

# Hilbert series, Howe duality, and branching rules

Thomas J. Enright\* and Jeb F. Willenbring†‡§

\*Department of Mathematics, University of California at San Diego, La Jolla, CA 92093; and †Mathematics Department, Yale University, New Haven, CT 06520

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Let  $\lambda$  be a partition, with  $l$  parts, and let  $F^\lambda$  be the irreducible finite dimensional representation of  $GL(m)$  associated to  $\lambda$  when  $l \leq m$ . The Littlewood Restriction Rule describes how  $F^\lambda$  decomposes when restricted to the orthogonal group  $O(m)$  or to the symplectic group  $Sp(m/2)$  under the condition that  $l \leq m/2$ . In this paper, this result is extended to all partitions  $\lambda$ . Our method combines resolutions of unitary highest weight modules by generalized Verma modules with reciprocity laws from the theory of dual pairs in classical invariant theory. Corollaries include determination of the Gelfand–Kirillov dimension of any unitary highest weight representation occurring in a dual pair setting, and the determination of their Hilbert series (as a graded module for  $\mathfrak{p}^-$ ). Let  $L$  be a unitary highest weight representation of  $\mathfrak{sp}(n, \mathbb{R})$ ,  $\mathfrak{so}^*(2n)$ , or  $\mathfrak{u}(p, q)$ . When the highest weight of  $L$  plus  $\rho$  satisfies a partial dominance condition called quasi-dominance, we associate to  $L$  a reductive Lie algebra  $\mathfrak{g}_L$  and a graded finite dimensional representation  $B_L$  of  $\mathfrak{g}_L$ . The representation  $B_L$  will have a Hilbert series  $P(q)$  that is a polynomial in  $q$  with positive integer coefficients. Let  $\delta(L) = \delta$  be the Gelfand–Kirillov dimension of  $L$  and set  $c_L$  equal to the ratio of the dimensions of the zeroth levels in the gradings of  $L$  and  $B_L$ . Then the Hilbert series of  $L$  may be expressed in the form

$$H_L(q) = c_L \frac{P(q)}{(1-q)^\delta}.$$

In the easiest example of the correspondence  $L \rightarrow B_L$ , the two components of the Weil representation of the symplectic group correspond to the two spin representations of an orthogonal group.

## 1. Extending the Littlewood Restriction Rule

Let  $V$  be a complex vector space of dimension  $n$  with a nondegenerate symmetric or skew symmetric form. The group leaving the form invariant is the group  $G$ , the orthogonal group  $O(n)$ , or the symplectic group  $Sp(n/2)$  when  $n$  is even. The representations  $F^\lambda$  of  $GL(V)$  are parameterized by the partitions  $\lambda$  with at most  $n$  parts. For any integer partition  $\lambda = (\lambda_1 \geq \dots \geq \lambda_l)$  with at most  $l$  parts, let  $F_{(l)}^\lambda$  be the irreducible representation of  $GL(l)$  indexed in the usual way by its highest weight (see ref. 1, section 5.2.1). Similarly, for each nonnegative integer partition  $\mu$  with at most  $l$  parts, let  $V_{(l)}^\mu$  be the irreducible representation of  $Sp(l)$  with highest weight  $\mu$ . Let  $E_{(l)}^\nu$  denote the irreducible representation of  $O(l)$  associated to the nonnegative integer partition  $\nu$  with at most  $l$  parts and having Young diagram whose first two columns have lengths that sum to  $l$  or less. The parameters in the  $O(l)$  case are somewhat more delicate than the earlier two cases due in part to the disconnectedness of the group. For more details consult ref. 1 (chapter 10).

In 1940 D. E. Littlewood gave a formula for the decomposition of  $F^\lambda$  as a representation of  $G$  by restriction.

**Theorem 1 [Littlewood Restriction (2, 3)].** Suppose that  $\lambda$  is a partition having at most  $n/2$  (positive) parts.

(i) Suppose  $n$  is even and set  $k = n/2$ . Then the multiplicity of the  $Sp(k)$  representation  $V^\mu$  in  $F^\lambda$  equals

$$\sum_{\xi} \dim \operatorname{Hom}_{GL(n)}(F^\lambda, F^\mu \otimes F^\xi), \quad [1.1]$$

where the sum is over all nonnegative integer partitions  $\xi$  with columns of even length.

(ii) Then, for all  $n$  the multiplicity of the  $O(n)$  representation  $E^\nu$  in  $F^\lambda$  equals

$$\sum_{\xi} \dim \operatorname{Hom}_{GL(n)}(F^\lambda, F^\nu \otimes F^\xi), \quad [1.2]$$

where the sum is over all nonnegative integer partitions  $\xi$  with rows of even length.

