

Unimetry: Phase-Space Reformulation of Special Relativity

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Abstract

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1 Introduction

1.1 Motivation

A Euclidean phase space (\mathcal{E}, δ) with a preferred unit vector field N provides a minimal geometric environment in which Minkowski spacetime and relativistic kinematics arise from projector-induced structure. This paper develops the kinematic core of unimetry.

1.2 Relation to previous work

Attempts to reinterpret Lorentzian geometry through Euclidean structures have a long history. Early geometric perspectives go back to the work of Karapetoff [1], who proposed Euclidean angle constructions for visualizing relativistic transformations. More recent studies by Brands [2], Akintsov et al. [3] and others investigate this correspondence.

A rigorous pointwise correspondence between Riemannian and Lorentzian metrics was established by Reddy, Sharma and Sivaramakrishnan [4]. Given a Riemannian manifold (M, h) and a unit vector field U , they introduce a Lorentzian metric via

$$g = h - 2U^b \otimes U_b,$$

which yields a Lorentzian signature and defines a local diffeomorphism between the Riemannian and induced Lorentzian geometries. This projector-based transformation shows that the specification of a unit direction field is sufficient to generate a Lorentzian metric from a Euclidean one; see also related constructions in the context of observer-based decompositions of pseudo-Riemannian metrics.

The present work combines these two lines of research. We construct a Euclidean phase-space counterpart of Minkowski spacetime by starting from a flat Euclidean phase space (\mathcal{E}, δ) equipped with a distinguished unit flow field N , and applying the equivalent projector transformation used in the Reddy–Sharma–Sivaramakrishnan framework, except we use a sign-flipped variant of the construction by Reddy et al. to achieve the $(+ - - -)$ signature preferred in particle physics. In addition to the geometric correspondence, we examine the physical interpretation of this Euclidean phase space in terms of constant-speed flows, phase calibrations and tilt geometry. The

resulting formulation provides a phase-space representation of special relativity which remains fully compatible with standard Lorentzian kinematics.

1.3 Outline

Brief summary of sections.

2 Lorentzian metric construction

2.1 Euclidean phase space

We consider a four-dimensional Euclidean manifold (\mathcal{E}, δ) equipped with the flat metric $\delta_{AB} = \text{diag}(1, 1, 1, 1)$.

2.2 Distinguished unit vector field

Let N be a smooth vector field on \mathcal{E} satisfying the unit condition

$$\delta(N, N) = 1.$$

In particular, N is nowhere vanishing. The field N singles out a preferred direction at each point of \mathcal{E} . We introduce the associated orthogonal projector onto the δ -orthogonal complement of N :

$$h_{AB} := \delta_{AB} - N_A N_B.$$

This defines a rank-3 projector satisfying $h_{AB} N^B = 0$ and $h_A{}^C h_{CB} = h_{AB}$.

2.3 Lorentzian metric definition

We now define a symmetric $(0, 2)$ -tensor field g on \mathcal{E} by

$$g_{AB} := 2N_A N_B - \delta_{AB}. \tag{2.1}$$

By construction, g is smooth and bilinear at each point of \mathcal{E} . The following properties are immediate:

- (i) N is unit vector with respect to g :

$$g(N, N) = 2(N \cdot N)^2 - \delta(N, N) = 2 \cdot 1^2 - 1 = 1.$$

- (ii) N is g -orthogonal to its δ -orthogonal complement. If X satisfies $\delta(X, N) = 0$, then

$$g(N, X) = 2(N \cdot N)(N \cdot X) - \delta(N, X) = 2 \cdot 1 \cdot 0 - 0 = 0.$$

- (iii) On vectors X, Y orthogonal to N with respect to δ (that is, in the image of h) one has

$$g(X, Y) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = -\delta(X, Y).$$

Thus at each point $p \in \mathcal{E}$ the tangent space decomposes as

$$T_p \mathcal{E} = \text{span}\{N_p\} \oplus \text{Im}(h_p),$$

and in this decomposition the bilinear form g_p takes the block form

$$g_p = (+1) \oplus (-\delta|_{\text{Im}(h_p)}).$$

Since $\delta|_{\text{Im}(h_p)}$ is positive definite of rank 3, the restriction $-\delta|_{\text{Im}(h_p)}$ is negative definite of rank 3. Consequently, g has Lorentzian signature $(+---)$, with N defining a distinguished orientation.

Equivalently, using $h_{AB} = \delta_{AB} - N_A N_B$, one may rewrite (2.1) in the form

$$g_{AB} = N_A N_B - h_{AB},$$

which makes explicit that g coincides with δ along N and equals $-\delta$ on the δ -orthogonal complement of N . This projector-based construction will be the starting point for the phase-space reformulation of special relativistic kinematics developed in the subsequent sections.

2.4 Hollow space and the no-source reference state

The projector construction requires an additional geometric datum beyond the Euclidean metric δ : a nowhere-vanishing unit vector field N . In the *no-source* reference state (discussed in §4) we assume a truly empty background: there is no vector field at all. Consequently, no projector h and no induced Lorentzian metric can be defined there. The construction in §2.3 is therefore conditional: whenever a unit vector field N is present (e.g. after emergence from a source), it induces h and g .

For notational convenience we denote this absence of a vector by the symbol

$$N_0 := 0,$$

understood as the zero section of $T\mathcal{E}$ used only as a bookkeeping reference. It does not represent a physical vector field and it is not meant to carry any geometric structure. In particular, N_0 cannot be normalized and cannot define either h or the Lorentzian metric g .

A nontrivial vector field N (hence h and g) emerges only when the background is populated by field sources. After coarse-graining over a sufficiently dense ensemble, the second moment develops a distinguished eigendirection, which defines the effective unit field N .

3 Lorentzian metric properties

Having constructed the Lorentzian metric g from the Euclidean metric δ and the distinguished unit field N , we now record several basic geometric consequences. Throughout this section, $p \in \mathcal{E}$ is arbitrary and all statements are understood pointwise.

3.1 Orthogonal decomposition of tangent vectors

For any $X \in T_p\mathcal{E}$ we define its δ -longitudinal and δ -transverse components relative to N by

$$X_{\parallel} := (N \cdot X) N, \quad X_{\perp} := h(X) = X - (N \cdot X) N.$$

Lemma 3.1. *For every $X \in T_p\mathcal{E}$,*

$$X = X_{\parallel} + X_{\perp},$$

where $X_{\parallel} \in \text{span}\{N_p\}$ and $X_{\perp} \in \text{Im}(h_p)$. The decomposition is unique.

Proof. Since h_p is a projector with kernel $\text{span}\{N_p\}$ and image $\text{Im}(h_p)$, the stated decomposition is the usual splitting associated with a complementary pair of subspaces. Uniqueness follows because their intersection is trivial. \square

3.2 Norm identities and classification of vectors

Using the decomposition above, the g -norm of any vector splits into explicit longitudinal and transverse parts.

Proposition 3.2. *For any $X \in T_p\mathcal{E}$,*

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp).$$

Proof. Insert $X = X_\parallel + X_\perp$ into $g(X, X)$ and use: $g(N, N) = 1$, $g(N, X_\perp) = 0$, and $g(X_\perp, X_\perp) = -\delta(X_\perp, X_\perp)$. \square

This makes the following classification immediate.

Corollary 3.3. *A vector X satisfies:*

- $g(X, X) > 0$ iff $(N \cdot X)^2 > \delta(X_\perp, X_\perp)$,
- $g(X, X) = 0$ iff $(N \cdot X)^2 = \delta(X_\perp, X_\perp)$,
- $g(X, X) < 0$ iff $(N \cdot X)^2 < \delta(X_\perp, X_\perp)$.

In particular, X is g -null if and only if $\delta(X_\perp, X_\perp) = (N \cdot X)^2$.

For each $p \in \mathcal{E}$ we may thus define three disjoint subsets of the tangent space $T_p\mathcal{E}$:

$$\mathcal{T}_p := \{X \in T_p\mathcal{E} : g(X, X) > 0\},$$

$$\mathcal{P}_p := \{X \in T_p\mathcal{E} : g(X, X) = 0\},$$

$$\mathcal{S}_p := \{X \in T_p\mathcal{E} : g(X, X) < 0\}.$$

3.3 Geometry of the null cone

The equation characterizing null vectors has a simple geometric form.

Proposition 3.4. *The set of g -null vectors at p is the quadratic cone*

$$\mathcal{C}_p = \{X \in T_p\mathcal{E} : \delta(X_\perp, X_\perp) = (N \cdot X)^2\}.$$

Under the decomposition $T_p\mathcal{E} = \text{span}\{N_p\} \oplus \text{Im}(h_p)$ the null cone is the graph of a quadratic form over $\text{Im}(h_p)$.

Proof. Immediate from the previous corollary. \square

Thus the null structure of g is encoded entirely by the Euclidean norm on the three-dimensional space $\text{Im}(h_p)$ and the linear functional $X \mapsto N \cdot X$.

3.4 Spatial rotations preserving δ and N

Let $\text{Aut}(\delta, N)$ denote the stabilizer of N in the Euclidean orthogonal group:

$$\text{Aut}(\delta, N) := \{L : T_p\mathcal{E} \rightarrow T_p\mathcal{E} \text{ linear} : \delta(LX, LY) = \delta(X, Y), LN = N\}.$$

Such transformations fix the distinguished direction N and act by ordinary Euclidean rotations/reflections on the transverse subspace $\text{Im}(h) = N^\perp$. In an adapted δ -orthonormal basis $\{e_0 = N, e_1, e_2, e_3\}$ one has

$$L = \text{diag}(1, R), \quad R \in O(3).$$

In particular, $\text{Aut}(\delta, N) \cong O(3)$ and does *not* contain boost-like transformations mixing N with $\text{Im}(h)$.

Lemma 3.5. *Every $L \in \text{Aut}(\delta, N)$ preserves g as well:*

$$g(LX, LY) = g(X, Y) \quad \text{for all } X, Y \in T_p\mathcal{E}.$$

Proof. Since $LN = N$ and L is δ -orthogonal,

$$g(LX, LY) = 2(N \cdot LX)(N \cdot LY) - \delta(LX, LY) = 2(N \cdot X)(N \cdot Y) - \delta(X, Y) = g(X, Y).$$

□

Thus $\text{Aut}(\delta, N)$ is a (spatial) subgroup of the Lorentz group $O(g)$: it preserves g and fixes N , but it generates only rotations in $\text{Im}(h)$.

3.5 Canonical coordinates adapted to N

At each point p one may choose a local frame $\{e_0, e_1, e_2, e_3\}$ with $e_0 = N_p$ and $\{e_i\}$ an orthonormal basis for $\text{Im}(h_p)$ with respect to δ . In this frame,

$$g = \text{diag}(1, -1, -1, -1),$$

recovering the standard Minkowski form of the metric.

This adapted representation will be used in subsequent sections to make explicit the hyperbolic-angle parametrizations and decomposition properties central to the phase-space formulation of relativistic kinematics developed below.

4 Field source

The construction of a Lorentzian metric in §2 assumes the presence of a smooth unit vector field N . In the no-source reference state, by contrast, there is no vector field at all; we denote this absence by the bookkeeping symbol $N_0 := 0$. In this section we model a field source as a finite (or countable) collection of localized orientation defects (streamlets) on top of this empty background. Only after coarse-graining over a sufficiently large ensemble of such defects within a closed four-volume does an effective smooth unit field N_{eff} emerge, and with it a well-defined induced Lorentzian metric g .

Indices are raised and lowered with the Euclidean metric δ unless explicitly stated otherwise; in particular $N_{aA} := \delta_{AB} N_a^B$ and $h^{AB} := \delta^{AC} \delta^{BD} h_{CD}$.

4.1 Streamlets on the background N_0

Fix $p \in \mathcal{E}$ and $R > 0$ and define

$$\Omega := B_R(p) := \{q \in \mathcal{E} : \|q - p\|_\delta \leq R\}, \quad \partial\Omega \cong S_R^3(p).$$

Let \mathcal{I} be a finite or countable index set (e.g. $\mathcal{I} = \{1, \dots, n\}$ in the finite case, or $\mathcal{I} \subset \mathbb{N}$ in the countable case) and index $a \in \mathcal{I}$.

Definition 4.1 (Streamlet). A *streamlet* is a triple (p_a, N_a, \mathbf{u}_a) consisting of

- a point $p_a \in \mathcal{E}$ (the location of a compact defect in 4D),
- a unit director $N_a \in T_{p_a}\mathcal{E}$ with $\delta(N_a, N_a) = 1$,
- an internal phase state $\mathbf{u}_a \in \overline{B^3} \subset \text{Im } \mathbb{H}$, where $\overline{B^3} := \{\mathbf{u} \in \text{Im } \mathbb{H} : \|\mathbf{u}\| \leq 1\}$.

The case $\|\mathbf{u}_a\| = 1$ corresponds to a *pure* internal state, while $\|\mathbf{u}_a\| < 1$ represents a *mixed* (coarse-grained) state. A concrete realization of \mathbf{u} via a Hopf lift, together with the phase parameter χ along the S^1 fibre, is given in Appendix A. The no-source background is the absence of streamlets (bookkeeping symbol $N_0 := 0$).

Definition 4.2 (Field source). A (monopole-type) *field source* supported in Ω is then an ensemble of streamlets

$$\mathcal{S}_\Omega := \{(p_a, N_a, w_a)\}_{a \in \mathcal{I}}, \quad p_a \in \Omega, \quad w_a \geq 0,$$

with total weight $W := \sum_a w_a \in (0, \infty)$.

In the flat phase space we may identify tangent spaces by translation and treat the vectors N_a as elements of a common \mathbb{R}^4 for the purpose of moment averages.

4.2 First moment and the centered (rest) configuration

The first moment (dipole moment) of the source is defined by the weighted average

$$M_{(1)}^A := \frac{1}{W} \sum_a w_a N_a^A. \quad (4.1)$$

Definition 4.3 (Centered / rest configuration). We say that the source is *centered* (or is described in its rest configuration) if its first moment vanishes:

$$M_{(1)}^A = 0. \quad (4.2)$$

This condition is the discrete analogue of a vanishing mean in the measure-theoretic formulation: there is no net preferred direction at first order, and the leading structural information is therefore carried by the second moment.

4.3 Second moment tensor of the source

The second moment tensor of the source is the symmetric tensor

$$M_{AB} := \frac{1}{W} \sum_a w_a N_{aA} N_{aB}. \quad (4.3)$$

By construction $M_{AB} = M_{BA}$ and for any $X \in \mathbb{R}^4$ one has

$$X^A X^B M_{AB} = \frac{1}{W} \sum_a w_a (N_a \cdot X)^2 \geq 0,$$

so M is positive semidefinite with respect to δ . Moreover, since $\delta(N_a, N_a) = 1$ for each streamlet,

$$\delta^{AB} M_{AB} = \frac{1}{W} \sum_a w_a \delta(N_a, N_a) = 1. \quad (4.4)$$

4.4 Field intensity from streamlet counts and second moments

So far we have extracted only the directional content of the source via the normalized second moment M_{AB} in (4.6), whose δ -trace is fixed to 1 by (4.7). To describe the strength of the source we also need an intensity scale measuring how many streamlets populate the region Ω .

Let $\text{Vol}_\delta(\Omega)$ denote the Euclidean four-volume of Ω induced by δ . We define the (coarse-grained) field intensity in Ω by

$$\mathcal{I}_\Omega := \frac{W}{\text{Vol}_\delta(\Omega)} = \frac{1}{\text{Vol}_\delta(\Omega)} \sum_{a \in \mathcal{I}} w_a. \quad (4.5)$$

If all weights are equal ($w_a \equiv 1$), then \mathcal{I}_Ω is simply the number density of streamlets (count per four-volume).

