Unimetry: A Phase-Space Reformulation of Special Relativity

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Abstract

We propose a compact reformulation of special relativity in which spacetime units (time and length) are treated as *phase velocities*—directional derivatives of a single underlying parameter, the phase $\vec{\chi} \in \mathbb{C}$. The observable Minkowski interval emerges as a conserved quantity under a change of parameter from the hidden phase coordinate χ to the observer's proper time τ. In this *unimetry* formalism, familiar relativistic effects—time dilation, Lorentz factor, Doppler shift, and relativistic velocity composition—arise as elementary projections and rotations in a Euclidean phase plane. Hyperbolic features of Lorentz kinematics reappear after a reparameterization of time, yielding the standard relations without altering empirical content. We provide closed-form derivations of the longitudinal/transverse Doppler factors, identify a simple lemma equating the total phase speed to the conserved Minkowski norm, and outline connections to gauge phases, rapidity, and a cosmological time gauge.

Keywords: special relativity; phase; rapidity; Doppler shift; Lorentz factor; phase parameterization.

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1 Introduction

We usually take time and space as primitive. The phase formalism introduced here suggests a different viewpoint: time and space are derived projections of a single parameter $\vec{\chi} \in \mathbb{C}$ ("phase"). In this picture, relativistic effects such as time dilation and the Doppler shift are geometric consequences of phase-flow rotations.

The proposal does not modify physics; it reorganizes familiar relations in a simpler language. In spirit it is akin to Lagrangian/Hamiltonian re-descriptions of classical mechanics: same empirical content, different coordinates. Throughout, Greek θ will denote the external rotation angle associated with relative motion, while ζ denotes an internal angle associated with the object's intrinsic state (mass/density heuristic). We emphasize that no modification of Einstein's dynamics is proposed; all results are kinematical identities obtained by a change of parameter.

Notation. Tildes, dots and primes indicate derivatives with respect to the phase parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \qquad \dot{X} := \frac{dX}{d\tau}, \qquad X' := \frac{dX}{dl}. \tag{1.1}$$

We use c for the speed of light; $\beta := V/c$, $\gamma := 1/\sqrt{1-\beta^2}$, rapidity $\tanh \eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

2 Time and space as phase derivatives

Phase as a 1-form. We model the phase as a differential 1-form $\boldsymbol{\chi}$ on an unlimited-dimensional ambient space (a proto-space). Its evaluation on a trajectory yields a scalar parameter χ; phase-flow is the pushforward of the worldline by this 1-form. Observables are projections of the phase-flow onto temporal and spatial directions.

Quaternion representation. For concrete kinematics we adopt a quaternion representation in which the scalar (real) part encodes the temporal projection and the vector (imaginary) part encodes the spatial projection. A unit *D-rotor* (boost) is

$$d(\hat{\mathbf{u}}, \psi) \;=\; \cos\frac{\psi}{2} \;+\; \hat{\mathbf{u}}\,\sin\frac{\psi}{2}, \tag{2.1}$$

with unit pure quaternion $\hat{\mathbf{u}} \in \mathrm{span}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$. An *R-rotor* (spatial rotation) is $r(\hat{\mathbf{n}}, \varphi) = \cos\frac{\varphi}{2} + \hat{\mathbf{n}}\,\sin\frac{\varphi}{2}$. Time and space then appear as derivatives (projections) of the phase-flow with respect to the chosen parameter:

$$\tilde{X} \equiv \frac{dX}{d\chi}, \qquad \dot{X} \equiv \frac{dX}{d\tau}, \qquad X' \equiv \frac{dX}{dl}, \tag{2.2}$$

and boosts act by $\mathbf{q} \mapsto d\,\mathbf{q}\,d$, while spatial rotations act by $\mathbf{q} \mapsto r\,\mathbf{q}\,r^{-1}$.

Quaternionic phase (replacing the quaternionic ansatz). Henceforth the phase is *quaternionic*, not quaternionic. We write a unit rotor as

$$d(\hat{\mathbf{u}}, \psi) = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\,\sin\frac{\psi}{2}, \tag{2.3}$$

with a unit pure quaternion $\hat{\mathbf{u}}$ (spatial direction) built from the basis $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$; the scalar part encodes the temporal projection, the vector part encodes the spatial projection. Any previous appearances of $\big[\cos(\psi) + \hat{\mathbf{u}}\,\sin(\psi)\big]$ or "quaternionic phase" are to be understood as their quaternionic counterparts via this rotor representation.

