Borcherds products learning seminar
Quadratic lattices, Hermitian symmetric domains, and vector-valued modular forms

Lecture 2 • Igor Dolgachev • February 12, 2016

1. Quadratic lattices

A *quadratic lattice* is a free abelian group M of finite rank equipped with a symmetric bilinear form
$$b : M \times M \to \mathbb{Z}.$$
The map
$$q : M \to \mathbb{Z},\ x \mapsto b(x, x)$$
is a quadratic form on M.[1] This means that
$$q(2x) = 4q(x)$$
and
$$b_q : (x, y) \mapsto q(x + y) - q(x) - q(y)$$
is a symmetric bilinear form on M. In our case,
$$b_q = 2b.$$
For brevity of notation, we denote the value of the bilinear form b on a pair of vectors x, y by $x \cdot y$. the value $x \cdot x$ will be denoted by x^2.

Choosing a basis $(e_1, \ldots, e_r)$ in M, the bilinear form b is defined by the symmetric integer matrix
$$A = (e_i \cdot e_j).$$
We have
$$q(\sum a_i e_i) = \sum_{i=1}^{r} a_{ii} + 2 \sum_{1 \le i < j \le r} a_{ij}.$$
Tensoring M by $\mathbb{R}$, we obtain a real vector space $M_{\mathbb{R}}$ equipped with a quadratic form and its associated symmetric bilinear form $b_q(x, y) = \mathfrak{h}(q(x + y) - q(x) - q(y))$. The *signature* of M is the signature of $M\mathbb{R}$, i.e. the triple (b_+, b_-, b_0), such that there exists a basis (e_i) in $M_{\mathbb{R}}$ with
$$q(\sum x_i e_i) = \sum_{i=1}^{b_+} x_i^2 - \sum_{i=1}^{b_-} x_{i+b_+}.$$
The lattice is called *non-degenerate* if $b_0 = 0$. In this case we write the signature in the form (b_+, b_-). We say that M is positive (resp. negative) definite if $b_- = 0$ (resp. $b_+ = 0$. If both b_+ and b_- are positive, we say that M is *indefinite*. Let
$$M^\vee = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z}).$$

[1] In other Lectures one may define $q = \frac{1}{2}b$

If M is non-degenerate the natural map

$$\iota : M \to M^\vee, x \mapsto (y \mapsto x \cdot y)$$

is injective. Thus

$$A_M = M^\vee/\iota(M)$$

is a finite abelian group. If A is the matrix of the bilinear form of M, then

$$|A_M| = |\det(A)|.$$

After tensoring with $\mathbb{Q}$, the map ι becomes an isomorphism of linear spaces over $\mathbb{Q}$

$$\iota_{\mathbb{Q}} : M_{\mathbb{Q}} \to M_{\mathbb{Q}}^\vee$$

. We transfer the quadratic form on $M_{\mathbb{Q}}$ to $M_{\mathbb{Q}}^\vee$ and restrict it to $M^\vee$ to equip $M^\vee$ with a quadratic form $q : M^\vee \to \mathbb{Q}$. Explicitly, for any $\tilde{x} \in M^\vee$, $d\tilde{x} \in M$, we have

$$\tilde{x} \cdot \tilde{y} := \frac{1}{d^2}(d\tilde{x} \cdot d\tilde{y}) \in \mathbb{Q}.$$

We define the *discriminant quadratic form*

$$q_{A_M} : A_M \to \mathbb{Q}/\mathbb{Z}$$

by setting

$$q_{A_M}(\tilde{x} + M) = \tilde{x}^2 \mod \mathbb{Z}.$$

We also have the *discriminant bilinear form*

$$b_{A_M}(\tilde{x} + M, \tilde{y} + M) := \tilde{x} + M, \tilde{y} \mod \mathbb{Z}$$

A quadratic lattice is called *even* if its values are even integers. In this case the discriminant quadratic form takes values in $\mathbb{Q}/2\mathbb{Z}$.

One of the usefulness of the discriminant quadratic form is explained by the following result of V. Nikulin:

Theorem 1. *Let M is an even indefinite quadratic lattice. Let $l(A_M)$ be the smallest number of generators of A_M. Suppose that*

(i) $b_+ + b_- \geq 3$,
(ii) $b_+ + b_+ \geq l(A_M) + 2$.

Then any lattice N with same signature as M for which there exists an isomorphism of the discriminant groups form preserving the quadratic forms is isomorphic to M.

