Unimetry: Proto-Space Reformulation of Special Relativity

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Abstract

We develop a reformulation of special relativity in which the Lorentzian causal structure is induced from a four–dimensional Euclidean proto–space $(\mathcal{E}, \delta)$ equipped with a distinguished unit vector field N. Defining the metric

$$g_{AB} \;=\; 2N_A N_B - \delta_{AB}$$

yields a Lorentzian signature $(+---)$ together with an observer–dependent orthogonal split into temporal and spatial parts. Introducing the Euclidean *tilt angle* ϑ between a timelike flow and N, we obtain $g(X,X) = \|X\|_\delta^2 \cos(2\vartheta)$ and the kinematic identity $\tanh\eta = \tan\vartheta$, which provides a bounded angular parametrization of velocities. Proper–time normalization is shown to be equivalent to a constant–speed Euclidean flow constrained to the calibrated sphere $S_c^3 \subset T\mathcal{E}$. For null rays we derive a canonical decomposition $K = \omega\,(N + E)$ and express the measured frequency as the projection $\omega_U = g(U, K)$, yielding concise derivations of Doppler shift and aberration. Finally, we formulate boosts within the Euclidean Clifford algebra $\mathcal{C}\ell_{4,0}$ and exhibit the corresponding rotor representation via an explicit change of bivector basis.

Keywords: special relativity; phase; rapidity; Doppler shift; aberration; Lorentz factor; phase parametrization.

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Contents

1 Introduction

1.1 Motivation

The transition from the definite signature of Euclidean geometry to the indefinite signature of Lorentzian spacetime has traditionally been treated as a fundamental postulate of physics. However, the rigidity and well-behaved spectral properties of elliptic operators suggest that the Euclidean signature may be more fundamental. This paper proposes *Unimetry*—a framework where the physical Lorentzian structure is not axiomatic but derived from a "proto-space": a four-dimensional Euclidean manifold $(\mathcal{E}, \delta)$ permeated by a distinguished flow field N.

Our goal is to provide a self–contained phase–space formulation of the kinematics of special relativity in which relativistic effects are traced to orthogonal decompositions and projections in a Euclidean background. The speed of light appears as the boundary of the physically admissible (timelike) sector on the calibrated sphere, and standard quantities such as β, γ, and rapidity admit simple expressions in terms of a bounded tilt angle. The formalism is designed to be compatible with geometric–algebra methods and to serve as a convenient kinematical foundation; dynamical field equations and source models are intentionally left outside the scope of the present paper.

1.2 Relation to previous work

Attempts to reinterpret Lorentzian geometry through Euclidean structures have a long history. Early geometric perspectives go back to the work of Karapetoff [1], who proposed Euclidean angle constructions for visualizing relativistic transformations. More recent studies by Brands [2], Akintsov et al. [3] investigate embeddings and correspondences between the two geometries.

A rigorous pointwise correspondence was established by Reddy, Sharma and Sivaramakrishnan [4]. Given a Riemannian manifold (M, h) and a unit vector field U, they introduce a Lorentzian metric via $g = h - 2\,U^\flat \otimes U^\flat$. We adopt a sign-flipped variant of this projector method to generate the $(+---)$ signature used in particle physics.

Algebraically, our work draws inspiration from the *Space-Time Algebra* (STA) pioneered by Hestenes [5, 6]. While STA traditionally operates within a Lorentzian Clifford algebra, we demonstrate that the physics can be fully described within a strictly Euclidean algebra $\mathcal{C}\ell_{4,0}$, where the "space-time split" is dynamic and observer-dependent. Additionally, we give an explicit Euclidean–algebra encoding of Lorentz boosts, complementing the standard Space–Time Algebra viewpoint by keeping the underlying quadratic form Euclidean and implementing the Lorentzian structure through the projector field N.

1.3 Outline

The paper is organized as follows:

- **Section 2** constructs the Lorentzian metric g from the Euclidean background (δ, N) and establishes the causal signature.
- **Section 3** explores the properties of the induced metric, including the orthogonal decomposition of tangent vectors and the geometry of the null cone.
- **Section 4** derives the central kinematic identity $\tanh \eta = \tan \vartheta$ and shows that the speed of light corresponds to a 45° Euclidean tilt.
- **Section 5** reformulates 4-velocity normalization as a constraint on the Euclidean flow, mapping admissible states to the sphere S^3.
- **Section 6** demonstrates the explanatory power of the framework. We derive optical effects (Doppler, aberration) as geometric projections, reformulate boosts using Euclidean Geometric Algebra, and reformulate boosts using Euclidean Geometric Algebra.

2 Lorentzian metric construction

2.1 Euclidean proto-space

We work on a four–dimensional Euclidean manifold $(\mathcal{E}, \delta)$, equipped with the flat metric

$$\delta_{AB} = \mathrm{diag}(1, 1, 1, 1).$$

Throughout, indices are raised and lowered with δ:

$$X_A := \delta_{AB} X^B, \qquad X^A := \delta^{AB} X_B,$$

and we use the δ–inner product notation

$$X \cdot Y := \delta(X, Y) = \delta_{AB} X^A Y^B.$$

Remark 2.1 (Index conventions: δ vs. g)*.* Throughout, δ is treated as the *background* Euclidean metric on $\mathcal{E}$, and we use δ to raise and lower abstract indices unless explicitly stated otherwise. The Lorentzian tensor g_{AB} introduced in §2.3 is regarded primarily as a derived bilinear form on $T\mathcal{E}$ (used to define interval–type scalars such as $g(X, X)$), and not as the default device for index gymnastics.

