Unimetry: A Phase-Space Reformulation of Special Relativity

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October 14, 2025

Abstract

We propose a compact reformulation of special relativity in which spacetime units (time and length) are treated as *phase velocities* - directional derivatives of a single underlying parameter, the phase $\vec{\chi} \in \mathbb{H}$. The observable Minkowski interval emerges as a conserved quantity under a change of parameter from the hidden phase coordinate χ to the observer's proper time τ. In this *unimetry* formalism, familiar relativistic effects - time dilation, Lorentz factor, Doppler shift, and relativistic velocity composition - arise as elementary projections and rotations in a Euclidean phase plane. Hyperbolic features of Lorentz kinematics reappear after a reparameterization of time, yielding the standard relations without altering empirical content. We provide closed-form derivations of the longitudinal/transverse Doppler factors, identify a simple lemma equating the total phase speed to the conserved Minkowski norm, and outline connections to gauge phases, rapidity, and a cosmological time gauge. Composition of non-collinear boosts (D-rotations) yields a Wigner rotation; in the continuous limit this gives Thomas precession.

Keywords: special relativity; phase; rapidity; Doppler shift; Lorentz factor; Wigner rotation; Thomas precession; phase parameterization.

MSC/PhCS: 83A05; 83-10; 70A05.

1 Introduction

We usually take time and space as primitive. The *phase formalism* introduced here suggests a different viewpoint: time and space are *derived projections* of a single parameter $\vec{\chi} \in \mathbb{H}$ ("phase"). In this picture, relativistic effects such as time dilation and the Doppler shift are geometric consequences of phase-flow rotations.

The proposal does not modify physics; it reorganizes familiar relations in a simpler language. In spirit it is akin to Lagrangian/Hamiltonian re-descriptions of classical mechanics: same empirical content, different coordinates. We will realize the phase kinematics with quaternionic rotors $d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\sin\frac{\psi}{2}$. Throughout, Greek θ will denote the *external* rotation angle associated with relative motion, while ζ denotes an *internal* angle associated with the object's intrinsic state (mass/density heuristic). We emphasize that no modification of Einstein's dynamics is proposed; all results are kinematical identities obtained by a change of parameter.

Notation. Tildes, dots and primes indicate derivatives with respect to the phase parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \qquad \dot{X} := \frac{dX}{d\tau}, \qquad X' := \frac{dX}{dl}.$$

We use $\mathtt{c}$ for the speed of light; $\beta := V/\mathtt{c}$, $\gamma := 1/\sqrt{1-\beta^2}$, rapidity $\tanh\eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

2 Time and space as phase derivatives

Why a complex slice of a quaternion? For local kinematics any unit direction $\hat{\mathbf{u}}$ singles out the two–dimensional subalgebra $\mathrm{Span}\{1, \hat{\mathbf{u}}\} \cong \mathbb{C} \subset \mathbb{H}$. Working in this complex *slice* preserves all boost/rotation algebra along $\hat{\mathbf{u}}$, but keeps formulas elementary. When the direction changes, one updates the slice; the full quaternionic structure is retained.

Let $\vec{\chi} \in \mathbb{C}$ be a variable whose change generates observable time-space effects. We treat the time and space units as directional derivatives (phase velocities) along the real and imaginary directions of a complex basis $(\hat{h}, \mathbf{l})$:

$$\hat{h}\, dx_0 = \frac{\partial \vec{\chi}}{\partial \chi_h} \frac{d\chi_h}{d\chi}\, d\chi = \tilde{H}\, d\chi, \qquad \mathbf{l}\, dx_l = \frac{\partial \vec{\chi}}{\partial \chi_l} \frac{d\chi_l}{d\chi}\, d\chi = \tilde{L}\, d\chi, \quad l = 1, 2, 3. \tag{2.1}$$