Let $\vec{\chi} \in \mathbb{C}$ be a variable whose change generates observable time-space effects. We treat the time and space units as directional derivatives (phase velocities) along the real and imaginary directions of a quaternionic basis $(\hat{h}, \mathbf{l})$:

$$\hat{h}\,dx_0 = \frac{\partial\vec{\chi}}{\partial\chi_h}\frac{d\chi_h}{d\chi}\,d\chi = \tilde{H}\,d\chi, \qquad \mathbf{l}\,dx_l = \frac{\partial\vec{\chi}}{\partial\chi_l}\frac{d\chi_l}{d\chi}\,d\chi = \tilde{L}\,d\chi, \quad l = 1, 2, 3. \tag{2.4}$$

Cheat sheet (actions & axes). *D-rotation (boost):* $d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\sin\frac{\psi}{2}$, $\;\mathbf{q} \mapsto d\,\mathbf{q}\,d$ (tilts the time axis to the 3-velocity).

R-rotation (spatial): $r = \cos\frac{\varphi}{2} + \hat{\mathbf{n}}\sin\frac{\varphi}{2}$, $\;\mathbf{q} \mapsto r\,\mathbf{q}\,r^{-1}$ (fixes the scalar/time axis).

Pullback to the observer's curvature: for observer rotor u, $\mathbf{q} \mapsto \bar{u}\,\mathbf{q}\,u$.

Introduce the phase speed of the SR interval $ds = \tilde{S}\,d\chi$. The interval conservation takes the form

$$\tilde{S}^2 = \frac{ds^2}{d\chi^2} = \frac{g_{ij}\,dx^i dx^j}{d\chi^2} = \tilde{H}^2 - \tilde{L}^2, \tag{2.5}$$

equivalently

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2. \tag{2.6}$$

Writing

$$\tilde{S} = \tilde{H}\cos\theta, \qquad \tilde{L} = \tilde{H}\sin\theta, \tag{2.7}$$

where θ is the angle of the phase speed relative to the real axis. Algebraically, (**??**) is a Euclidean decomposition of a single speed into orthogonal projections; physically, we will see that under reparameterization the *projection* $\tilde{S}$, not the Euclidean norm $\tilde{H}$, is the conserved Minkowski quantity.

3 Phase space (*khōra*)

Let the phase vector space ("*khōra*", after Plato) be $\mathbb{C}$ with orthonormal basis $(\hat{h}, \mathbf{l})$. For a phase vector $\vec{\chi} = R\, e^{\theta \mathbf{l}}$ with $\theta \in [-\pi, \pi]$,

$$\tilde{H} = R, \qquad \tilde{S} = R\cos\theta, \qquad \tilde{L} = R\,\mathbf{l}\sin\theta. \tag{3.1}$$

Choosing coordinates where the projectors onto $(\hat{h}, \mathbf{l})$ are unit, (**??**) simplifies to

$$\hat{h}\, dx_0 = \frac{d\chi_h}{d\chi}\, d\chi = \tilde{H}\, d\chi, \qquad \mathbf{l}\, dx_l = \frac{d\chi_l}{d\chi}\, d\chi = \tilde{L}\, d\chi. \tag{3.2}$$

The map from phase to observables is an integral transform:

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}^i(u)\, du, \qquad i = 0, 1, 2, 3, \tag{3.3}$$

where $\tilde{X}^i$ are projections of $d\vec{\chi}/d\chi$ onto $(\hat{h}, \mathbf{l})$ and $x^i(\chi_0)$ fix initial conditions.

4 Objects

A *fundamental particle* is an *elementary object* with nonzero phase $\vec{\chi} \neq 0$. Composite *objects* are phase configurations; to represent them *in phase space* one may require additional dimensions, except for the *photon*, whose phase is always aligned with the imaginary axis:

$$\mathbf{p} = \frac{d\vec{\chi}}{d\chi_l} = p\,\mathbf{l} \in \Im. \tag{4.1}$$

Non-photonic phenomena are associated with nonzero real projection and nonzero mass. A quaternionic object can be identified with an *event* or worldline; the photon corresponds to a null-interval point encoding information about the event.

