A PROOF OF THE HOWE DUALITY CONJECTURE

WEE TECK GAN AND SHUICHIRO TAKEDA

to Professor Roger Howe
who started it all
on the occasion of his 70th birthday

Abstract. We give a proof of the Howe duality conjecture in the theory of local theta correspondence for symplectic-orthogonal or unitary dual pairs in arbitrary residual characteristic.

1. Introduction

Let F be a nonarchimedean local field of characteristic not 2 and residue characteristic p. Let E be F itself or a quadratic field extension of F. For $\epsilon = \pm$, we consider a $-\epsilon$-Hermitian space W over E of dimension n and an ϵ-Hermitian space V of dimension m. We shall write W_n or V_m if there is a need to be specific about the dimension of the space in question.

Let $G(W)$ and $H(V)$ denote the isometry group of W and V respectively. Then the group $G(W) \times H(V)$ forms a dual reductive pair and possesses a Weil representation ω_ψ which depends on a nontrivial additive character ψ of F (and some other auxiliary data which we shall suppress for now). To be precise, when $E = F$ and one of the spaces, say V, is odd dimensional, one needs to consider the metaplectic double cover of $G(W)$; we shall simply denote this double cover by $G(W)$ as well. The various cases are tabulated in [GI, §3].

In the theory of local theta correspondence, one is interested in the decomposition of ω_ψ into irreducible representations of $G(W) \times H(V)$. More precisely, for any irreducible admissible representation π of $G(W)$, one may consider the maximal π-isotypic quotient of ω_ψ. This has the form $\pi \otimes \Theta_{W,V,\psi}(\pi)$ for some smooth representation $\Theta_{W,V,\psi}(\pi)$ of $H(V)$; we shall frequently suppress (W, V, ψ) from the notation if there is no cause for confusion. It was shown by Kudla [K1] that $\Theta(\pi)$ has finite length (possibly zero), so we may consider its maximal semisimple quotient $\theta(\pi)$. One has the following fundamental conjecture due to Howe [H]:

Howe Duality Conjecture for $G(W) \times H(V)$

(i) $\theta(\pi)$ is either 0 or irreducible.

(ii) If $\theta(\pi) = \theta(\pi') \neq 0$, then $\pi = \pi'$.

A concise reformulation is: for any irreducible π and π',

$$\text{(HD)} \qquad \dim \operatorname{Hom}_{H(V)}(\theta(\pi), \theta(\pi')) \le \delta_{\pi,\pi'} := \begin{cases} 1, & \text{if } \pi \cong \pi'; \\ 0, & \text{if } \pi \ncong \pi'. \end{cases}$$

We take note of the following theorem:

Theorem 1.1. *(i) If π is supercuspidal, then $\Theta(\pi)$ is either zero or irreducible (and thus is equal to $\theta(\pi)$). Moreover, for any irreducible supercuspidal π and π',*

$$\Theta(\pi) \cong \Theta(\pi') \neq 0 \Longrightarrow \pi \cong \pi'.$$

(ii) $\theta(\pi)$ is multiplicity-free.

(iii) If $p \neq 2$, the Howe duality conjecture holds.

The statement (i) is a classic theorem of Kudla [K1] (see also [MVW]), whereas (iii) is a well-known result of Waldspurger [W]. The statement (ii), on the other hand, is a result of Li-Sun-Tian [LST]. We note that the techniques for proving the three statements in the theorem are quite disjoint from each other. For example, the proof of (i) is based on arguments using the doubling see-saw and Jacquet modules of the Weil representation: these have become standard tools in the study of the local theta correspondence. The proof of (iii) is based on K-type analysis and uses various lattice models of the Weil representation. Finally, the proof of (ii) is based on an argument using the Gelfand-Kazhdan criterion for the (non-)existence of equivariant distributions.

In this paper, we shall not assume any of the statements in Theorem 1.1. Indeed, the purpose of this paper is to give a simple proof of the Howe duality conjecture using essentially the same tools in the proof of Theorem 1.1(i), as developed further in [MVW]. Thus, our main theorem is:

Theorem 1.2. *The Howe duality conjecture* (HD) *holds for the pair $G(W) \times H(V)$.*

Let us make a few remarks:

(1) The above setup makes sense even when $E = F \times F$ is a split quadratic algebra, in which case the groups $G(W)$ and $H(V)$ are general linear groups. In that case, the Howe duality conjecture has been shown by Minguez in [M]. As we shall see, the proof of Theorem 1.2 is essentially analogous to the one given by Minguez.

(2) In an earlier paper [GT], we had extended Theorem 1.1(i) (with $\Theta(\pi)$ replaced by $\theta(\pi)$) from supercuspidal to tempered representations. Using this, we had shown the Howe duality conjecture for almost equal rank dual pairs. The argument in Section 3 of this paper (doubling see-saw) is the same as that in [GT, §2], but pushed to the limit beyond tempered representations. On the other hand, the argument in Section 4 (Kudla's filtration) is entirely different from that in [GT, §3] and uses a key technique of Minguez [M].

(3) We remark that in the papers [M1, M2, M3, M4], Muić has conducted detailed studies of the local theta correspondence for symplectic-orthogonal dual pairs. In [M1], for example, he explicitly determined the theta lift of discrete series representations π in terms of the Moeglin-Tadić classification and observed as a consequence the irreducibility of $\theta(\pi)$. The Moeglin-Tadić classification was conditional at that point, and we are not sure where it stands today. In [M3, M4], Muić proved various general properties of the theta lifting of tempered representations (such as the issue of whether $\Theta(\pi) = \theta(\pi)$), and obtained very explicit information about the theta lifting under the assumption of the Howe duality conjecture. The main tools he used are Jacquet modules analysis and Kudla's filtration. Since the Howe duality conjecture is a simple statement without reference to classification, it seems desirable to have a classification-free proof. Indeed, our result renders most results in [M3, M4] unconditional.

As is well-known, there is another family of dual pairs associated to quaternionic Hermitian and skew-Hermitian spaces. (See [W] or [K2] for more details.) Our proof, unfortunately, does not apply to these quaternionic dual pairs, because we have made use of the MVW-involution $\pi \mapsto \pi^{MVW}$ on the category of smooth representations of $G(W)$ and $H(V)$. For the same reason, the result of [LST] in Theorem 1.1(ii) is not known for these quaternionic dual pairs. Unlike the contragredient functor, which is contravariant in nature, the MVW-involution is covariant and has the property that $\pi^{MVW} = \pi^\vee$ if π is irreducible. It was shown in [LnST] that such an involution does not exist for quaternionic unitary groups.

Nonetheless, even in the quaternionic case, our proof gives a partial result which is often sufficient for global applications. Namely, if π is an irreducible Hermitian representation (i.e. $\overline{\pi} = \pi^\vee$) and we let $\theta_{her}(\pi) \subset \theta(\pi)$ denote the submodule generated by irreducible Hermitian summands, then the results of Theorem 1.2 hold for Hermitian π's and with $\theta(\pi)$ replaced by $\theta_{her}(\pi)$. Namely we have:

Theorem 1.3. *Consider a quaternionic dual pair $G(W) \times H(V)$ and let π and π' be irreducible Hermitian representations of $G(W)$. Let $\theta_{her}(\pi) \subset \theta(\pi)$ be the submodule generated by irreducible Hermitian summands. Then we have*

$$\dim \mathrm{Hom}_{H(V)}(\theta_{her}(\pi), \theta_{her}(\pi')) \leq \delta_{\pi,\pi'}.$$

In particular, if π and π' are unitary, we have

$$\dim \mathrm{Hom}_{H(V)}(\theta_{unit}(\pi), \theta_{unit}(\pi')) \leq \delta_{\pi,\pi'},$$

where $\theta_{unit}(\pi) \subset \theta_{her}(\pi)$ consists of irreducible unitary summands of $\theta(\pi)$.