Introduce the phase speed of the SR interval $ds = \tilde{S}\, d\chi$. The interval conservation takes the form

$$\tilde{S}^2 = \frac{ds^2}{d\chi^2} = \frac{g_{ij}\, dx^i dx^j}{d\chi^2} = \tilde{H}^2 - \tilde{L}^2, \tag{2.2}$$

equivalently

$$\tilde{H}^2 = \tilde{S}^2 + \tilde{L}^2. \tag{2.3}$$

Writing

$$\tilde{S} = \tilde{H} \cos\theta, \qquad \tilde{L} = \tilde{H} \sin\theta, \tag{2.4}$$

where θ is the angle of the phase speed relative to the real axis. Algebraically, (2.3) is a Euclidean decomposition of a single speed into orthogonal projections; physically, we will see that under reparameterization the *projection* $\tilde{S}$, not the Euclidean norm $\tilde{H}$, is the conserved Minkowski quantity.

3 Phase space (*khōra*)

Let the phase vector space ("*khōra*", after Plato) be $\mathbb{C}$ with orthonormal basis $(\hat{h}, \mathbf{l})$. For a phase vector $\vec{\chi} = R\, e^{\theta \mathbf{l}}$ with $\theta \in [-\pi, \pi]$,

$$\tilde{H} = R, \qquad \tilde{S} = R \cos\theta, \qquad \tilde{L} = R\, \mathbf{l} \sin\theta. \tag{3.1}$$

Choosing coordinates where the projectors onto $(\hat{h}, \mathbf{l})$ are unit, (2.1) simplifies to

$$\hat{h}\, dx_0 = \frac{d\chi_h}{d\chi}\, d\chi = \tilde{H}\, d\chi, \qquad \mathbf{l}\, dx_l = \frac{d\chi_l}{d\chi}\, d\chi = \tilde{L}\, d\chi. \tag{3.2}$$

The map from phase to observables is an integral transform:

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}^i(u)\, du, \qquad i = 0, 1, 2, 3, \tag{3.3}$$

where $\tilde{X}^i$ are projections of $d\vec{\chi}/d\chi$ onto $(\hat{h}, \mathbf{l})$ and $x^i(\chi_0)$ fix initial conditions.

4 Objects

A *fundamental particle* is an *elementary object* with nonzero phase $\vec{\chi} \neq 0$. Composite *objects* are phase configurations; to represent them *in phase space* one may require additional dimensions, except for the *photon*, whose phase is always aligned with the imaginary axis:

$$\mathbf{p} = \frac{d\vec{\chi}}{d\chi_l} = p\, \mathbf{l} \in \Im. \tag{4.1}$$

Non-photonic phenomena are associated with nonzero real projection and nonzero mass. A complex object can be identified with an *event* or worldline; the photon corresponds to a null-interval point encoding information about the event.

Any object's phase can be rotated to the *zero* (purely real) direction,

$$\vec{\chi}_0 = R \in \Re. \tag{4.2}$$

An object A moving with speed V relative to a rest observer has

$$\vec{\chi}_A = R\, e^{\theta_A \mathbf{1}}, \qquad \sin\theta_A = \frac{V}{\mathsf{c}} \equiv \beta. \tag{4.3}$$

4.1 Space as a symmetric phase pair

From (2.4), a naive zero-angle limit would remove the imaginary projection, contradicting observability. We enforce a nonvanishing spatial projection by pairing opposite-phase tilts:

$$\vec{\chi}^{\pm} = R\, e^{\pm\zeta\mathbf{1}}, \qquad \vec{\chi}_l := \frac{\vec{\chi}^{+} - \vec{\chi}^{-}}{2} = R\,\mathbf{1}\sin\zeta, \tag{4.4}$$

where ζ is an *internal angle* (intrinsic to the object; heuristically linked to mass/density). The local decomposition is

$$\vec{\chi}_0 = \vec{\chi}_\tau + \vec{\chi}_l = R\cos\zeta + R\,\mathbf{1}\sin\zeta, \tag{4.5}$$

with unit components (normalized by R): the real component is $\cos\zeta$ and the imaginary component is $\sin\zeta$.

