

Unimetry: A Phase-Space Reformulation of Special Relativity

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October 14, 2025

Abstract

We propose a compact reformulation of special relativity in which spacetime units (time and length) are treated as *phase velocities*—directional derivatives of a single underlying parameter, the phase $\vec{\chi} \in \mathbb{H}$. The observable Minkowski interval emerges as a conserved quantity under a change of parameter from the hidden phase coordinate χ to the observer’s proper time τ . In this *unimetry* formalism, familiar relativistic effects—time dilation, Lorentz factor, Doppler shift, and relativistic velocity composition—arise as elementary projections and rotations in a Euclidean phase plane. Hyperbolic features of Lorentz kinematics reappear after a reparameterization of time, yielding the standard relations without altering empirical content. We provide closed-form derivations of the longitudinal/transverse Doppler factors, identify a simple lemma equating the total phase speed to the conserved Minkowski norm, and outline connections to gauge phases, rapidity, and a cosmological time gauge. Composition of non-collinear boosts (D-rotations) produces a Wigner rotation, and in the continuous limit this yields Thomas precession; both arise here as purely kinematical consequences of the quaternionic phase formalism (see §4.7).

Keywords: special relativity; phase; rapidity; Doppler shift; Lorentz factor; phase parameterization.

MSC/PhCS: 83A05; 83-10; 70A05.

1 Introduction

We usually take time and space as primitive. The *phase formalism* introduced here suggests a different viewpoint: time and space are *derived projections* of a single parameter $\vec{\chi} \in \mathbb{H}$ (“phase”). In this picture, relativistic effects such as time dilation and the Doppler shift are geometric consequences of phase-flow rotations.

The proposal does not modify physics; it reorganizes familiar relations in a simpler language. In spirit it is akin to Lagrangian/Hamiltonian re-descriptions of classical mechanics: same empirical content, different coordinates. We will realize the phase kinematics with quaternionic rotors $d = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}$. Throughout, Greek θ will denote the *external* rotation angle associated with relative motion, while ζ denotes an *internal* angle associated with the object’s intrinsic state (mass/density heuristic). We emphasize that no modification of Einstein’s dynamics is proposed; all results are kinematical identities obtained by a change of parameter.

Notation. Tildes, dots and primes indicate derivatives with respect to the phase parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \quad \dot{X} := \frac{dX}{d\tau}, \quad X' := \frac{dX}{dl}.$$

We use c for the speed of light; $\beta := V/c$, $\gamma := 1/\sqrt{1-\beta^2}$, rapidity $\tanh \eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

2 Time and space as phase derivatives

Let $\vec{\chi} \in \mathbb{C}$ be a variable whose change generates observable time-space effects. We treat the time and space units as directional derivatives (phase velocities) along the real and imaginary directions of a complex basis (\hat{h}, \mathbf{l}) :

$$\hat{h} dx_0 = \frac{\partial \vec{\chi}}{\partial \chi_h} \frac{d\chi_h}{d\chi} d\chi = \tilde{H} d\chi, \quad \mathbf{l} dx_l = \frac{\partial \vec{\chi}}{\partial \chi_l} \frac{d\chi_l}{d\chi} d\chi = \tilde{L} d\chi, \quad l = 1, 2, 3. \quad (2.1)$$

Introduce the phase speed of the SR interval $ds = \tilde{S} d\chi$. The interval conservation takes the form

$$\tilde{S}^2 = \frac{ds^2}{d\chi^2} = \frac{g_{ij} dx^i dx^j}{d\chi^2} = \tilde{H}^2 - \tilde{L}^2, \quad (2.2)$$

equivalently

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2. \quad (2.3)$$

Writing

$$\tilde{S} = \tilde{H} \cos \theta, \quad \tilde{L} = \tilde{H} \sin \theta, \quad (2.4)$$

where θ is the angle of the phase speed relative to the real axis. Algebraically, (??) is a Euclidean decomposition of a single speed into orthogonal projections; physically, we will see that under reparameterization the *projection* \tilde{S} , not the Euclidean norm \tilde{H} , is the conserved Minkowski quantity.