Another use of discriminant quadratic forms is explained by the following construction. Suppose M is a sublattice of some other lattice N of the same rank. We have the following obvious inclusions

$$M \subset N \subset N^\vee \subset M^\vee.$$

The group N/M is a subgroup of $A_M = M^\vee/M$ (we identify M with its image in $M^\vee$ under ι). One can check that the restriction of q_{A_M} to the subgroup N/M is identical zero (we say that N/M is an *isotropic subgroup* of A_M). Conversely, given an isotropic subgroup H of A_M we can construct an *overlattice* N of M by taking N to be the pre-image of A_M/H

under the homomorphism $M^\vee \to M^\vee/M = A_M$. One checks that it is indeed a quadratic lattice.

A quadratic lattice is called *unimodular* if $A_M = \{1\}$. of course this means that the symmetric matrix of the bilinear form in any basis has determinant equal to ± 1.

Let U (or $II_{1,1}$) denote the quadratic lattice of rank 2 defined by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is called the *hyperbolic plane*. Let E_8 denote the unique rank 8 definite unimodular lattice (isomorphic to the root lattice of a root system of type E_8).

The following useful result belongs to J. Milnor:

Theorem 2. *Let M be an even, indefinite unimodular lattice. Suppose that $b_+ \geq b_-$. Then*

$$M \cong U^{\oplus k} \oplus E_8^{\oplus m},$$

where the direct sum is the orthogonal direct sum.

For any lattice M and an integer k, we denote by $M(k)$ the lattice obtained from M by multiplying its quadratic form by k. Since $U(-1) \cong U$, we obtain that any even indefinite unimodular lattice with $b_- \geq b_+$ is isomorphic to the orthogonal sum of lattices isomorphic to U and $E_8(-1)$. In particular, we have always

$$|b_+ - b_-| = 0 \mod 8.$$

2. Hermitian symmetric domain of orthogonal type

The upper half plane

$$\mathbb{H} = \{x + iy \in \mathbb{C} : y > 0\}$$

can be considered as a very special case of a *tube domain* in a complex vector space defined by

$$\Omega = V + iC \subset V_{\mathbb{C}} \cong \mathbb{C}^n,$$

where V is a real vector space and C is a cone in V that does not contains lines. An additional property is that Ω admits a transitive group action of its group of holomorphic automorphisms. This will imply that the tube domain is a Hermitian symmetric space.

An example of such a tube domain is the *Siegel space*

$$\mathcal{Z}_g = \mathrm{Sym}_g(\mathbb{R}) + i\,\mathrm{Sym}_g^+(\mathbb{R}) \subset \mathrm{Sym}_g(\mathbb{C}),$$

where $V = \mathrm{Sym}_g(\mathbb{R})$ is the linear space of symmetric real matrices of size $g \times g$, and $\mathrm{Sym}_g^+(\mathbb{C})$ is the cone of positive definite matrices.

The Siegel space $\mathcal{Z}_g$ serves as the period space for abelian varieties of dimension g over $\mathbb{C}$. As a complex manifold, such a variety is isomorphic to a complex torus $A = \mathbb{C}^g/\Lambda$, where Λ is a free abelian subgroup of $\mathbb{C}^g$ generated by some $2g$-linear independent (over $\mathbb{Z}$) vectors

$v_1, \ldots, v_{2g}$. If $\omega_1, \ldots, \omega_g$ is a basis of holomorphic 1-forms on A and $\gamma_1, \ldots, \gamma_{2g}$ is a basis of the group $H_1(A, \mathbb{Z}) \cong \Lambda$, then

$$v_j = (\int_{\gamma_1} \omega_j, \ldots, \int_{\gamma_1} \omega_j).$$

Changing the coordinates in $\mathbb{C}^g$ and changing a basis, we may assume that the matrix with columns $v_1, \ldots, v_{2g}$ is of the form $[\tau \ I_g]$. The conditions that the torus can be embedded in a projective space are expressed by the *Riemann-Frobenius conditions*

$$\tau = {}^t\tau, \quad \mathrm{Im}(\tau) > 0.$$

This assigns to A a point in the Siegel space $\mathcal{Z}_G$, the *period point* of A. An embedding of A in a projective space is defined by some ample line bundle L over A. Its first Chern class is an element of $H^2(A, \mathbb{Z})$ which can be canonically identified with $\bigwedge^2 \Lambda^\vee$, or, in coordinates, with a skew-symmetric integral matrix of size $2g \times 2g$. As is well-known, after a change of a basis in Λ, we may assume that this matrix is a block-matrix

$$J_{D_g} := \begin{pmatrix} 0_g & D_g \\ -D_g & 0 \end{pmatrix},$$

where D_g is the diagonal matrix with diagonal elements $d_1 | d2 + \cdots | d_g$. The vector $(d_1, \ldots, d_g)$ is the *type* of L. A *marking* of (A, L) is a choice of a basis in $H^1(A, \mathbb{Z}) = \mathrm{Hom}(\Lambda, \mathbb{Z})$ such that $c_1(L)$ can be expressed by a matrix D_g as above. Two such bases differ by an automorphism of Λ represented by a matrix M written as a block-matrix in the form

$$M = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

such that $M \cdot J_{D_g} \cdot {}^tM = J_{D_g}$.

