Unimetry: Proto-Space Reformulation of Special Relativity

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Abstract

We present a kinematic reformulation of special relativity in a four–dimensional Euclidean proto–space $(\mathcal{E}, \delta)$ by splitting tangent vectors with respect to a chosen unit field N into longitudinal and transverse parts. Writing $\alpha := \delta(N, \cdot)$ and $h := \delta - \alpha\otimes\alpha$, we show that the operational reparametrization of a timelike worldline by proper time,

$$c^2 d\tau^2 := g(dX, dX), \qquad g = \alpha\otimes\alpha - h = 2\,\alpha\otimes\alpha - \delta,$$

induces the standard Lorentzian interval $c^2 d\tau^2 = c^2 dt^2 - d\ell^2$ with $dt := \alpha(dX)/c$ and $d\ell^2 := h(dX, dX)$. In the inertial sector we assume $\nabla^\delta N = 0$, so g is globally Minkowskian and the construction isolates pure SR kinematics without introducing field dynamics.

A complementary calibration uses a proto–parameter χ such that $\delta(\tilde{X}, \tilde{X}) = c^2$, placing flow states on the sphere $S^3_c \subset T\mathcal{E}$; the physically timelike sector corresponds to the open subset where $g(\tilde{X}, \tilde{X}) > 0$. This yields a compact tilt parametrization $\beta = v/c = \tan\vartheta$ (hence $\tanh\eta = \tan\vartheta$) and enables transparent geometric derivations of the relativistic Doppler shift and aberration as projection identities relative to N.

Keywords: special relativity; phase; rapidity; Doppler shift; aberration; Lorentz factor; phase parametrization.

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1 Introduction

1.1 Motivation

The transition from the definite signature of Euclidean geometry to the indefinite signature of Lorentzian spacetime has traditionally been treated as a fundamental postulate of physics. However, the rigidity and well-behaved spectral properties of elliptic operators suggest that the Euclidean signature may be more fundamental. This paper proposes kinematical part of *Unimetry* — a framework where the physical Lorentzian structure is not axiomatic but derived from a "proto-space": a four-dimensional Euclidean manifold $(\mathcal{E}, \delta)$ permeated by a distinguished vector field N.

Our goal is to provide a self–contained proto–space formulation of the kinematics of special relativity in which relativistic effects are traced to orthogonal decompositions and projections in a Euclidean background. The speed of light appears as the boundary of the physically admissible (timelike) sector on the calibrated sphere, and standard quantities such as β, γ, and rapidity admit simple expressions in terms of a bounded tilt angle. The formalism is designed to be compatible with geometric–algebra methods and to serve as a convenient kinematical foundation; dynamical field equations and source models are intentionally left outside the scope of the present paper.

1.2 Relation to previous work

Attempts to recast Lorentzian kinematics in Euclidean terms have a long history. Early geometric constructions already appear in Karapetoff [1], where relativistic transformations are visualized by Euclidean angle geometry. More recent works study various embeddings and correspondences between Euclidean and Lorentzian structures, including [2, 3].

A distinct line of literature aims at an explicitly *Euclidean* reformulation of special relativity. In particular, Euclidean SR can be obtained by a change of variables in which Lorentz transformations are represented as rotations in a Euclidean space (e.g. Gersten [9]). Other proposals postulate an absolute Euclidean background and then reinterpret relativistic observables in terms of proper time (e.g. Montanus [10, 11]), or develop related "four–dimensional optics" frameworks in which τ plays a central operational role (e.g. Almeida [12, 13]). For broader context on Euclidean viewpoints beyond SR, see also Atkinson [14].

Our construction differs in emphasis: we do not identify the Euclidean norm with the Lorentz interval by a coordinate trick, nor do we postulate an "absolute" Euclidean time. Instead, the Lorentzian interval emerges as an operational quadratic form associated with a distinguished time 1–form $\alpha := N^\flat = \delta(N, \cdot)$, together with the observer's proper–time reparametrization along worldlines. This standpoint is conceptually aligned with the relational-time viewpoint in generally covariant physics, where the physical time variable is a chosen clock observable (Rovelli's "partial observables" perspective [7, 8]).

A rigorous pointwise projector correspondence between a Riemannian metric and a Lorentzian one was established by Reddy, Sharma and Sivaramakrishnan [4]. Given a Riemannian manifold (M, h) and a unit vector field U, they define a Lorentzian metric by $g = h - 2\,U^\flat \otimes U^\flat$. We adopt the sign–flipped variant adapted to the particle–physics convention $(+ - --)$, namely $g = 2\,N^\flat \otimes N^\flat - \delta$, and then make explicit how proper time (hence SR kinematics) arises through phase reparametrization relative to α.

Inertial specialization (SR kinematics). Although the construction of §2 applies to any smooth unit field N on $(\mathcal{E}, \delta)$, the present paper focuses on special–relativistic kinematics. Accordingly, we restrict to the inertial case and assume that N is parallel with respect to the flat Euclidean connection,

$$\nabla^\delta N = 0. \tag{1.1}$$

Equivalently, in global Cartesian coordinates on $\mathcal{E} \simeq \mathbb{R}^4$ the components N^A are constant. Under this assumption the induced metric $g = 2\,N^\flat \otimes N^\flat - \delta$ is constant (globally Minkowskian), and the SR effects derived in this paper arise from the phase reparametrization and tilt geometry rather than from curvature.

Algebraically, our presentation is close in spirit to the geometric–algebra approach to relativity, in which boosts and rotations are treated uniformly as rotors (see, e.g., Hestenes' Space–Time Algebra [5, 6] and the discussion of spacetime algebra versus "imaginary time" in [15]). The difference is that we work throughout in the strictly Euclidean Clifford algebra $\mathcal{C}\ell_{4,0}$: the "space–time split" is encoded by the choice of N (hence α) and is therefore observer–dependent, while Lorentzian boosts arise from Euclidean rotations after an appropriate change of bivector basis.

1.3 Contributions

The main contributions of the present paper are:

(C1) **Operational origin of the Lorentz interval.** Starting from a Euclidean proto–space $(\mathcal{E}, \delta)$ and a unit field N, we introduce the associated time 1–form $\alpha := N^\flat = \delta(N, \cdot)$ and the spatial projector $h = \delta - \alpha \otimes \alpha$. We make explicit that the observer's interval is the induced quadratic form

$$g = \alpha \otimes \alpha - h = 2\,\alpha \otimes \alpha - \delta,$$

and that proper time arises as an operational reparametrization along worldlines via $c^2 d\tau^2 = g(dX, dX)$ (Sections 2–3 and §4).

(C2) **Reparameterization equivalence on the calibrated sphere.** We show that proper–time normalization in the induced Lorentzian metric is equivalent to a constant–speed Euclidean flow constrained to the calibrated sphere $S^3_c \subset T\mathcal{E}$ (Theorem 6.5 and §6).

(C3) **Geometric characterization of the physical sector.** The timelike condition $g(X, X) > 0$ selects an open subset of S^3_c (two polar caps), with the lightlike boundary reached at a finite Euclidean tilt $\vartheta = \pi/4$ (cf. §5.3 and Remark 6.6).

(C4) **Bounded angular parametrization of boosts.** We establish the kinematic identity $\tanh \eta = \tan \vartheta$ and use it to map the unbounded rapidity $\eta \in [0, \infty)$ to a bounded Euclidean tilt variable $\vartheta \in [0, \pi/4)$, providing a compact geometric description of boost kinematics (Section 5).

(C5) **Projection-based optics and Euclidean GA implementation.** Frequency and direction of null rays are obtained as g–projections, yielding concise derivations of the Doppler shift and aberration. We also encode boosts in the strictly Euclidean Clifford algebra $\mathcal{C}\ell_{4,0}$ via a change of bivector basis, so that Lorentzian boosts are represented by Euclidean rotors relative to the N–split (Section 7).

