Annals of Mathematics, **146** (1997), 545–646

Central derivatives of Eisenstein series and height pairings

By Stephen S. Kudla*

Introduction

The Siegel Eisenstein series on the symplectic group and on its metaplectic cover are well known to be connected with the arithmetic of quadratic forms, via the Siegel-Weil formula, which expresses certain special values of these series as theta functions. In particular, the nonsingular Fourier coefficients of the Siegel Eisenstein series have a product formula and, at special values, this factorization, together with the Siegel-Weil formula, yields Siegel's celebrated formula for global representation numbers of quadratic forms in terms of local representation densities. In a previous paper ([23]) the Fourier coefficients of the restriction of the Siegel Eisenstein series to certain subgroups were shown to involve the intersection numbers of certain algebraic cycles on the Shimura varieties associated to orthogonal groups of signature $(n, 2)$. An example of such a phenomenon was noted earlier by Gross and Keating [10] in the case of genus 2. Moreover, the same restrictions were used by Garrett [7] and Böcherer [1] in the classical case, and by Rallis and Piatetski-Shapiro [39], [42], in general, in the Rankin-Selberg integral representations of certain Langlands L-functions.

The present paper provides evidence that the derivatives at the center of symmetry of certain of these Siegel Eisenstein series are related to arithmetical algebraic geometry. More precisely, the nonsingular Fourier coefficients of such derivatives can be expressed as a sum of terms associated to the primes (including infinity). We give evidence, at least in the case of genus 2, that the restrictions of these sums to certain subgroups give the expression of the global height pairing of algebraic cycles (0-cycles on Shimura curves attached to indefinite quaternion algebras over $\mathbb{Q}$, in our case) in terms of local height pairings. Because of the occurrence of our Siegel Eisenstein series in an in-

*Work on this project was partially supported by NSF Grants, #DMS-9302539 and #DMS-9622987 for the summers 1993–1995 and 1996, respectively.

tegral representation of the L-functions attached to modular forms of weight 2 (having odd functional equation), our results, suitably completed, should have a Gross-Zagier formula for the central derivative of such L-functions as a consequence. We plan to pursue this application in a subsequent paper.

Our results suggest that it might be possible to establish Gross-Zagier type formulas for Shimura varieties of type $O(2n-1,2)$ and $U(2n-1,1)$, and in certain other sporadic cases, along the same lines. One of these sporadic cases — that of the central derivative of the triple product L-function attached to a triple of modular forms of weight 2 — was considered jointly with Gross and Zagier some time ago [12], [13], and was the origin of this program. Many very serious problems currently block the way, but one might at least hope to obtain further evidence for our conjectures in the higher dimensional cases.

In Part I, we review the definition and basic properties of the Siegel Eisenstein series and state the Siegel-Weil formula which describes their behavior at $s=0$, the center of the critical strip. Our results are a mild extension of some of the results of [30], and we give only a few indications of the proofs. We then consider the nonsingular Fourier coefficients of the derivative at $s=0$ of an incoherent Eisenstein series, i.e., of a series with an odd functional equation.

In Part II, we refine the general formula of Part I for derivatives of nonsingular Fourier coefficents in the special case of incoherent Siegel Eisenstein series of genus 2 and weight $\frac{3}{2}$. We make essential use of results of Shimura [45] and Kitaoka [19] on the archimedean and nonarchimedean local factors of such coefficients.

In Part III, we define weighted 0-cycles, equipped with Green currents, on Shimura curves over $\mathbb{Q}$. We then compute the local components of their Gillet-Soulé height pairing at the primes of good reduction of the Shimura curve and at $p=\infty$. In the nonarchimedean case, the essential part of the height pairing is obtained via the work of Gross-Keating, [10] and of Keating [18]. The resulting quantities coincide, up to a simple constant, with certain corresponding terms in the nonsingular Fourier coefficients of the central derivatives of our incoherent Siegel Eisenstein series (Theorem 12.6, for $p=\infty$, and Theorem 14.11, for $p<\infty$).

The introductions to Parts I–III contain a much more detailed summary of the results of this paper. Speculations about possible higher dimensional generalizations, and additional remarks, can be found in Section 16.

The results of this paper have taken shape over a number of years, and I owe a debt of gratitude to a number of people and institutions for support and assistance. I would particularly like to thank Dick Gross, Michael Harris, Kevin Keating, Steve Rallis, and Michael Rapoport for their help and advice at various stages this project. The most basic ideas about the structure of the central derivative of the incoherent Eisenstein series emerged during my collaboration with Michael Harris and with Dick Gross and Don Zagier on the central value of the triple product L-function [15], [12], [13]. The first main steps in the archimedean calculation were accomplished at the Isaac Newton Institute, Cambridge, during their special program on L-functions and Arithmetic April of 1993, and I would like to thank both the organizers of that program, Don Blasius, John Coates, and Richard Taylor as well as the staff of the Institute for their hospitality and for providing a particularly productive atmosphere. The final steps of the height calculation at a finite prime were accomplished in Wuppertal in January, 1995, where I was a guest of M. Rapoport, and I would like to thank him for his hospitality during that period. Finally, I would like to thank David Rohrlich and the referee for detailed and very helpful comments on the initial version of the manuscript.

Table of Contents

Notation

If F is a number field, define a character ψ of $\mathbb{A}_F/F$ as follows. Let $\psi_0 = \otimes_v \psi_v$ be the 'standard' additive character of $\mathbb{Q}_{\mathbb{A}}$, so that ψ is unramified everwhere and $\psi_\infty(x) = e(x) = \exp(2\pi i x)$. Let $\psi = \psi_0 \circ \mathrm{tr}_{F/\mathbb{Q}}$, where $\mathrm{tr}_{F/\mathbb{Q}}$ denotes the trace from F to $\mathbb{Q}$.

If F is a local field, $(\ ,\ )_F$ will denote the quadratic Hilbert symbol of F. For a nontrivial additive character ψ of F, and $x \in F^\times$, $\gamma_F(x,\psi)$ will denote the Weil index [43], [22]. It satisfies

$$\gamma_F(xy,\psi) = (x,y)_F\, \gamma_F(x,\psi)\, \gamma_F(y,\psi). \tag{0.1}$$

For a number field F, $(\ ,\ )_{\mathbb{A}}$ will denote the global Hilbert symbol on the ideles $F_{\mathbb{A}}^\times$.

Let W, $\langle\ ,\ \rangle$, be a symplectic vector space of dimension $2n$ over any field F of characteristic not 2. Fix a standard symplectic basis for W, i.e., a basis $e_1, \ldots, e_n, e'_1, \ldots, e'_n$ such that $\langle e_i, e'_i \rangle = 1$ and all other products are zero. We view $W \simeq F^{2n}$, (row vectors) with a right action of $\mathrm{Sp}(W) = \mathrm{Sp}_n(F)$. Let X (resp. Y) be the subspace of W spanned by the e_i's (resp. e'_i's) and let $P \subset \mathrm{Sp}(W)$ (the Siegel parabolic) be the stabilizer of Y. Then, a Levi decomposition $P = MN$ is given by:

$$M = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t a^{-1} \end{pmatrix} \mid a \in \mathrm{GL}_n(F) \right\}, \tag{0.2}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \mid b = {}^t b \in M_n(F) \right\}. \tag{0.3}$$

For $0 \le j \le n$, let

$$w_j = \begin{pmatrix} 0 & & 1_j & \\ & 1_{n-j} & & \\ -1_j & & & \\ & & & 1_{n-j} \end{pmatrix}. \tag{0.4}$$

We frequently write $w = w_n$. Then there is a Bruhat decomposition

$$\mathrm{Sp}_n(F) = \coprod_{j=0}^{n} P w_j P, \tag{0.5}$$

and the double coset $P w_j P$ is precisely the set of elements $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{Sp}(W)$ with $\mathrm{rank}(c) = j$.

If F is a local field of characteristic other than 2, let $\mathrm{Mp}(W)$ be the metaplectic extension of $\mathrm{Sp}(W) = \mathrm{Sp}_n(F)$,

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1, \tag{0.6}$$

obtained from the usual twofold cover by the inclusion of ± 1 in the circle $\mathbb{C}^1$. Our choice of a standard symplectic basis for W determines a Rao isomorphism

$$\mathrm{Mp}(W) \simeq \mathrm{Sp}(W) \times \mathbb{C}^1, \tag{0.7}$$

where the multiplication on the right-hand side is given by

$$(g_1, z_1) \cdot (g_1, z_2) = (g_1 g_2, c_{\mathrm{Rao}}(g_1, g_2) \cdot z_1 z_2). \tag{0.8}$$

The ± 1 valued cocycle here is given explicitly in [43] or [22]. In particular, if $g_1 = n_1 m(a_1)$ and $g_2 = n_2 m(a_2)$ lie in P, then

$$c_{\mathrm{Rao}}(g_1, g_2) = (\det(a_1), \det(a_2))_F. \tag{0.9}$$

We will occasionally abuse notation and write g for the element $(g, 1) \in \mathrm{Sp}(W) \times \mathbb{C}^1 \simeq \mathrm{Mp}(W)$. Note that the map $N \to \mathrm{Mp}(W)$ given by $n \mapsto (n, 1)$ is a homomorphism.

If F is nonarchimedean, we let $\mathcal{O}_F$ be the ring of integers of F and let L be the $\mathcal{O}_F$ lattice in W spanned by the standard basis. The stabilizer of L in $\mathrm{Sp}(W)$ is $\mathrm{Sp}_n(\mathcal{O}_F)$. We will write G for the group $\mathrm{Sp}(W)$ or $\mathrm{Mp}(W)$ (depending, ultimately, on the parity of n), and we write K for the maximal compact subgroup $\mathrm{Sp}_n(\mathcal{O}_F)$ or for its inverse image in $\mathrm{Mp}(W)$. If $F = \mathbb{R}$, we let K be either the subgroup:

$$\left\{ k = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid \mathbf{k} = a + ib \in U(n) \right\} \simeq U(n), \tag{0.10}$$

or for its inverse image in $\mathrm{Mp}(W)$.

If F is a number field, we let $\mathrm{Mp}(W_{\mathbb{A}})$ denote the global metaplectic extension

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(W_{\mathbb{A}}) \longrightarrow \mathrm{Sp}(W_{\mathbb{A}}) \longrightarrow 1. \tag{0.11}$$

Again, the choice of a standard basis yields a Rao isomorphism $\mathrm{Mp}(W_{\mathbb{A}}) \simeq \mathrm{Sp}(W_{\mathbb{A}}) \times \mathbb{C}^1$ where the multiplication on the right-hand side is defined by a global Rao cocycle valued in ± 1. The map $(g, z) \mapsto z^2$ defines a character μ: $\mathrm{Mp}(W_{\mathbb{A}}) \to \mathbb{C}^1$. There are unique splitting homomorphisms $\mathrm{Sp}(W_F) \to \mathrm{Mp}(W_{\mathbb{A}})$ and $N(\mathbb{A}) \to \mathrm{Mp}(W_{\mathbb{A}})$, subject to the condition that their composition with the Rao isomorphism take values in $\mathrm{Sp}(W_{\mathbb{A}}) \times \{\pm 1\} = \ker(\mu)$. These homomorphisms agree on $N(F) = N(\mathbb{A}) \cap \mathrm{Sp}(W_F)$.

For each place v of F, there is a unique homomorphism $\mathrm{Mp}(W_v) \to \mathrm{Mp}(W_{\mathbb{A}})$ of extensions, restricting to the identity map on the central $\mathbb{C}^1$ and lifting the natural inclusion $\mathrm{Sp}(W_v) \to \mathrm{Sp}(W_{\mathbb{A}})$. Since the standard local Rao cocycle is not trivial on $K_v \times K_v$ for almost all nonarchimedean places v, the local and global Rao isomorphisms differ by a twist [47]. This twist is trivial on N_v and on the elements w_j, so that the abuse of notation mentioned above will be compatible in the local and global cases. Moreover, for all finite places v of F with $v \nmid 2$, the map $\mathrm{Sp}_n(\mathcal{O}_{F_v}) \to \mathrm{Sp}(W_{\mathbb{A}}) \times \mathbb{C}^1 \simeq \mathrm{Mp}(W_{\mathbb{A}})$ given by $k \mapsto (k, 1)$ is a homomorphism.

Finally, for a totally real number field F, we write K_∞ (resp. K) for the inverse image in $\mathrm{Mp}(W_\infty)$ (resp. $\mathrm{Mp}(W_{\mathbb{A}_f})$) of the maximal compact subgroup of $\prod_{v\in S_\infty}\mathrm{Sp}(W_v)$ (resp. $\mathrm{Sp}(W_{\mathbb{A}_f})$) which is the product of the standard maximal compact subgroups fixed above.

PART I. Central derivatives of Siegel Eisenstein series

In Part I, we review the definition and basic properties of the Siegel Eisenstein series, and, in particular, their behavior at $s = 0$, the center of the critical strip. Our results are a mild extension of some of the results of [30], and we give only a few indications of the proofs. We consider a more general case than is needed for the application to Shimura curves, since the more general results are no more complicated to state and will be needed in higher dimensional cases.

More precisely, let F be a totally real number field and let W be a symplectic vector space of dimension $2n$ over F. Let $\mathrm{Sp}(W)$ be the symplectic group of W and let P be the maximal parabolic subgroup of $\mathrm{Sp}(W)$ which is the stabilizer of a maximal isotropic subspace of W. Let

$$G_{\mathbb{A}} = \begin{cases} \mathrm{Sp}(W_{\mathbb{A}}) & \text{if } n \text{ is odd} \\ \mathrm{Mp}(W_{\mathbb{A}}) & \text{if } n \text{ is even,} \end{cases} \tag{I.1}$$

where $\mathrm{Mp}(W_{\mathbb{A}})$ is the metaplectic extension of $\mathrm{Sp}(W_{\mathbb{A}})$ (cf. (0.11)). For a quadratic character χ of $F_{\mathbb{A}}^\times/F^\times$, let $I_n(s,\chi) = I_{P_{\mathbb{A}}}^{G_{\mathbb{A}}}(\chi|\,|_{\mathbb{A}}^s)$ be the degenerate principal series representation of $G_{\mathbb{A}}$ induced from $P_{\mathbb{A}}$ (cf. (2.4)). Given a standard section $\Phi(s) \in I_n(s,\chi)$ (see the paragraph following (2.5) for the definition), and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > \frac{n+1}{2}$, there is a Siegel Eisenstein series

$$E(g,s,\Phi) = \sum_{\gamma\in P_F\backslash G_F} \Phi(\gamma g, s). \tag{I.2}$$

These series have a meromorphic analytic continuation to the whole s plane and are holomorphic at $s = 0$. Their values at $s = 0$ define an intertwining map,

$$E(0)\colon\ I_n(0,\chi) \longrightarrow \mathcal{A}(G), \tag{I.3}$$

from the degenerate principal series at $s = 0$ to the space of automorphic forms $\mathcal{A}(G)$ on $G_{\mathbb{A}}$.

For any place v of F, the local degenerate principal series $I_{n,v}(s,\chi_v)$ of G_v which occur in the factorization $I_n(s,\chi) = \otimes_v I_{n,v}(s,\chi_v)$ have been well studied, and, in Section 1, we review their decomposition at $s = 0$ (Proposition 1.1). For example, if v is a nonarchimedean place of F, then

$$I_{n,v}(0,\chi_v) = R_n(V^+) \oplus R_n(V^-). \tag{I.4}$$

Here $V^{\pm}$ is the quadratic space over F_v of dimension $n+1$, character χ_v and Hasse invariant ± 1, and $R_n(V)$ is an irreducible representation of G_v which corresponds to the trivial representation of $O(V)$ under the local theta correspondence. For an archimedean place, there is an analogous decomposition

$$I_{n,v}(0,\chi) = \oplus R_n(V) \tag{I.5}$$

where V runs over quadratic forms of signature (p,q), $p+q=n+1$ with $\chi_v(-1) = (-1)^{\frac{n(n+1)}{2}+q}$. These local decompositions give rise to a global decomposition

$$I_n(0,\chi) = \left(\oplus_V \Pi_n(V)\right) \oplus \left(\oplus_{\mathcal{C}} \Pi_n(\mathcal{C})\right), \tag{I.6}$$

where V runs over quadratic spaces over F of dimension $n+1$ and character χ, and $\mathcal{C}$ runs over incoherent collections (Definition 2.1), $\{\mathcal{C}_v\}$, of local quadratic spaces of the type just described, such that the product formula for the Hasse invariant fails:

$$\prod_v \varepsilon_v(\mathcal{C}_v) = -1. \tag{I.7}$$

In this decomposition, the restricted products $\Pi_n(V) = \otimes_v R_n(V_v)$ and $\Pi_n(\mathcal{C}) = \otimes_v R_n(\mathcal{C}_v)$ are irreducible representations of $G_{\mathbb{A}}$.

A fundamental fact is that (Theorem 2.2)

$$\ker(E(0)) = \left(\oplus_{\mathcal{C}} \Pi_n(\mathcal{C})\right). \tag{I.8}$$

Moreover, for a global quadratic space V, there is also a (regularized) theta integral, (3.3) and (3.4),

$$I\colon\ S(V(\mathbb{A})^n) \longrightarrow \mathcal{A}(G), \tag{I.9}$$

and a natural map

$$\lambda\colon\ S(V(\mathbb{A})^n) \longrightarrow \Pi_n(V) \subset I_n(0,\chi). \tag{I.10}$$

Then, for a suitable normalization of measures, (Theorem 3.1)

$$E(0) \circ \lambda = 2 \cdot I. \tag{I.11}$$

These results together constitute a special case of the extended Siegel-Weil formula [30], [27], [47]. They lead to the following natural question: If $\Phi(s)$ is a standard section with $\Phi(0) \in \Pi_n(\mathcal{C})$ for some incoherent collection $\mathcal{C}$, what is the nature of the leading term $E'(g,0,\Phi)$ of the Laurent expansion of $E(g,s,\Phi)$ at $s=0$? The results of the remainder of the present paper suggest that these central derivatives of incoherent Eisenstein series have some connection with arithmetical algebraic geometry and, in particular, involve the height pairing of certain algebraic cycles on Shimura varieties.

In Sections 4–6 of Part I, we discuss the derivatives at $s = 0$ of the nonsingular Fourier coefficients of an incoherent Eisenstein series for general n. A Siegel Eisenstein series on $G_{\mathbb{A}}$ has a Fourier expansion

$$E(g,s,\Phi) = \sum_{T\in \mathrm{Sym}_n(F)} E_T(g,s,\Phi). \tag{I.12}$$

Moreover, if $\det(T) \neq 0$ and if $\Phi(s) = \otimes_v \Phi_v(s)$ is a factorizable section, then there is a product expansion ((4.4) and (4.5))

$$E_T(g,s,\Phi) = \prod_v W_{T,v}(g,s,\Phi_v), \tag{I.13}$$

where almost all factors are give by the inverse of an Euler factor (Proposition 4.1) which is nonzero and finite at $s = 0$. In general, for $\det(T) \neq 0$, the map $\Phi_v(0) \mapsto W_{T,v}(e,0,\Phi_v)$ defines a functional on $I_{n,v}(0,\chi_v)$. The crucial fact is that this functional is nonvanishing on the summand $R_n(V)$ of $I_{n,v}(0,\chi_v)$ if and only if the local quadratic space V of dimension $n+1$ over F_v represents T. Since the determinant of V is fixed by the requirement that $\chi_V = \chi_v$, a necessary (and in the nonarchimedean case, sufficient) condition for V to represent T is that the Hasse invariants $\varepsilon_v(V)$ and $\varepsilon_v(T)$ be related by (Proposition 1.3)

$$\varepsilon_v(V) = \chi_v(\det T)(\det T, -(-1)^{\frac{n(n+1)}{2}})_v \varepsilon_v(T). \tag{I.14}$$

For an incoherent Eisenstein series associated to a collection $\mathcal{C}$, this condition must fail to hold at a set of places $\mathrm{Diff}(T,\mathcal{C})$ with $|\,\mathrm{Diff}(T,\mathcal{C})|$ odd (Corollary 5.2), and so the only nonsingular T's which can contribute to $E'(g,0,\Phi)$ are those for which $|\,\mathrm{Diff}(T,\mathcal{C})| = 1$. For such a T, with $\mathrm{Diff}(T,\mathcal{C}) = \{v\}$,

$$E'_T(g,0,\Phi) = W'_{T,v}(g_v,0,\Phi_v) \cdot \prod_{u\neq v} W_{T,u}(g_u,0,\Phi_u). \tag{I.15}$$

Finally, as we will explain in Section 6, an application of the Siegel-Weil formula of Section 3 yields an expression (Theorem 6.1):

$$\begin{aligned} E'(g,0,\Phi) \quad &= \quad \sum_v \sum_{\substack{T \quad \det T\neq 0 \\ \mathrm{Diff}(T,\mathcal{C})=\{v\}}} \frac{W'_{T,v}(g_v,0,\Phi_v)}{W_{T,v}(g'_v,0,\Phi'_v)} \cdot I_T(g',\varphi^{(v)}) \\ &\quad + E'(g,0,\Phi)_{\mathrm{sing}}, \end{aligned} \tag{I.16}$$

for the derivative at $s = 0$. Here, $I_T(g',\varphi^{(v)})$ is the T^{th} Fourier coefficient of a regularized theta integral, as in (I.9), (3.3), and (3.4), and $E'(g,0,\Phi)_{\mathrm{sing}}$ is the contribution of the Fourier coefficients for singular T's.

In Parts II and III, we will show that, at least in the case $n = 2$ (genus 2) and $F = \mathbb{Q}$, the restriction of this expansion to a certain subgroup expresses the height pairing, in the sense of Gillet and Soulé, of 0-cycles on Shimura curves. In Section 16, we speculate on the possibilty of such a relationship in general.

1. *Some local theory.* In this section we recall some relations between Weil representations and induced representations which might be viewed as a local Siegel-Weil theory. For this section, we write F for a local field of characteristic 0, with a fixed nondegenerate additive character ψ.

Let V, $(\ ,\ )$, be a nondegenerate inner product space of dimension m over F, and let $\det V$ denote the image in $F^\times/F^{\times,2}$ of the determinant of the matrix

$$Q = \frac{1}{2}((v_i, v_j)), \tag{1.1}$$

where $v_1, \ldots, v_m$ is any F-basis for V. Note that this is the matrix for the quadratic form, $Q[x] = \frac{1}{2}(x, x)$, associated to V. We define a quadratic character χ_V of $F^\times$ by setting

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det V)_F, \tag{1.2}$$

where, as usual, $(\ ,\ )_F$ denotes the quadratic Hilbert symbol for F. We also let

$$\chi_V^\psi(x) = \chi_V(x)\gamma_F(x,\psi)^{-1}, \tag{1.3}$$

where $\gamma_F(x,\psi) = \gamma_F(x\psi)/\gamma_F(\psi)$ is the Weil index, an 8^{th} root of unity ([43], [22]).

Let W be a symplectic vector space of dimension $2n$ over F with a fixed standard basis; let $\mathrm{Sp}(W)$ be the symplectic group, and let $\mathrm{Mp}(W)$ be the metaplectic extension of $\mathrm{Sp}(W)$:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(W) \longrightarrow \mathrm{Sp}(W) \longrightarrow 1. \tag{1.4}$$

The choice of a standard basis determines an isomorphism $\mathrm{Mp}(W) \simeq \mathrm{Sp}(W) \times \mathbb{C}^1$, where the multiplication on the right-hand side is given by the Rao cocycle, valued in ± 1 ([43], [22]). Let

$$G = \begin{cases} \mathrm{Sp}(W) & \text{if } m \text{ is even,} \\ \mathrm{Mp}(W) & \text{if } m \text{ is odd.} \end{cases} \tag{1.5}$$

Associated to the quadratic space V and the fixed additive character ψ of F, there is a Weil representation $(\omega_V, S(V^n))$ of G. It will be useful to recall the following formulas: for $\varphi \in S(V^n)$, $x = (x_1, \ldots, x_n) \in V^n$ (row vector with entries in V), $a \in \mathrm{GL}_n(F)$ and $b \in \mathrm{Sym}_n(F)$,

$$\omega(m(a))\varphi(x) = \chi_V(a)\,|a|^{\frac{m}{2}}\varphi(xa)\begin{cases} 1 & \text{if } m \text{ is even} \\ \gamma(\det(a),\psi)^{-1} & \text{if } m \text{ is odd,} \end{cases} \tag{1.6}$$

$$\omega(n(b))\varphi(x) = \psi(\mathrm{tr}(bQ[x]))\,\varphi(x), \tag{1.7}$$

and, when m is odd, the central $\mathbb{C}^1$ in $G = \mathrm{Mp}(W)$ acts by

$$\omega((1,t))\varphi = t \cdot \varphi, \tag{1.8}$$

for all $t \in \mathbb{C}^1$. When m is even, the central $\mathbb{C}^1$ acts trivially. Here and elsewhere, we use the convention that, for $a \in \mathrm{GL}_n(F)$, $\chi(a) = \chi(\det a)$.

The Weil representation of G on $S(V^n)$ commutes with the natural action of $O(V)$ on this space. Let $R_n(V)$ be the maximal quotient of $S(V^n)$ on which $O(V)$ acts trivially. These representations of G are the constitutents of certain degenerate principal series representations [29], [48], [28], as we will recall in a moment in the case of interest to us.

From now on we *assume that* $m = \dim(V) = n+1$, so that

$$G = \begin{cases} \mathrm{Sp}(W) & \text{if } n \text{ is odd, and} \\ \mathrm{Mp}(W) & \text{if } n \text{ is even.} \end{cases} \tag{1.9}$$

For a fixed quadratic character χ of $F^\times$ and for $s \in \mathbb{C}$, we have the usual degenerate principal series representation $I_n(s,\chi)$ of G consisting of all smooth functions $\Phi(s)$ on G such that

$$\begin{aligned} \Phi(n\,m(a)g,s) &= \chi(a)\,|a|^{s+\rho_n}\Phi(g,s) \\ &\quad\times \begin{cases} 1 & \text{if } n \text{ is odd} \\ & \text{(so } m = n+1 \text{ is even)} \\ \gamma(\det(a),\psi)^{-1} & \text{if } n \text{ is even} \\ & \text{(so } m \text{ is odd).} \end{cases} \end{aligned} \tag{1.10}$$

Here $\rho_n = \frac{n+1}{2}$. If n is even, then $\Phi(tg,s) = t \cdot \Phi(g,s)$ for all t in the central $\mathbb{C}^1$ in G. Also, if $F = \mathbb{R}$, we add the condition that $\Phi(s)$ be K_∞ finite, where K_∞ is our standard maximal compact subgroup of G.

By the formulas above, if $\chi_V = \chi$, there is a G intertwining map:

$$\lambda\colon\ S(V^n) \longrightarrow I_n(0,\chi) \qquad \varphi \mapsto \Phi, \tag{1.11}$$

where

$$\Phi(g) = \lambda(\varphi)(g) = \omega(g)\varphi(0). \tag{1.12}$$

This map factors through the quotient $R_n(V)$ of $S(V^n)$.

We extend Φ to a section $\Phi(s) \in I_n(s,\chi)$ by setting

$$\Phi(g,s) = \omega(g)\varphi(0) \cdot |a(g)|^s, \tag{1.13}$$

where $|a(g)|$ is defined by writing $g = nm(a)k$ for $a \in \mathrm{GL}_n(F)$ and $k \in K$, our fixed maximal compact subgroup of G, and taking $|a(g)| = |\det a|_F$.

A section $\Phi(s) \in I_n(s,\chi)$ will be called standard if its restriction to K is independent of s. Note that the sections defined by (1.12) and (1.13) are standard.

Recall that, if F is nonarchimedean, the space V of dimension $m = n+1$ is determined up to isomorphism by the character χ_V together with its Hasse invariant $\varepsilon(V)$, while, in the case $F = \mathbb{R}$, V is determined by its signature (p,q).

PROPOSITION 1.1. *Fix a quadratic character* χ *of* $F^\times$.

(i) ([40], [28]) *The map* (1.11) *factors through an inclusion* $R_n(V) \hookrightarrow I_n(0,\chi)$, *where* $\chi = \chi_V$.

(ii) *The representations* $R_n(V)$ *are irreducible and*

$$I_n(0,\chi) = \bigoplus_{\substack{\dim V = n+1 \\ \chi_V = \chi}} R_n(V).$$

In particular, if F *is nonarchimedean,*

$$I_n(0,\chi) = R_n(V^+) \oplus R_n(V^-),$$

where $V^\pm$ *is the quadratic space of dimension* $n+1$, *character* $\chi_V = \chi$ *and Hasse invariant* $\varepsilon(V^\pm) = \pm 1$. *If* $F = \mathbb{R}$, *then*

$$I_n(0,\chi) = \bigoplus_{\substack{p+q=n+1 \\ (-1)^q = (-1)^{\frac{n(n+1)}{2}} \chi(-1)}} R_n(V(p,q)).$$

Note that the invariants $\chi = \chi_V$, $\dim V$, and $\varepsilon(V)$ are subject to the usual constraints ([44]): if $\dim V = 1$, then $\varepsilon(V) = 1$, and if $\dim V = 2$ and $\chi_V = 1$, then $\varepsilon(V) = 1$.

The summands of $I_n(0,\chi)$ are distinguished by the generalized Whittaker functionals.

For a moment, we suppose that F is nonarchimedean. For $T \in \mathrm{Sym}_n(F)$, let ψ_T be the character of N defined by

$$\psi_T(n(b)) = \psi(\mathrm{tr}(Tb)), \tag{1.14}$$

and let $R_n(V)_{N,T}$ (resp. $S(V^n)_{N,T}$) be the maximal quotient of $R_n(V)$ (resp. $S(V^n)$) on which N acts by the character ψ_T (the twisted Jacquet module). Then the analogue of Lemma 2.3 of [30] is the following:

PROPOSITION 1.2. (i) *Let*

$$\Omega_T = \{x \in V^n \mid Q[x] = T\}.$$

Then the map $S(V^n) \to S(V^n)_{N,T}$ *can be realized as the restriction* $S(V^n) \to S(\Omega_T)$. *If* $\det T \neq 0$, *then* Ω_T *is a single* $O(V)$ *orbit, and the map*

$$S(V^n) \longrightarrow (S(V^n)_{N,T})_{O(V)} = R_n(V)_{N,T}$$

can be realized by

$$\varphi \mapsto \int_{\Omega_T} \varphi(x)\, d\mu_T(x),$$

for the unique (up to scalar) $O(V)$-*invariant measure* $d\mu_T$ *on* Ω_T.