Any object's phase can be rotated to the *zero* (purely real) direction,

$$\vec{\chi}_0 = R \in \Re. \tag{4.2}$$

An object A moving with speed V relative to a rest observer has

$$\vec{\chi}_A = R\, e^{\theta_A \mathbf{l}}, \qquad \sin\theta_A = \frac{V}{\mathsf{c}} \equiv \beta. \tag{4.3}$$

4.1 Space as a symmetric phase pair

From (**??**), a naive zero-angle limit would remove the imaginary projection, contradicting observability. We enforce a nonvanishing spatial projection by pairing opposite-phase tilts:

$$\vec{\chi}^{\pm} = R\, e^{\pm\zeta \mathbf{l}}, \qquad \vec{\chi}_l := \frac{\vec{\chi}^{+} - \vec{\chi}^{-}}{2} = R\,\mathbf{l}\sin\zeta, \tag{4.4}$$

where ζ is an *internal angle* (intrinsic to the object; heuristically linked to mass/density). The local decomposition is

$$\vec{\chi}_0 = \vec{\chi}_\tau + \vec{\chi}_l = R\cos\zeta + R\,\mathbf{l}\sin\zeta, \tag{4.5}$$

with unit components (normalized by R): the real component is $\cos\zeta$ and the imaginary component is $\sin\zeta$.

4.2 Absolute, local, and observed time

Define *absolute* time $t = t(\tilde{H})$ at the zero phase direction; it is the fastest clock and useful for normalization between different phase speeds. Along the local real direction,

$$dx_0 = \frac{d}{d\chi}\Re(\vec{\chi})\,d\chi = \frac{\vec{\chi}^{+} + \vec{\chi}^{-}}{2}\,d\chi = \cos\zeta\,d\chi =: d\tau. \tag{4.6}$$

Here $d\chi_0 := \cos\zeta\,d\chi$ is the projection of $d\chi$ onto the local real axis; in Sec. **??** we calibrate $d\tau = (1/\nu_0)\,d\chi_0$. The observed proper time of A relative to the rest observer is

$$\tilde{H}_A = \Re\left(\frac{d\vec{\chi}_A}{d\vec{\chi}_0}\right) = \cos\theta_A = \sqrt{1 - \sin^2\theta_A} = \sqrt{1 - \frac{V^2}{\mathsf{c}^2}} = \frac{1}{\gamma}. \tag{4.7}$$

4.3 Normalization

Let local time be parameterized by *phase*; introduce a reference frequency ν_0 and set

$$d\tau = \frac{1}{\nu_0}\,d\chi_0. \tag{4.8}$$

By the chain rule,

$$dx_0 = \tilde{H}\,d\chi = \frac{dx_0}{d\chi_0}\frac{d\chi_0}{d\tau}\,d\tau = \tilde{H}\,\dot{\chi}\,d\tau =: \dot{H}\,d\tau, \tag{4.9}$$

where $\nu := d\chi/d\tau$, $\dot{\chi} := \nu/\nu_0$, and $\dot{H} := \tilde{H}\,\dot{\chi}$. Choosing the calibration $\dot{H} \equiv \mathsf{c}$ gives $dx_0 = \mathsf{c}\,d\tau$. Similarly for space,

$$dx_l = \tilde{L}\,d\chi = \frac{dx_l}{d\chi_0}\frac{d\chi_0}{dl}\,dl = \tilde{L}\,\chi'\,dl =: L'\,dl, \qquad \chi' := \frac{d\chi}{dl}. \tag{4.10}$$

From $dx_0 = dx_l$ for light one gets

$$\mathsf{c} = \tilde{L}'\,\frac{dl}{d\tau}, \tag{4.11}$$

hence with temporal calibration to c the spatial scale becomes unit: $\tilde{L}' = 1$.

4.4 Light and c as a calibration constant

From the normalized forms,

$$\frac{\mathsf{c}}{\dot{\chi}}\,d\chi = \frac{1}{\chi'}\,d\chi \quad \Rightarrow \quad \mathsf{c} = \frac{\dot{\chi}}{\chi'} = \frac{dl}{d\tau}, \tag{4.12}$$

i.e. c is a *calibration constant* tying temporal and spatial measures, independent of local phase variation. Equation (**??**) also reads

$$\mathsf{c} = \left(\frac{d\chi}{d\tau}\right)\left[\frac{dl}{d\chi}\right] \sim (\nu)\,[\lambda], \tag{4.13}$$

matching frequency and wavelength of a photon, with χ as its phase. For a lightlike trajectory,

$$ds^2 = \mathsf{c}^2\left(\frac{d\chi^2}{\dot{\chi}^2} - \frac{d\chi^2}{\dot{\chi}^2}\right) = 0. \tag{4.14}$$

At unit frequency, $\tau = \chi$: the photon's "proper time" is its phase, and the length of its phase-speed vector equals its wavelength, $\tilde{H}_p = \lambda$. Finally, the kinematic slope in phase coordinates is

$$\frac{dx_l}{dx_0} = \frac{\tilde{L}\,d\chi}{\tilde{H}\,d\chi} = \sin\theta = \frac{V}{\mathsf{c}} \equiv \beta, \tag{4.15}$$

so $\theta = \pi/2$ implies $V = \mathsf{c}$.