Equivalently, intensity is obtained directly from the unnormalized second moment, i.e. from the dyadic squares $N_{aA}N_{aB}$:

$$\mathcal{M}_{AB} := \frac{1}{\text{Vol}_\delta(\Omega)} \sum_{a \in \mathcal{I}} w_a N_{aA} N_{aB} = \mathcal{I}_\Omega M_{AB}. \quad (4.6)$$

Taking the δ -trace and using $\delta(N_a, N_a) = 1$ gives the identity

$$\delta^{AB} \mathcal{M}_{AB} = \frac{1}{\text{Vol}_\delta(\Omega)} \sum_a w_a \delta(N_a, N_a) = \frac{W}{\text{Vol}_\delta(\Omega)} = \mathcal{I}_\Omega. \quad (4.7)$$

Thus the scalar intensity is the trace of the second-moment density tensor, reflecting that it is built from the squared orientation vectors.

For an isotropic centered source, where $M_{AB} = \cos^2 \zeta N_A N_B + \frac{\sin^2 \zeta}{3} h_{AB}$ (see (4.12)), the density tensor factorizes as

$$\mathcal{M}_{AB} = \mathcal{I}_\Omega \left(\cos^2 \zeta N_A N_B + \frac{\sin^2 \zeta}{3} h_{AB} \right),$$

so \mathcal{I}_Ω controls the overall strength, while ζ encodes the directional budget between the longitudinal and transverse sectors.

4.5 Transversely isotropic source and canonical form

To speak about a physically isotropic source we require that, after coarse-graining over many streamlets, the second moment tensor is invariant under rotations in the subspace orthogonal to the distinguished direction. In the language of continuum mechanics, such a tensor is called transversely isotropic with respect to the vector N .

Definition 4.4 (Transversely isotropic field source). A centered source is *transversely isotropic* (or spatially isotropic in its rest frame) if there exists a unit vector N such that M_{AB} admits the spectral decomposition

$$\text{spec}(M) = \{\lambda_\parallel, \lambda_\perp, \lambda_\perp, \lambda_\perp\}, \quad \lambda_\parallel > \lambda_\perp \geq 0,$$

with N being the eigenvector corresponding to the simple eigenvalue λ_\parallel . Equivalently,

$$M_{AB} = \alpha N_A N_B + \beta h_{AB}, \quad (4.8)$$

where $h_{AB} = \delta_{AB} - N_A N_B$ is the transverse projector and $\alpha, \beta \geq 0$.

In this form the longitudinal and transverse traces are

$$M_\parallel := M_{AB} N^A N^B = \alpha, \quad M_\perp := h^{AB} M_{AB} = 3\beta,$$

and by (4.7) they satisfy $\alpha + 3\beta = 1$.

4.6 Structural angle ζ from the second moment

The second moment of an isotropic source naturally defines a single structural (budget) angle ζ measuring how the unit second-moment trace is split between the distinguished direction and its orthogonal complement. We define

$$\boxed{\cos^2 \zeta := M_\parallel = M_{AB} N^A N^B, \quad \sin^2 \zeta := M_\perp = h^{AB} M_{AB}.} \quad (4.9)$$

Because $\alpha + 3\beta = 1$, this is equivalent to the parametrization

$$\alpha = \cos^2 \zeta, \quad \beta = \frac{1}{3} \sin^2 \zeta, \quad M_{AB} = \cos^2 \zeta N_A N_B + \frac{\sin^2 \zeta}{3} h_{AB}.$$

Once N is identified, the induced Lorentzian metric $g_{AB} = 2N_A N_B - \delta_{AB}$ yields an additional scalar contraction:

$$g^{AB} M_{AB} = \alpha - 3\beta = \cos^2 \zeta - \sin^2 \zeta = \cos(2\zeta). \quad (4.10)$$

Thus, for an isotropic centered source, the angle ζ is equivalently recovered from the pair of invariants

$$\delta^{AB} M_{AB} = 1, \quad g^{AB} M_{AB} = \cos(2\zeta).$$

4.7 Emergence of the Lorentzian metric

For a sparse set of streamlets, the discrete data $\{(p_a, N_a)\}$ does not define a smooth unit field. However, assuming a sufficient density of sources, we may consider a local coarse-graining where $M_{AB}(x)$ is computed via a smoothing kernel around each point x .

Crucially, while the local first moment vanishes ($M_{(1)} \approx 0$) for a centered source distribution, the second moment $M_{AB}(x)$ does not. Provided the sources possess local orientational order, the principal eigenvector of $M_{AB}(x)$ uniquely identifies a smooth unit vector field $N(x)$. This emergent field then determines the projector $h(x)$ and induces the Lorentzian metric field

$$g_{AB}(x) = 2N_A(x)N_B(x) - \delta_{AB}.$$

In this framework, the causal structure of spacetime is not fundamental but emerges as a statistical property of the underlying streamlet ensemble.

4.8 Time as an unwrapped phase

The phase parameter χ introduced above is an angular variable on the Hopf fiber and is therefore cyclic: $\chi \sim \chi + 2\pi$. To obtain a non-compact time variable we use the standard unwrapping procedure.

Unwrapped phase. Along any connected physical history of a streamlet (i.e. along any chosen continuous evolution of its Hopf lift), we fix a continuous branch and define the *unwrapped phase*

$$\tilde{\chi} \in \mathbb{R}, \quad \tilde{\chi} \bmod 2\pi = \chi.$$

Equivalently, $\tilde{\chi}$ counts how many 2π -cycles have been completed since a reference state.

Phase time (proper time). A streamlet returns to the same internal state after $\Delta\chi = 2\pi$; hence each full phase cycle defines one “tick” of its intrinsic clock. We define the (local) proper time τ of the streamlet by calibrating the unwrapped phase with a structural angular frequency $\omega_0 > 0$:

$$d\tau := \frac{1}{\omega_0} d\tilde{\chi}, \quad \text{so that} \quad \tau = \frac{\tilde{\chi} - \tilde{\chi}_*}{\omega_0}. \quad (4.11)$$

Thus one period $\Delta\chi = 2\pi$ corresponds to the proper-time increment

$$\Delta\tau = \frac{2\pi}{\omega_0}.$$

The parameter ω_0 is a structural property of the streamlet (to be specified later, e.g. in terms of coarse-grained invariants such as intensity and the structural angle ζ). Equation (4.14) is the definition of time in the phase-space picture: time is the calibrated count of phase cycles.

Geometric rationale. In phase space the internal phase cycle closes, returning the streamlet to the same state after 2π . Because the defect is compact (a point in 4D), its small boundary is compact (S^3), and the Hopf fibration decomposes this boundary into closed S^1 fibers over a base direction. The existence of a distinguished fiber coordinate χ is therefore intrinsic, and the time variable τ arises as its unwrapped and calibrated counterpart.

5 Local quaternionic algebra and phase dynamics

5.1 Hopf lift and phase without punctures

To avoid introducing punctures or excised cores in order to define a phase potential, we represent phase as a fiber variable of a Hopf-type lift. Let

$$q : \mathcal{E} \rightarrow S^3 \subset \mathbb{H}, \quad |q| = 1,$$

be a unit quaternion (equivalently, an $SU(2)$ field). Fix an imaginary unit $\mathbf{i} \in \text{Im } \mathbb{H}$ and define the associated vector field

$$\mathbf{n} := q \mathbf{i} q^{-1} \in S^2 \subset \text{Im } \mathbb{H}.$$

This is the Hopf map $S^3 \rightarrow S^2$ applied pointwise. The fiber action $q \mapsto e^{i\theta} q$ leaves \mathbf{n} invariant and plays the role of a $U(1)$ gauge symmetry; a scalar phase Φ exists only locally after a gauge choice.

The gauge-covariant phase object is the connection 1-form

$$\alpha := -2 \text{Im}(\bar{q} dq),$$

which transforms as $\alpha \mapsto \alpha + 2 d\theta$ under $q \mapsto e^{i\theta} q$. Its curvature $F := d\alpha$ is gauge invariant and encodes the topological twisting of the phase fibers without any excised set.

6 Tilt angle geometry and hyperbolic parametrization

In this section we introduce a Euclidean tilt angle ϑ for nonzero vectors $X \in T_p \mathcal{E}$ and derive from it a natural hyperbolic parametrization on the g -positive subset $\mathcal{T}_p = \{X : g(X, X) > 0\}$. This parametrization will later encode the additive structure of the one-parameter family of Lorentz boosts in the (N, E) -plane, i.e. of a boost subgroup of the Lorentz group $O(g)$.

6.1 Euclidean tilt angle

For any nonzero $X \in T_p \mathcal{E}$ we define its Euclidean tilt angle $\vartheta \in [0, \pi]$ by

$$\cos \vartheta := \frac{N \cdot X}{\|X\|_\delta}, \quad \sin \vartheta := \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

where $X_\parallel := (N \cdot X) N$ and $X_\perp := h(X)$ is the δ -orthogonal projection onto $\text{Im}(h_p)$.

Whenever $X_\perp \neq 0$ we introduce the δ -unit transverse direction

$$E := \frac{X_\perp}{\|X_\perp\|_\delta}.$$

Thus the decomposition becomes

$$X = \|X\|_\delta \cos \vartheta N + \|X\|_\delta \sin \vartheta E.$$

Lemma 6.1. *For any nonzero $X \in T_p \mathcal{E}$,*

$$\|X\|_\delta^2 = (N \cdot X)^2 + \delta(X_\perp, X_\perp), \quad \cos^2 \vartheta + \sin^2 \vartheta = 1.$$

Proof. Immediate from $\delta(X_\parallel, X_\perp) = 0$. □

6.2 Lorentzian norm expressed via ϑ

From Proposition 2.1,

$$g(X, X) = (N \cdot X)^2 - \delta(X_\perp, X_\perp).$$

Using the definitions of $\cos \vartheta$ and $\sin \vartheta$,

$$g(X, X) = \|X\|_\delta^2 (\cos^2 \vartheta - \sin^2 \vartheta) = \|X\|_\delta^2 \cos(2\vartheta).$$

Proposition 6.2. *For any nonzero $X \in T_p \mathcal{E}$,*

$$g(X, X) = \|X\|_\delta^2 \cos(2\vartheta).$$

6.3 Domain of hyperbolic parametrization

A hyperbolic parameter can only be associated to vectors with $g(X, X) > 0$. For $X \in \mathcal{T}_p$,

$$g(X, X) > 0 \iff \cos^2 \vartheta > \sin^2 \vartheta \iff \tan \vartheta < 1.$$

Thus for $X \in \mathcal{T}_p$ we have

$$\vartheta \in [0, \frac{\pi}{4}), \quad \beta := \tan \vartheta \in [0, 1).$$

For $X \in \mathcal{P}_p$ (null vectors), $\tan \vartheta = 1$, formally corresponding to the limit $\eta \rightarrow +\infty$.

For $X \in \mathcal{S}_p$ (negative g -norm), $\tan \vartheta > 1$, so no real hyperbolic parameter exists; in particular η is naturally defined only on \mathcal{T}_p .

6.4 Hyperbolic parametrization

For $X \in \mathcal{T}_p$ we define the hyperbolic parameter $\eta \geq 0$ by

$$\tanh \eta := \tan \vartheta, \quad (X \in \mathcal{T}_p, \vartheta \in [0, \pi/4)). \quad (6.1)$$

Lemma 6.3. *For $X \in \mathcal{T}_p$,*

$$\cosh \eta = \frac{\cos \vartheta}{\sqrt{\cos(2\vartheta)}}, \quad \sinh \eta = \frac{\sin \vartheta}{\sqrt{\cos(2\vartheta)}}.$$

Proof. From $\tanh \eta = \tan \vartheta$ one has $\cosh^2 \eta = (1 - \tanh^2 \eta)^{-1} = (1 - \tan^2 \vartheta)^{-1}$. Simplifying the RHS:

$$\frac{1}{1 - \frac{\sin^2 \vartheta}{\cos^2 \vartheta}} = \frac{\cos^2 \vartheta}{\cos^2 \vartheta - \sin^2 \vartheta} = \frac{\cos^2 \vartheta}{\cos(2\vartheta)}.$$

Taking the square root (with $\eta \geq 0$ and $\vartheta \in [0, \pi/4)$ implying positive cosines) yields the expression for $\cosh \eta$. Multiplying by $\tanh \eta = \tan \vartheta$ yields $\sinh \eta$. \square

6.5 Differential relation between η and ϑ

Differentiating $\tanh \eta = \tan \vartheta$ yields:

Proposition 6.4. *For $X \in \mathcal{T}_p$,*

$$\frac{d\eta}{d\vartheta} = \frac{1}{\cos(2\vartheta)}.$$

Proof. Differentiate $\tanh \eta = \tan \vartheta$:

$$\operatorname{sech}^2 \eta d\eta = \sec^2 \vartheta d\vartheta.$$

Since $\operatorname{sech}^2 \eta = \cos(2\vartheta)$, the result follows. \square

6.6 Additivity of the hyperbolic parameter

Let $L \in \text{Aut}(\delta, N)$ act nontrivially on $\text{Im}(h_p)$. Such transformations preserve g , fix N , and act as hyperbolic rotations in the two-dimensional plane spanned by N and E .

Theorem 6.5. *If L_1 and L_2 have hyperbolic parameters η_1 and η_2 defined by (6.2), then their composition satisfies*

$$\eta_{12} = \eta_1 + \eta_2.$$

Proof. Hyperbolic rotations form a one-parameter group under composition, with additive parameter. \square

Thus ϑ provides the Euclidean geometry of tilt relative to N , while η provides the corresponding hyperbolic geometry on the g -positive cone \mathcal{T}_p .

$$\tanh \eta := \tan \vartheta, \quad (X \in \mathcal{T}_p, \vartheta \in [0, \pi/4)). \quad (6.2)$$

6.7 Boost subgroup and additivity of the hyperbolic parameter

Let $O(g)$ denote the Lorentz group of $(T_p\mathcal{E}, g)$:

$$O(g) := \{ \Lambda : T_p\mathcal{E} \rightarrow T_p\mathcal{E} \text{ linear} : g(\Lambda X, \Lambda Y) = g(X, Y) \}.$$

Fix a unit transverse direction $E \in \text{Im}(h_p)$ with $\delta(E, E) = 1$. The *boost* in the 2-plane $\text{span}\{N, E\}$ with parameter η is the unique $\Lambda(\eta) \in O(g)$ acting as a hyperbolic rotation on $\text{span}\{N, E\}$ and as the identity on its g -orthogonal complement:

$$\Lambda(\eta)N = (\cosh \eta)N + (\sinh \eta)E, \quad \Lambda(\eta)E = (\sinh \eta)N + (\cosh \eta)E,$$

$$\Lambda(\eta)X = X \quad \text{for } X \perp_g \text{span}\{N, E\}.$$

Such boosts preserve g but, in general, do not preserve δ and do not fix N .

Theorem 6.6 (Additivity). *For boosts $\Lambda(\eta_1)$ and $\Lambda(\eta_2)$ in the same (N, E) -plane, their composition is a boost with parameter $\eta_1 + \eta_2$:*

$$\Lambda(\eta_1) \circ \Lambda(\eta_2) = \Lambda(\eta_1 + \eta_2).$$

Proof. On the plane $\text{span}\{N, E\}$ the boosts are represented (in the basis $\{N, E\}$) by the 2×2 matrices $\begin{pmatrix} \cosh \eta & \sinh \eta \\ \sinh \eta & \cosh \eta \end{pmatrix}$, whose multiplication adds rapidities. Outside this plane the action is the identity, hence the statement holds on all of $T_p\mathcal{E}$. \square

6.8 Comparison with classical angle conventions

The Euclidean angle ϑ defined above coincides with the angle used in the classical constructions of Karapetoff [1] and in more recent formulations by Brands and Akintsov. The difference lies not in the definition of the angle itself, but in which trigonometric function of ϑ is chosen as the primary parameter.