3 Phase space (*khōra*)

Let the phase vector space (“*khōra*”, after Plato) be \mathbb{C} with orthonormal basis (\hat{h}, \mathbf{l}) . For a phase vector $\vec{\chi} = R e^{\theta \mathbf{l}}$ with $\theta \in [-\pi, \pi]$,

$$\tilde{H} = R, \quad \tilde{S} = R \cos \theta, \quad \tilde{L} = R \sin \theta. \quad (3.1)$$

Choosing coordinates where the projectors onto (\hat{h}, \mathbf{l}) are unit, (??) simplifies to

$$\hat{h} dx_0 = \frac{d\chi_h}{d\chi} d\chi = \tilde{H} d\chi, \quad \mathbf{l} dx_l = \frac{d\chi_l}{d\chi} d\chi = \tilde{L} d\chi. \quad (3.2)$$

The map from phase to observables is an integral transform:

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}^i(u) du, \quad i = 0, 1, 2, 3, \quad (3.3)$$

where \tilde{X}^i are projections of $d\vec{\chi}/d\chi$ onto (\hat{h}, \mathbf{l}) and $x^i(\chi_0)$ fix initial conditions.

4 Objects

A *fundamental particle* is an *elementary object* with nonzero phase $\vec{\chi} \neq 0$. Composite *objects* are phase configurations; to represent them *in phase space* one may require additional dimensions, except for the *photon*, whose phase is always aligned with the imaginary axis:

$$\mathbf{p} = \frac{d\vec{\chi}}{d\chi_l} = p \mathbf{l} \in \mathfrak{S}. \quad (4.1)$$

Non-photonic phenomena are associated with nonzero real projection and nonzero mass. A complex object can be identified with an *event* or worldline; the photon corresponds to a null-interval point encoding information about the event.

Any object's phase can be rotated to the *zero* (purely real) direction,

$$\vec{\chi}_0 = R \in \Re. \quad (4.2)$$

An object A moving with speed V relative to a rest observer has

$$\vec{\chi}_A = R e^{\theta_A \mathbf{l}}, \quad \sin \theta_A = \frac{V}{c} \equiv \beta. \quad (4.3)$$

4.1 Space as a symmetric phase pair

From (??), a naive zero-angle limit would remove the imaginary projection, contradicting observability. We enforce a nonvanishing spatial projection by pairing opposite-phase tilts:

$$\vec{\chi}^{\pm} = R e^{\pm \zeta \mathbf{l}}, \quad \vec{\chi}_l := \frac{\vec{\chi}^+ - \vec{\chi}^-}{2} = R \mathbf{l} \sin \zeta, \quad (4.4)$$

where ζ is an *internal angle* (intrinsic to the object; heuristically linked to mass/density). The local decomposition is

$$\vec{\chi}_0 = \vec{\chi}_\tau + \vec{\chi}_l = R \cos \zeta + R \mathbf{l} \sin \zeta, \quad (4.5)$$

with unit components (normalized by R): the real component is $\cos \zeta$ and the imaginary component is $\sin \zeta$.

4.2 Absolute, local, and observed time

Define *absolute* time $t = t(\tilde{H})$ at the zero phase direction; it is the fastest clock and useful for normalization between different phase speeds. Along the local real direction,

$$dx_0 = \frac{d}{d\chi} \Re(\vec{\chi}) d\chi = \frac{\vec{\chi}^+ + \vec{\chi}^-}{2} d\chi = \cos \zeta d\chi =: d\tau. \quad (4.6)$$

Here $d\chi_0 := \cos \zeta d\chi$ is the projection of $d\chi$ onto the local real axis; in Sec. ?? we calibrate $d\tau = (1/\nu_0) d\chi_0$. The observed proper time of A relative to the rest observer is

$$\tilde{H}_A = \Re\left(\frac{d\vec{\chi}_A}{d\vec{\chi}_0}\right) = \cos \theta_A = \sqrt{1 - \sin^2 \theta_A} = \sqrt{1 - \frac{V^2}{c^2}} = \frac{1}{\gamma}. \quad (4.7)$$