4.5 Lorentz factor via reparameterization

A change of direction of the phase speed transforms

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2 \;\longmapsto\; \dot{H}^2 = \dot{S}^2 + \dot{L}^2. \tag{4.16}$$

Lemma (parameter-change identity). The transition $\tilde{H} \to \dot{S}$ is the manifestation of evolving phase speed under the parameter change $\chi \mapsto \tau(\chi)$, with local Jacobian

$$\frac{d\tau}{d\chi} = \cos\zeta(\chi)\cos\theta(\chi) \quad \Rightarrow \quad \mathcal{J}(\zeta,\theta) := \frac{d\chi}{d\tau} = \frac{1}{\cos\zeta\,\cos\theta}. \tag{4.17}$$

Then

$$\dot{H} = \tilde{H}\,\mathcal{J}, \qquad \dot{L} = \tilde{L}\,\mathcal{J}. \tag{4.18}$$

In differential form,

$$d\ln\dot{H} = d\ln\mathcal{J} = \tan\zeta\,d\zeta + \tan\theta\,d\theta. \tag{4.19}$$

For a *pure boost* ($d\zeta = 0$) one has $d\dot{H} = \dot{H}\tan\theta\,d\theta$. Absorbing a constant $\cos\zeta$ into the calibration (set $\zeta = 0$ henceforth), we obtain

$$\tilde{H}^2 = \dot{H}^2 - \dot{L}^2 = \sec^2\theta\,(\tilde{H}^2 - \tilde{L}^2) = \gamma^2(\tilde{H}^2 - \tilde{L}^2). \tag{4.20}$$

Corollary. In phase space the Euclidean norm $\tilde{H}$ is conserved; in observed time the Minkowski norm $\dot{S}$ is conserved; they are identical as quantities:

$$\boxed{\tilde{H} = \dot{S}}. \tag{4.21}$$

4.6 Rapidity and the phase angle

By definition,

$$\beta = \frac{V}{\mathsf{c}} = \sin\theta, \qquad \tanh\eta = \beta, \qquad d\eta = \frac{d\beta}{1-\beta^2}. \tag{4.22}$$

With $d\beta = \cos\theta\,d\theta$ and $1-\beta^2 = \cos^2\theta$,

$$d\eta = \sec\theta\,d\theta, \qquad \eta(\theta) = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| = \tfrac{1}{2}\ln\frac{|1+\sin\theta|}{|1-\sin\theta|}. \tag{4.23}$$

Fixing $\eta(0) = 0$,

$$e^{\eta(\theta)} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \sec\theta = \cosh\eta. \tag{4.24}$$

Remark (groups). Observables satisfy $\beta = \sin\theta = \tanh\eta$ and $\gamma = \sec\theta = \cosh\eta$. Thus Euclidean rotations in the phase circle ($U(1)$ with angle θ) reproduce the numerical factors of hyperbolic boosts in $SO^{+}(1,1)$ (rapidity η) *after* reparameterizing time. We do not claim an isomorphism $U(1) \cong SO(1,1)$; only the equality of observable combinations under the change of parameter.

4.7 Velocity addition

Notation. In the unimetry formalism, an inertial boost is a *D-rotation*

$$\mathcal{B}(\hat{\mathbf{u}},\psi): \quad \mathbf{q} \;\mapsto\; d\,\mathbf{q}\,d, \qquad d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\,\sin\frac{\psi}{2}, \tag{4.25}$$

and a spatial rotation is an *R-rotation*

$$\mathcal{R}(\hat{\mathbf{n}},\varphi):\quad \mathbf{q}\;\mapsto\; r\,\mathbf{q}\,r^{-1},\qquad r=\cos\frac{\varphi}{2}+\hat{\mathbf{n}}\,\sin\frac{\varphi}{2}. \tag{4.26}$$

We parametrize kinematics by the unimetry angle ψ via

$$\beta\equiv v/c=\sin\psi,\qquad \gamma=\frac{1}{\cos\psi},\qquad \tan\frac{\psi}{2}=\frac{\sin\psi}{1+\cos\psi}=\frac{\gamma\beta}{\gamma+1}. \tag{4.27}$$

(Recovering the standard SR formulas is achieved by the replacement $\sin\leftrightarrow\sinh$, $\cos\leftrightarrow\cosh$.)