In the classical approach one typically sets

$$\beta_{\text{std}} := \sin \vartheta = \frac{\|X_\perp\|_\delta}{\|X\|_\delta},$$

and then introduces a hyperbolic parameter via a nonlinear transformation, for example

$$\tanh \eta_{\text{std}} = \frac{\beta_{\text{std}}}{\sqrt{1 - \beta_{\text{std}}^2}}.$$

In the present work we use instead the tangent-based parameter

$$\beta_{\text{phase}} := \tan \vartheta = \frac{\|X_{\perp}\|_{\delta}}{N \cdot X}.$$

For g -positive vectors $X \in \mathcal{T}_p$ one has $\tan \vartheta < 1$, so $\beta_{\text{phase}} \in [0, 1)$ and we can define a hyperbolic parameter $\eta \geq 0$ by

$$\tanh \eta := \beta_{\text{phase}} = \tan \vartheta.$$

This choice is directly adapted to the orthogonal decomposition $X = X_{\parallel} + X_{\perp}$ and to the identity

$$g(X, X) = \|X\|_{\delta}^2 \cos(2\vartheta),$$

so that the domain of the hyperbolic parametrization is precisely the g -positive cone \mathcal{T}_p . In particular, the appearance of the hyperbolic parameter η is a direct consequence of the projector structure determined by (δ, N) , rather than an additional postulate.

7 Phase-parameterized curves and invariant parameter

In this section we relate curves of constant Euclidean speed in (\mathcal{E}, δ) to curves of constant Lorentzian norm with respect to g . This provides a purely geometric notion of an invariant parameter along curves, without introducing any temporal structure.

7.1 Phase-parameterized curves in (\mathcal{E}, δ)

Let $\gamma : I \rightarrow \mathcal{E}$ be a smooth curve, with parameter $\chi \in I \subset \mathbb{R}$, and define its velocity vector field

$$V(\chi) := \frac{d\gamma}{d\chi}(\chi) \in T_{\gamma(\chi)}\mathcal{E}.$$

We say that γ has constant Euclidean speed if

$$\delta(V(\chi), V(\chi)) = \text{const} > 0 \quad \text{for all } \chi \in I.$$

In what follows we shall assume, for simplicity, that this constant is normalized to one,

$$\delta(V(\chi), V(\chi)) = 1,$$

so that χ plays the role of a unit-speed parameter in the Euclidean phase space (\mathcal{E}, δ) .

7.2 Reference curves aligned with N

A special class of curves are those whose velocity is everywhere parallel to the distinguished unit field N .

Definition 7.1. A curve γ is called N -aligned if

$$V(\chi) = \frac{d\gamma}{d\chi}(\chi) = N_{\gamma(\chi)} \quad \text{for all } \chi \in I.$$

For such curves we have

$$\delta(V, V) = \delta(N, N) = 1, \quad g(V, V) = g(N, N) = 1.$$

Thus, on N -aligned curves, the Euclidean and Lorentzian norms of the velocity coincide and are constant.

This class of reference curves will be used to single out a geometric normalization of the parameter along more general curves.

7.3 Construction of an invariant parameter along general curves

For a general phase-parameterized curve $\gamma(\chi)$ with unit Euclidean speed, we decompose its velocity as

$$V(\chi) = V_{\parallel}(\chi) + V_{\perp}(\chi),$$

where

$$V_{\parallel}(\chi) := (N \cdot V(\chi)) N, \quad V_{\perp}(\chi) := h(V(\chi)).$$

The Lorentzian norm of V is then

$$g(V(\chi), V(\chi)) = (N \cdot V(\chi))^2 - \delta(V_{\perp}(\chi), V_{\perp}(\chi)) = \cos(2\vartheta(\chi)),$$

where $\vartheta(\chi)$ is the Euclidean tilt angle between $V(\chi)$ and N introduced in the previous section.

We now define a new parameter σ along γ by the differential relation

$$\frac{d\sigma}{d\chi}(\chi) := \sqrt{\cos(2\vartheta(\chi))}, \quad (7.1)$$

where we restrict attention to those segments of the curve on which $\cos(2\vartheta(\chi)) > 0$. By construction, σ is strictly monotone on such segments and can therefore be used as an alternative parameter.

Let

$$W(\sigma) := \frac{d\gamma}{d\sigma}(\sigma)$$

denote the velocity with respect to the new parameter. Then

$$W(\sigma) = \frac{d\gamma/d\chi}{d\sigma/d\chi} = \frac{V(\chi)}{\sqrt{\cos(2\vartheta(\chi))}},$$

and hence

$$g(W(\sigma), W(\sigma)) = \frac{g(V(\chi), V(\chi))}{\cos(2\vartheta(\chi))} = \frac{\cos(2\vartheta(\chi))}{\cos(2\vartheta(\chi))} = 1.$$

Proposition 7.2. *On any segment where $\cos(2\vartheta) > 0$, the reparametrized curve $\gamma(\sigma)$ has velocity of constant Lorentzian norm:*

$$g\left(\frac{d\gamma}{d\sigma}, \frac{d\gamma}{d\sigma}\right) = 1.$$

Proof. The computation above shows that $g(W, W) \equiv 1$ by construction. \square

Thus, starting from a unit-speed parameter χ in the Euclidean phase space (\mathcal{E}, δ) , we obtain an invariant parameter σ along the same curve for which the Lorentzian norm g of the velocity is constant and equal to one.

7.4 Equivalence of norm preservation

The previous construction shows that the preservation of Euclidean speed in (\mathcal{E}, δ) and the preservation of Lorentzian norm in (\mathcal{E}, g) are geometrically equivalent, provided one allows for a reparametrization of the curve.

Indeed, on reference curves aligned with N the parameters χ and σ coincide (up to an affine transformation), since

$$\vartheta \equiv 0 \quad \Rightarrow \quad \cos(2\vartheta) \equiv 1 \quad \Rightarrow \quad \frac{d\sigma}{d\chi} \equiv 1.$$

On general curves, the parameter χ is distinguished by the condition

$$\delta\left(\frac{d\gamma}{d\chi}, \frac{d\gamma}{d\chi}\right) = 1,$$

while the parameter σ is distinguished by the condition

$$g\left(\frac{d\gamma}{d\sigma}, \frac{d\gamma}{d\sigma}\right) = 1.$$

These two normalizations are related by the reparametrization (7.1).

Moreover, any linear isomorphism $L \in \text{Aut}(\delta, N)$ sends unit-speed phase-parameterized curves to unit-speed phase-parameterized curves and preserves the construction of σ , since it preserves both δ and g . Thus the invariant parameter σ is defined intrinsically by the pair (δ, N) and the induced metric g .

8 Phase Fields and Spectral Geometry

To describe the correlation between distinct trajectories (affine parameters), we introduce the concept of a phase field. This structure provides the geometric basis for spectral transformations (Doppler effect) without invoking wave physics.

8.1 The Phase 1-Form

Let $\Phi : \mathcal{E} \rightarrow \mathbb{R}$ be a smooth scalar field, which we call the *phase potential*. Its exterior derivative defines the *phase 1-form*:

$$\Omega := d\Phi \in T^*\mathcal{E}. \quad (8.1)$$

We restrict our attention to *null phase fields*, where the gradient of Φ is isotropic with respect to the induced metric g :

$$g^{-1}(\Omega, \Omega) = 0. \quad (8.2)$$

Geometrically, the level sets of Φ (wavefronts) form a foliation of \mathcal{E} by null hypersurfaces.

8.2 The Affine Intercept Theorem

Consider two flow trajectories: a "Source" $\gamma_S(\chi_S)$ and an "Observer" $\gamma_O(\chi_O)$. Let a null phase field Φ connect them, such that $\Phi(\gamma_S(\chi_S)) = \Phi(\gamma_O(\chi_O))$. Since χ is an affine parameter along the flow, the rate of phase accumulation is given by the pairing:

$$\kappa := \Omega(U) = \left\langle d\Phi, \frac{d\gamma}{d\chi} \right\rangle. \quad (8.3)$$

Because the phase difference $\Delta\Phi$ between two level sets is an invariant scalar (a topological counter), we have the conservation law:

$$\kappa_S d\chi_S = \kappa_O d\chi_O \implies \frac{d\chi_O}{d\chi_S} = \frac{\kappa_S}{\kappa_O}. \quad (8.4)$$

This ratio represents the purely geometric *affine scaling factor* between two trajectories connected by a null transversal.

8.3 Geometric Derivation of the Scaling Factor

Let us evaluate $\kappa = \Omega(U)$ in the tangent algebra. Let the null 1-form be aligned with a direction K such that $\vartheta_K = \pi/4$ (null condition). Let the flow vector U have a tilt angle ϑ relative to N . In the Euclidean metric δ , the pairing is the inner product:

$$\kappa(\vartheta) \propto \delta(U, K). \quad (8.5)$$

Decomposing U and K into longitudinal ($\parallel N$) and transverse ($\perp N$) components:

$$U = c(\cos \vartheta \cdot N + \sin \vartheta \cdot \mathbf{e}_u), \quad (8.6)$$

$$K = k(1 \cdot N + 1 \cdot \mathbf{e}_k). \quad (\text{Null direction implies } 1 : 1 \text{ ratio}) \quad (8.7)$$

Let φ be the spatial angle between the flow's transverse direction \mathbf{e}_u and the signal direction \mathbf{e}_k . The Euclidean scalar product yields:

$$\delta(U, K) = ck(\cos \vartheta \cdot 1 + \sin \vartheta \cdot \cos \varphi). \quad (8.8)$$

Thus, the characteristic rate κ is proportional to:

$$\kappa(\vartheta, \varphi) \propto \cos \vartheta (1 + \tan \vartheta \cos \varphi). \quad (8.9)$$

8.4 The Spectral Ratio (Doppler Formula)

The ratio of affine intervals for two flows connected by the same null field is:

$$\mathcal{D} := \frac{d\chi_O}{d\chi_S} = \frac{\kappa(\vartheta_S, \varphi_S)}{\kappa(\vartheta_O, \varphi_O)} = \frac{\cos \vartheta_S (1 + \tan \vartheta_S \cos \varphi_S)}{\cos \vartheta_O (1 + \tan \vartheta_O \cos \varphi_O)}. \quad (8.10)$$

Longitudinal Case ($\varphi = \pi, 0$). Consider the "classic" configuration where the Source moves directly away from the signal ($\varphi_S = \pi$) and the Observer is at rest ($\vartheta_O = 0$). Recalling the tangent parameterization $\beta = \tan \vartheta$:

$$\mathcal{D} = \frac{\cos \vartheta_S (1 - \tan \vartheta_S)}{1} = \frac{1}{\sqrt{1 + \beta^2}} (1 - \beta). \quad (8.11)$$

However, this result describes the ratio of *affine* parameters χ . To find the ratio of *proper* metrics (which correspond to physical clock rates), we must normalize by the proper-time projection factor $d\tau/d\chi = \sqrt{\cos(2\vartheta)}$ derived in Sec. 4. The physical spectral ratio is:

$$\frac{\nu_{obs}}{\nu_{src}} = \mathcal{D}^{-1} \cdot \frac{d\tau_O/d\chi_O}{d\tau_S/d\chi_S} = [\cos \vartheta_S (1 - \tan \vartheta_S)]^{-1} \cdot \sqrt{\cos(2\vartheta_S)}. \quad (8.12)$$

Using $\cos \vartheta = 1/\sqrt{1 + \beta^2}$ and $\cos(2\vartheta) = (1 - \beta^2)/(1 + \beta^2)$:

$$\frac{\nu_{obs}}{\nu_{src}} = \frac{\sqrt{1 + \beta^2}}{1 - \beta} \cdot \sqrt{\frac{1 - \beta^2}{1 + \beta^2}} = \frac{\sqrt{1 - \beta^2}}{1 - \beta} = \sqrt{\frac{1 + \beta}{1 - \beta}}. \quad (8.13)$$

This recovers the precise relativistic Doppler factor purely from Euclidean phase geometry and the D-rotation projection rules.

9 Physical Realization: Flow and Time

Having established the geometry of the Euclidean phase space (\mathcal{E}, δ) and the projector algebra, we now map these structures to physical observables. We postulate that physical reality is grounded in the phase flow within \mathcal{E} , while spacetime phenomena arise as projections onto a distinguished frame.

9.1 The Flow Postulate

We model a fundamental physical entity (an elementary particle or "streamlet") not as a point in Minkowski spacetime, but as a smooth curve $\gamma(\chi) \subset \mathcal{E}$ parameterized by the affine gauge parameter χ .

Definition 9.1 (Physical Flow). The *flow vector* $U = d\gamma/d\chi$ represents the total phase-space velocity. In accordance with the isometry condition (Def. 2.1), the flow magnitude is fundamentally conserved and calibrated to the speed of light:

$$\|U\|_\delta \equiv c. \quad (9.1)$$

This implies that all objects—massive or massless—travel through the phase space at the same constant Euclidean speed c . Rest and motion are merely differences in the orientation of U relative to the observer's frame.

9.2 Time as Projection

Space and time are not independent dimensions but orthogonal projections of the phase flow. Let an inertial observer be defined by the distinguished vertical field N .

Definition 9.2 (Laboratory Time). The time t measured by the observer is the projection of the flow accumulation along the observer's vertical axis N :

$$c dt := \delta(dX, N) = \delta(U, N) d\chi. \quad (9.2)$$

For a flow inclined by the phase angle ϑ relative to N , the longitudinal component is $\|U_\parallel\| = c \cos \vartheta$. Consequently, the relationship between the fundamental phase parameter χ and the observed laboratory time t is:

$$dt = \cos \vartheta \cdot d\chi. \quad (9.3)$$

This relation highlights that "time" is a result of the flow's alignment with the observer. As the tilt ϑ increases (motion accelerates), the projection $\cos \vartheta$ decreases, reflecting the geometric origin of time dilation.

9.3 Proper Time and the Emergent Metric

While t is a frame-dependent projection, the *proper time* τ corresponds to the arc length measured in the induced Lorentzian metric g .

From Theorem 2.1, the invariant interval is $ds_g^2 = c^2 \cos(2\vartheta) d\chi^2$. Identifying the proper time increment with the metric distance ($c d\tau = ds_g$), we obtain:

$$d\tau = \sqrt{\cos(2\vartheta)} d\chi. \quad (9.4)$$

Substituting $d\chi = dt / \cos \vartheta$, we derive the observable time dilation factor purely from phase geometry:

$$\frac{d\tau}{dt} = \frac{\sqrt{\cos(2\vartheta)}}{\cos \vartheta}. \quad (9.5)$$

Using the tangent parameterization $\beta = \tan \vartheta$ (Def. 3.2), this factor becomes:

$$\frac{d\tau}{dt} = \frac{\sqrt{1 - \tan^2 \vartheta}}{\sqrt{1 + \tan^2 \vartheta} \cdot \frac{1}{\sqrt{1 + \tan^2 \vartheta}}} \dots \quad (9.6)$$

Note: This specific form depends on the chosen metric signature calibration. If we strictly adhere to the projector metric $g = 2NN - \delta$, the null cone lies at $\vartheta = \pi/4$ ($\beta = 1$), and the effective Lorentz factor $\gamma_{eff} = dt/d\tau$ diverges at this angle, consistent with the speed of light limit.