In particular, we distinguish the δ–raised components

$$g^{AB}_{(\delta)} := \delta^{AC} \delta^{BD} g_{CD}$$

from the inverse metric $(g^{-1})^{AB}$ defined by $(g^{-1})^{AC} g_{CB} = \delta^A{}_B$. For the special form $g_{AB} = 2N_A N_B - \delta_{AB}$ with $\delta(N, N) = 1$, one indeed has $(g^{-1})^{AB} = g^{AB}_{(\delta)} = 2N^A N^B - \delta^{AB}$, but the two notions remain conceptually distinct.

2.2 Distinguished unit vector field

Let N be a smooth vector field on $\mathcal{E}$ satisfying the unit condition

$$\delta(N, N) = 1.$$

In particular, N is nowhere vanishing and defines at each point a distinguished δ–unit direction.

Remark 2.2 (Fiducial observer field). In the unimetrical interpretation, the distinguished unit field N plays the role of a fiducial inertial observer field (a "laboratory" or vacuum rest frame). It selects, at each $p \in \mathcal{E}$, the canonical δ–orthogonal splitting

$$T_p\mathcal{E} = \mathbb{R}\, N_p \ \oplus \ N_p^{\perp_\delta},$$

which we use consistently to interpret longitudinal components as temporal and transverse components as spatial. When light rays are represented by null tangents K (see §6.1), the coefficient ω in the parametrization $K = \omega(N + E)$ is precisely the angular frequency measured by these N–observers, i.e. $\omega = g(N, K)$; Doppler shifts for other observers are encoded by $\omega_U = g(U, K)$ (Definition 6.3).

We introduce the δ–orthogonal projector onto the complement of N:

$$h_{AB} := \delta_{AB} - N_A N_B. \tag{2.1}$$

Then h has rank 3 and satisfies

$$h_{AB}N^B = 0, \qquad h_A{}^C h_{CB} = h_{AB}.$$

We write $\mathrm{Im}(h_p) = N_p^{\perp_\delta} \subset T_p\mathcal{E}$ for the δ–orthogonal complement at p.

2.3 Lorentzian metric definition

Define a symmetric $(0, 2)$–tensor field g on $\mathcal{E}$ by

$$g_{AB} := 2N_A N_B - \delta_{AB}. \tag{2.2}$$

Equivalently, using (2.1),

$$g_{AB} = N_A N_B - h_{AB}. \tag{2.3}$$

(i) N is g–unit:

$$g(N, N) = 2(\delta(N, N))^2 - \delta(N, N) = 2 \cdot 1 - 1 = 1.$$

(ii) N is g–orthogonal to $N^{\perp_\delta}$: if $X \in N^{\perp_\delta}$, i.e. $N \cdot X = 0$, then

$$g(N, X) = 2(N \cdot N)(N \cdot X) - \delta(N, X) = 0.$$

(iii) On $N^{\perp_\delta}$ one has $g = -\delta$: if $X, Y \in N^{\perp_\delta}$, then

$$g(X, Y) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = -\delta(X, Y).$$

Hence, at each $p \in \mathcal{E}$,

$$T_p\mathcal{E} = \mathrm{span}\{N_p\} \oplus N_p^{\perp_\delta},$$

and in an adapted δ–orthonormal basis $\{e_0 = N, e_1, e_2, e_3\}$ the bilinear form g_p has the Minkowski block form

$$g_p = (+1) \oplus (-1) \oplus (-1) \oplus (-1).$$

In particular, g has Lorentzian signature $(+ - - -)$.

3 Lorentzian metric properties

Throughout this section, $p \in \mathcal{E}$ is arbitrary and all statements are understood pointwise at p.

3.1 Orthogonal decomposition of tangent vectors

For any $X \in T_p\mathcal{E}$ we define the δ–longitudinal and δ–transverse components relative to N by

$$X_\parallel := (N \cdot X)\, N, \qquad X_\perp := h(X) = X - (N \cdot X)\, N. \tag{3.1}$$

Lemma 3.1. *For every $X \in T_p\mathcal{E}$,*

$$X = X_\parallel + X_\perp,$$

where $X_\parallel \in \operatorname{span}\{N_p\}$ and $X_\perp \in N_p^{\perp_\delta}$. The decomposition is unique.

Proof. Since h_p is a projector with $\ker(h_p) = \operatorname{span}\{N_p\}$ and $\operatorname{Im}(h_p) = N_p^{\perp_\delta}$, the splitting is the standard direct sum decomposition associated with complementary subspaces. □

3.2 Norm identities and classification of vectors

Proposition 3.2. *For any $X \in T_p\mathcal{E}$,*

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp).$$

Proof. Insert (3.1) into $g(X, X)$ and use: $g(N, N) = 1$, $g(N, X_\perp) = 0$ (since $X_\perp \in N^{\perp_\delta}$), and $g(X_\perp, X_\perp) = -\delta(X_\perp, X_\perp)$ from (iii). □

Corollary 3.3. *A vector X satisfies:*

- $g(X, X) > 0$ *iff* $(N \cdot X)^2 > \delta(X_\perp, X_\perp)$,
- $g(X, X) = 0$ *iff* $(N \cdot X)^2 = \delta(X_\perp, X_\perp)$,
- $g(X, X) < 0$ *iff* $(N \cdot X)^2 < \delta(X_\perp, X_\perp)$.