Such matrices form a group denoted by $\mathrm{Sp}(J_{D_g}, \mathbb{Z})$, the symplectic group of type D_g. When D_g is the identity matrix I_g, a choice of L is called a *principal polarization*. In this case $\mathrm{Sp}(J_{D_g}, \mathbb{Z})$ is just the group of $\mathbb{Z}$-points of the symplectic group Sp_{2g}. One can show that the moduli space of abelian varieties with polarization L of type D_g is isomorphic to the quotient space

$$\mathcal{A}_{d_1, \ldots, d_g} = \mathcal{Z}_g / \mathrm{Sp}(J_{D_g}, \mathbb{Z}).$$

In Cartan's classification of Hermitian symmetric spaces, the Siegel space goes under the name type III.

A Hermitian symmetric space of orthogonal type is of type IV in Cartan's classification. This time we take V to be an n-dimensional real vector space equipped with a quadratic form of signature $(1, n-1)$ (or $(n-1, 1)$). In an appropriate basis in V, this form is given by $x_1^2 - x_2^2 - \cdots - x_n^2$. We tqke for C the *light cone*, one of the two connected components of the set of vectors v with $v^2 > 0$. In coordinates, we may take C to be the set C^+ of vectors $(x_1, \ldots, x_n)$ with $x_1^2 - x_2^2 - \cdots - x_n^2 > 0$ and $x_1 > 0$. We denote by $\mathbb{R}^{1,n-1}$ a real vector space with a chosen basis and a quadratic form as above. So, in coordinates

$$\Omega_n = \mathbb{R}^{1,n-1} + iC^+ \subset \mathbb{C}^n.$$

There is another model of this space as a subset of a complex quadric in a projective space. let

$$L = V \oplus U \cong \mathbb{R}^{1,n-1} \oplus \mathbb{R}^{1,1} \cong \mathbb{R}^{2,n}.$$

(all direct sums are orthogonal direct sums). We assign to a vector $z \in \Omega$ the line in $L_{\mathbb{C}}$ generated by the vector

$$\alpha(z) = z + f - \frac{z^2}{2} g,$$

where $z \in V_{\mathbb{C}}$ and f, g is the basis in U such that $f^2 = g^2 = 0, f \cdot g = 1$. This gives a map from Ω_n to the projective space $\mathbb{P}(L_{\mathbb{C}}) \cong \mathbb{P}^{n+1}$ of lines in $L_{\mathbb{C}}$. A choice of a representative ω of $\alpha(z)$ of the line $\mathbb{C}\alpha(z)$ is determined by the property that $\omega \cdot g = 1$. One verified immediately that

$$\alpha(z)^2 = 0, \quad \alpha(z) \cdot \overline{\alpha(z)} > 0.$$

Thus the image of Ω_n in $\mathbb{P}(L_{\mathbb{C}})$ is contained in the quadric hypersurface Q with equation

$$z_1^2 - z_2^2 - \cdots - z_n^2 + z_{n+1} z_{n+2} = 0,$$

where $z_1, \ldots, z_n$ are coordinates in $(\mathbb{R}^{1,n})_{\mathbb{C}}$ and (z_{n+1}, z_{n+2}) are coordinates in $(\mathbb{R}^{1,1})_{\mathbb{C}}$. The second condition tells us that the image of Ω_n is contained in an open subset Q^0 of Q defined by the inequality

$$z_1 \bar{z}_1 - z_2 \bar{z}_2 - \cdots - z_n \bar{z}_n + z_{n+1} \bar{z}_{n+2} + \bar{z}_{n+2} \bar{z}_{n+1} > 0, \quad Re(z_1) > 0.$$

