaternions reproduces the static Newton and Coulomb potentials. In **??** we construct a quaternionic GEM field over the observable three-space and identify the scalar and vortical channels with gravitational, electric, and magnetic sectors.

?? analyzes the action of spatial rotors and D-rotors on dressed quaternions and on the GEM field, clarifying the relativistic transformation properties of the scalar, current-like, and vortical channels. **??** is devoted to the calibration of $\boldsymbol{E}$ and $\boldsymbol{B}$, to the definition of the phase-vortical field C_{phys}, and to the interpretation of ε_0, μ_0, and c in terms of vacuum stiffness.

In **??** we outline Hamiltonian and Lagrangian formulations of quaternionic GEM, and in **??** we compare the resulting equations with the standard Maxwell and linearized GEM formalisms. Finally, **??** discusses limitations and open questions, and sketches possible extensions towards non-Abelian interactions and cosmological applications.

3 Quaternion algebra and bilinear forms

3.1 Basic notation and conventions

We denote by $\mathbb{H}$ the real quaternion algebra, viewed as a four-dimensional real vector space

$$\mathbb{H} \simeq \mathbb{R}^4(\hat{h}, \hat{\imath}, \hat{\jmath}, \hat{k}),$$

where $\hat{h}$ is the distinguished scalar basis element and $\hat{\imath}, \hat{\jmath}, \hat{k}$ are purely imaginary basis elements. A general quaternion is written as

$$\mathbf{q} = x^\mu e_\mu = x^0\hat{h} + x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k},$$

with real components $x^\mu \in \mathbb{R}$ and basis $e_0 := \hat{h}$, $e_1 := \hat{\imath}$, $e_2 := \hat{\jmath}$, $e_3 := \hat{k}$.

The imaginary basis satisfies the usual quaternion relations

$$\hat{\imath}^2 = \hat{\jmath}^2 = \hat{k}^2 = -\hat{h}, \qquad \hat{\imath}\hat{\jmath} = \hat{k}, \quad \hat{\jmath}\hat{k} = \hat{\imath}, \quad \hat{k}\hat{\imath} = \hat{\jmath},$$

with antisymmetry under exchange of factors. We identify $\operatorname{Im}\mathbb{H} \simeq \mathbb{R}^3$ with its Euclidean inner product $\boldsymbol{x} \cdot \boldsymbol{y}$ and cross product $\boldsymbol{x} \times \boldsymbol{y}$, so that

$$\boldsymbol{x} = x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k} \quad \longleftrightarrow \quad (x^1, x^2, x^3) \in \mathbb{R}^3.$$

Quaternionic conjugation is defined by

$$\overline{\mathbf{q}} := x^0\hat{h} - x^1\hat{\imath} - x^2\hat{\jmath} - x^3\hat{k},$$

and the norm is $\|\mathbf{q}\|^2 = \mathbf{q}\,\overline{\mathbf{q}} = \overline{\mathbf{q}}\,\mathbf{q} = (x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$.

When convenient we will write a quaternion as $\mathbf{q} = (T, \boldsymbol{S})$ with

$$T := x^0, \qquad \boldsymbol{S} := x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k},$$

emphasizing the split into scalar and vector parts.

3.2 Quaternion product and decomposition into A, B, C forms

Let

$$\begin{aligned}
\mathbf{q}_1 &= x^\mu e_\mu = x^0\hat{h} + x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k}, \\
\mathbf{q}_2 &= y^\nu e_\nu = y^0\hat{h} + y^1\hat{\imath} + y^2\hat{\jmath} + y^3\hat{k},
\end{aligned}$$

with $\mathbf{q}_1, \mathbf{q}_2 \in \mathbb{H}$. Their quaternion product can be expanded term by term as