1.4 Outline

The paper is organized as follows:

- **Section 2** constructs the Lorentzian metric g from the Euclidean background (δ, N), introduces the time 1–form $\alpha = N^\flat$, and establishes the causal signature.
- **Section 3** explores the properties of the induced metric, including the orthogonal decomposition of tangent vectors, the null cone, and the characterization of timelike and lightlike directions.
- **Section 4** formulates the phase/clock viewpoint explicitly: we introduce a δ–calibrated proto–parameter χ, define coordinate time and proper time via α, and prove the equivalence between constant Euclidean flow budget in $(\mathcal{E}, \delta)$ and proper–time normalization in $(\mathcal{E}, g)$.
- **Section 5** derives the central kinematic identity $\tanh \eta = \tan \vartheta$ and shows that the speed of light corresponds to a 45° Euclidean tilt.
- **Section 6** reformulates 4–velocity normalization as a constraint on the Euclidean flow, mapping admissible states to the calibrated sphere $S^3_c \subset T\mathcal{E}$ and identifying the physical sector.
- **Section 7** demonstrates the explanatory power of the framework. We derive optical effects (Doppler, aberration) as geometric projections and reformulate boosts using Euclidean Geometric Algebra.

2 Lorentzian metric construction

2.1 Euclidean proto-space

We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$, equipped with the flat metric

$$\delta_{AB} = \mathrm{diag}(1, 1, 1, 1).$$

Throughout, indices are raised and lowered with δ:

$$X_A := \delta_{AB} X^B, \qquad X^A := \delta^{AB} X_B,$$

and we use the δ–inner product notation

$$X \cdot Y := \delta(X, Y) = \delta_{AB} X^A Y^B.$$

Remark 2.1 (Index conventions: δ vs. g). Throughout, δ is treated as the *background* Euclidean metric on $\mathcal{E}$, and we use δ to raise and lower abstract indices unless explicitly stated otherwise. The Lorentzian tensor g_{AB} introduced in §2.3 is regarded primarily as a derived bilinear form on $T\mathcal{E}$ (used to define interval–type scalars such as $g(X, X)$), and not as the default device for index gymnastics.

In particular, we distinguish the δ–raised components

$$g^{AB}_{(\delta)} := \delta^{AC} \delta^{BD} g_{CD}$$

from the inverse metric $(g^{-1})^{AB}$ defined by $(g^{-1})^{AC} g_{CB} = \delta^A{}_B$. For the special form $g_{AB} = 2N_A N_B - \delta_{AB}$ with $\delta(N, N) = 1$, one indeed has $(g^{-1})^{AB} = g^{AB}_{(\delta)} = 2N^A N^B - \delta^{AB}$, but the two notions remain conceptually distinct.

2.2 Distinguished unit vector field

Let N be a smooth vector field on $\mathcal{E}$ satisfying the unit condition

$$\delta(N, N) = 1.$$

In particular, N is nowhere vanishing and defines at each point a distinguished δ–unit direction.

We introduce the δ–orthogonal projector onto the complement of N:

$$h_{AB} := \delta_{AB} - N_A N_B. \tag{2.1}$$

Then h has rank 3 and satisfies

$$h_{AB} N^B = 0, \qquad h_A{}^C h_{CB} = h_{AB}.$$

We write $\mathrm{Im}(h_p) = N_p^{\perp_\delta} \subset T_p\mathcal{E}$ for the δ–orthogonal complement at p.

Remark 2.2 (Status of the reference field N). In this paper the unit field N is *not* introduced as a dynamical "aether" variable and is not assigned an action, field equations, or an energy–momentum tensor. Rather, N is a geometric datum that selects the 1+3 splitting of the Euclidean proto–space $(\mathcal{E}, \delta)$ and thereby fixes the associated projectors and the induced Lorentzian metric

$$g \;=\; 2\, N^\flat \otimes N^\flat \;-\; \delta.$$

In the special–relativistic (inertial) sector studied here we assume N is parallel with respect to the flat Euclidean connection, $\nabla^\delta N = 0$, so g is globally Minkowskian.

Lorentz covariance is recovered as an *equivariance* property of the construction: any other constant unit field N' corresponds to a different choice of inertial observer and is related to N by a Euclidean isometry $R \in O(4)$, $N' = R_* N$. Under this change of reference direction the induced metric transforms tensorially, $g' = R^* g$, and all observable statements in the present framework are required to be formulated in g–covariant (relational) quantities (invariants and g–projections). Hence the physical content does not depend on the particular representative N used to label an observer, but only on the relational geometry encoded by g.

Remark 2.3 (Nature of N and observer dependence). Unlike vector fields in spontaneous symmetry breaking models, N here does not carry dynamic energy-momentum. It serves as a geometric reference required to define the projection from the proto-space to the physical manifold. Lorentz covariance is recovered essentially because the theory allows any choice of N (corresponding to any inertial observer), and the resulting physics depends only on the relational quantities defined by the induced metric g.

Formally, one may view localized states as 3-spheres embedded in the proto-space $\mathcal{E}$. In this topological picture, N arises naturally as the radial unit field (the normal vector) of the sphere. Since such a field flows outward and inward in all directions relative to the center of the 3-sphere, the specific direction of N at any point is determined by the orientation of the local shell. Thus, the kinetically significant N is always a local choice of the observer, rather than a global background field.

2.3 Lorentzian metric definition

Define a symmetric $(0, 2)$–tensor field g on $\mathcal{E}$ by

$$g_{AB} := 2N_A N_B - \delta_{AB}. \tag{2.2}$$

Equivalently, using (2.2),

$$g_{AB} = N_A N_B - h_{AB}. \tag{2.3}$$

(i) N is g–unit:

$$g(N, N) = 2(\delta(N, N))^2 - \delta(N, N) = 2 \cdot 1 - 1 = 1.$$

(ii) N is g–orthogonal to $N^{\perp_\delta}$: if $X \in N^{\perp_\delta}$, i.e. $N \cdot X = 0$, then

$$g(N, X) = 2(N \cdot N)(N \cdot X) - \delta(N, X) = 0.$$

(iii) On $N^{\perp_\delta}$ one has $g = -\delta$: if $X, Y \in N^{\perp_\delta}$, then

$$g(X, Y) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = -\delta(X, Y).$$

Hence, at each $p \in \mathcal{E}$,

$$T_p\mathcal{E} = \operatorname{span}\{N_p\} \oplus N_p^{\perp_\delta},$$

and in an adapted δ–orthonormal basis $\{e_0 = N, e_1, e_2, e_3\}$ the bilinear form g_p has the Minkowski block form

$$g_p = (+1) \oplus (-1) \oplus (-1) \oplus (-1).$$

In particular, g has Lorentzian signature $(+ - - -)$.

Proposition 2.4 (One–form representation of g)**.** *Let N be a smooth δ–unit vector field on $\mathcal{E}$, $\delta(N, N) = 1$, and let*

$$\alpha := N^\flat := \delta(N, \cdot) \qquad (\textit{so } \alpha_A = \delta_{AB} N^B = N_A).$$

Define the spatial projector

$$h := \delta - \alpha \otimes \alpha \qquad (\textit{i.e. } h_{AB} = \delta_{AB} - N_A N_B).$$

Then the Lorentzian metric constructed in §2 admits the equivalent closed form

$$g \;=\; \alpha \otimes \alpha - h \;=\; 2\,\alpha \otimes \alpha - \delta, \qquad \textit{i.e.} \qquad g_{AB} = 2N_A N_B - \delta_{AB}. \tag{2.4}$$

Moreover, for every $X \in T_p\mathcal{E}$ one has

$$g(X, X) = \alpha(X)^2 - h(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp) = 2(N \cdot X)^2 - \delta(X, X), \tag{2.5}$$

where $X_\perp := X - (N \cdot X)\, N$ is the δ–orthogonal projection of X onto $N^{\perp_\delta}$.