(ii) *If* $\det(T) \neq 0$,

$$\dim R_n(V)_{N,T} = \begin{cases} 1 & \text{if } \Omega_T(F) \neq \phi \\ 0 & \text{otherwise.} \end{cases}$$

The following 'dichotomy' plays a fundamental role. Recall that V is said to represent T if the hyperboloid $\Omega_T(F) \neq \phi$.

PROPOSITION 1.3. *Fix a quadratic character χ of $F^\times$. Then, for each $T \in \mathrm{Sym}_n(F)$ with $\det T \neq 0$, there is a unique space V of dimension $n+1$ and character $\chi_V = \chi$ such that V represents T. The Hasse invariant of V is given by*

$$\varepsilon_F(V) = \chi(\det T)\,(\det T, -(-1)^{\frac{n(n+1)}{2}})_F\, \varepsilon_F(T).$$

Proof. If V is a quadratic space of dimension $n+1$ with character $\chi_V = \chi$ which represents T, then the matrix for the quadratic form on V must have the form

$$Q = \begin{pmatrix} T & \\ & \det Q \cdot (\det T)^{-1} \end{pmatrix}, \tag{1.15}$$

for a suitable basis. The Hasse invariant of this form is:

$$\begin{aligned} \varepsilon_F(V) &= \varepsilon_F(T)\,(\det T, \det Q \cdot \det T)_F \\ &= \chi(\det T)\,(\det T, -(-1)^{\frac{n(n+1)}{2}})_F\; \varepsilon_F(T), \end{aligned} \tag{1.16}$$

as claimed. □

The analogue of Proposition 1.2 for $F = \mathbb{R}$ with $S(V^n)$ the Schwartz space of V^n and appropriate continuity conditions is discussed in detail in [31, pp. 492–493] and in [30, pp. 23–26]. We will not repeat that discussion here.

The 'dichotomy' conditions on Jacquet functors, just described, give rise to analogous conditions for the nonvanishing of certain generalized Whittaker integrals at $s = 0$. For arbitrary F and for $\mathrm{Re}(s) > \rho_n = \frac{n+1}{2}$, the generalized Whittaker integral on $I_n(s,\chi)$ is defined by

$$W_T(g, s, \Phi) = \int_N \Phi(w^{-1}ng, s)\,\psi_T(n)^{-1}\,dn, \tag{1.17}$$

where dn is the Haar measure on $N \simeq \mathrm{Sym}_n(F)$ which is self dual with respect to the pairing $N \times N \longrightarrow \mathbb{C}^1$ defined by $\langle n(b_1), n(b_2)\rangle = \psi(\mathrm{tr}(b_1 b_2))$. Here

$$w = \begin{pmatrix} & 1_n \\ -1_n & \end{pmatrix} \in \mathrm{Sp}(W). \tag{1.18}$$

We write $W_T(s)$ for the linear functional on $I_n(s,\chi)$ given by

$$W_T(s)(\Phi) = W_T(e, s, \Phi). \tag{1.19}$$

Note that this functional satisfies

$$W_T(s)(r(n)\Phi) = \psi_T(n)\, W_T(s)(\Phi), \tag{1.20}$$

for all $n \in N$, where $r(g)$ denotes the action of g on $I_n(s,\chi)$ by right multiplication.

PROPOSITION 1.4. (i) (Karel [16], Wallach [50]). *If* $\det T \neq 0$, $W_T(g,s,\Phi)$ *has an entire analytic continuation.*

(ii) *If* $\det T \neq 0$, *then the restriction of* $W_T(0)$ *to the summand* $R_n(V)$ *is nonzero if and only if* V *represents* T.

(iii) *Suppose that* F *is nonarchimedean. Then, by Proposition* 1.3, *the restriction of* $W_T(0)$ *to the summand* $R_n(V)$ *is nonzero if and only if*

$$\varepsilon_F(V) = \chi(\det T)\,(\det T, -(-1)^{\frac{n(n+1)}{2}})_F\, \varepsilon_F(T).$$

(iv) *Suppose that* $F = \mathbb{R}$. *Then the condition of* (iii) *is necessary for the nonvanishing of the restriction of* $W_T(0)$ *to the summand* $R_n(V)$. *The precise condition for nonvanishing, when* $\mathrm{sig}(T) = (a,b)$, *is*

$$\mathrm{sig}(V) = \begin{cases} (a+1,b) & \text{if } (-1)^b = (-1)^{\frac{n(n+1)}{2}}\chi(-1), \\ (a,b+1) & \text{if } (-1)^b = -(-1)^{\frac{n(n+1)}{2}}\chi(-1). \end{cases}$$

2. *Siegel Eisenstein series.* We now turn to the global situation. Let F be a totally real number field with $|F:\mathbb{Q}| = d$, and with a fixed nondegenerate additive character ψ of $F_{\mathbb{A}}/F$. For a symplectic vector space W of dimension $2n$ over F, equipped with a standard basis, let $\mathrm{Sp}(W)$ be the symplectic group and let $\mathrm{Mp}(W_{\mathbb{A}})$ be the metaplectic extension of $\mathrm{Sp}(W_{\mathbb{A}})$:

$$1 \longrightarrow \mathbb{C}^1 \longrightarrow \mathrm{Mp}(W_{\mathbb{A}}) \longrightarrow \mathrm{Sp}(W_{\mathbb{A}}) \longrightarrow 1. \tag{2.1}$$

As in the local case, set

$$G_{\mathbb{A}} = \begin{cases} \mathrm{Sp}(W_{\mathbb{A}}) & \text{if } n \text{ is odd} \\ \mathrm{Mp}(W_{\mathbb{A}}) & \text{if } n \text{ is even,} \end{cases} \tag{2.2}$$

and, for a place v of F, set

$$G_v = \begin{cases} \mathrm{Sp}(W_v) & \text{if } n \text{ is odd} \\ \mathrm{Mp}(W_v) & \text{if } n \text{ is even.} \end{cases} \tag{2.3}$$

Also, let $G_F = \mathrm{Sp}(W_F)$, which we identify with a subgroup of $G_{\mathbb{A}}$ (via the canonical splitting, when n is even).

For a character χ of $F_{\mathbb{A}}^{\times}/F^{\times}$, the global induced representation $I_n(s,\chi)$ consists of smooth, K_∞-finite functions $\Phi(s)$ on $G_{\mathbb{A}}$ which satisfy

$$(2.4)\quad \Phi(n\,m(a)g,s) = \chi(a)|a|_{\mathbb{A}}^{s+\rho_n}\Phi(g,s)\cdot\begin{cases}1 & \text{if } n \text{ is odd}\\ \gamma_{\mathbb{A}}(\det(a),\psi)^{-1} & \text{if } n \text{ is even.}\end{cases}$$

Here $\rho_n = \frac{n+1}{2}$. When n is even, the central $\mathbb{C}^1$ in $G_{\mathbb{A}}$ acts by

$$(2.5)\qquad \Phi(t\,g,s) = t\cdot\Phi(g,s),$$

for all $t\in\mathbb{C}^1$. A section will be called *standard* if its restriction to K is independent of s, where $K = K_\infty K_f$ is our fixed standard maximal compact subgroup of $G_{\mathbb{A}}$. A standard section $\Phi(s)$ is determined by it value $\Phi(0)\in I_n(0,\chi)$, and any element of $I_n(0,\chi)$ has a unique extension to a standard section in $I_n(s,\chi)$. Note that the global degenerate principal series is a restricted tensor product of the local ones

$$(2.6)\qquad I_n(s,\chi)\simeq \otimes_v' I_{n,v}(s,\chi_v),$$

in a suitable sense [47].

The Eisenstein series associated to a section $\Phi(s)$ is defined, as usual, by

$$(2.7)\qquad E(g,s,\Phi) = \sum_{\gamma\in P_F\backslash G_F}\Phi(\gamma g,s),$$

where we identify $G_F = \mathrm{Sp}(W_F)$ with it image in $G_{\mathbb{A}}$. The series converges absolutely for $\mathrm{Re}(s) > \rho_n$, and it has a meromorphic analytic continuation and functional equation

$$(2.8)\qquad E(g,s,\Phi) = E(g,-s,M(s)\Phi),$$

where

$$(2.9)\qquad M(s,\chi)\colon\ I_n(s,\chi)\longrightarrow I_n(-s,\chi^{-1})$$

is the usual intertwining operator, defined by the global analogue of (1.17), with $T=0$; see (4.1) below.

Now suppose that χ is a quadratic character of $F_{\mathbb{A}}^{\times}/F^{\times}$. The results on the local constituents of $I_{n,v}(0,\chi_v)$, described in the previous section, imply that there is a direct sum decomposition

$$(2.10)\qquad I_n(0,\chi)\simeq\Big(\bigoplus_V \Pi_n(V)\Big)\oplus\Big(\bigoplus_{\mathcal{C}}\Pi_n(\mathcal{C})\Big)$$

into irreducible constitutents defined as follows. First, V runs over all global quadratic spaces of dimension $n+1$ with $\chi_V = \chi$, and

$$(2.11)\qquad \Pi_n(V) := \otimes_v' R_{n,v}(V_v)$$

is the associated irreducible genuine representation of $G_{\mathbb{A}}$. Similarly, $\mathcal{C}$ runs over all *incoherent collections* of dimension $n+1$ and character χ:

Definition 2.1. For fixed n and χ, an *incoherent collection* $\mathcal{C} = \{\mathcal{C}_v\}$ of quadratic spaces is a set of quadratic spaces $\mathcal{C}_v$, indexed by the places of F, such that

(i) For all v, $\dim_{F_v}(\mathcal{C}_v) = n+1$, and $\chi_{\mathcal{C}_v} = \chi_v$.

(ii) The collection $\mathcal{C}$ is almost everywhere unramified, i.e., for some (any) global quadratic space V of dimension $n+1$ and character $\chi_V = \chi$, $\mathcal{C}_v \simeq V_v$, for almost all v.

(iii) (Incoherence condition) The product formula fails for the Hasse invariants:

$$\prod_v \varepsilon_v(\mathcal{C}_v) = -1.$$

The restricted product

$$\Pi_n(\mathcal{C}) := \otimes'_v R_{n,v}(\mathcal{C}_v) \tag{2.12}$$

is well defined and is an irreducible admissible representation of $G_{\mathbb{A}}$.

The following result is Theorem 4.10 of [30] when n is odd, and, in the metaplectic case, when n is even, it is proved in the same way.

THEOREM 2.2. (i) *For any standard section $\Phi(s) \in I_n(s,\chi)$, the Eisenstein series $E(g,s,\Phi)$ is holomorphic at $s=0$.*

(ii) *The map*

$$E(0)\colon\ I_n(0,\chi) \longrightarrow \mathcal{A}(G) \qquad\qquad \Phi(0) \mapsto E(\cdot,0,\Phi)$$

induces an injective intertwining operator

$$E(0)\colon\ \left(\bigoplus_V \Pi_n(V)\right) \hookrightarrow \mathcal{A}(G),$$

and so has kernel precisely

$$\left(\bigoplus_{\mathcal{C}} \Pi_n(\mathcal{C})\right).$$

Here $\mathcal{A}(G)$ is the space of (genuine, if n is even) automorphic forms on $G_F\backslash G_{\mathbb{A}}$.

Remark 2.3. (i) The Siegel-Weil formula, to be reviewed in the next section, identifies the space $E(0)(\Pi_n(V))$ as a space of theta functions associated to V.

(ii) The main goal of this paper is to obtain information about the functions

$$E'(g, 0, \Phi) = \frac{\partial}{\partial s}\{E(g, s, \Phi)\}\Big|_{s=0},$$

in the case of an *incoherent section* $\Phi(0) \in \Pi_n(\mathcal{C})$.

(iii) Note that, by a result of Langlands [34] the constituents $\Pi_n(\mathcal{C})$ are also automorphic representations.

3. *The Siegel-Weil formula.* In this section we describe the 'extended' Siegel-Weil formula, [30]. We restrict ourselves, however, to the center of the critical strip. Recall that we have fixed a global additive character ψ.

For a global quadratic space V of dimension $n+1$ and character χ, there is a global Weil representation $(\omega_V, S(V_{\mathbb{A}}^n))$ of $G_{\mathbb{A}}$. It is the restricted product of the local representations described in Section 2:

$$S(V_{\mathbb{A}}^n) \simeq \otimes_v' S(V_v^n), \tag{3.1}$$

and there are formulas analogous to (1.6), (1.7), and (1.8) describing the action of $n(b)$, $m(a)$ and $t \in \mathbb{C}^1$.

As usual, for $g \in G_{\mathbb{A}}$, $h \in O(V_{\mathbb{A}})$, and $\varphi \in S(V_{\mathbb{A}}^n)$, we have the theta function

$$\theta(g, h; \varphi) = \sum_{x \in V_F^n} (\omega(g)\varphi)\,(h^{-1}x), \tag{3.2}$$

and the theta integral

$$I(g, \varphi) = \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \theta(g, h; \varphi)\, dh, \tag{3.3}$$

where we normalize the measure dh to have $\mathrm{vol}(O(V_F)\backslash O(V_{\mathbb{A}})) = 1$. In general, the theta integral is absolutely convergent when V is anisotropic over F, or when $m - r > n+1$, where r is the Witt index of V (dimension of a maximal F-isotropic subspace). Since we have assumed that $m = n+1$, the theta integral is only absolutely convergent when V anisotropic over F. When V is isotropic, the theta integral must be defined by a regularization process:

$$I(g, \varphi) := \frac{1}{P(0)} \int_{O(V_F)\backslash O(V_{\mathbb{A}})} \theta(g, h; \omega(z)\varphi)\, dh, \tag{3.4}$$

where $z \in \mathfrak{z}(\mathfrak{sp}(n, \mathbb{R}))$ is the 'regularizing differential operator' and $P(0)$ is its 'eigenvalue' at $s = 0$ (see [30] for more details).

The theta integral (defined by regularization for V isotropic) induces a $G_{\mathbb{A}}$ intertwining map

$$S(V_{\mathbb{A}}^n) \xrightarrow{\ I\ } \mathcal{A}(G) \tag{3.5}$$

which factors through the maximal quotient $\Pi_n(V)$ of $S(V_{\mathbb{A}}^n)$ on which $O(V)(\mathbb{A})$ acts trivially. Let λ: $S(V_{\mathbb{A}}^n) \to I_n(0,\chi)$ be given by the analogue of (1.11).

THEOREM 3.1 (Extended Siegel-Weil Formula). *On* $S(V_{\mathbb{A}}^n)$,

$$E(0) \circ \lambda = 2 \cdot I.$$

Remark 3.2. When V is anisotropic, this is proved in [47], by the method of [27]. When V is isotropic, it follows by the argument of [30].

Comparing Fourier coefficients, we have

COROLLARY 3.3. *For* $T \in \mathrm{Sym}_n(F)$,

$$E_T(g,0,\Phi) = 2 \cdot I_T(g,\varphi),$$

where $\Phi(0) \in I_n(0,\chi)$ *is the image of* $\varphi \in S(V_{\mathbb{A}}^n)$, $I_T(g,\varphi)$ *is the* T^{th} *Fourier coefficient of the* (*regularized*) *theta integral* (3.3) *or* (3.4), *and* $E_T(g,0,\Phi)$ *is the value at* $s = 0$ *of the* T^{th} *Fourier coefficient of the Eisenstein series* $E(g,s,\Phi)$ (*cf.* (4.1)).

4. *Nonsingular Fourier coefficients of Siegel Eisenstein series.* In this section we collect some facts about the nonsingular Fourier coefficients of the Eisenstein series $E(g,s,\Phi)$ on the group $G_{\mathbb{A}}$.

Recall that there are natural homomorphisms $\mathrm{Sym}_n(\mathbb{A}) \simeq N(\mathbb{A}) \to G_{\mathbb{A}}$, $b \mapsto n(b)$ and $\mathrm{Sp}(W_F) = G_F \to G_{\mathbb{A}}$ which agree on $N(F)$. We will usually identify G_F with its image in $G_{\mathbb{A}}$. We take the Tamagawa measure dn on $N(\mathbb{A})$, and recall that this measure is self dual with respect to the pairing $\langle n(b_1), n(b_2)\rangle = \psi(\mathrm{tr}(b_1 b_2))$, where we write $dn(b) = db$. We also write $db = \prod_v d_v b_v$ where the local measures are self dual with respect to the local pairings $\langle b_1, b_2\rangle = \psi_v(tr(b_1 b_2))$.

For $T \in \mathrm{Sym}_n(F)$ and $g \in G_{\mathbb{A}}$, let

$$E_T(g,s,\Phi) = \int_{N(F)\backslash N(\mathbb{A})} E(ng,s,\Phi)\, \psi_T(n)^{-1}\, dn, \tag{4.1}$$

where, as before, $\psi_T(n(b)) = \psi(\mathrm{tr}(Tb))$. For $\mathrm{Re}(s) > \rho_n$, this integral may be unfolded in the usual way [41], and, if $\det T \neq 0$, it reduces to the single term

$$\int_{\mathrm{Sym}_n(\mathbb{A})} \Phi(w^{-1}n(b)g,s)\, \psi(-\mathrm{tr}(Tb))\, db, \tag{4.2}$$

where

$$w = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \in \mathrm{Sp}(W_F). \tag{4.3}$$

Assuming that $\Phi(s) = \otimes_v \Phi_v(s)$, we obtain, for $\mathrm{Re}(s) > \rho_n$ and for $g \in G_{\mathbb{A}}$,

$$E_T(g,s,\Phi) = \prod_v W_{T,v}(g_v,s,\Phi_v), \tag{4.4}$$

where

$$W_{T,v}(g_v,s,\Phi_v)=\int_{\mathrm{Sym}_n(F_v)}\Phi_v(w^{-1}n(b)g_v,s)\,\psi_v(-\operatorname{tr}(Tb))\,db, \tag{4.5}$$

is the local generalized Whittaker function (1.17). Note that

$$W_{T,v}(g_v,s,\Phi_v)=W_{T,v}(e,s,r(g_v)\Phi_v), \tag{4.6}$$

where $r(g_v)$ denotes the action by right translation of $g_v\in G_v$ in the local induced representation $I_{n,v}(s,\chi_v)$. Recall that, for T nonsingular, these local integrals have an entire analytic continuation.

PROPOSITION 4.1 ([41, p. 102]). *If v is an odd nonarchimedean place at which χ_v and ψ_v are unramified,* $\mathrm{ord}_v\det T=0$, *and $\Phi_v(s)=\Phi_v^0(s)$ is the normalized spherical vector in $I_{n,v}(s,\chi_v)$, then*

$$W_{T,v}(e,s,\Phi_v)=\begin{cases} L(s+\frac{n+1}{2},\chi_v)^{-1}\prod_{j=1}^{\frac{n-1}{2}}L_v(2s+n+1-2j,\chi_v^2)^{-1} & \textit{if } n \textit{ is odd},\\ \prod_{j=1}^{\frac{n}{2}}L_v(2s+n+2-2j,\chi_v^2)^{-1} & \textit{if } n \textit{ is even}.\end{cases}$$

COROLLARY 4.2. *Fix χ, $g\in G_{\mathbb{A}}$ and $T\in\mathrm{Sym}_n(F)$, and a standard factorizable section $\Phi(s)=\otimes_v\Phi_v(s)\in I_n(s,\chi)$. Suppose that S is a finite set of places of F such that, for $v\notin S$, v is a nonarchimedean place with odd residue characteristic, χ_v and ψ_v are unramified,* $\mathrm{ord}_v\det(T)=0$, *$\Phi_v(s)=\Phi_v^0(s)$ and $g_v\in K_v$. Then*

$$E_T(g,s,\Phi)=\left(\prod_{v\in S}W_{T,v}(e,s,r(g_v)\Phi_v)\right)$$
$$\begin{cases} L^S(s+\frac{n+1}{2},\chi)^{-1}\prod_{j=1}^{\frac{n-1}{2}}L^S(2s+n+1-2j,\chi^2)^{-1} & \textit{if } n \textit{ is odd},\\ \prod_{j=1}^{\frac{n}{2}}L^S(2s+n+2-2j,\chi^2)^{-1} & \textit{if } n \textit{ is even},\end{cases}$$

where $L^S(s,\chi^2)$ is the partial L-function, with the Euler factors of places in S omitted.

COROLLARY 4.3. *For nonsingular T, the functions $E_T(g,s,\Phi)$ have a meromorphic analytic continuation and are holomorphic at $s=0$.*

Thus the vanishing of the nonsingular coefficients at $s=0$ is controlled by that of the local factors $W_{T,v}(e,s,\Phi_v)$.

5. *Vanishing of local factors.* We now turn to the incoherent case and consider the Eisenstein series $E(g,s,\Phi)$ for a standard section $\Phi(s)=\otimes_v\Phi_v(s)$ with $\Phi(0)\in\Pi_n(\mathcal{C})$ for an incoherent family $\mathcal{C}$. In this section, we describe the order of vanishing at $s=0$ of the nonsingular Fourier coefficients.

First recall that the local factors in (4.4) and (4.5) define linear functionals

$$W_{T,v}(0) := W_{T,v}(e,0,\cdot)\colon\ I_{n,v}(0,\chi) \longrightarrow \mathbb{C}. \tag{5.1}$$

If v is a nonarchimedean place, these functionals satisfy

$$W_{T,v}(0)\,(r(n(b))\Phi) = \psi_v(\mathrm{tr}(Tb))\cdot W_{T,v}(0)\,(\Phi)\,, \tag{5.2}$$

while, if v is an archimedean place, $W_{T,v}(0)$ extends to a continuous linear functional ([50]) on the smooth induced representation $I^{\infty}_{n,v}(0,\chi)$, which satisfies

$$W_{T,v}(0)\,(r(X)\Phi) = d\psi_{T,v}(X)\,W_{T,v}(0)\,(\Phi)\,, \tag{5.3}$$

where $d\psi_{T,v}\colon \mathfrak{n}_v = \mathrm{Lie}(N_v) \to \mathbb{C}$ is the differential of the character $\psi_{T,v}(n(b)) = \psi_v(\mathrm{tr}(Tb))$, and $X \in \mathfrak{n}_v$.

By the results reviewed in Section 1, the restriction of such a linear functional to the subspace $R_n(\mathcal{C}_v) \subset I_{n,v}(0,\chi_v)$ is unique up to a scalar, and vanishes identically unless the local quadratic space $\mathcal{C}_v$ of dimension $n+1$ and character χ_v represents T. Moreover, the condition for $\mathcal{C}_v$ to represent T is given by the 'dichotomy' of Proposition 1.3.

Definition 5.1. For an incoherent collection $\mathcal{C}$, and for $T \in \mathrm{Sym}_n(F)$ with $\det(T) \neq 0$, let

$$\mathrm{Diff}(T,\mathcal{C}) := \{\ v \mid \varepsilon_v(\mathcal{C}_v) \neq \varepsilon_v(T)\,(\det T, -(-1)^{\frac{n(n+1)}{2}})_v\,\chi_v(\det T)\ \}.$$

Note that

$$\mathrm{Diff}(T,\mathcal{C}) \subset \{\ v \mid T \text{ is not represented by } \mathcal{C}_v\ \}. \tag{5.4}$$

The only difference between these sets arises from the fact that an archimedean place ∞_j can occur on the right side, even if the Hasse invariants of T and $\mathcal{C}_{\infty_j}$ are compatible, if an additional incompatibility in the signatures of T and $\mathcal{C}_\infty$ occurs (cf. (iv) of Proposition 1.4).

COROLLARY 5.2. *$|\mathrm{Diff}(T,\mathcal{C})|$ is odd*!

Proof. The key fact is that the set of places where

$$\varepsilon_v(\mathcal{C}_v) = -\varepsilon_v(T)(\det T, -(-1)^{\frac{n(n+1)}{2}})_v\chi_v(\det T) \tag{5.5}$$

has *odd* cardinality. In fact,

$$\begin{aligned}\prod_v \varepsilon_v(T)(\det T, -(-1)^{\frac{n(n+1)}{2}})_v\chi_v(\det T)&\\ = (\det T, -(-1)^{\frac{n(n+1)}{2}})_{\mathbb{A}}\,\chi(\det T)\prod_v \varepsilon_v(T) &= 1,\end{aligned} \tag{5.6}$$

whereas

$$\prod_v \varepsilon_v(\mathcal{C}_v) = -1, \tag{5.7}$$

by the incoherence condition (iii) of Definition 2.1. □

COROLLARY 5.3. *If* $\det T \neq 0$, *then*

$$\operatorname*{ord}_{s=0} E_T(g,s,\Phi) \geq |\operatorname{Diff}(T,\mathcal{C})|.$$

In particular, the only nonsingular Fourier coefficients which can contribute to $E'(g,0,\Phi)$ *are those for which* $|\operatorname{Diff}(T,\mathcal{C})| = 1$.

6. *A formula for the derivative* $E'(g,0,\Phi)$. In this section, we fix an incoherent collection $\mathcal{C}$ and a factorizable standard section $\Phi(s) = \otimes_v \Phi_v(s)$, with $\Phi(0) \in \Pi_n(\mathcal{C})$, and we consider the derivative $E'(g,0,\Phi)$ of the Eisenstein series $E(g,s,\Phi)$ at its center of symmetry, $s = 0$. Recall that the value $E(g,0,\Phi) = 0$ by Theorem 2.2.

It will be convenient to define the Schwartz space $S(\mathcal{C}^n_{\mathbb{A}})$ as follows. We fix a global quadratic space U of dimension $n+1$ and character χ, and a finite set of places S_0 of F, including all archimedean places, and such that, for $v \notin S_0$, $\mathcal{C}_v \simeq U_v$. At each place $v \notin S_0$, we fix such an isomorphism. We also choose a $\mathcal{O}_F$ lattice L in U and, for each finite place $v \notin S_0$, we let $L_v \subset \mathcal{C}_v$ be the image of the completion $L \otimes \mathcal{O}_{F_v}$ via the fixed isomorphism of U_v and $\mathcal{C}_v$. If U' is another quadratic space of dimension $n+1$ and character χ, we say that a family of isomorphisms $\mathcal{C}_v \simeq U'_v$ for all places v outside of some finite set S'_0, is *allowable* if, for any $\mathcal{O}_F$ lattice L' in U', the lattices L_v and L'_v in $\mathcal{C}_v$ coincide for all places outside of some finite set. Note that, for fixed U' and S'_0, two allowable families of isomorphisms differ by an element of $O(U')(\mathbb{A}^{S'_0})$. We may then define the Schwartz space

$$S(\mathcal{C}^n_{\mathbb{A}}) = \otimes'_v S(\mathcal{C}^n_v) \tag{6.1}$$

to be the restricted product with respect to the characteristic functions of the lattices L^n_v. This space is independent of the choice of U and L. For any U' with an allowable family of isomorphisms, there is an isomorphism

$$S(\mathcal{C}^n_{\mathbb{A}}) \simeq \Big(\otimes_{v\in S'_0} S(\mathcal{C}^n_v)\Big) \otimes S((U'_{\mathbb{A}^{S'_0}})^n). \tag{6.2}$$

Given a standard section $\Phi(s) = \otimes_v \Phi_v(s)$ with $\Phi(0) \in \Pi_n(\mathcal{C})$, we choose a Schwartz function $\varphi = \otimes_v \varphi_v \in S(\mathcal{C}^n_{\mathbb{A}})$ whose image in $I_{n,v}(0,\chi_v)$ is $\Phi_v(0)$.

By Corollary 5.3, the only nonsingular T's which can contribute to the Fourier expansion of $E'(g,0,\Phi)$ are those for which $|\operatorname{Diff}(T,\mathcal{C})| = 1$. We write

$$E(g,s,\Phi) = \sum_T E_T(g,s,\Phi). \tag{6.3}$$

Note that, since the function $b \mapsto E(n(b)g, s, \Phi)$ is a smooth function on $\mathrm{Sym}_n(\mathbb{A}_f)/\mathrm{Sym}_n(F)$ which is holomorphic in a neighborhood of $s = 0$, this series is absolutely convergent, uniformly for s is a closed disk around $s = 0$. Hence, it may be differentiated termwise with respect to s, and the resulting series for the derivative is again absolutely convergent. Upon differentiating termwise with respect to s, evaluating at $s = 0$, and collecting terms for nonsingular T's according to $\{v\} = Diff(T, \mathcal{C})$ and terms for singular T's we have

$$(6.4)\qquad E'(g,0,\Phi) = \sum_v \sum_{\substack{T \\ \mathrm{Diff}(T,\mathcal{C})=\{v\}}} E'_T(g,0,\Phi) + (\text{ terms from singular } T\ ).$$

For a fixed v, the coefficients $E'_T(g,0,\Phi)$ for nonsingular T's with $\mathrm{Diff}(T,\mathcal{C}) = \{v\}$, are related to the coefficients of certain (regularized) theta integrals attached to the global quadratic space obtained from $\mathcal{C}$ by twisting (changing the Hasse invariant) at v. More precisely, for each place v of F, we obtain from $\mathcal{C}$ a global quadratic space $V^{(v)}$ with character χ by the requirement that the localizations of $V^{(v)}$ coincide with the components of $\mathcal{C}$ at all places other than v, but that, at the place v the localization $V_v^{(v)}$ has

$$(6.5)\qquad \varepsilon_v(V_v^{(v)}) = -\varepsilon_v(\mathcal{C}_v).$$

The choice of such a space is unique if v is nonarchimedean, but, if v is real, the signature of $V^{(v)}$ can be adjusted arbitrarily, subject to the condition (6.5) on the Hasse invariant and the requirement that $\chi_v(-1) = (-1)^{q+\frac{n(n+1)}{2}}$. There is, however, a unique choice of this signature for which the resulting space $V_v^{(v)}$ represents T, given by Proposition 1.4. By the Hasse principle, the space $V^{(v)}$ then represents T globally, and it is the unique quadratic space of dimension $n+1$, character χ and 'distance one' from $\mathcal{C}$ which does so. We fix an allowable family of isomorphisms $\mathcal{C}_u \simeq V_u^{(v)}$ for $u \neq v$.