We give a proof of this theorem in the last section of the paper.

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2. **Basic Notations and Conventions**

2.1. **Fields.** Throughout the paper, F denotes a nonarchimedean local field of characteristic different from 2 and residue characteristic p. Once and for all, we fix a non-trivial additive character ψ on F. Let E be F itself or a quadratic field extension of F. For $\epsilon = \pm 1$, we set

$$\epsilon_0 = \begin{cases} \epsilon \text{ if } E = F; \\ 0 \text{ if } E \neq F. \end{cases}$$

2.2. **Spaces.** Let

$$\begin{aligned} W = W_n &= \text{a } -\epsilon\text{-Hermitian space over } E \text{ of dimension } n \text{ over } E \\ V = V_m &= \text{an } \epsilon\text{-Hermitian space over } E \text{ of dimension } m \text{ over } E. \end{aligned}$$

We also set:

$$s_{m,n} = \frac{m - (n + \epsilon_0)}{2}.$$

2.3. **Groups.** We will consider the isometry groups associated to the pair (V, W) of $\pm\epsilon$-Hermitian spaces. More precisely, we set:

$$G(W) = \begin{cases} \text{the metaplectic group } \mathrm{Mp}(W), \text{ if } W \text{ is symplectic and } \dim V \text{ is odd;} \\ \text{the isometry group of } W, \text{ otherwise.} \end{cases}$$

We define $H(V)$ similarly by switching the roles of W and V. Occasionally we write

$$\begin{aligned} G_n &:= G(W_n) \\ H_m &:= H(V_m). \end{aligned}$$

For the general linear group, we shall write GL_n for the group $\mathrm{GL}_n(E)$. Also for a vector space X over E, we write $\det_{\mathrm{GL}(X)}$ or sometimes simply $\det_X$ for the determinant on $\mathrm{GL}(X)$.

2.4. **Representations.** For a p-adic group G, let $\mathrm{Rep}(G)$ denote the category of smooth representations of G and denote by $\mathrm{Irr}(G)$ the set of equivalence classes of irreducible smooth representations of G.

For a parabolic $P = MN$ of G, we have the normalized induction functor

$$\mathrm{Ind}_P^G : \mathrm{Rep}(M) \longrightarrow \mathrm{Rep}(G).$$

On the other hand, we have the normalized Jacquet functor

$$R_P : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(M).$$

If $\overline{P} = M\overline{N}$ denotes the opposite parabolic subgroup to P, we likewise have the functor $R_{\overline{P}}$. We shall frequently exploit the following two Frobenius' reciprocity formulas:

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_P^G \sigma) \cong \mathrm{Hom}_M(R_P(\pi), \sigma) \quad \text{(standard Frobenius reciprocity)}$$

and

$$\mathrm{Hom}_G(\mathrm{Ind}_P^G \sigma, \pi) \cong \mathrm{Hom}_M(\sigma, R_{\overline{P}}(\pi)) \quad \text{(Bernstein's Frobenius reciprocity).}$$

Moreover, Bernstein's Frobenius reciprocity is equivalent to the statement:

$$R_P(\pi^\vee)^\vee \cong R_{\overline{P}}(\pi)$$

for any smooth representation π with contragredient $\pi^\vee$, where $\overline{P}$ denotes the opposite parabolic subgroup to P.

2.5. **Parabolic subgroups and induced representations.** When G is a classical group, we shall use Tadić's notation for induced representations. Namely, for general linear groups, we set

$$\rho_1 \times \cdots \times \rho_a := \mathrm{Ind}_Q^{\mathrm{GL}_{n_1+\cdots+n_a}} \rho_1 \otimes \cdots \otimes \rho_a$$

where Q is the standard parabolic subgroup with Levi subgroup $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a}$. For a classical group such as $G(W)$, its parabolic subgroups are given as the stabilizers of flags of isotropic spaces. If X_t is a t-dimensional isotropic space of $W = W_n$ and we decompose

$$W = X_t \oplus W_{n-2t} \oplus X_t^*,$$

the corresponding maximal parabolic subgroup $Q(X_t) = L(X_t) \cdot U(X_t)$ has Levi factor $L(X_t) = \mathrm{GL}(X_t) \times G(W_{n-2t})$. If ρ is a representation of $\mathrm{GL}(X_t)$ and σ is a representation of $G(W_{n-2t})$, we write

$$\rho \rtimes \sigma = \mathrm{Ind}_{Q(X_t)}^{G(W)} \rho \otimes \sigma.$$

More generally, a standard parabolic subgroup Q of $G(W)$ has the Levi factor of the form $\mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_a} \times G(W_{n'})$ and we set

$$\rho_1 \times \cdots \times \rho_a \rtimes \sigma := \mathrm{Ind}_Q^{G(W)} \rho_1 \otimes \cdots \otimes \rho_a \otimes \sigma,$$

where ρ_i is a representation of GL_{n_i} and σ is a representation of $G(W_{n'})$. When $G(W) = \mathrm{Mp}(W)$ is a metaplectic group, we will follow the convention of [GS, §2.2-2.5] for normalized parabolic induction.

We have the analogous convention for parabolic subgroups and induced representations of $H(V_m)$. For example, a maximal parabolic subgroup of $H(V_m)$ has the form $P(Y_t) = M(Y_t) \cdot N(Y_t)$ and is the stabilizer of a t-dimensional isotropic subspace Y_t of V_m.

To distinguished between representations of $G(W)$ and $H(V)$, we will normally use lower case Greek letters such as π, σ etc to denote representations of $G(W)$, and upper case Greek letters such as Π, Σ etc to denote representations of $H(V)$.

Let c be the generator of $\mathrm{Gal}(E/F)$. For a representation ρ of $\mathrm{GL}_n(E)$, we define ${}^c\rho$ to be the c-conjugate of ρ, namely

$$ {}^c\rho(g) := \rho(c(g)) $$

for $g \in \mathrm{GL}_n(E)$. Of course if $E = F$, then ${}^c\rho = \rho$.

2.6. **Weil representations.** To consider the Weil representation of the pair $G(W) \times H(V)$, we need to specify extra data to give a splitting $G(W) \times H(V) \to \mathrm{Mp}(W \otimes V)$ of the dual pair. Such splittings were constructed and parametrized by Kudla [K2] and we shall use his convention here, as described in [GI, §3.2-3.3]. In particular, a splitting is specified by fixing a pair of splitting characters $\boldsymbol{\chi} = (\chi_V, \chi_W)$, which are certain unitary characters of $E^\times$. Pulling back the Weil representation of $\mathrm{Mp}(W \otimes V)$ to $G(W) \times H(V)$ via the splitting, we obtain the associated Weil representation $\omega_{W,V,\boldsymbol{\chi},\psi}$ of $G(W) \times H(V)$. Note that the character χ_V satisfies

$$ {}^c\chi_V^{-1} = \chi_V \tag{2.1} $$

and likewise for χ_W. We shall frequently suppress $\boldsymbol{\chi}$ and ψ from the notation, and simply write $\omega_{W,V}$ for the Weil representation.