4.2 Absolute, local, and observed time

Define *absolute* time $t = t(\tilde{H})$ at the zero phase direction; it is the fastest clock and useful for normalization between different phase speeds. Along the local real direction,

$$dx_0 = \frac{d}{d\chi}\Re(\vec{\chi})\, d\chi = \frac{\vec{\chi}^{+} + \vec{\chi}^{-}}{2}\, d\chi = \cos\zeta\, d\chi =: d\tau. \tag{4.6}$$

Here $d\chi_0 := \cos\zeta\, d\chi$ is the projection of $d\chi$ onto the local real axis; in Sec. 4.3 we calibrate $d\tau = (1/\nu_0)\, d\chi_0$. The observed proper time of A relative to the rest observer is

$$\tilde{H}_A = \Re\!\left(\frac{d\vec{\chi}_A}{d\vec{\chi}_0}\right) = \cos\theta_A = \sqrt{1-\sin^2\theta_A} = \sqrt{1-\frac{V^2}{\mathsf{c}^2}} = \frac{1}{\gamma}. \tag{4.7}$$

4.3 Normalization

Let local time be parameterized by *phase*; introduce a reference frequency ν_0 and set

$$d\tau = \frac{1}{\nu_0}\, d\chi_0. \tag{4.8}$$

By the chain rule,

$$dx_0 = \tilde{H}\, d\chi = \frac{dx_0}{d\chi_0}\frac{d\chi_0}{d\tau}\, d\tau = \tilde{H}\,\dot{\chi}\, d\tau =: \dot{H}\, d\tau, \tag{4.9}$$

where $\nu := d\chi/d\tau$, $\dot{\chi} := \nu/\nu_0$, and $\dot{H} := \tilde{H}\,\dot{\chi}$. Choosing the calibration $\dot{H} \equiv \mathsf{c}$ gives $dx_0 = \mathsf{c}\, d\tau$. Similarly for space,

$$dx_l = \tilde{L}\, d\chi = \frac{dx_l}{d\chi_0}\frac{d\chi_0}{dl}\, dl = \tilde{L}\,\chi'\, dl =: L'\, dl, \qquad \chi' := \frac{d\chi}{dl}. \tag{4.10}$$

From $dx_0 = dx_l$ for light one gets

$$\mathsf{c} = \tilde{L}'\,\frac{dl}{d\tau}, \tag{4.11}$$

hence with temporal calibration to c the spatial scale becomes unit: $\tilde{L}' = 1$.

4.4 Light and c as a calibration constant

From the normalized forms,

$$\frac{\mathsf{c}}{\dot\chi}\,d\chi = \frac{1}{\chi'}\,d\chi \quad \Rightarrow \quad \mathsf{c} = \frac{\dot\chi}{\chi'} = \frac{dl}{d\tau}, \tag{4.12}$$

i.e. c is a *calibration constant* tying temporal and spatial measures, independent of local phase variation. Equation (4.12) also reads

$$\mathsf{c} = \left(\frac{d\chi}{d\tau}\right)\left[\frac{dl}{d\chi}\right] \sim (\nu)\,[\lambda], \tag{4.13}$$

matching frequency and wavelength of a photon, with χ as its phase. For a lightlike trajectory,

$$ds^2 = \mathsf{c}^2\left(\frac{d\chi^2}{\dot\chi^2} - \frac{d\chi^2}{\dot\chi^2}\right) = 0. \tag{4.14}$$

At unit frequency, $\tau = \chi$: the photon's "proper time" is its phase, and the length of its phase-speed vector equals its wavelength, $\tilde H_p = \lambda$. Finally, the kinematic slope in phase coordinates is

$$\frac{dx_l}{dx_0} = \frac{\tilde L\,d\chi}{\tilde H\,d\chi} = \sin\theta = \frac{V}{\mathsf{c}} \equiv \beta, \tag{4.15}$$

so $\theta = \pi/2$ implies $V = \mathsf{c}$.