For fixed v and for a nonsingular T with $\mathrm{Diff}(T,\mathcal{C}) = \{v\}$, we have

$$(6.6)\qquad W_{T,v}(g_v, 0, \Phi_v) = W_{T,v}(e, 0, r(g_v)\Phi_v) = 0,$$

by Proposition 1.4. Choose a standard section $\Phi'_v(s)$ with $\Phi'_v(0) \in R_n(V_v^{(v)})$ and an element $g'_v \in G_v$ such that

$$(6.7)\qquad W_{T,v}(g'_v, 0, \Phi'_v) \neq 0.$$

This choice may depend on T. Also choose $\varphi'_v \in S((V_v^{(v)})^n)$ whose image in $R_n(V_v^{(v)})$ is $\Phi'_v(0)$. Let

$$(6.8)\qquad \varphi^{(v)} = \varphi'_v \otimes \varphi^v \in S((V_{\mathbb{A}}^{(v)})^n),$$

where $\varphi^v = \otimes_{u\neq v}\varphi_u$ is viewed as a function on $(V_{\mathbb{A}^v}^{(v)})^n$ via our fixed allowable family of isomorphisms and (6.2). Let

$$(6.9)\qquad \Phi^{(v)}(s) = \Phi'_v(s) \otimes (\otimes_{u\neq v}\Phi_u(s)),$$

and note that $\Phi^{(v)}(0) \in \Pi_n(V^{(v)})$. Thus $\Phi^{(v)}(s)$ is a coherent standard section. Then, using Corollary 4.2, we have

$$E'_T(g,0,\Phi) = \frac{W'_{T,v}(g_v,0,\Phi_v)}{W_{T,v}(g'_v,0,\Phi'_v)} \cdot E_T(g^{(v)},0,\Phi^{(v)}), \tag{6.10}$$

where $g^{(v)} = g'_v g^v \in G_{\mathbb{A}}$. Summarizing these constructions and applying the Siegel-Weil formula, we obtain the following result.

THEOREM 6.1. *Fix an incoherent collection $\mathcal{C}$ and a factorizable standard section $\Phi(s) = \otimes_v \Phi_v(s)$ such that $\Phi(0) \in \Pi_n(\mathcal{C})$. Choose $\varphi = \otimes_v \varphi_v \in S(\mathcal{C}^n_{\mathbb{A}})$ with image $\Phi(0)$ in $\Pi_n(\mathcal{C}) \subset I_n(0,\chi)$. For $T \in \mathrm{Sym}_n(F)$ with $\det T \neq 0$, let $\mathrm{Diff}(T,\mathcal{C})$ be the set of places of F of Definition 5.1 above.*

(i) *If $|\mathrm{Diff}(T,\mathcal{C})| > 1$, then, for any $g \in G_{\mathbb{A}}$,*

$$E'_T(g,0,\Phi) = 0.$$

(ii) *Suppose that $\mathrm{Diff}(T,\mathcal{C}) = \{v\}$, and let $V^{(v)}$ be the global quadratic space obtained from $\mathcal{C}$ by switching the Hasse invariant at v, as explained above. If v is an archimedean place, assume, furthermore that $V^{(v)}_v$ represents T. These conditions determine $V^{(v)}$ uniquely, and $V^{(v)}$ represents T globally. Choose $\varphi'_v \in S((V^{(v)}_v)^n)$, $\Phi'_v(s) \in I_{n,v}(s,\chi_v)$ and $g'_v \in G_v$ as above. Then*

$$E'_T(g,0,\Phi) = 2 \cdot \frac{W'_{T,v}(g_v,0,\Phi_v)}{W_{T,v}(g'_v,0,\Phi'_v)} \cdot I_T(g^{(v)},\varphi^{(v)}),$$

where $I_T(g^{(v)},\varphi^{(v)})$ is the T^{th} Fourier coefficient of the theta integral associated to $\varphi^{(v)} \in S((V^{(v)}_{\mathbb{A}})^n)$, as defined in Section 3, and $g^{(v)}$ is the element of $G_{\mathbb{A}}$ whose local component at v has been changed to g'_v. Note that g'_v, and hence $g^{(v)}$, may depend on T.

(iii) *Thus,*

$$\begin{aligned} E'(g,0,\Phi) \quad &= \quad \sum_v \sum_{\substack{T \\ \mathrm{Diff}(\mathcal{C},T)=\{v\}}} 2 \cdot \frac{W'_{T,v}(g_v,0,\Phi_v)}{W_{T,v}(g'_v,0,\Phi'_v)} \cdot I_T(g^{(v)},\varphi^{(v)}) \\ &\qquad + (\text{ terms from singular } T\ ). \end{aligned}$$

Note that the theta integrals in all of the terms in the sum for a fixed nonarchimedean place v are associated to the same quadratic space $V^{(v)}$, while in the case of an archimedean place v, the space $V^{(v)}$ depends on T.

Theorem 6.1 describes the basic structure of $E'(g,0,\Phi)$, the central derivative of an incoherent Eisenstein series. Note that it involves the infinite family of global quadratic spaces 'adjacent to' $\mathcal{C}$. Also, the nonvanishing nonsingular

Fourier coefficients of $E'(g,0,\Phi)$ are given as a product of a global representation number, $I_T(g^{(v)},\varphi^{(v)})$, times a term, $\frac{W'_{T,v}(g_v,0,\Phi_v)}{W_{T,v}(g'_v,0,\Phi'_v)}$ which is 'local' at v. It should be noted that, since the theta integral $I(g^{(v)},\varphi^{(v)})$ is an $O(V^{(v)})(\mathbb{A})$ invariant functional on $S((V_{\mathbb{A}}^{(v)})^n)$, the quantity $I_T(g^{(v)},\varphi^{(v)})$ does not depend on the various choices made in defining $\varphi^{(v)}$. For example, the function $\varphi^{(v)} \in S(\mathcal{C}_{\mathbb{A}}^n)$ depends on the choice of a family of allowable isomorphisms $\mathcal{C}_u \simeq V_u^{(v)}$ for $u \neq v$, but a change in the choice of this family changes $\varphi^{(v)}$ by an element of $O(V^{(v)})(\mathbb{A}^v)$, and the resulting function $I(g^{(v)},\varphi^{(v)})$ is unaffected by such a change.

Finally, we describe the pullback of $E'(g,0,\Phi)$ to certain subgroups. Associated to a decomposition

$$W = W_1 + W_2 \tag{6.11}$$

with nondegenerate symplectic spaces W_1 and W_2 of dimensions n_1 and n_2, $n_1+n_2=n$, we have a homomorphism

$$\iota\colon\ G'_{1,\mathbb{A}} \times G'_{2,\mathbb{A}} \longrightarrow G_{\mathbb{A}}, \tag{6.12}$$

where we take G'_i of the same type as G; i.e.,

$$G'_{i,\mathbb{A}} = \begin{cases} \mathrm{Sp}(W_{i,\mathbb{A}}) & \text{if } n \text{ is odd} \\ \mathrm{Mp}(W_{i,\mathbb{A}}) & \text{if } n \text{ is even.} \end{cases} \tag{6.13}$$

For $g'_1 \in G'_{1,\mathbb{A}}$ and $g'_2 \in G'_{2,\mathbb{A}}$, let

$$F(g'_1,g'_2,\Phi) := E'(\iota(g'_1,g'_2),0,\Phi). \tag{6.14}$$

Then, for $d_1 \in \mathrm{Sym}_{n_1}(F)$ and $d_2 \in \mathrm{Sym}_{n_2}(F)$, we can consider the Fourier coefficient

$$\begin{aligned}(6.15)\ & F_{d_1,d_2}(g'_1,g'_2,\Phi) \\ &:= \int_{N_1(F)\backslash N_1(\mathbb{A})} \int_{N_2(F)\backslash N_2(\mathbb{A})} F(n_1g'_1,n_2g'_2,\Phi)\,\psi_{d_1}(n_1)^{-1}\psi_{d_2}(n_2)^{-1}\,dn_1\,dn_2.\end{aligned}$$

By Theorem 6.1, this function has the following expansion:

COROLLARY 6.2.

$$F_{d_1,d_2}(g'_1,g'_2,\Phi) = \sum_v F_{d_1,d_2}(g'_1,g'_2,\Phi)_v + F_{d_1,d_2}(g'_1,g'_2,\Phi)_{\mathrm{sing}},$$

where

$$\begin{aligned}&F_{d_1,d_2}(g'_1,g'_2,\Phi)_v \\ &\quad= \sum_{\substack{T\in\mathrm{Sym}_n(F),\ \det(T)\neq 0\\ T=\begin{pmatrix} d_1 & m\\ {}^tm & d_2\end{pmatrix}\\ \mathrm{Diff}(T,\mathcal{C})=\{v\}}} 2\cdot\frac{W'_{T,v}(\iota(g'_{1,v},g'_{2,v}),0,\Phi_v)}{W_{T,v}(\iota(g'_{1,v},g'_{2,v}),0,\Phi'_v)}\cdot I_T(\iota(g'_1,g'_2),\varphi^{(v)}).\end{aligned}$$

Also,

$$F_{d_1,d_2}(g_1',g_2',\Phi)_{\mathrm{sing}} = \sum_{\substack{T\in \mathrm{Sym}_n(F),\ \det(T)=0 \\ T=\begin{pmatrix} d_1 & m \\ {}^tm & d_2\end{pmatrix}}} E_T'(\iota(g_1',g_2'),0,\Phi).$$

is the part of $F_{d_1,d_2}(g_1',g_2',\Phi)$ *associated to singular* T*'s.*

In the ratio $\frac{W'_{T,v}(\iota(g'_{1,v},g'_{2,v}),0,\Phi_v)}{W_{T,v}(\iota(g'_{1,v},g'_{2,v}),0,\Phi'_v)}$, we have taken the same $\iota(g'_{1,v},g'_{2,v})$ as argument in both numerator and denominator and we assume that Φ'_v is chosen accordingly. This was done for notational convenience; a change of argument could have been included as well. Indeed, such a change will be used in the next section.

We expect that the functions $F_{d_1,d_2}(g_1',g_2',\Phi)$ should be related to the Bloch-Beilinson-Gillet-Soulé height pairing of certain elements in the arithmetic Chow groups of certain Shimura varieties associated to orthogonal groups of signature $(n-1,2)$. In particular, the sum decomposition of Corollary 6.2 should correspond to the decomposition of the height into a sum of local heights. In the rest of this paper, we will obtain evidence for such a relationship in the simplest case $n=2$ and $n_1=n_2=1$ with $F=\mathbb{Q}$. Some speculations about the general case are included in Section 16.

PART II. Derivatives of Fourier coefficients of metaplectic Eisenstein series of genus 2

We now specialize the discussion of Part I to the case $n=2$, $n_1=n_2=1$ and $F=\mathbb{Q}$.

We fix an indefinite quaternion algebra B over $\mathbb{Q}$, and let V be the space of elements of B of trace 0. For a fixed quadratic character χ, as above, with $\chi_\infty(-1)=-1$, a suitable multiple of the reduced norm gives us a quadratic form on V of character χ and of signature $(1,2)$. We define an incoherent collection $\mathcal{C}$, in the sense of Definition 2.1, by taking $\mathcal{C}_p=V_p$ for all finite primes p and by taking $\mathcal{C}_\infty$ of signature $(3,0)$. For a Schwartz function $\varphi\in S(V(\mathbb{A}_f)^2)$ on the finite adeles of V, we let

$$\text{(II.1)} \qquad \lambda(\varphi)=\Phi_f(0)\in \Pi_2(V)_f=\Pi_2(\mathcal{C})_f,$$

and let $\Phi_f(s)\in I_{2,f}(s,\chi_f)$ be the standard section extending $\Phi_f(0)$. Let $\Phi_\infty^{\frac{3}{2}}(s)\in I_{2,\infty}(s,\chi_\infty)$ be the standard section of weight $\frac{3}{2}$. Then $\Phi(s)=\Phi_\infty^{\frac{3}{2}}(s)\otimes\Phi_f(s)$ is a standard section with $\Phi(0)\in\Pi_2(\mathcal{C})$, and we consider the nonsingular Fourier coefficients of $E(g,s,\Phi)$, the corresponding incoherent Eisenstein series. These are indexed by $T\in\mathrm{Sym}_2(\mathbb{Q})$ with $\det(T)\neq 0$.

As explained in Section 6, only T's with $|\operatorname{Diff}(T,\mathcal{C})| = 1$ can contribute to the central derivative, and hence

$$\text{(II.2)}\quad E'(g,0,\Phi) = \sum_{p\leq\infty}\left(\sum_{\substack{T\\ \operatorname{Diff}(T,\mathcal{C})=\{p\}}} W'_{T,p}(g_p,0,\Phi_p)\cdot\prod_{\ell\neq p} W_{T,\ell}(g_\ell,0,\Phi_\ell)\right) + E'(g,0,\Phi)_{\text{sing}}.$$

This expression specializes that given in Theorem 6.1 for general n.

For each prime p and nonsingular T with $\operatorname{Diff}(T,\mathcal{C}) = \{p\}$, we choose a standard section $\Phi'_p(s) \in I_{2,p}(s,\chi_p)$ such that $\Phi'_p(0) \in R_2(\mathcal{C}'_p)$, where $\mathcal{C}'_p$ has Hasse invariant $\varepsilon_p(\mathcal{C}'_p) = -\varepsilon_p(\mathcal{C}_p)$. For $p = \infty$ we require that $\mathcal{C}'_\infty$ represent T. We also take Φ'_p so that $W_{T,p}(g_p,0,\Phi'_p) \neq 0$. Then, applying the extended Siegel-Weil formula, we may write the term associated to p and T in (II.2) as

$$\text{(II.3)}\quad \frac{W'_{T,p}(g_p,0,\Phi_p)}{W_{T,p}(g_p,0,\Phi'_p)}\cdot W_{T,p}(g_p,0,\Phi'_p)\cdot\prod_{\ell\neq p} W_{T,\ell}(g_\ell,0,\Phi_\ell) = \frac{W'_{T,p}(g_p,0,\Phi_p)}{W_{T,p}(g_p,0,\Phi'_p)}\cdot 2\cdot I_T(g,\varphi^{(p)})$$

where $I_T(g,\varphi^{(p)})$ is the T^{th} Fourier coefficient of the (regularized) theta integral $I(g,\varphi^{(p)})$ determined by a function

$$\text{(II.4)}\quad \varphi^{(p)} = \varphi'_p\otimes(\otimes_{\ell\neq p}\varphi_\ell) \in S(V^{(p)}(\mathbb{A})).$$

Here $V^{(p)}$ is the global quadratic space with $V^{(p)}_\ell \simeq \mathcal{C}_\ell$ for all $\ell\neq p$ and $V^{(p)}_p \simeq \mathcal{C}'_p$, and $\varphi'_p \in S(V^{(p)}_p)$ is a function whose image in $I_{2,p}(0,\chi_p)$ is $\Phi'_p(0)$. In turn, the quantity $I_T(g,\varphi^{(p)})$ can be identified as a global (weighted) representation density, Lemmas 7.1 and 7.2, and (7.29).

For $p<\infty$, the quantity $W'_{T,p}(g_p,0,\Phi_p)$ is considered in Section 8, where, for good primes p, an explicit formula (Corollary 8.7) is obtained by differentiating the formulas of Kitaoka [19] for the local representation densities (Proposition 8.1).

For $p = \infty$, the local generalized Whittaker integral $W_{T,\infty}(g,s,\Phi^{\frac{3}{2}}_\infty)$ is studied in Section 9. For general tube domains, such functions were considered by Shimura [45]. We show that

$$\text{(II.5)}\quad \operatorname*{ord}_{s=0} W_{T,\infty}(g,s,\Phi^{\frac{3}{2}}_\infty) = \begin{cases} 0 & \text{if } T>0\\ 1 & \text{if } \operatorname{sig}(T) = (1,1) \text{ or } (0,2),\end{cases}$$

and, for $T>0$, we compute the value $W_{T,\infty}(g,0,\Phi^{\frac{3}{2}}_\infty)$ (Proposition 9.3). The case of $\operatorname{sig}(T) = (0,2)$ will play no role, since, in this case, $\infty\notin\operatorname{Diff}(T,\mathcal{C})$, so that the vanishing of the archimedean factor in the product for $E_T(g,s,\Phi)$

occurs in addition to that of the factor for at least one finite prime, and so such terms make no contribution to $E'(g,0,\Phi)$.

We then study the derivative $W'_{T,\infty}(g,0,\Phi^{\frac{3}{2}}_{\infty})$ in the case $\mathrm{sig}(T)=(1,1)$. It turns out that, after a suitable manipulation, (9.24) and (9.27), the zero of this function arises from a factor $\Gamma(s)^{-1}$. It therefore remains to 'evaluate' a certain function $\eta(2y,T;\alpha,\beta)$ at $s=0$. This is accomplished by a long series of manipulations (9.34)–(9.55), the result of which is give in Theorem 9.5. It is noteworthy that the exponential integral Ei arises here. In order to apply this formula later, a further peculiar change of variables is performed, (9.57)–(9.63), resulting in the expression of Corollary 9.8.

Thus we obtain a fairly explicit expression for the 'nonsingular' part of $E'(g,0,\Phi)$ in (II.2).

As a final step on the 'analytic side', we consider the following restriction of $E'(g,0,\Phi)$. Let $W_1=W_2$ be two copies of a 2 dimensional symplectic space over $\mathbb{Q}$, and let $W=W_1+W_2$, so that there is a natural embedding

$$\iota\colon\ G'_{\mathbb{A}}\times G'_{\mathbb{A}}\longrightarrow G_{\mathbb{A}}, \tag{II.6}$$

where $G'_{\mathbb{A}}=\mathrm{Mp}(W_{1,\mathbb{A}})=\mathrm{Mp}(W_{2,\mathbb{A}})$ is the metaplectic group. For g'_1 and $g'_2\in G'_{\mathbb{A}}$, we define

$$F(g'_1,g'_2;\Phi)=E'(\iota(g'_1,g'_2),0,\Phi), \tag{II.7}$$

the pullback of the central derivative of the incoherent Eisenstein series, and we consider the Fourier expansion of this function

$$F(g'_1,g'_2;\Phi)=\sum_{d_1,d_2\in\mathbb{Q}}F_{d_1,d_2}(g'_1,g'_2;\Phi). \tag{II.8}$$

By restricting the expansion (II.2), we can write

$$F_{d_1,d_2}(g'_1,g'_2;\Phi)=\sum_{p\le\infty}F_{d_1,d_2}(g'_1,g'_2;\Phi)_p+F_{d_1,d_2}(g'_1,g'_2;\Phi)_{\mathrm{sing}}, \tag{II.9}$$

where, for $p\le\infty$,

$$F_{d_1,d_2}(g'_1,g'_2;\Phi)_p=\sum_{\substack{T,\quad \det(T)\neq 0\\ T=\begin{pmatrix}d_1&m\\m&d_2\end{pmatrix}}}\frac{W'_{T,p}(g_p,0,\Phi_p)}{W_{T,p}(g_p,0,\Phi'_p)}\cdot 2\cdot I_T(\iota(g'_1,g'_2),\varphi^{(p)}). \tag{II.10}$$

Note that the singular part $F_{d_1,d_2}(g'_1,g'_2;\Phi)_{\mathrm{sing}}$ vanishes automatically when $d_1d_2\neq 0$ is not a square in $\mathbb{Q}^{\times}$, since, in this case, only T's with $\det(T)\neq 0$ can contribute to the restriction. Actually, we refine this expression slightly (Proposition 7.3); this refined expression, with the additional information from Sections 8 and 9 about local factors, will be compared, in Part III, with the height pairing of certain 0-cycles on Shimura curves over $\mathbb{Q}$.

7. *Siegel Eisenstein series of genus* 2. We now restrict the construction of the previous sections to the simplest case: $n = 2$ and $F = \mathbb{Q}$, so that $G_{\mathbb{A}} = \mathrm{Mp}(W_{\mathbb{A}})$ where $\dim_{\mathbb{Q}} W = 4$.

First, we define our incoherent collection. We fix an indefinite quaternion algebra B over $\mathbb{Q}$, and let $D(B)$ be the product of the primes p at which $B_p = B \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is division. Note that the case $B = M_2(\mathbb{Q})$, $D(B) = 1$ is allowed unless explicitly excluded. We write our quadratic character χ as

$$\chi(x) = (x, \kappa)_{\mathbb{Q}}, \tag{7.1}$$

for $\kappa \in \mathbb{Z}$, $\kappa \neq 0$. We assume, without loss of generality, that κ is square free. Let

$$V = V_{\chi}^{B} = \{ x \in B \mid \mathrm{tr}(x) = 0 \}, \tag{7.2}$$

with bilinear form

$$(x, y) = -\kappa \cdot \mathrm{tr}(xy^{\iota}). \tag{7.3}$$

Note that the associated quadratic form is

$$Q[x] = -\kappa \cdot \nu(x), \tag{7.4}$$

where $\nu(x)$ is the reduced norm of $x \in B$. We assume that V has signature $(1, 2)$, so that

$$\chi_{\infty}(-1) = -1 \tag{7.5}$$

and $\kappa < 0$.

We define an incoherent collection $\mathcal{C}$ of local ternary quadratic spaces by taking $\mathcal{C}_\ell = V_\ell$ for all finite primes ℓ, and $\mathcal{C}_\infty =$ the positive definite ternary space. For each prime $p \leq \infty$, there is a unique global ternary quadratic space $V^{(p)}$, with character χ, which is 'adjacent' to $\mathcal{C}$; the localizations of $V^{(p)}$ are isomorphic to those of $\mathcal{C}$ at all primes $\ell \neq p$. More explicitly, if $p \neq \infty$, let $B^{(p)}$ be the quaternion algebra over $\mathbb{Q}$ whose invariants are given by

$$\operatorname*{inv}_{\ell}(B^{(p)}) = \begin{cases} \mathrm{inv}_\ell(B) & \text{if } \ell \neq p,\ \infty \\ -\mathrm{inv}_\ell(B) & \text{if } \ell = p,\ \infty. \end{cases} \tag{7.6}$$

Then

$$V^{(p)} = \{ x \in B^{(p)} \mid \mathrm{tr}(x) = 0 \} \tag{7.7}$$

with $(x, y) = -\kappa \cdot \mathrm{tr}(xy^{\iota})$. If $p = \infty$, $B^{(\infty)} = B$ and $V^{(\infty)} = V$. We let

$$H^{(p)} = B^{(p),\times}, \tag{7.8}$$

and for each finite prime p, we fix an identification

$$B^{(p)}(\mathbb{A}_f^p) \simeq B(\mathbb{A}_f^p). \tag{7.9}$$

Thus, we obtain identifications

$$V(\mathbb{A}_f^p) \simeq V^{(p)}(\mathbb{A}_f^p) \qquad \text{and} \qquad H^{(p)}(\mathbb{A}_f^p) \simeq H(\mathbb{A}_f^p). \tag{7.10}$$

We choose a Schwartz function $\varphi \in S(V(\mathbb{A}_f)^2)$ which we assume to be factorizable, $\varphi = \otimes_p \varphi_p$ and locally even, i.e., we assume that, for all p,

$$\varphi_p(-x) = \varphi_p(x). \tag{7.11}$$

The reason for this assumption is the following. Let

$$\lambda_p \colon S(V_p^2) \longrightarrow I_{2,p}(0, \chi_p), \qquad \lambda_p(\varphi_p)(g) = \omega(g)\varphi_p(0), \tag{7.12}$$

be the intertwining map, as in (1.11) and (1.12). If $\varphi_p(-x) = -\varphi_p(x)$, then $\lambda_p(\varphi_p) = 0$. Thus the map λ_p factors through the even functions in $S(V_p^2)$, and the global map $\lambda = \otimes_p \lambda_p$ factors through the space of locally even functions. At ∞, we take $\varphi_\infty \in S(\mathcal{C}_\infty^2)$ to be the Gaussian

$$\varphi_\infty(x) = e^{-\pi \operatorname{tr}(x,x)}. \tag{7.13}$$

Let $\Phi(s) \in I_2(s, \chi)$ be the standard section determined by $\varphi_\infty \otimes \varphi \in S(\mathcal{C}_{\mathbb{A}}^2)$. Note that $\Phi(s) = \otimes_{p \le \infty} \Phi_p(s)$, with $\Phi_p(s) \in I_{2,p}(s, \chi_p)$ the standard section associated to φ_p and with

$$\Phi_\infty(s) = \Phi_\infty^{\frac{3}{2}}(s), \tag{7.14}$$

where $\Phi_\infty^{\frac{3}{2}}(s)$ is the standard section whose restriction to K_∞ is the character $\det^{\frac{3}{2}}$. Let $E(g, s, \Phi)$ be the incoherent Eisenstein series of weight $\frac{3}{2}$ constructed from $\Phi(s)$, and let $E'(g, 0, \Phi)$ be its derivative at $s = 0$.

Let $G'_{\mathbb{A}} = \mathrm{Mp}(W_{1,\mathbb{A}}) = \mathrm{Mp}(W_{2,\mathbb{A}})$, in the notation of (6.12) and (6.13), be the metaplectic cover of $\mathrm{Sp}_1(\mathbb{A})$. For d_1 and $d_2 \in \mathbb{Q}$ and for g'_1 and $g'_2 \in G'_{\mathbb{A}}$, we want to obtain more explicit information about the function

$$F_{d_1,d_2}(g'_1, g'_2, \Phi) = \sum_{p \le \infty} F_{d_1,d_2}(g'_1, g'_2, \Phi)_p + F_{d_1,d_2}(g'_1, g'_2, \Phi)_{\mathrm{sing}}, \tag{7.15}$$

whose summands $F_{d_1,d_2}(g'_1, g'_2, \Phi)_p$ and $F_{d_1,d_2}(g'_1, g'_2, \Phi)_{sing}$ are as in Corollary 6.2.

First, for each $p \le \infty$, we want to give a more explicit formula for the theta integral $I_T(\iota(g'_1, g'_2), \varphi^{(p)})$ which occurs in $F_{d_1,d_2}(g'_1, g'_2, \Phi)_p$. Here note that the modified component φ'_p of (6.8) is assumed to be an even function on $(V_p^{(p)})^2$. As explained in [30, p. 60, (6.21)], for any $g \in G_{\mathbb{A}}$,

$$I_T(g, \varphi^{(p)}) = \int_{O(V^{(p)})(\mathbb{Q}) \backslash O(V^{(p)})(\mathbb{A})} \sum_{\substack{x \in V^{(p)}(\mathbb{Q})^2 \\ \frac{1}{2}(x,x) = T}} \left(\omega(g)\varphi^{(p)}\right)(h^{-1}x)\, dh, \tag{7.16}$$

whether or not regularization was required in the definition of the whole theta integral $I(g, \varphi^{(p)})$. Here the Haar measure dh is normalized so that

$$\text{vol}\left(O(V^{(p)})(\mathbb{Q})\backslash O(V^{(p)})(\mathbb{A})\right) = 1. \tag{7.17}$$

Fix $p \leq \infty$. Let Z be the center of $H^{(p)}$. Recall that there is an exact sequence

$$1 \longrightarrow Z \longrightarrow H^{(p)} \xrightarrow{\text{pr}} \text{SO}(V^{(p)}) \longrightarrow 1, \tag{7.18}$$

arising from the action of $H^{(p)}$ on $V^{(p)}$ by conjugation.

It will be useful to fix our measures in the following way. For our fixed additive character $\psi = \otimes_{\ell\leq\infty}\psi_\ell$ and for each prime $\ell \leq \infty$, let dx_ℓ denote the additive Haar measure on $B_\ell^{(p)}$ which is self dual with respect to the pairing $\psi_\ell(\text{tr}(xy^\iota))$. The product measure $dx = \prod_{\ell\leq\infty} dx_\ell$ is then Tamagawa measure on $B_{\mathbb{A}}^{(p)}$ and is self dual with respect to the global pairing $\psi(\text{tr}(xy^\iota))$. On the group $H^{(p)}(\mathbb{A}) = (B^{(p)})_{\mathbb{A}}^\times$, let $\omega_{\mathbb{A}}$ denote the Tamagawa measure defined by the local measures $\omega_\ell = \frac{dx_\ell}{|\nu(x)|_\ell^2}$ with convergence factors $\lambda_\ell = (1-\ell^{-1})$. On $\mathbb{A}$ we take the Tamagawa measure, which we write as a product of local self dual measures with respect to the pairing $\langle x_\ell, y_\ell\rangle = \psi_\ell(x_\ell y_\ell)$. Then on $Z(\mathbb{A}) \simeq \mathbb{A}^\times$, we take the measure defined by the local measures $\frac{dx_\ell}{|x|_\ell}$ and the convergence factors λ_ℓ. The quotient, dh, of the measure $\omega_{\mathbb{A}}$ by the corresponding measure on $Z(\mathbb{A}) \simeq \mathbb{Q}_{\mathbb{A}}^\times$ is the Tamagawa measure on $Z(\mathbb{A})\backslash H^{(p)}(\mathbb{A}) = \mathbb{Q}_{\mathbb{A}}^\times\backslash(B^{(p)})_{\mathbb{A}}^\times \simeq \text{SO}(V^{(p)})(\mathbb{A})$, and

$$\text{vol}\left(H^{(p)}(\mathbb{Q})Z(\mathbb{A})\backslash H^{(p)}(\mathbb{A}), dh\right) = 2. \tag{7.19}$$

Also recall that $O(V^{(p)}) = \text{SO}(V^{(p)}) \times \mu_2$. We take the measure dc on $\mu_2(\mathbb{A})$ with total volume 1, so that

$$\text{vol}\left(\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A}), dc\right) = \frac{1}{2}, \tag{7.20}$$

and

$$\text{vol}\left(O(V^{(p)})(\mathbb{Q})\backslash O(V^{(p)})(\mathbb{A}), dh\, dc\right) = 1. \tag{7.21}$$

Note that these conventions determine a factorization of the Tamagawa measure $dh = \prod_{\ell\leq\infty} dh_\ell$, where dh_ℓ is the quotient of $\frac{dx_\ell}{|\nu(x)|_p^2}$ by the measure $\frac{dz_\ell}{|z|_\ell}$ on $Z(\mathbb{Q}_\ell)$. We also take $dc = \prod_{\ell\leq\infty} dc_\ell$ with $\text{vol}(\mu_2(\mathbb{Q}_\ell), dc_\ell) = 1$.