Proof. By definition, $h = \delta - \alpha \otimes \alpha$, hence $\alpha \otimes \alpha - h = 2\alpha \otimes \alpha - \delta$, giving (2.5) and the component form. Evaluating on X yields $g(X, X) = \alpha(X)^2 - h(X, X)$. Using the orthogonal decomposition $X = (N \cdot X)N + X_\perp$ one has $\delta(X, X) = (N \cdot X)^2 + \delta(X_\perp, X_\perp)$, which gives (2.6). $\square$

3 Lorentzian metric properties

Throughout this section, $p \in \mathcal{E}$ is arbitrary and all statements are understood pointwise at p.

3.1 Orthogonal decomposition of tangent vectors

For any $X \in T_p\mathcal{E}$ we define the δ–longitudinal and δ–transverse components relative to N by

$$X_\parallel := (N \cdot X)\, N, \qquad X_\perp := h(X) = X - (N \cdot X)\, N. \tag{3.1}$$

Lemma 3.1. *For every $X \in T_p\mathcal{E}$,*

$$X = X_\parallel + X_\perp,$$

where $X_\parallel \in \operatorname{span}\{N_p\}$ and $X_\perp \in N_p^{\perp_\delta}$. The decomposition is unique.

Proof. Since h_p is a projector with $\ker(h_p) = \operatorname{span}\{N_p\}$ and $\operatorname{Im}(h_p) = N_p^{\perp_\delta}$, the splitting is the standard direct sum decomposition associated with complementary subspaces. □

3.2 Norm identities and classification of vectors

Proposition 3.2. *For any $X \in T_p\mathcal{E}$,*

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp).$$

Proof. Insert (3.1) into $g(X, X)$ and use: $g(N, N) = 1$, $g(N, X_\perp) = 0$ (since $X_\perp \in N^{\perp_\delta}$), and $g(X_\perp, X_\perp) = -\delta(X_\perp, X_\perp)$ from (iii). □

Corollary 3.3. *A vector X satisfies:*

- $g(X, X) > 0$ *iff* $(N \cdot X)^2 > \delta(X_\perp, X_\perp)$,
- $g(X, X) = 0$ *iff* $(N \cdot X)^2 = \delta(X_\perp, X_\perp)$,
- $g(X, X) < 0$ *iff* $(N \cdot X)^2 < \delta(X_\perp, X_\perp)$.

Define the three disjoint subsets of $T_p\mathcal{E}$:

$$\mathcal{T}_p := \{X \in T_p\mathcal{E} :\ g(X, X) > 0\}, \qquad \mathcal{P}_p := \{X \in T_p\mathcal{E} :\ g(X, X) = 0\}, \qquad \mathcal{S}_p := \{X \in T_p\mathcal{E} :\ g(X, X) < 0\}.$$

We also single out the *future* time cone (relative to N):

$$\mathcal{T}_p^+ := \{X \in \mathcal{T}_p :\ N \cdot X > 0\}. \tag{3.2}$$

3.3 Geometry of the null cone

Proposition 3.4. *The set of g–null vectors at p is the quadratic cone*

$$\mathcal{C}_p = \{\, X \in T_p\mathcal{E} :\ \delta(X_\perp, X_\perp) = (N \cdot X)^2 \,\}.$$

Under the decomposition $T_p\mathcal{E} = \operatorname{span}\{N_p\} \oplus N_p^{\perp_\delta}$, it is a double cone given by

$$N \cdot X = \pm \|X_\perp\|_\delta.$$

Proof. Immediate from Corollary 3.3. □

Remark 3.5 (Why light corresponds to a 45° Euclidean tilt). A common intuition is to regard a photon as a "purely spatial" entity, propagating orthogonally to the time-like flow N (i.e. tangent to the spatial 3–slices) as it happens under standard angle parametrization in 5.7. However, an observation is a causal relation between an emission event and a detection event. During the signal's flight the observer necessarily advances along N, so the measured separation between emission and detection in the proto–space is not a purely spatial displacement but a superposition of a spatial part and a temporal part:

$$X \;=\; X_{\parallel} + X_{\perp}, \qquad X_{\parallel} := (N\!\cdot\! X)\, N, \qquad X_{\perp} := X - (N\!\cdot\! X)\, N.$$

The operational content of "light propagation" is precisely that the induced Lorentzian interval vanishes,

$$g(X, X) = 0.$$

Using the split identity $g(X, X) = (N \cdot X)^2 - \delta(X_{\perp}, X_{\perp})$, this imposes the equality of the longitudinal and transverse δ–magnitudes,

$$\|X_{\parallel}\|_{\delta} \;=\; \|X_{\perp}\|_{\delta}. \tag{3.3}$$

Thus, although the photon is not "a spatial object evolving in time" in the same sense as a timelike body, the null separation between emission and detection corresponds in the Euclidean picture to an equal allocation of the total displacement budget between the N–direction and its orthogonal complement. Geometrically, (3.3) means that the null direction lies at a Euclidean tilt of $\vartheta = \pi/4$, which is the origin of the 45° light cone in the present parametrization.

3.4 Spatial rotations preserving δ and N

Let $\mathrm{Aut}(\delta, N)$ denote the stabilizer of N in the Euclidean orthogonal group:

$$\mathrm{Aut}(\delta, N) := \{\, L : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} :\; \delta(LX, LY) = \delta(X, Y),\; LN = N \,\}.$$

In an adapted δ–orthonormal basis $\{e_0 = N, e_1, e_2, e_3\}$ one has

$$L = \mathrm{diag}(1, R), \qquad R \in O(3),$$

so $\mathrm{Aut}(\delta, N) \cong O(3)$ and contains no boost–like maps mixing N with $N^{\perp_\delta}$.

Lemma 3.6. *Every $L \in \mathrm{Aut}(\delta, N)$ preserves g:*

$$g(LX, LY) = g(X, Y) \qquad \textit{for all } X, Y \in T_p\mathcal{E}.$$

Proof. Since $LN = N$ and L is δ–orthogonal,

$$g(LX, LY) = 2(N \cdot LX)(N \cdot LY) - \delta(LX, LY) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = g(X, Y).$$

□

Thus $\mathrm{Aut}(\delta, N)$ is a spatial subgroup of $\mathrm{O}(g)$: it preserves g and fixes N, but generates only Euclidean rotations on $N^{\perp_\delta}$.

4 Phase reparametrization and the operational origin of the Lorentz interval

This section makes explicit the logical bridge between the Euclidean proto–metric δ and the Lorentzian metric g constructed in §2. The key point is that an observer does *not* have direct access to an arbitrary curve parameter; instead, physical clocks implement a specific *phase* (or time) parametrization along worldlines. The Lorentzian interval arises as the quadratic form naturally associated with that operational parametrization.

4.1 A δ–affine proto–parameter χ

Let $X : I \to \mathcal{E}$ be a C^1 curve. For a parameter λ on I we set

$$X'(\lambda) := \frac{dX}{d\lambda} \in T_{X(\lambda)}\mathcal{E}.$$

Because $(\mathcal{E}, \delta)$ is Euclidean, one may always reparametrize X by a scaled δ–arc length. We single out the following convenient choice.

Definition 4.1 (δ–calibrated (proto–affine) parameter)**.** A parameter χ along X is called *δ–calibrated* (or *proto–affine*) if the δ–speed is constant and equal to c, i.e.

$$\delta(\tilde{X}, \tilde{X}) = c^2, \qquad \tilde{X} := \frac{dX}{d\chi}. \tag{4.1}$$

Remark 4.2. In flat Euclidean geometry this is simply a scaled arc–length parametrization: $d\chi = \frac{1}{c}\,\|dX\|_\delta$. It plays the role of a convenient *proto–phase clock* in $\mathcal{E}$, fixing the *total* flow budget $\|\tilde{X}\|_\delta = c$.

4.2 The time 1–form of N and the induced spatial projector

Throughout, N denotes the smooth δ–unit field of §2. Its δ–dual 1–form (the "time form" of the field) is

$$\alpha := \delta(N, \cdot), \qquad \text{i.e.} \qquad \alpha_A = \delta_{AB} N^B = N_A. \tag{4.2}$$

The spatial projector h was defined in (2.2) by $h_{AB} = \delta_{AB} - N_A N_B$. For any tangent vector V we therefore have the orthogonal split

$$\delta(V, V) = \alpha(V)^2 + h(V, V), \tag{4.3}$$

which is precisely the Pythagorean identity behind Lemma 5.1 ("full budget" minus "spatial budget" yields the squared longitudinal flow).