4.5 Lorentz factor via reparameterization

A change of direction of the phase speed transforms

$$\tilde H^2 = \tilde S^2 + \tilde L^2 \;\longmapsto\; \dot H^2 = \dot S^2 + \dot L^2. \tag{4.16}$$

Lemma (parameter-change identity). The transition $\tilde H \to \dot S$ is the manifestation of evolving phase speed under the parameter change $\chi \mapsto \tau(\chi)$, with local Jacobian

$$\frac{d\tau}{d\chi} = \cos\zeta(\chi)\cos\theta(\chi) \quad \Rightarrow \quad \mathcal{J}(\zeta,\theta) := \frac{d\chi}{d\tau} = \frac{1}{\cos\zeta\,\cos\theta}. \tag{4.17}$$

Then

$$\dot H = \tilde H\,\mathcal{J}, \qquad \dot L = \tilde L\,\mathcal{J}. \tag{4.18}$$

In differential form,

$$d\ln\dot H = d\ln\mathcal{J} = \tan\zeta\,d\zeta + \tan\theta\,d\theta. \tag{4.19}$$

For a *pure boost* ($d\zeta = 0$) one has $d\dot H = \dot H\tan\theta\,d\theta$. Absorbing a constant $\cos\zeta$ into the calibration (set $\zeta = 0$ henceforth), we obtain

$$\tilde H^2 = \dot H^2 - \dot L^2 = \sec^2\theta\,(\tilde H^2 - \tilde L^2) = \gamma^2(\tilde H^2 - \tilde L^2). \tag{4.20}$$

Corollary. In phase space the Euclidean norm $\tilde H$ is conserved; in observed time the Minkowski norm $\dot S$ is conserved; they are identical as quantities:

$$\boxed{\tilde H = \dot S}. \tag{4.21}$$

4.6 Rapidity and the phase angle

By definition,

$$\beta = \frac{V}{\mathsf{c}} = \sin\theta, \qquad \tanh\eta = \beta, \qquad d\eta = \frac{d\beta}{1-\beta^2}. \tag{4.22}$$

With $d\beta = \cos\theta\, d\theta$ and $1-\beta^2 = \cos^2\theta$,

$$d\eta = \sec\theta\, d\theta, \qquad \eta(\theta) = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| = \tfrac{1}{2}\ln\frac{|1+\sin\theta|}{|1-\sin\theta|}. \tag{4.23}$$

Fixing $\eta(0) = 0$,

$$e^{\eta(\theta)} = \sqrt{\frac{1+\sin\theta}{1-\sin\theta}}, \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \sec\theta = \cosh\eta. \tag{4.24}$$

Remark (groups). Observables satisfy $\beta = \sin\theta = \tanh\eta$ and $\gamma = \sec\theta = \cosh\eta$. Thus Euclidean rotations in the phase circle ($U(1)$ with angle θ) reproduce the numerical factors of hyperbolic boosts in $SO^{+}(1,1)$ (rapidity η) *after* reparameterizing time. We do not claim an isomorphism $U(1) \cong SO(1,1)$; only the equality of observable combinations under the change of parameter.

4.7 Velocity addition

Notation. In unimetry, an inertial boost is a *D-rotation*

$$\mathcal{B}(\hat{\mathbf{u}}, \psi): \quad \mathbf{q} \mapsto d\,\mathbf{q}\,d, \qquad d = \cos\frac{\psi}{2} + \hat{\mathbf{u}}\sin\frac{\psi}{2}, \tag{4.25}$$

and a spatial rotation is an *R-rotation*

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi): \quad \mathbf{q} \mapsto r\,\mathbf{q}\,r^{-1}, \qquad r = \cos\frac{\varphi}{2} + \hat{\mathbf{n}}\sin\frac{\varphi}{2}. \tag{4.26}$$

Kinematic mapping: $\beta \equiv v/c = \sin\psi$, $\gamma = 1/\cos\psi$, $\tan\dfrac{\psi}{2} = \dfrac{\gamma\beta}{\gamma+1}$. For quaternionic/GA accounts of rotors and Lorentz boosts see [5, 6, 7].