4.3 Normalization

Let local time be parameterized by *phase*; introduce a reference frequency ν_0 and set

$$d\tau = \frac{1}{\nu_0} d\chi_0. \quad (4.8)$$

By the chain rule,

$$dx_0 = \tilde{H} d\chi = \frac{dx_0}{d\chi_0} \frac{d\chi_0}{d\tau} d\tau = \tilde{H} \dot{\chi} d\tau =: \dot{H} d\tau, \quad (4.9)$$

where $\nu := d\chi/d\tau$, $\dot{\chi} := \nu/\nu_0$, and $\dot{H} := \tilde{H} \dot{\chi}$. Choosing the calibration $\dot{H} \equiv c$ gives $dx_0 = c d\tau$. Similarly for space,

$$dx_l = \tilde{L} d\chi = \frac{dx_l}{d\chi_0} \frac{d\chi_0}{dl} dl = \tilde{L} \chi' dl =: L' dl, \quad \chi' := \frac{d\chi}{dl}. \quad (4.10)$$

From $dx_0 = dx_l$ for light one gets

$$c = \tilde{L}' \frac{dl}{d\tau}, \quad (4.11)$$

hence with temporal calibration to c the spatial scale becomes unit: $\tilde{L}' = 1$.

4.4 Light and c as a calibration constant

From the normalized forms,

$$\frac{c}{\dot{\chi}} d\chi = \frac{1}{\chi'} d\chi \quad \Rightarrow \quad c = \frac{\dot{\chi}}{\chi'} = \frac{dl}{d\tau}, \quad (4.12)$$

i.e. c is a *calibration constant* tying temporal and spatial measures, independent of local phase variation. Equation (??) also reads

$$c = \left(\frac{d\chi}{d\tau} \right) \left[\frac{dl}{d\chi} \right] \sim (\nu) [\lambda], \quad (4.13)$$

matching frequency and wavelength of a photon, with χ as its phase. For a lightlike trajectory,

$$ds^2 = c^2 \left(\frac{d\chi^2}{\dot{\chi}^2} - \frac{d\chi^2}{\dot{\chi}^2} \right) = 0. \quad (4.14)$$

At unit frequency, $\tau = \chi$: the photon's “proper time” is its phase, and the length of its phase-speed vector equals its wavelength, $\tilde{H}_p = \lambda$. Finally, the kinematic slope in phase coordinates is

$$\frac{dx_l}{dx_0} = \frac{\tilde{L} d\chi}{\tilde{H} d\chi} = \sin \theta = \frac{V}{c} \equiv \beta, \quad (4.15)$$

so $\theta = \pi/2$ implies $V = c$.

4.5 Lorentz factor via reparameterization

A change of direction of the phase speed transforms

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2 \mapsto \dot{H}^2 = \dot{S}^2 + \dot{L}^2. \quad (4.16)$$

Lemma (parameter-change identity). The transition $\tilde{H} \rightarrow \dot{S}$ is the manifestation of evolving phase speed under the parameter change $\chi \mapsto \tau(\chi)$, with local Jacobian

$$\frac{d\tau}{d\chi} = \cos \zeta(\chi) \cos \theta(\chi) \quad \Rightarrow \quad \mathcal{J}(\zeta, \theta) := \frac{d\chi}{d\tau} = \frac{1}{\cos \zeta \cos \theta}. \quad (4.17)$$

Then

$$\dot{H} = \tilde{H} \mathcal{J}, \quad \dot{L} = \tilde{L} \mathcal{J}. \quad (4.18)$$

In differential form,

$$d \ln \dot{H} = d \ln \mathcal{J} = \tan \zeta d\zeta + \tan \theta d\theta. \quad (4.19)$$

For a *pure boost* ($d\zeta = 0$) one has $d\dot{H} = \dot{H} \tan \theta d\theta$. Absorbing a constant $\cos \zeta$ into the calibration (set $\zeta = 0$ henceforth), we obtain

$$\tilde{H}^2 = \dot{H}^2 - \dot{L}^2 = \sec^2 \theta (\tilde{H}^2 - \tilde{L}^2) = \gamma^2 (\tilde{H}^2 - \tilde{L}^2). \quad (4.20)$$

Corollary. In phase space the Euclidean norm \tilde{H} is conserved; in observed time the Minkowski norm \dot{S} is conserved; they are identical as quantities:

$$\boxed{\tilde{H} = \dot{S}}. \quad (4.21)$$

4.6 Rapidity and the phase angle

By definition,

$$\beta = \frac{V}{c} = \sin \theta, \quad \tanh \eta = \beta, \quad d\eta = \frac{d\beta}{1 - \beta^2}. \quad (4.22)$$