Define the three disjoint subsets of $T_p\mathcal{E}$:

$$\mathcal{T}_p := \{X \in T_p\mathcal{E} : \ g(X, X) > 0\}, \qquad \mathcal{P}_p := \{X \in T_p\mathcal{E} : \ g(X, X) = 0\}, \qquad \mathcal{S}_p := \{X \in T_p\mathcal{E} : \ g(X, X) < 0\}.$$

We also single out the *future* time cone (relative to N):

$$\mathcal{T}_p^+ := \{X \in \mathcal{T}_p : \ N \cdot X > 0\}. \tag{3.2}$$

3.3 Geometry of the null cone

Proposition 3.4. *The set of g–null vectors at p is the quadratic cone*

$$\mathcal{C}_p = \{\, X \in T_p\mathcal{E} : \ \delta(X_\perp, X_\perp) = (N \cdot X)^2 \,\}.$$

Under the decomposition $T_p\mathcal{E} = \operatorname{span}\{N_p\} \oplus N_p^{\perp_\delta}$, it is a double cone given by

$$N \cdot X = \pm \|X_\perp\|_\delta.$$

Proof. Immediate from Corollary 3.3. □

3.4 Spatial rotations preserving δ and N

Let $\mathrm{Aut}(\delta, N)$ denote the stabilizer of N in the Euclidean orthogonal group:

$$\mathrm{Aut}(\delta, N) := \{\, L : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \ \delta(LX, LY) = \delta(X, Y),\ LN = N \,\}.$$

In an adapted δ–orthonormal basis $\{e_0 = N, e_1, e_2, e_3\}$ one has

$$L = \mathrm{diag}(1, R), \qquad R \in O(3),$$

so $\mathrm{Aut}(\delta, N) \cong O(3)$ and contains no boost–like maps mixing N with $N^{\perp_\delta}$.

Lemma 3.5. *Every $L \in \mathrm{Aut}(\delta, N)$ preserves g:*

$$g(LX, LY) = g(X, Y) \qquad \textit{for all } X, Y \in T_p\mathcal{E}.$$

Proof. Since $LN = N$ and L is δ–orthogonal,

$$g(LX, LY) = 2(N \cdot LX)(N \cdot LY) - \delta(LX, LY) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = g(X, Y).$$

□

Thus $\mathrm{Aut}(\delta, N)$ is a spatial subgroup of $\mathrm{O}(g)$: it preserves g and fixes N, but generates only Euclidean rotations on $N^{\perp_\delta}$.

4 Tilt angle geometry and hyperbolic parametrization

4.1 Euclidean tilt angle

For any nonzero $X \in T_p\mathcal{E}$ define its Euclidean tilt angle $\vartheta \in [0, \pi]$ by

$$\cos\vartheta := \frac{N \cdot X}{\|X\|_\delta}, \qquad \sin\vartheta := \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

where $X_\perp$ is defined by (3.1). Whenever $X_\perp \neq 0$, define the δ–unit transverse direction

$$E := \frac{X_\perp}{\|X_\perp\|_\delta} \in N_p^{\perp_\delta}.$$

Then

$$X = \|X\|_\delta \cos\vartheta\, N + \|X\|_\delta \sin\vartheta\, E. \tag{4.1}$$

Lemma 4.1. *For any nonzero $X \in T_p\mathcal{E}$,*

$$\|X\|_\delta^2 = (N \cdot X)^2 + \delta(X_\perp, X_\perp), \qquad \cos^2\vartheta + \sin^2\vartheta = 1.$$

Proof. Immediate from $\delta(X_\parallel, X_\perp) = 0$. □

4.2 Lorentzian norm expressed via ϑ

From Proposition 3.2,

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp) = \|X\|_\delta^2(\cos^2\vartheta - \sin^2\vartheta) = \|X\|_\delta^2 \cos(2\vartheta).$$

Proposition 4.2. *For any nonzero $X \in T_p\mathcal{E}$,*

$$g(X, X) = \|X\|_\delta^2 \cos(2\vartheta).$$

4.3 Domain of hyperbolic parametrization

A real hyperbolic parameter is naturally attached to vectors in the future time cone $\mathcal{T}_p^+$ defined in (3.2). For $X \in \mathcal{T}_p^+$ one has

$$g(X,X) > 0 \quad \Longleftrightarrow \quad \cos(2\vartheta) > 0 \quad \Longleftrightarrow \quad \vartheta \in \Big[0, \frac{\pi}{4}\Big),$$

and moreover $N \cdot X > 0$ implies $\cos\vartheta > 0$, so

$$\beta := \tan\vartheta \in [0,1).$$

Null vectors satisfy $\tan\vartheta = 1$ (equivalently $\vartheta = \pi/4$), while g–negative vectors have $\tan\vartheta > 1$.

