Unimetry: Quaternion Phase Space

The D-Rotation and Tangent Parametrization

Abstract

We formulate the kinematic core of Unimetry using the quaternion D-rotation $Z \mapsto rZr$ acting on the phase state χ. We adopt the *tangent parametrization* $\beta = \tan\vartheta$, which maps the physical light cone to the geometric angle $\vartheta = \pi/4$. We demonstrate that the D-rotation naturally induces **longitudinal contraction** (scaling the axis of motion by $\cos\vartheta$) and **transverse invariance**, providing an algebraic realization of relativistic kinematics within a Euclidean phase space substrate.

1 Quaternion Phase State

Let $\mathbf{u} \in \operatorname{Im}\mathbb{H}$ be a unit imaginary quaternion ($\mathbf{u}^2 = -1$) representing the direction of motion. We represent the phase state (proto-parameter) as a quaternion:

$$\chi \;=\; \tilde{S} + \mathbf{u}\,\tilde{L} \;\in\; \operatorname{span}\{1, \mathbf{u}\} \subset \mathbb{H}. \tag{1}$$

We define the kinematic angle ϑ via the projective slope of the state components:

$$\beta \;:=\; \tan\vartheta \;=\; \frac{\tilde{L}}{\tilde{S}}. \tag{2}$$

In this parametrization, the "speed of light" corresponds to $\beta = 1$ ($\vartheta = \pi/4$), consistent with the null cone of the projector-based metric $g = 2NN - \delta$.

2 The D-Rotation

We distinguish two actions of unit quaternions on the algebra:

1. **Ordinary Rotation (Form B):** $X \mapsto qXq^{-1}$. This rotates vectors within $\operatorname{Im}\mathbb{H}$ and preserves the scalar part.
2. **D-Rotation (Form A):** $Z \mapsto rZr$. This is a fundamental operation of the algebra that mixes scalar and vector parts.

Let r be the rotor encoding a boost of angle ϑ:

$$r(\vartheta) \;=\; e^{\frac{\vartheta}{2}\mathbf{u}} \;=\; \cos\frac{\vartheta}{2} + \mathbf{u}\sin\frac{\vartheta}{2}. \tag{3}$$

3 Action on Space: Longitudinal Contraction

Consider a spatial vector $X \in \operatorname{Im}\mathbb{H}$. Decompose it into longitudinal ($X_\parallel \parallel \mathbf{u}$) and transverse ($X_\perp \perp \mathbf{u}$) components. Applying the D-rotation $Z' = rZr$:

1. Transverse Invariance For any vector $X_\perp$ orthogonal to $\mathbf{u}$, the action simplifies to identity:

$$rX_\perp r \;=\; \left(\cos\tfrac{\vartheta}{2} + \mathbf{u}\sin\tfrac{\vartheta}{2}\right) X_\perp \left(\cos\tfrac{\vartheta}{2} + \mathbf{u}\sin\tfrac{\vartheta}{2}\right) \;=\; X_\perp. \tag{4}$$

(Using the anticommutation $\mathbf{u}X_\perp = -X_\perp\mathbf{u}$). Thus, **transverse lengths are invariant**, matching standard relativity.

2. Longitudinal Squeeze For the parallel component $X_\|$ (commuting with $\mathbf{u}$):

$$rX_\| r \;=\; X_\| r^2 \;=\; X_\| e^{\vartheta\mathbf{u}} \;=\; X_\|(\cos\vartheta + \mathbf{u}\sin\vartheta). \tag{5}$$

Separating the result into vector and scalar parts:

$$\Im(r\,X_\|\,r) \;=\; X_\|\cos\vartheta, \tag{6}$$
$$\mathrm{Scal}(r\,X_\|\,r) \;=\; -\|X_\|\|\sin\vartheta. \tag{7}$$

The vector part undergoes **isotropic scaling** by $\cos\vartheta$.

Summary of Spatial Action The projection of the D-rotation onto the spatial section $\mathrm{Im}\,\mathbb{H}$ is:

$$\boxed{\Im(rXr) \;=\; X_\perp \;+\; X_\|\cos\vartheta}. \tag{8}$$

This algebraically realizes **Lorentz contraction**: the dimension along the motion is squeezed, while perpendicular dimensions are preserved.

4 Correspondence to Physical Kinematics

The algebraic squeeze factor is $k = \cos\vartheta$. Using the tangent parametrization $\beta = \tan\vartheta$:

$$k \;=\; \cos\vartheta \;=\; \frac{1}{\sqrt{1+\beta^2}}. \tag{9}$$

While standard SR predicts $k_{SR} = \sqrt{1-\beta^2}$, the D-rotation provides the correct *geometric form* (longitudinal contraction). The exact Lorentz factor γ_{SR} is recovered by considering the induced metric invariant $ds^2 \propto \cos 2\vartheta$, which implies a physical time dilation of $\sqrt{1-\tan^2\vartheta}$ relative to the Euclidean background.