2.7. **MVW.** In [MVW, p. 91], Moeglin, Vigneras and Waldspurger introduced a functor

$$ MVW : \mathrm{Rep}(G(W)) \longrightarrow \mathrm{Rep}(G(W)) $$

which is an involution and satisfies

$$ \pi^{MVW} = \pi^\vee \quad \text{if } \pi \text{ is irreducible.} $$

Unlike the contragredient functor, this MVW involution is covariant. It will be useful to observe that

$$ (\rho \rtimes \sigma)^{MVW} = {}^c\rho \rtimes \sigma^{MVW}. \tag{2.2} $$

2.8. **Boundary.** A notion which plays an important role in this paper is that of an irreducible representation π of $G(W)$ occurring on the boundary of $\omega_{W,V}$. For $\pi \in \mathrm{Irr}(G(W))$, we shall say that π occurs on the boundary of $\omega_{W,V}$ if there exists $t > 0$ such that

$$ \pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n} - \frac{t}{2}} \rtimes \sigma \tag{2.3} $$

for some irreducible representation σ of $G(W_{n-2t})$. Dualizing and using the MVW involution and Bernstein's Frobenius reciprocity, this is equivalent to

$$ \mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V |\det|^{-s_{m,n} + \frac{t}{2}}, R_{\overline{Q}(X_t)}(\pi)) \neq 0, \tag{2.4} $$

where $\overline{Q}(X_t) = L(X_t) \cdot \overline{U}(X_t)$ stands for the parabolic subgroup opposite to $Q(X_t)$. This terminology is due to Kudla and Rallis and the reader may consult [KR, Definition 1.3] for the explanation of the use of "boundary".

2.9. **Outline of proof.** With the basic notations introduced, we can now give a brief outline of the proof of Theorem 1.2. First of all, there is no loss of generality in assuming that

$$m \leq n + \epsilon_0,$$

because otherwise one can switch the roles of $G(W)$ and $H(V)$. We shall assume this henceforth.

The proof proceeds by induction on $\dim W$, with the base case with $\dim W = 0$ being trivial. The inductive step is divided into two different parts.

The first part, which is given in Section 3, deals with the case when π does not "occur on the boundary", which cover "almost all" representations. The argument for this case is close to that of [KR], using the doubling see-saw and a well-known filtration of degenerate principal series. For this part, the induction hypothesis is not used.

The second part of the proof, which is given in Section 4, deals with the case when π occurs on the boundary. To use the induction hypothesis, we use the Jacquet module of the Weil representation along with a key idea of Minguez [M] in his proof of the Howe duality conjecture for general linear groups.

3. **Special Case of Theorem 1.2**

Henceforth, we shall assume that

$$m \leq n + \epsilon_0, \tag{3.1}$$

so that

$$s_{m,n} = \frac{m - (n + \epsilon_0)}{2} \leq 0. \tag{3.2}$$

There is no loss of generality in assuming this, since we may switch the roles of $G(W)$ and $H(V)$ otherwise.

As mentioned at the end of the last section, we shall prove (HD) by induction on $\dim W$. In this section, we will give a proof of (HD) when at least one of π or $\pi' \in \mathrm{Irr}(G(W))$ does not occur on the boundary. Thus, we assume that π does not satisfy (2.3) or (2.4). It turns out that for this case, one can prove (HD) without appealing to the induction hypothesis. Namely, we shall prove

Theorem 3.1. *Assume that $m \leq n + \epsilon_0$ and suppose that $\pi \in \mathrm{Irr}(G(W))$ does not occur on the boundary of $\omega_{W,V}$. Then for any $\pi' \in \mathrm{Irr}(G(W))$,*

$$\dim \mathrm{Hom}_{H(V)}(\theta(\pi), \theta(\pi')) \leq \delta_{\pi,\pi'}.$$

In particular, $\theta(\pi)$ is either zero or irreducible, and moreover for any irreducible π',

$$0 \neq \theta(\pi) \subset \theta(\pi') \Longrightarrow \pi \cong \pi'.$$

Proof. First, we consider the following see-saw diagram

$$\begin{tikzcd} G(W+W^-) \arrow[d, dash] \arrow[dr, dash] & H(V)\times H(V) \arrow[d, dash] \arrow[dl, dash] \\ G(W)\times G(W^-) & H(V)^{\Delta}, \end{tikzcd}$$

where W^- denotes the space obtained from W by multiplying the form by -1, so that $G(W^-)=G(W)$. Given irreducible representations π and π' of $G(W)$, the see-saw identity [GI, §6.1] gives:

$$\mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,W+W^-}(\chi_W),\pi'\otimes\pi^\vee\chi_V)=\mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi')\otimes\Theta(\pi)^{MVW},\mathbb{C}). \tag{3.3}$$

Here $\Theta_{V,W+W^-}(\chi_W)$ denotes the big theta lift of the character χ_W of $H(V)$ to $G(W+W^-)$.

We analyze each side of the see-saw identity in turn. For the RHS, one has

$$\mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi')\otimes\Theta(\pi)^{MVW},\mathbb{C})\supseteq\mathrm{Hom}_{H(V)^\Delta}(\theta(\pi')\otimes\theta(\pi)^\vee,\mathbb{C})=\mathrm{Hom}_{H(V)}(\theta(\pi'),\theta(\pi)). \tag{3.4}$$

For the LHS of the see-saw identity (3.3), we need to understand $\Theta_{V,W+W^-}(\chi_W)$. It is known that $\Theta_{V,W+W^-}(\chi_W)$ is irreducible (see [GI, Prop. 7.2]). Moreover, it was shown by Rallis that

$$\Theta_{V,W+W^-}(\chi_W)\hookrightarrow\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^{s_{m,n}} \tag{3.5}$$

where

- $\Delta W\subset W+W^-$ is diagonally embedded and is a maximal isotropic subspace;
- $Q(\Delta W)$ is the maximal parabolic subgroup of $G(W+W^-)$ which stabilizes ΔW and has Levi factor $\mathrm{GL}(\Delta W)$;
- $\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^s$ denotes the degenerate principal series representation induced from the character $\chi_V|\det|^s$ of $Q(\Delta W)$ (normalized induction).

Since $s_{m,n}\le 0$, there is a surjective map (see [GI, Prop. 8.2])

$$\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^{-s_{m,n}}\longrightarrow\Theta_{V,W+W^-}(\chi_W).$$

Hence the see-saw identity (3.3) gives:

$$\mathrm{Hom}_{G(W)\times G(W)}(\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^{-s_{m,n}},\pi'\otimes\pi^\vee\chi_V)\supseteq\mathrm{Hom}_{H(V)}(\theta(\pi'),\theta(\pi)).$$

To prove the theorem, it suffices to show that the LHS has dimension ≤ 1, with equality only if $\pi=\pi'$.

For this, we need the following lemma (see [KR]):

Lemma 3.2. *As a representation of $G(W)\times G(W^-)$, $\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V\cdot|\det|^s$ possesses an equivariant filtration*

$$0\subset I_0\subset I_1\subset\cdots\subset I_q=\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V\cdot|\det|^s$$

with successive quotients

$$\begin{aligned} R_t &= I_t/I_{t-1} \\ &= \mathrm{Ind}_{Q_t\times Q_t}^{G(W)\times G(W^-)}\left(\left(\chi_V|\det_{X_t}|^{s+\frac{t}{2}}\boxtimes\chi_V|\det_{X_t}|^{s+\frac{t}{2}}\right)\otimes\left((\chi_V\circ\det_{W^-_{n-2t}})\otimes C_c^\infty(G(W_{n-2t}))\right)\right). \end{aligned}$$

Here, the induction is normalized and

- $\cdot$ *q is the Witt index of W;*
- $\cdot$ *Q_t is the maximal parabolic subgroup of $G(W)$ stabilizing a t-dimensional isotropic subspace X_t of W, with Levi subgroup $\mathrm{GL}(X_t)\times G(W_{n-2t})$, where $\dim W_{n-2t}=n-2t$.*
- $\cdot$ *$G(W_{n-2t})\times G(W_{n-2t})$ acts on $C_c^\infty(G(W_{n-2t}))$ by left-right translation.*

In particular,

$$R_0=(\chi_V\circ\det_{W^-})\otimes C_c^\infty(G(W)).$$

Applying the lemma, we claim that if π is not on the boundary, the natural restriction map

$$\mathrm{Hom}_{G(W)\times G(W)}(\mathrm{Ind}_{Q(\Delta W)}^{G(W+W^-)}\chi_V|\det|^{-s_{m,n}},\pi'\otimes\pi^\vee\chi_V)\longrightarrow\mathrm{Hom}_{G(W)\times G(W)}(R_0,\pi'\otimes\pi^\vee\chi_V)$$

is injective. This will imply the theorem since the RHS has dimension ≤ 1, with equality if and only if $\pi=\pi'$.