4.3 Coordinate time t versus proper time τ

The 1–form α defines the field–adapted coordinate time t along a worldline by the operational rule

$$dt := \frac{1}{c}\,\alpha(dX) \qquad \Longleftrightarrow \qquad \frac{dt}{d\lambda} = \frac{1}{c}\,\alpha(X'(\lambda)). \tag{4.4}$$

In general, (4.4) defines t only along a given worldline; a global time function t on an open set exists only under an integrability condition on α (e.g. α exact / N hypersurface–orthogonal).

Likewise, the induced spatial line element is defined by

$$d\ell^2 := h(dX, dX) \qquad \Longleftrightarrow \qquad \Big(\frac{d\ell}{d\lambda}\Big)^2 = h(X'(\lambda), X'(\lambda)). \tag{4.5}$$

Proposition 4.3 (Operational form of the Lorentz interval)**.** *For every curve X and every parameter λ one has*

$$g(dX, dX) = c^2\, dt^2 - d\ell^2, \tag{4.6}$$

where dt and $d\ell$ are given by (4.4)–(4.5)*.*

Proof. Using (2.4) and (4.2) we obtain for any V

$$g(V, V) = \alpha(V)^2 - h(V, V).$$

Apply this to $V = dX$ and substitute the definitions of dt and $d\ell$. □

We now define the *proper time* τ of the worldline as the g–arc length parameter:

Definition 4.4 (Proper time)**.** Along a g–timelike curve (i.e. $g(dX, dX) > 0$) the proper time is defined by

$$c^2\, d\tau^2 := g(dX, dX). \tag{4.7}$$

Equivalently, combining (4.7) with (4.6) yields

$$d\tau^2 = dt^2 - \frac{1}{c^2}\, d\ell^2. \tag{4.8}$$

4.4 Equivalence of δ–budget and the proper–time interval

Assume henceforth that χ is δ–calibrated in the sense of Definition 4.1. Set $\tilde{X} = dX/d\chi$. Define the *longitudinal* and *spatial* flow rates (per unit χ) by

$$S(\chi) := \alpha(\tilde{X}) = N\cdot\tilde{X}, \qquad L(\chi)^2 := h(\tilde{X}, \tilde{X}) = \delta(\tilde{X}_\perp, \tilde{X}_\perp), \tag{4.9}$$

so that (4.3) gives the exact budget identity

$$\underbrace{\delta(\tilde{X}, \tilde{X})}_{c^2} = \underbrace{S^2}_{\text{“observable longitudinal flow”}} + \underbrace{L^2}_{\text{“spatial expenditure”}}. \tag{4.10}$$

In particular, $S^2 = c^2 - L^2$ is precisely the statement emphasized in the interpretation of Lemma 5.1: the observable flow is reduced by allocating part of the total budget to motion in $N^{\perp_\delta}$.

Theorem 4.5 (Phase reparametrization $\chi \mapsto \tau$)**.** *Let $X(\chi)$ be δ–calibrated and g–*timelike *along the curve, i.e.*

$$g(\tilde{X}, \tilde{X}) > 0.$$

Then the proper time satisfies

$$\frac{d\tau}{d\chi} = \frac{1}{c}\sqrt{g(\tilde{X}, \tilde{X})} = \frac{1}{c}\sqrt{S^2 - L^2}. \tag{4.11}$$

Equivalently, reparametrizing the same geometric curve by τ (i.e. setting $\dot{X} := dX/d\tau$) yields the unit–speed condition

$$g(\dot{X}, \dot{X}) = c^2. \tag{4.12}$$

Proof. By Definition 4.4, $c^2(d\tau/d\chi)^2 = g(\tilde{X}, \tilde{X})$, which is (4.11). Then $\dot{X} = (d\chi/d\tau)\tilde{X}$ implies

$$g(\dot{X}, \dot{X}) = \left(\frac{d\chi}{d\tau}\right)^2 g(\tilde{X}, \tilde{X}) = \frac{c^2}{g(\tilde{X}, \tilde{X})}\, g(\tilde{X}, \tilde{X}) = c^2,$$

giving (4.12). □

Remark 4.6 (Why g is observed rather than δ)*.* The proto–metric δ measures the *total* flow budget with respect to the auxiliary parameter χ (a convenient but unobservable calibration). Physical clocks, however, implement the proper–time parametrization τ, defined by the Lorentzian interval (4.7). In this sense the observer “lives by its own phase”: the operational time variable is τ, and the quadratic form that controls it is g, not δ.

4.5 The tilt angle and the explicit χ–ϑ link

For completeness, define the tilt angle ϑ of $\tilde{X}$ by

$$\cos(2\vartheta) := \frac{g(\tilde{X}, \tilde{X})}{\delta(\tilde{X}, \tilde{X})}. \tag{4.13}$$

Under the χ–calibration $\delta(\tilde{X}, \tilde{X}) = c^2$, this becomes $g(\tilde{X}, \tilde{X}) = c^2 \cos(2\vartheta)$ and (4.11) reads

$$\frac{d\tau}{d\chi} = \sqrt{\cos(2\vartheta)}. \tag{4.14}$$

This is the explicit χ–ϑ relation.

5 Tilt angle geometry and hyperbolic parametrization

5.1 Euclidean tilt angle

For any nonzero $X \in T_p\mathcal{E}$ define its Euclidean tilt angle $\vartheta \in [0, \pi]$ by

$$\cos\vartheta := \frac{N \cdot X}{\|X\|_\delta}, \qquad \sin\vartheta := \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

where $X_\perp$ is defined by (3.1). Whenever $X_\perp \neq 0$, define the δ–unit transverse direction

$$E := \frac{X_\perp}{\|X_\perp\|_\delta} \in N_p^{\perp_\delta}.$$

Then

$$X = \|X\|_\delta \cos\vartheta\, N + \|X\|_\delta \sin\vartheta\, E. \tag{5.1}$$

Lemma 5.1. *For any nonzero $X \in T_p\mathcal{E}$,*

$$\|X\|_\delta^2 = (N \cdot X)^2 + \delta(X_\perp, X_\perp), \qquad \cos^2\vartheta + \sin^2\vartheta = 1.$$

Proof. Immediate from $\delta(X_\|, X_\perp) = 0$. □

5.2 Lorentzian norm expressed via ϑ

From Proposition 3.2,

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp) = \|X\|_\delta^2(\cos^2\vartheta - \sin^2\vartheta) = \|X\|_\delta^2 \cos(2\vartheta).$$

Proposition 5.2. *For any nonzero $X \in T_p\mathcal{E}$,*

$$g(X, X) = \|X\|_\delta^2 \cos(2\vartheta).$$

5.3 Domain of hyperbolic parametrization

A real hyperbolic parameter is naturally attached to vectors in the future time cone $\mathcal{T}_p^+$ defined in (3.2). For $X \in \mathcal{T}_p^+$ one has

$$g(X, X) > 0 \iff \cos(2\vartheta) > 0 \iff \vartheta \in \left[0, \frac{\pi}{4}\right),$$

and moreover $N \cdot X > 0$ implies $\cos\vartheta > 0$, so

$$\beta := \tan\vartheta \in [0, 1).$$

Null vectors satisfy $\tan\vartheta = 1$ (equivalently $\vartheta = \pi/4$), while g–negative vectors have $\tan\vartheta > 1$.

5.4 Hyperbolic parameter (rapidity)

For $X \in \mathcal{T}_p^+$ define $\eta \geq 0$ by

$$\tanh\eta := \tan\vartheta. \tag{5.2}$$

Equivalently, one may define η invariantly by

$$\cosh\eta := \frac{N \cdot X}{\sqrt{g(X, X)}}, \qquad \sinh\eta := \frac{\|X_\perp\|_\delta}{\sqrt{g(X, X)}}, \qquad (X \in \mathcal{T}_p^+), \tag{5.3}$$

which immediately implies $\tanh\eta = \|X_\perp\|_\delta/(N \cdot X) = \tan\vartheta$.