4.4 Hyperbolic parameter (rapidity)

For $X \in \mathcal{T}_p^+$ define $\eta \geq 0$ by

$$\tanh\eta := \tan\vartheta. \tag{4.2}$$

Equivalently, one may define η invariantly by

$$\cosh\eta := \frac{N \cdot X}{\sqrt{g(X,X)}}, \qquad \sinh\eta := \frac{\|X_\perp\|_\delta}{\sqrt{g(X,X)}}, \qquad (X \in \mathcal{T}_p^+), \tag{4.3}$$

which immediately implies $\tanh\eta = \|X_\perp\|_\delta / (N \cdot X) = \tan\vartheta$.

Lemma 4.3. *For $X \in \mathcal{T}_p^+$,*

$$\cosh\eta = \frac{\cos\vartheta}{\sqrt{\cos(2\vartheta)}}, \qquad \sinh\eta = \frac{\sin\vartheta}{\sqrt{\cos(2\vartheta)}}.$$

Proof. From (4.2), $\tanh\eta = \tan\vartheta$ gives

$$\cosh^2\eta = \frac{1}{1-\tanh^2\eta} = \frac{1}{1-\tan^2\vartheta} = \frac{\cos^2\vartheta}{\cos^2\vartheta - \sin^2\vartheta} = \frac{\cos^2\vartheta}{\cos(2\vartheta)}.$$

Taking the positive square root (since $\eta \geq 0$ and $\vartheta \in [0, \pi/4)$) yields the expression for $\cosh\eta$, and multiplying by $\tanh\eta = \tan\vartheta$ yields $\sinh\eta$. □

4.5 Differential relation between η and ϑ

Proposition 4.4. *For $X \in \mathcal{T}_p^+$,*

$$\frac{d\eta}{d\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

Proof. Differentiate $\tanh\eta = \tan\vartheta$:

$$\operatorname{sech}^2\eta \, d\eta = \sec^2\vartheta \, d\vartheta.$$

Using $\operatorname{sech}^2\eta = 1 - \tanh^2\eta = 1 - \tan^2\vartheta = \dfrac{\cos(2\vartheta)}{\cos^2\vartheta}$, we obtain

$$\frac{d\eta}{d\vartheta} = \frac{\sec^2\vartheta}{\operatorname{sech}^2\eta} = \frac{1/\cos^2\vartheta}{\cos(2\vartheta)/\cos^2\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

□

4.6 Boost subgroup and additivity of the hyperbolic parameter

Let $\mathrm{O}(g)$ denote the Lorentz group of $(T_p\mathcal{E}, g)$:

$$\mathrm{O}(g) := \{\, \Lambda : T_p\mathcal{E} \to T_p\mathcal{E} \text{ linear} : \; g(\Lambda X, \Lambda Y) = g(X, Y)\,\}.$$

Fix a δ–unit transverse direction $E \in N_p^{\perp_\delta}$ with $\delta(E, E) = 1$. The *boost* in the 2–plane $\operatorname{span}\{N, E\}$ with parameter η is the unique $\Lambda(\eta) \in \mathrm{O}(g)$ acting as a hyperbolic rotation on $\operatorname{span}\{N, E\}$ and as the identity on its g–orthogonal complement:

$$\Lambda(\eta)N = (\cosh\eta)\, N + (\sinh\eta)\, E, \qquad \Lambda(\eta)E = (\sinh\eta)\, N + (\cosh\eta)\, E,$$

$$\Lambda(\eta)X = X \quad \text{for } X \perp_g \operatorname{span}\{N, E\}.$$

Such boosts preserve g but, in general, do not preserve δ and do not fix N.

Theorem 4.5 (Additivity)**.** *For boosts $\Lambda(\eta_1)$ and $\Lambda(\eta_2)$ in the same (N, E)–plane, their composition is a boost with parameter $\eta_1 + \eta_2$:*

$$\Lambda(\eta_1) \circ \Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2).$$

Proof. On $\operatorname{span}\{N, E\}$ the boosts are represented (in the basis $\{N, E\}$) by the matrices

$$\begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix},$$

whose multiplication adds rapidities. On the g–orthogonal complement the action is the identity, hence the statement holds on all of $T_p\mathcal{E}$. $\square$

4.7 Comparison with classical angle conventions

The Euclidean angle ϑ above coincides with the geometric tilt angle used in classical constructions (e.g. Karapetoff) and in later reformulations; the difference lies in which trigonometric function is taken as the primary dimensionless parameter.

A common choice is

$$\beta_{\text{std}} := \sin\vartheta = \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

whereas in the present work we use the tangent–based parameter

$$\beta_{\text{phase}} := \tan\vartheta = \frac{\|X_\perp\|_\delta}{N \cdot X}.$$

On $\mathcal{T}_p^+$ one has $\beta_{\text{phase}} \in [0, 1)$, and the rapidity η is introduced directly by (4.2).

Remark 4.6 (Photon limit and the 45° Euclidean tilt). The null cone is characterized by $g(X, X) = 0$, equivalently $\cos(2\vartheta) = 0$, so that the lightlike limit corresponds to $\vartheta \to \pi/4$ in the Euclidean picture. In this limit one has

$$\beta_{\text{phase}} = \tan\vartheta \to 1, \qquad \beta_{\text{std}} = \sin\vartheta \to \frac{1}{\sqrt{2}}.$$

Thus a light ray is reached at a finite Euclidean tilt of 45° relative to N (not at 90°). This is precisely why the tangent parameterization is better adapted to the projector identity $g(X, X) = \|X\|_\delta^2 \cos(2\vartheta)$: it saturates at the speed–of–light barrier as $\vartheta \to \pi/4$, while the sine parameter does not.