To deduce the claim, it suffices to to show that for each $0<t\leq q$,

$$\mathrm{Hom}_{G(W)\times G(W)}(R_t,\pi'\otimes\pi^\vee\chi_V)=0. \tag{3.6}$$

By Bernstein's Frobenius reciprocity, this Hom space is equal to

$$\begin{aligned}\mathrm{Hom}_{L(X_t)\times L(X_t)}&\Big(\big(\chi_V|\det|^{-s_{m,n}+\frac{t}{2}}\boxtimes\chi_V|\det|^{-s_{m,n}+\frac{t}{2}}\big)\\ &\otimes\Big((\chi_V\circ\det_{W^-_{n-2t}})\otimes C_c^\infty(G(W_{n-2t}))\Big),\ R_{\overline{Q}_t}(\pi')\otimes R_{\overline{Q}_t}(\pi^\vee\chi_V)\Big).\end{aligned}$$

Hence we deduce that the equation (3.6) holds if

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V|\det|^{-s_{m,n}+\frac{t}{2}},R_{\overline{Q}_t}(\pi^\vee\chi_V))=0.$$

By dualizing and Bernstein's Frobenius reciprocity, this is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(R_{Q_t}(\pi\cdot(\chi_V^{-1}\circ\det_W)),(\chi_V^{-1}\circ\det_{\mathrm{GL}(X_t)})\cdot|\det|^{s_{m,n}-\frac{t}{2}})=0.$$

Now note that

$$\chi_V\circ\det_W|_{\mathrm{GL}(X_t)}=\chi_V^2\circ\det_{\mathrm{GL}(X_t)}.$$

Hence the above condition is equivalent to

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(R_{Q_t}(\pi),\chi_V|\det|^{s_{m,n}-\frac{t}{2}})=0.$$

Since this condition holds when π is not on the boundary of $\omega_{W,V}$, the equation (3.6) is proved. This completes the proof of Theorem 3.1. □

Finally, let us note that the above argument also gives the following proposition, which we will use later.

Proposition 3.3. *Assume $m \leq n + \epsilon_0$. If $\pi \neq \pi' \in \mathrm{Irr}(G(W))$ are such that*

$$\mathrm{Hom}_{H(V)}(\theta_{W,V}(\pi), \theta_{W,V}(\pi')) \neq 0, \tag{3.7}$$

then there exists $t > 0$ such that

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n} - \frac{t}{2}} \rtimes \tau$$

and

$$\pi' \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n} - \frac{t}{2}} \rtimes \tau'$$

for some τ and $\tau' \in \mathrm{Irr}(G(W_{n-2t}))$.

Proof. If $\pi \neq \pi'$ but $\mathrm{Hom}_{H(V)}(\theta_{W,V}(\pi), \theta_{W,V}(\pi')) \neq 0$, then arguing as above one must have $\mathrm{Hom}_{G(W)\times G(W)}(R_t, \pi \otimes \pi'^{\vee}\chi_V) \neq 0$ for some $t > 0$, which gives the conclusion of the proposition. □

Of course, the hypothesis of this proposition contradicts the Howe duality (HD), and hence is never satisfied. So what this proposition is saying is that if (HD) is to be violated as in (3.7), it must be violated by representations π and π' which occur on the boundary for "the same t".

4. Proof of Theorem 1.2

To prove Theorem 1.2 or equivalently (HD), it remains to consider the case when both π and π' occur on the boundary of $\omega_{W,V}$. Recall that we are arguing by induction on $\dim W$. Thus, by induction hypothesis, we assume that Theorem 1.2 is known for dual pairs $G(W') \times H(V')$ with $\dim V' \leq \dim W' + \epsilon_0 < n + \epsilon_0$. Unlike the non-boundary case treated in Section 3, we shall make crucial use of the induction hypothesis in the boundary case treated in this section.

4.1. **An idea of Minguez.** Since π occurs on the boundary, we have

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}_{(X_t)}}|^{s_{m,n} - \frac{t}{2}} \rtimes \pi_0$$

for some $\pi_0 \in \mathrm{Irr}(G(W_{n-2t}))$ and some $t > 0$. By induction in stages, we have

$$\pi \hookrightarrow (\chi_V |-|^{s_{m,n} - t + \frac{1}{2}} \times \cdots \times \chi_V |-|^{s_{m,n} - \frac{1}{2}}) \rtimes \pi_0.$$

For simplicity, let us set

$$s_{m,n,t} = s_{m,n} - t + \frac{1}{2} < 0.$$

Now let us use a key idea of Minguez [M]. Let $a > 0$ be maximal such that

$$\pi \hookrightarrow \chi_V |-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma,$$

where

$$1^{\times a} = \underbrace{1 \times \cdots \times 1}_{a \text{ times}} = \mathrm{Ind}_B^{\mathrm{GL}_a} 1 \otimes \cdots \otimes 1$$

and σ is an irreducible representation of $G(W_{n-2a})$. To simplify notation, let us set

$$\sigma_a := \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma. \tag{4.1}$$

Note that the representation $1^{\times a}$ is irreducible and generic by [Z, Theorem 9.7].

Observe that if $\pi \hookrightarrow \sigma_a$ with $a > 0$ maximal, then

$$\sigma \not\subseteq \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'$$

for any σ'. In fact, the converse is also true, as we shall verify in Corollary 4.3(ii) below.

The chief reason for considering σ_a with $a > 0$ maximal is the following proposition:

Proposition 4.1. *Suppose $\sigma_a = \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$ is such that*

$$\sigma \not\subseteq \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'$$

for any σ'. Then σ_a has a unique irreducible submodule.

The proof of Proposition 4.1 relies on the following key technical lemma, which we state in slightly greater generality as it may be useful for other purposes. The proof of the lemma is deferred to the appendix.

Lemma 4.2. *Let ρ be a supercuspidal representation of $\mathrm{GL}_r(E)$ and consider the induced representation*

$$\sigma_{\rho,a} = \rho^{\times a} \rtimes \sigma$$

of $G(W_n)$ where σ is an irreducible representation of $G(W_{n-ra})$. Assume that

(a) ${}^c\rho^\vee \neq \rho$;
(b) $\sigma \not\subseteq \rho \rtimes \sigma_0$ *for any* σ_0.

Then we have the following:

(i) One has a natural short exact sequence

$$0 \longrightarrow T \longrightarrow R_{\overline{Q}(X_{ra})}\sigma_{\rho,a} \longrightarrow ({}^c\rho^\vee)^{\times a} \otimes \sigma \longrightarrow 0$$

and T does not contain any irreducible subquotient of the form $({}^c\rho^\vee)^{\times a} \rtimes \sigma'$ for any σ'. In particular, $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ contains $({}^c\rho^\vee)^{\times a} \otimes \sigma$ with multiplicity one, and does not contain any other subquotient of the form $({}^c\rho^\vee)^{\times a} \otimes \sigma'$. Likewise, $R_{Q(X_{ra})}\sigma_{\rho,a}$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $\rho^{\times a} \otimes \sigma'$.