Lemma 5.3. *For $X \in \mathcal{T}_p^+$,*

$$\cosh\eta = \frac{\cos\vartheta}{\sqrt{\cos(2\vartheta)}}, \qquad \sinh\eta = \frac{\sin\vartheta}{\sqrt{\cos(2\vartheta)}}.$$

Proof. From (5.2), $\tanh\eta = \tan\vartheta$ gives

$$\cosh^2\eta = \frac{1}{1-\tanh^2\eta} = \frac{1}{1-\tan^2\vartheta} = \frac{\cos^2\vartheta}{\cos^2\vartheta - \sin^2\vartheta} = \frac{\cos^2\vartheta}{\cos(2\vartheta)}.$$

Taking the positive square root (since $\eta \geq 0$ and $\vartheta \in [0, \pi/4)$) yields the expression for $\cosh\eta$, and multiplying by $\tanh\eta = \tan\vartheta$ yields $\sinh\eta$. $\square$

5.5 Differential relation between η and ϑ

Proposition 5.4. *For $X \in \mathcal{T}_p^+$,*

$$\frac{d\eta}{d\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

Proof. Differentiate $\tanh\eta = \tan\vartheta$:

$$\operatorname{sech}^2\eta\, d\eta = \sec^2\vartheta\, d\vartheta.$$

Using $\operatorname{sech}^2\eta = 1 - \tanh^2\eta = 1 - \tan^2\vartheta = \dfrac{\cos(2\vartheta)}{\cos^2\vartheta}$, we obtain

$$\frac{d\eta}{d\vartheta} = \frac{\sec^2\vartheta}{\operatorname{sech}^2\eta} = \frac{1/\cos^2\vartheta}{\cos(2\vartheta)/\cos^2\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

$\square$

5.6 Boost subgroup and additivity of the hyperbolic parameter

Let $\mathrm{O}(g)$ denote the Lorentz group of $(T_p\mathcal{E}, g)$:

$$\mathrm{O}(g) := \{\, \Lambda : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \; g(\Lambda X, \Lambda Y) = g(X, Y) \,\}.$$

Fix a δ–unit transverse direction $E \in N_p^{\perp_\delta}$ with $\delta(E, E) = 1$. The *boost* in the 2–plane $\operatorname{span}\{N, E\}$ with parameter η is the unique $\Lambda(\eta) \in \mathrm{O}(g)$ acting as a hyperbolic rotation on $\operatorname{span}\{N, E\}$ and as the identity on its g–orthogonal complement:

$$\Lambda(\eta)N = (\cosh\eta)\, N + (\sinh\eta)\, E, \qquad \Lambda(\eta)E = (\sinh\eta)\, N + (\cosh\eta)\, E,$$

$$\Lambda(\eta)X = X \quad \text{for } X \perp_g \operatorname{span}\{N, E\}.$$

Such boosts preserve g but, in general, do not preserve δ and do not fix N.

Theorem 5.5 (Additivity)**.** *For boosts $\Lambda(\eta_1)$ and $\Lambda(\eta_2)$ in the same (N, E)–plane, their composition is a boost with parameter $\eta_1 + \eta_2$:*

$$\Lambda(\eta_1) \circ \Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2).$$

Proof. On $\operatorname{span}\{N, E\}$ the boosts are represented (in the basis $\{N, E\}$) by the matrices

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix},$$

whose multiplication adds rapidities. On the g–orthogonal complement the action is the identity, hence the statement holds on all of $T_p\mathcal{E}$. $\square$

5.7 Comparison with classical angle conventions

The Euclidean angle ϑ above coincides with the geometric tilt angle used in classical constructions (e.g. Karapetoff) and in later reformulations; the difference lies in which trigonometric function is taken as the primary dimensionless parameter.

A common choice is

$$\beta_{\text{sin}} := \sin\vartheta = \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

whereas in the present work we use the tangent–based parameter

$$\beta_{\text{phase}} := \tan\vartheta = \frac{\|X_\perp\|_\delta}{N\cdot X}.$$

On $\mathcal{T}_p^+$ one has $\beta_{\text{phase}} \in [0,1)$, and the rapidity η is introduced directly by (5.2).

Remark 5.6 (Photon limit and the 45° Euclidean tilt)*.* The null cone is characterized by $g(X,X)=0$, equivalently $\cos(2\vartheta)=0$, so that the lightlike limit corresponds to $\vartheta\to\pi/4$ in the Euclidean picture. In this limit one has

$$\beta_{\text{phase}} = \tan\vartheta \to 1, \qquad \beta_{\text{sin}} = \sin\vartheta \to \frac{1}{\sqrt{2}}.$$

We emphasize that the physical speed parameter in this section is $\beta := v/c = \tan\vartheta$; the sine–parameter is a bounded reparameterization,

$$\beta_{\text{sin}} = \sin\vartheta = \frac{\beta}{\sqrt{1+\beta^2}}.$$

Thus a light ray is reached at a finite Euclidean tilt of 45° relative to N (not at 90°). This is precisely why the tangent parameterization is better adapted to the projector identity $g(X,X) = \|X\|_\delta^2\cos(2\vartheta)$: it saturates at the speed–of–light barrier as $\vartheta\to\pi/4$, while the sine parameter does not.

Remark 5.7 (Nonadditivity of the Euclidean tilt angle ϑ)*.* The tilt angle ϑ is a convenient Euclidean coordinate on the calibrated sphere S_c^3, defined by $\beta = v/c = \tan\vartheta$ via the orthogonal decomposition $\tilde{X} = \tilde{X}_\parallel + \tilde{X}_\perp$. It is therefore not a group parameter for boosts. Even in the collinear case, where rapidities add, $\eta_{12} = \eta_1 + \eta_2$, the corresponding tilt angles do not add:

$$\vartheta_{12} \neq \vartheta_1 + \vartheta_2.$$

Indeed, the collinear velocity–composition law

$$\beta_{12} = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

translates into

$$\tan\vartheta_{12} = \frac{\tan\vartheta_1 + \tan\vartheta_2}{1+\tan\vartheta_1\tan\vartheta_2},$$

so ϑ is a nonlinear reparameterization of the additive rapidity.

This choice is adapted to the orthogonal splitting (3.1) and to the identity of Proposition 5.2,

$$g(X,X) = \|X\|_\delta^2\cos(2\vartheta),$$

so that the domain of the hyperbolic parametrization is exactly the future g–time cone $\mathcal{T}_p^+$, without additional postulates.

6 Flow invariants and the emergence of S^3 in the Euclidean proto-space

This section makes precise a key equivalence of the proto–space approach: invariance of the Minkowski interval in an observer's local time is equivalent to constancy of the full proto–space flow vector with respect to a calibrated proto–parameter.

6.1 Worldlines, proto–parameter, and the full flow vector

Let $X : I \to \mathcal{E}$ be a smooth timelike worldline. A *proto–parameter* χ along X is any smooth parameter with nowhere–vanishing derivative. We define the associated *full flow vector* (the proto–space tangent) by

$$\tilde{X} \;:=\; \frac{dX}{d\chi} \;\in T_{X(\chi)}\mathcal{E}. \tag{6.1}$$

Definition 6.1 (Calibrated proto–parameter)**.** A proto–parameter χ is called *calibrated* (with scale c) if

$$\delta(\tilde{X}, \tilde{X}) \;=\; c^2 \qquad \text{along } X. \tag{6.2}$$

Remark 6.2 (Existence and gauge nature). For any timelike worldline X, a calibrated proto–parameter always exists. Indeed, if τ denotes proper time, one may define χ by $d\chi/d\tau = \|\dot{X}\|_\delta/c$. Fixing the time orientation ($d\chi/d\tau > 0$), the resulting calibrated parameter is unique up to an additive constant.