4.7.1 Wigner rotation

Let d_1, d_2 be D-rotors of two successive boosts. The raw action on any unimetry 4-object is

$$\mathbf{q}' = d_2 d_1\,\mathbf{q}\,d_1 d_2 \equiv L_{12}\,\mathbf{q}\,L_{21}, \qquad L_{12} = d_2 d_1, \quad L_{21} = d_1 d_2. \tag{4.27}$$

Define d_{12} to be the unique D-rotor reproducing the combined spatio–temporal tilt of L_{12}:

$$\boxed{d_{12}\,\mathbf{e}_t\,d_{12} \;=\; L_{12}\,\mathbf{e}_t\,L_{21}, \qquad \Re(d_{12}) \ge 0} \tag{4.28}$$

(the sign choice removes the trivial two-fold ambiguity). Then the *Wigner rotor* is the residual R-rotation in the symmetric D–R factorization:

$$\boxed{L_{12} = d_{12}\,r_W, \qquad L_{21} = r_W^{-1}\,d_{12}} \tag{4.29}$$

equivalently,

$$\boxed{r_W = \bar{d}_{12}\,L_{12} = L_{21}\,\bar{d}_{12}}. \tag{4.30}$$

Hence the observed map after compensating the tilt is $\bar{d}_{12}\,\mathbf{q}'\,\bar{d}_{12} = r_W\,\mathbf{q}\,r_W^{-1}$.

$$\begin{tikzcd}
\mathbf{q} \arrow[r, "d_1"] \arrow[rr, bend left, dashed, "d_{12}"] & d_1\,\mathbf{q}\,d_1 \arrow[r, "d_2"] & d_2 d_1\,\mathbf{q}\,d_1 d_2 \arrow[d, "\bar{d}_{12}\text{ on both sides}"] \\
 & & r_W\,\mathbf{q}\,r_W^{-1}
\end{tikzcd}$$

Figure 1: Two successive D-rotations (boosts) and compensation of the net spatio–temporal angle by the conjugate of d_{12}, leaving a pure R-rotation r_W.

4.7.2 Thomas precession

The continuous limit of Wigner rotation for a time-dependent velocity direction $\hat{\mathbf{u}}(t)$ yields

$$\boldsymbol{\omega}_T = (\gamma - 1)\left(\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}\right) = \frac{\gamma^2}{\gamma + 1}\,\frac{\mathbf{a} \times \mathbf{v}}{c^2}, \qquad \gamma = \frac{1}{\cos\psi}. \tag{4.31}$$

For uniform circular motion ($|\mathbf{v}| = \text{const}$) with $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega} \times \hat{\mathbf{u}}$ one has $|\boldsymbol{\omega}_T| = (\gamma - 1)\,\Omega$.

4.8 Doppler shift

Define the observed frequency as the phase growth rate in the observer's proper time:

$$\nu := \frac{d\chi}{d\tau}. \tag{4.32}$$

For two successive wavefronts the phase increment is identical, hence

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{d\chi/d\tau_{\text{obs}}}{d\chi/d\tau_{\text{src}}} = \frac{d\tau_{\text{src}}}{d\tau_{\text{obs}}}. \tag{4.33}$$

Longitudinal case: during $\gamma\, d\tau_{\text{src}}$ in the observer frame the source displaces by $\pm V\,\gamma\, d\tau_{\text{src}}$ ("+" receding, "−" approaching). Then

$$d\tau_{\text{obs}} = \gamma\, d\tau_{\text{src}}(1 \pm \beta), \qquad \Rightarrow \quad \boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 \pm \beta)}}. \tag{4.34}$$

Equivalent forms (with $\beta = \sin\theta$, $\gamma = \sec\theta$ and rapidity η):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec\theta\,(1 \mp \sin\theta) = e^{\mp\eta}. \tag{4.35}$$

Transverse Doppler ($\varphi = 90^\circ$ in the observer's frame):

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma} = \cos\theta. \tag{4.36}$$

General line-of-sight (LOS) angle φ in the observer's frame:

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \gamma\,(1 - \beta\cos\varphi)}. \tag{4.37}$$

Wavelength ratios are inverse to frequency ratios.

