1. Time normalization as a function of phase speed

Consider the 3–sphere $S^3_R \subset \mathbb{R}^4$ in Hopf coordinates (ξ_0, ξ_1, ξ_2):

$$x_1 = R\cos\xi_0\cos\xi_1, \quad x_2 = R\cos\xi_0\sin\xi_1, \quad x_3 = R\sin\xi_0\cos\xi_2, \quad x_4 = R\sin\xi_0\sin\xi_2.$$

Fixing $\xi_0 = \xi_0^\star$ yields a flat torus $\mathbb{T}^2$ with radii

$$R_1 = R\cos\xi_0^\star, \qquad R_2 = R\sin\xi_0^\star,$$

and metric $ds^2 = R_1^2\,\mathrm{d}\xi_1^2 + R_2^2\,\mathrm{d}\xi_2^2$. Introduce arc–lengths along the two circles,

$$\chi := R_1\xi_1, \qquad \zeta := R_2\xi_2,$$

with $\dot\chi = R_1\dot\xi_1$ and $\dot\zeta = R_2\dot\xi_2$. Free (geodesic) motion on $\mathbb{T}^2$ has Lagrangian $L = \frac{1}{2}(R_1^2\dot\xi_1^2 + R_2^2\dot\xi_2^2)$, hence conserved momenta

$$p_1 = R_1^2\dot\xi_1 = \text{const}, \qquad p_2 = R_2^2\dot\xi_2 = \text{const}.$$

Define the phase speed in local time $\omega_\chi := \mathrm{d}\chi/\mathrm{d}\tau = \dot\chi$ and the inter–fiber ("proto") speed $\tilde H := \mathrm{d}\zeta/\mathrm{d}\chi$. Then

$$\frac{p_1}{p_2} = \frac{R_1^2\dot\xi_1}{R_2^2\dot\xi_2} = \frac{R_1}{R_2}\frac{\dot\chi}{\dot\zeta} = \frac{R_1}{R_2}\frac{\omega_\chi}{\dot\zeta} = \frac{R_1}{R_2}\frac{\omega_\chi}{\tilde H\,\omega_\chi} = \frac{R_1}{R_2}\frac{1}{\tilde H}.$$

Equivalently,

$$\boxed{\omega_\chi \;=\; k\,\tilde H, \qquad k := \frac{p_1}{p_2}\frac{R_1}{R_2}}. \tag{1}$$

The *time–normalization factor* (a 1–form) is, therefore,

$$\boxed{\Theta_\chi := \frac{\mathrm{d}\tau}{\mathrm{d}\chi} = \frac{1}{\omega_\chi} = \frac{1}{k\,\tilde H}}. \tag{2}$$

2. Equivalence of gauges (no contradictions)

There are two natural, but mutually exclusive, gauge choices:

$$(\chi\text{-gauge})\quad \tilde H \equiv c, \qquad (\tau\text{-gauge})\quad \dot H \equiv c \quad (H \equiv \zeta).$$

They are related by the invariant identity

$$\boxed{\dot H \;=\; \tilde H\,\dot\chi \;=\; k\,\tilde H^2}. \tag{3}$$

Thus one must not set c for both $\tilde{H}$ and $\dot{H}$ simultaneously (except in the special $k = 1$ case). From (**??**):

$$\begin{cases} \chi\text{-gauge: } \tilde{H} \equiv c \;\Rightarrow\; \dot{H} = k\,c^2, \\ \tau\text{-gauge: } \dot{H} \equiv c \;\Rightarrow\; \tilde{H} = \sqrt{c/k}. \end{cases}$$

3. Flow speed and energy in both gauges

Define the *phase–space energy* as the gauge–invariant scalar

$$\boxed{E \;:=\; k\,\tilde{H}^2}. \tag{4}$$

Using (**??**) we have $E = \dot{H}$ precisely in the χ-gauge, while in the τ-gauge $E = k\,c^2$:

$$\begin{cases} \chi\text{-gauge: } \tilde{H} \equiv c \;\Rightarrow\; E = k\,c^2 = \dot{H}, \\ \tau\text{-gauge: } \dot{H} \equiv c \;\Rightarrow\; E = k\,c^2, \quad \dot{H} = c. \end{cases}$$

Hence $E = kc^2$ in either gauge, and E is independent of which variable is held fixed by convention.

Interpretation. k plays the role of an invariant mass scalar (winding/structure on the Hopf torus), while c is the fixed inter–fiber speed. The observable flux along the time fiber is $\dot{H}$; it equals E only in the χ-gauge.

4. Generalized energy in phase space

In curved or inhomogeneous settings k and $\tilde{H}$ may vary with position along the flow on S^3; the local energy field is

$$\boxed{E(x) = k(x)\,\tilde{H}(x)^2}. \tag{5}$$

For simple (photon–like) flows one has $k = 0$; the energy is then carried by the phase frequency $\omega_\chi = \mathrm{d}\chi/\mathrm{d}\tau$. Introducing an action constant σ (to be calibrated empirically),

$$\boxed{E_\gamma \;=\; \sigma\,\omega_\chi}, \tag{6}$$

so that the massive and simple branches are recovered as the limits of the unified ansatz

$$\boxed{E \;=\; k\,\tilde{H}^2 \;+\; \sigma\,\omega_\chi}. \tag{7}$$

In the massive regime the first term dominates; for $k = 0$ we get the pure frequency law $E_\gamma = \sigma\omega_\chi$.

Remark. The geodesic Hamiltonian on $\mathbb{T}^2$ reads $H = \frac{1}{2}(\dot{\chi}^2 + \dot{\zeta}^2) = \frac{1}{2}(1 + k^2)\tilde{H}^2$, which is a kinematic invariant of the free motion; it should not be confused with the physical energy scalar E in (**??**).