In words: in a calibrated proto–parameter, the full flow vector $\tilde{X}$ has a fixed Euclidean norm in the proto–space. This is the proto–space counterpart of the standard SR statement that the 4–velocity has fixed Minkowski norm in proper time.

6.2 Observer splitting and the interval–rate identity

Fix the distinguished unit field N (hence g and h) as in §2–§3. Pointwise along X, decompose $\tilde{X}$ into δ–longitudinal and δ–transverse parts relative to N:

$$\tilde{X} \;=\; \tilde{H}\,N + \tilde{X}_\perp, \qquad \tilde{H} := N \cdot \tilde{X}, \qquad \tilde{X}_\perp := h(\tilde{X}) \in \mathrm{Im}(h). \tag{6.3}$$

Let $\tilde{L} := \|\tilde{X}_\perp\|_\delta$ and, when $\tilde{L} \neq 0$, $E := \tilde{X}_\perp/\tilde{L} \in \mathrm{Im}(h)$ so that $\tilde{X} = \tilde{H}N + \tilde{L}E$ with $\delta(E, E) = 1$.

Lemma 6.3 (Euclidean and Lorentzian norms of the flow)**.** *Along X one has the identities*

$$\delta(\tilde{X}, \tilde{X}) = \tilde{H}^{\,2} + \tilde{L}^{\,2}, \qquad g(\tilde{X}, \tilde{X}) = \tilde{H}^{\,2} - \tilde{L}^{\,2}. \tag{6.4}$$

Proof. Since $\tilde{X}_\perp \in N^{\perp_\delta}$, we have $\delta(N, \tilde{X}_\perp) = 0$. Hence $\delta(\tilde{X}, \tilde{X}) = \tilde{H}^2 + \delta(\tilde{X}_\perp, \tilde{X}_\perp) = \tilde{H}^2 + \tilde{L}^2$. For g, use $g(N, N) = 1$, $g(N, \tilde{X}_\perp) = 0$, and $g(\tilde{X}_\perp, \tilde{X}_\perp) = -\delta(\tilde{X}_\perp, \tilde{X}_\perp) = -\tilde{L}^2$. □

Motivated by the phase–formalism viewpoint, we introduce the *interval rate* with respect to χ:

$$\tilde{S}^{\,2} \;:=\; g(\tilde{X}, \tilde{X}) \;=\; \tilde{H}^{\,2} - \tilde{L}^{\,2}. \tag{6.5}$$

This is the precise proto–space analogue of writing $ds^2 = g(dX, dX) = \tilde{S}^{\,2}\, d\chi^2$ and viewing $\tilde{S}$ as a "Minkowski projection" of the full Euclidean flow.

6.3 Equivalence: invariant interval in local time $\Longleftrightarrow$ calibrated full flow

Let τ denote the *local time* (proper time) along the timelike curve X, i.e. a parameter such that the tangent $\dot{X} := dX/d\tau$ satisfies

$$g(\dot{X}, \dot{X}) = c^2. \tag{6.6}$$

Equivalently, $ds^2 = g(dX, dX) = c^2\, d\tau^2$ along X.

Remark 6.4 (Units and normalization). The proper–time tangent $\dot{X} = dX/d\tau$ has the physical dimension of a speed and is normalized by (6.6). For kinematical formulas it is convenient to use the associated *dimensionless* unit 4–velocity

$$U \;:=\; \frac{1}{c}\,\dot{X}, \qquad \text{so that} \qquad g(U, U) = 1.$$

In the remainder of the paper, an "observer" will be represented by such a unit timelike U; whenever U arises from a worldline it is implicitly understood as the normalized proper–time tangent.

Theorem 6.5 (Reparameterization equivalence)**.** *Let X be a regular g–timelike worldline (geometric curve). Then the following statements are equivalent up to reparameterization:*

(A) X is parameterized by proper time τ so that $g(\dot{X}, \dot{X}) = c^2$.

(B) X is parameterized by a calibrated proto–parameter χ so that $\delta(\tilde{X}, \tilde{X}) = c^2$.

Moreover, when both parameters are used on the same curve, they are related by

$$\frac{d\tau}{d\chi} \;=\; \frac{\sqrt{g(\tilde{X}, \tilde{X})}}{c} \;=\; \frac{\tilde{S}}{c}, \qquad \frac{d\chi}{d\tau} \;=\; \frac{\|\dot{X}\|_\delta}{c}. \tag{6.7}$$

Proof. Assume (A). Define χ (up to an additive constant) by the ODE

$$\frac{d\chi}{d\tau} := \frac{\|\dot{X}\|_\delta}{c},$$

which is smooth and positive since $\dot{X} \neq 0$. Then $\tilde{X} = dX/d\chi = (d\tau/d\chi)\dot{X}$, so

$$\delta(\tilde{X}, \tilde{X}) = \Big(\frac{d\tau}{d\chi}\Big)^2 \delta(\dot{X}, \dot{X}) = \frac{c^2}{\|\dot{X}\|_\delta^2}\, \|\dot{X}\|_\delta^2 = c^2,$$

which is (B).

Conversely, assume (B). Define τ (up to an additive constant) by

$$\frac{d\tau}{d\chi} := \frac{\sqrt{g(\tilde{X}, \tilde{X})}}{c},$$

which is well–defined and positive for timelike $\tilde{X}$ since $g(\tilde{X}, \tilde{X}) > 0$. Then $\dot{X} = dX/d\tau = (d\chi/d\tau)\tilde{X}$, hence

$$g(\dot{X}, \dot{X}) = \Big(\frac{d\chi}{d\tau}\Big)^2 g(\tilde{X}, \tilde{X}) = \frac{c^2}{g(\tilde{X}, \tilde{X})}\, g(\tilde{X}, \tilde{X}) = c^2,$$

which is (A). The relations (6.7) are exactly the two defining ODEs. $\square$

Operational meaning. Statement (A) is the standard SR normalization of the 4–velocity in proper time, $g(\dot{X}, \dot{X}) = c^2$. Statement (B) is the corresponding calibration of the proto–parameter, $\delta(\tilde{X}, \tilde{X}) = c^2$. Theorem 6.5 shows these are equivalent and amount to a change of parameter ("norm vs projection"), rather than an additional dynamical assumption.

6.4 The S^3 of admissible flow states

Fix $p \in \mathcal{E}$. The set of all calibrated flow vectors at p is the Euclidean 3–sphere of radius c inside $T_p\mathcal{E}$:

$$S^3_c(p) \;:=\; \big\{\, V \in T_p\mathcal{E} : \; \delta(V,V) = c^2 \,\big\} \;\cong\; S^3. \tag{6.8}$$

Thus, once the calibration (6.2) is imposed, every instantaneous *kinematic state* in the proto–space is a point of an S^3 in the tangent space.

Relative to a chosen time direction N_p, each $V \in S^3_c(p)$ admits the decomposition

$$V = \tilde{H}\, N + \tilde{L}\, E, \qquad \tilde{H}^2 + \tilde{L}^2 = c^2, \qquad E \in \mathrm{Im}(h_p),\; \delta(E,E) = 1,$$

so that the observable "spatial direction" is carried by $E \in S^2 \subset \mathrm{Im}(h_p)$ while the pair $(\tilde{H}, \tilde{L})$ lies on a circle $\tilde{H}^2 + \tilde{L}^2 = c^2$. When $\tilde{L} = 0$, the direction E is irrelevant (any unit choice in $\mathrm{Im}(h_p)$ yields the same vector), and the state reduces to $V = \pm c\, N_p$. This is precisely the geometric reason why S^3 is the natural state manifold for calibrated flows in the Euclidean proto–space: it is the locus of constant full flow magnitude, whereas Lorentzian interval effects arise from the g–projection (6.5) and the reparameterization (6.7).