Recently, Gavarini (4) used Brauer algebra methods to reprove and slightly extend Littlewood's formula. In this announcement, we describe some new results in character theory and an interpretation of these results through Howe duality. This will yield yet another proof of the Littlewood Restriction Rule and, more importantly, a generalization valid for all parameters  $\lambda$ . Our formula is rather more complex in form (see Section 5) but reduces to the above formula when  $\lambda$  is a partition having at most  $n/2$  (positive) parts.

## 2. Gelfand–Kirillov Dimension

The character theory mentioned above is for unitarizable highest weight representations of a real reductive group. Much is known about these representations. They were classified in 1980 (5, 6). Their characters and cohomology formulas were determined a decade later (7–9). Here we report some refinements to this character theory and focus on a set of coarser invariants that can be determined more precisely. These are the Hilbert series, Gelfand–Kirillov dimension, and Bernstein degree.

Let  $L$  denote a unitarizable highest weight representation for  $\mathfrak{g}$ , one of the classical Lie algebras  $\mathfrak{su}(p, q)$ ,  $\mathfrak{sp}(n, \mathbb{R})$ , or  $\mathfrak{so}^*(2n)$ . These Lie algebras occur as part of the reductive dual pairs (10):

- (i)  $Sp(k) \times \mathfrak{so}^*(2n)$  acting on  $\mathcal{P}(M_{2k \times n})$
- (ii)  $O(k) \times \mathfrak{sp}(n)$  acting on  $\mathcal{P}(M_{k \times n})$
- (iii)  $U(k) \times \mathfrak{u}(p, q)$  acting on  $\mathcal{P}(M_{k \times n})$ ,

[2.1]

where  $n = p + q$ . Let  $GKdim$  denote the Gelfand–Kirillov dimension. Our first main result is as follows.

**Theorem 2.** Suppose that  $L$  is a unitarizable highest weight representation occurring in one of the dual pair settings (2.1).

- If  $\mathfrak{g}$  is  $\mathfrak{so}^*(2n)$ , then  $GKdim(L)$  equals  $k(2n - 2k - 1)$  for  $1 \leq k \leq [n - 2/2]$  and equals  $\binom{n}{2}$  otherwise.
- If  $\mathfrak{g}$  is  $\mathfrak{sp}(n)$ , then  $GKdim(L)$  equals  $k/2(2n - k + 1)$  for  $1 \leq k \leq n - 1$  and equals  $\binom{n+1}{2}$  otherwise.
- If  $\mathfrak{g}$  is  $\mathfrak{u}(p, q)$ , then  $GKdim(L)$  equals  $k(n - k)$  for  $1 \leq k \leq \min\{p, q\}$  and equals  $pq$  otherwise.

Note that in all cases the  $GK$  dimension depends only on the dual pair setting given by  $k$  and  $n$  and is independent of  $\lambda$  otherwise. It is of course convenient to compute  $GK$  dimension of  $L$  directly from the highest weight. Let  $\beta$  denote the maximal root of  $\mathfrak{g}$ .

†J.F.W. conducted this research in part for the Clay Mathematics Institute.

§To whom correspondence should be addressed. E-mail: willenbring@math.yale.edu.

**Corollary 3.** Set  $s = -(2(\lambda, \beta)/(\beta, \beta))$ . Then for  $\mathfrak{so}^*(2n)$ ,

$$GKdim(L) = \begin{cases} \frac{s}{2}(2n - s - 1) & \text{for } 2 \leq s \leq 2\left[\frac{n}{2}\right] - 2 \\ \binom{n}{2} & \text{otherwise,} \end{cases}$$

for  $\mathfrak{sp}(n)$ ,

$$GKdim(L) = \begin{cases} s(2n - 2s + 1) & \text{for } 1 \leq 2s \leq n \\ \binom{n+1}{2} & \text{otherwise} \end{cases}$$

and for  $\mathfrak{u}(p, q)$  with  $n = p + q$ ,

$$GKdim(L) = \begin{cases} s(n - s) & \text{for } 1 \leq s \leq \min\{p, q\} \\ pq & \text{otherwise.} \end{cases}$$

The referee has noted that *Theorem 2* is known to experts.

### 3. Hilbert Series

For any Hermitian symmetric pair  $\mathfrak{g}, \mathfrak{k}$ , let  $\mathfrak{h}$  be a Cartan subalgebra of both  $\mathfrak{k}$  and  $\mathfrak{g}$  and let  $\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+$  be a  $\mathfrak{k}$ -module decomposition of  $\mathfrak{g}$ . Choose  $\Delta^+$  a set of positive roots for the root system  $\Delta$  such that  $\mathfrak{p}^+$  is the span of the positive noncompact root spaces. Set  $\rho$  equal to half the sum of all the positive roots and let  $\mathcal{W}$  (resp.  $\mathcal{W}_{\mathfrak{k}}$ ) denote the Weyl group of  $\mathfrak{g}$  (resp.  $\mathfrak{k}$ ).