Choose a compact open subgroup $K^{(p)} \subset H^{(p)}(\mathbb{A}_f)$ such that $\varphi^{(p)}$ is $K^{(p)}$-invariant. We may (and do) assume that $K^{(p)} = \prod_\ell K_\ell^{(p)}$ and that $K_\ell^{(p)} \cap Z(\mathbb{Q}_\ell) \simeq \mathbb{Z}_\ell^\times$, since Z acts trivially on $V^{(p)}$. Then

$$H^{(p)}(\mathbb{A}) = \coprod_j H^{(p)}(\mathbb{Q})H^{(p)}(\mathbb{R})^+ h_j K^{(p)} \tag{7.22}$$

for a finite set of double coset representatives $h_j \in H^{(p)}(\mathbb{A}_f)$. Our assumptions on $K^{(p)}$ imply that each double coset is stable under multiplication by the center $Z(\mathbb{A}) \simeq \mathbb{Q}_{\mathbb{A}}^{\times}$. Hence, via (7.18) and (7.22),

$$\text{(7.23)} \qquad \mathrm{SO}(V^{(p)})(\mathbb{A}) = \coprod_j \mathrm{SO}(V^{(p)})(\mathbb{Q})\ \mathrm{SO}(V^{(p)})(\mathbb{R})^+\ \mathrm{pr}(h_j)\ \mathrm{pr}(K^{(p)}).$$

Let

$$\text{(7.24)} \qquad \Gamma'_j = H^{(p)}(\mathbb{Q}) \cap H^{(p)}(\mathbb{R})^+ h_j K^{(p)} h_j^{-1}.$$

Note that $\Gamma'_j \cap Z(\mathbb{Q}) = \{\pm 1\}$. Let

$$\text{(7.25)} \qquad \mathrm{ev}\colon\ S((V_{\mathbb{A}}^{(p)})^2) \longrightarrow S((V_{\mathbb{A}}^{(p)})^2), \qquad \varphi \mapsto \mathrm{ev}(\varphi) = \int_{\mu_2(\mathbb{A})} \varphi(cx)\, dc$$

be the projection onto the invariants of the compact group $\mu_2(\mathbb{A}) \subset O(V^{(p)})(\mathbb{A})$, i.e., onto the subspace of locally even functions. Note that $\mathrm{ev}(\varphi) = \varphi$ if φ is already locally even.

Assume, for convenience, that $g \in G_{\mathbb{R}}$. (The finite part of g can be absorbed into $\varphi^{(p)}$.) Also write

$$\text{(7.26)} \qquad Z_{K^{(p)}} = Z(\mathbb{A}_f) \cap K^{(p)} \simeq \prod_{\ell} \mathbb{Z}_{\ell}^{\times},$$

and note that

$$\begin{aligned}\text{(7.27)} \qquad & Z(\mathbb{A})H^{(p)}(\mathbb{Q})\backslash H^{(p)}(\mathbb{Q})H^{(p)}(\mathbb{R})^+ h_j K^{(p)} \\ &= Z(\mathbb{R})^+ Z_{K^{(p)}} H^{(p)}(\mathbb{Q})\backslash H^{(p)}(\mathbb{Q})H^{(p)}(\mathbb{R})^+ h_j K^{(p)} \\ &\simeq \left(\Gamma'_j Z(\mathbb{R})^+\backslash H^{(p)}(\mathbb{R})^+\right) \times Z_{K^{(p)}}\backslash h_j K^{(p)}.\end{aligned}$$

Then, for an arbitrary function $\varphi = \varphi_\infty \otimes \varphi_f \in S(V(\mathbb{A})^2)$,

$$\begin{aligned}\text{(7.28)} \qquad & I_T(g, \varphi^{(p)}) \\ &= \int_{\mathrm{SO}(V^{(p)})(\mathbb{Q})\backslash \mathrm{SO}(V^{(p)})(\mathbb{A})} \int_{\mu_2(\mathbb{Q})\backslash\mu_2(\mathbb{A})} \sum_x \left(\omega(g)\varphi^{(p)}\right)(h^{-1}cx)\, dh\, dc \\ &= \frac{1}{2}\int_{Z(\mathbb{A})H^{(p)}(\mathbb{Q})\backslash H^{(p)}(\mathbb{A})} \sum_x \left(\omega(g)\mathrm{ev}(\varphi^{(p)})\right)(h^{-1}x)\, dh \\ &= \frac{1}{2}\sum_j \int_{Z(\mathbb{A})H^{(p)}(\mathbb{Q})\backslash H^{(p)}(\mathbb{Q})H^{(p)}(\mathbb{R})^+ h_j K^{(p)}} \\ &\qquad\qquad \times \sum_x \left(\omega(g)\mathrm{ev}(\varphi^{(p)})\right)(h^{-1}x)\, dh \\ &= \frac{1}{2}\mathrm{vol}(K^{(p)}) \sum_j \int_{\Gamma'_j Z(\mathbb{R})^+\backslash H^{(p)}(\mathbb{R})^+} \sum_x \left(\omega(g)\mathrm{ev}(\varphi_\infty^{(p)})\right)(h_\infty^{-1}x)\, dh_\infty \\ &\qquad\qquad \times\ \mathrm{ev}(\varphi_f^{(p)})(h_j^{-1}x)\end{aligned}$$

$$= \frac{1}{2} \left(\int_{Z(\mathbb{R})\backslash H^{(p)}(\mathbb{R})^+} \left(\omega(g)\mathrm{ev}(\varphi_\infty^{(p)})\right)(h_\infty^{-1}x_0)\, dh_\infty \right) \times \mathrm{vol}(K^{(p)}) \sum_j \sum_{\substack{x \in V^{(p)}(\mathbb{Q})^2 \\ \frac{1}{2}(x,x)=T \\ \mathrm{mod}\, \Gamma_j}} \frac{1}{|\Gamma_{j,x}|} \cdot \mathrm{ev}(\varphi_f^{(p)})(h_j^{-1}x).$$

Here, $x_0 \in V^{(p)}(\mathbb{R})^2$ is any element with $\frac{1}{2}(x_0, x_0) = T$, and $\Gamma_{j,x}$ is the image of $\Gamma'_{j,x}$ in $Z(\mathbb{Q})\backslash H^{(p)}(\mathbb{Q}) \simeq \mathrm{SO}(V^{(p)})(\mathbb{Q})$, so that $|\Gamma_{j,x}| = \frac{1}{2}|\Gamma'_{j,x}|$. Also note the change from $Z(\mathbb{R})^+$ to $Z(\mathbb{R})$, which absorbs the extra $-1 \in \Gamma'_{j,x}$, in the last step.

At this point, we also observe that, since $\det(T) \neq 0$ in our situation, the components of $x \in V^{(p)}(\mathbb{Q})^2$ span a two dimensional subspace of the three dimensional space $V^{(p)}(\mathbb{Q})$. Thus, the pointwise stabilizer $\mathrm{SO}(V^{(p)})(\mathbb{Q})_x$ of x is trivial. Therefore, in particular, $\Gamma_{j,x} = 1$.

Returning to our locally even function $\varphi^{(p)}$, we set

$$\mathrm{Rep}(T, \varphi^{(p)}, V^{(p)}) := \mathrm{vol}(K^{(p)}) \sum_j \sum_{\substack{x \in V^{(p)}(\mathbb{Q})^2 \\ \frac{1}{2}(x,x)=T \\ \mathrm{mod}\, \Gamma_j}} \varphi_f^{(p)}(h_j^{-1}x). \tag{7.29}$$

Note that $\mathrm{vol}(K^{(p)}) = \mathrm{vol}(K^{(p)}/Z_{K^{(p)}})$ and that

$$2 \cdot \mathrm{vol}(K^{(p)})^{-1} = \sum_j \mathrm{vol}(\Gamma'_j Z(\mathbb{R})\backslash H^{(p)}(\mathbb{R})^+), \tag{7.30}$$

in general.

In the case $p < \infty$, so that $B^{(p)}$ is definite, we have $H^{(p)}(\mathbb{R})^+ = H^{(p)}(\mathbb{R})$,

$$\mathrm{vol}(Z(\mathbb{R})\backslash H^{(p)}(\mathbb{R})) = (2\pi)^2, \tag{7.31}$$

and hence

$$\mathrm{vol}(K^{(p)})^{-1} = 2\pi^2 \sum_j \frac{1}{|\Gamma_j|}. \tag{7.32}$$

Here, note that $|\Gamma'_j/\Gamma'_j \cap Z(\mathbb{R})| = |\Gamma_j|$. Thus, $\mathrm{Rep}(T, \varphi^{(p)}, V^{(p)})$ is $\frac{1}{2\pi^2}$ times the classical representation number. Moreover, in this case, $\varphi_\infty = \varphi_\infty^{(p)}$ is the Gaussian for $V^{(p)}(\mathbb{R})$, and is invariant under $O(V^{(p)})(\mathbb{R})$. Thus

$$\int_{Z(\mathbb{R})\backslash H^{(p)}(\mathbb{R})^+} \left(\omega(g)\mathrm{ev}(\varphi_\infty^{(p)})\right)(h_\infty^{-1}x_0)\, dh_\infty = (2\pi)^2\, \omega(g)\varphi_\infty(x_0). \tag{7.33}$$

For $g \in G_{\mathbb{R}}$, we set

$$W_T^{\frac{3}{2}}(g) := \omega(g)\varphi_\infty(x_0), \tag{7.34}$$

and we note that

$$W_T^{\frac{3}{2}}(\iota(g'_1, g'_2)) = W_{d_1}^{\frac{3}{2}}(g'_1)\, W_{d_2}^{\frac{3}{2}}(g'_2), \tag{7.35}$$

where $W_d^{\frac{3}{2}}(g')$ is defined by the analogue of (7.34) for the Weil representation of $G'_{\mathbb{R}}$ on $S(V_\infty^{(p)})$ with a vector $x_{01} \in V_\infty^{(p)}$ with $\frac{1}{2}(x_{01}, x_{01}) = d$.

LEMMA 7.1. *Assume that* g'_1 *and* $g'_2 \in G'_{\mathbb{R}}$. *Then, if* $p < \infty$,

$$I_T(\iota(g'_1,g'_2),\varphi^{(p)}) = 2\pi^2 \cdot W_{d_1}^{\frac{3}{2}}(g'_1)\, W_{d_2}^{\frac{3}{2}}(g'_2) \cdot \mathrm{Rep}(T,\varphi^{(p)},V^{(p)}),$$

where $\mathrm{Rep}(T,\varphi^{(p)},V^{(p)})$ *is given by* (7.29).

Next, suppose that $p = \infty$, so that $V^{(\infty)} = V$ and $B^{(\infty)} = B$. In this case, the function

$$\int_{Z(\mathbb{R})\backslash H(\mathbb{R})^+} (\omega(g)\varphi'_\infty)\,(h_\infty^{-1}x_0)\,dh_\infty \tag{7.36}$$

on $G_{\mathbb{R}}$ may be quite unpleasant. (Note that φ'_∞ is even.) However, we can simplify things by a good choice of φ'_∞. First of all, we take φ'_∞ to be invariant under the element $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{R}) = H(\mathbb{R})$, so that (7.36) is half of the analogous integral over $Z(\mathbb{R})\backslash H(\mathbb{R})$. Next, suppose that φ'_∞ is chosen to have compact support in the submersive set

$$V(\mathbb{R})^2_{\mathrm{sub}} = \{\; x \in V(\mathbb{R})^2 \;\mid\; \dim \mathrm{span}(x) = 2 \;\}. \tag{7.37}$$

The proof of Proposition 2.7 of [30] shows that, if $\Phi'_\infty(s)$ is the corresponding standard section of the induced representation $I_{2,\infty}(s,\chi_\infty)$, then

$$W_{T,\infty}(e,0,\Phi'_\infty) = \gamma_\infty(V)\cdot M_{\varphi'_\infty}(T), \tag{7.38}$$

where $M_{\varphi'_\infty}$ is the test function on $\mathrm{Sym}_2(\mathbb{R})$ defined by the condition that, for any locally L^1 function ϕ on $\mathrm{Sym}_2(\mathbb{R})$,

$$\int_{V(\mathbb{R})^2} \varphi'_\infty(x)\,\phi\left(\frac{1}{2}(x,x)\right)dx = \int_{\mathrm{Sym}_2(\mathbb{R})} M_{\varphi'_\infty}(b)\,\phi(b)\,db. \tag{7.39}$$

Here the measure dx, is self dual with respect to the pairing $\psi_\infty(\mathrm{tr}(x,y))$ on $V(\mathbb{R})^2$, the measure db is self dual with respect to the pairing $\psi_\infty(\mathrm{tr}(ab))$ on $\mathrm{Sym}_2(\mathbb{R})$, and $\gamma_\infty(V)$ is the factor which enters into the action in the Weil representation of the element w^{-1}, i.e., for $f \in S(V(\mathbb{R})^2)$,

$$\omega(w^{-1})f(y) = \gamma_\infty(V)\int_{V(\mathbb{R})^2} \psi(-\mathrm{tr}(x,y))\,f(x)\,dx. \tag{7.40}$$

It has the simple expression $\gamma_\infty(V) = i$, (see [22, p. 380, (3.4)]). On the other hand, by [41],

$$M_{\varphi'_\infty}(T) = C_\infty \cdot \int_{O(V)(\mathbb{R})} \varphi'_\infty(h_\infty^{-1}x_0)\,dh_\infty, \tag{7.41}$$

for the measure dh_∞ on $O(V)(\mathbb{R}) = \mathrm{SO}(V)(\mathbb{R}) \times \mu_2$ defined above. A simple computation yields

$$C_\infty = \sqrt{2}. \tag{7.42}$$

Thus, by (7.38), (7.41), and (7.18), the quantity (7.36) is equal to

$$\frac{1}{2iC_\infty}\cdot W_{T,\infty}(e,0,\Phi'_\infty). \tag{7.43}$$

LEMMA 7.2. *With the choice of φ'_∞ described above,*

$$I_T(e,\varphi^{(\infty)}) = \frac{1}{4iC_\infty}\cdot W_{T,\infty}(e,0,\Phi'_\infty)\cdot \mathrm{Rep}(T,\varphi,V).$$

Substituting the results of Lemmas 7.1 and 7.2 into the expressions of Corollary 6.2, we obtain the following more precise description of the functions $F_{d_1,d_2}(g'_1, g'_2, \Phi)_p$.

PROPOSITION 7.3. *Assume that g'_1 and g'_2 are in $G'_{\mathbb{R}}$, i.e., that the components at the finite places are trivial. For $g' = (n(b)m(a)k_\theta,t) \in G'_{\mathbb{R}}$, with $a > 0$ and $\theta \in (-\pi,\pi]$, and for $d \in \mathbb{Q}$, the function $W_d^{\frac{3}{2}}(g')$ defined above is given explicitly by*

$$W_d^{\frac{3}{2}}(g') = t\, e(bd)\, |a|^{\frac{3}{2}} (e^{i\theta})^{\frac{3}{2}} e^{-2\pi a^2 d}.$$

Here, $e(x) = e^{2\pi i x}$.

(i) *For $p < \infty$,*

$$F_{d_1,d_2}(g'_1,g'_2,\Phi)_p = (2\pi)^2\cdot W_{d_1}^{\frac{3}{2}}(g'_1)\, W_{d_2}^{\frac{3}{2}}(g'_2) \sum_{\substack{T,\ \det T\neq 0\\ T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}}} \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}\cdot \mathrm{Rep}\left(T,\varphi^{(p)},V^{(p)}\right).$$

Here $\mathrm{Rep}\left(T,\varphi^{(p)},V^{(p)}\right)$ *is as in* (7.29). *Also, $\varphi^{(p)} = \varphi'_p \otimes \varphi^p$, where $\varphi^p \in S(V^{(p)}(\mathbb{A}_f^p)^2)$ is the image of $\otimes_{\ell\neq p}\varphi_\ell$ under the fixed identification $V(\mathbb{A}_f^p) \simeq V^{(p)}(\mathbb{A}_f^p)$ and $\varphi'_p \in S(V^{(p)}(\mathbb{Q}_p)^2)$ is chosen as in Theorem* 6.1. *Finally, $\Phi'_p(s)$ is the standard section of $I_{2,p}(s,\chi_p)$ determined by φ'_p.*

(ii) *For $p = \infty$,*

$$F_{d_1,d_2}(g'_1,g'_2,\Phi)_\infty = \frac{1}{2iC_\infty}\cdot \sum_{\substack{T,\ \det T\neq 0\\ T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}}} W'_{T,\infty}(\iota(g'_1,g'_2),0,\Phi_\infty)\cdot \mathrm{Rep}\left(T,\varphi^{(\infty)},V^{(\infty)}\right).$$

Here, recall that $B^{(\infty)} = B$ and $\varphi^{(\infty)} = \varphi$.

Remark 7.4. (a) In part (ii) of this result, we have taken the modified local component g_∞, allowed by Theorem 6.1, to be the identity. By Lemma 7.2, this yields

$$(7.44)\qquad \frac{W'_{T,\infty}(\iota(g'_1,g'_2),0,\Phi_\infty)}{W_{T,\infty}(e,0,\Phi'_\infty)}\cdot I_T(e,\varphi^{(\infty)}) \\ = \frac{1}{4iC_\infty}\cdot W'_{T,\infty}(\iota(g'_1,g'_2),0,\Phi_\infty)\cdot \mathrm{Rep}(T,\varphi_f,V).$$

(b) Note that, for any $p\le\infty$,

$$(7.45)\qquad \mathrm{Rep}\left(T,\varphi^{(p)},V^{(p)}\right)\neq 0 \Longrightarrow \mathrm{Diff}(T,\mathcal{C})=\{p\}.$$

Thus, no condition on $\mathrm{Diff}(T,\mathcal{C})$ need be written in the sums. Also note that, for $p<\infty$, only positive definite T's contribute to $F_{d_1,d_2}(g'_1,g'_2,\Phi)_p$, while, only T's of signature $(1,1)$ contribute to $F_{d_1,d_2}(g'_1,g'_2,\Phi)_\infty$. The sets of nonsingular T's which contribute for different primes are disjoint.

In Section 8, we will obtain a more explicit formula for the quantity

$$(7.46)\qquad \frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)},$$

under the assumption that φ_p is 'unramified', in a sense define there. In Section 9, we will give an explicit formula for $W'_{T,\infty}(\iota(g'_1,g'_2),0,\Phi^{\frac{3}{2}}_\infty)$.

8. *The local factor* $\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi'_p)}$ *in the unramified case.* In this section we determine the quantity (7.46), under the following assumptions:

(i) $p\nmid 2D(B)$.

(ii) The local component χ_p of the character χ is unramified.

(iii) φ_p is 'unramified' in the following sense. Let R be a maximal order in B, and let $L_p=R_p\cap V_p$ be the $\mathbb{Z}_p$ lattice of trace zero elements in the completion $R_p=R\otimes_{\mathbb{Z}}\mathbb{Z}_p$. Let $\varphi_p\in S(V_p^2)$ be the characteristic function of the lattice $L_p^2\subset V_p^2$.

Recall that $\chi(x)=(\kappa,x)_{\mathbb{A}}$ where $\kappa\in\mathbb{Z}$ is a square free negative integer. Let

$$(8.1)\qquad S_0=\kappa\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix}.$$

For $r\in\mathbb{Z}_{\ge 0}$, we also let

$$(8.2)\qquad S_r=S_0\perp\begin{pmatrix}1_r&\\&-1_r\end{pmatrix}.$$

Note that S_0 is the matrix for the quadratic form $-\kappa\cdot\nu$ on the lattice $L_p = R_p \cap V_p$ with respect to the usual trace zero part of a basis for the maximal order R_p in $B_p = M_2(\mathbb{Q}_p)$. By Corollary A.5 of the Appendix,

$$W_{T,p}(e, r, \Phi_p) = \gamma_p \cdot |\det(S_0)| \cdot \alpha_p(S_r, T), \tag{8.3}$$

where $\alpha_p(S_r, T)$ is the classical representation density of T by the form S_r, and

$$\gamma_p = \chi_p(-1)\,\gamma_p(-1, \psi). \tag{8.4}$$

Thus, if we write the representation density $\alpha_p(S_r, T)$ as a polynomial in $X = p^{-r}$, we obtain

$$W'_{T,p}(e, 0, \Phi_p) = -\log p \cdot \gamma_p \cdot |\det(S_0)| \cdot \frac{\partial}{\partial X}\{\alpha_p(S_r, T)\}\,|_{X=1}. \tag{8.5}$$

Next, let $R^{(p)}$ be a maximal order in $B^{(p)}$, chosen so that its localizations at all finite primes $\ell \neq p$ coincide with those of R under the identification (7.9). Let $\varphi'_p \in S((V_p^{(p)})^2)$ be the characteristic function of the lattice $(L_p^{(p)})^2 \subset (V_p^{(p)})^2$, where $L_p^{(p)} = R_p^{(p)} \cap V_p^{(p)}$. Let

$$S'_0 = \kappa \begin{pmatrix} \beta & & \\ & p & \\ & & -p\beta \end{pmatrix}, \tag{8.6}$$

where we take $\beta \in \mathbb{Z}_p^\times$ with $(\beta, p)_p = -1$. Again, note that S'_0 is the matrix for the quadratic form $-\kappa\cdot\nu$ on the lattice $L_p^{(p)}$ with respect to the usual trace zero part of a basis for the maximal order $R_p^{(p)}$ in $B_p^{(p)}$, the division quaternion algebra over $\mathbb{Q}_p$. Then, in Proposition 7.3, we have

$$\frac{W'_{T,p}(e, 0, \Phi_p)}{W_{T,p}(e, 0, \Phi'_p)} = -\log p \cdot \frac{|\det(S_0)|}{|\det S'_0|} \cdot \frac{\frac{\partial}{\partial X}\{\alpha_p(S_r, T)\}\,|_{X=1}}{\alpha_p(S'_0, T)}, \tag{8.7}$$

provided $\alpha_p(S'_0, T) \neq 0$. Note that the quantity γ_p of (8.4) has cancelled, since it does not depend on the Hasse invariant. Moreover, the quantity $\mathrm{Rep}(T, \varphi_f^{(p)}, V^{(p)})$ of Proposition 7.3 also depends on the choice of φ'_p just made. The derivation of (8.7) makes no use of the assumption that χ be unramified at p. Note that

$$\frac{|\det(S_0)|}{|\det S'_0|} = p^2. \tag{8.8}$$

To proceed further, we need an explicit expression for $\alpha_p(S_r, T)$. Since $\det S_r$ is a unit, such a formula is given by Kitaoka [19]. He proved the following:

PROPOSITION 8.1 (Kitaoka's formulas). *Assume that* $\det S$ *is a unit and that* p *is odd. If* $T \notin \mathrm{Sym}_2(\mathbb{Z}_p)$, *then* $\alpha_p(S,T) = 0$. *Moreover,* $\alpha_p(S,T)$ *depends only on the* $\mathrm{GL}_2(\mathbb{Z}_p)$ *orbit of* T, *so it suffices to consider*

$$T = \begin{pmatrix} \varepsilon_1 p^a & 0 \\ 0 & \varepsilon_2 p^b \end{pmatrix}$$

where $0 \leq a \leq b \in \mathbb{Z}$ *and* ε_1, *and* $\varepsilon_2 \in \mathbb{Z}_p^\times$. *Write* $X = p^{-r}$, *where* $\dim(S) = m = 2r+3$.

(i) *If* a *is odd,*

$$\frac{\alpha_p(S,T)}{(1-p^{-2}X^2)} = \sum_{j=0}^{\frac{a-1}{2}} p^j \left(X^{2j} + \nu X^{a+b-2j}\right),$$

where

$$\nu = \begin{cases} (p, -\varepsilon_1\varepsilon_2)_p & \text{if } b \text{ is odd} \\ (p, -\varepsilon_2 \det S)_p & \text{if } b \text{ is even.} \end{cases}$$

(ii) *If* a *is even, then*

$$\frac{\alpha_p(S,T)}{(1-p^{-2}X^2)} = \sum_{j=0}^{\frac{a}{2}-1} p^j \left(X^{2j} + \nu X^{a+b-2j}\right) + p^{\frac{a}{2}} X^a \sum_{j=0}^{b-a} (\nu_0 X)^j,$$

where

$$\nu = \begin{cases} 1 & \text{if } b \text{ is even} \\ (p, -\varepsilon_1 \det S)_p & \text{if } b \text{ is odd.} \end{cases}$$

and

$$\nu_0 = (p, -\varepsilon_1 \det S)_p.$$

Here we have rewritten Kitaoka's expressions to make the functional equation relating s and $-s$, i.e., relating values at X and X^{-1}, more apparent.

Specializing Proposition 1.3, we have the following characterization:

LEMMA 8.2. *In the nonarchimedean case,* T *is represented by* V_p *if and only if*

$$\varepsilon_p(V_p) = \varepsilon_p(T)\chi_p(\det T).$$

In particular, this is always the case if $p \nmid 2D(B)\kappa \det T$.

On the other hand, evaluating Kitaoka's formulas for $S = S_0$, we have:

LEMMA 8.3. *If $p \nmid 2D(B)\kappa$ and if $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$, then $\alpha_p(S_0,T) \neq 0$ if and only if $\mu_p(T) = 1$, where*

$$\mu_p(T) = \begin{cases} (-\varepsilon_1\varepsilon_2,p)_p & \text{if } a \text{ and } b \text{ are odd} \\ (-\varepsilon_2 \det S_0,p)_p & \text{if } a \text{ is odd and } b \text{ is even} \\ (-\varepsilon_1 \det S_0,p)_p & \text{if } a \text{ is even and } b \text{ is odd} \\ 1 & \text{if } a \text{ and } b \text{ are both even.} \end{cases}$$

Moreover, by Lemma 8.2, $\mu_p(T) = 1$ if and only if V_p represents T.

Proof. The first statement here can be obtained by inspecting Kitaoka's formulas. Since $p \nmid 2\kappa$, we have $\det(S_0) = -\kappa$ and $\varepsilon(S_0) = 1$. Thus, by Lemma 8.2, T is represented by V_p if and only

$$\begin{aligned} 1 = \varepsilon(S_0) &= \varepsilon(T)\chi(\det T) \\ &= (\varepsilon_1 p^a, \varepsilon_2 p^b)\,(\kappa, \varepsilon_1\varepsilon_2 p^{a+b}) \\ &= (\varepsilon_1, p)^b(\varepsilon_2, p)^a(-1, p)^{ab}(\kappa, p)^{a+b}. \end{aligned} \tag{8.9}$$

If a and b are odd, this comes to $(-\varepsilon_1\varepsilon_2, p) = 1$. If a is odd and b is even, we get $(\varepsilon_2, p)(-\det S_0, p) = 1$. If a is even and b is odd, we get $(\varepsilon_1, p)(-\det S_0, p) = 1$. Finally, if a and b are both even, we get 1 identically. □

Note that this proof gives some insight into the structure of Kitaoka's formulas.

COROLLARY 8.4. *In the cases in which $\alpha_p(S_0,T) \neq 0$, i.e., when $\mu_p(T) = 1$ in Lemma 8.3,*

$$\frac{\alpha_p(S_0,T)}{(1-p^{-2})} = \begin{cases} 2\sum_{j=0}^{\frac{a-1}{2}} p^j & \text{if } a \text{ is odd} \\ 2\sum_{j=0}^{\frac{a}{2}-1} p^j + p^{\frac{a}{2}} & \text{if } a \text{ and } b \text{ are even} \\ & \text{and } (-\varepsilon_1 \det S_0,p) = -1 \\ 2\sum_{j=0}^{\frac{a}{2}-1} p^j + p^{\frac{a}{2}}(b-a+1) & \text{if } a \text{ is even} \\ & \text{and } (-\varepsilon_1 \det S_0,p) = 1. \end{cases}$$

COROLLARY 8.5. *In the cases in which $\alpha_p(S_0,T) = 0$, i.e., when $\mu_p(T) = -1$ in Lemma 8.3,*

$$\frac{\partial}{\partial X}\left\{\frac{\alpha_p(S_r,T)}{(1-p^{-2}X^2)}\right\}\Bigg|_{X=1} = -\begin{cases} \sum_{j=0}^{\frac{a-1}{2}}(a+b-4j)p^j & \text{if } a \text{ is odd} \\ \sum_{j=0}^{\frac{a}{2}-1}(a+b-4j)p^j + \frac{1}{2}(b-a+1)p^{\frac{a}{2}} & \\ & \text{if } a \text{ is even and } b \text{ is odd.} \end{cases}$$

We also need some information on the denominator in (8.7).

PROPOSITION 8.6 (Myers). *Suppose that $p \nmid 2D(B)\kappa$, and that $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ with $\det T \neq 0$. Then, writing T as in Proposition 8.1, and taking $\mu_p(T)$ as in Lemma 8.3, the density $\alpha_p(S_0',T)$ vanishes when $\mu_p(T) = 1$. If $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ and $\mu_p(T) = -1$, then*

$$\alpha_p(S_0',T) = 2(p+1).$$

Finally, if $T \notin \mathrm{Sym}_2(\mathbb{Z}_p)$, then $\alpha_p(S_0',T) = 0$.

This result is a very special case of the results of Bruce Myer's thesis [36], which also contains information about the derivative $\alpha_p'(S_0', T)$ in the cases where the density $\alpha_p(S_0', T)$ vanishes. The claims of Proposition 8.6 are not difficult to obtain directly.