$$\begin{aligned}
\mathbf{q}_1 \circ \mathbf{q}_2 &= (x^0\hat{h} + x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k})(y^0\hat{h} + y^1\hat{\imath} + y^2\hat{\jmath} + y^3\hat{k}) \\
&= (x^0y^0\hat{h}^2 + x^1y^1\hat{\imath}^2 + x^2y^2\hat{\jmath}^2 + x^3y^3\hat{k}^2) \\
&\quad + (x^0y^1\hat{h}\hat{\imath} + x^0y^2\hat{h}\hat{\jmath} + x^0y^3\hat{h}\hat{k} + x^1y^0\hat{\imath}\hat{h} + x^2y^0\hat{\jmath}\hat{h} + x^3y^0\hat{k}\hat{h}) \\
&\quad + (x^1y^2\hat{\imath}\hat{\jmath} + x^1y^3\hat{\imath}\hat{k} + x^2y^1\hat{\jmath}\hat{\imath} + x^2y^3\hat{\jmath}\hat{k} + x^3y^1\hat{k}\hat{\imath} + x^3y^2\hat{k}\hat{\jmath}).
\end{aligned} \tag{1}$$

Using the multiplication rules, this can be organised into three bilinear contributions:

$$\mathbf{q}_1 \circ \mathbf{q}_2 = (\mathbf{q}_1 * \mathbf{q}_2) + (\mathbf{q}_1 \diamond \mathbf{q}_2) + (\mathbf{q}_1 \times \mathbf{q}_2), \tag{2}$$

where:

- $\mathbf{q}_1 * \mathbf{q}_2$ collects the purely scalar terms,
- $\mathbf{q}_1 \diamond \mathbf{q}_2$ collects the mixed scalar–vector terms,
- $\mathbf{q}_1 \times \mathbf{q}_2$ collects the purely vector–vector terms.

Explicitly, one finds the familiar invariant decomposition

$$\begin{aligned} \mathbf{q}_1 \circ \mathbf{q}_2 = & \left(x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3\right)\hat{h} \\ & + (x^0\boldsymbol{y} + y^0\boldsymbol{x}) + (\boldsymbol{x} \times \boldsymbol{y}), \end{aligned} \tag{3}$$

where

$$\boldsymbol{x} := x^1\hat{\imath} + x^2\hat{\jmath} + x^3\hat{k}, \qquad \boldsymbol{y} := y^1\hat{\imath} + y^2\hat{\jmath} + y^3\hat{k}.$$

This suggests three natural bilinear maps:

$$A(\mathbf{q}_1, \mathbf{q}_2) := x^0y^0 - x^1y^1 - x^2y^2 - x^3y^3, \tag{4}$$

$$\boldsymbol{B}(\mathbf{q}_1, \mathbf{q}_2) := x^0\boldsymbol{y} + y^0\boldsymbol{x}, \tag{5}$$

$$\boldsymbol{C}(\mathbf{q}_1, \mathbf{q}_2) := \boldsymbol{x} \times \boldsymbol{y}. \tag{6}$$

In terms of these,

$$\mathbf{q}_1 \circ \mathbf{q}_2 = (A(\mathbf{q}_1, \mathbf{q}_2))\,\hat{h} + \boldsymbol{B}(\mathbf{q}_1, \mathbf{q}_2) + \boldsymbol{C}(\mathbf{q}_1, \mathbf{q}_2). \tag{7}$$

It is often convenient to view (3) in a tensor-like form. We can write

$$\mathbf{q}_1 \circ \mathbf{q}_2 = \sum_{\mu,\nu=0}^{3} \left(A_{\mu\nu}\,\hat{h} + B_{\mu\nu} + C_{\mu\nu}\right) x^\mu y^\nu,$$

with three 4×4 coefficient matrices:

$$A_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{8}$$

$$B_{\mu\nu} = \begin{pmatrix} 0 & \hat{\imath} & \hat{\jmath} & \hat{k} \\ \hat{\imath} & 0 & 0 & 0 \\ \hat{\jmath} & 0 & 0 & 0 \\ \hat{k} & 0 & 0 & 0 \end{pmatrix}, \tag{9}$$

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{k} & -\hat{\jmath} \\ 0 & -\hat{k} & 0 & \hat{\imath} \\ 0 & \hat{\jmath} & -\hat{\imath} & 0 \end{pmatrix}. \tag{10}$$

Here $A_{\mu\nu}$ is the usual Minkowski-like bilinear form $\mathrm{diag}(1, -1, -1, -1)$ acting on the coordinate components, while $B_{\mu\nu}$ and $C_{\mu\nu}$ collect the symmetric and antisymmetric vector-valued pieces of the product.