(ii) The induced representation $\sigma_{\rho,a}$ has a unique irreducible submodule.

Proof of Proposition 4.1. We shall apply this lemma with

$$r = 1, \quad \rho = \chi_V|-|^{s_{m,n,t}} \quad \text{and} \quad \sigma_a = \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma.$$

Here, condition (a) holds since $s_{m,n,t} < 0$, whereas condition (b) holds by the maximality of a. This proves Proposition 4.1. $\square$

We shall have another occasion to use Lemma 4.2 later on. We also note the following corollary:

Corollary 4.3. *Suppose that* $\pi \hookrightarrow \sigma_a$ *and* $\sigma \not\subseteq \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'$ *for any* σ'.

(i) If $\pi \hookrightarrow \delta_a := \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \delta$ *for some* δ, *then* $\delta \cong \sigma$.

(ii) Moreover, a *is maximal with respect to the property that* $\pi \hookrightarrow \delta_a$ *for some irreducible* δ.

Proof. By the exactness of the Jacquet functor, $R_{Q(X_a)}(\pi)$ is a submodule of $R_{Q(X_a)}(\sigma_a)$. By Lemma 4.2, it follows that $R_{Q(X_a)}(\pi)$ contains $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma$ with multiplicity one, and does not contain any other subquotient of the form $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma'$ with $\sigma' \neq \sigma$. This key fact will imply both (i) and (ii).

(i) If $\pi \hookrightarrow \delta_a$, then $R_{Q(X_a)}(\pi)$ contains $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \delta$ as a quotient. By the key fact observed above, it follows that $\delta \cong \sigma$.

(ii) Suppose for the sake of contradiction that

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times (a+1)} \rtimes \sigma'$$

for some irreducible σ'. Then by induction in stages, one has

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \delta \quad \text{ with } \delta = \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'.$$

By the Frobenius reciprocity, one has a nonzero equivariant map

$$R_{Q(X_a)}(\pi) \longrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \delta.$$

By the key fact observed above, the image of this nonzero map must be isomorphic to $\chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma$. Hence,

$$\sigma \hookrightarrow \delta = \chi_V|-|^{s_{m,n,t}} \rtimes \sigma'$$

which is a contradiction to the hypothesis of the corollary. $\square$

4.2. **Proof for boundary cases.** We are now ready to launch into a relatively long computation needed to complete the proof of Theorem 1.2 for the representations on the boundary. The following is the key proposition:

Proposition 4.4. *Assume* $0 \neq \Pi \subset \theta(\pi)$ *and* Π *is irreducible.*

(i) If

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a *maximal (and for some* σ*), then*

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma$$

for some Σ *and where* a *is also maximal for* Π.

(ii) Moreover, whenever Π *is presented as a submodule as above, one has*

$$0 \neq \mathrm{Hom}_{G_n \times H_m}(\omega_{W_n,V_m}, \pi \otimes \Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a} \times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma \otimes \Sigma),$$

so that $\Sigma \subseteq \theta_{W_{n-2a},V_{m-2a}}(\sigma)$.

The main ingredient for the proof of this proposition is the following by-now well-known theorem due to Kudla [K1] on the Jacquet module of the Weil representation; see also [MVW].

Lemma 4.5. *The Jacquet module* $R_{Q(X_a)}(\omega_{W_n,V_m})$ *has an equivariant filtration*

$$R_{Q(X_a)}(\omega_{W_n,V_m}) = R^0 \supset R^1 \supset \cdots \supset R^a \supset R^{a+1} = 0$$

whose successive quotient $J^k = R^k/R^{k+1}$ *is described in* [GI, Lemma C.2]. *More precisely,*

$$J^k = \mathrm{Ind}_{Q(X_{a-k},X_a)\times G(W_{n-2a})\times P(Y_k)}^{\mathrm{GL}(X_a)\times G(W_{n-2a})\times H(V_m)} \left(\chi_V|\det_{X_{a-k}}|^{\lambda_{a-k}} \otimes C_c^\infty(\mathrm{GL}_k) \otimes \omega_{W_{n-2a},V_{m-2k}}\right),$$

where

- $\lambda_{a-k} = s_{m,n} + \frac{a-k}{2}$;
- $V_m = Y_k + V_{m-2k} + Y_k^*$ *with* Y_k *a* k*-dimensional isotropic space;*
- $X_a = X_{a-k} + X_k'$ *and* $Q(X_{a-k}, X_a)$ *is the maximal parabolic subgroup of* $\mathrm{GL}(X_a)$ *stabilizing* X_{a-k};
- $\mathrm{GL}(X_k) \times \mathrm{GL}(Y_k)$ *acts on* $C_c^\infty(\mathrm{GL}_k)$ *as*

  $$((b,c)\cdot f)(g) = \chi_V(\det b)\chi_W(\det c) f(c^{-1}gb)$$

  for $(b,c) \in \mathrm{GL}(X_k) \times \mathrm{GL}(Y_k)$, $f \in C_c^\infty(\mathrm{GL}_k)$ *and* $g \in \mathrm{GL}_k$.
- $J^k = 0$ *for* $k > \min\{a, q\}$, *where* q *is the Witt index of* V_m.

In particular, the bottom piece of the filtration (if nonzero) is:

$$J^a \cong \mathrm{Ind}_{\mathrm{GL}(X_a)\times G(W_{n-2a})\times P(Y_a)}^{\mathrm{GL}(X_a)\times G(W_{n-2a})\times H(V_m)} (C_c^\infty(\mathrm{GL}_a) \otimes \omega_{W_{n-2a},V_{m-2a}}).$$

With this one can prove the proposition.

Proof of Proposition 4.4. (i) Since $0 \neq \Pi \subseteq \theta(\pi)$, we have

$$\begin{aligned} 0 &\neq \mathrm{Hom}_{G_n\times H_m}(\omega_{W,V}, \pi \otimes \Pi) \\ &\hookrightarrow \mathrm{Hom}_{G_n\times H_m}(\omega_{W,V}, \sigma_a \otimes \Pi) \\ &= \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(R_{Q(X_a)}(\omega_{W,V}), \chi_V|-|^{s_{m,n,t}} 1^{\times a} \otimes \sigma \otimes \Pi), \end{aligned}$$

where we used the Frobenius reciprocity for the last step. Now the Jacquet module $R_{Q(X_a)}(\omega_{W,V})$ of the Weil representation is computed as in the lemma, which implies that there is a natural restriction map

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(R_{Q(X_a)}(\omega_{W,V}), \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma \otimes \Pi) \\ &\longrightarrow \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^a, \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma \otimes \Pi). \end{aligned}$$

We claim that this map is injective. To see this, it suffices to show that for all $0 \leq k < a$,

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^k, \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \otimes \sigma \otimes \Pi) = 0.$$

By the above lemma, this Hom space is equal to

$$\mathrm{Hom}_{M(X_{a-k},X_a)\times G_{n-2a}\times H_m}(\mathrm{Ind}_{P(Y_k)}^{H(V_m)}\chi_V|\det_{X_{a-k}}|^{\lambda_{a-k}}\otimes C_c^\infty(\mathrm{GL}_k)\otimes\omega_{W_{n-2a},V_{m-2k}},$$
$$R_{\overline{Q}(X_{a-k},X_a)}(\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a})\otimes\sigma\otimes\Pi),$$

where $M(X_{a-k},X_a)$ is the Levi factor of the parabolic subgroup of $\mathrm{GL}(X_a)$ stabilizing X_{a-k}. Because $1^{\times a}$ is generic, the second representation in this Hom space has a nonzero Whittaker functional when viewed as a representation of $\mathrm{GL}(X_{a-k})$ and hence the first one must also have a non-zero Whittaker functional, which is possible only when $a-k=1$. Therefore, if this Hom space is nonzero, we must have $a-k=1$. But in that case, one has:

$$\lambda_1 = s_{m,n}+\frac{1}{2} > s_{m,n}-t+\frac{1}{2} = s_{m,n,t},$$

so that the above Hom space is zero even when $a-k=1$.