This choice is adapted to the orthogonal splitting (3.1) and to the identity of Proposition 4.2,

$$g(X, X) = \|X\|_\delta^2 \cos(2\vartheta),$$

so that the domain of the hyperbolic parametrization is exactly the future g–time cone $\mathcal{T}_p^+$, without additional postulates.

5 Flow invariants and the emergence of S^3 in the Euclidean proto-space

This section makes precise a key equivalence of the proto–space approach: invariance of the Minkowski interval in an observer's local time is equivalent to constancy of the full proto–space flow vector with respect to a calibrated proto–parameter.

5.1 Worldlines, proto–parameter, and the full flow vector

Let $X : I \to \mathcal{E}$ be a smooth timelike worldline. A *proto–parameter* χ along X is any smooth parameter with nowhere–vanishing derivative. We define the associated *full flow vector* (the proto–space tangent) by

$$\tilde{X} \;:=\; \frac{dX}{d\chi} \;\in T_{X(\chi)}\mathcal{E}. \tag{5.1}$$

Definition 5.1 (Calibrated proto–parameter)**.** A proto–parameter χ is called *calibrated* (with scale c) if

$$\delta(\tilde{X}, \tilde{X}) \;=\; c^2 \qquad \text{along } X. \tag{5.2}$$

Remark 5.2 (Existence and gauge nature). For any timelike worldline X, a calibrated proto–parameter always exists. Indeed, if τ denotes proper time, one may define χ by $d\chi/d\tau = \|\dot{X}\|_\delta/c$. Fixing the time orientation ($d\chi/d\tau > 0$), the resulting calibrated parameter is unique up to an additive constant.

In words: in a calibrated proto–parameter, the full flow vector $\tilde{X}$ has a fixed Euclidean norm in the proto–space. This is the proto–space counterpart of the standard SR statement that the 4–velocity has fixed Minkowski norm in proper time.

5.2 Observer splitting and the interval–rate identity

Fix the distinguished unit field N (hence g and h) as in §2–§3. Pointwise along X, decompose $\tilde{X}$ into δ–longitudinal and δ–transverse parts relative to N:

$$\tilde{X} \;=\; \tilde{H}\,N + \tilde{X}_\perp, \qquad \tilde{H} := N \cdot \tilde{X}, \qquad \tilde{X}_\perp := h(\tilde{X}) \in \mathrm{Im}(h). \tag{5.3}$$

Let $\tilde{L} := \|\tilde{X}_\perp\|_\delta$ and, when $\tilde{L} \neq 0$, $E := \tilde{X}_\perp/\tilde{L} \in \mathrm{Im}(h)$ so that $\tilde{X} = \tilde{H}N + \tilde{L}E$ with $\delta(E, E) = 1$.

Lemma 5.3 (Euclidean and Lorentzian norms of the flow)**.** *Along X one has the identities*

$$\delta(\tilde{X}, \tilde{X}) = \tilde{H}^2 + \tilde{L}^2, \qquad g(\tilde{X}, \tilde{X}) = \tilde{H}^2 - \tilde{L}^2. \tag{5.4}$$

Proof. Since $\tilde{X}_\perp \in N^{\perp_\delta}$, we have $\delta(N, \tilde{X}_\perp) = 0$. Hence $\delta(\tilde{X}, \tilde{X}) = \tilde{H}^2 + \delta(\tilde{X}_\perp, \tilde{X}_\perp) = \tilde{H}^2 + \tilde{L}^2$. For g, use $g(N, N) = 1$, $g(N, \tilde{X}_\perp) = 0$, and $g(\tilde{X}_\perp, \tilde{X}_\perp) = -\delta(\tilde{X}_\perp, \tilde{X}_\perp) = -\tilde{L}^2$. $\square$

Motivated by the phase–formalism viewpoint, we introduce the *interval rate* with respect to χ:

$$\tilde{S}^2 \;:=\; g(\tilde{X}, \tilde{X}) \;=\; \tilde{H}^2 - \tilde{L}^2. \tag{5.5}$$

This is the precise proto–space analogue of writing $ds^2 = g(dX, dX) = \tilde{S}^2\, d\chi^2$ and viewing $\tilde{S}$ as a "Minkowski projection" of the full Euclidean flow.

5.3 Equivalence: invariant interval in local time $\Longleftrightarrow$ calibrated full flow

Let τ denote the *local time* (proper time) along the timelike curve X, i.e. a parameter such that the tangent $\dot{X} := dX/d\tau$ satisfies

$$g(\dot{X}, \dot{X}) = c^2. \tag{5.6}$$

Equivalently, $ds^2 = g(dX, dX) = c^2\, d\tau^2$ along X.

Theorem 5.4 (Reparameterization equivalence)**.** *For any timelike worldline X, the following statements are equivalent:*

(A) X is parameterized by local time τ so that $g(\dot{X}, \dot{X}) = c^2$.

(B) X is parameterized by a calibrated proto–parameter χ so that $\delta(\tilde{X}, \tilde{X}) = c^2$.