4.7.1 Wigner rotation

Statement (boost composition). The composition of two non-collinear unimetry boosts factorizes into a net boost followed by a spatial rotation (the Wigner rotation):

$$\boxed{\mathcal{B}(\hat{\mathbf{u}}_2,\psi_2)\,\mathcal{B}(\hat{\mathbf{u}}_1,\psi_1)\;=\;\mathcal{R}(\hat{\mathbf{n}}_W,\phi_U)\,\mathcal{B}(\hat{\mathbf{u}}_{12},\psi_{12})} \tag{4.28}$$

Quaternionic pullback (observer's curvature). Let u denote the observer's D-rotor (so that passing to the observer's curvature amounts to the sandwich $X\mapsto\bar{u}\,X\,u$). Consider two boosts with rotors $d_i=\cos\frac{\psi_i}{2}+\hat{\mathbf{u}}_i\sin\frac{\psi_i}{2}$ and let

$$L\;\equiv\;d_2d_1,\qquad L_u\;\equiv\;\bar{u}\,L\,u\;=\;\bar{u}\,d_2d_1\,u, \tag{4.29}$$

be the composed transformation as *seen by the observer*. Let $d_{12}^{(u)}$ be the unique unimetry boost obtained from the velocity-addition rule in the observer frame, i.e.

$$d_{12}^{(u)}=\cos\frac{\psi_{12}}{2}+\hat{\mathbf{u}}_{12}\,\sin\frac{\psi_{12}}{2}, \tag{4.30}$$

with $(\hat{\mathbf{u}}_{12},\psi_{12})$ computed from $(\hat{\mathbf{u}}_1,\psi_1)$, $(\hat{\mathbf{u}}_2,\psi_2)$ in §**??** but expressed in the observer's frame. Then the *observed* Wigner rotation is simply the residual rotor in the polar factorization of L_u:

$$\boxed{r_W^{(u)}\;=\;L_u\,\big(d_{12}^{(u)}\big)^{-1}\;=\;\bar{u}\,d_2d_1\,u\,\big(\cos\frac{\psi_{12}}{2}-\hat{\mathbf{u}}_{12}\,\sin\frac{\psi_{12}}{2}\big)} \tag{4.31}$$

(a unit quaternion). It is purely spatial in the observer frame, i.e. it fixes the scalar subspace: $r_W^{(u)}\,\lambda\,\big(r_W^{(u)}\big)^{-1}=\lambda$ for all scalars λ. Writing $r_W^{(u)}=\cos\frac{\phi_U}{2}+\hat{\mathbf{n}}_W^{(u)}\sin\frac{\phi_U}{2}$ identifies the observed axis $\hat{\mathbf{n}}_W^{(u)}$ and angle ϕ_U.

Proof sketch. In the observer frame, $d_{12}^{(u)}$ is the unique D-rotor that maps the time axis to the composed 3-velocity (Sec. **??**). Therefore $L_u\big(d_{12}^{(u)}\big)^{-1}$ must leave the time axis invariant and hence is a pure R-rotation—the Wigner rotation.

with the Wigner axis along the cross product of the boost directions:

$$\hat{\mathbf{n}}_W=\frac{\hat{\mathbf{u}}_2\times\hat{\mathbf{u}}_1}{\sin\theta},\qquad \cos\theta=\hat{\mathbf{u}}_2\!\cdot\!\hat{\mathbf{u}}_1. \tag{4.32}$$

Angle (unimetry half-angles). In terms of unimetry half-angles, the Wigner angle is

$$\boxed{\tan\frac{\phi_U}{2}=\frac{\tan\frac{\psi_1}{2}\,\tan\frac{\psi_2}{2}\,\sin\theta}{1+\tan\frac{\psi_1}{2}\,\tan\frac{\psi_2}{2}\,\cos\theta}} \tag{4.33}$$

equivalently,

$$\tan\frac{\phi_U}{2}=\frac{\sin\frac{\psi_1}{2}\sin\frac{\psi_2}{2}\,\sin\theta}{\cos\frac{\psi_1}{2}\cos\frac{\psi_2}{2}+\sin\frac{\psi_1}{2}\sin\frac{\psi_2}{2}\,\cos\theta}. \tag{4.34}$$