5 Discussion: links to known structures

Gauge phases. A global shift $\chi \mapsto \chi + \chi_0$ is unobservable. Allowing local reparameterizations $\chi \mapsto \chi + \alpha(x)$ induces a connection when comparing phases at different points. On wavefunctions $\psi \sim e^{i\chi}$ this is the familiar $U(1)$ gauge freedom $\psi \to e^{i\alpha(x)}\psi$ with $D_\mu = \partial_\mu - iA_\mu$ as the *phase-transport connection.*

Mass and the internal angle. With the decomposition by ζ, mass heuristically correlates with an irreducible real projection: massless objects have $\zeta = \pm\pi/2$ (no proper time; photon subspace), while massive objects have $|\zeta| < \pi/2$ (proper time exists). In the present paper we set $\zeta = 0$ in boost kinematics by calibration; a detailed mass-generation mechanism is left for future work.

Cosmological gauge. A natural global calibration of "absolute" time is the comoving frame with vanishing CMB dipole. This fixes a cosmological time t (FLRW) as a gauge, without affecting local Lorentz invariance; Doppler factors are then operationally referenced to that frame.

6 Conclusion

In unimetry, time and space are integrals of phase velocities; the Minkowski interval appears as a conserved quantity under parameter change. The core relations of SR—γ, rapidity, velocity addition, and Doppler factors—follow from elementary phase-plane geometry with a single rotation angle θ, while hyperbolic structure re-emerges upon reparameterizing time. The formalism is empirically equivalent to standard SR but can clarify causality and composition by treating all effects as projections of a single flow.

Outlook. Future directions include (i) a more explicit group-theoretic embedding, (ii) a rigorous treatment of the internal angle ζ and its relation to mass, and (iii) exploration of curved metrics as spatially varying Jacobians $\mathcal{J}(x)$ in the phase-to-observable map.

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A Equivalence to the classical Wigner rotation

We sketch an intrinsic quaternionic proof that the unimetry expression for the Wigner rotation coincides with the standard special-relativistic formula.

Step 1: product of two D-rotors. For $d_i = \cos\frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin\frac{\psi_i}{2}$,

$$d_2 d_1 = (c_2 c_1 - s_2 s_1 \cos\theta) + \Big(c_2 s_1 \hat{\mathbf{u}}_1 + s_2 c_1 \hat{\mathbf{u}}_2 + s_2 s_1\, \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1\Big), \tag{A.1}$$

with $c_i = \cos(\psi_i/2)$, $s_i = \sin(\psi_i/2)$ and $\cos\theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1$.

Step 2: symmetric D–R factorization. Define d_{12} by $d_{12}\,\mathbf{e}_t\,d_{12} = L_{12}\,\mathbf{e}_t\,L_{21}$ and set $r_W = \bar{d}_{12}\,L_{12} = L_{21}\,\bar{d}_{12}$. Then r_W fixes $\mathbf{e}_t$ and is a pure spatial rotor, so $r_W = \cos\frac{\phi}{2} + \hat{\mathbf{n}} \sin\frac{\phi}{2}$ with $\hat{\mathbf{n}} \parallel \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. Matching scalar and bivector parts gives

$$\tan\frac{\phi}{2} = \frac{s_1 s_2\, \sin\theta}{c_1 c_2 + s_1 s_2\, \cos\theta}. \tag{A.2}$$

Step 3: map to rapidities. With the substitutions $\sin(\psi/2) \mapsto \sinh(\eta/2)$, $\cos(\psi/2) \mapsto \cosh(\eta/2)$, $\tan(\psi/2) \mapsto \tanh(\eta/2)$ (where $\tanh\eta = \beta$, $\cosh\eta = \gamma$), (A.2) becomes the textbook Wigner angle:

$$\tan\frac{\phi}{2} = \frac{\sinh\frac{\eta_1}{2}\, \sinh\frac{\eta_2}{2}\, \sin\theta}{\cosh\frac{\eta_1}{2}\, \cosh\frac{\eta_2}{2} + \sinh\frac{\eta_1}{2}\, \sinh\frac{\eta_2}{2}\, \cos\theta}, \tag{A.3}$$

with axis along $\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. This circular–hyperbolic correspondence is classical; cf. Grassmann [4].