Suppose  $L$  is a unitarizable highest weight representation with highest weight  $\lambda$ . In ref. 9, one associates to  $\lambda$  a subgroup  $\mathcal{W}_{\lambda}$  of  $\mathcal{W}$ , a root system  $\Delta_{\lambda}$ , and a Hermitian symmetric pair  $\mathfrak{g}_{\lambda}, \mathfrak{k}_{\lambda}$  as follows. Let  $\Delta_n^+$  denote the roots with root spaces in  $\mathfrak{p}^+$ . For any root  $\alpha$  let  $\alpha^\vee$  denote the coroot defined by  $(\alpha^\vee, \xi) = 2(\alpha, \xi)/(\alpha, \alpha)$ . Suppose  $\lambda$  is  $\mathfrak{k}$ -dominant and define  $\mathcal{W}_{\lambda}$  to be the subgroup of the Weyl group  $\mathcal{W}$  generated by the identity and all the reflections  $r_{\alpha}$  that satisfy the following three conditions:

- (i)  $\alpha \in \Delta_n^+$  and  $(\lambda + \rho, \alpha^\vee) \in \mathbb{N}^*$ .
- (ii) If  $\delta \in \Delta$  and  $(\lambda + \rho, \delta) = 0$  then  $(\alpha, \delta) = 0$ .
- (iii) If  $\delta \in \Delta$  is long and  $(\lambda + \rho, \delta) = 0$  then  $\alpha$  is short.

[3.1]

When the root system  $\Delta$  contains only one root length, we call the roots short. Let  $\Delta_{\lambda}$  equal the subset of  $\Delta$  of elements  $\delta$  for which  $r_{\delta} \in \mathcal{W}_{\lambda}$  and let  $\Delta_{\lambda, \mathfrak{k}} = \Delta_{\lambda} \cap \Delta_{\mathfrak{k}}$ ,  $\Delta_{\lambda}^+ = \Delta_{\lambda} \cap \Delta^+$  and  $\Delta_{\lambda, \mathfrak{k}}^+ = \Delta_{\lambda, \mathfrak{k}} \cap \Delta^+$ . Then in our setting  $\Delta_{\lambda}$  and  $\Delta_{\lambda, \mathfrak{k}}$  are abstract root systems and we let  $\mathfrak{g}_{\lambda}$  (resp.  $\mathfrak{k}_{\lambda}$ ) denote the reductive Lie algebra with root system  $\Delta_{\lambda}$  (resp.  $\Delta_{\lambda, \mathfrak{k}}$ ) and Cartan subalgebra equal to  $\mathfrak{h}$ . Then the pair  $(\mathfrak{g}_{\lambda}, \mathfrak{k}_{\lambda})$  is a Hermitian symmetric pair [although not necessarily of the same type as  $(\mathfrak{g}, \mathfrak{k})$  as is indicated by the example of the Weil representation below in Section 4, where the first pair is  $(D_n, A_{n-1})$  and the second is  $(C_n, A_{n-1})$ ]. Set  $\mathcal{W}_{\lambda, \mathfrak{k}} = \mathcal{W}_{\lambda} \cap \mathcal{W}_{\mathfrak{k}}$  and define:

$$\begin{aligned} \mathcal{W}_{\lambda}^{\mathfrak{k}} &= \{x \in \mathcal{W}_{\lambda} \mid x\Delta_{\lambda}^+ \supset \Delta_{\lambda, \mathfrak{k}}^+\} \text{ and} \\ \mathcal{W}_{\lambda}^{i, \mathfrak{k}} &= \{x \in \mathcal{W}_{\lambda}^{\mathfrak{k}} \mid \text{card}(x\Delta_{\lambda}^+ \cap -\Delta_n^+) = i\}. \end{aligned} \quad [3.2]$$