Moreover, when both parameters are used on the same curve, they are related by

$$\frac{d\tau}{d\chi} = \frac{\sqrt{g(\tilde{X}, \tilde{X})}}{c} = \frac{\tilde{S}}{c}, \qquad \frac{d\chi}{d\tau} = \frac{\|\dot{X}\|_\delta}{c}. \tag{5.7}$$

Proof. Assume (A). Define χ (up to an additive constant) by the ODE

$$\frac{d\chi}{d\tau} := \frac{\|\dot{X}\|_\delta}{c},$$

which is smooth and positive since $\dot{X} \neq 0$. Then $\tilde{X} = dX/d\chi = (d\tau/d\chi)\dot{X}$, so

$$\delta(\tilde{X}, \tilde{X}) = \Big(\frac{d\tau}{d\chi}\Big)^2 \delta(\dot{X}, \dot{X}) = \frac{c^2}{\|\dot{X}\|_\delta^2}\, \|\dot{X}\|_\delta^2 = c^2,$$

which is (B).

Conversely, assume (B). Define τ (up to an additive constant) by

$$\frac{d\tau}{d\chi} := \frac{\sqrt{g(\tilde{X}, \tilde{X})}}{c},$$

which is well–defined and positive for timelike $\tilde{X}$ since $g(\tilde{X}, \tilde{X}) > 0$. Then $\dot{X} = dX/d\tau = (d\chi/d\tau)\tilde{X}$, hence

$$g(\dot{X}, \dot{X}) = \Big(\frac{d\chi}{d\tau}\Big)^2 g(\tilde{X}, \tilde{X}) = \frac{c^2}{g(\tilde{X}, \tilde{X})}\, g(\tilde{X}, \tilde{X}) = c^2,$$

which is (A). The relations (5.7) are exactly the two defining ODEs. $\square$

Operational meaning. Statement (A) is the standard SR normalization of the 4–velocity in proper time, $g(\dot{X}, \dot{X}) = c^2$. Statement (B) is the corresponding calibration of the proto–parameter, $\delta(\tilde{X}, \tilde{X}) = c^2$. Theorem 5.4 shows these are equivalent and amount to a change of parameter ("norm vs projection"), rather than an additional dynamical assumption.

5.4 The S^3 of admissible flow states

Fix $p \in \mathcal{E}$. The set of all calibrated flow vectors at p is the Euclidean 3–sphere of radius c inside $T_p\mathcal{E}$:

$$S_c^3(p) \;:=\; \big\{\, V \in T_p\mathcal{E} :\; \delta(V,V) = c^2 \,\big\} \;\cong\; S^3. \tag{5.8}$$

Thus, once the calibration (5.2) is imposed, every instantaneous *kinematic state* in the proto–space is a point of an S^3 in the tangent space.

Relative to a chosen time direction N_p, each $V \in S_c^3(p)$ admits the decomposition

$$V = \tilde{H}\,N + \tilde{L}\,E, \qquad \tilde{H}^2 + \tilde{L}^2 = c^2, \qquad E \in \mathrm{Im}(h_p),\ \delta(E,E) = 1,$$

so that the observable "spatial direction" is carried by $E \in S^2 \subset \mathrm{Im}(h_p)$ while the pair $(\tilde{H}, \tilde{L})$ lies on a circle $\tilde{H}^2 + \tilde{L}^2 = c^2$. When $\tilde{L} = 0$, the direction E is irrelevant (any unit choice in $\mathrm{Im}(h_p)$ yields the same vector), and the state reduces to $V = \pm c\,N_p$. This is precisely the geometric reason why S^3 is the natural state manifold for calibrated flows in the Euclidean proto–space: it is the locus of constant full flow magnitude, whereas Lorentzian interval effects arise from the g–projection (5.5) and the reparameterization (5.7).

Remark 5.5 (The physical sector of S^3). While the calibrated flow states form the full sphere

$$S_c^3(p) = \{\, \tilde{X} \in T_p\mathcal{E} :\; \delta(\tilde{X}, \tilde{X}) = c^2 \,\},$$

the requirement that a worldline be timelike with respect to g,

$$g(\tilde{X}, \tilde{X}) > 0,$$

restricts the physically realizable states to the open subset

$$|\tilde{H}| > |\tilde{L}|, \qquad \tilde{H} := N \cdot \tilde{X}, \quad \tilde{L} := \|h(\tilde{X})\|_\delta.$$

Geometrically, this is the union of two polar caps on $S_c^3(p)$ centered at $\pm cN$. The boundary $|\tilde{H}| = |\tilde{L}|$ is the null locus (lightlike curves); equivalently, $g(\tilde{X}, \tilde{X}) = 0$ implies $d\tau/d\chi = 0$ in (5.7).

6 Kinematical consequences in the Euclidean proto-space

The Lorentzian metric g constructed from (δ, N) endows each tangent space $(T_p\mathcal{E}, g)$ with the usual causal structure. The additional advantage of the Euclidean proto–space viewpoint is that it keeps, in the same object, both (i) the observable *spatial* direction data encoded in $\mathrm{Im}(h_p) = N_p^{\perp_\delta}$ and (ii) the *calibration* data (frequencies, time rates) encoded by g–projections onto time directions. In this sense, "light" is not an extra entity but a geometric slice of the null cone by the Euclidean layers $\Sigma_\chi \simeq S^3$ introduced later.