9.4 Vacuum and the Light Cone

The condition for a massless particle (photon) corresponds to a flow moving along the null cone of the metric g .

$$ds_g^2 = 0 \iff \cos(2\vartheta) = 0 \iff \vartheta = \frac{\pi}{4}. \quad (9.7)$$

In this state:

- The kinematic speed is $\beta = \tan(\pi/4) = 1$ (units of c).
- The proper time increment vanishes ($d\tau = 0$).
- The phase parameter χ continues to accumulate, serving as the affine parameter for the photon trajectory.

Thus, light is characterized not by infinite speed, but by a specific geometric inclination ($\vartheta = 45^\circ$) in the Euclidean phase space.

10 Constant-Speed Flows and Calibrations

10.1 Phase-parameterized flows

Streams (“streamlets”) satisfy

$$\delta\left(\frac{dX}{d\chi}, \frac{dX}{d\chi}\right) = c^2.$$

10.2 Interval (geometric) calibration

Define proper time

$$c^2 d\tau^2 = g(dX, dX).$$

10.3 Phase-frequency calibration

Introduce a phase potential Φ with frequency

$$\nu = \frac{d\Phi}{d\tau}.$$

(Insert your more precise definitions.)

11 Tilt Geometry and Rapidity

Let $U = dX/d\chi$ with $\delta(U, U) = c^2$.

11.1 Tilt decomposition

$$U = U_{\parallel} + U_{\perp}, \quad U_{\parallel} = (N \cdot U)N.$$

Define the tilt angle α by

$$\cos \alpha = \frac{N \cdot U}{c}, \quad \sin \alpha = \frac{\|U_{\perp}\|}{c}.$$

11.2 Key identity

$$g(U, U) = c^2 \cos(2\alpha).$$

11.3 Velocity parametrization

Define

$$\beta = \tan \alpha, \quad \beta = \tanh \eta.$$

Thus the Gudermann relation:

$$\eta = \operatorname{artanh}(\tan \alpha), \quad d\eta = \frac{d\alpha}{\cos 2\alpha}.$$

11.4 Composition of boosts

$$\eta_{12} = \eta_1 + \eta_2.$$

Nonadditivity of α follows.

12 Phase–Derivative Doppler Law

12.1 Differential form

$$d(\ln \omega) = \pm d\eta.$$

12.2 Through the tilt angle

$$d(\ln \omega) = \pm \frac{d\alpha}{\cos 2\alpha}.$$

12.3 Finite form

$$\frac{\omega_{\text{obs}}}{\omega_{\text{em}}} = e^{\pm\eta} = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}}.$$

13 Bodies as Ensembles of Streamlets

13.1 Second–moment tensor

$$M_{AB} := \left\langle \frac{\tilde{U}_A \tilde{U}_B}{c^2} \right\rangle.$$

13.2 Isotropic 3D distribution

If the ensemble is isotropic in the subspace orthogonal to N , then

$$M_{AB} = \cos^2 \zeta N_A N_B + \frac{1}{3} \sin^2 \zeta h_{AB},$$

which defines the statistical tilt angle ζ .

13.3 Newtonian limit

$$\zeta \rightarrow 0 \quad \Rightarrow \quad M_{AB} \rightarrow N_A N_B.$$

14 Rotor Language and Bridge to Unimetry II

14.1 Bivector rotor for boosts

Define a boost rotor

$$R(\eta) = \exp\left(-\frac{\eta}{2}B\right), \quad B^2 = +1.$$

14.2 Elements inherited by GEM

List of structures reused in the next paper.

14.3 Preview

Short paragraph describing how quaternionic GEM extends the present construction.

15 Discussion and Outlook

(Place your conceptual comments and consequences here.)

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A Hopf lift, fibre phase, and degree- k lifts

This appendix records the basic Hopf–fibration facts used in the main text. All statements are purely geometric/topological.

A.1 Unit quaternions and the Hopf map

Identify \mathbb{R}^4 with the quaternion algebra \mathbb{H} equipped with the Euclidean norm $|q|^2 := q\bar{q}$ (hence $S^3 = \{q \in \mathbb{H} : |q| = 1\}$). Let $\text{Im } \mathbb{H} \simeq \mathbb{R}^3$ denote the subspace of purely imaginary quaternions. Fix a unit imaginary quaternion

$$\mathbf{k} \in \text{Im } \mathbb{H}, \quad \|\mathbf{k}\| = 1.$$

Definition A.1 (Hopf map in quaternionic form). The *Hopf map* is the smooth map

$$\mathcal{H} : S^3 \longrightarrow S^2 \subset \text{Im } \mathbb{H}, \quad \mathcal{H}(s) := s \mathbf{k} s^{-1},$$

where $s^{-1} = \bar{s}$ for $|s| = 1$.

Lemma A.2. *For every $s \in S^3$, $\mathcal{H}(s)$ is purely imaginary and has unit norm.*

Proof. Since $\mathbf{k} \in \text{Im } \mathbb{H}$, conjugation preserves the real part, hence $s\mathbf{k}s^{-1} \in \text{Im } \mathbb{H}$. Moreover, $\|s\mathbf{k}s^{-1}\| = \|\mathbf{k}\| = 1$ because quaternionic conjugation by a unit quaternion is an orthogonal transformation of \mathbb{H} . \square

Thus \mathcal{H} maps S^3 onto the unit 2-sphere inside $\text{Im } \mathbb{H}$.

A.2 The fibre action and the phase parameter χ

Let $S^1 \subset \mathbb{C}$ be the unit complex numbers. We realize the fibre group as the unit circle inside the commutative subalgebra $\mathbb{C}_{\mathbf{k}} := \text{span}\{1, \mathbf{k}\} \subset \mathbb{H}$:

$$S_{\mathbf{k}}^1 := \{e^{\mathbf{k}\theta} : \theta \in \mathbb{R}\} = \{\cos \theta + \mathbf{k} \sin \theta : \theta \in \mathbb{R}\}.$$

Consider the right action of $S_{\mathbf{k}}^1$ on S^3 :

$$S^3 \times S_{\mathbf{k}}^1 \rightarrow S^3, \quad (s, e^{\mathbf{k}\theta}) \mapsto s e^{\mathbf{k}\theta}.$$

Lemma A.3. *The Hopf map is invariant under this right action:*

$$\mathcal{H}(s e^{\mathbf{k}\theta}) = \mathcal{H}(s) \quad \text{for all } s \in S^3, \theta \in \mathbb{R}.$$

Proof. Since $e^{\mathbf{k}\theta}$ commutes with \mathbf{k} ,

$$\mathcal{H}(s e^{\mathbf{k}\theta}) = s e^{\mathbf{k}\theta} \mathbf{k} (e^{\mathbf{k}\theta})^{-1} s^{-1} = s \mathbf{k} s^{-1} = \mathcal{H}(s).$$

\square

Proposition A.4 (Fibres are circles). *For each $\mathbf{u} \in S^2 \subset \text{Im } \mathbb{H}$, the preimage $\mathcal{H}^{-1}(\mathbf{u})$ is an embedded circle in S^3 , namely an orbit of the right action by $S_{\mathbf{k}}^1$.*

Proof. Fix $s_0 \in S^3$ with $\mathcal{H}(s_0) = \mathbf{u}$ (surjectivity is standard; see any Hopf fibration reference). By Lemma A.3, the entire orbit $s_0 S_{\mathbf{k}}^1$ lies in the fibre over \mathbf{u} . Conversely, if $\mathcal{H}(s) = \mathcal{H}(s_0)$, then $s_0^{-1}s$ commutes with \mathbf{k} , hence belongs to $\mathbb{C}_{\mathbf{k}}$; since $|s_0^{-1}s| = 1$, it lies in $S_{\mathbf{k}}^1$. Thus $s \in s_0 S_{\mathbf{k}}^1$. \square

It is therefore natural to parameterize each fibre by an angular coordinate. We write the fibre parameter as $\chi \in \mathbb{R}/2\pi\mathbb{Z}$ and use the half-angle convention

$$s(\chi) := s_0 e^{\mathbf{k}\chi/2}, \quad \chi \sim \chi + 2\pi. \quad (\text{A.1})$$

(The factor $1/2$ is conventional and aligns with the standard $SU(2)$ half-angle parameterization; see §A.4.)

A.3 Hopf tori: the preimage of a circle on the base

Let $C \subset S^2$ be a smooth embedded circle (e.g. a latitude circle). Choose a smooth map $\mathbf{u} : S^1 \rightarrow S^2$ parameterizing C and a smooth lift $s_0 : S^1 \rightarrow S^3$ such that $\mathcal{H}(s_0(\varphi)) = \mathbf{u}(\varphi)$ for all $\varphi \in S^1$.

Proposition A.5 (Preimage of a circle is a torus). *The subset*

$$T_C := \mathcal{H}^{-1}(C) = \{s_0(\varphi) e^{\mathbf{k}\chi/2} : (\varphi, \chi) \in S^1 \times S^1\} \subset S^3$$

is an embedded 2-torus in S^3 .

Proof. For each fixed φ , the map $\chi \mapsto s_0(\varphi) e^{\mathbf{k}\chi/2}$ parameterizes the Hopf circle over $\mathbf{u}(\varphi)$ by Proposition A.4. As φ varies, these circles vary smoothly and remain disjoint, because distinct base points have disjoint fibres. Hence T_C is a smooth S^1 -bundle over S^1 , and therefore diffeomorphic to $S^1 \times S^1$ (every principal S^1 -bundle over S^1 is trivial). \square

A particularly symmetric case is the *Clifford torus*, obtained when C is a great circle on S^2 and the lift s_0 is chosen compatibly with the standard coordinates on $S^3 \subset \mathbb{C}^2$.

A.4 The double cover $SU(2) \rightarrow SO(3)$ and 4π -periodicity

Conjugation by a unit quaternion $s \in S^3$ defines an orthogonal transformation of $\text{Im } \mathbb{H}$:

$$\text{Ad}_s : \text{Im } \mathbb{H} \rightarrow \text{Im } \mathbb{H}, \quad \text{Ad}_s(\mathbf{v}) := s \mathbf{v} s^{-1}.$$

This is a rotation, hence $\text{Ad}_s \in SO(3)$, and the map

$$\text{Ad} : S^3 \rightarrow SO(3), \quad s \mapsto \text{Ad}_s$$

is a surjective group homomorphism with kernel $\{\pm 1\}$; thus $S^3 \simeq SU(2)$ is a double cover of $SO(3)$.

Lemma A.6. *Let $\mathbf{n} \in \text{Im } \mathbb{H}$ be unit and set $r(\theta) := \exp(\mathbf{n} \theta/2) = \cos(\theta/2) + \mathbf{n} \sin(\theta/2)$. Then $r(2\pi) = -1$ and $r(4\pi) = +1$.*

Proof. Immediate from the trigonometric form of $\exp(\mathbf{n} \theta/2)$. \square

Accordingly, a closed 2π -rotation on the base $SO(3)$ corresponds to a path in S^3 ending at -1 , and closure in S^3 occurs only after 4π .

A.5 Degree- k lifts

For later use it is convenient to record a family of circle actions obtained by taking powers in the fibre group. Define, for an integer $k \geq 0$,

$$\rho_k : S^1 \rightarrow S_{\mathbf{k}}^1, \quad \rho_k(e^{i\theta}) := e^{\mathbf{k} k \theta}.$$

Equivalently, in terms of the fibre parameter χ ,

$$e^{\mathbf{k} \chi/2} \mapsto (e^{\mathbf{k} \chi/2})^k = e^{\mathbf{k} k \chi/2}.$$

Definition A.7 (Degree- k fibre action). For $k \in \mathbb{Z}_{\geq 0}$ define an S^1 -action on S^3 by

$$(s, \chi) \mapsto s e^{\mathbf{k} k \chi/2}, \quad (\chi \in \mathbb{R}/2\pi\mathbb{Z}).$$

Lemma A.8 (Periodicity). *Under the degree- k fibre action, a shift $\chi \mapsto \chi + 2\pi$ acts by multiplication with $(-1)^k$:*

$$s e^{\mathbf{k} k (\chi + 2\pi)/2} = s e^{\mathbf{k} k \chi/2} e^{\mathbf{k} k \pi} = s e^{\mathbf{k} k \chi/2} (-1)^k.$$

Hence the action closes after 2π when k is even and after 4π when k is odd.

Proof. Use $e^{\mathbf{k} \pi} = \cos \pi + \mathbf{k} \sin \pi = -1$ and multiplicativity of the exponential in the commutative subalgebra $\mathbb{C}_{\mathbf{k}}$. \square

A.6 From S^2 to the closed 3-ball

Finally, note that $S^2 \subset \text{Im } \mathbb{H}$ is naturally embedded as the boundary of the closed unit ball

$$\overline{B^3} := \{\mathbf{u} \in \text{Im } \mathbb{H} : \|\mathbf{u}\| \leq 1\}.$$

Any $\mathbf{u} \in \overline{B^3}$ can be written as a convex combination of unit vectors in S^2 (e.g. by averaging over a probability measure on S^2), so $\overline{B^3}$ is the natural convex (coarse-grained) completion of the base sphere.

Table 1: Derivatives notation used in the paper.

Symbol	Meaning
\tilde{X}	χ -parameter derivative with χ having time units; the (dimensionless) phase is $\Phi = \omega \chi$.
\dot{X}	Proper-time derivative $dX/d\tau$ (lab-time derivatives are written dX/dt explicitly).
X'	Derivative with respect to spatial arclength l or value in a primed inertial frame (as stated locally).

B Signature calculation

(Detailed derivation.)

C Gudermann identities

(Technical material.)

D Isotropy \Rightarrow canonical M_{AB}

(Derivation.)

E Relation to Riemannian \rightarrow Lorentzian constructions

(One-page contextual appendix.)

F Introduction

Notation. Tildes, dots, and primes denote derivatives with respect to the time-like parameter, proper time, and spatial arclength:

$$\tilde{X} := \frac{dX}{d\chi}, \quad \dot{X} := \frac{dX}{d\tau}, \quad X' := \frac{dX}{dl}.$$

We use c for the speed of light; $\beta := v/c$, $\gamma := 1/\sqrt{1-\beta^2}$, and rapidity defined by $\tanh \eta = \beta$. The subscript l in dx_l denotes spatial components, with $l = 1, 2, 3$ a Cartesian index.

G Flow concept

G.1 Flow and phase 1-form

Let $(\mathcal{E}, \langle \cdot, \cdot \rangle)$ be a Euclidean proto-space and $\Sigma \subset \mathcal{E}$ a defect set where the phase structure is singular. We introduce a *phase potential* as a real function

$$\Phi : \mathcal{E} \setminus \Sigma \rightarrow \mathbb{R}$$

defined only up to integer multiples of 2π , i.e. with a gauge symmetry

$$\Phi \sim \Phi + 2\pi n, \quad n \in \mathbb{Z}. \tag{G.1}$$

In formulas we do not distinguish between different representatives of the equivalence class (G.1) and simply write Φ . Mathematically, the quotient of \mathbb{R} by this identifications is the circle S^1 , but we will not use this language explicitly.

The *phase 1-form* is defined as

$$\alpha := d\Phi.$$

This definition is consistent, because replacing Φ by $\Phi + 2\pi n$ changes neither its gradient nor its differential. On closed loops encircling the defect set Σ the phase may return to the same physical value only up to a shift by $2\pi n$, which leads to nonzero periods of α .