This established a holomorphic isomorphism between the tube domain Ω_n and the open subset Q^0 of the quadric Q in $\mathbb{P}(V_{\mathbb{C}})$. The inverse map $Q^0 \to \Omega_n$ is defined as follows. Take the point $[g] \in Q^0$ with projective coordinates $[0, \ldots, 0, 0, 1]$. We choose a representative of a point $[w] \in Q^0$ given by the condition $w \cdot g = 1$. Then we project from $Q^0 \subset \mathbb{P}(L_{\mathbb{C}})$ to the projective space $\mathbb{P}(L_{\mathbb{C}}/[g])$ (recall that $[g]$ is a line in $L_{\mathbb{C}}$) from the point $[g]$. The image of the quadric Q lies in the affine part of $\mathbb{P}(L_{\mathbb{C}}/[g])$ represented by cosets of vectors $w \in L_{\mathbb{C}}$ with the condition $w \cdot f = 1$. It can be identifies with the vector space $[g]^{\perp}/[g] = V_{\mathbb{C}}$. The image of Q^0 is equal to Ω_n.

For example, take $n = 1$. Then $V = \mathbb{R}^{1,0} = \mathbb{R}$ and $C^+ = \mathbb{R}_{>0}$. We have $L = \mathbb{R}^{2,1}$ and Q is the conic in $\mathbb{P}(\mathbb{R}^{2,1}) \cong \mathbb{P}^2$ with the equation $z_1^2 + z_2 z_3 = 0$. The open subset Q^0 is the subset of the conic defined by the condition $|z_1|^2 + z_2 \bar{z}_3 + \bar{z}_2 z_3 > 0, Re(z_2) > 0$. We project the conic from the point $[g]$ with projective coordinates $[0, 0, 1]$. The image of the projection lies in the projective line $\mathbb{P}^1$ with coordinates $[z_1, z_2]$. The inequality condition says that $z_2 \neq 0$ and $\mathrm{Im}(z_1/z_2) > 0$. So, we get the upper half-plane.

There is still another model for a Hermitian symmetric space of orthogonal type. By assigning to $[w] \in \Omega_n = Q^0$ the real and imaginary parts $Re(w), \mathrm{Im}(w)$ in this order, we obtain a real 2-dimensional subspace P of L. The condition $w \cdot \bar{w} > 0$ translates into the condition that P is positive definite, and the choice of the basis $(Re(w), \mathrm{Im}(w)))$ defines an orientation. In this way Ω_n is mapped bijectively onto the Grassmannian $G(2, L)^+$ of positive definite oriented planes in the real vector space L.

To locate discrete groups acting totally discontinuous on the domain Ω_n by holomorphic automorphisms we put an *integral structure* on the complex vector spaces L. Namely, we fix a free abelian subgroup of L of rank equal to $n + 2$, so that we can identify L with $T_{\mathbb{R}}$

and $L_{\mathbb{C}}$ with $T_{\mathbb{C}}$. Then we let $\Gamma_T \subset \mathrm{O}(2,n)$ to be the group of isometries of T or any its subgroup Γ'_T of finite index. The quotient space Ω_n/Γ'_T is a complex analytic space, which, by a theorem of Baily-Borel can be embedded into a projective space as a quasi-projective algebraic variety of dimension n.

For $n \leq 19$ (and some conditions on the lattice T), the algebraic variety $\Gamma_T \subset \mathrm{O}(2,n)$ can be realized as the (coarse) moduli space of complex *algebraic K3 surfaces* which admit T as a sublattice of the lattice of transcendental 2-cycles. Let us recall the definitions.

An algebraic K3 surface X is a nonsingular projective surface satisfying the conditions that the canonical class K_X and the first Betti number (in étale cohomology if the ground field is not $\mathbb{C}$) are equal to zero. An example of such a surface is a hypersurface of degree 4 in $\mathbb{P}^3$. Another example is a *Kummer surface*, a nonsingular minimal model of the quotient of an abelian surface A by the involution $[-1] : a \mapsto -a$ (here one has assume that A is not supersingular if the characteristic is equal to 2).

We assume here that the ground field is $\mathbb{C}$. One can show that the second Betti number $b_2(X)$ is equal to 22 and $H^2(X,\mathbb{Z})$ is a free abelian group of rank 22. The Poincaré Duality, the cup-product equips $H^2(X,\mathbb{Z})$ with a structure of unimodular quadratic lattice. It identifies $H^2(X,\mathbb{Z})$ with its dual lattice $H_2(X,\mathbb{Z})$ One can also show, using the Hodge decomposition for $H^2(X,\mathbb{C})$ that its signature is equal to $(3,19)$. Since $K_X = 0$, Wu's Formula in algebraic topology implies that $H^2(X,\mathbb{Z})$ is an even lattice. Applying Milnor's Theorem we obtain that

$$H^2(X,\mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1).$$

We denote the right-hand side by L_{K3} and call it the *K3-lattice*.