6.1 Light rays: Derivation of frequency and direction

Fix $p \in \mathcal{E}$. A light ray is represented by a nonzero null vector $K \in T_p\mathcal{E}$, satisfying $g(K,K) = 0$ and future-directed ($N \cdot K > 0$). Rather than postulating a parametrization, we derive it from the geometry of the N-split.

Definition 6.1 (Proto-frequency)**.** The frequency of the ray K measured by the distinguished observer N is defined as the g-projection:

$$\omega := g(K, N). \tag{6.1}$$

Note that due to the structure of the metric (2.2), this coincides with the Euclidean projection:

$$g(K,N) = 2(N\cdot K)(N\cdot N) - \delta(K,N) = 2(N\cdot K) - (N\cdot K) = N\cdot K.$$

Thus, $\omega = N\cdot K > 0$.

Now, decompose K into longitudinal and transverse parts relative to N:

$$K = \omega N + K_\perp, \qquad K_\perp \in N^{\perp_\delta}.$$

The null condition $g(K,K) = 0$ implies:

$$0 = g(K,K) = \omega^2 g(N,N) + g(K_\perp, K_\perp) = \omega^2 - \|K_\perp\|_\delta^2.$$

Hence $\|K_\perp\|_\delta = \omega$. We can therefore write $K_\perp = \omega E$, where E is a unique δ-unit spatial direction ($E \in N^{\perp_\delta}$, $\|E\|_\delta = 1$).

Lemma 6.2 (Canonical form)**.** *Every future-directed null vector K admits the unique decomposition:*

$$K = \omega\,(N + E), \tag{6.2}$$

where $\omega = g(K,N)$ is the frequency and E is the observable propagation direction.

6.2 Observers as unit timelike states; measured frequency as a projection

In the proto–space formalism, a (local) observer is represented by a unit future timelike vector

$$U \in T_p\mathcal{E}, \qquad g(U,U) = 1, \qquad g(U,N) > 0.$$

All measurable scalars are obtained by taking g–contractions.

Definition 6.3 (Frequency measured by an observer)**.** For a null ray $K \neq 0$, the frequency measured by the observer U is

$$\omega_U := g(U,K). \tag{6.3}$$

This is the standard invariant definition used in geometric optics in SR. In the present framework, however, (6.3) will become an explicit function of Euclidean angles in $\mathrm{Im}(h_p)$.

6.3 Doppler shift

Let $E_v \in \mathrm{Im}(h_p)$ be a δ–unit direction,

$$\delta(E_v, E_v) = 1,$$

and let U be the observer obtained from N by a boost of rapidity $\eta \geq 0$ in the (N, E_v)–plane:

$$U := (\cosh\eta)\,N + (\sinh\eta)\,E_v. \tag{6.4}$$

Let the null ray be given by (6.2),

$$K = \omega\,(N+E), \qquad \delta(E,E) = 1, \quad E \in \mathrm{Im}(h_p).$$

Then, using bilinearity and the basic identities $g(N,N) = 1$, $g(N,E) = 0$, $g(E_v,N) = 0$, and $g(E_v,E) = -\delta(E_v,E)$, we obtain

$$\begin{aligned}\omega_U &= g(U,K)\\ &= \omega\, g\big((\cosh\eta)N + (\sinh\eta)E_v,\ N+E\big)\\ &= \omega\Big(\cosh\eta - \sinh\eta\,\delta(E_v,E)\Big).\end{aligned} \tag{6.5}$$

Introduce

$$\beta := \tanh\eta, \qquad \gamma := \cosh\eta,$$

and define the Euclidean angle $\psi \in [0,\pi]$ between the velocity axis E_v and the ray direction E in $\mathrm{Im}(h_p)$ by

$$\cos\psi := \delta(E_v, E). \tag{6.6}$$

Then (6.5) becomes the standard relativistic Doppler law:

$$\frac{\omega_U}{\omega} = \gamma\big(1 - \beta\cos\psi\big). \tag{6.7}$$

Interpretation. Equation (6.7) is obtained here as a direct function of an ordinary Euclidean angle on the δ–unit sphere inside $\mathrm{Im}(h_p)$ (the "S^3 layers" $\Sigma_\chi \simeq S^3$ will be introduced later). In this sense the Doppler shift is simply "projection geometry" in the proto–space: the measured frequency is the g–projection of a null direction onto an observer state.

6.4 Aberration as projection plus normalization

The direction of the ray measured by U is the normalized spatial part of K in the g–orthogonal complement of U. Define the g–spatial component of K relative to U by

$$K_{\perp U} := K - (g(U,K))\,U. \tag{6.8}$$

Then $g(U, K_{\perp U}) = 0$, so $K_{\perp U} \in U^{\perp_g}$, and its g–norm is

$$g(K_{\perp U}, K_{\perp U}) = -\,\omega_U^2 \qquad (\text{since } g(K,K) = 0,\ g(U,U) = 1).$$

Hence the *unit* spatial direction of the ray in the U–frame may be taken as

$$E_U := \frac{1}{\omega_U}\,K_{\perp U} = \frac{1}{g(U,K)}\Big(K - (g(U,K))U\Big), \qquad g(E_U,E_U) = -1,\ \ g(U,E_U) = 0. \tag{6.9}$$

To extract the usual aberration formula, specialize to the same kinematics as in §6.3, i.e. U as in (6.4) and $K = \omega(N+E)$. Let ψ' denote the angle between the ray and the velocity axis in the *U–rest space*. Equivalently, $\cos\psi'$ is the (Euclidean) cosine of the angle between the spatial ray direction measured by U and the axis of motion, which is encoded invariantly by the contraction of E_U with the boost axis.