Limits: for collinear boosts $\theta=0$ one has $\phi_U=0$; for $\beta\ll 1$,

$$\phi_U\;\approx\;\tfrac{1}{2}\,\big|\boldsymbol{\beta}_2\times\boldsymbol{\beta}_1\big|,\qquad \boldsymbol{\beta}_i\equiv\beta_i\,\hat{\mathbf{u}}_i. \tag{4.35}$$

Operator form. If $d_i = \cos\frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin\frac{\psi_i}{2}$, then

$$d_2\, d_1 \;=\; r_W\, d_{12}, \qquad r_W = \cos\frac{\phi_U}{2} + \hat{\mathbf{n}}_W \sin\frac{\phi_U}{2}, \tag{4.36}$$

and the action on any unimetry 4-object is

$$\mathbf{q}' \;=\; r_W\,\big(d_{12}\,\mathbf{q}\,d_{12}\big)\,r_W^{-1}.$$

Quaternionic pullback as angle compensation. Let d_1 and d_2 be the D-rotors of two successive boosts. Their raw action on any unimetry 4-object is

$$\mathbf{q}' \;=\; d_2 d_1\,\mathbf{q}\,d_1 d_2. \tag{4.38}$$

Let d_{12} denote the unique D-rotor that reproduces the combined space–time tilt (the change of the spatio–temporal angle) of $d_2 d_1$, i.e. it maps the time axis to the composite 3-velocity given by the velocity-addition law in §??. Pulling back by the conjugate rotor $\bar{d}_{12}$ on both sides removes this tilt:

$$\mathbf{q}^{(u)} \;=\; \bar{d}_{12}\,\mathbf{q}'\,\bar{d}_{12} \;=\; (\bar{d}_{12} d_2 d_1)\,\mathbf{q}\,(d_1 d_2 \bar{d}_{12}). \tag{4.39}$$

Define the residual rotor

$$\boxed{r_W \;\equiv\; \bar{d}_{12}\, d_2 d_1} \qquad \Rightarrow \qquad \mathbf{q}^{(u)} \;=\; r_W\,\mathbf{q}\,r_W^{-1}, \tag{4.40}$$

Uniqueness of d_{12} and factorization lemma. Let $\mathbf{e}_t$ denote the unit temporal basis (observer's time axis). The composite D-rotor d_{12} is uniquely fixed by requiring it to map the time axis to that of the product $d_2 d_1$,

$$d_{12}\,\mathbf{e}_t\,d_{12} \;=\; d_2 d_1\,\mathbf{e}_t\,d_1 d_2, \qquad \text{with } \Re(d_{12}) \geq 0, \tag{4.41}$$

(the sign choice $\Re(d_{12}) \geq 0$ removes the trivial two-fold ambiguity $d \to -d$).

Lemma (D–R factorization). For any product of D-rotors $L = d_2 d_1$ there exist unique rotors d_{12} (of type D) and r_W (of type R) such that

$$\boxed{L \;=\; d_{12}\, r_W}, \qquad r_W \;=\; \bar{d}_{12}\, L \;=\; \bar{d}_{12}\, d_2 d_1. \tag{4.42}$$

Order dependence (noncommutativity). Write

$$L_{12} \equiv d_2 d_1 \;=\; d_{12}\, r_W, \qquad L_{21} \equiv d_1 d_2 \;=\; d_{21}\, r_W^{-1}, \tag{4.43}$$

where d_{12} and d_{21} are the unique D-rotors specified by (??) for L_{12} and L_{21}, respectively, and r_W is the Wigner rotor defined in (??). In general,

$$d_{12} \neq d_{21}, \qquad r_W^{-1} \neq r_W, \tag{4.44}$$

except for the collinear (or trivial) case. Swapping the order flips the Wigner rotation ($r_W \mapsto r_W^{-1}$) and changes the net 3-velocity, reflecting the noncommutativity of velocity addition ($\mathbf{v}_2 \oplus \mathbf{v}_1 \neq \mathbf{v}_1 \oplus \mathbf{v}_2$). The quaternionic compensation picture remains the same: for the $(1 \to 2)$ order the observed map is $r_W\,\mathbf{q}\,r_W^{-1}$, for $(2 \to 1)$ it is $r_W^{-1}\,\mathbf{q}\,r_W$.