Combining the above information, we obtain:

COROLLARY 8.7. *Assume that $p \nmid 2D(B)\kappa$ and write $T \in \mathrm{Sym}_2(\mathbb{Z}_p)$ as in Proposition 8.1. Assume that $\alpha_p(S_0,T) = 0$; i.e., $\mu_p(T) = -1$, as in Lemma 8.3. Then,*

$$\frac{W'_{T,p}(e,0,\Phi_p)}{W_{T,p}(e,0,\Phi_p')} = \frac{1}{2}(p-1)\log(p)\cdot\delta_p(T),$$

where

$$\delta_p(T) = \begin{cases} \sum_{j=0}^{\frac{a-1}{2}}(a+b-4j)p^j & \text{if } a \text{ is odd} \\ \sum_{j=0}^{\frac{a}{2}-1}(a+b-4j)p^j + \frac{1}{2}(b-a+1)p^{\frac{a}{2}} & \text{if } a \text{ is even} \\ & \text{and } b \text{ is odd.} \end{cases}$$

9. *The archimedean local factors.* We now turn to the archimedean local case, and, dropping the subscript ∞ to lighten notation, we consider the function

$$W_T(g,s,\Phi) = \int_N \Phi(w^{-1}ng,s)\,\psi_T(n)^{-1}\,dn. \tag{9.1}$$

Here $g \in G_{\mathbb{R}} = \mathrm{Mp}_2(\mathbb{R})$, $T \in \mathrm{Sym}_2(\mathbb{R})$, and $\Phi(s) \in I_2(s,\chi)$. We now assume that $\det T \neq 0$. Recall that $K_\infty \subset G$ is the inverse image of the subgroup

$$\left\{ k = \begin{pmatrix} c & d \\ -d & c \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R}) \mid \mathbf{k} := c + \mathrm{i}d \in U(2) \right\}. \tag{9.2}$$

We take $\Phi(s) = \Phi^\ell(s)$, where, for $\ell \in \frac{1}{2} + \mathbb{Z}$, $\Phi^\ell(s)$ is determined by the condition that its restriction to K_∞ is the character $\det^\ell$. Recall that a standard section $\Phi(s)$ is determined by its restriction to K_∞. Since

$$W_T(ngk,s,\Phi) = \psi_T(n)\cdot W_T(g,s,\Phi)\cdot(\det\mathbf{k})^\ell, \tag{9.3}$$

it will suffice to determine the function $W_T(m(a),s,\Phi)$, for $a \in \mathrm{GL}_2(\mathbb{R})^+$.

Remark 9.1. There is a consistency condition between ℓ and the character χ_∞. The element

$$k_0 = \left(m\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right), 1\right) \in K_\infty \tag{9.4}$$

has

$$(\det \mathbf{k}_0)^\ell = (e^{i\pi})^\ell = e^{i\pi\ell}, \tag{9.5}$$

whereas

$$\Phi\left(m\left(\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}\right) g, s\right) = \chi_\infty(-1)\,\gamma(-1,\psi)^{-1}\cdot\Phi(g,s). \tag{9.6}$$

Thus, using [22, p. 380], we require

$$\chi_\infty(-1)\,\gamma(-1,\psi)^{-1} = \chi_\infty(-1)\cdot i = e^{i\pi\ell}. \tag{9.7}$$

Since we assume that $\chi_\infty(-1) = -1$, this amounts to the condition that $2\ell \equiv 3 \bmod 4$. Eventually, we will take $\ell = \frac{3}{2}$.

Now, setting $X = \mathrm{Sym}_2(\mathbb{R})$, we want to compute,

$$\frac{1}{\sqrt{2}}\cdot W_T(m(a), s, \Phi^\ell) = \int_X \Phi^\ell(w^{-1}n(x)m(a), s)\, e(-\operatorname{tr}(Tx))\, dx, \tag{9.8}$$

where we assume that $\det a > 0$. Here, for $x = \begin{pmatrix} x_1 & x_2 \\ x_2 & x_3\end{pmatrix} \in \mathrm{Sym}_2(\mathbb{R})$, we take $dx = dx_1\, dx_2\, dx_3$, as in Shimura [45, p. 273]. The self dual measure on $\mathrm{Sym}_2(\mathbb{R})$ is $\sqrt{2}\, dx$. We will write $C_\infty^{-1} = \frac{1}{\sqrt{2}}$ for the factor which results on the left-hand side.

LEMMA 9.2. *In the decomposition*

$$w^{-1}n(x)\, m(a) = m(a^\vee)\, w^{-1}n(x') = n\, m(\eta)k,$$

with $x' = a^{-1}x^ta^{-1}$ *and* $\eta \in \mathrm{GL}_2(\mathbb{R})^+$, *set* $z = x + iy$, *for* $y = a\cdot {}^ta$. *Also let* $k \in K_\infty$ *and* $\mathbf{k} = c + \mathrm{i}d$ *be as in* (9.2). *Then*

$$\det\eta = \frac{\det y}{|\det z|} \qquad \textit{and} \qquad \det\mathbf{k} = \frac{\det\bar{z}}{|\det z|}.$$

Proof. First, in $\mathrm{Sp}_2(\mathbb{R})$, we write

$$w^{-1}n(x)m(a) = m(a^\vee)w^{-1}n(x') = m(a^\vee)\begin{pmatrix} & -1 \\ 1 & x'\end{pmatrix}, \tag{9.9}$$

where $x' = a^{-1}x^ta^{-1}$. Here $a^\vee = {}^ta^{-1}$. Setting

$$w^{-1}n(x') = nm(\eta)k, \tag{9.10}$$

we consider

$$(9.11)\qquad (0,1_2)\cdot nm(\eta)k\cdot\begin{pmatrix}1\\ i\end{pmatrix} = i\,{}^t\eta^{-1}\cdot(c+id) = i\,{}^t\eta^{-1}\cdot\mathbf{k}.$$

On the other hand,

$$(9.12)\qquad (0,1_2)\,w^{-1}n(x')\cdot\begin{pmatrix}1\\ i\end{pmatrix} = i(x'-i).$$

Thus

$$(9.13)\qquad \eta^\vee\cdot\mathbf{k} = x'-i,$$

and we conclude that

$$(9.14)\qquad \eta^\vee\cdot\eta^{-1} = \eta^\vee\mathbf{k}\cdot{}^t\bar{\mathbf{k}}\eta^{-1} = (x'-i)\cdot(x'+i).$$

But

$$(9.15)\qquad x'-i = a^{-1}(x-ia\cdot{}^ta)a^\vee = a^{-1}(x-iy)a^\vee,$$

where $y = a\cdot{}^ta$. Thus, from (9.14) and (9.15),

$$(9.16)\qquad \det\eta^{-2} = (\det a)^{-4}|\det z|^2,$$

and

$$(9.17)\qquad \mathbf{k} = {}^t\eta\cdot a^{-1}\cdot\bar{z}\cdot a^\vee.$$

□

Now, since $\det a > 0$ and $\det(\eta) > 0$, the decomposition of Lemma 9.2 also holds in $G_{\mathbb{R}} = \mathrm{Mp}_2(\mathbb{R})$, i.e., when we write g for $(g,1)\in \mathrm{Sp}_2(\mathbb{R})\times\mathbb{C}^1 \simeq \mathrm{Mp}_2(\mathbb{R})$, and we have

$$(9.18)\qquad \begin{aligned}\Phi^\ell\left(w^{-1}n(x)m(a),s\right) &= \Phi^\ell\left(m(a^\vee)n\,m(\eta)\,k,s\right)\\ &= |a|^{-s-\frac{3}{2}}\cdot|a|^{2s+3}|\det z|^{-s-\frac{3}{2}}\cdot\left(\frac{\det\bar{z}}{|\det z|}\right)^\ell\\ &= |a|^{s+\frac{3}{2}}(\det z)^{-\alpha}\cdot(\det\bar{z})^{-\beta},\end{aligned}$$

where

$$(9.19)\qquad \alpha = \frac{1}{2}\left(s+\frac{3}{2}+\ell\right),\qquad \text{and}\qquad \beta = \frac{1}{2}\left(s+\frac{3}{2}-\ell\right).$$

Here we use the sign conventions in Shimura [45, p. 272, (1.11)], and set

$$(9.20)\qquad \begin{aligned}\det(x+iy)^\alpha &= (e^{i\alpha\pi})\det(y-ix)^\alpha \quad\text{and}\\ \det(x-iy)^\beta &= (e^{-i\beta\pi})\det(y+ix)^\beta,\end{aligned}$$

with the standard branch of the logarithm used to define $\det(y\pm ix)^s$. Using this expression in the integral, we obtain:

LEMMA 9.3. *If* $a \in \mathrm{GL}_2(\mathbb{R})^+$ *and* $y = a \cdot {}^t a$, *then, with the branch conventions of* (9.19),

$$\begin{aligned} C_\infty^{-1} \cdot W_T(m(a),s,\Phi^\ell) &= |a|^{s+\frac{3}{2}} \int_X \det(x+iy)^{-\alpha} \det(x-iy)^{-\beta} e(-\mathrm{tr}(Tx))\, dx \\ &=: |a|^{s+\frac{3}{2}} \xi(y,T;\alpha,\beta). \end{aligned}$$

Remark 9.4. The integral function $\xi(y, T; \alpha, \beta)$ in Lemma 9.3 is precisely as in [45, (1.25)].

We are interested in the values and derivatives of $W_T(m(a), s, \Phi^\ell)$ with respect to s, at the point $s = 0$, particularly in the case $\ell = \frac{3}{2}$.

Recall [45] that for $n \geq 1$, and $\mathrm{Re}(s) > \frac{n-1}{2}$,

$$\text{(9.21)} \qquad \begin{aligned} \Gamma_n(s) : &= \int_{\substack{x \in \mathrm{Sym}_n(\mathbb{R}) \\ x>0}} e^{-\mathrm{tr}(x)} \det(x)^{s-\frac{n+1}{2}}\, dx \\ &= \pi^{\frac{n(n-1)}{4}} \prod_{k=0}^{n-1} \Gamma\left(s - \frac{k}{2}\right). \end{aligned}$$

We will also need the following integral formulas [45]:

$$\text{(9.22)} \qquad \Gamma_n(s) \det(z)^{-s} = \int_{x>0} e^{-\mathrm{tr}(xz)} \det(x)^{s-\frac{n+1}{2}}\, dx,$$

when $\mathrm{Re}(s) > \frac{n-1}{2}$ and $\mathrm{Re}(z) > 0$, and

$$\text{(9.23)} \qquad \begin{aligned} &\int_{\mathrm{Sym}_n(\mathbb{R})} e(\mathrm{tr}(ux)) \det(b + 2\pi i x)^{-s}\, dx \\ &\quad = \begin{cases} 2^{-\frac{n(n-1)}{2}} \Gamma_n(s)^{-1} \cdot e^{-\mathrm{tr}(ub)} \det(u)^{s-\frac{n+1}{2}} & \text{if } u > 0 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This last formula is valid for $b \in \mathrm{Sym}_n(\mathbb{R})$, $b > 0$ and for $\mathrm{Re}(s) > n$, [45, p. 274 (1.23)]. It is obtained from the first by Fourier inversion.

Recall from Shimura [45], that, setting $\rho = \frac{3}{2}$,

$$\text{(9.24)} \qquad \begin{aligned} &\xi(y, T; \alpha, \beta) \\ &= \int_X \det(x+iy)^{-\alpha} \det(x-iy)^{-\beta} e(-\mathrm{tr}(Tx))\, dx \\ &= e^{-i\pi(\alpha-\beta)} \int_X \det(y - ix)^{-\alpha} \det(y+ix)^{-\beta} e(-\mathrm{tr}(Tx))\, dx \\ &= \frac{e^{-i\pi(\alpha-\beta)}}{2\Gamma_2(\alpha)} \\ &\quad \times \int_X \int_{u>0} e^{-\mathrm{tr}(u(y-ix))} \det(u)^{\alpha-\rho}\, du \det(y+ix)^{-\beta} e(-\mathrm{tr}(Tx))\, dx \end{aligned}$$

$$= \frac{e^{-i\pi(\alpha-\beta)}}{2\Gamma_2(\alpha)} \int_{u>0} e^{-\operatorname{tr}(uy)} \det(u)^{\alpha-\rho}$$
$$\times \int_X e\left(x\left(\frac{u}{2\pi}-T\right)\right) (2\pi)^{2\beta} \det(2\pi y+2\pi i x)^{-\beta}\,dx$$
$$= \frac{e^{-i\pi(\alpha-\beta)}(2\pi)^{2\rho}}{2\Gamma_2(\alpha)\Gamma_2(\beta)} \int\limits_{\substack{u>0\\ u>2\pi T}} e^{-\operatorname{tr}((2u-2\pi T)y)} \det(u)^{\alpha-\rho}\det(u-2\pi T)^{\beta-\rho}\,du.$$

This transformation is valid if $\mathrm{Re}(\alpha) > \rho - 1 = \frac{1}{2}$ and $\mathrm{Re}(\beta) > 2\rho - 1 = 2$, [45].

Next, we substitute $u + \pi T$ for u, and obtain

(9.25)
$$\xi(y,T;\alpha,\beta) = \frac{e^{-i\pi(\alpha-\beta)}(2\pi)^{2\rho}}{2\Gamma_2(\alpha)\Gamma_2(\beta)} \times \int\limits_{\substack{u>-\pi T\\ u>\pi T}} e^{-\operatorname{tr}(2uy)} \det(u+\pi T)^{\alpha-\rho}\det(u-\pi T)^{\beta-\rho}\,du.$$
$$=: \frac{e^{-i\pi(\alpha-\beta)}(2\pi)^{2\rho}}{2\Gamma_2(\alpha)\Gamma_2(\beta)}\eta(2y,T;\alpha,\beta).$$

Here, again, the notation $\eta(2y,T;\alpha,\beta)$ is as in Shimura [45, (1.26)].

We now assume that $\ell = \frac{3}{2}$, and we consider the behavior of $W_T(g,s,\Phi^{\frac{3}{2}})$ at $s=0$. Note that

(9.26)
$$\alpha = \frac{1}{2}(s+3) \qquad \text{and} \qquad \beta = \frac{1}{2}s,$$

so that

(9.27)
$$\Gamma_2(\alpha)\Gamma_2(\beta) = \pi\,\Gamma\left(\frac{s+3}{2}\right)\Gamma\left(\frac{s+2}{2}\right)\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s-1}{2}\right).$$

This quantity has a simple pole at $s=0$, and so $W_T(m(a),s,T)$ will vanish at $s=0$ if $\eta(2y,T;\frac{1}{2}(s+3),\frac{1}{2}s)$ remains finite at that point. The behavior of $\eta(2y,T;\alpha,\beta)$ will depend on the signature of T.

PROPOSITION 9.5. (i) *If $T>0$, then, for $a\in \mathrm{GL}_2(\mathbb{R})^+$ and $y = a^t a$,*

$$C_\infty^{-1}\cdot W_T\left(m(a),0,\Phi^{\frac{3}{2}}\right) = 2i(2\pi)^2\cdot\det(y)^{\frac{3}{4}}\cdot e^{-2\pi\operatorname{tr}(Ty)}.$$

(ii) *If* $\mathrm{sig}(T) = (1,1)$ *or* $(0,2)$, *then*

$$W_T\left(m(a),0,\Phi^{\frac{3}{2}}\right) = 0.$$

Of course, this result can be obtained by specializing the results of Shimura to the present case. For example, (i) follows from (4.35.K) of [45]. But we prefer to give some details of the proofs in this case, since these will be needed as background for the computation of the derivative. Specifically, we want to consider the derivative:

$$\frac{\partial}{\partial s}\left\{W_T(m(a), s, \Phi^{\frac{3}{2}})\right\}\Big|_{s=0}, \tag{9.28}$$

in case (ii). In fact, it will turn out that, when $T < 0$, this derivative will not play a role in what follows.

Proof. First suppose that $T > 0$. In the integral $\eta(2y, T; \alpha, \beta)$, we substitute $u + \pi T$ for u and obtain

$$e^{-2\pi\,\mathrm{tr}(Ty)} \int\limits_{u>0} e^{-\,\mathrm{tr}(2uy)} \det(u + 2\pi T)^{\frac{1}{2}s} \det(u)^{\frac{1}{2}(s-3)}\, du. \tag{9.29}$$

Noting that the factor $\det(u + 2\pi T)^{\frac{1}{2}s}$ has the expansion

$$\det(u + 2\pi T)^{\frac{1}{2}s} = 1 + O(s), \tag{9.30}$$

in a neighborhood of $s = 0$, where the $O(s)$ terms are bounded as any eigenvalue of u goes to zero and slowly increasing as any eigenvalue goes to ∞, we obtain an expansion

$$\eta\left(2y, T; \frac{1}{2}(s+3), \frac{1}{2}s\right) = \Gamma_2\left(\frac{s}{2}\right) \cdot \det(2y)^{-\frac{1}{2}s} \cdot e^{-2\pi\,\mathrm{tr}(Ty)} + O(1), \tag{9.31}$$

in a neighborhood of $s = 0$. This yields (i).

Next suppose that $T < 0$, and substitute $u - \pi T$ for u. We find

$$e^{2\pi\,\mathrm{tr}(Ty)} \int\limits_{u>0} e^{-\,\mathrm{tr}(2uy)} \det(u)^{\frac{1}{2}s} \det(u - 2\pi T)^{\frac{1}{2}(s-3)}\, du. \tag{9.32}$$

Since the function $\det(u - 2\pi T)^{\frac{1}{2}(s-3)}$ is now bounded, uniformly in a neighborhood of $s = 0$, this integral is dominated by

$$\int\limits_{u>0} e^{-\,\mathrm{tr}(2uy)} \det(u)^{\frac{1}{2}\mathrm{Re(s)}}\, du = \Gamma_2\left(\frac{s+3}{2}\right) \cdot \det(2y)^{-\frac{s+3}{2}}; \tag{9.33}$$

hence it is finite at $s = 0$. Thus, we have proved the $T < 0$ part of (ii).

We will prove the $\mathrm{sig}(T) = (1,1)$ part of (ii) below. □

We now assume that $\mathrm{sig}(T) = (1,1)$, and we analyze the function $\eta(2y, T; \frac{1}{2}(s+3), \frac{1}{2}s)$ in several steps, based, initially, on the method of Kaufold-Shimura ([17], [45]). It should be noted that we essentially are in the case already covered by [17], whereas generalizations to high dimensional situations will require the extension proved in [45]. Our first goal is to show that this function remains holomorphic in a neighborhood of $s = 0$. This will finish the proof of part (ii) of Proposition 9.5. Next, we want to obtain an 'explicit' formula for $\eta(2y, T; \frac{3}{2}, 0)$. This amounts to finding an explicit formula for the derivative of W_T at $s = 0$.

Step 1. First, we use the $\mathrm{GL}_2(\mathbb{R})^+$ equivariance in the variables y and T.

LEMMA 9.6. *For* $a \in \mathrm{GL}_2(\mathbb{R})^+$,

$$\eta(2ay^ta, T; \alpha, \beta) = (\det a)^{-2(\alpha+\beta-\rho)}\, \eta(2y, {}^taTa; \alpha, \beta).$$

Proof. Write $y = a \cdot {}^ta$, for $a \in \mathrm{GL}_2(\mathbb{R})^+$, as above. Then

$$\begin{aligned}(9.34)\ \eta(2y, T; \alpha, \beta) &= \int\limits_{\substack{u+\pi T>0\\ u-\pi T>}} e^{-2\,\mathrm{tr}(yu)} \det(u+\pi T)^{\alpha-\rho} \det(u-\pi T)^{\beta-\rho}\, du \\ &= \pi^{2(\alpha+\beta-\rho)} \int\limits_{\substack{u+T>0\\ u-T>0}} e^{-2\pi\,\mathrm{tr}(yu)} \det(u+T)^{\alpha-\rho} \det(u-T)^{\beta-\rho}\, du \\ &= (\det(y)^{-1}\pi^2)^{(\alpha+\beta-\rho)} \int\limits_{\substack{u+T'>0\\ u-T'>0}} e^{-2\pi\,\mathrm{tr}(u)} \det(u+T')^{\alpha-\rho} \\ &\qquad\qquad \times \det(u-T')^{\beta-\rho}\, du,\end{aligned}$$

where $T' = T[a] = {}^taTa$. This proves what we want and also gives a useful expression. □

Step 2. We now consider the integral in the last expression in the proof of Lemma 9.6. First, set

$$(9.35)\qquad T' = \begin{pmatrix} \delta_+ & \\ & -\delta_- \end{pmatrix} [k_\theta], \quad \Delta = \begin{pmatrix} \delta_+ & \\ & \delta_- \end{pmatrix}, \quad \text{and}$$
$$I = I_{1,1} = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},$$

with $k_\theta = \left(\begin{smallmatrix} c & s \\ -s & c \end{smallmatrix}\right)$ and $c = \cos(\theta)$, $s = \sin(\theta)$. Then we get:

$$\begin{aligned}(9.36)\quad &\int\limits_{\substack{u+T'>0\\ u-T'>0}} e^{-2\pi\,\mathrm{tr}(u)} \det(u+T')^{\alpha-\rho} \det(u-T')^{\beta-\rho}\, du \\ &= \int\limits_{\substack{u+\Delta I>0\\ u-\Delta I>0}} e^{-2\pi\,\mathrm{tr}(u)} \det(u+\Delta I)^{\alpha-\rho} \det(u-\Delta I)^{\beta-\rho}\, du \\ &= (\delta_+\delta_-)^{\alpha+\beta-\rho} \cdot \int\limits_{\substack{u+I>0\\ u-I>0}} e^{-2\pi\,\mathrm{tr}(u\Delta)} \det(u+I)^{\alpha-\rho} \det(u-I)^{\beta-\rho}\, du\end{aligned}$$

$$= c(s,T') \int\limits_{\substack{U_+>0\\U_->0}} e^{-2\pi\,\mathrm{tr}(u\Delta)} \det(U_+)^{\alpha-\rho}\det(U_-)^{\beta-\rho}\,du$$

$$= 2^{2s}c(s,T') \int\limits_{\substack{V_+>0\\V_->0}} e^{-4\pi\,\mathrm{tr}(V\Delta)} \det(V_+)^{\alpha-\rho}\det(V_-)^{\beta-\rho}\,dV,$$

where

$$U_+ = u + \begin{pmatrix} 2 & \\ & 0 \end{pmatrix}, \qquad U_- = u + \begin{pmatrix} 0 & \\ & 2 \end{pmatrix},$$

$$V_+ = V + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \qquad V_- = V + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix},$$

and

$$c(s,T') = e^{-2\pi(\delta_+ + \delta_-)}(\delta_+\delta_-)^s. \tag{9.37}$$

Step 3. Now we can make the Kaufhold-Shimura substitution ([45], [17]). Write

$$V = \begin{pmatrix} x & z \\ z & y \end{pmatrix} \tag{9.38}$$

and let

$$\begin{aligned} x &= u + w^2 \\ y &= v + w^2 \\ z &= (u+1+w^2)^{\frac12}(1+w^2)^{-\frac12} w (v+1+w^2)^{\frac12}. \end{aligned} \tag{9.39}$$

Then

$$\frac{\partial(x,y,x)}{\partial(u,v,w)} = (1+x)^{\frac12}(1+y)^{\frac12}(1+w^2)^{-\rho}, \tag{9.40}$$

and

$$\begin{aligned} \det\left(V + \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}\right) &= (u+1+w^2)v(1+w^2)^{-1} \\ \det\left(V + \begin{pmatrix} 0 & \\ & 1 \end{pmatrix}\right) &= (v+1+w^2)u(1+w^2)^{-1}. \end{aligned} \tag{9.41}$$

The integral in the last line of (9.36) becomes:

$$\begin{aligned} &\int_{u>0}\int_{v>0}\int_{-\infty}^{\infty} e^{-4\pi[\delta_+(u+w^2)+\delta_-(v+w^2)]} \\ &\times \Big((u+1+w^2)v(1+w^2)^{-1}\Big)^{\alpha-\rho}\Big((v+1+w^2)u(1+w^2)^{-1}\Big)^{\beta-\rho} \\ &\times (u+1+w^2)^{\frac12}(v+1+w^2)^{\frac12}(1+w^2)^{-\rho}\,du\,dv\,dw. \end{aligned} \tag{9.42}$$

This is

$$(9.43)\int_{u>0}\int_{v>0} e^{-4\pi(\delta_+u+\delta_-v)}u^{\beta-\rho}v^{\alpha-\rho}\int_{-\infty}^{\infty} e^{-4\pi w^2(\delta_++\delta_-)}(u+1+w^2)^{\alpha+\frac{1}{2}-\rho}$$
$$\times\,(v+1+w^2)^{\beta+\frac{1}{2}-\rho}(1+w^2)^{\rho-\alpha-\beta}\,dw\,du\,dv$$
$$=\int_{-\infty}^{\infty} e^{-4\pi w^2(\delta_++\delta_-)}$$
$$\times\left(\int_{u>0}(u+1+w^2)^{\alpha+\frac{1}{2}-\rho}\,u^{\beta-\rho}\,e^{-4\pi\delta_+u}\,du\right.$$
$$\left.\times\int_{v>0}(v+1+w^2)^{\beta+\frac{1}{2}-\rho}\,v^{\alpha-\rho}\,e^{-4\pi\delta_-v}\,dv\right)\,(1+w^2)^{\rho-\alpha-\beta}\,dw.$$

Substituting $(1+w^2)u$ for u and $(1+w^2)v$ for v, we get

$$(9.44)\qquad \int_{-\infty}^{\infty} e^{-4\pi w^2(\delta_++\delta_-)}$$
$$\times\int_{u>0}(u+1)^{\alpha+\frac{1}{2}-\rho}\,u^{\beta-\rho}\,e^{-4\pi\delta_+(1+w^2)u}\,du$$
$$\times\int_{v>0}(v+1)^{\beta+\frac{1}{2}-\rho}\,v^{\alpha-\rho}\,e^{-4\pi\delta_-(1+w^2)v}\,dv\;\cdot(1+w^2)^{\alpha+\beta-\rho}\,dw.$$

Step 4. Recall that $\alpha=\frac{1}{2}(s+3)$ and $\beta=\frac{1}{2}s$, so that the previous manipulation is valid in the region $\mathrm{Re}(s)>1$ (i.e., $\mathrm{Re}(\beta)>\frac{1}{2}$).

We improve the region of convergence by applying integration by parts to the integral with respect to u.

$$(9.45)\quad \int_{u>0}(u+1)^{\alpha+\frac{1}{2}-\rho}u^{\beta-\rho}e^{-4\pi\delta_+(1+w^2)u}\,du$$
$$=\int_0^{\infty}(u+1)^{\alpha-1}\frac{1}{\beta-\frac{1}{2}}\cdot\frac{\partial}{\partial u}\left\{u^{\beta-\frac{1}{2}}\right\}\,e^{-4\pi\delta_+(1+w^2)u}\,du,$$

and then, the integral here gives:

$$(9.46)\quad \frac{1}{\beta-\frac{1}{2}}\cdot\left[(u+1)^{\alpha-1}u^{\beta-\frac{1}{2}}e^{-4\pi\delta_+(1+w^2)u}\Big|_0^{\infty}\right.$$
$$-(\alpha-1)\int_0^{\infty}(u+1)^{\alpha-2}u^{\beta-\frac{1}{2}}e^{-4\pi\delta_+(1+w^2)u}\,du$$
$$\left.+\,4\pi\delta_+(1+w^2)\int_0^{\infty}(u+1)^{\alpha-1}u^{\beta-\frac{1}{2}}e^{-4\pi\delta_+(1+w^2)u}\,du\right].$$

Slightly rearranged, this becomes:

$$(9.47)\quad \frac{\pi}{(\beta-\frac{1}{2})}\int_0^{\infty}\left[4\delta_+(1+w^2)(u+1)-\frac{1}{\pi}(\alpha-1)\right]\,(u+1)^{\alpha-2}\,u^{\beta-\frac{1}{2}}e^{-4\pi\delta_+(1+w^2)u}\,du.$$

The whole integral (9.44) is then

$$\begin{aligned}(9.48)\quad &\frac{\pi}{(\beta-\frac{1}{2})}\int_{-\infty}^{\infty} e^{-4\pi w^2(\delta_++\delta_-)}\\ &\times\int_{u>0}\left[4\delta_+(1+w^2)(u+1)-\frac{1}{\pi}(\alpha-1)\right]\\ &\qquad\times (u+1)^{\alpha-2}\,u^{\beta-\frac{1}{2}}e^{-4\pi\delta_+(1+w^2)u}\,du\\ &\times\int_{v>0}(v+1)^{\beta+\frac{1}{2}-\rho}\,v^{\alpha-\rho}\,e^{-4\pi\delta_-(1+w^2)v}\,dv\cdot(1+w^2)^{\alpha+\beta-\rho}\,dw.\end{aligned}$$

The integral here is absolutely convergent in the half plane $\mathrm{Re}(s)>-1$ (i.e., $\mathrm{Re}(\beta)>-\frac{1}{2}$). Moreover, the holomorphy of this expression at $s=0$ implies that $\eta(2y,T;\frac{1}{2}(s+3),\frac{1}{2}s)$ is holomorphic at $s=0$, and thus completes the proof of (ii) of Proposition 9.5.

Step 5. We may now set $s=0$ in the integral in (9.48). First we consider the dv integral when $\beta=0$:

$$\begin{aligned}(9.49)\quad \int_0^{\infty}(v+1)^{-1}e^{-4\pi\delta_-(1+w^2)v}\,dv &= \int_1^{\infty} v^{-1}e^{-4\pi\delta_-(1+w^2)v}\,dv\; e^{4\pi\delta_-(1+w^2)}\\ &= -Ei\left(-4\pi(1+w^2)\delta_-\right)\; e^{4\pi\delta_-(1+w^2)},\end{aligned}$$

where Ei is the exponential integral ([35]) defined by:

$$(9.50)\qquad -Ei(-z)=\int_0^{\infty}\frac{e^{-z(t+1)}}{t+1}\,dt.$$

Similarly, the du integral reduces to

$$(9.51)\qquad \int_0^{\infty}\left[4\delta_+(1+w^2)(u+1)-\frac{1}{2\pi}\right]e^{-4\pi\delta_+(1+w^2)u}\,(u+1)^{-\frac{1}{2}}\,u^{-\frac{1}{2}}\,du,$$

while the factor $\frac{\pi}{\beta-\frac{1}{2}}$ becomes -2π. Collecting terms, we obtain:

$$\begin{aligned}(9.52)\quad &2\pi\int_w e^{-4\pi w^2(\delta_++\delta_-)}Ei\left(-4\pi(1+w^2)\delta_-\right)\; e^{4\pi\delta_-(1+w^2)}\\ &\times\left(\int_0^{\infty}\left[4\delta_+(1+w^2)(u+1)-\frac{1}{2\pi}\right]e^{-4\pi\delta_+(1+w^2)u}\,(u+1)^{-\frac{1}{2}}\,u^{-\frac{1}{2}}\,du\right)dw;\end{aligned}$$

that is,

$$\begin{aligned}(9.53)\quad 2\pi\,e^{4\pi(\delta_++\delta_-)}\int_w\int_{u>0} &Ei\left(-4\pi(1+w^2)\delta_-\right)\left[4\delta_+(1+w^2)(u+1)-\frac{1}{2\pi}\right]\\ &\times e^{-4\pi\delta_+(1+w^2)(u+1)}\,(u+1)^{-\frac{1}{2}}\,u^{-\frac{1}{2}}\,du\,dw.\end{aligned}$$

Gathering all of this information, we obtain the main result of this section.