A direct computation (substituting (6.4) and $K = \omega(N+E)$ into (6.9) and taking the component along the U–spatial image of E_v) yields the standard aberration law:

$$\cos\psi' = \frac{\cos\psi - \beta}{1 - \beta\cos\psi}, \tag{6.10}$$

with $\cos\psi = \delta(E_v, E)$ as in (6.6).

Interpretation. Aberration and Doppler are the same operation in two steps:

- Doppler: take the g–projection $g(U,K)$ (a scalar).
- Aberration: subtract the time component $(g(U,K))U$ and normalize the remaining U–spatial part.

In the proto–space picture, both effects are thus immediate consequences of the null cone geometry together with the observer–dependent splitting induced by U.

6.5 Geometric Algebra perspective: The Euclidean origin of Lorenz rotation

The Euclidean proto-space framework admits a natural algebraic structure provided by the Clifford algebra $\mathcal{C}\ell(\mathcal{E},\delta)$. Let the geometric product of vectors $u, v \in T_p\mathcal{E}$ be defined by the fundamental relation

$$uv = \delta(u,v) + u \wedge v.$$

Unlike the standard Spacetime Algebra which postulates a Lorentzian signature, here the algebra is strictly Euclidean ($\mathcal{C}\ell_{4,0}$). The distinguished field N serves as the generator of the *space-time split.*

Observable algebra. Multiplication of any vector X by the distinguished element N decomposes it into a scalar part (time) and a bivector part (space) relative to N:

$$XN = \underbrace{X \cdot N}_{\text{scalar}} + \underbrace{X \wedge N}_{\text{bivector}}. \tag{6.11}$$

Identifying the bivectors containing N with the spatial vectors of the observer, the algebra of observables corresponds to the even subalgebra $\mathcal{C}\ell^{+}_{4,0}$, which is isomorphic to the Pauli algebra (and the quaternions).

The kinematic results of Section 5 find their most natural expression in the language of Geometric Algebra $\mathcal{C}\ell(\mathcal{E},\delta)$, applying the principles of Space-Time Algebra pioneered by Hestenes [5, 6]. Specifically, we utilize the observer-dependent "space-time split" technique. This formalism clarifies that the transition from a Euclidean rotation to a Lorentzian boost is not an ad-hoc analytic continuation, but a direct consequence of the metric signature changing the algebraic properties of the basis vectors.

Euclidean rotation. In the Euclidean proto-space $(\mathcal{E},\delta)$, both the distinguished vector N and any transverse spatial direction E square to $+1$:

$$N^2 = 1, \qquad E^2 = 1.$$

The plane spanned by them is described by the bivector $\mathbf{I} = NE$. Due to the Euclidean signature, this generator squares to -1:

$$\mathbf{I}^2 = (NE)(NE) = -N^2E^2 = -(1)(1) = -1.$$

Consequently, the exponential of this bivector generates standard trigonometric rotations. The kinematic relationship between two flow vectors $\tilde{X}_1, \tilde{X}_2 \in S^3_c$ is governed by the rotor $R = e^{-\mathbf{I}\vartheta/2}$:

$$\tilde{X}_2 = R\tilde{X}_1\tilde{R}^{-1} = \tilde{X}_1 \cos\vartheta + \tilde{X}_{1\perp} \sin\vartheta.$$

Lorentzian boost via vector substitution. The construction of the Lorentzian metric g effectively replaces the Euclidean spatial vector E with a physical spatial vector $\mathbf{e}$ which, while parallel to E, squares to -1 in the metric g:

$$\mathbf{e}^2 \stackrel{g}{=} -1, \qquad \text{while } N^2 \stackrel{g}{=} 1.$$

This change in the basis vector fundamentally alters the bivector describing the time-space plane. The new generator (the boost bivector) $\mathbf{K} = N\mathbf{e}$ satisfies:

$$\mathbf{K}^2 = (N\mathbf{e})(N\mathbf{e}) = -N^2\mathbf{e}^2 = -(1)(-1) = +1.$$

Because the generator now squares to $+1$, its exponential series produces hyperbolic functions instead of trigonometric ones. Using the rapidity η (related to tilt by $\tanh\eta = \tan\vartheta$), the physical transformation becomes a boost $L = e^{-\mathbf{K}\eta/2}$:

$$L = \cosh\frac{\eta}{2} - \mathbf{K}\sinh\frac{\eta}{2}.$$

Applying this to the rest vector N yields the 4-velocity U:

$$U = LNL^{-1} = N\cosh\eta + \mathbf{e}\sinh\eta.$$

Thus, the phenomenon of "Lorentzian" kinematics emerges algebraically simply because the spatial part of the basis becomes "imaginary" (squares to -1) relative to the time direction, flipping the sign of the bivector square and transitioning the geometry from circular to hyperbolic.

6.6 Scope and limitations

The present work is purely kinematical. We do not introduce dynamical equations for the field N, nor do we model localized sources or topological defects. Such extensions may be investigated separately, once the induced–metric framework is combined with appropriate field dynamics.

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