3.3 Geometric interpretation and tensor structure of A, B, C

The decomposition (7) and the matrices (8)–(10) make explicit that the quaternion product can be regarded as the contraction of a rank-$(0,2)$ object with two four-vectors:

$$\mathbf{q}_1 \circ \mathbf{q}_2 = \big(A + B + C\big)_{\mu\nu}\, x^\mu y^\nu,$$

with three structurally distinct blocks:

- $A_{\mu\nu}$ is a symmetric scalar bilinear form of signature $(+,-,-,-)$. In the unimetrical context it will play the role of an energy-like invariant and will generate both Newtonian and Coulomb potentials once we pass to dressed body quaternions.

- $B_{\mu\nu}$ is symmetric and vector-valued; it couples the scalar component to the spatial components. It will later be interpreted as a current-like channel, encoding the coupling between temporal and spatial parts of dressed flow quaternions.

- $C_{\mu\nu}$ is antisymmetric and vector-valued; it encodes the cross product $\boldsymbol{x} \times \boldsymbol{y}$ of the spatial parts and thus represents a vorticity (axial) channel. This will underlie the magnetic and gravitomagnetic sectors of the GEM field.

In summary, the elementary quaternion product already contains, in a rigid algebraic way, the three channels that we will later reinterpret as

(i) an energy-like scalar invariant A,

(ii) a current-like symmetric vector channel $\boldsymbol{B}$,

(iii) a vortical (axial) vector channel $\boldsymbol{C}$.

In the next section we recall how unimetry associates physical flows and an effective space–time structure to quaternions, so that these three forms can be given a gravito–electromagnetic meaning.

4 Phase proto-space and flow: brief unimetry overview

4.1 Proto-space, phase potential and phase 1-form

In unimetry the basic kinematical arena is a Euclidean (or, more generally, Hilbert) proto-space $(\mathcal{E}, \langle\cdot,\cdot\rangle)$. Points of $\mathcal{E}$ will be denoted by X, and the inner product $\langle\cdot,\cdot\rangle$ is used to identify tangent and cotangent spaces via the Riesz isomorphism. For the purposes of this paper one may think of $\mathcal{E}$ as a finite- or countable-dimensional real Hilbert space.

The fundamental scalar field of unimetry is a dimensionless *phase potential*

$$\Phi : \mathcal{E} \to \mathbb{R}.$$

From Φ we obtain the *phase 1-form*

$$\alpha := d\Phi,$$

which is a smooth 1-form on $\mathcal{E}$. At each point $X \in \mathcal{E}$, the value α_X is a linear functional on the tangent space $T_X\mathcal{E}$:

$$\alpha_X : T_X\mathcal{E} \to \mathbb{R}, \qquad \alpha_X(V) = d\Phi_X(V).$$

Using the inner product, we define the gradient $\nabla\Phi(X) \in T_X\mathcal{E}$ by the standard relation

$$\alpha_X(V) = d\Phi_X(V) = \langle \nabla\Phi(X), V\rangle, \qquad \forall\, V \in T_X\mathcal{E}.$$

Thus α and $\nabla\Phi$ carry the same information; the former is covariant, the latter contravariant.

Physically, the phase potential Φ encodes the global phase structure of the underlying flow, while the phase 1-form α and the gradient $\nabla\Phi$ encode local directions in which the phase changes most rapidly. The key idea of unimetry is to use this structure to define a canonical flow through $\mathcal{E}$.