Therefore we have $J^a\neq 0$ and

$$\begin{aligned}0 &\neq \mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_m}(J^a,\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\otimes\sigma\otimes\Pi)\\ &= \mathrm{Hom}_{H_m}(\chi_W|-|^{-s_{m,n,t}}\cdot 1^{\times a}\rtimes\Theta_{W_{n-2a},V_{m-2a}}(\sigma),\Pi).\end{aligned}$$

Dualizing and applying MVW along with (2.2) and (2.1), this shows that

$$\Pi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes(\Theta_{W_{n-2a},V_{m-2a}}(\sigma)^\vee)^{MVW},$$

and hence

$$\Pi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes\Sigma \tag{4.2}$$

for some irreducible representation Σ of $H(V_{m-2a})$ which is a subquotient of the representation $(\Theta_{W_{n-2a},V_{m-2a}}(\sigma)^\vee)^{MVW}$ and hence of $\Theta_{W_{n-2a},V_{m-2a}}(\sigma)$.

To prove (i), it remains to show that in (4.2), the integer a is maximal for Π. Let $b\geq a$ be maximal such that

$$\Pi\hookrightarrow\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes\Sigma_0$$

for some irreducible representation Σ_0 of $H(V_{m-2b})$. Then we have

$$\begin{aligned}0 &\neq \mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m},\pi\otimes\Pi)\\ &\hookrightarrow \mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m},\pi\otimes(\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes\Sigma_0))\\ &= \mathrm{Hom}_{G_n\times\mathrm{GL}(Y_b)\times H_{m-2b}}(R_{P(Y_b)}(\omega_{W_n,V_m}),\pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes\Sigma_0).\end{aligned} \tag{4.3}$$

We can compute the Jacquet module $R_{P(Y_b)}(\omega_{W_n,V_m})$ by using Lemma 4.5 with the roles of $H(V_m)$ and $G(W_n)$ switched. But for this, it should be noted that the exponent λ_{b-k} (for $k<b$) in Lemma 4.5 satisfies:

$$\lambda_{b-k} = -s_{m,n}+\frac{b-k}{2} > 0 > s_{m,n,t}. \tag{4.4}$$

Keeping this in mind, the last Hom space in (4.3) can be computed as

$$\begin{aligned}
&\mathrm{Hom}_{G_n\times \mathrm{GL}(Y_b)\times H_{m-2b}}(R_{P(Y_b)}(\omega_{W_n,V_m}),\pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes\Sigma_0)\\
\hookrightarrow\,&\mathrm{Hom}_{G_n\times \mathrm{GL}(Y_b)\times H_{m-2b}}(J^b,\pi\otimes\chi_W|-|^{s_{m,n,t}}\cdot 1^{\times b}\otimes\Sigma_0)\\
=\,&\mathrm{Hom}_{\mathrm{GL}(X_b)\times G_{n-2b}\times H_{m-2b}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\otimes\omega_{W_{n-2b},V_{m-2b}},R_{\overline{Q}(X_b)}(\pi)\otimes\Sigma_0)\\
=\,&\mathrm{Hom}_{\mathrm{GL}(X_b)\times G_{n-2b}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\otimes\Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0),R_{\overline{Q}(X_b)}(\pi))\\
=\,&\mathrm{Hom}_{G_n}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\rtimes\Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0),\pi)
\end{aligned}$$

where to obtain the second injection, we used the genericity of $1^{\times b}$ and (4.4) as before. Then again by dualizing and applying MVW, we have

$$\pi\hookrightarrow\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times b}\rtimes\sigma_0$$

for some σ_0 which is a subquotient of $\Theta_{W_{n-2b},V_{m-2b}}(\Sigma_0)$. By the maximality of a, we conclude that $b\le a$ and hence $b=a$. This completes the proof of (i).

(ii) Suppose that Π is given as in (4.2) with a maximal and some Σ. Now that we know that $b=a$ in the proof of (i), we revisit the computations starting from (4.3):

$$\begin{aligned}
0\neq\,&\mathrm{Hom}_{G_n\times H_m}\left(\omega_{W_n,V_m},\pi\otimes\Pi\right)\\
\hookrightarrow\,&\mathrm{Hom}_{\mathrm{GL}(X_b)\times G_{n-2b}\times H_{m-2b}}(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times b}\otimes\omega_{W_{n-2b},V_{m-2b}},R_{\overline{Q}(X_b)}(\pi)\otimes\Sigma)\\
\hookrightarrow\,&\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}\big(\chi_V|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\omega_{W_{n-2a},V_{m-2a}},\\
&\qquad\qquad R_{\overline{Q}(X_a)}(\chi_V|-|^{s_{m,n,t}}\cdot 1^{\times a}\rtimes\sigma)\otimes\Sigma\big).
\end{aligned}$$

To show the proposition, it suffices to show that the last Hom space embeds into

$$\mathrm{Hom}_{G_{n-2a}\times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}},\sigma\otimes\Sigma).$$

To show this inclusion, we shall make use of Lemma 4.2. In Lemma 4.2(i), set $\sigma_{\rho,a}=\sigma_a$, namely set $\rho=\chi_V|-|^{s_{m,n,t}}$. Tensoring the short exact sequence with Σ and then applying the functor

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}(\chi_V\cdot|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\omega_{W_{n-2a},V_{m-2a}},-),$$

one sees that the desired inclusion follows from the assertion:

$$\mathrm{Hom}_{\mathrm{GL}(X_a)\times G_{n-2a}\times H_{m-2a}}(\chi_V\cdot|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\omega_{W_{n-2a},V_{m-2a}},T\otimes\Sigma)=0.$$

But this follows from Lemma 4.2(ii) which asserts that T does not contain any irreducible subquotient of the form

$$\chi_W|-|^{-s_{m,n,t}}\cdot 1^{\times a}\otimes\Sigma'\quad\text{for any }\Sigma'.$$

This completes the proof of (ii). $\square$

4.3. **Proof of Theorem 1.2.** Now we are ready to finish our proof of Theorem 1.2.

Proof of Theorem 1.2. By Theorem 3.1, we assume π occurs on the boundary, so that

$$\pi \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \sigma$$

for some $t>0$ and some σ. First we will show that $\theta(\pi)$ (if nonzero) is irreducible. Let us write

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a maximal. Let $\Pi \subseteq \theta(\pi)$ be an irreducible submodule. But by Proposition 4.4,

$$0 \neq \mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m}, \pi\otimes\Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a}\times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma\otimes\Sigma),$$

where $\Sigma \subset \theta_{W_{n-2a},V_{m-2a}}(\sigma)$ is an irreducible representation such that

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma$$

where a is also maximal for Π. By induction hypothesis, we have $\Sigma = \theta_{W_{n-2a},V_{m-2a}}(\sigma)$, and hence

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \theta_{W_{n-2a},V_{m-2a}}(\sigma).$$

By Proposition 4.1, this induced representation has a unique submodule. This shows that $\theta(\pi)$ is an isotypic representation. Further, since

$$\dim \mathrm{Hom}_{G_{n-2a}\times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma\otimes\Sigma) = 1,$$

by the induction hypothesis, we conclude by Proposition 4.4(ii) that

$$\dim \mathrm{Hom}_{G_n\times H_m}(\omega_{W_n,V_m}, \pi\otimes\Pi) = 1.$$

This shows that Π occurs with multiplicity one in $\theta(\pi)$, so that $\theta(\pi)$ is irreducible.