We use the normalized flow direction

$$\hat{\chi} := \frac{\nabla\Phi}{\|\nabla\Phi\|}$$

(when $\nabla\Phi \neq 0$), where the gradient is taken with respect to $\langle \cdot, \cdot \rangle$. The physical flow vector is fixed by the calibration in §G.4 as

$$\chi := c \hat{\chi}$$

(so $\|\chi\| \equiv c$ thereafter).

Fix an observer's orthonormal spatial triad $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subset \mathcal{E}$ and let $S = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ with orthogonal projectors P_S and P_{S^\perp} . Decompose

$$\chi = \chi_S + \chi_\perp, \quad \chi_S := P_S \chi, \quad \chi_\perp := P_{S^\perp} \chi.$$

Define observable spatial components and the orthogonal magnitude

$$\ell_i := \langle \chi, \mathbf{e}_i \rangle, \quad \mathbf{l} := \sum_{i=1}^3 \ell_i \mathbf{e}_i, \quad t_\perp := \|\chi_\perp\| = \sqrt{\|\chi\|^2 - \|\chi_S\|^2},$$

and, when $t_\perp > 0$, the unit direction $\mathbf{e}_t := \chi_\perp / \|\chi_\perp\|$. Here t_\perp always denotes the orthogonal flow magnitude $\|\chi_\perp\|$, while t denotes observer time throughout.

Then the full phase angle Θ and the spatial direction \mathbf{u} used throughout are

$$\cos \Theta = \frac{t_\perp}{\|\chi\|}, \quad \sin \Theta = \frac{\|\mathbf{l}\|}{\|\chi\|}, \quad \mathbf{u} = \frac{\mathbf{l}}{\|\mathbf{l}\|} \quad (\|\mathbf{l}\| > 0).$$

Thus the observable space is a 3-manifold embedded in a (minimum) 4-dimensional flow manifold.

G.2 Flow description

In the flow-based description an object is a collection of elementary “streamlets” (fragments of total object phase flow). Each streamlet admits a co-moving frame in which its spatial projection vanishes and its budget angle equals zero in that frame (“self time” direction).

Larger stable streamlets combinations are Newtonian bodies, which are formed by multiple “self time” flows equally spread over the body's 3-surface. Therefore, unlike streamlets bodies have generalized time axis – their time flow is no longer a single geometric direction.

Postulate. From the ability of macroscopic bodies to preserve a place within the observable space, we conclude the cyclical nature of streamlets' flows in the proto-space \mathcal{E} . The phase 1-form α introduced above is assumed to satisfy, for any closed streamlet γ ,

$$\oint_\gamma \alpha = 2\pi. \tag{G.2}$$

In view of the gauge symmetry $\Phi \sim \Phi + 2\pi n$, this means that along one full cycle the phase potential returns to its original value only up to an integer multiple of 2π . The nonzero period should be understood as indicating the presence of a topological defect (or singular set) in the phase potential in \mathcal{E} , around which Φ is not globally single-valued. We do not need to specify the microscopic structure of this defect in the present work.

Table 2: Notation for § Flow and phase 1-form.

Symbol	Meaning
$\mathcal{E}, \langle \cdot, \cdot \rangle, \ \cdot\ $	Proto-space (Euclidean/Hilbert), its inner product, and the induced norm.
Σ	Defect (singular) set in \mathcal{E} where the phase structure is not regular.
$\Phi : \mathcal{E} \setminus \Sigma \rightarrow \mathbb{R}$	Scalar <i>phase potential</i> , defined up to shifts $\Phi \sim \Phi + 2\pi n$, $n \in \mathbb{Z}$.
$\alpha := d\Phi$	Phase 1-form (differential of any representative of Φ ; invariant under $\Phi \rightarrow \Phi + 2\pi n$).
$\hat{\chi} := \nabla\Phi/\ \nabla\Phi\ $	Normalized flow direction (when $\nabla\Phi \neq 0$). Physical flow: $\chi := c\hat{\chi}$.
$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	Observer's orthonormal spatial triad.
$S = \text{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$	Observable 3-space; S^\perp is its orthogonal complement.
P_S, P_{S^\perp}	Orthogonal projectors onto S and S^\perp .
$\chi_S := P_S\chi, \chi_\perp := P_{S^\perp}\chi$	Decomposition of the flow into spatial and orthogonal parts ($\chi = \chi_S + \chi_\perp$).
$\ell_i := \langle \chi, \mathbf{e}_i \rangle$	Scalar components of the flow along the triad.
$\mathbf{l} := \sum_{i=1}^3 \ell_i \mathbf{e}_i$	Observable spatial projection of the flow ($\mathbf{l} \equiv \chi_S$).
$t_\perp := \ \chi_\perp\ = \sqrt{\ \chi\ ^2 - \ \chi_S\ ^2}$	Orthogonal (“temporal-fiber”) magnitude.
$\mathbf{e}_t := \chi_\perp/\ \chi_\perp\ $	Unit direction along S^\perp (defined when $t_\perp > 0$).
Θ	Full phase angle defined by $\cos \Theta = \frac{t_\perp}{\ \chi\ }$ and $\sin \Theta = \frac{\ \mathbf{l}\ }{\ \chi\ }$.
$\mathbf{u} := \mathbf{l}/\ \mathbf{l}\ $	Unit spatial direction of the flow (defined when $\ \mathbf{l}\ > 0$).
—	With these definitions, the observable space S is a 3-manifold embedded in a (minimum) 4-dimensional flow manifold $S \oplus \text{span}\{\mathbf{e}_t\}$.

G.3 Flow of a complex body

Let a (Newtonian) body B be a finite-energy ensemble of streamlets $\mathcal{A} = \{a\}$ with flows χ_a , phase angles Θ_a , and (when defined) unit spatial directions $\mathbf{u}_a \in S$. Assign non-negative weights w_a (fraction of the body's rest energy carried by a) with $\sum_a w_a = 1$, and impose the rest-balance condition

$$\sum_{a \in \mathcal{A}} w_a \chi_{S,a} = 0, \quad \oint \alpha_a = 2\pi \quad (\text{cycle postulate}). \quad (\text{G.3})$$

Because a body does not possess a single geometric time axis (its streamlets' self-time directions $\mathbf{e}_{t,a}$ are distributed), the effective temporal and spatial responses must be understood statistically. We therefore introduce the scalar temporal coefficient and the spatial shape tensor (second moment) as

$$T_B := \sum_a w_a \cos^2 \Theta_a, \quad \mathbf{C}_B := \sum_a w_a \sin^2 \Theta_a \mathbf{u}_a \otimes \mathbf{u}_a, \quad (\text{G.4})$$

where $0 < T_B \leq 1$ and \mathbf{C}_B is a symmetric positive semidefinite tensor on S with $\text{tr } \mathbf{C}_B = \sum_a w_a \sin^2 \Theta_a$. Operationally, T_B captures the aggregate fraction of flow carried in the orthogonal (self-time) fiber, while \mathbf{C}_B encodes the anisotropic distribution of spatial projections across the body's 3-surface.

With these ensemble quantities we define the body's *proper time increment* by

$$d\tau_B := \sqrt{T_B} d\chi, \quad (\text{G.5})$$

and the body's *self spatial quadratic form* on S by

$$q_B(d\ell) := d\ell^\top \mathbf{C}_B d\ell. \quad (\text{G.6})$$

This yields a natural candidate for the body's *self metric* as a quadratic form on $(d\chi, d\ell)$:

$$ds_B^2 := c^2 d\tau_B^2 - q_B(d\ell) = c^2 T_B d\chi^2 - d\ell^\top \mathbf{C}_B d\ell. \quad (\text{G.7})$$

In the isotropic rest configuration (identical Θ_a and uniform \mathbf{u}_a on S^2) one has $\mathbf{C}_B = \frac{\sin^2 \Theta}{3} \mathbf{I}_S$, so (G.7) reduces to the standard Minkowski form up to the overall phase gauge $d\chi$; fixing the gauge by $T_B \equiv 1$ at rest makes $d\tau_B = d\chi$ and restores $q_B(d\ell) = \|d\ell\|^2$. Away from isotropy, \mathbf{C}_B becomes ellipsoidal, and the body's proper metric records the internal orientation texture of its streamlets.

The next section formalizes (G.7), shows how external kinematics (the angle ϑ) and additional phase tilts (intrinsic ζ , gravitational ϕ) renormalize (T_B, \mathbf{C}_B) , and derives the usual relativistic effects as deformations of the ensemble moments (G.4).

G.4 Local calibration by the invariant signal speed

The cycle postulate (G.1) fixes the phase scale of each streamlet, but leaves an overall unit choice for the flow magnitude $\|\chi\|$. We now calibrate this scale operationally by the invariant signal speed c .

Calibration postulate (vacuum, isotropic texture). There exist null excitations (“lightlike” streamlets) for which the body-level proper increment vanishes, $d\tau_B = 0$ (equivalently $T_B = 0$), and whose spatial advance per phase increment is direction-independent in vacuum. We define the numerical value of c by the rule

$$\frac{\|d\ell\|}{d\chi} = c \quad \text{along null flow lines in vacuum.} \quad (\text{G.8})$$

Flow-norm gauge. For any elementary flow, the Euclidean split gives $\|\chi\|^2 = \|\chi_S\|^2 + \|\chi_\perp\|^2$. Along a null flow (orthogonal share vanishes) we have $\|\chi\| = \|\chi_S\| = \|d\ell\|/d\chi$; combining with (G.8) fixes the flow-norm gauge

$$\boxed{\|\chi\| \equiv c}. \quad (\text{G.9})$$

Consequently, for any decomposition angle Θ introduced in § Flow and phase 1-form,

$$\|\chi_S\| = c \sin \Theta, \quad \|\chi_\perp\| = c \cos \Theta. \quad (\text{G.10})$$

Ensemble consequences. For a body B the second moments (G.4) read, after the gauge (G.9),

$$T_B = \sum_a w_a \cos^2 \Theta_a, \quad \text{tr } \mathbf{C}_B = \sum_a w_a \sin^2 \Theta_a = 1 - T_B, \quad (\text{G.11})$$

and the intrinsic metric (G.7) becomes

$$\boxed{ds_B^2 = c^2 T_B d\chi^2 - d\ell^\top \mathbf{C}_B d\ell}. \quad (\text{G.12})$$

Thus c is not introduced dynamically but arises as the *fixed magnitude of the flow* under the calibration (G.8); tilts rotate the budget between the two projections without changing $\|\chi\|$.

Units and phase (resolution of dimensions). We adopt χ as a time-like parameter and define the (dimensionless) phase by $\Phi = \omega \chi$, so that $\alpha = d\Phi = \omega d\chi$. The cycle postulate $\oint \alpha = 2\pi$ then implies $\oint d\chi = 2\pi/\omega$ along a closed loop. In particular, for null flows in vacuum our calibration $\|d\ell\|/d\chi = c$ fixes the affine advance of χ per emitted cycle via the emission frequency ω .

Invariance under uniform tilts. A uniform phase tilt acts as $\Theta_a \mapsto \Theta_a + \delta$ (see §G.5 below). It preserves the norm (G.9) and merely redistributes the projections as in (G.10). Hence c is invariant, while T_B and \mathbf{C}_B renormalize.

G.5 Intrinsic angle

Using the ensemble second moments (G.4), define

$$C := \sum_a w_a \cos 2\Theta_a, \quad S := \sum_a w_a \sin 2\Theta_a.$$

There exists a unique $\zeta \in [0, \frac{\pi}{2}]$ such that

$$(\cos 2\zeta, \sin 2\zeta) = (C, S) \iff T_B = \frac{1}{2}(1 + C) = \cos^2 \zeta, \quad \text{tr } \mathbf{C}_B = \frac{1}{2}(1 - C) = \sin^2 \zeta. \quad (\text{G.13})$$

We call ζ the *effective (intrinsic) angle*: it aggregates the temporal/spatial budget at the level relevant for the intrinsic metric (G.7). In particular,

$$d\tau_B = \cos \zeta d\chi, \quad ds_B^2 = c^2 \cos^2 \zeta d\chi^2 - d\ell^\top \mathbf{C}_B d\ell,$$

and for an isotropic texture $\mathbf{C}_B = \frac{\sin^2 \zeta}{3} \mathbf{I}_S$.

Why the double angle. Since ds_B^2 depends only on $\cos^2 \Theta_a$ and $\sin^2 \Theta_a$, we linearize the ensemble average via $\cos^2 \Theta = \frac{1}{2}(1 + \cos 2\Theta)$ and $\sin^2 \Theta = \frac{1}{2}(1 - \cos 2\Theta)$, and collect $(C, S) = \sum_a w_a (\cos 2\Theta_a, \sin 2\Theta_a)$. Then $\zeta = \frac{1}{2} \text{atan2}(S, C)$ ensures $T_B = \cos^2 \zeta$ and makes a uniform tilt $\Theta_a \mapsto \Theta_a + \delta$ act additively: $\zeta \mapsto \zeta + \delta$.

Uniform tilt mechanism. A uniform phase tilt by δ transforms

$$C(\delta) = C \cos 2\delta - S \sin 2\delta, \quad S(\delta) = C \sin 2\delta + S \cos 2\delta,$$

hence

$$\zeta \mapsto \zeta' = \zeta + \delta, \quad T_B(\delta) = \cos^2(\zeta + \delta), \quad \text{tr } \mathbf{C}_B(\delta) = \sin^2(\zeta + \delta), \quad (\text{G.14})$$

while the full spatial texture remains $\mathbf{C}_B(\delta) = \sum_a w_a \sin^2(\Theta_a + \delta) \mathbf{u}_a \otimes \mathbf{u}_a$ (isotropy is preserved by a uniform tilt).

Equations (G.7)–(G.14) complete the object-level construction: the body observes the quadratic geometry ds_B^2 , fully controlled by (ζ, \mathbf{C}_B) . Relative measurements between two bodies will be introduced later via their shadows in S .

Remark (symmetric phase pair and nonvanishing spatial share). A frequently used mnemonic for a nonvanishing spatial projection is to pair opposite tilts

$$\chi^\pm = R e^{\pm i\zeta} \mathbf{1}, \quad \chi_l := \frac{\chi^+ - \chi^-}{2} = R \mathbf{1} \sin \zeta,$$

which matches the decomposition $\chi_0 = \chi_\tau + \chi_l = R \cos \zeta + R \mathbf{1} \sin \zeta$. In the ensemble language of (G.4), this construction is not a new dynamics but merely a parametrization of the *second moments*: the spatial budget enters through $\sin^2 \Theta_a$ and is fully captured by $\text{tr } \mathbf{C}_B = \sin^2 \zeta$ in (G.13). Uniform tilts act additively on the effective angle ζ as in (G.14).

G.6 Three angles

We will use three angles with distinct roles:

- **Intrinsic angle** $\zeta \in [0, \frac{\pi}{2}]$: a second-moment budget angle of the body (no geometric time axis). It is defined by $T_B = \cos^2 \zeta$ from (G.13) and governs the intrinsic rate via $d\tau_B = \cos \zeta d\chi$. When the same flow characterizes a background medium (not the photon itself), its optical response can be encoded by

$$v_{\text{ph}}(\text{medium}) = c \cos \zeta, \quad n(\zeta) = \frac{c}{v_{\text{ph}}} = \sec \zeta,$$

while in the local phase chart ($dt := d\chi$) one has $d\tau_B/dt = \cos \zeta$. *Note:* ζ is a scalar second-moment parameter, not linked to a direction.