4.2 Flow vector and normalization

Whenever $\nabla\Phi(X) \neq 0$, we define the *normalized flow direction* at X by

$$\widehat{\boldsymbol{\chi}}(X) := \frac{\nabla\Phi(X)}{\|\nabla\Phi(X)\|}, \qquad \|\nabla\Phi(X)\| := \sqrt{\langle\nabla\Phi(X), \nabla\Phi(X)\rangle}. \tag{11}$$

Thus $\widehat{\boldsymbol{\chi}}(X)$ is a unit vector in $T_X\mathcal{E}$ pointing along the steepest phase ascent. We then introduce the *physical flow vector* by a global calibration

$$\boldsymbol{\chi}(X) := c\,\widehat{\boldsymbol{\chi}}(X), \qquad \|\boldsymbol{\chi}(X)\| \equiv c, \tag{12}$$

where c is the speed of light. In other words, in unimetry the physical flow is a unit-speed curve in $\mathcal{E}$ with respect to the fixed scale c.

A flow line (or *stream*) is then a curve $\gamma : \lambda \mapsto X(\lambda) \in \mathcal{E}$ whose tangent vector is everywhere aligned with the physical flow:

$$\dot{X}(\lambda) := \frac{dX}{d\lambda} = \boldsymbol{\chi}(X(\lambda)), \qquad \|\dot{X}(\lambda)\| = c. \tag{13}$$

The parameter λ is a proto-space parameter, not yet identified with any observed time. The geometric content of (13) is simply that physical objects are represented by flows of constant Euclidean speed c in the proto-space.

4.3 Intrinsic angle, proper time and correspondence with SR

In unimetry a macroscopic body B is represented not by a single flow line, but by an ensemble of streamlets with weights w_a and tilt angles Θ_a relative to the body's self–time fibre.[1] On this ensemble one defines the temporal second moment and the spatial shape tensor as

$$T_B := \sum_a w_a \cos^2\Theta_a, \qquad \mathbf{C}_B := \sum_a w_a \sin^2\Theta_a\, \mathbf{u}_a \otimes \mathbf{u}_a, \tag{14}$$

where $0 < T_B \leq 1$, $\mathbf{C}_B$ is a symmetric positive semidefinite tensor on the body's three-surface, and $\mathbf{u}_a$ are unit spatial directions of the streamlets' projections. Operationally, T_B captures the aggregate fraction of flow carried in the orthogonal (self–time) fibre, while $\mathbf{C}_B$ encodes the anisotropic distribution of spatial projections across the body.

From these second moments one can define an *intrinsic angle* $\zeta \in [0, \frac{\pi}{2}]$ as an effective statistical parameter of the ensemble. Introducing

$$C := \sum_a w_a \cos 2\Theta_a, \qquad S := \sum_a w_a \sin 2\Theta_a,$$

there exists a unique ζ such that

$$(\cos 2\zeta,\ \sin 2\zeta) = (C, S) \quad \Longleftrightarrow \quad T_B = \tfrac{1}{2}(1 + C) = \cos^2\zeta, \qquad \operatorname{tr}\mathbf{C}_B = \tfrac{1}{2}(1 - C) = \sin^2\zeta. \tag{15}$$

We call ζ the *intrinsic angle* of the body. It aggregates the second–moment information $(T_B, \mathbf{C}_B)$ into a single scalar and should be thought of as a *statistical* internal parameter: it is *not* a geometric direction and is not attached to any particular flow line.

The intrinsic angle controls the rate at which the body's own proper time τ_B accumulates with respect to the phase parameter χ used to parametrize the flow in proto-space. In the calibrated gauge $\|\boldsymbol{\chi}\| = c$, one has

$$d\tau_B = \cos\zeta\, d\chi, \tag{16}$$

[1] For the detailed construction see Paper I, §?? there.

so that the temporal second moment T_B appears as $T_B = \cos^2\zeta = (d\tau_B/d\chi)^2$. The corresponding intrinsic metric of the body, as a quadratic form on $(d\chi, d\boldsymbol{\ell})$, reads

$$ds_B^2 := c^2\, d\tau_B^2 - d\boldsymbol{\ell}^\top \mathbf{C}_B\, d\boldsymbol{\ell} = c^2 T_B\, d\chi^2 - d\boldsymbol{\ell}^\top \mathbf{C}_B\, d\boldsymbol{\ell}. \tag{17}$$

For an isotropic texture one has $\mathbf{C}_B = \frac{\sin^2\zeta}{3}\, \mathbf{I}_S$, and with the rest gauge $T_B \equiv 1$ this reduces to the familiar Minkowski form in the body's rest frame (up to the overall phase gauge $d\chi$).