Let  $\rho_{\lambda}$  denote half the sum of the positive roots of  $\mathfrak{g}_{\lambda}$  and set  $\delta_{\lambda} = \rho - \rho_{\lambda}$ . For any  $\mathfrak{k}$ -integral  $\xi \in \mathfrak{h}^*$ , let  $\xi^{++}$  denote the unique element in the  $\mathcal{W}_{\mathfrak{k}}$ -orbit of  $\xi$  which is  $\Delta_{\mathfrak{k}}^+$  dominant. For any  $\mathfrak{k}$ -dominant integral weight  $\lambda$  define the generalized Verma module with highest weight  $\lambda$  to be the induced module defined by:  $N(\lambda + \rho) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k} \oplus \mathfrak{p}^+)} E_{\lambda}$ , with  $E_{\lambda}$  is the irreducible  $\mathfrak{k}$  representation with highest weight  $\lambda$ . For any  $\mathfrak{k}$ -dominant integral weights  $\xi$  and  $\zeta$ , let  $m_{\xi}^{\zeta}$  denote the multiplicity of the  $\mathfrak{k}$  representation  $E_{\xi}$  in the  $\mathfrak{g}$  generalized Verma module  $N(\zeta + \rho)$ . If  $S(\mathfrak{p}^-)$  denotes the symmetric tensor algebra of  $\mathfrak{p}^-$ , then

$$m_{\xi}^{\zeta} = \dim \text{Hom}_{\mathfrak{k}}(E_{\xi}, E_{\zeta} \otimes S(\mathfrak{p}^-)). \quad [3.3]$$

We let  $L(\lambda + \rho)$  denote the unique irreducible quotient of  $N(\lambda + \rho)$ , which is the irreducible highest weight representation of  $\mathfrak{g}$  with highest weight  $\lambda$ .

**Remark 1:** We have used three separate notations that overlap in the case of finite dimensional representations. The predominant notation for irreducible finite dimensional representations is by highest weight. However, in some classical settings, we switch to coordinate versions of highest weights that are then identified with integer partitions. Finally, when we are considering highest weight  $\mathfrak{g}$ -modules we use the generalized Verma module parameters of highest weight plus  $\rho$ .

Suppose  $\lambda$  is  $\Delta_{\mathfrak{k}}^+$ -dominant and let  $\Sigma_{\lambda}$  denote the set of all positive roots  $\alpha$  where  $(\alpha, \lambda + \rho) = 0$ . Then  $\Sigma_{\lambda}$  is a set of strongly orthogonal noncompact roots. For such  $\lambda$ , we say  $\lambda + \rho$  is *quasi-dominant* if  $(\alpha, \lambda + \rho) > 0$  for all positive roots  $\alpha$  that are orthogonal to all elements of  $\Sigma_{\lambda}$ .

For any Hermitian symmetric pair  $\mathfrak{g}, \mathfrak{k}$  and irreducible highest weight  $\mathfrak{g}$ -module  $M$ , let  $M_0$  denote the  $\mathfrak{k}$ -submodule generated by any highest weight vector. Set  $M_j = \mathfrak{p}^- \cdot M_{j-1}$  for  $j > 0$ . Define the Hilbert series  $H_M(q)$  of  $M$  by

$$H_M(q) = \sum_{j \geq 0} \dim M_j q^j. \quad [3.4]$$

Because the enveloping algebra of  $\mathfrak{p}^-$  is Noetherian, there is a unique integer  $d$  and a unique polynomial  $R_M(q)$  such that

$$H_M(q) = \frac{R_M(q)}{(1 - q)^d}, \quad \text{where } R_M(q) = \sum_{0 \leq j \leq e} a_j q^j. \quad [3.5]$$

In this setting, the integer  $d$  is the Gelfand–Kirillov dimension,  $d = GKdim M$ . Expanding the denominator as a series, we obtain

$$H_M(q) = \sum_{s, j} a_j \binom{d - 1 + s}{d - 1} q^{s+j}. \quad [3.6]$$

The coefficient of  $q^t$  can be expressed as a polynomial in  $t$  for large values of  $t$ . The leading coefficient of this polynomial is  $1/(d - 1)! \sum_{0 \leq j \leq e} a_j = R_M(1)/(d - 1)!$  and is called the Bernstein degree of  $M$  and denoted  $B \deg(M)$ .

For any  $\mathfrak{g}_{\lambda}$ -dominant integral  $\mu$  we let  $B_{\mu}$  denote the irreducible finite dimensional  $\mathfrak{g}_{\lambda}$ -module with highest weight  $\mu$ . Let  $\mathfrak{p}_{\lambda}^-$  denote the subalgebra of  $\mathfrak{p}^-$  spanned by the root spaces for roots in  $-\Delta_n^+ \cap \Delta_{\lambda}$  and let  $B_{\mu}^i$  denote the grading of  $B_{\mu}$  as a  $\mathfrak{p}_{\lambda}^-$ -module. Define the Hilbert series by

$$P(q) = P_{\lambda}(q) = \sum \dim B_{\lambda}^i q^i. \quad [3.7]$$

This is of course a polynomial. For any  $\mathfrak{k}$ -dominant integral weight  $\mu$ , let  $E_{\mu}$  (resp.  $E_{\mathfrak{k}_{\lambda, \mu}}$ ) denote the irreducible finite dimensional  $\mathfrak{k}$  (resp.  $\mathfrak{k}_{\lambda}$ )-module with highest weight  $\mu$ .