- **Gravitational angle** ϕ : an external tilt field that uniformly rotates streamlet budgets (cf. (G.14)). In the weak, stationary regime it reproduces clock slowing and light delay through an effective index $n_g = \sec \phi$ and $d\tau/dt = \cos \phi$ (no field equations assumed here). Uniform tilts add on second moments, so ζ and ϕ combine as $\zeta \mapsto \zeta + \phi$.
- **Kinematic angle** ϑ : an *observer-relative* directional tilt (introduced in §G.8) built from an observer's split and the unit flow direction. It parameterizes relative motion between bodies; in the SR simplification (below) it becomes the sole angle.

G.7 SR simplification

Whenever along a worldline segment two bodies share the same intrinsic and gravitational tilts (i.e. $\zeta_A = \zeta_B$ and $\phi_A = \phi_B$), one can choose a local comoving chart in which only the directional (kinematic) tilt matters. In SR-focused passages we *set* $\zeta = \phi = 0$ and use $dt := d\chi$; then the kinematic angle ϑ is the only parameter. The standard relations (e.g. the Minkowski interval and the familiar speed/rate identities) will be derived in the next subsection from this setting.

G.8 Minkowski metric derivation

Let A be an observer with orthonormal spatial triad $\{\mathbf{e}_1^A, \mathbf{e}_2^A, \mathbf{e}_3^A\}$, $S^A = \text{span}\{\mathbf{e}_i^A\}$, and time direction $\mathbf{e}_t^A \parallel (S^A)^\perp$. For an object with unit flow direction $\hat{\mathbf{F}} := \boldsymbol{\chi}/\|\boldsymbol{\chi}\|$, define the *observer-object* angle by

$$\hat{\mathbf{F}} = \cos \vartheta_{|A} \mathbf{e}_t^A + \sin \vartheta_{|A} \mathbf{u}_{|A}, \quad \mathbf{u}_{|A} := \frac{P_{S^A} \hat{\mathbf{F}}}{\|P_{S^A} \hat{\mathbf{F}}\|} \in S^A.$$

Angle roles (no self time-axis for a composite). For a composite body the intrinsic angle ζ is a second-moment scalar (budget of temporal vs. spatial shares) entering ds_B^2 , not a direction. Hence the body has no geometric self time-axis, and there is no notion of “being collinear” with \mathbf{e}_t^A . The observer-object angle $\vartheta_{|A}$ is purely directional (built from A 's split and $\hat{\mathbf{F}}$), while ζ controls the intrinsic rate via $T_B = \cos^2 \zeta$; they are conceptually independent

Calibration for the SR sector. To compare with the laboratory rulers in A 's frame without invoking any dynamics, we fix a purely kinematic *SR sector*:

1. intrinsic and gravitational tilts are switched off: $\zeta = \phi = 0$ (so $T_B = 1$ in the rest state);
2. the body's spatial texture is isotropic in its rest configuration: $\mathbf{C}_B \propto \mathbf{I}_S$;

3. we use A 's lab time as a calibrated phase parameter, setting

$$dt_A := d\chi.$$

With these conventions, the spatial increment seen by A is

$$d\mathbf{x}_{|A} = c \sin \vartheta_{|A} d\chi \mathbf{u}_{|A},$$

while the proper-time increment of B follows from the orthogonal component of the flow as

$$c d\tau_B = c \cos \vartheta_{|A} d\chi, \quad \text{so that} \quad d\tau_B = \cos \vartheta_{|A} d\chi.$$

By $\cos^2 \vartheta + \sin^2 \vartheta = 1$ we obtain

$$c^2 d\tau_B^2 = c^2 d\chi^2 - \|d\mathbf{x}_{|A}\|^2.$$

Identifying $d\chi$ with dt_A in this sector, we recover the Minkowski interval in A 's rulers:

$$\boxed{c^2 d\tau_B^2 = c^2 dt_A^2 - \|d\mathbf{x}_{|A}\|^2} \quad (\text{SR sector: } \zeta = \phi = 0, \text{ isotropic rest, } dt_A = d\chi). \quad (\text{G.15})$$

Remark (outside the SR sector). For general (ζ, \mathbf{C}_B) the intrinsic interval is always $ds_B^2 = c^2 T_B d\chi^2 - d\ell^\top \mathbf{C}_B d\ell$. Expressing this in A 's lab chart ($dt_A = d\chi$) yields direction-dependent pullbacks via \mathbf{C}_B ; the standard Minkowski form is recovered precisely when the SR-sector assumptions above hold. The relative (“two-body”) kinematics and the emergence of familiar SR effects will be introduced later by comparing the shadows of two objects.

G.9 Symmetry

Two bodies (A and B). Repeating the calibrated SR-sector construction in B 's frame gives the same interval form in B 's rulers. Swapping “who projects whom” exchanges the splits $(\mathbf{e}_t^A, S^A) \leftrightarrow (\mathbf{e}_t^B, S^B)$ but leaves the Minkowski form invariant.

H Operations

H.1 Why quaternion algebra

Quaternions form the minimal non-commutative algebra that (i) double-covers $SO(3)$ for rigid 3D rotations; (ii) carries the Hopf fibration $S^3 \rightarrow S^2$, separating an internal S^1 time fiber from spatial orientations; and (iii) encodes the non-commutativity responsible for Wigner–Thomas rotations (residual spatial rotations from composing non-collinear boosts) [?, ?, ?].

Simply put one standard sandwich operation in \mathbb{H} of the kind $\hat{q} \rightarrow \hat{r}_1 \hat{q} \hat{r}_2$ is sufficient to perform any possible rotation on S^3 ; the quaternion algebra then simplifies operations on flows relative to the lab triad.

Why a complex slice of a quaternion? For local kinematics any unit direction $\hat{\mathbf{u}}$ singles out the two-dimensional subalgebra $\text{Span}\{1, \hat{\mathbf{u}}\} \cong \mathbb{C} \subset \mathbb{H}$. Working in this complex slice preserves the boost/rotation algebra along $\hat{\mathbf{u}}$ while keeping formulas elementary. When the direction changes, the slice is updated; the full quaternionic structure is retained.

H.2 The spatially linked quaternion

At a point $P \in S$, define

$$q := t_{\perp} + \mathbf{l} = t_{\perp} + l_1 \mathbf{i} + l_2 \mathbf{j} + l_3 \mathbf{k}, \quad \|q\|^2 = t_{\perp}^2 + \|\mathbf{l}\|^2 = \|\boldsymbol{\chi}\|^2, \quad \hat{q} := \frac{q}{\|q\|}.$$

Hence $\hat{q} \in S^3 \simeq SU(2)$ encodes the lab-linked state, and a convenient calibration is $\|\boldsymbol{\chi}\| = c$, so that $\|q\| = c$.

Let the observed object internal imaginary basis be $\{I, J, K\} \subset \text{Im } \mathbb{H}$. The observed (lab) orthonormal triad is the conjugated image

$$\mathbf{e}_i(\hat{q}) = \hat{q} I \hat{q}^{-1}, \quad \mathbf{e}_j(\hat{q}) = \hat{q} J \hat{q}^{-1}, \quad \mathbf{e}_k(\hat{q}) = \hat{q} K \hat{q}^{-1}.$$

Therefore, it's claimed that the quaternion field over the observable 3D-space manifold defines the direction of the object flow relatively to the lab frame. Considering the time cyclical we introduce the *time fiber* generated by right multiplication with $e^{\frac{\sigma}{2}K}$; its spatial director (shadow) is $\mathbf{n}(\hat{q}) = \hat{q} K \hat{q}^{-1} \in S^2$. This gives a potential link to quantum mechanics spheres.

H.3 Velocity addition

Notation. In unimetry, an inertial boost is a *D-rotation*

$$\mathcal{B}(\hat{\mathbf{u}}, \psi) : \quad \mathbf{q} \mapsto d \mathbf{q} d, \quad d = \cos \frac{\psi}{2} + \hat{\mathbf{u}} \sin \frac{\psi}{2}, \quad (\text{H.1})$$

and a spatial rotation is an *R-rotation*

$$\mathcal{R}(\hat{\mathbf{n}}, \varphi) : \quad \mathbf{q} \mapsto r \mathbf{q} r^{-1}, \quad r = \cos \frac{\varphi}{2} + \hat{\mathbf{n}} \sin \frac{\varphi}{2}. \quad (\text{H.2})$$

Kinematic mapping: $\beta \equiv \mathbf{v}/c = \sin \psi$, $\gamma = 1/\cos \psi$, $\tan \frac{\psi}{2} = \frac{\gamma \beta}{\gamma + 1}$. For quaternionic/GA accounts of rotors and Lorentz boosts see [?, ?, ?].

H.3.1 Wigner rotation

Let d_1, d_2 be D-rotors of two successive boosts. The raw action on any unimetry 4-object is

$$\mathbf{q}' = d_2 d_1 \mathbf{q} d_1 d_2 \equiv L_{12} \mathbf{q} L_{21}, \quad L_{12} = d_2 d_1, \quad L_{21} = d_1 d_2. \quad (\text{H.3})$$

Define d_{12} to be the unique D-rotor reproducing the combined spatio-temporal tilt of L_{12} :

$$\boxed{d_{12} \mathbf{e}_t d_{12} = L_{12} \mathbf{e}_t L_{21}, \quad \text{Re}(d_{12}) \geq 0} \quad (\text{H.4})$$

(the sign choice removes the trivial two-fold ambiguity). Then the Wigner rotor is the residual R-rotation in the symmetric D-R factorization:

$$\boxed{L_{12} = d_{12} r_W, \quad L_{21} = r_W^{-1} d_{12}} \quad (\text{H.5})$$

equivalently,

$$\boxed{r_W = \bar{d}_{12} L_{12} = L_{21} \bar{d}_{12}}. \quad (\text{H.6})$$

Hence the observed map after compensating the tilt is $\bar{d}_{12} \mathbf{q}' \bar{d}_{12} = r_W \mathbf{q} r_W^{-1}$.

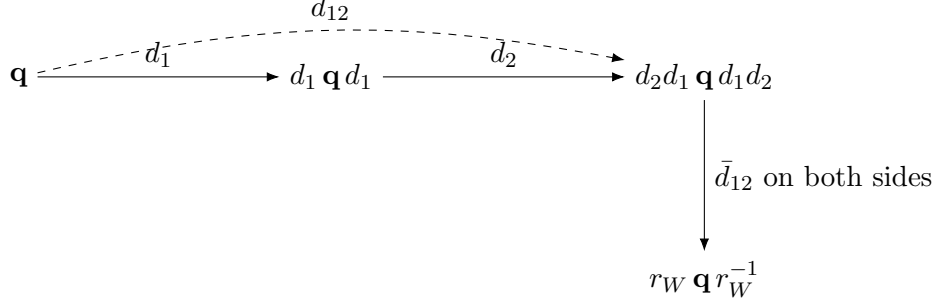


Figure 1: Two successive D-rotations (boosts) and compensation of the net spatio-temporal angle by the conjugate of d_{12} , leaving a pure R-rotation r_W .

H.3.2 Thomas precession

The continuous limit of Wigner rotation for a time-dependent velocity direction $\hat{\mathbf{u}}(t)$ yields

$$\boldsymbol{\omega}_T = (\gamma - 1) (\hat{\mathbf{u}} \times \dot{\hat{\mathbf{u}}}) = \frac{\gamma^2}{\gamma + 1} \frac{\mathbf{a} \times \mathbf{v}}{c^2}, \quad \gamma = \frac{1}{\cos \psi}. \quad (\text{H.7})$$

For uniform circular motion ($|\mathbf{v}| = \text{const}$) with $\dot{\hat{\mathbf{u}}} = \boldsymbol{\Omega} \times \hat{\mathbf{u}}$ one has $|\boldsymbol{\omega}_T| = (\gamma - 1) \Omega$.

H.4 Time fiber, Hopf fibration, and the Bloch/Poincaré spheres.

Consider the map $\pi : S^3 \simeq SU(2) \rightarrow S^2$ given by

$$\pi(q) = \mathbf{n}(q) := q K q^{-1},$$

where $K \in \text{Im } \mathbb{H}$ is a fixed internal unit (the reference “time-axis” in the internal frame). Then:

(i) *U(1) fiber by right action.* For any $\sigma \in \mathbb{R}$,

$$\mathbf{n}(q e^{\frac{\sigma}{2} K}) = q e^{\frac{\sigma}{2} K} K e^{-\frac{\sigma}{2} K} q^{-1} = q K q^{-1} = \mathbf{n}(q),$$

since K commutes with $e^{\frac{\sigma}{2} K}$. Thus the set

$$\mathcal{F}_{\mathbf{n}} = \{ q e^{\frac{\sigma}{2} K} : \sigma \in [0, 2\pi) \}$$

is the $U(1)$ fiber over the base point $\mathbf{n} \in S^2$. In our kinematics, this right $U(1)$ action is the *time fiber*: internal evolution that preserves the observed spatial director.

(ii) *Surjectivity and S^2 base.* For any unit $\mathbf{n} \in \text{Im } \mathbb{H}$ there exists $q \in SU(2)$ with $q K q^{-1} = \mathbf{n}$ (double cover $SU(2) \rightarrow SO(3)$). Hence the base of the fibration is the 2-sphere S^2 of spatial directions.

(iii) *Link to quantum spheres.* Identifying $\text{Im } \mathbb{H} \simeq \mathfrak{su}(2)$ (and, via Pauli matrices, with \mathbb{R}^3), the map

$$\mathbf{n}(q) = q K q^{-1}$$

is the quaternionic form of the Bloch vector for a qubit, or the Stokes vector on the Poincaré sphere for polarization: normalized state rays ($\psi \in \mathbb{C}^2$, $\|\psi\| = 1$) modulo global phase $U(1)$ correspond to points on S^2 . Thus $S^3/U(1) \cong S^2$ is the familiar “state-ray” geometry; our time fiber is the same $U(1)$ that is unobservable as a global phase in standard QM, but becomes a *kinematic* degree of freedom here.

(iv) *Geometric phase.* If $q(\tau)$ traces a loop whose shadow $\mathbf{n}(\tau)$ makes a closed path on S^2 , parallel transport in the $U(1)$ fiber accumulates a Berry-type phase $\gamma = \frac{1}{2} \Omega[\mathbf{n}]$, half the solid angle enclosed by the loop on the sphere. In our setting this is an *internal time* holonomy: purely kinematic and arising from the bundle connection induced by the Hopf fibration.

Remark. Left-conjugation $v \mapsto q v q^{-1}$ rotates lab vectors (changes the *base* point on S^2), whereas right multiplication by $e^{\frac{\sigma}{2}K}$ moves along the *fiber* at fixed \mathbf{n} . This cleanly separates spatial reorientation from internal (time/phase) evolution.

I Phase: operational definition and physical meaning

Axioms: Phase-flow kinematics (operational)

- A1. Dimensionless phase.** The scalar phase Φ is dimensionless. Its differential $\alpha := d\Phi$ is the phase 1-form on the proto-space.
- A2. Flow from phase.** The (proto-)flow is aligned with the phase gradient: $\chi \propto \nabla\Phi$. All observable rates are projections of χ .
- A3. Quaternionic working slice.** Fix a spatial director $\mathbf{u} \in S^2$ and work on the quaternionic slice $\mathbb{C}_{\mathbf{u}} := \text{span}\{1, \mathbf{u}\} \subset \mathbb{H}$.
- A4. Calibration.** Choose the scale so that the flow norm is $\|\chi\| = c$ (or, equivalently, absorb c into the definition of Φ).
- A5. Circular kinematics.** Parametrize motion by a circular angle ϑ on $\mathbb{C}_{\mathbf{u}}$ via the unit rotor $\hat{q}(\vartheta, \mathbf{u}) = \cos \vartheta + \mathbf{u} \sin \vartheta$ and use the dictionary $\beta = \sin \vartheta$, $\gamma = \sec \vartheta$.
- A6. Domain.** Work with $\vartheta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ so that $|\beta| < 1$ and the map $\vartheta \mapsto \beta$ is monotone; the photon limit corresponds to $|\vartheta| \rightarrow \frac{\pi}{2}$.