Since X is a projective algebraic variety it contains *algebraic cycles*, the cohomology classes of the form $\sum n_\gamma[\gamma]$, where γ is the fundamental class of an irreducible algebraic curve on X. The subgroup $H^2_{\mathrm{alg}}(X,\mathbb{Z})$ of such cycles is isomorphic to the Picard group $\mathrm{Pic}(X)$ of isomorphism classes of line bundles (or invertible sheaves of $\mathcal{O}_X$-modules) on X. The isomorphism is defined by the first Chern class. The quadratic form on $H^2(X,\mathbb{Z})$ equips $H^2_{\mathrm{alg}}(X,\mathbb{Z})$ with a structure of an even lattice. By *Hodge Index Theorem*, its signature is equal to $(1,\rho-1)$, where $\rho = \mathrm{rk}\, H^2_{\mathrm{alg}}(X,\mathbb{Z})$. We denote this lattice by S_X. It is an important invariant of X. Note that, in general, the lattice S_X is not a unimodular lattice. It rank ρ take all possible values between 1 and 20.

Let $T_X = S_X^\perp$ be the orthogonal complement of S_X in $H^2(X,\mathbb{Z})$. It is a sublattice of $H^2(X,\mathbb{Z})$ of signature $(2, 20-\rho)$. It is called the lattice of *transcendental cycles*. Since $K_X = 0$, we have the space of holomorphic 2-forms on X is one-dimensional. Choose a generator ω of this space. By *Lefschetz Theorem*, a 2-cycle γ is algebraic if and only if

$$\int_\gamma \omega = 0.$$

Thus ω can be identified with a linear function $H_2(X,\mathbb{Z}) \to \mathbb{C}$ vanishing on the subspace of algebraic cycles, hence with an element of $(T_X)_{\mathbb{C}}$. Replacing ω with $\lambda\omega$ multiplies this function by λ. Thus we have a canonical choice of a line in the space $(T_X)_{\mathbb{C}} \subset H^2(X,\mathbb{Z})_{\mathbb{C}}$, i.e. a point $[\omega]$ in the projective space $\mathbb{P}(H^2(X,\mathbb{C})) \cong \mathbb{P}^{21}$. Since ω is a holomorphic form of type $(2,0)$, we obtain $\omega \wedge \omega = 0$. Using the De Rham Theorem, this can be viewed as the condition $[\omega]^2 = 0$. Also, we can choose the orientation on X such that $\omega \wedge \bar{\omega}$ is a volume form. This gives us a point $[\omega]$ in $\Omega_{T_X} \subset \Omega_n$, the *period point* of X.

Now suppose we have an algebraic family (X_t) of K3 surfaces together with a primitive embedding of a fixed even sublattice M of L_{K3} signature $(1, \rho - 1)$ in $\mathrm{Pic}(X_t)$.[2] We also assume that there is a vector in M whose image in $\mathrm{Pic}(X_t)$ is realized by the class of a hyperplane section in some projective embedding of X_t). Choose an isomorphism $\phi : L_{K3} \to H^2(X_t, \mathbb{Z})$ such that its restriction to M defines an embedding $j_t : M \to \mathrm{Pic}(X_t)$. Let $T = M^\perp$ in L_{K3}. Then we can assign to X_t the image of the period point $\omega_t] \in \Omega_{T_{X_t}}$ in Ω_T. This defines a map from the parameter space of the family to Ω_T. To get rid of the choice of a basis in T_X defined by the map $\phi|_T : T \to T_X \subset H^2(X, \mathbb{Z})$, we have to consider the action of the discrete group

$$\Gamma_T = \{\sigma \in \mathrm{O}(L_{K3}) : \sigma(T) \subset T\} \cong \mathrm{Ker}(\mathrm{O}(T) \to \mathrm{O}(A_T, q_{A_T}).$$

on Ω_T and take the quotient space Ω_T/Γ_T. Not all points in this quotient space correspond to the period points. In fact, for any $\delta \in T, \delta^2 = -2$ let

$$H_\delta = \{x \in T_{\mathbb{C}} : x \cdot \delta\}/\mathbb{C}^* \subset \Omega_T.$$

Suppose $x = [\omega]$ for some period point of a K3 surface X. Then $x \cdot \delta = 0$ implies that x is an algebraic cycle. By Riemann-Roch on X, x or $-x$ is effective. We could choose the connected component of C^+ in such a way that x is effective. Since $x \in (j(M))^\perp$, and $j(M)$ contains the class of a hyperplane section of X in some projective embedding, we obtain a contradiction.