**Theorem 4.** Suppose that  $L = L(\lambda + \rho)$  is a unitarizable representation occurring in one of the dual pair settings (2.1) and that  $\lambda + \rho$  is quasi-dominant. Set  $d$  equal to the GK dimension of  $L$  as given by Theorem 2. Then the Hilbert series of  $L$  is

$$H_L(q) = \frac{\dim E_{\lambda}}{\dim E_{\mathfrak{k}_{\lambda, \lambda + \delta_{\lambda}}}} \frac{P_{\lambda + \delta_{\lambda}}(q)}{(1 - q)^d}. \quad [3.8]$$

In particular, the Bernstein degree of  $L$  is given by

$$B \deg(L) = \frac{\dim E_{\lambda}}{\dim E_{\mathfrak{k}_{\lambda, \lambda + \delta_{\lambda}}}} \dim B_{\lambda + \delta_{\lambda}}. \quad [3.9]$$

To illustrate through an example all the parameters, roots, Hermitian symmetric pairs, and finite dimensional representations involved, we consider a special set of representations, the Wallach representations (see ref. 11). Let  $r$  equal the split rank of  $\mathfrak{g}$ , and let  $\zeta$  be the fundamental weight of  $\mathfrak{g}$  that is orthogonal to all the roots of  $\mathfrak{k}$ . Suppose  $\mathfrak{g}$  is isomorphic to  $\mathfrak{so}^*(2n)$ ,  $\mathfrak{sp}(n)$ , or  $\mathfrak{su}(p, q)$  and set  $c = 2, 1/2$ , or  $1$ , depending on which of the three cases we are in. For  $1 \leq j < r$  define the  $j$ th Wallach representation  $W_j$  to be the unitarizable highest weight representation with highest weight  $-jc\zeta$ .

Suppose  $L$  is a Wallach representation for  $\mathfrak{sp}(n)$  having highest weight  $\lambda = -(j/2)(1, 1, \dots, 1)$  for some  $j$ ,  $1 \leq j \leq n - 1$ . Then

$$\lambda + \rho = \left( n - \frac{j}{2}, n - 1 - \frac{j}{2}, \dots, 1 - \frac{j}{2} \right),$$

the set of strongly orthogonal roots

$$\Sigma_\lambda = \left\{ e_{n+1-j+l} + e_{n+1-l} \mid 1 \leq l \leq \left\lfloor \frac{j}{2} \right\rfloor \right\}$$

and  $\Delta_\lambda$  is the root system of type  $D_{n+1-j}$  with simple roots

$$\{e_1 - e_2, \dots, e_{n-j} - e_{n+1-j}, e_{n-j} + e_{n+1-j}\},$$

and the Hermitian symmetric pair  $\mathfrak{g}_\lambda, \mathfrak{k}_\lambda$  is  $\mathfrak{so}^*(2n + 2 - 2j)$ . Here  $\Sigma_\lambda$  contains a long root precisely when  $j$  is even and for all  $j$ ,  $\lambda + \rho$  is quasi-dominant. The first  $n + 1 - j$  coordinates of  $\lambda + \rho - \rho_\lambda$  all equal  $j/2$  and so this weight  $\lambda + \rho - \rho_\lambda$  equals  $j$  times the last fundamental weight of  $D_{n+1-j}$  on  $[\mathfrak{g}_\lambda, \mathfrak{g}_\lambda] \cap \mathfrak{h}$  plus a central character of  $\mathfrak{g}_\lambda$ . Also in this example the dimensions of  $E_\lambda$  and  $E_{\mathfrak{k}_\lambda, \lambda + \delta_\lambda}$  are both one.