We will systematically replace hyperbolic functions and nested square roots by the circular trigonometry of a single *phase angle* ϑ , interpreting $\cos \vartheta$ as the temporal projection and $\sin \vartheta$ as the spatial one. This keeps all kinematic identities while avoiding hyperbolic parametrization.

Convention. In this section we track a single body, so $\tau \equiv \tau_B$.

We define the *kinematic angle* $\vartheta \in [0, \frac{\pi}{2})$ of a system with respect to a fixed observer as an **operationally measurable split** of a constant “flow budget” c between temporal and spatial projections:

$$\cos \vartheta \equiv \frac{d\tau}{dt}, \quad \sin \vartheta \equiv \frac{\|\mathbf{v}\|}{c} = \beta, \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \quad (\text{I.1})$$

where t is the observer’s time, τ is the proper time, and \mathbf{v} is the 3-velocity. The identity $\cos^2 \vartheta + \sin^2 \vartheta = 1$ then restates the empirical invariance of the Minkowski interval.

Phase state and quaternionic slice. An (inertial) phase state is the pair (ϑ, \mathbf{u}) , with $\mathbf{u} \in S^2$. We associate to it the unit quaternion

$$q(\vartheta, \mathbf{u}) = \cos \vartheta + \mathbf{u} \sin \vartheta, \quad \mathbf{u} := u_x \mathbf{i} + u_y \mathbf{j} + u_z \mathbf{k}, \quad \|\mathbf{u}\| = 1, \quad (\text{I.2})$$

that is, a complex slice $\mathbb{C}_{\mathbf{u}} := \text{Span}\{1, \mathbf{u}\} \subset \mathbb{H}$ aligned with \mathbf{u} . This is the minimal structure that linearly encodes collinear compositions and naturally induces the Wigner–Thomas rotation for non-collinear boosts via quaternion multiplication.

J Mapping to standard SR variables

The phase angle ϑ is *not* the rapidity η ; they are related by a Gudermann-type [?] bridge

$$\beta = \tanh \eta = \sin \vartheta, \quad \gamma = \cosh \eta = \sec \vartheta, \quad \tan \frac{\vartheta}{2} = \tanh \frac{\eta}{2}. \quad (\text{J.1})$$

Consequently, all standard kinematic relations follow from circular trigonometry in ϑ :

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}} = \sec \vartheta, \quad k_{\parallel} = e^{\pm \eta} = \frac{1 + \tan(\vartheta/2)}{1 - \tan(\vartheta/2)}. \quad (\text{J.2})$$

All collinear compositions reduce to angle addition inside the slice $\mathbb{C}_{\mathbf{u}}$.

Collinear composition. For \mathbf{u} fixed,

$$q(\vartheta_2, \mathbf{u}) q(\vartheta_1, \mathbf{u}) = q(\vartheta_1 \oplus \vartheta_2, \mathbf{u}), \quad \cos(\vartheta_1 \oplus \vartheta_2) = \cos \vartheta_1 \cos \vartheta_2 - \sin \vartheta_1 \sin \vartheta_2, \quad (\text{J.3})$$

which implies Einstein's velocity addition

$$\beta_{12} = \sin(\vartheta_1 \oplus \vartheta_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2}. \quad (\text{J.4})$$

Non-collinear composition and Wigner–Thomas rotation. For $\mathbf{u}_1 \neq \mathbf{u}_2$ one has the factorization

$$q(\vartheta_2, \mathbf{u}_2) q(\vartheta_1, \mathbf{u}_1) = R_W q(\vartheta_{12}, \mathbf{u}_{12}), \quad (\text{J.5})$$

where $R_W \in \text{SO}(3)$ is the Wigner–Thomas rotation (a genuine 3D rotation), while $q(\vartheta_{12}, \mathbf{u}_{12})$ is the effective boost in the slice $\mathbb{C}_{\mathbf{u}_{12}}$. The mechanism can be found refined in H.3.1. The rotation angle and axis can be extracted from the vector part of the quaternion product; a closed expression equivalent to the standard formulas is provided in Appendix A.

K SR–phase correspondences

Below is a minimal dictionary of correspondences between the hyperbolic SR picture and the circular phase picture.

Quantity	Standard SR (hyperbolic)	Phase picture (circular)
Rapidity	$\eta = \text{artanh } \beta$	$\tanh \eta = \sin \vartheta$
Lorentz factor	$\gamma = \cosh \eta$	$\gamma = \sec \vartheta$
Speed	$\beta = \tanh \eta$	$\beta = \sin \vartheta$
Doppler (longitudinal)	$k = e^{\pm \eta}$	$k = \frac{1 + \tan(\vartheta/2)}{1 - \tan(\vartheta/2)}$
Temporal projection	$\text{sech } \eta$	$\cos \vartheta$

Table 3: One-to-one correspondences between hyperbolic (rapidity) and circular (phase) parametrizations.

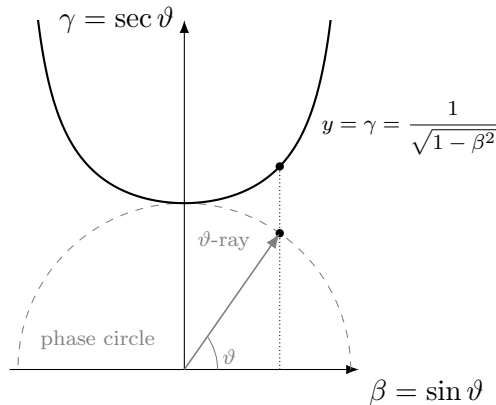


Figure 2: Phase circle vs. Lorentz hyperbola at common $\beta = \sin \vartheta$. Vertical mapping at fixed β illustrates the Gudermann bridge: $\gamma = \sec \vartheta = \cosh \eta$.

L Phase space

We represent a flow state on a quaternionic *complex slice* $\mathbb{C}_{\mathbf{u}} := \text{Span}\{1, \mathbf{u}\} \subset \mathbb{H}$, where $\mathbf{u} \in S^2$ is the unit spatial direction selected by the observer's split. The unit rotor on this slice is

$$\widehat{q}(\vartheta, \mathbf{u}) := \cos \vartheta + \mathbf{u} \sin \vartheta \in S^3 \simeq SU(2),$$

and, under the calibration $\|\chi\| \equiv c$ (cf. §G.4), we parameterize the flow vector as

$$\chi(\vartheta, \mathbf{u}) = c \widehat{q}(\vartheta, \mathbf{u}) = c(\cos \vartheta + \mathbf{u} \sin \vartheta). \quad (\text{L.1})$$

Phase-rate projections (observer A). Given an observer A with split (\mathbf{e}_t^A, S^A) and observer-object angle $\vartheta_{|A}$ (see §G.8), the phase-rate components with respect to A are

$$\tilde{X}_{|A}^0 := \frac{dx_{|A}^0}{d\chi} = c \cos \vartheta_{|A}, \quad \tilde{\mathbf{X}}_{|A} := \frac{d\mathbf{x}_{|A}}{d\chi} = c \sin \vartheta_{|A} \mathbf{u}_{|A}, \quad (\text{L.2})$$

where $\mathbf{u}_{|A} := P_{S^A} \widehat{\mathbf{F}} / \|P_{S^A} \widehat{\mathbf{F}}\| \in S^A$ is the unit spatial direction as seen by A and $\widehat{\mathbf{F}} := \chi / \|\chi\|$.

These rates satisfy the identity

$$\left(\frac{ds}{d\chi}\right)^2 = (\tilde{X}_{|A}^0)^2 - \|\tilde{\mathbf{X}}_{|A}\|^2 = c^2(1 - \sin^2 \vartheta_{|A}) = c^2 \cos^2 \vartheta_{|A}. \quad (\text{L.3})$$

which, in the SR simplification ($\zeta = \phi = 0$ and $dt_A := d\chi$), is just the operational split of the fixed budget c into temporal/spatial projections.

Phase-to-observable map (integral form). For any chart x^i adapted to A we have the integral transform

$$x^i(\chi) = x^i(\chi_0) + \int_{\chi_0}^{\chi} \tilde{X}_{|A}^i(u) du, \quad i = 0, 1, 2, 3, \quad (\text{L.4})$$

with $\tilde{X}_{|A}^0$ and $\tilde{\mathbf{X}}_{|A}$ from (L.2). In the SR sector this becomes $t_A(\chi) = t_A(\chi_0) + \int_{\chi_0}^{\chi} du$ and $\mathbf{x}_{|A}(\chi) = \mathbf{x}_{|A}(\chi_0) + \int_{\chi_0}^{\chi} c \sin \vartheta_{|A}(u) \mathbf{u}_{|A}(u) du$.

Intrinsic vs. directional factors. The intrinsic second-moment angle ζ (§G.5) enters the body's own rate via $d\tau_B/d\chi = \cos \zeta$, while (L.2) is purely directional and built from the observer's split. Outside the SR sector the intrinsic interval is $ds_B^2 = c^2 \cos^2 \zeta d\chi^2 - d\ell^\top \mathbf{C}_B d\ell$, and (L.4) remains valid as a kinematic reconstruction of observables from phase rates.

M Objects: operational catalogue and inference

Scope. This section is a look-up summary. The physics and proofs live in §G.3, §G.5, §G.4, and §N.

Canonical cases

1. **Photon (vacuum).** $T_B = 0$, $d\tau_B = 0$, $\|d\ell\|/d\chi = c$ (Def. (G.8)).
2. **Massive isotropic body.** $\mathbf{C}_B = \frac{\sin^2 \zeta}{3} \mathbf{I}$, $d\tau_B = \cos \zeta d\chi$, $ds_B^2 = c^2 \cos^2 \zeta d\chi^2 - \frac{\sin^2 \zeta}{3} \|d\ell\|^2$.
3. **Anisotropic body.** $\mathbf{C}_B = \sum_{i=1}^3 \lambda_i \hat{\mathbf{e}}_i \otimes \hat{\mathbf{e}}_i$, $\lambda_i \geq 0$, $\sum_i \lambda_i = \sin^2 \zeta$. Pullbacks along $\hat{\mathbf{e}}_i$ probe λ_i .
4. **Effective medium (ζ only).** $d\tau/dt = \cos \zeta$, $n(\zeta) = \sec \zeta$ (cf. §O).

Parameter inference

- *Rate split at rest.* With $\vartheta = 0$, $\phi = 0$: $\cos \zeta = (d\tau_B/dt)_{\text{meas}}$.
- *Directional pullbacks.* Measure ds_B^2 for small $d\ell$ along several lab directions; fit the quadratic form $q_B = d\ell^\top \mathbf{C}_B d\ell \Rightarrow$ eigenpairs $(\lambda_i, \hat{\mathbf{e}}_i)$.
- *Factorization with gravity/kinematics.* Use $d\tau/dt = \cos \zeta \cos \phi \cos \vartheta$ to separate ϕ (clock angle) from ζ via e.g. static vs moving protocols (see §O).

Ensemble composition

For streamlets a with weights w_a : $T_B = \sum_a w_a \cos^2 \Theta_a$, $\mathbf{C}_B = \sum_a w_a \sin^2 \Theta_a \mathbf{u}_a \otimes \mathbf{u}_a$ (Eq. (G.4)).

Non-collinear boosts

Composition induces a spatial Wigner rotation on orientations without altering the scalar rate factorization; see §H.3.1, §H.3.2.

N Phase-derivative view

Scope. This section repackages core identities in the language of *phase derivatives*, using only notions already introduced (§G.4–G.8). No new kinematics is postulated.

Notation recap (from §L). Phase-rate components for an observer A are

$$\tilde{X}_{|A}^0 = \frac{dx_{|A}^0}{d\chi} = c \cos \vartheta_{|A}, \quad \tilde{\mathbf{X}}_{|A} = \frac{d\mathbf{x}_{|A}}{d\chi} = c \sin \vartheta_{|A} \mathbf{u}_{|A}, \quad (\text{L.2})$$

and obey the identity

$$\left(\frac{ds}{d\chi}\right)^2 = (\tilde{X}_{|A}^0)^2 - \|\tilde{\mathbf{X}}_{|A}\|^2 = c^2(1 - \sin^2 \vartheta_{|A}) = c^2 \cos^2 \vartheta_{|A}. \quad (\text{L.3})$$

A change of parameter $\chi \mapsto \tau$ acts by the Jacobian $\mathcal{J} := \frac{d\chi}{d\tau}$:

$$\dot{X}^i = \frac{dx^i}{d\tau} = \mathcal{J} \tilde{X}^i, \quad \dot{X} \equiv d/d\tau, \quad \tilde{X} \equiv d/d\chi. \quad (\text{N.1})$$

N.1 Minkowski–phase equality of invariants

Theorem. For any worldline segment and any observer A ,

$$\boxed{\tilde{H} = \dot{S}}, \quad \tilde{H}^2 := (\tilde{X}_{|A}^0)^2 + \|\tilde{\mathbf{X}}_{|A}\|^2, \quad \dot{S}^2 := (\dot{X}_{|A}^0)^2 - \|\dot{\mathbf{X}}_{|A}\|^2. \quad (\text{N.2})$$

Proof. By (N.1), $\dot{X}^i = \mathcal{J} \tilde{X}^i$, hence

$$\dot{S}^2 = (\mathcal{J} \tilde{X}^0)^2 - \|\mathcal{J} \tilde{\mathbf{X}}\|^2 = \mathcal{J}^2 (\tilde{X}^{02} - \|\tilde{\mathbf{X}}\|^2) = \mathcal{J}^2 \left(\frac{ds}{d\chi}\right)^2 = \left(\frac{ds}{d\tau}\right)^2.$$

At the same time, $\tilde{H}^2 = \tilde{X}^{02} + \|\tilde{\mathbf{X}}\|^2 = c^2$ in the SR gauge (§G.7), and $ds/d\tau = c$ by the operational split $\cos \vartheta = d\tau/dt$, $\sin \vartheta = \|\mathbf{v}\|/c$ (§I). Therefore $\tilde{H} = \dot{S} = c$. \square

Reading. In words: the Euclidean norm of the phase-rate vector equals the Minkowski norm of the observable rate vector. The equality (N.2) is the derivative form of “the same budget seen in two parametrizations”.

N.2 Photon time: null interval and phase parameter

For a lightlike flow in vacuum, the body’s intrinsic quadratic form gives $ds_B^2 = 0$ (§G.4), hence the body-level proper increment vanishes:

$$\boxed{d\tau_B = 0 \quad (\text{null flow, vacuum})}. \quad (\text{N.3})$$

Nevertheless the phase parameter χ advances and serves as an affine parameter along the ray. With the calibration (G.8) we have

$$\frac{\|d\ell\|}{d\chi} = c, \quad \Rightarrow \quad \chi = (\text{cycle counter}). \quad (\text{N.4})$$

Under the convention $\Phi = \omega \chi$, one cycle corresponds to $\Delta\chi = 2\pi/\omega$, so χ counts wavefronts affinely while the intrinsic metric remains null ($d\tau_B = 0$).

At unit frequency, $\nu := d\chi/d\tau = 1$, the phase equals the proper-time parameter used as an affine label of wavefronts; no contradiction with (N.3) arises because τ here is a *chosen* parameter for counting cycles, while the intrinsic metric length remains null.