Remark 6.6 (The physical sector of S^3). While the calibrated flow states form the full sphere

$$S^3_c(p) = \{\, \tilde{X} \in T_p\mathcal{E} : \; \delta(\tilde{X}, \tilde{X}) = c^2 \,\},$$

the requirement that a worldline be timelike with respect to g,

$$g(\tilde{X}, \tilde{X}) > 0,$$

restricts the physically realizable states to the open subset

$$|\tilde{H}| > |\tilde{L}|, \qquad \tilde{H} := N \cdot \tilde{X}, \quad \tilde{L} := \|h(\tilde{X})\|_\delta.$$

Geometrically, this is the union of two polar caps on $S^3_c(p)$ centered at $\pm cN$. The boundary $|\tilde{H}| = |\tilde{L}|$ is the null locus (lightlike curves). In the timelike sector one has $d\tau/d\chi = \sqrt{g(\tilde{X}, \tilde{X})}/c$; hence, along any timelike sequence approaching the null locus, $d\tau/d\chi \to 0$, consistently with (6.7).

7 Why the Euclidean proto-space viewpoint

The Lorentzian metric g constructed from (δ, N) endows each tangent space $(T_p\mathcal{E}, g)$ with the usual causal structure. The additional advantage of the Euclidean proto–space viewpoint is that it keeps, in the same object, both (i) the observable *spatial* direction data encoded in $\mathrm{Im}(h_p) = N_p^{\perp_\delta}$ and (ii) the *calibration* data (frequencies, time rates) encoded by g–projections onto time directions. In this sense, "light" is not an extra entity but a geometric slice of the null cone by the calibrated Euclidean sphere $S^3_c \subset T_p\mathcal{E}$ defined by $\delta(V,V) = c^2$.

7.1 Light rays: Derivation of frequency and direction

Fix $p \in \mathcal{E}$. A light ray is represented by a nonzero null vector $K \in T_p\mathcal{E}$, satisfying $g(K,K) = 0$ and future-directed $(N \cdot K > 0)$. Rather than postulating a parametrization, we derive it from the geometry of the N-split.

Definition 7.1 (Proto-frequency)**.** The frequency of the ray K measured by the distinguished observer N is defined as the g-projection:

$$\omega := g(K, N). \tag{7.1}$$

Note that due to the structure of the metric (2.3), this coincides with the Euclidean projection:

$$g(K, N) = 2(N \cdot K)(N \cdot N) - \delta(K, N) = 2(N \cdot K) - (N \cdot K) = N \cdot K.$$

Thus, $\omega = N \cdot K > 0$.

Now, decompose K into longitudinal and transverse parts relative to N:

$$K = \omega N + K_\perp, \qquad K_\perp \in N^{\perp_\delta}.$$

The null condition $g(K, K) = 0$ implies:

$$0 = g(K, K) = \omega^2 g(N, N) + g(K_\perp, K_\perp) = \omega^2 - \|K_\perp\|_\delta^2.$$

Hence $\|K_\perp\|_\delta = \omega$. We can therefore write $K_\perp = \omega E$, where E is a unique δ-unit spatial direction ($E \in N^{\perp_\delta}, \|E\|_\delta = 1$).

Lemma 7.2 (Canonical form)**.** *Every future-directed null vector K admits the unique decomposition:*

$$K = \omega\,(N + E), \tag{7.2}$$

where $\omega = g(K, N)$ is the frequency and E is the observable propagation direction.

7.2 Observers as unit timelike states; measured frequency as a projection

In the proto–space formalism, a (local) observer is represented by a unit future timelike vector (dimensionless, cf. Remark 6.4)

$$U \in T_p\mathcal{E}, \qquad g(U, U) = 1, \qquad g(U, N) > 0.$$

All measurable scalars are obtained by taking g–contractions.

Definition 7.3 (Frequency measured by an observer)**.** For a null ray $K \neq 0$, the frequency measured by the observer U is

$$\omega_U := g(U, K). \tag{7.3}$$

Remark 7.4 (Ray vector vs. wave covector)*.* We represent a light ray by a future–directed null vector $K \in T_p\mathcal{E}$. Its metric dual 1–form

$$k \;:=\; g(K, \cdot) \;\in T_p^*\mathcal{E}$$

is the standard wave covector of geometric optics. With this notation, the measured frequency can be written equivalently as $\omega_U = k(U) = g(U, K)$.

This is the standard invariant definition used in geometric optics in SR. In the present framework, however, (7.3) will become an explicit function of Euclidean angles in $\mathrm{Im}(h_p)$.

7.3 Doppler shift

Let $E_v \in \mathrm{Im}(h_p)$ be a δ–unit direction,

$$\delta(E_v, E_v) = 1,$$

and let U be the observer obtained from N by a boost of rapidity $\eta \geq 0$ in the (N, E_v)–plane:

$$U := (\cosh \eta)\, N + (\sinh \eta)\, E_v. \tag{7.4}$$

Let the null ray be given by (7.2),

$$K = \omega\,(N + E), \qquad \delta(E, E) = 1, \quad E \in \mathrm{Im}(h_p).$$

Then, using bilinearity and the basic identities $g(N, N) = 1$, $g(N, E) = 0$, $g(E_v, N) = 0$, and $g(E_v, E) = -\delta(E_v, E)$, we obtain

$$\begin{aligned} \omega_U &= g(U, K) \\ &= \omega\, g\big((\cosh \eta)N + (\sinh \eta)E_v,\ N + E\big) \\ &= \omega\Big(\cosh \eta - \sinh \eta\, \delta(E_v, E)\Big). \end{aligned} \tag{7.5}$$

Introduce

$$\beta := \tanh \eta, \qquad \gamma := \cosh \eta,$$

and define the Euclidean angle $\psi \in [0, \pi]$ between the velocity axis E_v and the ray direction E in $\mathrm{Im}(h_p)$ by

$$\cos \psi := \delta(E_v, E). \tag{7.6}$$

Then (7.5) becomes the standard relativistic Doppler law:

$$\frac{\omega_U}{\omega} = \gamma\big(1 - \beta \cos \psi\big). \tag{7.7}$$

Interpretation. Equation (7.7) is obtained here as a direct function of an ordinary Euclidean angle on the δ–unit sphere inside $\mathrm{Im}(h_p)$ (i.e. on the δ–unit sphere in $\mathrm{Im}(h_p)$). In this sense the Doppler shift is simply "projection geometry" in the proto–space: the measured frequency is the g–projection of a null direction onto an observer state.

7.4 Aberration as projection plus normalization

The direction of the ray measured by U is the normalized spatial part of K in the g–orthogonal complement of U. Define the g–spatial component of K relative to U by

$$K_{\perp U} := K - (g(U, K))\, U. \tag{7.8}$$

Then $g(U, K_{\perp U}) = 0$, so $K_{\perp U} \in U^{\perp_g}$, and its g–norm is

$$g(K_{\perp U}, K_{\perp U}) = -\,\omega_U^2 \qquad (\text{since } g(K, K) = 0,\ g(U, U) = 1).$$

Hence the *unit* spatial direction of the ray in the U–frame may be taken as

$$E_U := \frac{1}{\omega_U}\, K_{\perp U} = \frac{1}{g(U, K)}\Big(K - (g(U, K))U\Big), \qquad g(E_U, E_U) = -1, \quad g(U, E_U) = 0. \tag{7.9}$$

To extract the usual aberration formula, specialize to the same kinematics as in §7.3, i.e. U as in (7.4) and $K = \omega(N + E)$. Let ψ' denote the angle between the ray and the velocity axis in the U*–rest space*. Equivalently, $\cos \psi'$ is the (Euclidean) cosine of the angle between the spatial ray direction measured by U and the axis of motion, which is encoded invariantly by the contraction of E_U with the boost axis.

A direct computation (substituting (7.4) and $K = \omega(N + E)$ into (7.9) and taking the component along the U–spatial image of E_v) yields the standard aberration law:

$$\cos\psi' = \frac{\cos\psi - \beta}{1 - \beta\cos\psi}, \tag{7.10}$$

with $\cos\psi = \delta(E_v, E)$ as in (7.6).

Interpretation. Aberration and Doppler are the same operation in two steps:

- Doppler: take the g–projection $g(U, K)$ (a scalar).
- Aberration: subtract the time component $(g(U, K))U$ and normalize the remaining U–spatial part.