#### 4. Examples

We next apply *Theorem 4* to some well known representations to determine their GKdim, Hilbert series and Bernstein degree. We begin with the Wallach representations. For  $\mathfrak{so}^*(2n)$  the Hilbert series for the first Wallach representation is

$$H_L(q) = \frac{R(q)}{(1-q)^{2n-3}} = \quad [4.1]$$

$$\frac{1}{(1-q)^{2n-3}} \frac{1}{n-2} \sum_{0 \leq s \leq n-3} \binom{n-2}{s} \binom{n-2}{s+1} q^s. \quad [4.2]$$

For  $\mathfrak{sp}(n)$  the Hilbert series for the first Wallach representation is

$$H_L(q) = \frac{1}{(1-q)^n} \sum_{0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2t} q^t. \quad [4.3]$$

This is the Hilbert series for the half of the Weil representation generated by a one dimensional representation of  $\mathfrak{k}$ . The other part of the Weil representation has Hilbert series

$$H_L(q) = \frac{1}{(1-q)^n} \sum_{0 \leq t \leq \left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2t+1} q^t. \quad [4.4]$$

For the components of the Weil representation of  $\mathfrak{sp}(n)$  there is an elementary derivation of the Hilbert series that we now give for comparison. Let  $L_e$  and  $L_o$  denote the even and odd components of the Weil representation acting respectively on the even and odd degree polynomials in  $n$  variables. Then

$$\frac{(1+q)^n}{(1-q^2)^n} = \frac{1}{(1-q)^n} = H_{L_e}(q^2) + qH_{L_o}(q^2). \quad [4.5]$$

Breaking the numerator  $(1+q)^n$  into its even and odd degree parts in  $q$  gives the expressions **4.2** and **4.3**.

For  $U(p, q)$  the Hilbert series for the first Wallach representation is:

$$H_L(q) = \frac{1}{(1-q)^{n-1}} \sum_{0 \leq t < \min\{p, q\}} \binom{p-1}{t} \binom{q-1}{t} q^t. \quad [4.6]$$

These examples are obtained from *Corollary 3* and *Theorem 4* by writing out, respectively, the Hilbert series of the  $n - 3$ rd exterior power of the standard representation of  $\mathfrak{so}^*(2n - 4)$ , the two components of the spin representation of  $\mathfrak{so}^*(2n)$ , and the  $p - 1$ st fundamental representation of  $U(p - 1, q - 1)$ . Combinatorial expressions for the Hilbert series of other Wallach representations can be given as well.

We say a highest weight representation  $L$  is positive if all the nonzero coefficients of the polynomial  $R_L(q)$  in **3.5** are positive. All Cohen–Macaulay  $S(\mathfrak{p}^-)$ -modules including the Wallach representations are positive but many unitary highest weight representations are not. So *Theorem 4* introduces a large class of positive highest weight representations that include all the examples in the previous paragraph. This theorem asserts that if  $\lambda + \rho$  is quasi-dominant then  $L(\lambda + \rho)$  is positive.

We conjecture that among positive highest weight modules occurring in the dual pair setting that are nontrivial quotients of generalized Verma modules, the modules with quasi-dominant highest weights prevail. However, the conditions of positivity and quasi-dominance are not equivalent. The former properly contain the latter. There are irreducible positive generalized Verma modules with highest weights which are not quasi-dominant. We would like to know to what extent the conditions of positivity and quasi-dominance are related.

#### 5. Reciprocity Laws and Resolutions

The refinements in the character theory for unitarizable highest weight representations discussed above lead to new finite dimensional branching formulas extending the work of Littlewood (2). Let  $\mathcal{S} = \mathcal{P}(M_{2k \times n})$  or  $\mathcal{P}(M_{k \times n})$  as in **2.1**. We consider the action of two dual pairs on  $\mathcal{S}$ . The first is  $GL(m) \times GL(n)$  with  $m = 2k$  or  $k$  and the second is  $G_1 \times \mathfrak{g}_2$ , one of the two pairs (i) or (ii) in **2.1**. In this setting  $G_1$  is contained in  $GL(m)$  while  $GL(n)$  is closely related to the maximal compact subgroup of  $\mathfrak{g}_2$ . We can calculate the multiplicity of an irreducible  $G_1 \times GL(n)$  representation in  $\mathcal{S}$  in two ways. The resulting identity is the branching formula. These reciprocity laws have been studied by Howe (12) where the two dual pairs are called see-saw pairs in the sense of Kudla.

The theory of dual pairs gives a decompositions of  $\mathcal{S}$  for each of the three actions.

As a  $GL(m) \times GL(n)$  representation,

$$\mathcal{S} = \sum_{\lambda} F_{(m)}^{\lambda} \otimes F_{(n)}^{\lambda}, \quad [5.1]$$

where the sum is over all nonnegative integer partitions having  $\min\{m, n\}$  or fewer parts.

As a  $Sp(k) \times \mathfrak{so}^*(2n)$  representation,

$$\mathcal{S} = \sum_{\mu} V_{(k)}^{\mu} \otimes V_{\mu}^{(n)}, \quad [5.2]$$

where the sum is over all nonnegative integer partitions  $\mu$  having  $\min\{k, n\}$  or fewer parts.