It remains to prove that if $\theta(\pi)=\theta(\pi')=\Pi\neq 0$, then $\pi=\pi'$. By Proposition 3.3, this holds unless both π and π' occur on the boundary for the same t, namely there exists $t>0$ such that

$$\pi \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau$$

and

$$\pi' \hookrightarrow \chi_V|\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-\frac{t}{2}} \rtimes \tau'.$$

This means that we may write

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

and

$$\pi' \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a'} \rtimes \sigma'$$

with a and a' maximal (and for some σ and σ'). But then by Proposition 4.4(i) one must have $a=a'$, where a is maximal such that

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma \quad \text{for some } \Sigma.$$

Moreover, with Proposition 4.4(ii) and the induction hypothesis, we have

$$\theta_{W_{n-2a},V_{m-2a}}(\sigma) = \Sigma = \theta_{W_{n-2a},V_{m-2a}}(\sigma'),$$

so that $\sigma\cong\sigma'$. We then deduce by Proposition 4.1 that $\pi\cong\pi'$ is the unique irreducible submodule of $\sigma_a = \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$. This completes the proof of Theorem 1.2.

□

5. **Quaternionic Dual Pairs**

In this final section, we consider the case of the quaternionic dual pairs. As mentioned in the introduction, due to the lack of the MVW involution, we are not able to prove the Howe duality conjecture in full generality. The best we can prove is Theorem 1.3, where we consider only Hermitian representations (i.e. those π such that $\overline{\pi} \cong \pi^\vee$, where $\overline{\pi}$ is the complex conjugate of π). The idea of the proof is essentially the same as for the non-quaternionic case. But in place of the MVW involution, we use the involution

$$\pi \mapsto \overline{\pi}.$$

In what follows, we will outline how to modify the proof.

5.1. **Setup.** Let us briefly recall the setup in the quaternionic case, with emphasis on the aspects which are different from before.

Let B be the unique quaternion division algebra over F. For $\epsilon = \pm$, let $W = W_n$ be a rank n B-module equipped with a $-\epsilon$-Hermitian form and V_m a rank m B-module equipped with an ϵ-Hermitian form. Then the product $G(W_n) \times H(V_m)$ of isometry groups is a dual pair, with a Weil representation $\omega_{W,V}$ associated to a pair of splitting characters (χ_V, χ_W). In this case, the characters χ_V and χ_W are simply (possibly trivial) quadratic characters determined by the discriminants of the corresponding spaces V and W.

For an isotropic subspace X_t of rank t over B, let $Q(X_t)$ be the stabilizer of X_t, which is a maximal parabolic subgroup of $G(W)$ with the Levi factor $L(X_t) \cong \mathrm{GL}(X_t) \times G(W_{n-2t})$, where $\mathrm{GL}(X_t) \cong \mathrm{GL}_t(B)$. We shall denote by $\det_{\mathrm{GL}(X_t)} : \mathrm{GL}(X_t) \longrightarrow F^\times$ the reduced norm map. Likewise, a maximal parabolic subgroup $P(Y_t)$ of $H(V_m)$ is the stabilizer of an isotropic subspace Y_t of V_m. As before, we have Tadić's notation for parabolic induction.

We set

$$s_{m,n} = m - n + \frac{\epsilon}{2}.$$

In the quaternionic case, we say that an irreducible representation π of $G(W_n)$ lies on the boundary of $\omega_{W,V}$ if there exists $t > 0$ such that

$$\pi \hookrightarrow \chi_V |\det_{\mathrm{GL}(X_t)}|^{s_{m,n}-t} \rtimes \sigma$$

for some irreducible representation σ of $G(W_{n-2t})$. If $\overline{\pi} \cong \pi^\vee$, i.e. if π is Hermitian, then by dualizing and complex conjugating, we see that this is equivalent to:

$$\mathrm{Hom}_{\mathrm{GL}(X_t)}(\chi_V |\det_{\mathrm{GL}(X_t)}|^{-s_{m,n}+t}, R_{\overline{Q}(X_t)}(\pi)) \neq 0.$$

5.2. **Non-boundary case.** We can now begin the proof of Theorem 1.3, starting with the case when the Hermitian representation π does not lie on the boundary. For the sake of proving Theorem 1.3, there is no loss of generality in assuming that $m < n - \frac{\epsilon}{2}$, so that $s_{m,n} < 0$.

Now one can verify that all the arguments in Section 3 continue to work for a Hermitian π, with the following modifications:

- The first place where the MVW involution is used in Section 3 is the see-saw identity (3.3). But one can see from the proof of the see-saw identity in [GI, §6.1] that one has
$$\mathrm{Hom}_{G(W)\times G(W)}(\Theta_{V,W+W^-}(\chi_W), \pi' \otimes \pi^\vee) = \mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi') \otimes \overline{\Theta(\pi)}, \mathbb{C}).$$
Then (3.4) can be written as
$$\mathrm{Hom}_{H(V)^\Delta}(\Theta(\pi')\otimes\overline{\Theta(\pi)}, \mathbb{C}) \supseteq \mathrm{Hom}_{H(V)^\Delta}(\theta_{her}(\pi')\otimes\theta_{her}(\pi)^\vee, \mathbb{C}) = \mathrm{Hom}_{H(V)}(\theta_{her}(\pi'), \theta_{her}(\pi)).$$
- There is an evident analogue of Lemma 3.2 in the quaternionic case. The statement is as given in Lemma 3.2, except that the terms $|\det_{\mathrm{GL}(X_t)}|^{s+\frac{t}{2}}$ should be replaced by $|\det_{\mathrm{GL}(X_t)}|^{s+t}$.

With these provisions, the rest of the argument in Section 3 does not use the MVW involution, and hence apply to the quaternionic case without any modification. One also has the analogue of Proposition 3.3, with the exponent $s_{m,n} - \frac{t}{2}$ replaced by $s_{m,n} - t$.

5.3. **Boundary case.** Suppose that π is an irreducible Hermitian representation of $G(W)$ which lies on the boundary of $\omega_{W,V}$. Then, for some $t > 0$, one has
$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$
with a maximal (and for some σ) and
$$s_{m,n,t} = s_{m,n} - 2t + 1 < 0.$$
Now one has an analogue of Lemma 4.2, based on the explicit Geometric Lemma in the quaternionic case (which is written down in the thesis of M. Hanzer [Ha, Theorem 2.2.5]). Using this, one deduces the analogue of Proposition 4.1 by the same argument. However, it is essential to note the following lemma:

Lemma 5.1. *Suppose that*
$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$
with a maximal and for some σ. If π is Hermitian, so is σ.