One-line algorithm for r_W. Compute $L = d_2 d_1$, find the unique d_{12} by $d_{12}\,\mathbf{e}_t\,d_{12} = L\,\mathbf{e}_t\,L$ with $\Re(d_{12}) \geq 0$, then $r_W = \bar{d}_{12}\,L$ and the observed action is $\mathbf{q} \mapsto r_W\,\mathbf{q}\,r_W^{-1}$.

$$
\begin{tikzcd}
\mathbf{q} \arrow[r, "d_1"] \arrow[rr, dashed, bend left, "d_{12}"] & d_1\,\mathbf{q}\,d_1 \arrow[r, "d_2"] & d_2 d_1\,\mathbf{q}\,d_1 d_2 \arrow[d, "\bar{d}_{12}\text{ on both sides}"] \\
 & & r_W\,\mathbf{q}\,r_W^{-1}
\end{tikzcd}
$$

Figure 1: Two successive D-rotations (boosts) and compensation of the net spatio–temporal angle by the conjugate of d_{12}, leaving a pure R-rotation r_W.

Passive vs active actions (disambiguation).

Name	**Map on q**	**Meaning**	**Fixes time?**
Passive pullback	$\bar{u}\,\mathbf{q}\,u$	change of frame / curvature (observer's view)	yes
Active D-cancel	$\bar{u}\,\mathbf{q}\,\bar{u}$	undo D-tilt (acts on the object)	no
R-rotation	$r\,\mathbf{q}\,r^{-1}$	pure spatial rotation	yes
D-rotation	$d\,\mathbf{q}\,d$	boost (tilt of time axis)	no

With this choice, r_W fixes $\mathbf{e}_t$ and acts as a pure spatial rotation in the observer frame, reproducing the Wigner rotation with axis/angle given by (**??**)–(**??**).

so the *observed* transformation is a pure spatial rotation. Writing $r_W = \cos\frac{\phi_U}{2} + \hat{\mathbf{n}}_W \sin\frac{\phi_U}{2}$ recovers the axis and angle formulas (**??**)–(**??**). This exhibits the Wigner rotation as the quaternionic *compensation* of the net spatio–temporal angle.

4.7.2 Thomas precession

Definition (continuous limit of Wigner rotation). For a worldline with time-dependent velocity direction $\hat{\mathbf{u}}(t)$, Thomas precession is the instantaneous angular velocity of the Wigner rotation accumulated by the sequence of infinitesimal boosts:

$$
\boxed{\boldsymbol{\omega}_T \;=\; (\gamma-1)\,\big(\hat{\mathbf{u}}\times\dot{\hat{\mathbf{u}}}\big) \;=\; \frac{\gamma^2}{\gamma+1}\,\frac{\mathbf{a}\times\mathbf{v}}{c^2}} \tag{4.45}
$$

where $\mathbf{v} = v\,\hat{\mathbf{u}}$, $\mathbf{a} = \dot{\mathbf{v}}$, and $\gamma = 1/\cos\psi$. Equivalently, in pure unimetry variables,

$$
\boldsymbol{\omega}_T = \frac{1-\cos\psi}{\cos\psi}\,\big(\hat{\mathbf{u}}\times\dot{\hat{\mathbf{u}}}\big). \tag{4.46}
$$

Special cases and limits. For uniform circular motion ($|\mathbf{v}| = \text{const}$) with orbital angular velocity $\boldsymbol{\Omega}$ (defined by $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega}\times\hat{\mathbf{u}}$),

$$
\boxed{|\boldsymbol{\omega}_T| = (\gamma-1)\,\Omega} \qquad \text{(axis opposite to } \boldsymbol{\Omega} \text{ in the standard convention).} \tag{4.47}
$$

In the nonrelativistic limit $\beta \ll 1$,

$$
\boldsymbol{\omega}_T \;\approx\; \tfrac{1}{2}\,\frac{\mathbf{a}\times\mathbf{v}}{c^2}. \tag{4.48}
$$

Remark on placement. Since Thomas precession is the differential (continuous) limit of the Wigner rotation for a sequence of infinitesimal non-collinear boosts, it is natural to present §**??** first and then §**??**.