In the full unimetrical construction the intrinsic angle ζ is combined with a kinematic angle ϑ (associated with the relative motion between bodies) and, when present, with a gravitational angle ϕ (associated with an external tilt field). The resulting time-rate factor factorises into intrinsic, kinematic, and gravitational contributions. For the purposes of the present GEM paper, we only need the following structural facts:

- The intrinsic angle ζ is a scalar *second-moment* parameter of a body, not a direction: it encodes how the flow budget is split between self–time and spatial channels in the ensemble of streamlets.
- The proper time τ_B along the body's worldline is related to the phase parameter χ by (16), and the body's intrinsic metric takes the Minkowski form (17) once the rest gauge is fixed.
- The relativistic kinematics of unimetry can therefore be formulated entirely in terms of phase flow and second-moment data, with the usual SR interval emerging as a derived object; we will reuse this structure when interpreting the scalar form A as an energy-like invariant for dressed quaternions.

4.4 Notation table

For reference, we collect here the main unimetrical symbols used in the remainder of the paper. A more extensive table can be found in Paper I; the subset below is chosen to make the present text self-contained.

Symbol	Meaning
$\mathcal{E}$	Euclidean/Hilbert proto-space with inner product $\langle\cdot,\cdot\rangle$
$\Phi : \mathcal{E} \to \mathbb{R}$	dimensionless phase potential
$\alpha = d\Phi$	phase 1-form, $\alpha_X(V) = \langle \nabla\Phi(X), V\rangle$
$\nabla\Phi(X)$	gradient of Φ at X, defined via the inner product
$\widehat{\boldsymbol{\chi}}(X)$	normalized flow direction, $\widehat{\boldsymbol{\chi}} = \nabla\Phi/\|\nabla\Phi\|$
$\boldsymbol{\chi}(X)$	physical flow vector, $\boldsymbol{\chi} = c\,\widehat{\boldsymbol{\chi}}$, $\|\boldsymbol{\chi}\| = c$
$\gamma(\lambda)$	flow line in $\mathcal{E}$ with tangent $\dot{X} = \boldsymbol{\chi}$
$\widehat{\boldsymbol{u}}$	unit rest direction associated with an observer (local temporal axis)
$\widehat{\boldsymbol{n}}$	unit spatial direction orthogonal to $\widehat{\boldsymbol{u}}$
ζ	flow angle between $\boldsymbol{\chi}$ and $\widehat{\boldsymbol{u}}$, see (**??**)
δT	effective temporal increment for the observer, see (**??**)
$\delta\boldsymbol{x}$	effective spatial increment in the observer's rest space, see (**??**)
δs^2	effective interval, $\delta s^2 = c^2\delta T^2 - \|\delta\boldsymbol{x}\|^2$, see (**??**)

Table 1: Key unimetrical quantities used in the quaternionic GEM construction.

In the next section we introduce two calibrations of the flow — one kinematic and one energetic — which will allow us to interpret the scalar form A as an energy-like invariant and to define metrically dressed body quaternions suitable for the gravito–electromagnetic setting.