Proof. To see this, starting from $\pi \hookrightarrow \rho^{\times a} \rtimes \sigma$ (with a maximal and ρ a real-valued 1-dimensional character for which $\rho^\vee \neq \rho$), one deduces (by dualizing and complex-conjugating) that
$$(\overline{\rho}^\vee)^{\times a} \rtimes \overline{\sigma}^\vee \twoheadrightarrow \overline{\pi}^\vee \cong \pi.$$
Now note that for the case at hand, the supercuspidal representation ρ satisfies $\overline{\rho} = \rho$; indeed, ρ is a real-valued character for our application. Thus, Bernstein's Frobenius reciprocity implies that
$$(\rho^\vee)^{\times a} \otimes \overline{\sigma}^\vee \hookrightarrow R_{\overline{Q}(X_{ra})}(\pi).$$

However, the analogue of Lemma 4.2(i) says that the only irreducible subquotient of $R_{\overline{Q}(X_{ra})}(\pi)$ of the form $(\rho^\vee)^{\times a} \otimes \sigma_0$ is $(\rho^\vee)^{\times a} \otimes \sigma$. Hence, we see that $\overline{\sigma}^\vee \cong \sigma$. $\square$

Then one has the following analogue of Proposition 4.4:

Proposition 5.2. *Assume that π is an irreducible Hermitian representation of $G(W)$ and and $0 \neq \Pi \subset \theta_{her}(\pi)$.*

(i) If

$$\pi \hookrightarrow \chi_V|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \sigma$$

with a maximal (and for some σ, necessarily Hermitian by Lemma 5.1), then

$$\Pi \hookrightarrow \chi_W|-|^{s_{m,n,t}} \cdot 1^{\times a} \rtimes \Sigma$$

for some Σ (necessarily Hermitian by Lemma 5.1) and where a is also maximal for Π.

(ii) Moreover, whenever Π is presented as a submodule as above, one has

$$0 \neq \mathrm{Hom}_{G_n \times H_m}(\omega_{W_n,V_m}, \pi \otimes \Pi) \hookrightarrow \mathrm{Hom}_{G_{n-2a} \times H_{m-2a}}(\omega_{W_{n-2a},V_{m-2a}}, \sigma \otimes \Sigma),$$

so that $\Sigma \subseteq \theta_{W_{n-2a},V_{m-2a},her}(\sigma)$.

Proposition 5.2 is proved by the same argument as that for Proposition 4.4, using the analogue of Lemma 4.5 (see [MVW, Chap. 3, Sect. IV, Thm. 5, Pg. 70]). We only take note that in the statement of Lemma 4.5, the quantity λ_{a-k} should be equal to $s_{m,n} + a - k$ in the quaternionic case.

With the above provisions, the rest of the proof goes through for the quaternionic case, which completes the proof of Theorem 1.3.

Appendix: Proof of Lemma 4.2

The goal of this appendix is to prove the technical Lemma 4.2. We restate the lemma here for the convenience of the reader.

Lemma 4.2. Let ρ be a supercuspidal representation of $\mathrm{GL}_r(E)$ and consider the induced representation

$$\sigma_{\rho,a} = \rho^{\times a} \rtimes \sigma$$

of $G(W_n)$ where σ is an irreducible representation of $G(W_{n-ra})$. Assume that

(a) ${}^c\rho^\vee \neq \rho$;
(b) $\sigma \nsubseteq \rho \rtimes \sigma_0$ for any σ_0.

Then we have the following:

(i) One has a natural short exact sequence

$$0 \longrightarrow T \longrightarrow R_{\overline{Q}(X_{ra})}\sigma_{\rho,a} \longrightarrow ({}^c\rho^\vee)^{\times a} \otimes \sigma \longrightarrow 0$$

and T does not contain any irreducible subquotient of the form $({}^c\rho^\vee)^{\times a} \rtimes \sigma'$ for any σ'. In particular, $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ contains $({}^c\rho^\vee)^{\times a} \otimes \sigma$ with multiplicity one, and does not contain any other subquotient of the form $({}^c\rho^\vee)^{\times a} \otimes \sigma'$. Likewise, $R_{Q(X_{ra})}\sigma_{\rho,a}$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one and does not contain any other subquotient of the form $\rho^{\times a} \otimes \sigma'$.

(ii) The induced representation $\sigma_{\rho,a}$ has a unique irreducible submodule.

Proof. We shall use an explication of the Geometric Lemma of Bernstein-Zelevinsky due to Tadić [T, Lemmas 5.1 and 6.3]. (See [HM] for the metaplectic group.) Tadić's results imply that any irreducible subquotient $\delta \otimes \sigma'$ of $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$ is obtained in the following way.

For any partition $k_1 + k_2 + k_3 = ra$, write the semisimplification of the normalized Jacquet module of $\rho^{\times a}$ to the Levi subgroup $\mathrm{GL}_{k_1} \times \mathrm{GL}_{k_2} \times \mathrm{GL}_{k_3}$ as a sum of $\delta_1 \otimes \delta_2 \otimes \delta_3$. Similarly, write the semisimplification of the normalized Jacquet module of σ to the Levi subgroup $\mathrm{GL}_{k_2} \times G(W_{n-2ra-2k_2})$ as a sum of $\delta_4 \otimes \sigma_5$. Then δ is a subquotient of $\delta_3 \times^c \delta_1^\vee \times^c \delta_4^\vee$ whereas σ' is a subquotient of $\delta_2 \otimes \sigma_5$.

For the case at hand, since ρ is supercuspidal, we can assume the partition of ra is of the form $rk_1 + rk_2 + rk_3 = ra$, and the (semisimplified) normalized Jacquet module of $\rho^{\times a}$ is the isotypic sum of $\rho^{\times k_1} \otimes \rho^{\times k_2} \otimes \rho^{\times k_3}$. Hence we see that for any irreducible subquotient $\delta \otimes \sigma'$ of $R_{\overline{Q}(X_{ra})}\sigma_{\rho,a}$, δ is a subquotient of $\rho^{\times k_3} \times ({}^c\rho^{\times k_1})^\vee \times^c \delta_4^\vee$.

Now the irreducible subquotients of T correspond to those partitions with $k_2 > 0$ or $k_3 > 0$. (Note that the case $k_2 = k_3 = 0$ corresponds to the closed cell in $Q\backslash G/\overline{Q}$, which gives the third term in the short exact sequence.) The conditions (a) and (b) then imply that $\delta \neq ({}^c\rho^\vee)^{\times a}$. This proves the statements about T in (i). Now $Q(X_{ra})$ and $\overline{Q}(X_{ra})$ are conjugate in $G(W)$ by an element w which normalizes the Levi subgroup $L(X_{ra}) = \mathrm{GL}(X_{ra}) \times G(W_{n-2ra})$, acting as the identity on $G(W_{n-2ra})$ and via $g \mapsto {}^c({}^tg^{-1})$ on $\mathrm{GL}(X_{ra})$. Then one has

$$^{w}R_{\overline{Q}(X_{ra})}(\sigma_{\rho,a}) = R_{Q(X_{ra})}(\sigma_{\rho,a})$$

where the LHS is the representation of the Levi $L(X_{ra})$ obtained by twisting $R_{\overline{Q}(X_{ra})}(\sigma_{\rho,a})$ by w. Hence, one deduces that $R_{Q(X_{ra})}(\sigma_{\rho,a})$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity one.

Finally, for (ii), let $\pi \subseteq \sigma_{\rho,a}$ be any irreducible submodule. Then Frobenius reciprocity implies that the semisimplification of $R_{Q(X_{ra})}(\pi)$ contains $\rho^{\times a} \otimes \sigma$. Thus, if $\sigma_{\rho,a}$ contains more than one irreducible submodule, the exactness of the Jacquet functor implies that $R_{Q(X_{ra})}(\sigma_{\rho,a})$ contains $\rho^{\times a} \otimes \sigma$ with multiplicity ≥ 2, which contradicts (i). □

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