4.8 Doppler shift

Define the observed frequency as the phase growth rate in the observer's proper time:

$$\nu := \frac{d\chi}{d\tau}. \tag{4.49}$$

For two successive wavefronts the phase increment is identical, hence

$$\frac{\nu_{\rm obs}}{\nu_{\rm src}} = \frac{d\chi/d\tau_{\rm obs}}{d\chi/d\tau_{\rm src}} = \frac{d\tau_{\rm src}}{d\tau_{\rm obs}}. \tag{4.50}$$

Longitudinal case: during $\gamma\, d\tau_{\rm src}$ in the observer frame the source displaces by $\pm V\,\gamma\, d\tau_{\rm src}$ ("+" receding, "−" approaching). Then

$$d\tau_{\rm obs} = \gamma\, d\tau_{\rm src}(1 \pm \beta), \qquad \Rightarrow \qquad \boxed{\frac{\nu_{\rm obs}}{\nu_{\rm src}} = \frac{1}{\gamma(1 \pm \beta)}}. \tag{4.51}$$

Equivalent forms (with $\beta = \sin\theta$, $\gamma = \sec\theta$ and rapidity η):

$$\frac{\nu_{\rm obs}}{\nu_{\rm src}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec\theta\,(1 \mp \sin\theta) = e^{\mp\eta}. \tag{4.52}$$

Transverse Doppler ($\varphi = 90^\circ$ in the observer's frame):

$$\frac{\nu_{\rm obs}}{\nu_{\rm src}} = \frac{1}{\gamma} = \cos\theta. \tag{4.53}$$

General line-of-sight (LOS) angle φ in the observer's frame:

$$\boxed{\frac{\nu_{\rm obs}}{\nu_{\rm src}} = \gamma\,(1 - \beta\cos\varphi)}. \tag{4.54}$$

Wavelength ratios are inverse to frequency ratios.

Differential pullback and Thomas precession. Let $d(t)$ be the instantaneous boost rotor and fix the observer's $u(t)$ as instantaneously comoving. Over a small interval Δt the observed residual rotation is

$$r_W^{(u)}(\Delta t) \;=\; \bar{u}(t)\, d(t+\Delta t)\, d(t)\, u(t)\, \left(d_{12}^{(u)}(t,\Delta t)\right)^{-1}, \tag{4.55}$$

whose first-order expansion is $r_W^{(u)}(\Delta t) \simeq 1 - \frac{1}{2}\,(\boldsymbol{\omega}_T\,\Delta t)\cdot\hat{\mathbf{N}}$, yielding (**??**)–(**??**) with $\boldsymbol{\omega}_T = (\gamma-1)\,\hat{\mathbf{u}}\times\dot{\hat{\mathbf{u}}}$. This makes explicit that the Thomas precession is the differential limit of the quaternionic pullback residual rotation.

5 Discussion: links to known structures

Gauge phases. A global shift $\chi \mapsto \chi + \chi_0$ is unobservable. Allowing local reparameterizations $\chi \mapsto \chi + \alpha(x)$ induces a connection when comparing phases at different points. On wavefunctions $\psi \sim \big[\cos(\chi) + \hat{\mathbf{u}}\,\sin(\chi)\big]$ this is the familiar $U(1)$ gauge freedom $\psi \to \big[\cos(\alpha(x)) + \hat{\mathbf{u}}\,\sin(\alpha(x))\big]\psi$ with $D_\mu = \partial_\mu - iA_\mu$ as the *phase-transport connection.*

Mass and the internal angle. With the decomposition by ζ, mass heuristically correlates with an irreducible real projection: massless objects have $\zeta = \pm\pi/2$ (no proper time; photon subspace), while massive objects have $|\zeta| < \pi/2$ (proper time exists). In the present paper we set $\zeta = 0$ in boost kinematics by calibration; a detailed mass-generation mechanism is left for future work.

Cosmological gauge. A natural global calibration of "absolute" time is the comoving frame with vanishing CMB dipole. This fixes a cosmological time t (FLRW) as a gauge, without affecting local Lorentz invariance; Doppler factors are then operationally referenced to that frame.

6 Conclusion

In unimetry, time and space are integrals of phase velocities; the Minkowski interval appears as a conserved quantity under parameter change. The core relations of SR—γ, rapidity, velocity addition, and Doppler factors—follow from elementary phase-plane geometry with a single rotation angle θ, while hyperbolic structure re-emerges upon reparameterizing time. The formalism is empirically equivalent to standard SR but can clarify causality and composition by treating all effects as projections of a single flow.

Outlook. Future directions include (i) a more explicit group-theoretic embedding, (ii) a rigorous treatment of the internal angle ζ and its relation to mass, and (iii) exploration of curved metrics as spatially varying Jacobians $\mathcal{J}(x)$ in the phase-to-observable map.

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