Let

$$D_T = \bigcup_{\delta \in T, \delta^2 = -2} H_\delta.$$

We call it the *discriminant locus*.

The fundamental result of I. Shafarevich and I.Pyatetsky-Shapiro asserts that, in this way, we obtain a coarse moduli space M-lattice polarized K3 surfaces

$$\mathcal{M}_{K3,M} = (\Omega_T \setminus D_T)/\Gamma_T.$$

We already know Ω_1 coincides with the upper half-plane $\mathbb{H}$ (in fact any one-dimensional Hermitian symmetric domain coincides with $\mathbb{H}$). Let us see that Ω_2 coincides with the product $\mathbb{H} \times \mathbb{H}$, and Ω_3 coincides with the Siegel space $\mathcal{Z}_2$. In fact, in the tube domain realization, $\Omega_2 = \mathbb{R}^{1,1} + iC^+$, where $C^+ = \{(y_1, y_2) : y_1^2 - y_2^2 > 0, y_1 > 0\}$. We map Ω_2 to $\mathbb{H} \times \mathbb{H}$ by assigning to a vector $z = x + iy$ the point $(x_1 + i(y_1 + y_2), x_2 + i(y_1 - y_2))$.

If $n = 3$, we have $\Omega_3 = \mathbb{R}^{2,1} + iC^+$, where $C^+ = \{(y_1, y_2, y_3) : y_1^2 + y_2 y_3 < 0, y_2 > 0\}$ (we change the sign because the signature is $(2,1)$ but not $(1,2)$). We assign to $z = x + iy$ the matrix

$$\begin{pmatrix} x_2 + iy_2 & x_1 + iy_1 \\ x_1 + iy_1 & x_3 + iy_3 \end{pmatrix} = \begin{pmatrix} x_2 & x_1 \\ x_1 & x_3 \end{pmatrix} + i \begin{pmatrix} y_2 & y_1 \\ y_1 & -y_3 \end{pmatrix}.$$

[2] Primitive means that the quotient group has no torsion.

It is obviously symmetric and its imaginary part satisfies $y_2 > 0, -(y_2^2 + y_1 y_3) > 0$, so it is positive definite.

Note that in the moduli interpretation this makes an isomorphism

$$\mathcal{A}_{1,d} \cong \mathcal{M}_{K3,M_d},$$

where $M_d^\perp = U \oplus U \oplus \langle -2d \rangle$ (see my Lecture notes "Endomorphisms of complex abelian varieties", Lecture 10).

3. The Weil representation

Let V be a complex manifold and let $\mu : \Gamma \to \mathrm{Aut}(V)$ be an totally discontinuous holomorphic action of Γ on V. Let $\pi : \mathbb{E} \to V$ be a holomorphic vector bundle of rank r over V equipped with a Γ-linearization, i.e. a lift $\tilde{\mu}(g) = \tilde{g}_{\mathbb{E}}$ of the action of Γ on V to an action on $\mathbb{E}$ that commutes with the projection, i.e.

$$g(\pi(e)) = \pi(\tilde{g}_{\mathbb{E}}(e)), \;\; g \in \Gamma, e \in \mathbb{E}.$$

We call the pair $J = (\mathbb{E}, \tilde{\mu})$ an *automorphy factor* on V with respect to Γ. Let $L = H^0(V, \mathbb{E})$ be the space of holomorphic sections of $\mathbb{E}$. Then an automorphy factor $\mathbb{E}$ defines a linear representation

$$\Gamma \to \mathrm{GL}(L), \;\; {}^g s(z) = \tilde{g}_{\mathbb{E}}(s(g^{-1}(z)).$$

A *weak modular form* of Γ with respect to $J = (\mathbb{E}, \tilde{\mu})$ is an element of the subspace of invariant sections

$$H^0(V, \mathbb{E})^\Gamma \;= \{s \in L : {}^g s = s\} = \{s \in L : s(g(z)) = \tilde{g}_{\mathbb{E}}(s(z))\}.$$

A special case of this definition which we will be dealing with is when $\mathbb{E} = \underline{W} \otimes L$ is the tensor product of the trivial bundle $\underline{W} = V \times W$, where W is a complex vector space of some dimension d and an automorphy facto $\mathbb{L}$ of rank 1. In this case, an automorphy factor $(\mathbb{E}, \tilde{\mu})$ is defined by a linear representation

$$\rho : \Gamma \to \mathrm{GL}(W)$$

such that the lift $\tilde{g}$ of g acting on V is defined by

$$\tilde{g}_{\mathbb{E}}(z, w) = (g(z), \rho(g)(w) \otimes \tilde{g}_{\mathbb{L}}.$$