5 Flow calibrations and energy-like functionals

In the previous section the phase proto-space $(\mathcal{E}, \langle\cdot,\cdot\rangle)$, the phase potential Φ and the physical flow vector $\boldsymbol{\chi}$ were introduced at a purely geometric level. In the unimetry convention we use here, the flow parameter χ carries *units of time* (seconds), while the phase potential Φ is dimensionless and related to χ through a frequency scale,

$$\Phi = \omega_\chi \, \chi, \qquad [\omega_\chi] = \mathrm{s}^{-1}, \tag{18}$$

along a given worldline. More generally, $\omega_\chi = d\Phi/d\chi$ may depend on the state, but the product $\omega_\chi \chi$ is always dimensionless.

To connect the flow structure to observable kinematics and to energy, unimetry uses two complementary calibrations of χ: a *phase (kinematic) calibration*, in which χ is tied to a coordinate time, and a *proper-time (energetic) calibration*, in which χ is tied to the body's proper time. Both will be important for the gravito–electromagnetic construction.

5.1 Phase (kinematic) calibration

Consider a macroscopic body B represented by an ensemble of streamlets in proto-space, as in Section 4.3. Let χ be the flow parameter with units of time. In the phase (kinematic) calibration we choose χ so that it coincides with the coordinate time t of a chosen inertial laboratory frame,

$$\chi = t, \qquad \frac{d\chi}{dt} = 1. \tag{19}$$

Along the worldline we may then write the phase potential as

$$\Phi(t) = \omega_\chi \, \chi(t) = \omega_\chi \, t,$$

with a frequency $\omega_\chi = d\Phi/d\chi = d\Phi/dt$ in the lab frame. The product $\omega_\chi t$ is dimensionless, as required.

With this choice, the flow vector $\boldsymbol{\chi}$ projected onto the lab frame splits into a temporal and a spatial component,

$$\boldsymbol{\chi} = c\,\widehat{\boldsymbol{\chi}} = c\big(\cos\zeta\,\widehat{\boldsymbol{u}} + \sin\zeta\,\widehat{\boldsymbol{n}}\big),$$

where $\widehat{\boldsymbol{u}}$ is the lab-frame time direction and $\widehat{\boldsymbol{n}}$ is a spatial unit vector. For an inertially moving body with constant lab three-velocity $\boldsymbol{v} = d\boldsymbol{x}/dt$, we impose the calibration

$$\sin\zeta = \beta := \frac{\|\boldsymbol{v}\|}{c}, \qquad \cos\zeta = \sqrt{1-\beta^2}. \tag{20}$$

This ensures that the spatial projection of the flow has magnitude $\|\boldsymbol{v}\| = c\sin\zeta$, while the temporal second moment $T_B = \cos^2\zeta$ coincides with the usual factor $1-\beta^2$ that appears in time dilation.

Using (16) and (19), the connection to SR kinematics can be summarized as

$$\frac{d\tau_B}{dt} = \frac{d\tau_B/d\chi}{dt/d\chi} = \cos\zeta = \sqrt{1-\beta^2},$$

so that the body's proper time τ_B along its worldline satisfies

$$d\tau_B = \sqrt{1-\beta^2}\, dt$$

in the usual way. The key point is that ζ is now a function of the observable speed β, and the flow parameter χ is tied to a physical time coordinate, both measured in seconds.

5.2 Proper-time (energetic) calibration

In many situations it is more natural to parametrize the flow by the body's proper time τ_B rather than by an external coordinate time. The proper-time (energetic) calibration chooses χ so that

$$\chi = \chi(\tau_B), \qquad \frac{d\tau_B}{d\chi} = \cos\zeta, \tag{21}$$

as in (16). Both χ and τ_B carry units of time, so $\cos\zeta$ is dimensionless, as it should be. Equivalently,

$$\frac{d\chi}{d\tau_B} = \frac{1}{\cos\zeta}.$$

Along the body's worldline we define a *proper phase frequency* ω_B by

$$\omega_B := \frac{d\Phi}{d\tau_B} = \frac{d\Phi}{d\chi}\frac{d\chi}{d\tau_B} = \omega_\chi \frac{1}{\cos\zeta}, \tag{22}$$

where $\omega_\chi := d\Phi/d\chi$ has dimension s^{-1}. In the simplest gauge one may take ω_χ to be a constant reference frequency ω_0 characteristic of the body's rest state, so that $\omega_B = \omega_0/\cos\zeta$. More generally, ω_χ may encode additional internal structure of the body, but for the present GEM construction we only use the fact that ω_B transforms as an inverse proper-time scale and is a monotone function of ζ.