THEOREM 9.7. *Suppose that* $\mathrm{sig}(T) = (1,1)$ *and, for* $a \in \mathrm{GL}_2(\mathbb{R})^+$, *let* δ_+ *and* $-\delta_-$ *be the eigenvalues of the matrix* $T' = {}^t aTa$. *Then*

$$\begin{aligned}
&\frac{1}{2iC_\infty} \cdot \frac{\partial}{\partial s}\left\{W_T(m(a),s,\Phi^{\frac{3}{2}})\right\}\Bigg|_{s=0} \\
&\quad = -2\pi^2 \,|a|^{\frac{3}{2}}\, e^{2\pi(\delta_+ + \delta_-)} \\
&\qquad \times \int_w \int_{u>0} Ei\left(-4\pi(1+w^2)\delta_-\right)\left[4\delta_+(1+w^2)(1+u) - \frac{1}{2\pi}\right] \\
&\qquad\qquad \times e^{-4\pi\delta_+(1+w^2)(u+1)}\,(u+1)^{-\frac{1}{2}}\,u^{-\frac{1}{2}}\,du\,dw.
\end{aligned}$$

Proof. We have:

$$\begin{aligned}
(9.54)\quad C_\infty^{-1} \cdot W_T(m(a), s, \Phi^{\frac{3}{2}}) &= |a|^{s+\frac{3}{2}} \cdot \xi(y, T; \alpha, \beta) \\
&= |a|^{s+\frac{3}{2}} \frac{-(2\pi i)^3}{2\Gamma_2(\alpha)\Gamma_2(\beta)} \cdot \eta(2y, T; \alpha, \beta) \\
&= |a|^{s+\frac{3}{2}} \frac{-(2\pi i)^3}{2\Gamma_2(\alpha)\Gamma_2(\beta)} \\
&\quad \cdot |a|^{-s} \cdot \eta(2 \cdot 1_2, T'; \alpha, \beta) \\
&= |a|^{\frac{3}{2}} \frac{-(2\pi i)^3 2^{2s}}{2\Gamma_2(\alpha)\Gamma_2(\beta)} \cdot c(s, T')\,\frac{\pi}{\beta - \frac{1}{2}} \\
&\quad \cdot \eta^*(2 \cdot 1_2, T'; \alpha, \beta),
\end{aligned}$$

where $\eta^*(2 \cdot 1_2, T'; \alpha, \beta)$ is the integral occurring in (9.48). This expression vanishes at $s = 0$, due to the factor $\Gamma(\frac{s}{2})^{-1}$ coming from $\Gamma_2(\beta)^{-1}$. Since

$$(9.55)\qquad \frac{\partial}{\partial s}\left(\frac{1}{\Gamma(\frac{s}{2})}\right)\Bigg|_{s=0} = \frac{1}{2},$$

we have

$$\begin{aligned}
(9.56)\quad & C_\infty^{-1} \cdot \frac{\partial}{\partial s}\left\{W_T(m(a), s, \Phi^{\frac{3}{2}})\right\}\Bigg|_{s=0} \\
&= \frac{1}{2} \cdot |a|^{\frac{3}{2}} \frac{-(2\pi i)^3}{2\Gamma_2(\frac{3}{2})\sqrt{\pi}\,\Gamma(-\frac{1}{2})}\, c(0, T')\, 2\pi\, e^{4\pi(\delta_+ + \delta_-)} \\
&\quad \times \int_w \int_{u>0} Ei\left(-4\pi(1+w^2)\delta_-\right)\left[4\delta_+(1+w^2)(1+u) - \frac{1}{2\pi}\right] \\
&\quad \times e^{-4\pi\delta_+(1+w^2)(u+1)}\,(u+1)^{-\frac{1}{2}}\,u^{-\frac{1}{2}}\,du\,dw.
\end{aligned}$$

This simplifies, slightly, to give the claimed expression. □

Finally, we make a rather strange change of variables, which will yield an expression most easily related to that arising in the archimedean height calculation. First we write the integral in Theorem 9.7 as

$$\begin{aligned}(9.57)\qquad \int_w \int_{z>0} & Ei\left(-4\pi(1+w^2)\delta_-\right) \\ & \times \left[4\delta_+(1+w^2)(z+1) - \frac{1}{2\pi}\right] \\ & \times e^{-4\pi\delta_+(1+w^2)(z+1)}\,(z+1)^{-\frac{1}{2}}\,z^{-\frac{1}{2}}\,dz\,dw.\end{aligned}$$

Now, for $\tau = u + iv \in D^+$, D^+ the upper half-plane, set

$$(9.58)\qquad w = \frac{u}{v} \qquad \text{and} \qquad z = \left(\frac{|\tau| - |\tau|^{-1}}{2}\right)^2. \qquad (!!)$$

This change of variables will have a singularity along $|\tau| = 1$ and will map D^+ two-to-one to the region $-\infty < w < \infty$, $0 < z < \infty$ over which we wish to integrate. We have

$$(9.59)\qquad \left|\det \frac{\partial(w,z)}{\partial(u,v)}\right| = \frac{1}{2v^2}\left||\tau|^2 - |\tau|^{-2}\right|,$$

and

$$(9.60)\qquad z+1 = \left(\frac{|\tau| + |\tau|^{-1}}{2}\right)^2, \qquad\qquad z(z+1) = \left(\frac{|\tau|^2 - |\tau|^{-2}}{4}\right)^2,$$

so that

$$(9.61)\qquad (z+1)^{-\frac{1}{2}} z^{-\frac{1}{2}}\,dz\,dw = 2\frac{du\,dv}{v^2}.$$

Also

$$(9.62)\qquad 4(1+w^2)(z+1) = v^{-2}(1+|\tau|^2)^2 \qquad \text{and} \qquad (1+w^2) = v^{-2}|\tau|^2.$$

Substituting these, we find that the previous integral is equal to

$$\begin{aligned}(9.63)\qquad \int_{D^+} Ei\left(-4\pi\delta_- v^{-2}|\tau|^2\right) \cdot & \left[\delta_+ v^{-2}(1+|\tau|^2)^2 - \frac{1}{2\pi}\right] \\ & \times e^{-\pi\delta_+ v^{-2}(1+|\tau|^2)^2} \cdot \frac{du\,dv}{v^2}.\end{aligned}$$

COROLLARY 9.8. *With the notation of Theorem* 9.7,

$$\frac{1}{2iC_\infty} \cdot \frac{\partial}{\partial s}\left\{W_T(m(a),s,\Phi^{\frac{3}{2}})\right\}\Bigg|_{s=0}$$

$$
\begin{aligned}
&= -2\pi^2\, |a|^{\frac{3}{2}} \cdot e^{-2\pi(\delta_+-\delta_-)} \\
&\quad\times \int_{D^+} Ei\left(-4\pi\delta_- v^{-2}|\tau|^2\right)\cdot\left[\delta_+ v^{-2}(1+|\tau|^2)^2-\frac{1}{2\pi}\right] \\
&\qquad\qquad\times e^{-\pi\delta_+[v^{-2}(1+|\tau|^2)^2-4]}\cdot\frac{du\,dv}{v^2}.
\end{aligned}
$$

Note that $\delta_+ - \delta_- = \mathrm{tr}(T') = \mathrm{tr}({}^t aTa) = \mathrm{tr}(Ty)$.

Part III. Height pairings of 0-cycles on Shimura curves

In this section, we construct certain weighted 0-cycles on the family of Shimura curves attached to the quaternion algebra B, and we show that the terms in their height pairing associated to primes of good reduction (including $p=\infty$) are closely related to the quantities $F_{d_1,d_2}(g_1', g_2';\Phi)_p$ of Proposition 7.3 (cf., also (II.9) and (II.10)).

For our fixed indefinite quaternion algebra B, let $D(B)$ be the product of the primes p such that B_p is division. We assume from now on that $D(B)>1$ so that B is a division algebra. Let $H=B^\times$, and, for a compact open subgroup $K\subset H(\mathbb{A}_f)$, let $X_K(\mathbb{C}) = H(\mathbb{Q})\backslash D\times H(\mathbb{A}_f)/K$ be the complex points of the associated Shimura curve X_K over $\mathbb{Q}$. Here $D\simeq \mathbb{P}^1(\mathbb{C})-\mathbb{P}^1(\mathbb{R})$.

In Section 10, for a 'weight function' $\varphi\in S(V(\mathbb{A}_f))^K$ and for each $d\in\mathbb{Q}_{>0}$, we define, (10.15), a weighted 0-cycle $Z(d,\varphi;K)$ on X_K. These cycles are special cases of those of [23]. They are rational over $\mathbb{Q}$ and have the property that, if $K'\subset K$ is another compact open subgroup and if pr: $X_{K'}\longrightarrow X_K$ is the natural projection, then

$$\text{(III.1)}\qquad \mathrm{pr}^*\left(Z(d,\varphi;K)\right) = Z(g,\varphi;K').$$

In Sections 11 and 12, we consider the divisors $Z(d,\varphi;K)(\mathbb{C})$ on $X_K(\mathbb{C})$, we equip them with Green functions [8], and we compute their archimedean height pairing in the sense of Gillet and Soulé. This is done in two steps. First, in Section 11, we recall that explicit Poincaré dual forms for cycles like $Z(d,\varphi;K)(\mathbb{C})$ were constructed in general in [24], [25], [26]. Specializing the construction of [24] we obtain, (11.16) and (11.17), a Schwartz function on $V(\mathbb{R})$, valued in $(1,1)$-forms on D,

$$\text{(III.2)}\qquad \varphi_{KM}\in\left[S(V(\mathbb{R}))\otimes A^{(1,1)}(D)\right]^{H(\mathbb{R})}.$$

For $x\in V(\mathbb{R})$, write $\varphi_{KM}(x) = e^{-\pi(x,x)}\cdot\varphi^0_{KM}(x)$. We identify D with the space of oriented negative 2-planes in $V(\mathbb{R})$. For a vector $x\in V(\mathbb{R})$ of positive length, let $D_x\subset D$ be the set of oriented negative 2-planes orthogonal to x; it consists of two points, $x^\perp$ with its two possible orientations. The form

$\varphi^0_{KM}(x) \in A^{(1,1)}(D)$ is rapidly decreasing at the boundary of D, and for any arithmetic group $\Gamma \subset Z(\mathbb{Q})\backslash H(\mathbb{Q})$ the sum

$$\omega^0 = \sum_{\gamma\in\Gamma} \gamma^*(\varphi^0_{KM}(x)) \tag{III.3}$$

converges and is a Poincaré dual form for the image of D_x in $\Gamma\backslash D$. For any $x \in V(\mathbb{R})$ of positive length, we explicitly solve the equation

$$dd^c[\xi^0(x)] + \delta_{D_x} = [\varphi^0_{KM}(x)] \tag{III.4}$$

on D, where the function $\xi^0(x)$, (11.24), on D has a logarithmic singularity at the two points in D_x, is smooth elsewhere, and is rapidly decreasing at the boundary of D (Proposition 11.1). The sum

$$\Xi^0 = \sum_{\gamma\in\Gamma} \gamma^*(\xi^0(x)) \tag{III.5}$$

converges to a Green function on $\Gamma\backslash D$ satisfying

$$dd^c[\Xi^0(x)] + \delta_{\Gamma\cdot D_x} = [\omega^0] \tag{III.6}$$

and having a logarithmic singularity on the cycle $\Gamma \cdot D_x$.

We are then in a position to calculate the archimedean height pairing in the sense of Gillet-Soulé, [8], [46]. Thanks to the rapid decay of ξ^0, this can be done on D itself, and we obtain a function, Definition 11.3 and (11.55):

$$Ht(x_1,x_2)_\infty := \Big\langle (D_{x_1},\xi^0(x_1))\,,\,(D_{x_2},\xi^0(x_2))\Big\rangle_\infty = \int_D \xi^0(x_1) * \xi^0(x_2), \tag{III.7}$$

where $\xi^0(x_1) * \xi^0(x_2)$ is the star product ([8], [46]). On $\Gamma\backslash D$, we have, by unfolding,

$$\Big\langle (Z_1,\Xi^0_1)\,,\,(Z_2,\Xi^0_2)\Big\rangle = \sum_{\gamma\in\Gamma} Ht(\gamma x_1, x_2)_\infty, \tag{III.8}$$

where $Z_i = \Gamma D_{x_i}$, and where, for convenience, we assume that Γ acts without fixed points on D (Lemma 12.1).

Initially, the function $Ht(x_1,x_2)_\infty$ is defined only for x_1 and $x_2 \in V(\mathbb{R})$ of positive length, but the resulting formulas involving ξ^0 make sense for any pair of nonzero vectors x_1 and $x_2 \in V(\mathbb{R})$ whose matrix of inner products

$$T = \frac{1}{2}\begin{pmatrix} (x_1,x_1) & (x_1,x_2) \\ (x_2,x_1) & (x_2,x_2) \end{pmatrix} \tag{III.9}$$

has $\det(T) \neq 0$. By the $H(\mathbb{R})$ invariance properties of ξ^0, the function

$$Ht(x_1,x_2)_\infty = Ht(T)_\infty \tag{III.10}$$

depends only on the matrix T, (11.58). This function has the following geometric interpretation. As the length of x_1 decreases to zero, i.e., as the vector

x_1 moves toward the light cone in $V(\mathbb{R})$, the corresponding point in the upper half-plane D^+ moves to a point on the boundary. When the length of x_1 becomes negative, there is no longer an associated point in D^+, but one might associate to x_1 the geodesic arc consisting of the negative 2-planes containing x_1. The same remarks apply to x_2, and the quantity $Ht(x_1, x_2)_\infty = Ht(T)_\infty$ might be viewed as giving an archimedean height pairing among these various types of objects!

The function $Ht(T)_\infty$ has the following crucial and somewhat startling property (Theorem 11.6):

$$Ht({}^t k_\theta T k_\theta) = Ht(T), \tag{III.11}$$

where k_θ is an element of $\mathrm{SO}(2)$. This says, in particular, that our height pairing does not change when the pair of vectors $[x_1, x_2]$ is replaced by the pair $[x_1(\theta), x_2(\theta)] := [x_1, x_2]k_\theta$. This transformation does not preserve lengths, and so (III.11) can be viewed as expressing a conservation of height under a transformation which may involve the passage of a point to the boundary of D and its subsequent return as an 'echo' (geodesic arc)! The proof of (III.11) is quite long and is given in Section 13.

The matrix T has signature $(1,1)$ and can be diagonalized by the action of $\mathrm{SO}(2)$. When θ is taken so that $T[k_\theta] = {}^t k_\theta T k_\theta$ is diagonal, the integral expressing the local archimedean height pairing $Ht(T)$ is particularly simple (Lemma 11.7) and *essentially coincides with the derivative of the archimedean local factor* (Theorem 11.8 and Corollary 11.10):

$$W'_{T,\infty}\left(\tilde{\iota}(g'_1, g'_2), 0, \Phi_\infty^{\frac{3}{2}}\right) = 2i\sqrt{2}\,\pi^2\; W_{d_1}^{\frac{3}{2}}(g'_1) W_{d_2}^{\frac{3}{2}}(g'_2) \cdot Ht(T[a])_\infty, \tag{III.12}$$

where g'_1 and $g'_2 \in G'_{\mathbb{R}}$, with $g'_j = n_j m(a_j) k_j$, $a_j \in \mathbb{R}_+^\times$, and $a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$! Here the function $W_{d_i}^{\frac{3}{2}}(g'_i)$ is given in Proposition 7.3 and (11.71). This result justifies, in some sense, our perhaps peculiar choice of Green function, and provides a link between the particular function φ_{KM} of [24] and a standard object arising from the Siegel Eisenstein series. It also should be emphasized that we are taking advantage of the flexibility provided by the Gillet-Soulé formulation of Arakelov theory for curves, since we are considering pairings of (Z, Ξ)'s where the Green function Ξ is not constrained by some choice of metric.

In Section 12, we define (Definition 12.4) the weighted cycles $\hat{Z}(g', d, \varphi; K)$ in $\widehat{CH}^1(X_K)_{\mathbb{C}}$,

$$\hat{Z}(g', d, \varphi; K) =: \begin{cases} \left(W_d^{\frac{3}{2}}(g')\, Z(d, \varphi; K)\,,\; \Xi(g', d, \varphi)\right) & \text{if } d > 0 \\ (0,\; \Xi(g', d, \varphi)) & \text{if } d \le 0, \end{cases} \tag{III.13}$$

with Green function obtained by summing translates of ξ, (12.21). The identity (III.12) together with an unfolding argument then yields (Theorem 12.6)

MAIN THEOREM ($p = \infty$). *Fix a compact open subgroup K and weight functions φ_1 and $\varphi_2 \in S(V(\mathbb{A}_f))^K$. Then, for g_1' and $g_2' \in G'_{\mathbb{R}}$ and for d_1 and $d_2 \in \mathbb{Q}^\times$,*

$$F_{d_1,d_2}(g_1',g_2',\Phi)_\infty = 2\pi^2 \mathrm{vol}(K) \left\langle \hat{Z}(g_1',d_1,\varphi_1;K), \hat{Z}(g_2',d_2,\varphi_2;K) \right\rangle_\infty^{\mathrm{ns.}}.$$

Here $\Phi(s) = \Phi_\infty^{\frac{3}{2}}(s) \otimes \Phi_f(s)$ where $\Phi_f(s)$ is the standard section of $I_{2,f}(s,\chi)$ associated to $\varphi_1 \otimes \varphi_2 \in S(V(\mathbb{A}_f)^2)$. If $d_1 d_2$ is not a square, then the superscript 'ns.' — which indicates the 'nonsingular part' (Definition 12.3) — can be dropped.

In Section 14, we turn to the (good) finite primes, that is, we assume that $p \nmid D(B)$ and that $K = K_p K^p$ with $K_p \simeq \mathrm{GL}_2(\mathbb{Z}_p)$ and with $K^p \subset H(\mathbb{A}_f^p)$ sufficiently small. As in [51], there is a model $\mathcal{X}_K$ of X_K which is smooth over $\mathbb{Z}_{(p)}$ and is defined as the solution to a moduli problem for abelian surfaces with $\mathcal{O}_B$ action. We then give a modular definition of divisors $\mathcal{Z}(d, \varphi^p; K^p)$ on $\mathcal{X}_K$ whose restrictions to the generic fiber are the divisors $Z(d, \varphi; K)$ of Section 10, where $\varphi = \varphi_p \otimes \varphi^p$ and φ_p is the characteristic function of $V(\mathbb{Q}_p) \cap M_2(\mathbb{Z}_p)$. Here $d > 0$.

We now assume that $d_1 d_2 \neq 0$ is not a square in $\mathbb{Q}^\times$, so that the cycles $\mathcal{Z}(d_1, \varphi_1^p; K^p)$ and $\mathcal{Z}(d_2, \varphi_2^p; K^p)$ do not meet in the generic fiber. There can be no intersection in the fiber at p if p splits in either of the quadratic fields $\mathrm{k}_{d_i} = \mathbb{Q}(\sqrt{\kappa d_i})$. Here the global character χ is given by $\chi(x) = (x, \kappa)_{\mathbb{A}}$ for a square free negative integer κ. If p does not split in either of the fields k_{d_i}, the intersections in the special fiber can only occur at points in the supersingular set $\mathcal{X}_K(\bar{\mathbb{F}}_p)^{\mathrm{ss.}}$. There is a well-known parameterization

$$\mathcal{X}_K(\bar{\mathbb{F}}_p)^{\mathrm{ss.}} \simeq H^{(p)}(\mathbb{Q}) \backslash H^{(p)}(\mathbb{A}_f)/K' \tag{III.14}$$

of this set, via the multiplicative group $H^{(p)} = B^{(p),\times}$ of the definite quaternion algebra $B^{(p)}$ with invariants $\mathrm{inv}_\ell B^{(p)} = \mathrm{inv}_\ell B$ for $\ell \neq p$, and $\mathrm{inv}_\ell B^{(p)} = -\mathrm{inv}_\ell B$ for $\ell = p$ or ∞. Here $K' = K_p' K^p$ where K_p' is the maximal compact subgroup of $H^{(p)}(\mathbb{Q}_p)$. Using this, we obtain a formula for the p part of height pairing of the cycles $\mathcal{Z}(d_1, \varphi_1^p; K^p)$ and $\mathcal{Z}(d_2, \varphi_2^p; K^p)$ as a weighted sum of a quantity $Ht(y_1, y_2)_p$ associated to pairs of vectors $y_1 \in \Omega'_{d_1}(\mathbb{Q}) \subset V^{(p)}(\mathbb{Q})$ and $y_2 \in \Omega'_{d_2}(\mathbb{Q}) \subset V^{(p)}(\mathbb{Q})$. This quantity was studied by Keating [18], who proved that it only depends on the matrix

$$T = \frac{1}{2} \begin{pmatrix} (y_1, y_1) & (y_1, y_2) \\ (y_2, y_1) & (y_2, y_2) \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q}_p). \tag{III.15}$$

By specializing his earlier work with Gross [10], Keating also gave a completely explicit formula for $Ht(T)_p$, Proposition 14.11, from which it follows that

$$Ht(T[k])_p = Ht(T)_p \tag{III.16}$$

for all $k \in \mathrm{GL}_2(\mathbb{Z}_p)$. This invariance is the analogue of (III.11) above.

Comparing Keating's formula with the derivative of Kitaoka's density, Corollary 8.7, we obtain the second part of our main result:

MAIN THEOREM ($p < \infty$). *Assume that $p \nmid 2\kappa D(B)$ and that $K = K_p K^p$ with $K_p = \mathrm{GL}_2(\mathbb{Z}_p)$ and with K^p sufficiently small. Also assume that $d_1 d_2$ is not a square. Then, for g_1' and $g_2' \in G_{\mathbb{R}}'$,*

$$\begin{aligned} F_{d_1,d_2}(g_1',g_2',\Phi)_p &= 2\pi^2 \operatorname{vol}(K) \cdot \left\langle \hat{\mathcal{Z}}(g_1',d_1,\varphi_1^p;K^p), \hat{\mathcal{Z}}(g_2',d_2,\varphi_2^p;K^p) \right\rangle_p \\ &= 2\pi^2 \operatorname{vol}(K) \cdot W_{d_1}^{\frac{3}{2}}(g_1') W_{d_2}^{\frac{3}{2}}(g_2') \cdot \left\langle \hat{\mathcal{Z}}(d_1,\varphi_1^p;K^p), \hat{\mathcal{Z}}(d_2,\varphi_2^p;K^p) \right\rangle_p . \end{aligned}$$

In Section 15, we combine the results on the individual $F_{d_1,d_2}(g_1', g_2', \Phi)_p$'s. Let $\mathcal{Y}_K$ be a regular minimal model of X_K over $\mathrm{Spec}(\mathbb{Z})$, and define cycles $\mathfrak{Z}(d_1, \varphi_1; K)$ and $\mathfrak{Z}(d_2, \varphi_2; K)$ by taking the closure in $\mathcal{Y}$ of $Z(d_1, \varphi_1; K)$ and $Z(d_2, \varphi_2; K)$. Define $\hat{\mathfrak{Z}}(g_1', d_1, \varphi_1; K)$ and $\hat{\mathfrak{Z}}(g_2', d_2, \varphi_2; K)$ in $\widehat{CH}^1(\mathcal{Y}) \otimes \mathbb{C}$ by analogy with the first part of (III.13). Assume that $K = \prod_p K_p$ with K_p our fixed maximal compact subgroup for all p with $p \nmid D(B)N$, for some N. Note that $\mathcal{Y}_K$ has good reduction at all $p \nmid D(B)N$. Then, for $d_1 d_2$ not a square, we have

$$\begin{aligned} \sum_{p \nmid 2D(B)N} & F_{d_1,d_2}(g_1', g_2', \Phi)_p \\ &= 2\pi^2 \sum_{p \nmid 2D(B)N} \left\langle \hat{\mathfrak{Z}}(g_1', d_1, \varphi_1; K), \hat{\mathfrak{Z}}(g_2', d_2, \varphi_2; K) \right\rangle_p . \end{aligned} \tag{III.16}$$

Finally, in Section 16, we make some brief remarks about possible higher dimensional generalizations.

Our results ought to extend in several ways.

First, the method of Section 14 can be applied to the case $p \mid D(B)$, and reduces the computation of the pairing at p to a computation of the function $Ht(T)_p$. Also, in the case $p = 2$, when $2 \nmid D(B)$, it only remains to suitably extend Kitaoka's representation density formulas (Proposition 8.1). It should also not be difficult to remove the condition that K^p be sufficiently small in Section 14 by working on a cover and then keeping track of the action of a finite group.

Next, when d_1 and d_2 are positive and $d_1 d_2$ is a square in $\mathbb{Q}^\times$, the cycles $Z(d_1, \varphi_1; K)$ and $Z(d_2, \varphi_2; K)$ may not be disjoint on the generic fiber. One would like to compare the function $F_{d_1,d_2}(g_1', g_2', \Phi)_{\mathrm{sing}}$ with the sum of the self pairings of the common components of $\mathfrak{Z}(g_1', d_1, \varphi_1; K)$ and $\mathfrak{Z}(g_2', d_2, \varphi_2; K)$.

A more difficult extension would involve the comparison of the quantities $F_{d_1,d_2}(g_1', g_2', \Phi)_p$ with the intersection numbers on the fiber at p in the case where K_p is arbitrary. This would be analogous to the problem of counting point on the reduction mod p of a Shimura variety at a prime p dividing the level. It seems that some essentially new ideas are necessary in this case.

Finally, it should be remarked that there are quite strong parallels between our computation of the height pairing associated with a prime $p < \infty$ and the well known computation of the number of points on quaternionic Shimura varieties via the trace formula [33], [4], [37], [32], etc.. This suggests that some relative trace formula analogue of the method of [33] might lie behind our calculation.

10. *Special* 0-*cycles on Shimura curves.* We retain the notation of Section 7, so that

$$V = V_\chi^B = \{\, x \in B \mid \mathrm{tr}(x) = 0 \,\}, \tag{10.1}$$

$$(x, y) = -\kappa \cdot \mathrm{tr}(xy^\iota), \tag{10.2}$$

and $(x,x) = -2\kappa\nu(x)$, where $\nu(x)$ is the reduced norm of x. Recall from (7.1) and (7.5) that $\chi(x) = (\kappa, x)_{\mathbb{A}}$, and that $\chi_\infty(-1) = -1$, so that $\kappa < 0$. We fix an isomorphism $B(\mathbb{R}) \simeq M_2(\mathbb{R})$, and we have

$$V(\mathbb{R}) = \{\, x = \begin{pmatrix} x_1 & x_2 \\ x_3 & -x_1 \end{pmatrix} \in M_2(\mathbb{R}) \,\}, \tag{10.3}$$

and so $(x,x) = 2\kappa(x_1^2 + x_2x_3)$. Thus $\mathrm{sig}(V) = (1,2)$. As usual, let D be the set of oriented negative 2-planes in $V(\mathbb{R})$, and note that it has two connected components. Let $H = B^\times \simeq \mathrm{G\,Spin}(V)$, so that

$$1 \longrightarrow Z \longrightarrow H \longrightarrow \mathrm{SO}(V) \longrightarrow 1 \tag{10.4}$$

is exact. Here $h \in H$ acts by $x \mapsto hxh^{-1}$.

For any compact open subgroup K in $H(\mathbb{A}_f)$, let

$$X_K(\mathbb{C}) = H(\mathbb{Q})\backslash D \times H(\mathbb{A}_f)/K. \tag{10.5}$$

We assume from now on that $K = \prod_\ell K_\ell$ and that $K_\ell \cap Z(\mathbb{Q}_\ell) = \mathbb{Z}_\ell^\times$, under the natural identification of $Z(\mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell^\times$. Write $H(\mathbb{R})^+ = \mathrm{GL}_2(\mathbb{R})^+$ for the identity component of $H(\mathbb{R})$, let $H(\mathbb{Q})^+ = H(\mathbb{Q}) \cap H(\mathbb{R})^+$, and write

$$H(\mathbb{A}_f) = \coprod_j H(\mathbb{Q})^+ h_j K. \tag{10.6}$$

Let

$$\Gamma_j' = H(\mathbb{Q})^+ \cap h_j K h_j^{-1}, \tag{10.7}$$

and let Γ_j be the image of Γ'_j in $\mathrm{SO}(V)(\mathbb{Q})$. Let D^+ be the connected component of D determined by the choice of base point defined in (11.2) and (11.3) below. Then

$$X_K(\mathbb{C}) \simeq \coprod_j \Gamma_j \backslash D^+. \tag{10.8}$$

If $x \in V(\mathbb{Q})$ with $\nu(x) > 0$, let H_x be the stabilizer of x in H, and let

$$D_x = \{ \, z \in D \mid z \perp x \, \}, \tag{10.9}$$

so that

$$D_x = D_x^+ \cup D_x^-, \tag{10.10}$$

consists of the two points given by $x^\perp$ with its two orientations.