A weak modular form becomes a holomorphic function $f : V \to \underline{W} \otimes \mathbb{L}$ satisfying

$$f(g(z)) = (\rho(g) \otimes \tilde{g}_{\mathbb{L}})(f(z)).$$

The familiar example of a weak modular form of weight $k \in \mathbb{Z}_{\geq 0}$ is the case where $V = \mathbb{H}$, the upper half-plane, Γ is a subgroup of finite index of the modular group $\mathrm{PSL}(2, \mathbb{Z})$, $\mathbb{E} = T^*\mathbb{H}^{\otimes k}$ is the cotangent line bundle and $\tilde{g}_{\mathbb{E}} = ({}^t dg^{-1})^k$, where $dg : T(\mathbb{H}) \to T(\mathbb{H})$ is the differential of the action $g : \mathbb{H} \to \mathbb{H}$. We can trivialize $T^*(\mathbb{H})$ by taking the basis $d\tau$. If $g(\tau) = \frac{a\tau+b}{c\tau+d}$, where $ad - bc = 1$, then $(dg)_\tau$ is the multiplication by $g'(\tau) = \frac{1}{(c\tau+d)^2}$, and

$$\tilde{g}(\tau, td\tau) = (\frac{a\tau + b}{c\tau + d}, (c\tau + d)^{-k} td\tau).$$

A weak modular form now can be identified with a holomorphic function $f:\mathbb{H}\to\mathbb{C}$ such that
$$f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(z).$$
Since Γ is of finite index in $\mathrm{PSL}(2,\mathbb{Z})$, there exists n such that $\tau\mapsto\tau+n$ belongs to Γ. We can choose the smallest positive n with this property. The modularity condition implies that $f(\tau+n)=f(\tau)$, hence we can expand $f(\tau)$ in the Fourier series
$$f(\tau)=\sum_{r\in\mathbb{Z}} a_r e^{2\pi rn\tau} = \sum_{r\in\mathbb{Z}} a_r q^{rn}, \quad q=e^{2\pi\tau}.$$
A weak modular form is a *modular form* if $a_r=0, r<0$ (a *cusp form* if additionally $a_0=0$).

Note that the action of Γ on V is not necessary faithful, however, we assume that its lift to $\mathbb{E}$ is a faithful action. This allows one to define modular forms of half-integer weight with respect to the double extension $\mathrm{Mp}_2(\mathbb{Z})$ of $\mathrm{PSL}(2,\mathbb{Z})$.

Recall that the the universal cover group $\tilde{G}=\widetilde{\mathrm{PSL}}(2,\mathbb{R})$ of the group $G=\mathrm{PSL}(2,\mathbb{R})$ is a Lie group zith the center $Z(\tilde{G})$ isomorphic to $\mathbb{Z}$. It is isomorphic to the group
$$\{(g,h)\in \mathrm{PSL}(2,\mathbb{R})\times \mathrm{Hol}(\mathbb{H},\mathbb{C}) : e^{2\pi i h(\tau)} = g'(\tau)\}$$
with group law
$$(g_2,h_2(\tau))\cdot(g_1,h_1(\tau)) := (g_2\cdot g_1, h_2(g_1(\tau))+h_2(\tau)).$$
Here $\mathrm{Hol}(\mathbb{H},\mathbb{C})$ is the space of holomorphic function on $\mathbb{H}$. The projection $(g,h(\tau))\to g$ is a homomorphism of groups and the kernel consists of elements $(1,h(\tau)$ such that $e^{2\pi h(\tau)}=1$. The function must be a constant equal to an integer, hence the kernel is isomorphic to $\mathbb{Z}$.