5.3 Energy of a flow and the self-form $A(q, q)$

The quaternionic algebra introduced in Section 3 provides a natural way to encode the split between temporal and spatial channels in a single object. For a normalized flow direction with intrinsic angle ζ we define the associated *dimensionless flow quaternion*

$$\widehat{\mathsf{q}} := \cos\zeta\,\hat{h} + \sin\zeta\,\widehat{\boldsymbol{n}}, \qquad \|\widehat{\boldsymbol{n}}\| = 1, \tag{23}$$

which has no physical dimension. The scalar self-form A introduced in Eqs. (4) and (7) then yields

$$A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}}) = \cos^2\zeta - \sin^2\zeta = \cos 2\zeta, \tag{24}$$

so $A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})$ is a pure number. All physical dimensions in the energy are carried by prefactors such as $m_B c^2$ or a universal scale κ, while A supplies the dimensionless kinematic factor.

In the kinematic calibration (20) we have $\cos\zeta = \sqrt{1-\beta^2}$, so

$$A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}}) = 1 - 2\beta^2, \qquad 1 - \beta^2 = \frac{1 + A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})}{2}. \tag{25}$$

The Lorentz factor can then be expressed as

$$\gamma(\beta) = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{\frac{2}{1 + A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})}}. \tag{26}$$

Conversely, given the scalar invariant $A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})$, one can reconstruct the usual relativistic energy per unit rest mass,

$$\frac{E_B}{m_B c^2} = \gamma(\beta) = \sqrt{\frac{2}{1 + A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})}}.$$

For small velocities $\beta \ll 1$, one has $A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}}) = 1 - 2\beta^2 + O(\beta^4)$, so that

$$1 - A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}}) = 2\beta^2 + O(\beta^4). \tag{27}$$

This suggests using the deviation of A from its rest value as a building block for an energy-like functional. For instance, for a body of rest mass m_B one may define a free-body Hamiltonian

$$H_{\mathrm{free}}[\widehat{\mathsf{q}}] := m_B c^2 + \frac{m_B c^2}{4}\left(1 - A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})\right), \tag{28}$$

so that, in the non-relativistic limit,

$$H_{\mathrm{free}}[\widehat{\mathsf{q}}] = m_B c^2 + \tfrac{1}{2} m_B v^2 + O(\beta^4).$$

The precise functional dependence on A is not unique; what matters for the present work is that:

- $A(\widehat{\mathsf{q}}, \widehat{\mathsf{q}})$ is a scalar invariant of the flow direction, monotonically related to the intrinsic angle ζ and thus to the proper-time rate and Lorentz factor;
- it can be used as a natural scalar argument in Hamiltonians and Lagrangians, with the standard SR expressions recovered by an appropriate calibration;
- in the gravito–electromagnetic setting we will apply the same scalar form A to *metrically dressed* body quaternions, where it will directly generate Newtonian and Coulomb potentials.

In the next section we introduce these metrically dressed body quaternions, which package mass and charge into an effective length split into a temporal and a spatial channel. The scalar form A, together with the vector forms $\boldsymbol{B}$ and $\boldsymbol{C}$, will then be used to define gravito–electromagnetic interactions as bilinear functionals of dressed quaternions.

5.4 Phase (kinematic) calibration

5.5 Proper-time (energetic) calibration

5.6 Energy of a flow and the self-form A(q,q)

6 Metrically dressed body quaternions

6.1 Free unit imaginary vector for isotropic Newtonian bodies

In the metrically dressed setting we write the spatial part of a body quaternion as

$$\boldsymbol{S}_i = L_{G,i}\, \widehat{\boldsymbol{n}}_i,$$

where $\widehat{\boldsymbol{n}}_i$ is a unit spatial direction associated with body i. For charged bodies it is natural to interpret $\widehat{\boldsymbol{n}}_i$ as an intrinsic flow direction (e.g. an orientation of the underlying streamlet structure), which will in general produce non-trivial contributions in both the symmetric vector form $\boldsymbol{B}$ and the vortical form $\boldsymbol{C}$.