With $d\beta = \cos \theta d\theta$ and $1 - \beta^2 = \cos^2 \theta$,

$$d\eta = \sec \theta d\theta, \quad \eta(\theta) = \int \sec \theta d\theta = \ln |\sec \theta + \tan \theta| = \frac{1}{2} \ln \frac{|1 + \sin \theta|}{|1 - \sin \theta|}. \quad (4.23)$$

Fixing $\eta(0) = 0$,

$$e^{\eta(\theta)} = \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}}, \quad \gamma = \frac{1}{\sqrt{1 - \beta^2}} = \sec \theta = \cosh \eta. \quad (4.24)$$

Remark (groups). Observables satisfy $\beta = \sin \theta = \tanh \eta$ and $\gamma = \sec \theta = \cosh \eta$. Thus Euclidean rotations in the phase circle ($U(1)$ with angle θ) reproduce the numerical factors of hyperbolic boosts in $SO^+(1, 1)$ (rapidity η) *after* reparameterizing time. We do not claim an isomorphism $U(1) \cong SO(1, 1)$; only the equality of observable combinations under the change of parameter.

4.7 Velocity addition

Notation. In unimetry, an inertial boost is a *D-rotation*

$$B(, \psi) : \quad \mathbf{q} \mapsto d \mathbf{q} d, \quad d = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}, \quad (4.25) \text{ and a spatial rotation is an } R\text{-rotation}$$

$$R(, \varphi) : \quad \mathbf{q} \mapsto r \mathbf{q} r^{-1}, \quad r = \cos \frac{\varphi}{2} + \hat{\mathbf{n}} \sin \frac{\varphi}{2}. \quad (4.26) \text{ Kinematic mapping: } \beta \equiv v/c = \sin \psi, \\ \gamma = 1/\cos \psi, \quad \tan \frac{\psi}{2} = \frac{\gamma \beta}{\gamma + 1}. \text{ For quaternionic/GA treatments of rotors and Lorentz boosts see } [?, ?, ?].$$

4.7.1 Wigner rotation

Let d_1, d_2 be D-rotors of two successive boosts. The raw action on any unimetry 4-object is

$$\mathbf{q}' = d_2 d_1 \mathbf{q} d_1 d_2 \equiv L_{12} \mathbf{q} L_{21}, \quad L_{12} = d_2 d_1, \quad L_{21} = d_1 d_2. \quad (4.27) \text{ Define } d_{12} \text{ to be the unique D-rotor reproducing the combined spatio-temporal tilt of } L_{12}:$$

$$\boxed{d_{12} \mathbf{e}_t d_{12} = L_{12} \mathbf{e}_t L_{21}, \quad \Re(d_{12}) \geq 0} \quad (4.28)$$

(the sign choice removes the trivial two-fold ambiguity). Then the *Wigner rotor* is the residual R-rotation in the symmetric D–R factorization:

$$\boxed{L_{12} = d_{12} r_W, \quad L_{21} = r_W^{-1} d_{12}} \quad (4.29)$$

equivalently,

$$\boxed{r_W = \bar{d}_{12} L_{12} = L_{21} \bar{d}_{12}}. \quad (4.30)$$

Hence the observed map after compensating the tilt is $\bar{d}_{12} \mathbf{q}' \bar{d}_{12} = r_W \mathbf{q} r_W^{-1}$.

Axis and angle. Let $c_i = \cos(\psi_i/2)$, $s_i = \sin(\psi_i/2)$, $\cos \theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1$, and $\hat{\mathbf{n}}_W = \frac{\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1}{\sin \theta}$. A direct quaternionic multiplication gives

$$\tan \frac{\phi_U}{2} = \frac{s_1 s_2 \sin \theta}{c_1 c_2 + s_1 s_2 \cos \theta} = \frac{\tan \frac{\psi_1}{2} \tan \frac{\psi_2}{2} \sin \theta}{1 + \tan \frac{\psi_1}{2} \tan \frac{\psi_2}{2} \cos \theta}, \quad (4.31)$$

with the axis $\hat{\mathbf{n}}_W \parallel \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. For collinear boosts $\theta = 0$ one has $\phi_U = 0$; in the nonrelativistic limit $\phi_U \simeq \frac{1}{2} |\boldsymbol{\beta}_2 \times \boldsymbol{\beta}_1|$. Swapping the order sends $r_W \mapsto r_W^{-1}$ (same axis, opposite angle).