And, as a  $O(k) \times \mathfrak{sp}(n)$  representation,



$$\mathcal{S} = \sum_{\nu} E_{(k)}^{\nu} \otimes E_{\nu}^{(n)}, \quad [5.3]$$

where the sum is over all nonnegative integer partitions  $\nu$  having  $\min\{k, n\}$  or fewer parts and having a Young diagram whose first two columns sum to  $k$  or less.

As remarked above computing the multiplicity of  $V^{\mu} \otimes F_{(n)}^{\lambda}$  in  $\mathcal{S}$  and  $E^{\nu} \otimes F_{(n)}^{\lambda}$  in  $\mathcal{S}$  we obtain the following.

**Theorem 5.** (i) The multiplicity of the  $Sp(k)$  representation  $V_{(k)}^{\mu}$  in  $F_{(2k)}^{\lambda}$  equals the multiplicity of  $F_{(n)}^{\lambda}$  in the unitarizable highest weight representation  $V_{\mu}^{(n)}$  of  $\mathfrak{so}^*(2n)$ .

(ii) The multiplicity of the  $O(k)$  representation  $E_{(k)}^{\nu}$  in  $F_{(k)}^{\lambda}$  equals the multiplicity of  $F_{(n)}^{\lambda}$  in the unitarizable highest weight representation  $E_{\nu}^{(n)}$  of  $\mathfrak{sp}(n)$ .

In the cases where the unitarizable highest weight representation is the full generalized Verma module, we call the parameter a generic point. A short calculation shows that the Littlewood hypothesis is included in the generic set.

**Corollary 6.** (i) Suppose  $\mu$  is a generic point. Then the multiplicity of the  $Sp(k)$  representation  $V_{(k)}^{\mu}$  in  $F_{(2k)}^{\sigma}$  equals

$$\sum_{\xi} \dim \operatorname{Hom}_{GL(n)}(F_{(n)}^{\sigma}, F_{(n)}^{\mu} \otimes F_{(n)}^{\xi}), \quad [5.4]$$

where the sum is over all nonnegative integer partitions  $\xi$  with columns of even length.

(ii) Suppose  $\nu$  is a generic point. Then the multiplicity of the  $O(k)$  representation  $E_{(k)}^{\nu}$  in  $F_{(k)}^{\sigma}$  equals

$$\sum_{\xi} \dim \operatorname{Hom}_{GL(n)}(F_{(n)}^{\sigma}, F_{(n)}^{\nu} \otimes F_{(n)}^{\xi}), \quad [5.5]$$

where the sum is over all nonnegative integer partitions  $\xi$  with rows of even length.

The formulas 5.4 and 5.5 are those of the Littlewood Restriction Theorem (ref. 2, p. 240). Littlewood's hypotheses correspond to the unitarizable highest weight representations lying in the discrete series or in some cases the limits of the discrete series.

For a general unitarizable highest weight representation  $L$  from ref. 9 we know the  $\mathfrak{p}^+$ -cohomology formulas for  $L$ . From this data we obtain explicit resolutions for  $L$  where the terms of this resolution are sums of generalized Verma modules.

**Theorem 7.** Suppose  $L = L(\lambda + \rho)$  is a unitarizable highest weight module. Then  $L$  admits a resolution in terms of generalized Verma modules. Specifically, for  $1 \leq i \leq r_{\lambda} = \operatorname{card}(\Delta_{\lambda} \cap \Delta_{n+})$ , set  $\mathbf{C}_i^{\lambda} = \sum_{x \in W_{\lambda}^{t_i}} N((x(\lambda + \rho))^{++})$ . Then there is a resolution of  $L$ :

$$0 \rightarrow \mathbf{C}_{r_{\lambda}}^{\lambda} \rightarrow \cdots \rightarrow \mathbf{C}_1^{\lambda} \rightarrow \mathbf{C}_0^{\lambda} \rightarrow L \rightarrow 0. \quad [5.6]$$

Combining Theorems 5 and 7 we obtain the following.

**Corollary 8.** Suppose that  $F^{\sigma} \otimes F^{\sigma}$  occurs in  $\mathcal{S}$  as in 5.1 and that  $V_{(k)}^{\mu}$  and  $E_{(k)}^{\nu}$ , respectively, occur as  $Sp(k)$  and  $O(k)$  subrepresentations of  $F^{\sigma}$ . Then (i) the multiplicity of the  $Sp(k)$  representation  $V_{(k)}^{\mu}$  in  $F_{(2k)}^{\sigma}$  equals the multiplicity of  $F_{(n)}^{\sigma}$  in  $V_{\mu}^{(n)}$ , which is given by