Define

$$c(x, j, K) = \Gamma_j \cdot D_x^+ \subset \Gamma_j \backslash D^+, \tag{10.11}$$

so that, as a set, $c(x, j, K)$ is a point, the image of D_x^+, on the j^{th} component of $X_K(\mathbb{C})$. Note that, if the stabilizer $\Gamma'_{j,x}$ of x in Γ'_j has image $\Gamma_{j,x}$ in $P\,\mathrm{GL}_2(\mathbb{R})$, then the cycle $c(x, j, K)$ consists of the point $\Gamma_j \cdot z^+(x)$ *counted with multiplicity* $|\Gamma_{j,x}|^{-1} = 2/|\Gamma'_{j,x}|$.

The weighted cycles, which will play a fundamental role, are defined as follows. First, for $x \in V(\mathbb{Q})$ with $\nu(x) > 0$, and for $g \in H(\mathbb{A}_f)$, define the cycle $Z(x, g; K)$:

(10.12)

$$H_x(\mathbb{Q}) \backslash \Big(D_x \times H_x(\mathbb{A}_f) / \left(H_x(\mathbb{A}_f) \cap gKg^{-1} \right) \Big) \longrightarrow H(\mathbb{Q}) \backslash D \times H(\mathbb{A}_f)/K,$$

by $(z, h) \mapsto (z, hg)$. Thus $Z(x, g; K)$ is a sum of a finite number of (special) points on $X_K(\mathbb{C})$.

Next, for $d \in \mathbb{Q}$, let

$$\Omega_d = \{ x \in V \mid -\kappa\nu(x) = d \} \tag{10.13}$$

be the corresponding hyperboloid in V. Assume that $\Omega_d(\mathbb{Q}) \neq \phi$, i.e., that the quadratic field $\mathbb{Q}(\sqrt{\kappa d}) = \Bbbk_{\kappa d}$ splits B. For $d \in \mathbb{Q}_{>0}$, $\Omega_d(\mathbb{A}_f)$ is a closed subset of $V(\mathbb{A}_f)$ on which $H(\mathbb{A}_f)$ acts continuously. For a compact open subgroup K, as above, and for a locally even function $\varphi \in S(V(\mathbb{A}_f))^K$ (a weight function), write

$$\Omega_d(\mathbb{A}_f) \cap \operatorname{supp} \varphi = \coprod_r K \cdot g_r^{-1} x_0 \tag{10.14}$$

where $x_0 \in \Omega_d(\mathbb{Q})$ and $g_r \in H(\mathbb{A}_f)$. Then define the weighted cycle

$$Z(d, \varphi; K) = \sum_r \varphi(g_r^{-1} x_0) \cdot Z(x_0, g_r; K). \tag{10.15}$$

Note that this expression is independent of the choice of the g_r's and of x_0. These weighted cycles have a number of nice properties, as shown in [23]. For example,

LEMMA 10.1.

$$Z(d,\varphi;K) = \sum_j \sum_{\substack{x\in\Omega_d(\mathbb{Q}) \\ \mathrm{mod}\,\Gamma_j}} \varphi(h_j^{-1}x)\cdot c(x,j,K).$$

Note that the above convention about stabilizers is taken into account here.

Lemma 10.1 describes the cycle $Z(d,\varphi;K)$ as a weighted sum of special points on the various components of $X_K(\mathbb{C})$.

11. Green functions and the archimedean height pairing (locally) In this section and the next we will give an explicit construction of a certain Green function in the sense of [8], [46], for the special zero cycle $Z(d,\varphi,K)$. In this section, we begin by working on D.

Let $A^{(p,q)}(D)$ be the space of smooth differential forms of type (p,q) on D. First recall that a certain element

$$\varphi_{KM} \in \Big[S(V(\mathbb{R}))\otimes A^{(1,1)}(D)\Big]^{H(\mathbb{R})} \tag{11.1}$$

was constructed in [24]. We now review its definition in the current situation.

Fix the base point $z_0 \in D$,

$$z_0 = \mathrm{span}\,\{z_1, z_2\}, \tag{11.2}$$

with

$$z_1 = |\kappa|^{-\frac12}\begin{pmatrix} 1 & \\ & -1\end{pmatrix} \quad \text{and} \quad z_2 = |\kappa|^{-\frac12}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{11.3}$$

Let D^+ be the component of D containing z_0. The line perpendicular to z_0 is spanned by the vector

$$x_0 = |\kappa|^{-\frac12}\begin{pmatrix} & 1 \\ -1 & \end{pmatrix}. \tag{11.4}$$

The stabilizer of z_0 in $H(\mathbb{R}) = \mathrm{GL}_2(\mathbb{R})$ is $K_\infty = \mathbb{R}^\times\,\mathrm{SO}(2)$, and we have an isomorphism

$$\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \simeq H(\mathbb{R})/K_\infty \xrightarrow{\sim} D, \tag{11.5}$$

with

$$\tau \mapsto hK_\infty \mapsto h\cdot z_0 := z(\tau). \tag{11.6}$$

It is more convenient to keep track of the vector $x(\tau) := h \cdot x_0$, so that

$$
\begin{aligned}
(11.7) \qquad x(\tau) &= |\kappa|^{-\frac{1}{2}} v^{-1} \begin{pmatrix} -u & u^2+v^2 \\ -1 & u \end{pmatrix} \\
&= |\kappa|^{-\frac{1}{2}} v^{-1} \begin{pmatrix} -\frac{1}{2}(\tau+\bar\tau) & \tau\bar\tau \\ -1 & \frac{1}{2}(\tau+\bar\tau) \end{pmatrix}.
\end{aligned}
$$

Here $\tau = u + iv$. Note that $x(\bar\tau) = -x(\tau)$, $(x(\tau), x(\tau)) = 2$, since $\det x(\tau) = \det x_0 = |\kappa|^{-1}$, and $x(i) = x_0$. Since $(x_0, x(\tau)) = v^{-1}(|\tau|^2 + 1)$, the component of D containing $z(\tau)$ is determined by the sign of this inner product. The isomorphism (11.5) induces a complex structure on D. Also note that

$$
(11.8) \qquad (x, x(\tau)) = -|\kappa|^{\frac{1}{2}} v^{-1}(x_3\tau\bar\tau - x_1(\tau+\bar\tau) - x_2),
$$

and that, if τ_1 and τ_2 lie in the same component of $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$, then

$$
(11.9) \qquad \frac{1}{2}(x(\tau_1), x(\tau_2)) = \frac{|\tau_1 - \tau_2|^2}{2v_1v_2} + 1 = \cosh(d(\tau_1, \tau_2)),
$$

where $d(\tau_1, \tau_2)$ is the hyperbolic distance between τ_1 and τ_2 with respect to the standard metric

$$
(11.10) \qquad ds^2 = v^{-2}(du^2 + dv^2).
$$

If $x \in V(\mathbb{R})$, the decomposition of x with respect to the orthogonal decomposition

$$
(11.11) \qquad V(\mathbb{R}) = z(\tau) + \mathbb{R}x(\tau)
$$

is

$$
(11.12) \qquad x = x' + \frac{(x, x(\tau))}{(x(\tau), x(\tau))} x(\tau),
$$

and so

$$
(11.13) \qquad (x, x) = (x', x') + \frac{1}{2}(x, x(\tau))^2.
$$

Let

$$
(11.14) \qquad R = R(x, \tau) := -(x', x') = \frac{1}{2}(x, x(\tau))^2 - (x, x),
$$

and let

$$
(11.15) \qquad (x, x)_\tau := (x, x) + 2R = (x, x(\tau))^2 - (x, x)
$$

be the majorant of $(\ ,\ )$ associated to the point $z(\tau)$. This is independent of the orientation of $z(\tau)$. Note that $R(x, \tau) = 0$ if and only if x lies in the line $\mathbb{R} \cdot x(\tau)$. For example, this can never occur if $(x, x) < 0$.

Following [23], [24], for $x \in V(\mathbb{R})$, let

$$\varphi^0_{KM}(x,\tau) := \left((x,x(\tau))^2 - \frac{1}{2\pi}\right) e^{-2\pi R(x,\tau)} \cdot \frac{i}{2}\omega, \tag{11.16}$$

and let

$$\begin{aligned} \varphi_{KM}(x,\tau) \;&:=\; \varphi^0_{KM}(x,\tau)\, e^{-\pi(x,x)} \\ &=\; \left((x,x(\tau))^2 - \frac{1}{2\pi}\right) e^{-\pi(x,x)_\tau} \cdot \frac{i}{2}\omega. \end{aligned} \tag{11.17}$$

Here we set

$$\omega = \frac{d\tau \wedge d\bar{\tau}}{v^2} = -2i \cdot \frac{du \wedge dv}{v^2}. \tag{11.18}$$

Then φ_{KM} is a Schwartz function on $V(\mathbb{R})$ valued in $(1,1)$-forms on D. We will sometimes write $\varphi_{KM}(x) \in A^{(1,1)}(D)$ for its value at x. This satisfies:

$$h^*\varphi_{KM}(h \cdot x) = \varphi_{KM}(x), \tag{11.19}$$

for $h \in H(\mathbb{R})$. In particular, $\varphi_{KM}(x)$ is $H_x(\mathbb{R})$ invariant, and $\varphi_{KM}(x,\bar{\tau}) = \varphi_{KM}(x,\tau)$.

We next want to write down a Green current related to $\varphi_{KM}(x)$. First recall that, for $z \in \mathbb{C}$, the exponential integral $Ei(z)$ is defined by ([35])

$$Ei(z) = \int_{-\infty}^{z} \frac{e^t}{t}\, dt, \tag{11.20}$$

where the path of integration lies in the plane cut along the line $\mathrm{Re}(z) \geq 0$. Then

$$Ei'(z) = \frac{e^z}{z}, \tag{11.21}$$

and

$$Ei(z) = \gamma + \log(-z) + \int_0^z \frac{e^t - 1}{t}\, dt, \tag{11.22}$$

where γ is Euler's constant [35]. In particular, the integral on the right hand side of the last equation defines an entire function of z, so that $Ei(z)$ has a logarithmic singularity at 0. The following integral representation will be useful:

$$-\,Ei(-z) = \int_0^\infty \frac{e^{-z(t+1)}}{t+1}\, dt = \int_1^\infty \frac{e^{-zt}}{t}\, dt. \tag{11.23}$$

For $x \in V(\mathbb{R})$, $x \neq 0$ define

$$\xi^0(x,\tau) = -Ei(-2\pi R(x,\tau)), \tag{11.24}$$

and let

$$\xi(x,\tau) = \xi^0(x,\tau)\, e^{-\pi(x,x)} = -Ei(-2\pi R)\, e^{-\pi(x,x)}, \tag{11.25}$$

with $R = R(x,\tau) = \frac{1}{2}(x, x(\tau))^2 - (x,x) = -(x', x')$, as above. For fixed x, $\xi(x) = \xi(x,\cdot)$ is a smooth function on $D - D_x$, where D_x is given by (10.9) above. When $(x,x) \leq 0$, $D_x = \phi$, while, if $(x,x) > 0$, D_x consists of two points. The function $\xi(x)$ has logarithmic growth 'along' D_x, so we may view it as a current $[\xi(x)]$ on D. The same remarks hold for ξ^0, since, as functions on D, ξ and ξ^0 differ by the constant factor $e^{-\pi(x,x)}$. Note that $R(x,\tau)$ is an even function of x and of $x(\tau)$ so that R, ξ^0, and ξ are invariant under $\tau \mapsto \bar{\tau}$.

Let d, ∂, and $\bar{\partial}$ be the usual exterior differentials on D, and set $d^c = \frac{1}{4\pi i}(\partial - \bar{\partial})$, as in [46]. Note that $dd^c = -\frac{1}{2\pi i}\partial\bar{\partial}$.

PROPOSITION 11.1. *As currents*:

$$dd^c[\xi^0(x)] + \delta_{D_x} = [\varphi^0_{KM}(x)].$$

In fact, $\xi^0(x,\cdot)$ is the unique **rapidly decreasing** *Green function associated to φ^0_{KM}.*

Proof. First we restrict to the space $D - D_x$ and compute $\partial\bar{\partial}\xi^0(x)$. We have:

$$-\bar{\partial}\{\xi^0(x,\tau)\} = \frac{e^{-2\pi R}}{R}\bar{\partial}R, \tag{11.26}$$

and

$$\begin{aligned} -\partial\bar{\partial}\{\xi^0(x,\tau)\} &= -2\pi \cdot \frac{e^{-2\pi R}}{R} \cdot \partial R \wedge \bar{\partial}R \\ &\quad + \frac{e^{-2\pi R}}{R^2} \cdot \left(-\partial R \wedge \bar{\partial}R + R\,\partial\bar{\partial}R\right). \end{aligned} \tag{11.27}$$

Now $\partial R = (x, x(\tau))\,(x, \partial x(\tau))$, and $\bar{\partial}R = (x, x(\tau))\,(x, \bar{\partial}x(\tau))$, so that

$$\partial R \wedge \bar{\partial}R = (x, x(\tau))^2\,(x, \partial x(\tau)) \wedge (x, \bar{\partial}x(\tau)). \tag{11.28}$$

Since $\partial v^{-1} = -\frac{1}{2i}v^{-2}\,d\tau$, and $\bar{\partial}v^{-1} = \frac{1}{2i}v^{-2}\,d\bar{\tau}$, we have

$$\partial x(\tau) = -\frac{1}{2i}v^{-1}x(\tau)\,d\tau + |\kappa|^{-\frac{1}{2}}v^{-1}\begin{pmatrix} -\frac{1}{2} & \bar{\tau} \\ 0 & \frac{1}{2}\end{pmatrix} d\tau, \tag{11.29}$$

and

$$\bar{\partial}x(\tau) = \frac{1}{2i}v^{-1}x(\tau)\,d\bar{\tau} + |\kappa|^{-\frac{1}{2}}v^{-1}\begin{pmatrix} -\frac{1}{2} & \tau \\ 0 & \frac{1}{2}\end{pmatrix} d\bar{\tau}. \tag{11.30}$$

Then,

$$(11.31)\quad \begin{aligned} \partial\bar{\partial}x(\tau) &= \frac{1}{4}v^{-2}x(\tau)\,d\tau\wedge d\bar{\tau} \\ &\quad + \frac{1}{2i}v^{-1}\left(-\frac{1}{2i}v^{-1}x(\tau)+|\kappa|^{-\frac{1}{2}}v^{-1}\begin{pmatrix} -\frac{1}{2} & \bar{\tau} \\ 0 & \frac{1}{2}\end{pmatrix}\right)d\tau\wedge d\bar{\tau} \\ &\quad - \frac{1}{2i}|\kappa|^{-\frac{1}{2}}v^{-2}\begin{pmatrix} -\frac{1}{2} & \tau \\ 0 & \frac{1}{2}\end{pmatrix}d\tau\wedge d\bar{\tau} \\ &\quad + |\kappa|^{-\frac{1}{2}}v^{-1}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}d\tau\wedge d\bar{\tau} \\ &= \frac{1}{2}x(\tau)\cdot\omega, \end{aligned}$$

where $\omega = \frac{d\tau\wedge d\bar{\tau}}{v^2}$, as before.

Using (11.8) and noting that

$$(11.32)\qquad \left(x, |\kappa|^{-\frac{1}{2}}v^{-1}\begin{pmatrix} -\frac{1}{2} & \tau \\ 0 & \frac{1}{2}\end{pmatrix}\right) = -|\kappa|^{\frac{1}{2}}v^{-1}(x_3\tau - x_1),$$

we have:

$$(11.33)\quad \begin{aligned} &(x,\partial x(\tau))\wedge(x,\bar{\partial}x(\tau)) \\ &= \Big(\frac{1}{4}(x,x(\tau))^2 + \frac{1}{2i}(x,x(\tau))\,|\kappa|^{\frac{1}{2}}(x_3\tau - x_1) \\ &\qquad - \frac{1}{2i}(x,x(\tau))\,|\kappa|^{\frac{1}{2}}(x_3\bar{\tau}-x_1) \\ &\qquad + |\kappa|(x_3\tau-x_1)(x_3\bar{\tau}-x_1)\Big)\,\omega \\ &= \Big(\frac{1}{4}(x,x(\tau))^2 + |\kappa|^{\frac{1}{2}}(x,x(\tau))x_3v \\ &\qquad + |\kappa|(x_3\tau-x_1)(x_3\bar{\tau}-x_1)\Big)\,\omega. \end{aligned}$$

But now,

$$(11.34)\quad \begin{aligned} (x_3\tau-x_1)(x_3\bar{\tau}-x_1) &= x_3^2\tau\bar{\tau} - x_3x_1(\tau+\bar{\tau}) + x_1^2 \\ &= -vx_3|\kappa|^{-\frac{1}{2}}(x,x(\tau)) + x_2x_3 + x_1^2 \\ &= -vx_3|\kappa|^{-\frac{1}{2}}(x,x(\tau)) - \frac{1}{2}|\kappa|^{-1}(x,x), \end{aligned}$$

and so our previous expression becomes

$$(11.35)\qquad (x,\partial x(\tau))\wedge(x,\bar{\partial}x(\tau)) = \left(\frac{1}{4}(x,x(\tau))^2 - \frac{1}{2}(x,x)\right)\omega = \frac{1}{2}R\cdot\omega.$$

This yields

$$\partial R\wedge\bar\partial R=\frac{1}{2}(x,x(\tau))^2\,R\cdot\omega, \tag{11.36}$$

and so the first term in our main expression becomes

$$-2\pi\cdot\frac{e^{-2\pi R}}{R}\cdot\partial R\wedge\bar\partial R=-\pi\cdot e^{-2\pi R}\,(x,x(\tau))^2\cdot\omega. \tag{11.37}$$

Next we have to consider

$$\begin{aligned}\partial\bar\partial R &= \partial\left[(x,x(\tau))(x,\bar\partial x(\tau))\right]\\ &= (x,\partial x(\tau))\wedge(x,\bar\partial x(\tau))+(x,x(\tau))(x,\partial\bar\partial x(\tau))\\ &= \frac{1}{2}R\cdot\omega+\frac{1}{2}(x,x(\tau))^2\cdot\omega.\end{aligned} \tag{11.38}$$

Here we have used the identity (11.31). Thus

$$\begin{aligned}-\partial R\wedge\bar\partial R+R\,\partial\bar\partial R &= -\frac{1}{2}(x,x(\tau))^2\cdot R\cdot\omega\\ &\quad+R\left(\frac{1}{2}R\cdot\omega+\frac{1}{2}(x,x(\tau))^2\right)\cdot\omega\\ &= \frac{1}{2}R^2\cdot\omega.\end{aligned} \tag{11.39}$$

Combining terms, we obtain:

$$\begin{aligned}-\partial\bar\partial\xi^0(x) &= \left[-\pi(x,x(\tau))^2+\frac{1}{2}\right]e^{-2\pi R}\cdot\omega,\\ &= -\pi\left[(x,x(\tau))^2-\frac{1}{2\pi}\right]e^{-2\pi R}\cdot(-2i)\frac{i}{2}\omega,\\ &= 2\pi i\varphi^0_{KM}(x). \qquad (!!)\end{aligned} \tag{11.40}$$

Thus we have shown that, away from the point $\tau_0=D_x^+$,

$$dd^c\xi^0(x)=-\frac{1}{2\pi i}\partial\bar\partial\xi^0(x)=\varphi^0_{KM}(x). \tag{11.41}$$

Next, observe the behavior of $\xi^0(x,\tau)$ in the neighborhood of the points in D_x.

LEMMA 11.2. *If $x=\pm\sqrt{d}\,x(\tau_0)$, then*

$$\xi^0(x,\tau)=-\log|\tau-\tau_0|^2+O(1)$$

in a neighborhood of τ_0. Also, writing $z=\tau-\tau_0=re^{i\theta}$,

$$\begin{aligned}d\xi^0 &= -\left(\frac{dz}{z}+\frac{d\bar z}{\bar z}\right)+\mathit{smooth}\\ &= -2\frac{dr}{r}+\mathit{smooth},\end{aligned}$$

and

$$\begin{aligned} d^c\xi^0 &= -\frac{1}{4\pi i}\left(\frac{dz}{z}-\frac{d\bar{z}}{\bar{z}}\right) + \mathit{smooth} \\ &= -\frac{1}{2\pi}d\theta + \mathit{smooth}, \end{aligned}$$

in a neighborhood of τ_0.

Proof. Note that

$$\begin{aligned} R(x,\tau) &= 2d\left[\left(\frac{|\tau-\tau_0|^2}{2vv_0}+1\right)^2-1\right] \\ &= 2d\left[\frac{|\tau-\tau_0|^2}{2vv_0}\right]\left[\frac{|\tau-\tau_0|^2}{2vv_0}+2\right]. \end{aligned} \tag{11.42}$$

Thus, by (11.22) and (11.24),

$$\xi^0(x,\tau) = -\log|\tau-\tau_0|^2 + \mathrm{O}(1) \tag{11.43}$$

in a neighborhood of τ_0, as claimed. The $\mathrm{O}(1)$ quantity here is a smooth function of $|\tau-\tau_0|^2$, so the remaining identities follow by differentiation. □

Now we compute the pairing of the current $dd^c[\xi^0]$ with a compactly supported smooth function f on D, which, for the moment, we assume has support in D^+ or D^-:

$$\begin{aligned} \langle f, dd^c[\xi^0]\rangle &:= \int_D \xi^0\cdot dd^c f \\ &= \int_{D-U_\varepsilon}\xi^0\cdot dd^c f + \int_{U_\varepsilon}\xi^0\cdot dd^c f, \end{aligned} \tag{11.44}$$

where U_ε is an ε-neighborhood of τ_0. The second term here goes to zero as ε goes to zero, via the logarithmic growth of $\xi^0(x)$ near τ_0. For the first term,

$$\int_{D-U_\varepsilon}\xi^0\cdot dd^c f = \int_{D-U_\varepsilon} f\cdot dd^c\xi^0 + \int_{\partial\{D-U_\varepsilon\}}\left(\xi^0\, d^c f - f\, d^c\xi^0\right). \tag{11.45}$$

The first term here is

$$\int_{D-U_\varepsilon} f\cdot\varphi^0_{KM} \tag{11.46}$$

and this goes to $\int_D f\cdot\varphi^0_{KM}$ as ε goes to zero. If we write $z=\tau-\tau_0=re^{i\theta}$ in U_ε, then the second term is

$$\begin{aligned} \int_{\partial\{D-U_\varepsilon\}}\left(\xi^0\, d^c f - f\, d^c\xi^0\right) &= -\int_{\partial\{U_\varepsilon\}}\left(\xi^0\, d^c f - f\, d^c\xi^0\right) \\ &= \int_{\partial\{U_\varepsilon\}} f\, d^c\xi^0 + \mathrm{O}(\varepsilon) \\ &= \int_0^{2\pi} f(re^{i\theta})\left(-\frac{1}{2\pi}d\theta\right) + \mathrm{O}(\varepsilon) \\ &\longrightarrow -f(\tau_0), \end{aligned} \tag{11.47}$$

as ε goes to zero. Here note that the term $\int_{\partial\{U_\varepsilon\}} \xi^0\, d^c f$ tends to 0 as $\varepsilon \to 0$, via the logarithmic growth of ξ^0 at τ_0. Thus, with no assumption on the support of f,

$$\langle f, dd^c[\xi^0]\rangle = \int_D f \cdot \varphi^0_{KM} - \langle f, \delta_{D_x}\rangle, \tag{11.48}$$

as claimed. This finishes the proof of Proposition 11.1.

Proposition 11.1 asserts that $\xi^0(x)$ is a Green function of logarithmic type [46] for the cycle D_x in D. Since $\xi^0(x)$ is actually very rapidly decreasing at infinity, i.e., at the boundary of D, we can define an archimedean local 'height pairing' on D as follows. Given $x_1 = \pm\sqrt{d_1}\cdot x(\tau_1)$ and $x_2 = \pm\sqrt{d_2}\cdot x(\tau_2)$ with $\tau_1 \neq \tau_2 \in D^+$, we form the star product [46, p. 50]:

$$[\xi^0(x_1)] * [\xi^0(x_2)] = [\xi^0(x_1)] \wedge \delta_{D_{x_2}} + [\varphi^0_{KM}(x_1)] \wedge [\xi^0(x_2)], \tag{11.49}$$

where, for a differential form η which is either smooth or of logarithmic type, $[\eta]$ denotes the corresponding current, [46]. The star product here is a $(1,1)$–current on D which is, in fact rapidly decreasing. It may then be evaluated on the constant function 1:

Definition 11.3. The *archimedean height pairing* of the 'arithmetic' cycles $(D_{x_1}, \xi^0(x_1))$ and $(D_{x_2}, \xi^0(x_2))$ in D is given by:

$$\begin{aligned}\Big\langle (D_{x_1}, \xi^0(x_1)), (D_{x_2}, \xi^0(x_2))\Big\rangle_\infty \quad &:= \quad \Big\langle 1, [\xi^0(x_1)] * [\xi^0(x_2)]\Big\rangle \\ &= \quad \xi^0(x_1, D_{x_2}) + \int_D \xi^0(x_2)\, \varphi^0_{KM}(x_1).\end{aligned}$$

It is instructive to check that this pairing is symmetric.

LEMMA 11.4. (i) *With the notation above, so that* $\tau_1 \neq \tau_2$,

$$\begin{aligned}&\xi^0(x_1,D_{x_2}) + \int_D \xi^0(x_2)\, \varphi^0_{KM}(x_1)\\ &\qquad = \xi^0(x_1,D_{x_2}) + \xi^0(x_2,D_{x_1}) - \int_D d\xi^0(x_1)\wedge d^c\xi^0(x_2)\\ &\qquad = \xi^0(x_2,D_{x_1}) + \int_D \xi^0(x_1)\, \varphi^0_{KM}(x_2).\end{aligned}$$

(ii) *In particular,*

$$\langle (D_{x_1},\xi^0(x_1)),(D_{x_2},\xi^0(x_2))\rangle_\infty \;=\; \langle (D_{x_2},\xi^0(x_2)),(D_{x_1},\xi^0(x_1))\rangle_\infty.$$

Proof. We consider the expression

$$\int_D d\xi^0(x_1)\wedge d^c\xi^0(x_2) = \int_D d\xi^0(x_2)\wedge d^c\xi^0(x_1). \tag{11.50}$$

Now, taking ε-neighborhoods U_1 of $D_{x_1} = \{\tau_1, \ \bar\tau_1\}$ and U_2 of $D_{x_2} = \{\tau_2, \bar\tau_2\}$,

$$\begin{aligned}
(11.51) \quad \int_D d\xi^0(x_1) \wedge d^c\xi^0(x_2) &= \lim_{\varepsilon_1,\varepsilon_2\to 0} \int_{D-U_1-U_2} d\xi^0(x_1) \wedge d^c\xi^0(x_2) \\
&= \lim_{\varepsilon_1,\varepsilon_2\to 0} \Bigg(-\int_{D-U_1-U_2} \xi^0(x_1)\, dd^c\xi^0(x_2) \\
&\qquad + \int_{\partial\{D-U_1-U_2\}} \xi^0(x_1)\, d^c\xi^0(x_2) \Bigg) \\
&= -\int_D \xi^0(x_1)\, \varphi^0_{KM}(x_2) + \xi^0(x_1, D_{x_2}).
\end{aligned}$$

Here we use the fact that, as ε_1 goes to zero

$$(11.52) \qquad \int_{\partial U_1} \xi^0(x_1)\, d^c\xi^0(x_2) \longrightarrow 0,$$

via the logarithmic growth of $\xi^0(x_1)$ near τ_1, and $\bar\tau_1$ while, as ε_2 goes to zero

$$(11.53) \qquad \int_{\partial U_2} \xi^0(x_1)\, d^c\xi^0(x_2) \longrightarrow -\xi^0(x_1, D_{x_2}),$$

as in the proof of Proposition 11.1, above. Thus

$$\begin{aligned}
(11.54) \quad \int_D d\xi^0(x_1) \wedge d^c\xi^0(x_2) &= -\int_D \xi^0(x_1)\, \varphi^0_{KM}(x_2) + \xi^0(x_1, D_{x_2}) \\
&= -\int_D \xi^0(x_2)\, \varphi^0_{KM}(x_1) + \xi^0(x_2, D_{x_1}),
\end{aligned}$$

so that we obtain (i) and (ii). □

Because of the equivariance $\xi^0(hx, h\tau) = \xi^0(x, \tau)$, for $h \in H(\mathbb{R})$, the pairing

$$(11.55) \qquad Ht(x_1, x_2)_\infty := \Big\langle (D_{x_1}, \xi^0(x_1)), (D_{x_2}, \xi^0(x_2)) \Big\rangle_\infty$$

satisfies

$$(11.56) \qquad Ht(hx_1, hx_2)_\infty = Ht(x_1, x_2)_\infty,$$

and thus depends only on the matrix of inner products

$$(11.57) \qquad T = \frac{1}{2} \begin{pmatrix} (x_1, x_1) & (x_1, x_2) \\ (x_2, x_1) & (x_2, x_2) \end{pmatrix} = \begin{pmatrix} d_1 & m \\ m & d_2 \end{pmatrix}.$$

We sometimes write

$$(11.58) \qquad Ht(T)_\infty = Ht(x_1, x_2)_\infty.$$

This quantity has the following more explicit expression, which will be used in Section 13:

LEMMA 11.5. *For $j = 1, 2$, let*

$$R_j = R(x_j,\tau) = \frac{1}{2}(x_j,x(\tau))^2 - 2d_j,$$

and let

$$M = \frac{1}{2}(x_1,x(\tau))(x_2,x(\tau)) - 2m.$$

Then

$$\begin{aligned} Ht(T)_\infty &= -2Ei\left(-4\pi\frac{|\det T|}{d_1}\right) \\ &\quad - \int_D Ei(-2\pi R_1)\cdot\left(2R_2 + 4d_2 - \frac{1}{2\pi}\right) e^{-2\pi R_2}\cdot\frac{du\,dv}{v^2} \\ &= -2Ei\left(-4\pi\frac{|\det T|}{d_1}\right) - 2Ei\left(-4\pi\frac{|\det T|}{d_2}\right) \\ &\quad - \frac{1}{\pi}\int_D \frac{M(M+2m)}{R_1\cdot R_2}\cdot e^{-2\pi(R_1+R_2)}\cdot\frac{du\,dv}{v^2} \\ &= -2Ei\left(-4\pi\frac{|\det T|}{d_2}\right) \\ &\quad - \int_D Ei(-2\pi R_2)\left(2R_1 + 4d_1 - \frac{1}{2\pi}\right) e^{-2\pi R_1}\cdot\frac{du\,dv}{v^2}. \end{aligned}$$

Proof. We continue to suppose that $x_1 = \pm\sqrt{d_1}\cdot x(\tau_1)$, and $x_2 = \pm\sqrt{d_2}\cdot x(\tau_2)$. Then

$$\begin{aligned} R(x_1,\tau_2) &= \frac{1}{2}(x_1, x(\tau_2))^2 - (x_1,x_1) \\ &= \frac{1}{2}\left(x_1, \pm\frac{1}{\sqrt{d_2}}x_2\right)^2 - (x_1,x_1) \\ &= \frac{(x_1,x_2)^2}{2d_2} - 2d_1 \\ &= \frac{(x_1,x_2)^2 - 4d_1d_2}{2d_2} \\ &= 2\frac{|\det T|}{d_2}. \end{aligned} \tag{11.59}$$

This yields:

$$\xi^0(x_1,\tau_2) = -Ei\left(-4\pi\frac{|\det T|}{d_2}\right). \tag{11.60}$$

The integral is treated similarly. □

We now remark that the function $\xi^0(x)$ and the $(1,1)$-form $\varphi^0_{KM}(x)$ on D are well defined for any $x \in V(\mathbb{R})$ ($x \neq 0$ for ξ^0). Here, of course, if $(x,x) > 0$, the function $\xi^0(x)$ on D has a logarithmic singularity at the points of D_x. On the other hand, if $(x,x) \le 0$ and $x \ne 0$, then $\xi^0(x)$ is smooth on all of D. Inspection of the proofs of Lemmas 11.4 and 11.5 reveals that we may extend the definition of $Ht(x_1,x_2)_\infty$ to all pairs of vectors x_1, x_2 provided $\det T \ne 0$. The resulting function is given by any of the formulas of Lemma 11.5, provided one takes care to omit any terms of the form $-Ei(-4\pi\frac{|\det T|}{d_j})$ when $2d_j = (x_j,x_j) \le 0$. Note that when $(x_j,x_j) \le 0$, there is no corresponding point τ_j, so that the disk U_j can be omitted in the proof of Lemma 11.4. Of course, the $H(\mathbb{R})$ equivariance persists, so that we may still write $Ht(x_1,x_2)_\infty = Ht(T)_\infty$, as before.