One-line algorithm.

Compute r_W in one line. Compute $L_{12} = d_2 d_1$ and $L_{21} = d_1 d_2$; find the unique d_{12} by $d_{12} \mathbf{e}_t d_{12} = L_{12} \mathbf{e}_t L_{21}$ with $\Re(d_{12}) \geq 0$. Then $r_W = \bar{d}_{12} L_{12} = L_{21} \bar{d}_{12}$ and the observed action is $\mathbf{q} \mapsto r_W \mathbf{q} r_W^{-1}$.

Diagram.

Diagram.

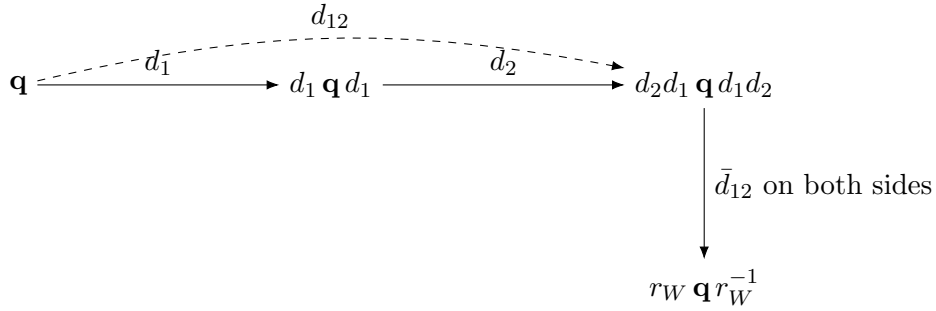


Figure 1: Two successive D-rotations (boosts) and compensation of the net spatio-temporal angle by the conjugate of d_{12} , leaving a pure R-rotation r_W .

Diagram.

4.7.2 Thomas precession

The continuous limit of Wigner rotation for a time-dependent velocity direction $\hat{\mathbf{u}}(t)$ yields

$$\boldsymbol{\omega}_T = (\gamma - 1) (\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}) = \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{a} \times \mathbf{v}}{c^2}, \quad \gamma = \frac{1}{\cos \psi}. \quad (4.32)$$

For uniform circular motion ($|\mathbf{v}| = \text{const}$) with $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega} \times \hat{\mathbf{u}}$ one has $|\boldsymbol{\omega}_T| = (\gamma - 1) \Omega$.

4.8 Doppler shift

Define the observed frequency as the phase growth rate in the observer's proper time:

$$\nu := \frac{d\chi}{d\tau}. \quad (4.33)$$

For two successive wavefronts the phase increment is identical, hence

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{d\chi/d\tau_{\text{obs}}}{d\chi/d\tau_{\text{src}}} = \frac{d\tau_{\text{src}}}{d\tau_{\text{obs}}}. \quad (4.34)$$

Longitudinal case: during $\gamma d\tau_{\text{src}}$ in the observer frame the source displaces by $\pm V \gamma d\tau_{\text{src}}$ (“+” receding, “−” approaching). Then

$$d\tau_{\text{obs}} = \gamma d\tau_{\text{src}}(1 \pm \beta), \quad \Rightarrow \quad \boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 \pm \beta)}}. \quad (4.35)$$

Equivalent forms (with $\beta = \sin \theta$, $\gamma = \sec \theta$ and rapidity η):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec \theta (1 \mp \sin \theta) = e^{\mp \eta}. \quad (4.36)$$

Transverse Doppler ($\varphi = 90^\circ$ in the observer’s frame):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma} = \cos \theta. \quad (4.37)$$

General line-of-sight (LOS) angle φ in the observer’s frame:

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \gamma (1 - \beta \cos \varphi)}. \quad (4.38)$$

Wavelength ratios are inverse to frequency ratios.