$$\sum_i \sum_{x \in W_{\mu}^{t_i}} (-1)^i m_{x(\mu^{\dagger} + \rho)^{++} - \rho}^{\sigma_i}, \quad [5.7]$$

where  $\mu^{\dagger}$  (resp.  $\sigma^{\dagger}$ ) is the highest weight of  $V_{\mu}^{(n)}$  (resp.  $F_{(n)}^{\sigma}$ ), (ii) and the multiplicity of the  $O(k)$  representation  $E_{(k)}^{\nu}$  in  $F_{(k)}^{\sigma}$  equals the multiplicity of  $F_{(n)}^{\sigma}$  in  $E_{\nu}^{(n)}$ , which is given by

$$\sum_i \sum_{x \in W_{\nu}^{t_i}} (-1)^i m_{x(\nu^{\dagger} + \rho)^{++} - \rho}^{\sigma_i}, \quad [5.8]$$

where  $\nu^{\dagger}$  (resp.  $\sigma^{\dagger}$ ) is the highest weight of  $E_{\nu}^{(n)}$  (resp.  $F_{(n)}^{\sigma}$ ).

The above corollary provides an algorithm for decomposing finite dimensional  $GL(n)$  representations into irreducible orthogonal or symplectic representations. Such algorithms have been implemented using the MAPLE software package.

As an example of this multiplicity formula we list some cases for the orthogonal group.

**Example 9:** Let  $\nu$  be a partition such that,

$$\nu = (\overbrace{d, 2, \dots, 2}^a, \overbrace{1, \dots, 1}^b, \overbrace{0, \dots, 0}^c) \quad k$$

with:  $d \geq 2$  and  $2 + 2a + b \leq k$ .

Then for all  $\sigma$ ,

$$\dim \operatorname{Hom}_{O(k)}(E^{\nu}, F^{\sigma}) = m_{\nu_1}^{\sigma_i} - m_{\nu_1^1}^{\sigma_i},$$

where  $\nu_1$  is the  $k$  tuple given by

$$\nu_1 = (\overbrace{d, 2, \dots, 2}^c, \overbrace{1, \dots, 1}^b, \overbrace{0, \dots, 0}^a)$$

## 6. Final Remarks

Theorem 7 is the technical backbone of the article. In addition to giving the extension to all parameters of Littlewood Restriction, the proof of Theorem 4 begins with two applications of Theorem 7, one for  $\mathfrak{g}$  and one for  $\mathfrak{g}_{\lambda}$ . The argument then passes to the computation of a ratio using the hypothesis of quasi-dominance.

During the time this announcement has been refereed, some related research has appeared (13). In this work, the authors begin with a highest weight module  $L$  and then consider the associated variety  $\mathcal{V}(L)$  as defined by Vogan. This variety is the union of  $K_{\mathbb{C}}$ -orbits and equals the closure of a single orbit. In ref. 13, the GK dimension and the Bernstein degree of  $L$  are recovered from the corresponding objects for the variety  $\mathcal{V}(L)$ . As an example of their technique, they obtain the GK dimension and the degree of the Wallach representations (ref. 13, pp. 149–150). Our results in this setting obtain these two invariants as well as the full Hilbert series because all of the highest weights are quasi-dominant. The results of these two very different approaches have substantial overlap although neither subsumes the other.

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- Goodman, R. & Wallach, N. (1998) *Representations and Invariants of the Classical Groups* (Cambridge Univ. Press, Cambridge, U.K.) pp. 406–463.
- Littlewood, D. E. (1940) *The Theory of Group Characters and Matrix Representations of Groups* (Oxford Univ. Press, New York).
- Littlewood, D. E. (1944) *Philos. Trans. R. Soc. London Ser. A* **239**, 387–417.
- Gavarini, F. (1999) *J. Algebra* **212**, 240–271.
- Enright, T. J., Howe, R. & Wallach, N. (1982) *Progr. Math.* **40**, 97–143.
- Jakobsen, H. P. (1983) *J. Funct. Anal.* **52**, 385–412.

- Adams, J. (1987) *Adv. Math.* **63**, 113–137.
- Collingwood, D. (1985) *Math. Ann.* **272**, 161–187.
- Enright, T. J. (1988) *J. Reine Angew. Math.* **392**, 27–36.
- Howe, R. (1989) *Trans. Am. Math. Soc.* **313**, 539–570.
- Wallach, N. R. (1979) *Trans. Am. Math. Soc.* **251**, 1–17, 19–37.
- Howe, R. (1982) *Progr. Math.* **40**, 159–175.
- Nishiyama, K., Ochiai, H., Taniguchi, K., Yamashita, H. & Kato, S. (2001) *Asterisque* **273**, 1–163.