For any integer m, let
$$\widetilde{\mathrm{PSL}}(2,\mathbb{R})_m = \widetilde{\mathrm{PSL}}(2,\mathbb{R})/m\cdot Z(\tilde{G}).$$
It is a finite central extension of G
$$1\to\mathbb{Z}/m\mathbb{Z}\to\widetilde{G}_m\to G\to 1.$$
It can be viewed as the group
$$\{(g,\delta(\tau))\in G\times \mathrm{Hol}(H,\mathbb{C}^*) : \delta(\tau)^m = g'(\tau)\}.$$
with the group law
$$(g_2,\delta_2(z))\cdot(g_1,\delta_1(\tau)) := (g_2\cdot g_1, \delta_2(g_1(\tau))\delta_1(\tau)).$$
If we take $m=1$, and use the Chain Rule, we obtain that the map $G\to\tilde{G}_1, g\mapsto (g,g'(\tau))$ is an isomorphism. If we take $m=2$, we obtain that $\tilde{G}_2\cong \mathrm{SL}(2,\mathbb{R})$. The Lie group $\tilde{G}_4$ is denoted by Mp_2 and is called the *metaplectic group*. Its elements are pairs $(g,\delta(\tau))$, where $g=\frac{a\tau+b}{c\tau+d}$ and $\delta(z)^2=\sqrt{c\tau+d}$, where we consider $\sqrt{c\tau+d}$ as holomorphic function by taking its branch with positive imaginary part.

Let $\mathrm{Mp}_2(\mathbb{Z})$ be the pre-image in Mp_2 of the modular group $\mathrm{PSL}(2,\mathbb{Z})$. It is known that $\mathrm{PSL}(2,\mathbb{Z})$ is generated by the transformations $S = -1/\tau$ and $T = \tau + 1$. The group $\mathrm{Mp}_2(\mathbb{Z})$ is generated by the elements

$$\tilde{S} = (S, 1), \quad \tilde{T} = (T, \sqrt{\tau}).$$

they satisfy the basic relations $\tilde{S}^2 = (\tilde{S}\tilde{T})^3 = Z = (1, \sqrt{-1})$. For any non-degenerate even quadratic lattice M with discriminant group A_M and signature (b_+, b_-), we let $\mathbb{C}[A_M]$ denote the group algebra of A_M. Its basis are elements $e_\gamma, \gamma \in A_M$, and the multiplication table is $e_\gamma \cdot e_{\gamma'} = e_{\gamma \cdot \gamma'}$. We equip the vector space $\mathbb{C}[A_M]$ with the unitary inner-product by declaring that (e_γ) is an orthonormal basis.

We define the *Weil representation*

$$\rho_M : \mathrm{Mp}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}[A_M])$$

by setting

$$\begin{aligned} \rho_M(T) \cdot e_\gamma &= e^{\pi i \gamma^2} e_\gamma, \\ \rho(S) \cdot e_\gamma &= \frac{e^{\pi i (b_- - b_+)/4}}{\sqrt{|A_M|}} \sum_{\delta \in A_M} e^{-2\pi i \gamma \cdot \delta} e_\delta. \end{aligned} \tag{1}$$

We leave to the reader to check that the Weil representation is a unitary representation. Via the Weil representation the trivial vector bundle $\mathbb{H} \times \mathbb{C}[A_M]$ tensored with the line bundle $T^*(\mathbb{H})^{\otimes k}, k \in \mathfrak{h}\mathbb{Z}$ becomes an automorphy factor, and we define a *weak modular form of weight* k *with respect to* $\mathrm{Mp}_2(\mathbb{Z})$ as a holomorphic vector-valued function $f : \mathbb{H} \to \mathbb{C}[A_M]$ satisfying

$$f(g(\tau)) = \delta(\tau)^{2k} \rho_M((g, \delta(\tau)) f(\tau)$$

for all $(g, \delta(\tau)) \in \mathrm{Mp}_2(\mathbb{Z})$. In coordinates with respect to the basis (e_γ), $f(\tau)$ is given by

$$f(\tau) = \sum_{\gamma \in A_M} f_\gamma(\tau) e_\gamma,$$

where

(i) $f_\gamma(\tau + 1) = (-1)^{\gamma^2}$,
(ii) $f(-1/\tau) = \frac{e^{\pi i (b_- - b_+)/4}}{\sqrt{|A_M|}} \sum_{\delta \in A_M} e^{-2\pi i \gamma \cdot \delta} f_\delta$.

The first condition implies that the function $q^{\gamma^2/2} f(\tau)$ is periodic with period 1. This gives a Fourier expansion of $f_\gamma(\tau)$

$$f_\gamma(\tau) = \sum_{n = m - \mathfrak{h}\gamma^2, m \in \mathbb{Z}} c(n, \gamma) q^n, \quad q = e^{2\pi i \tau}.$$

We say that $(f_\gamma)_{\gamma \in A_M}$ is *modular form of weight* k *with respect to* $\mathrm{Mp}_2(\mathbb{Z})$ if $c(n, \gamma) = 0$ for $n < 0$ and all γ. We say that it is a *cusp form* if $c(0, \gamma) = 0$ for all γ.