5 Discussion: links to known structures

Gauge phases. A global shift $\chi \mapsto \chi + \chi_0$ is unobservable. Allowing local reparameterizations $\chi \mapsto \chi + \alpha(x)$ induces a connection when comparing phases at different points. On wavefunctions $\psi \sim [\cos(\chi) + \hat{\mathbf{u}} \sin(\chi)]$ *this is the familiar U(1) gauge freedom* $\psi \rightarrow [\cos(\alpha(x)) + \hat{\mathbf{u}} \sin(\alpha(x))]\psi$ with $D_\mu = \partial_\mu - iA_\mu$ as the *phase-transport connection*.

Mass and the internal angle. With the decomposition by ζ , mass heuristically correlates with an irreducible real projection: massless objects have $\zeta = \pm\pi/2$ (no proper time; photon subspace), while massive objects have $|\zeta| < \pi/2$ (proper time exists). In the present paper we set $\zeta = 0$ in boost kinematics by calibration; a detailed mass-generation mechanism is left for future work.

Cosmological gauge. A natural global calibration of “absolute” time is the comoving frame with vanishing CMB dipole. This fixes a cosmological time t (FLRW) as a gauge, without affecting local Lorentz invariance; Doppler factors are then operationally referenced to that frame.

6 Conclusion

In unimetry, time and space are integrals of phase velocities; the Minkowski interval appears as a conserved quantity under parameter change. The core relations of SR— γ , rapidity, velocity addition, and Doppler factors—follow from elementary phase-plane geometry with a single rotation angle θ , while hyperbolic structure re-emerges upon reparameterizing time. The formalism is empirically equivalent to standard SR but can clarify causality and composition by treating all effects as projections of a single flow.

Outlook. Future directions include (i) a more explicit group-theoretic embedding, (ii) a rigorous treatment of the internal angle ζ and its relation to mass, and (iii) exploration of curved metrics as spatially varying Jacobians $\mathcal{J}(x)$ in the phase-to-observable map.

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A Equivalence to the classical Wigner rotation

We sketch an intrinsic quaternionic proof that the unimetry expression for the Wigner rotation coincides with the standard special-relativistic formula.

Step 1: product of two D-rotors. For $d_i = \cos \frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin \frac{\psi_i}{2}$,

$$d_2 d_1 = (c_2 c_1 - s_2 s_1 \cos \theta) + \left(c_2 s_1 \hat{\mathbf{u}}_1 + s_2 c_1 \hat{\mathbf{u}}_2 + s_2 s_1 \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1 \right), \quad (\text{A.1}) \text{ with } c_i = \cos(\psi_i/2), \\ s_i = \sin(\psi_i/2) \text{ and } \cos \theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1.$$

Step 2: symmetric D–R factorization. Define d_{12} by $d_{12} \mathbf{e}_t d_{12} = L_{12} \mathbf{e}_t L_{21}$ and set $r_W = \bar{d}_{12} L_{12} = L_{21} \bar{d}_{12}$. Then r_W fixes \mathbf{e}_t and is a pure spatial rotor, so $r_W = \cos \frac{\phi}{2} + \hat{\mathbf{n}} \sin \frac{\phi}{2}$ with $\hat{\mathbf{n}} \parallel \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. Matching scalar and bivector parts gives

$$\tan \frac{\phi}{2} = \frac{s_1 s_2 \sin \theta}{c_1 c_2 + s_1 s_2 \cos \theta}. \quad (\text{A.2})$$

Step 3: map to rapidities. With the substitutions $\sin(\psi/2) \mapsto \sinh(\eta/2)$, $\cos(\psi/2) \mapsto \cosh(\eta/2)$, $\tan(\psi/2) \mapsto \tanh(\eta/2)$ (where $\tanh \eta = \beta$, $\cosh \eta = \gamma$), (??) becomes the textbook Wigner angle:

$$\tan \frac{\phi}{2} = \frac{\sinh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \sin \theta}{\cosh \frac{\eta_1}{2} \cosh \frac{\eta_2}{2} + \sinh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \cos \theta}, \quad (\text{A.3}) \text{ with axis along } \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1. \text{ This circular–hyperbolic} \\ \text{correspondence is classical; cf. Grassmann [?]}. \quad (\text{A.3})$$