N.3 Relativistic Doppler as a phase-kinematic theorem

We derive the Doppler law purely from the phase split, without hyperbolic parametrization or dynamics.

Setup. Define frequency as the phase growth rate in proper time, $\nu := d\chi/d\tau$, and consider two successive wavefronts separated by the same phase increment $\Delta\chi$ for the source and the observer. Then

$$\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{d\chi/d\tau_{\text{obs}}}{d\chi/d\tau_{\text{src}}} = \frac{d\tau_{\text{src}}}{d\tau_{\text{obs}}}. \quad (\text{N.5})$$

In the SR sector ($\zeta = \phi = 0$) the observer at rest has $d\tau_{\text{obs}} = dt_{\text{obs}}$, and for the (possibly moving) source $dt_{\text{src}} = \gamma d\tau_{\text{src}}$ with $\gamma = \sec \vartheta$, $\beta = \sin \vartheta$.

Longitudinal case. During the emission interval $dt_{\text{src}} = \gamma d\tau_{\text{src}}$ (measured in the observer’s lab chart) the source shifts by $\pm\beta c dt_{\text{src}}$ along the line of sight (“+” receding, “−” approaching). The reception interval is the emission interval plus the change in light flight time:

$$dt_{\text{obs}} = dt_{\text{src}} (1 \pm \beta) = \gamma d\tau_{\text{src}} (1 \pm \beta).$$

With (N.5) and $d\tau_{\text{obs}} = dt_{\text{obs}}$ this gives the Doppler factor

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 \pm \beta)}} = \sqrt{\frac{1 \mp \beta}{1 \pm \beta}} = \sec \vartheta (1 \mp \sin \vartheta) = e^{\mp \eta}, \quad (\text{N.6})$$

where the equivalent forms use $\beta = \sin \vartheta$, $\gamma = \sec \vartheta$, and the Gudermann bridge (J.1).

Transverse case. For $\varphi = 90^\circ$ (emission orthogonal to the relative velocity in the observer’s frame), the geometric flight-time correction vanishes, hence

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma} = \cos \vartheta}. \quad (\text{N.7})$$

General line-of-sight angle. Let φ be the angle between the source velocity and the line of sight in the observer's frame. Only the LOS component $\beta \cos \varphi$ changes the flight time; repeating the above,

$$\boxed{\frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \frac{1}{\gamma(1 - \beta \cos \varphi)} = \gamma(1 - \beta \cos \varphi)^{-1}}. \quad (\text{N.8})$$

Equivalently, in phase variables, $\nu_{\text{obs}}/\nu_{\text{src}} = \sec \vartheta (1 - \sin \vartheta \cos \varphi)^{-1}$.

Corollary (with intrinsic/gravitational tilts). Outside the pure SR sector, the total redshift factorizes:

$$1 + z_{\text{tot}} = \frac{\nu_{\text{src}}}{\nu_{\text{obs}}} = \underbrace{\gamma(1 - \beta \cos \varphi)}_{\text{kinematic}} \times \underbrace{\frac{\cos \zeta_{\text{obs}} \cos \phi_{\text{obs}}}{\cos \zeta_{\text{src}} \cos \phi_{\text{src}}}}_{\text{intrinsic \& gravitational}},$$

which reproduces (P.3) upon identifying the kinematic part with (N.8).

Infinitesimal form (longitudinal). Taking the differential of $\ln(\nu_{\text{obs}}/\nu_{\text{src}})$,

$$\boxed{d \ln \frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = \mp d\eta = \mp \sec \vartheta d\vartheta = \mp \gamma^2 d\beta}, \quad (\text{N.9})$$

with the upper sign for receding (“+” in (N.6)) and lower for approaching. Equation (N.9) is the *phase-derivative* Doppler law.

Transverse case. For $\varphi = 90^\circ$ in the observer's frame the flight-time correction vanishes, so $\nu_{\text{obs}}/\nu_{\text{src}} = 1/\gamma$ and

$$d \ln \frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = -d \ln \gamma = -\beta \gamma^2 d\beta. \quad (\text{N.10})$$

General LOS angle (fixed geometry). If φ is the angle between the velocity and the line of sight in the observer's frame and $d\varphi = 0$, then from $\nu_{\text{obs}}/\nu_{\text{src}} = 1/[\gamma(1 - \beta \cos \varphi)]$ one gets

$$d \ln \frac{\nu_{\text{obs}}}{\nu_{\text{src}}} = -\left[\beta \gamma^2 - \frac{\cos \varphi}{1 - \beta \cos \varphi}\right] d\beta = -d\eta \quad \text{when } \varphi = 0, \pi. \quad (\text{N.11})$$

Together, (N.2), (N.3)–(N.4), and (N.9)–(N.11) show how *phase derivatives* encode: (i) equality of observable and phase invariants; (ii) the null nature of photon proper time with χ as an affine counter; (iii) the Doppler law, both finite and in instantaneous differential form.

O Unified temporal effects in phase space

We factor all time-related effects through a single phase tilt structure. Let the intrinsic (ζ), gravitational (ϕ), and kinematic (ϑ) angles be independent contributors to the phase tilt at a point. Define the local *time-rate factor*

$$\frac{d\tau}{dt} = \cos \zeta \cos \phi \cos \vartheta, \quad n_{\text{tot}} := \sec \zeta \sec \phi \sec \vartheta, \quad (\text{O.1})$$

where n_{tot} plays the role of an effective (phase) refractive index for light propagation. Accordingly, the coordinate light-travel time along a spatial curve Γ is

$$t_\Gamma = \frac{1}{c} \int_\Gamma n_{\text{tot}} ds. \quad (\text{O.2})$$

Lemma (Unified time-rate). In the unimetry phase representation, the proper-to-lab time rate and the optical length factorize as in (O.1)–(O.2). Setting any subset of angles to zero recovers the corresponding standard effects:

- **Special relativity (vacuum SR):** $\zeta = 0, \phi = 0 \Rightarrow d\tau/dt = \cos \vartheta = 1/\gamma$ (kinematic time dilation).
- **Gravitational clocks (static weak field):** $\zeta = 0, \vartheta = 0 \Rightarrow d\tau/dt = \cos \phi$ (gravitational redshift/clock rate); calibration of ϕ reproduces the usual $\sqrt{g_{00}}$.
- **Shapiro delay (gravitational contribution to TOF):** with $\zeta = 0, \vartheta = 0$, (O.2) yields $t_{\Gamma} = (1/c) \int \sec \phi ds$; the excess over $\int ds/c$ is the Shapiro time delay.
- **Intrinsic/evolutionary tilt:** $\phi = 0, \vartheta = 0 \Rightarrow d\tau/dt = \cos \zeta, n_{\text{tot}} = \sec \zeta$ (useful as a time gauge in cosmological/effective-medium contexts).

Remarks. (i) The factorization in (O.1) is the phase-space counterpart of composing independent tilts; it preserves the conserved Minkowski norm once time is reparameterized by τ . (ii) For vacuum SR we simply set $\zeta = 0$ and work with ϑ ; for pure gravitational timing we set $\vartheta = 0$ and use ϕ . (iii) Non-collinear boost composition acts on ϑ via quaternionic rotors and induces a Wigner/Thomas residual *spatial* rotation without altering (O.1).

P Gravity as a phase rotation: local tetrads and clock angle

On a curved background (\mathcal{M}, g) we work with orthonormal tetrads e_a^μ such that $g_{\mu\nu}e_a^\mu e_b^\nu = \eta_{ab}$ and take the observer’s time leg to be e_0^μ . We apply the gravitational (clock) angle ϕ as

$$\cos \phi := e^0_\mu u^\mu = \frac{d\tau_{\text{stat}}}{dt} = \sqrt{-g_{00}} \quad (\text{stationary case}). \quad (\text{P.1})$$

Kinematics remains encoded by the phase angle ϑ in the slice $\mathbb{C}_{\mathbf{u}}$, with $\cos \vartheta = d\tau/d\tau_{\text{stat}}$. Therefore the proper time factorizes as

$$d\tau = dt \cos \phi \cos \vartheta, \quad d\mathbf{x} = c dt \sin \vartheta \mathbf{u}, \quad (\text{P.2})$$

and the total redshift factorizes into kinematic and gravitational parts:

$$1 + z_{\text{tot}} = \frac{\cos \vartheta_{\text{em}}}{\cos \vartheta_{\text{obs}}} \times \frac{\cos \phi(x_{\text{em}})}{\cos \phi(x_{\text{obs}})}. \quad (\text{P.3})$$

For static emitter/observer ($\vartheta_{\text{em}} = \vartheta_{\text{obs}} = 0$) one recovers the standard gravitational redshift $1 + z_g = \sqrt{\frac{-g_{00}(x_{\text{obs}})}{-g_{00}(x_{\text{em}})}}$. In the weak-field limit $g_{00} \simeq -(1 + 2\Phi/c^2)$ this gives $z_g \simeq (\Phi_{\text{obs}} - \Phi_{\text{em}})/c^2$.

Beyond static fields. In a 3+1 split $ds^2 = -N^2 c^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$ one may identify $\cos \phi := N$ in the observer’s tetrad, which keeps (P.2)–(P.3) coordinate-agnostic. Uniform acceleration (Rindler) accumulates phase according to $d\vartheta = \kappa dt$ inside the local slice, consistently reproducing accelerated-frame kinematics when combined with the clock angle ϕ .

Q Discussion

The phase-space reformulation of special relativity presented in this work provides an original and conceptually unifying framework, recasting all standard kinematic effects as geometric projections and rotations within a Euclidean (as opposed to pseudo-Euclidean) phase space structured by the algebra of quaternions. This approach enables closed-form derivations of time

dilation, Lorentz contraction, all Doppler effects, and the composition of boosts—including the emergence of Wigner and Thomas rotations—as pure consequences of phase kinematics and composition rules for rotors. Notably, the speed of light appears not as a postulate but as a derived invariant: the magnitude of the calibrated flow.

One of the main strengths of the unimetry approach is its ability to coherently and transparently separate intrinsic, gravitational, and kinematic effects through the introduction of independent angular parameters. This structural modularity allows for clear generalizations beyond the regime of special relativity (SR). The use of quaternionic algebra naturally incorporates the noncommutative composition of boosts (and hence Wigner rotation), and the direct operational mapping from phase parameters to observable quantities avoids the hyperbolic constructs and nested root expressions traditional to the Lorentzian metric, potentially offering both conceptual and computational simplifications.

The revisit of the kinematical foundations of relativity within a Euclidean metric is reminiscent of various efforts to find deeper group-theoretic or geometric underpinnings for known physics, such as the reinterpretation of quantum spinors, twistors, or the generalized use of Clifford and quaternionic structures.

Despite these advantages, there are several limitations and open issues:

- The present work, while offering a full reformulation of SR, leaves the dynamical aspects—especially those tied to gravitational fields and interactions—outside its direct scope. The approach to gravity as a phase tilt field is promising, but it will require a precise translation of Einsteinian field dynamics into the quaternionic, phase-based language.
- Similarly, the connection to quantum mechanics, though structurally motivated by the phase-space formulation and inherent links to phase and geometric quantization, remains undeveloped at the level of operator theory or Hilbert space structures. Concrete examples or models connecting unimetry directly to quantum phenomena would reinforce the physical scope and relevance of the approach.
- A key question for future study is the physical interpretability and experimental accessibility of the various angular parameters (intrinsic, gravitational, kinematic) in more complex, curved, or quantum-influenced scenarios.
- The mathematical tools used—particularly the phase decomposition and modularization—have strong implications for the construction of algorithms in numerical relativity and simulation of accelerated particle beams, but explicit comparisons with existing computational methods remain to be presented.

Outlook

The unimetry framework developed here provides a versatile foundation for future theoretical developments in both relativity and quantum physics. The following directions emerge as particularly promising:

- **Extension to General Relativity.** The modular phase decomposition is well suited for the introduction of curved metrics, potentially allowing gravitational redshift, time delay (Shapiro effect), and even curvature-driven precessions to be recast in the language of phase tilts and generalized rotors. Realizing Einstein’s dynamics for gravitational fields in this formalism is a natural next step, especially via the local tetrad and clock angle structures already sketched.
- **Quantum Connections.** Given the central role of phase, rotors, and symmetry groups (notably the demonstration of Hopf fibration structure), the unimetry formalism can serve as

a stepping stone toward a deeper unification between relativistic and quantum frameworks, possibly via geometric quantization or generalized qubit/Bloch-sphere analogues.

- **Physical and Experimental Applications.** The structure is already suited for reinterpretation and numerical simulation of phenomena involving non-collinear accelerations, such as in advanced particle beam physics or optical systems with phase media. This computational potential deserves further methodological and comparative exploration.
- **Generalizations and Group Theory.** The embedding of Lorentzian kinematics within a more general Euclidean group-theoretic framework (potentially encompassing de Sitter or conformal transformations) may suggest further extensions, including nontrivial topologies, higher dimensions, or self-dual (instanton-like) solutions.

In summary, by replacing the traditional spacetime paradigm with a geometric flow in phase space, unimetry opens avenues for new mathematical structures and connections among relativity, geometry, and quantum theory. Its broader success will depend on future progress in developing the gravitational and quantum sectors and on the demonstration of concrete computational and experimental advantages.

A Equivalence to the classical Wigner rotation

We sketch an intrinsic quaternionic proof that the unimetry expression for the Wigner rotation coincides with the standard special-relativistic formula.

Step 1: product of two D-rotors. For $d_i = \cos \frac{\psi_i}{2} + \hat{\mathbf{u}}_i \sin \frac{\psi_i}{2}$,

$$d_2 d_1 = (c_2 c_1 - s_2 s_1 \cos \theta) + \left(c_2 s_1 \hat{\mathbf{u}}_1 + s_2 c_1 \hat{\mathbf{u}}_2 + s_2 s_1 \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1 \right), \quad (\text{A.1})$$

with $c_i = \cos(\psi_i/2)$, $s_i = \sin(\psi_i/2)$ and $\cos \theta = \hat{\mathbf{u}}_2 \cdot \hat{\mathbf{u}}_1$.

Step 2: symmetric D–R factorization. Define d_{12} by $d_{12} \mathbf{e}_t d_{12} = L_{12} \mathbf{e}_t L_{21}$ and set $r_W = \bar{d}_{12} L_{12} = L_{21} \bar{d}_{12}$. Then r_W fixes \mathbf{e}_t and is a pure spatial rotor, so $r_W = \cos \frac{\phi}{2} + \hat{\mathbf{n}} \sin \frac{\phi}{2}$ with $\hat{\mathbf{n}} \parallel \hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. Matching scalar and bivector parts gives

$$\tan \frac{\phi}{2} = \frac{s_1 s_2 \sin \theta}{c_1 c_2 + s_1 s_2 \cos \theta}. \quad (\text{A.2})$$

Step 3: map to rapidities. With the substitutions $\sin(\psi/2) \mapsto \sinh(\eta/2)$, $\cos(\psi/2) \mapsto \cosh(\eta/2)$, $\tan(\psi/2) \mapsto \tanh(\eta/2)$ (where $\tanh \eta = \beta$, $\cosh \eta = \gamma$), (A.2) becomes the textbook Wigner angle:

$$\tan \frac{\phi}{2} = \frac{\sinh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \sin \theta}{\cosh \frac{\eta_1}{2} \cosh \frac{\eta_2}{2} + \sinh \frac{\eta_1}{2} \sinh \frac{\eta_2}{2} \cos \theta}, \quad (\text{A.3})$$

with axis along $\hat{\mathbf{u}}_2 \times \hat{\mathbf{u}}_1$. This circular–hyperbolic correspondence is classical; cf. Gudermann [?].