In the proto–space picture, both effects are thus immediate consequences of the null cone geometry together with the observer–dependent splitting induced by U.

7.5 Geometric Algebra perspective: The Euclidean origin of Lorentz rotation

The Euclidean proto-space framework admits a natural algebraic structure provided by the Clifford algebra $\mathcal{C}\ell(\mathcal{E}, \delta)$. Let the geometric product of vectors $u, v \in T_p\mathcal{E}$ be defined by the fundamental relation

$$uv = \delta(u, v) + u \wedge v.$$

Throughout this subsection we distinguish the Euclidean Clifford algebra $\mathcal{C}\ell(T_p\mathcal{E}, \delta)$ from the Lorentzian one $\mathcal{C}\ell(T_p\mathcal{E}, g)$ induced by (δ, N). Concretely, the metric extensor $G := 2\,N \otimes N - \mathrm{Id}$ satisfies $g(u, v) = \delta(Gu, v)$ and transports the δ–product to the g–product, making precise in which algebra a given rotor (circular vs. hyperbolic) is computed.

Unlike the standard Spacetime Algebra which postulates a Lorentzian signature, here the algebra is strictly Euclidean ($\mathcal{C}\ell_{4,0}$). The distinguished field N serves as the generator of the *space-time split*.

Observable algebra. Multiplication of any vector X by the distinguished element N decomposes it into a scalar part (time) and a bivector part (space) relative to N:

$$XN = \underbrace{X \cdot N}_{\text{scalar}} + \underbrace{X \wedge N}_{\text{bivector}}. \tag{7.11}$$

Identifying the bivectors containing N with the spatial vectors of the observer, the algebra of observables corresponds to the even subalgebra $\mathcal{C}\ell^{+}_{4,0}$, which is isomorphic to the Pauli algebra (and the quaternions).

The kinematic results of Section 6 find their most natural expression in the language of Geometric Algebra $\mathcal{C}\ell(\mathcal{E}, \delta)$, applying the principles of Space-Time Algebra pioneered by Hestenes [5, 6]. Specifically, we utilize the observer-dependent "space-time split" technique. This formalism clarifies that the transition from a Euclidean rotation to a Lorentzian boost is not an ad-hoc analytic continuation, but a direct consequence of the metric signature changing the algebraic properties of the basis vectors.

Euclidean rotation. In the Euclidean proto-space $(\mathcal{E}, \delta)$, both the distinguished vector N and any transverse spatial direction E square to $+1$:

$$N^2 = 1, \qquad E^2 = 1.$$

The plane spanned by them is described by the bivector $\mathbf{I} = NE$. Due to the Euclidean signature, this generator squares to -1:

$$\mathbf{I}^2 = (NE)(NE) = -N^2E^2 = -(1)(1) = -1.$$

Consequently, the exponential of this bivector generates standard trigonometric rotations. The kinematic relationship between two flow vectors $\tilde{X}_1, \tilde{X}_2 \in S_c^3$ is governed by the rotor $R = e^{-\mathbf{I}\vartheta/2}$:

$$\tilde{X}_2 = R\tilde{X}_1\tilde{R}^{-1} = \tilde{X}_1 \cos\vartheta + \tilde{X}_{1\perp} \sin\vartheta.$$

Lorentzian boost via vector substitution. The construction of the Lorentzian metric g effectively replaces the Euclidean spatial vector E with a physical spatial vector $\mathbf{e}$ which, while parallel to E, squares to -1 in the metric g:

$$\mathbf{e}^2 \overset{g}{=} -1, \qquad \text{while } N^2 \overset{g}{=} 1.$$

This change in the basis vector fundamentally alters the bivector describing the time-space plane. The new generator (the boost bivector) $\mathbf{K} = N\mathbf{e}$ satisfies:

$$\mathbf{K}^2 = (N\mathbf{e})(N\mathbf{e}) = -N^2\mathbf{e}^2 = -(1)(-1) = +1.$$

Because the generator now squares to $+1$, its exponential series produces hyperbolic functions instead of trigonometric ones. Using the rapidity η (related to tilt by $\tanh\eta = \tan\vartheta$), the physical transformation becomes a boost $L = e^{-\mathbf{K}\eta/2}$:

$$L = \cosh\frac{\eta}{2} - \mathbf{K}\sinh\frac{\eta}{2}.$$

Applying this to the rest vector N yields the 4-velocity U:

$$U = LNL^{-1} = N\cosh\eta + \mathbf{e}\sinh\eta.$$

Thus, the phenomenon of "Lorentzian" kinematics emerges algebraically simply because the spatial part of the basis becomes "imaginary" (squares to -1) relative to the time direction, flipping the sign of the bivector square and transitioning the geometry from circular to hyperbolic.

7.6 Scope and limitations

The present work is purely kinematical. We do not introduce dynamical equations for the field N, nor do we model localized sources or topological defects. Such extensions may be investigated separately, once the induced–metric framework is combined with appropriate field dynamics.

8 Discussion

8.1 What this reformulation does (and does not) claim

The primary contribution of this paper is a coordinate–free geometric reformulation of special–relativistic kinematics in a Euclidean phase space $(\mathcal{E}, \delta)$ equipped with an induced Lorentzian metric $g = 2\,\alpha\otimes\alpha - \delta$ generated by a unit field N (or $\alpha := \delta(N, \cdot)$). The reformulation is designed to make the kinematic content of SR visible as simple Euclidean geometry on calibrated spheres S_c^3 and in the orthogonal decomposition $X = X_\parallel + X_\perp$. At the same time, we emphasize that this construction is not a "Euclideanization" of spacetime physics: physical statements are made with respect to g, and δ serves as an underlying bookkeeping geometry for calibration and decomposition.

8.2 On the proto–parameter χ, proper time τ, and the locality of t

A central technical device is the calibrated proto–parameter χ, defined by $\delta(\tilde{X}, \tilde{X}) = c^2$, which provides a uniform Euclidean arclength normalization in $\mathcal{E}$. Proper time τ remains the g–invariant parameter along timelike worldlines, related to χ by $d\tau/d\chi = \sqrt{g(\tilde{X}, \tilde{X})}/c$ (Theorem 4.5). The quantity t introduced via $dt = \alpha(dX)/c$ should be understood as a line integral along a given worldline; a global time coordinate exists only under additional integrability conditions on α (e.g. hypersurface orthogonality).

8.3 Angle conventions and comparison with standard SR

The reformulation naturally introduces a Euclidean tilt angle ϑ via $\tan\vartheta = \|X_\perp\|_\delta/(N\cdot X)$, so that the physical speed parameter is $\beta = v/c = \tan\vartheta$, while alternative bounded parameters such as $\sin\vartheta$ may be used as auxiliary reparameterizations. This choice is adapted to the orthogonal decomposition and yields compact identities such as $g(X, X) = \|X\|_\delta^2 \cos(2\vartheta)$, making the timelike/null/spacelike sectors transparent in the calibrated geometry.

8.4 Advantages, limitations, and potential points of confusion

The main advantage is conceptual: several standard SR relations (Doppler, aberration) reduce to Euclidean projection identities once g is induced by (δ, N) and the calibration is fixed. A limitation is that, for general (nonintegrable) N, the "coordinate time" t is not globally defined, and the formalism should be interpreted locally or along worldlines. Another limitation is that the present paper focuses on inertial kinematics; extensions to accelerated motion and to dynamical fields require additional structure (e.g. evolution laws for N and/or α).

8.5 Relation to other constructions and outlook

The induced–metric mechanism is compatible with known recipes for obtaining Lorentzian metrics from Riemannian data (cf. [4]), but the present emphasis is different: we treat S_c^3 calibration and the tilt geometry as the primary kinematic arena. Natural next steps include: (i) a systematic treatment of accelerated observers in the calibrated picture and the emergence of Wigner–Thomas rotation; (ii) coupling to a field sector where N (or α) becomes dynamical; and (iii) identifying observationally accessible invariants that remain compact in the calibrated geometry while reproducing the standard SR predictions.

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