For purely Newtonian, isotropic mass distributions, however, we may and should distinguish between an intrinsic direction and the *interaction* direction. To reflect this, we introduce the notion of a *free unit imaginary vector* for the gravitational channel.

Definition 6.1 (Free unit imaginary vector)**.** A free unit imaginary vector is a unit spatial quaternion $\widehat{\boldsymbol{u}} \in \operatorname{Im}\mathbb{H}$ whose orientation is not fixed by the internal structure of the body, but is freely assigned at the level of the interaction. For an isotropic Newtonian body i we take the gravitational spatial part of its dressed quaternion to be

$$\boldsymbol{S}_i^{(G)}(\boldsymbol{x}) := L_{G,i}\, \widehat{\boldsymbol{u}}_i(\boldsymbol{x}),$$

where, for a field point $\boldsymbol{x}$, the free unit vector is chosen to be aligned with the radius vector from the body to that point,

$$\widehat{\boldsymbol{u}}_i(\boldsymbol{x}) := \widehat{\boldsymbol{r}}_i(\boldsymbol{x}) = \frac{\boldsymbol{x} - \boldsymbol{x}_i}{\|\boldsymbol{x} - \boldsymbol{x}_i\|}.$$

In other words, for an isotropic Newtonian source the gravitational channel of the dressed quaternion is always taken to be *radial* with respect to the field point, and carries no intrinsic "spin" information. This has an important structural consequence for the bilinear forms.

Consider two isotropic masses m_1, m_2 at positions $\boldsymbol{x}_1, \boldsymbol{x}_2$, and evaluate their gravitational spatial parts at a common field point $\boldsymbol{x}$. By construction,

$$\boldsymbol{S}_1^{(G)}(\boldsymbol{x}) \parallel \widehat{\boldsymbol{r}}_1(\boldsymbol{x}), \qquad \boldsymbol{S}_2^{(G)}(\boldsymbol{x}) \parallel \widehat{\boldsymbol{r}}_2(\boldsymbol{x}).$$

In the static two-body configuration the interaction is along the line joining the bodies, so that effectively

$$\boldsymbol{S}_1^{(G)} \parallel \boldsymbol{S}_2^{(G)},$$

and therefore their contribution to the vortical form

$$\boldsymbol{C}(\widetilde{\mathrm{q}}_1, \widetilde{\mathrm{q}}_2) = \boldsymbol{S}_1 \times \boldsymbol{S}_2$$

vanishes in the purely gravitational, isotropic limit:

$$\boldsymbol{C}^{(G)} = \boldsymbol{S}_1^{(G)} \times \boldsymbol{S}_2^{(G)} = \boldsymbol{0}.$$

Thus, by assigning the gravitational spatial part of an isotropic body to a free unit imaginary vector that is always chosen radial, we ensure that:

- the gravitational interaction of isotropic masses is purely scalar and radial, as in Newtonian gravity;
- there is *no* spurious contribution of the gravitational channel to the vortical form $\boldsymbol{C}$ in the static limit;
- all non-trivial vortical contributions in $\boldsymbol{C}$ are genuinely associated with anisotropy and/or motion (currents), i.e. with the EM and gravitomagnetic sectors rather than with static isotropic gravity.

In contrast, for charged bodies we will keep an intrinsic unit direction $\widehat{\boldsymbol{n}}_i$ in the electric channel, which can contribute to both the symmetric form $\boldsymbol{B}$ and the vortical form $\boldsymbol{C}$. This separation between a free gravitational direction and an intrinsic electromagnetic direction will be important when we analyze the GEM field and its magnetic and gravitomagnetic components in **????**.