For any $T \in \mathrm{Sym}_2(\mathbb{R})$ with $\det T < 0$, we can diagonalize by the action of $\mathrm{SO}(2)$ and write

$$T = {}^t k_{\theta_0} \cdot \begin{pmatrix} \delta_+ & \\ & -\delta_- \end{pmatrix} \cdot k_{\theta_0}, \tag{11.61}$$

where, for any θ,

$$k_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}. \tag{11.62}$$

THEOREM 11.6. *The function $Ht(T)_\infty$ depends only on the eigenvalues δ_+ and δ_-. More precisely, $Ht(T)_\infty$ is $\mathrm{SO}(2)$ invariant:*

$$Ht(T[k_\theta])_\infty = Ht(T)_\infty$$

for all θ. Here

$$T[k_\theta] = {}^t k_\theta \cdot T \cdot k_\theta.$$

We will give the, somewhat long, proof of this essential result in Section 13 below.

Since, for any $a \in \mathrm{GL}_2(\mathbb{R})$ and for any pair of vectors $x = [x_1,x_2] \in V(\mathbb{R})^2$,

$$\frac{1}{2}(xa,xa) = {}^t a\, \frac{1}{2}(x,x)\, a = T[a], \tag{11.63}$$

Theorem 11.6 asserts that the quantity $Ht(x_1,x_2)_\infty = Ht(x)_\infty$ depends only on the $\mathrm{SO}(2)$ orbit of x. If our original $x \in V(\mathbb{R})^2$ has $Q[x_1] = d_1 > 0$ and $Q[x_2] = d_2 > 0$, and if τ_1 and $\tau_2 \in D^+$ are the corresponding points in the upper half plane, then we recall that

$$m = \frac{1}{2}(x_1,x_2) = \pm\sqrt{d_1 d_2}\cdot\frac{1}{2}(x(\tau_1),x(\tau_2)) = \pm\sqrt{d_1 d_2}\cdot\cosh(d(\tau_1,\tau_2)), \tag{11.64}$$

where $d(\tau_1,\tau_2)$ is the hyperbolic distance. The point τ_j depends only on the *line* spanned by x_j, but our Green's function $\xi^0(x_j)$ depends on d_j as well as on τ_j. Thus our height pairing $Ht(x_1,x_2)$ depends on both the hyperbolic distance between the τ_j's and on the lengths d_1 and d_2. As we move x along an SO(2) orbit; i.e., consider

$$[x_1(\theta), x_2(\theta)] := [x_1, x_2] \cdot k_\theta, \tag{11.65}$$

both the hyperbolic distance and the lengths vary. As θ approaches a critical value θ_2 (say), the length of $x_2(\theta)$ goes to zero; i.e., $x_2(\theta)$ approaches the light cone in $V(\mathbb{R})$, and the point τ_2 moves to the corresponding boundary point of D^+. As θ increases past θ_2, the vector $x_2(\theta)$ has negative length, and there is no corresponding point in D. In fact, it is natural to associate to such a negative vector the whole hyperbolic arc of negative 2-planes $z \in D$ such that $x_2(\theta) \in z$! One might view the function $Ht(x \cdot k_\theta)$, for various values of theta, as giving an archimedean height pairing among these various objects. This viewpoint should be compatible with [5]. Note that the hyperbolic arcs play an essential role in [11].

Now suppose that T has been diagonalized, and consider a pair $y \in V(\mathbb{R})^2$ of vectors y_1 and y_2 such that

$$\frac{1}{2}(y, y) = \begin{pmatrix} \delta_+ & \\ & -\delta_- \end{pmatrix}. \tag{11.66}$$

Note that the $H(\mathbb{R})$ orbit of y is unique so that we may as well take

$$y_1 = \sqrt{\delta_+} \cdot x(i) \qquad \text{and} \qquad y_2 = \sqrt{\delta_-} \cdot |\kappa|^{-\frac{1}{2}} \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}. \tag{11.67}$$

We then use the last expression in Lemma 11.5, remembering to omit the first term, to obtain:

LEMMA 11.7.

$$\begin{aligned} Ht(y_1, y_2)_\infty \;\; &= \;\; -\int_D Ei(-4\pi\delta_- v^{-2}|\tau|^2) \\ &\qquad \times \left[\delta_+ v^{-2}(1+|\tau|^2)^2 - \frac{1}{2\pi}\right] e^{-\pi\delta_+\left[v^{-2}(1+|\tau|^2)^2-4\right]} \frac{du\,dv}{v^2}. \end{aligned}$$

Proof. For our particular choice of y, we have

$$(y_1, x(\tau)) = \sqrt{\delta_+} \cdot v^{-1}(1+|\tau|^2), \qquad \text{and} \qquad (y_2, x(\tau)) = \sqrt{\delta_-} \cdot 2 \cdot \frac{u}{v}. \tag{11.68}$$

Thus, in the last expression in Lemma 11.5, we substitute

$$2 \cdot R_1 = \delta_+ \left[v^{-2}(1+|\tau|^2)^2 - 4\right], \tag{11.69}$$

and

$$2 \cdot R_2 = 4 \cdot \delta_- v^{-2}|\tau|^2. \tag{11.70}$$

Comparing this expression with that given in Corollary 9.8, at the end of Section 9, we obtain the following, rather startling, conclusion, one of the main results of our paper. This result relates the derivative at $s = 0$ of Shimura's confluent hypergeometric function to the value of the function Ht_∞ on the matrix T.

THEOREM 11.8. *Suppose that $T \in \mathrm{Sym}_2(\mathbb{R})$ with $\det T \neq 0$ and with* $\mathrm{sig}(T) = (1,1)$. *Then, for $a \in \mathrm{GL}_2(\mathbb{R})^+$, and for $T[a] = {}^t a \cdot T \cdot a$*

$$\frac{1}{2iC_\infty} \cdot \frac{\partial}{\partial s} \left\{ W_T(m(a),s,\Phi^{\frac{3}{2}}) \right\} \Bigg|_{s=0} = \pi^2 \, |a|^{\frac{3}{2}} \cdot e^{-2\pi \,\mathrm{tr}(T[a])} \cdot Ht(T[a])_\infty,$$

where $Ht(\cdot)_\infty$ is the archimedean height function defined above. Here $C_\infty = \sqrt{2}$.

We now want to take into account the action of the metaplectic group, which we write as $G'_{\mathbb{R}} \simeq \mathrm{Sp}(1,\mathbb{R}) \times \mathbb{C}^1$, as in the notation section. For $g' = (n(b)m(a)k_\theta, t) \in G'_{\mathbb{R}}$, with $a > 0$, recall from Proposition 7.3 that

$$W_d^{\frac{3}{2}}(g') = t\, e(bd)\, |a|^{\frac{3}{2}} \, (e^{i\theta})^{\frac{3}{2}} e^{-2\pi a^2 d}, \tag{11.71}$$

where, for $z = re^{i\theta} \in \mathbb{C}$, with $\theta \in (-\pi, \pi]$, we take

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\theta/2}. \tag{11.72}$$

It is easy to check that the map

$$(k_\theta, t) \mapsto (e^{i\theta})^{\frac{3}{2}} t = \chi_{\frac{3}{2}}((k_\theta, t)) \tag{11.73}$$

is a character of the maximal compact subgroup K'_∞ of $G'_{\mathbb{R}}$.

As observed at the outset, it is the function $\varphi_{KM}(x) = e^{-\pi(x,x)} \varphi^0_{KM}(x)$, rather than φ^0_{KM}, which is a Schwartz function on $V(\mathbb{R})$. It is known, [24], and easily checked directly in the present case, that φ_{KM} is an eigenfunction of K'_∞:

$$\omega((k_\theta, t))\varphi_{KM}(x) = \chi_{\frac{3}{2}}((k_\theta, t))\varphi_{KM}(x), \tag{11.74}$$

and hence,

$$\varphi_{KM}(g', x) := \left(\omega(g')\varphi_{KM}\right)(x) = W_d^{\frac{3}{2}}(g') \cdot \varphi^0_{KM}(ax), \tag{11.75}$$

with $d = \frac{1}{2}(x,x)$. We set

$$\xi(g', x) := W_d^{\frac{3}{2}}(g') \cdot \xi^0(ax). \tag{11.76}$$

In all of these expressions, we have omitted the dependence on $\tau \in D$ from the notation. By Proposition 11.1,

$$dd^c \xi(g', x) + W_d^{\frac{3}{2}}(g') \cdot \delta_{D_x} = [\varphi_{KM}(g', x)] = W_d^{\frac{3}{2}}(g') \cdot [\varphi^0_{KM}(ax)], \tag{11.77}$$

so that $\xi(g',x)$ is a Green form of logarithmic type for the weighted 0-cycle $W_d^{\frac{3}{2}}(g') \cdot D_x$. We write

$$\hat{z}(g',x) := \Big(W_d^{\frac{3}{2}}(g') \cdot D_x\,,\ \xi(g',x) \Big) \tag{11.78}$$

for this weighted 0-cycle with Green form associated to the vector $x \in V(\mathbb{R})$ with $d = \frac{1}{2}(x,x) > 0$.

We observe that, for $x \in V(\mathbb{R})$, $x \neq 0$, with $d = \frac{1}{2}(x,x) \leq 0$, the function $\xi(g',x)$ is still well defined, and is, in fact, smooth on all of D. In this case, we set

$$\hat{z}(g',x) = \big(0,\ \xi(g',x)\big)\,. \tag{11.79}$$

We can extend the height pairing to all of the $\hat{z}(g',x)$'s, using the star product and Definition 11.3.

COROLLARY 11.9. *Assume that x_1 and $x_2 \in V(\mathbb{R})$ are that* $\det(T) \neq 0$. *Then, for g_1' and $g_2' \in G_{\mathbb{R}}'$,*

$$\langle \hat{z}(g_1',x_1), \hat{z}(g_2',x_2)\rangle_\infty = W_{d_1}^{\frac{3}{2}}(g_1') W_{d_2}^{\frac{3}{2}}(g_2') \cdot Ht(a_1x_1,a_2x_2)_\infty.$$

Using weighted 0-cycles on D in Theorem 11.8, we obtain the following result, which interprets all of the signature $(1,1)$ factors as archimedean heights:

COROLLARY 11.10. *For all $T \in \mathrm{Sym}_2(\mathbb{R})$ with signature* (1,1), *and for all g_1' and $g_2' \in G_{\mathbb{R}}'$,*

$$\begin{aligned} \frac{1}{2iC_\infty} \cdot \frac{\partial}{\partial s} \Big\{ W_T(\iota(g_1',g_2'),s,\Phi^{\frac{3}{2}}) \Big\}\Big|_{s=0} &= \pi^2\, W_{d_1}^{\frac{3}{2}}(g_1') W_{d_2}^{\frac{3}{2}}(g_2') \cdot Ht(T[a])_\infty \\ &= \pi^2 \,\langle \hat{z}(g_1',x_1), \hat{z}(g_2',x_2)\rangle_\infty\,, \end{aligned}$$

where $a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$, and $g_j' = (n(b_j)m(a_j)k_{\theta_j},t) \in G_{\mathbb{R}}'$, with $a_j > 0$.

12. The archimedean height pairing (globally) In this section we use the results of Section 11 to construct a Green function $\Xi(d,\varphi,K)$ for the weighted cycle $Z(d,\varphi;K)$ of Section 10. Recall that we are assuming that $D(B) > 1$, so that X_K is a projective curve over $\mathbb{Q}$. The pair

$$(Z(d,\varphi;K), \Xi(d,\varphi;K)) \tag{12.1}$$

then defines an element of the arithmetic Chow group $\widehat{CH^1(X_K)}$ in the sense of Gillet and Soulé [8], [46]. Our goal is to 'calculate' the arithmetic intersection pairing

$$\langle (Z(d_1,\varphi_1;K),\Xi(d_1,\varphi_1;K)), (Z(d_2,\varphi_2;K),\Xi(d_2,\varphi_2;K))\rangle \in \mathbb{C} \tag{12.2}$$

of a pair of such cycles. In fact, we will relate the generating function for such height pairings to the derivative at the center of symmetry of our Eisenstein series on the metaplectic group $G_{\mathbb{A}}$. In the present section we will calculate the archimedean part of the height pairing in terms of the function $Ht(T)_\infty$ defined in Section 11.

First we recall some generalities, from Chapter III of [46]. If X is a smooth complex projective variety of dimension n, and if Y and Z are algebraic cycles on X of pure codimensions p and q respectively, then a Green current for Y, say, is a current $g_Y \in D^{(p-1,p-1)}(X)$ such that

$$dd^c g_Y + \delta_Y = [\omega_Y] \tag{12.3}$$

for a smooth (p,p) form ω_Y. The pair (Y, g_Y) then defines an element $[(Y, g_Y)]$ in the arithmetic Chow group $\widehat{CH}^p(X)$. There is a pairing

$$\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \longrightarrow \widehat{CH}^{p+q}(X)_{\mathbb{Q}} \tag{12.4}$$

given by

$$[(Y, g_Y)] \cdot [(Z, g_Z)] = [([Y] \cdot [Z], g_Y * g_Z)], \tag{12.5}$$

where $[Y] \cdot [Z]$ is the cycle product in the usual Chow ring, and $g_Y * g_Z$ is the $*$-product of the Green currents. If g_Y is of logarithmic type for Y, then

$$[g_Y] * g_Z = [g_Y] \wedge \delta_Z + [\omega_Y] \wedge g_Z. \tag{12.6}$$

Here, if Z is nonsingular and $i\colon\ Z \hookrightarrow X$, then

$$[g_Y] \wedge \delta_Z := i_*[i^* g_Y]. \tag{12.7}$$

Finally, if $p + q = n + 1$, we define the archimedean height pairing to be

$$\langle (Y, g_Y), (Z, g_Z) \rangle_\infty := \langle 1_X, g_Y * g_Z \rangle\,, \tag{12.8}$$

where 1_X is the constant function 1 on X, with which the (n,n) current $g_Y * g_Z$ is naturally paired. Assuming that g_Y is of logarithmic type, we get

$$\langle (Y, g_Y), (Z, g_Z) \rangle_\infty = \int_Z i^* g_Y + \langle \omega_Y, g_Z \rangle\,. \tag{12.9}$$

If, in addition, g_Z is of logarithmic type, then

$$\langle (Y, g_Y), (Z, g_Z) \rangle_\infty = \int_Z i^* g_Y + \int_X \omega_Y \wedge g_Z = \int_X g_Y * g_Z. \tag{12.10}$$

We will make use of this last formula.

We begin by taking $X = \Gamma \backslash D^+$ (this will later be a single component of our Shimura curve), and 0-cycles Z_1 and Z_2 of the form

$$Z_i = \mathrm{pr}_\Gamma(\tau_i) \qquad\qquad \tau_i \in D^+. \tag{12.11}$$

Here $\mathrm{pr}_\Gamma\colon\ D^+ \longrightarrow \Gamma\backslash D^+ = X$ is the natural projection, and we take $\Gamma \subset \mathrm{PGL}_2(\mathbb{R})$. We also let

$$x_i = \sqrt{d_i} \cdot x(\tau_i) \in V(\mathbb{R}), \tag{12.12}$$

with $(x_i, x_i) > 0$. We note that, if the stablizer Γ_{x_i} of x_i in Γ has nontrivial image in $\mathrm{PGL}_2(\mathbb{R})$ (or, equivalently, in $O(V)$), then the point $\mathrm{pr}_\Gamma(\tau_i)$ must be counted with multiplicity $e_{x_i}^{-1}$, where e_{x_i} is the order of the image of Γ_{x_i} in $\mathrm{PGL}_2(\mathbb{R})$ (or in $O(V)$). Thus, in general,

$$Z_i = e_{x_i}^{-1} \cdot \mathrm{pr}_\Gamma(\tau_i). \tag{12.13}$$

We set

$$\omega_{Z_i} = e_{x_i}^{-1} \cdot \sum_\gamma \gamma^* \varphi_{KM}(x_i) = e_{x_i}^{-1} \cdot \sum_\gamma \varphi_{KM}(\gamma^{-1} x_i), \tag{12.14}$$

and

$$\Xi_{Z_i} = e_{x_i}^{-1} \cdot \sum_\gamma \gamma^* \xi(x_i) = e_{x_i}^{-1} \cdot \sum_\gamma \xi(\gamma^{-1} x_i). \tag{12.15}$$

These sums converge by the rapid decay properties of φ_{KM} and ξ. The function Ξ_{Z_i} on X is of logarithmic type, i.e., has a logarithmic singularity on Z_i, while ω_{Z_i} is a smooth $(1,1)$-form on X. As currents, these satisfy

$$dd^c[\Xi_{Z_i}] + e^{-\pi(x_i,x_i)}\delta_{Z_i} = [\omega_{Z_i}]. \tag{12.16}$$

LEMMA 12.1. *Assume that $Z_1 \cap Z_2 = \phi$ and write $2d_i = (x_i,x_i)$, $\Xi_i = \Xi_{Z_i}$, and $\omega_i = \omega_{Z_i}$. Then*

$$\langle (Z_1,\Xi_1),(Z_2,\Xi_2)\rangle_\infty = \frac{1}{e_{x_1} \cdot e_{x_2}} \cdot e^{-\pi(d_1+d_2)} \sum_{\gamma\in\Gamma} \frac{1}{2} Ht(\gamma^{-1}x_1,x_2)_\infty,$$

where $Ht(x_1,x_2)_\infty$ is as in the previous section.

Remark 12.2. The factor $\frac{1}{2}$ occurs with Ht_∞ here since we are working on D^+ rather than on D.

Proof. We calculate

$$\begin{aligned} \langle (Z_1, \Xi_1), (Z_2, \Xi_2)\rangle_\infty & \\ &= e^{-2\pi d_2} \int_{Z_2} \Xi_1 + \int_X \omega_1 \cdot \Xi_2 \\ &= \frac{1}{e_{x_1} \cdot e_{x_2}} \\ &\quad \cdot \left(e^{-2\pi d_2} \sum_{\gamma\in\Gamma} \xi(\gamma^{-1}x_1, \tau_2) + \int_{D^+} \sum_{\gamma\in\Gamma} \xi(x_2)\, \varphi_{KM}(\gamma^{-1}x_1) \right) \\ &= \frac{1}{e_{x_1} \cdot e_{x_2}} \cdot e^{-\pi(d_1+d_2)} \sum_{\gamma\in\Gamma} \frac{1}{2} Ht(\gamma^{-1}x_1, x_2)_\infty. \end{aligned} \tag{12.17}$$

$\square$

Even when Z_1 and Z_2 are not disjoint, this formula can be modified to give part of the height pairing.

Definition 12.3. The *nonsingular* part of the height is

$$\begin{aligned}\langle(Z_1,\Xi_1),(Z_2,\Xi_2)\rangle_\infty^{\text{ns.}}\\ := \frac{1}{e_{x_1}\cdot e_{x_2}}\cdot e^{-\pi(d_1+d_2)} \sum_{\substack{\gamma\in\Gamma\\ (\gamma\cdot D_{x_1})\cap D_{x_2}=\phi}} \frac{1}{2}Ht(\gamma^{-1}x_1,x_2)_\infty.\end{aligned}$$

In the case of Lemma 12.1, where d_1 and d_2 are positive, the nonsingular part is contributed by the interaction of distinct components. In this case, the condition that $D_{x_1}\cap D_{x_2}=\phi$, is equivalent to the condition that $\det(T)\neq 0$. (In Lemma 12.1, we apply this condition with γx_1 in place of x_1.) In the case of arbitrary d_1 and d_2, we use this later condition in the definition of the nonsingular part.

We now apply this discussion to our weighted cycles $Z(d,\varphi;K)$ in X_K. First, for $g'\in G'_\mathbb{R}$ and for $d>0$, we use Lemma 10.1 and define

$$(12.18)\qquad \begin{aligned}\omega(g',d,\varphi;K)(\tau,h_j) &:= \sum_{\substack{x\in\Omega_d(\mathbb{Q})\\ \operatorname{mod}\Gamma_j}} e_{j,x}^{-1}\cdot\varphi(h_j^{-1}x)\\ &\qquad\times\sum_{\gamma\in\Gamma_j}\varphi_{KM}(g',\gamma^{-1}x,\tau)\\ &= \sum_{x\in\Omega_d(\mathbb{Q})}\varphi(h_j^{-1}x)\cdot\varphi_{KM}(g',x,\tau),\end{aligned}$$

and

$$(12.19)\qquad \begin{aligned}\Xi(g',d,\varphi;K)(\tau,h_j) &:= \sum_{\substack{x\in\Omega_d(\mathbb{Q})\\ \operatorname{mod}\Gamma_j}} e_{j,x}^{-1}\cdot\varphi(h_j^{-1}x)\\ &\qquad\times\sum_{\gamma\in\Gamma_j}\xi(g',\gamma^{-1}x,\tau)\\ &= \sum_{x\in\Omega_d(\mathbb{Q})}\varphi(h_j^{-1}x)\cdot\xi(g',x,\tau).\end{aligned}$$

Here the second expressions do not involve Γ_j, so we may write simply

$$(12.20)\qquad \omega(g',d,\varphi)(\tau,h)=\sum_{x\in\Omega_d(\mathbb{Q})}\varphi(h^{-1}x)\cdot\varphi_{KM}(g',x,\tau),$$

and

$$(12.21)\qquad \Xi(g',d,\varphi)(\tau,h)=\sum_{x\in\Omega_d(\mathbb{Q})}\varphi(h^{-1}x)\cdot\xi(g',x,\tau).$$

The point here is that the $(1,1)$–form $\omega(g',d,\varphi)$ and the Green current $\Xi(g',d,\varphi)$ are well defined on $H(\mathbb{Q})\backslash(D\times H(\mathbb{A}_f))$ and descend to any quotient $X_K=H(\mathbb{Q})\backslash(D\times H(\mathbb{A}_f))/K$, provided the weight function $\varphi\in S(V(\mathbb{A}_f)^2)$ is K-invariant.

If $d \le 0$, we again define $\omega(g', d, \varphi)$ and $\Xi(g', d, \varphi)$ by (12.20) and (12.21), where, if $d = 0$, we sum over $x \in \Omega_d(\mathbb{Q}) - \{0\}$.

Recall that the weighted cycles have the following property. If $K' \subset K$, let pr: $X_{K'} \to X_K$ be the natural projection. Then ([23]),

$$\mathrm{pr}^*(Z(d, \varphi; K)) = Z(d, \varphi; K'). \tag{12.22}$$

Thus, for fixed d and φ, the cycles $Z(d, \varphi; K)$ define an element

$$Z(d, \varphi) \in CH^1(X)_{\mathbb{C}} := \varinjlim_{K} CH^1(X_K)_{\mathbb{C}}. \tag{12.23}$$

This is the nice advantage of the weighted cycles. Expression (12.21) implies that we may define elements of

$$\widehat{CH}^1(X)_{\mathbb{C}} := \varinjlim_{K} \widehat{CH}^1(X_K)_{\mathbb{C}} \tag{12.24}$$

as follows:

Definition 12.4. For $g' \in G'_{\mathbb{R}}$, $d \in \mathbb{Q}$, and $\varphi \in S(V(\mathbb{A}_f))$, define the element $\hat{Z}(g', d, \varphi) \in \widehat{CH}^1(X) \otimes \mathbb{C}$ by

$$\hat{Z}(g', d, \varphi) := \begin{cases} \Big(W_d^{\frac{3}{2}}(g')\, Z(d, \varphi),\ \Xi(g', d, \varphi) \Big) & \text{if } d > 0 \\ (0,\ \Xi(g', d, \varphi)\,) & \text{if } d \le 0. \end{cases}$$

For a fixed K, let

$$\hat{Z}(g', d, \varphi, K) := \begin{cases} \Big(W_d^{\frac{3}{2}}(g')\, Z(d, \varphi, K),\ \Xi(g', d, \varphi) \Big) & \text{if } d > 0 \\ (0,\ \Xi(g', d, \varphi)\,) & \text{if } d \le 0. \end{cases}$$

PROPOSITION 12.5. *For g'_1 and $g'_2 \in G'_{\mathbb{R}}$, the archimedean height pairing of the 'arithmetic' weighted cycles $\hat{Z}(g'_1,d_1,\varphi_1,K)$ and $\hat{Z}(g'_2,d_2,\varphi_2,K)$ on X_K is given by*

$$\begin{aligned} &\mathrm{vol}(K) \Big\langle \hat{Z}(g'_1,d_1,\varphi_1,K),\ \hat{Z}(g'_2,d_2,\varphi_2,K) \Big\rangle_{\infty}^{\mathrm{ns.}} \\ &\quad = \frac{1}{2} W_{d_1}^{\frac{3}{2}}(g'_1) W_{d_2}^{\frac{3}{2}}(g'_2) \\ &\quad \times \sum_{\substack{T \in \mathrm{Sym}_2(\mathbb{Q}) \\ T = \begin{pmatrix} d_1 & m \\ m & d_2 \end{pmatrix},\, \det T < 0}} Ht(T[a])_{\infty} \cdot \mathrm{Rep}(T,\varphi_1 \otimes \varphi_2,V) \end{aligned}$$

where $a = \begin{pmatrix} a_1 & \\ & a_2 \end{pmatrix}$, the function $Ht(x_1,x_2)_{\infty}$ is as in the previous section, and

$$\mathrm{Rep}(T,\varphi_1 \otimes \varphi_2,V) = \mathrm{vol}(K) \cdot \sum_j \sum_{\substack{x \in V(\mathbb{Q})^2 \\ \frac{1}{2}(x,x) = T \\ \mathrm{mod}\, \Gamma_j}} (\varphi_1 \otimes \varphi_2)\,(h_j^{-1} x).$$

Note that the stabilizer $\Gamma_{j,x}$ of x in Γ_j, for x occurring in this last sum, is trivial.

If d_1d_2 is not a square, the weighted cycles $Z(d_1,\varphi_1,K)$ and $Z(d_1,\varphi_1,K)$ are disjoint, and the superscript "ns." can be omitted. Moreover, in this case, $\det T \neq 0$ for all T in the sum. If a solution to $\frac{1}{2}(x,x) = T$ exists in $V(\mathbb{Q})^2$, then $\mathrm{sig}(T) = (1,1)$, so that only T's of this signature contribute to the sum.

Proof. First assume that d_1d_2 is not a square, so that our weighted cycles are disjoint. Then we have:

$$
\begin{aligned}
(12.25)\ \Big\langle \hat{Z}(g_1',d_1,\varphi_1),\ \hat{Z}(g_2',d_2,\varphi_2)\Big\rangle_\infty
&= \int_X \Xi_1 * \Xi_2 \\
&= \int_{H(\mathbb{Q})\backslash D\times H(\mathbb{A}_f)/K} \Xi_1 * \Xi_2 \\
&= \int_{H(\mathbb{Q})^+\backslash D^+\times H(\mathbb{A}_f)/K} \Xi_1 * \Xi_2 \\
&= \sum_j \int_{\Gamma_j\backslash D^+} \sum_{x_1\in\Omega_{d_1}(\mathbb{Q})}\ \sum_{x_2\in\Omega_{d_2}(\mathbb{Q})} \varphi_1(h_j^{-1}x_1)\,\varphi_2(h_j^{-1}x_2) \\
&\quad \times \xi(g_1',x_1) * \xi(g_2',x_2) \\
&= \sum_j \sum_{\substack{x\in(\Omega_{d_1}(\mathbb{Q})\times\Omega_{d_2}(\mathbb{Q}))\\ \mathrm{mod}\,\Gamma_j}} \varphi(h_j^{-1}x)\cdot \int_{D^+} \xi(g_1',x_1) * \xi(g_2',x_2) \\
&= \sum_{T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}} \sum_j \sum_{\substack{x\in\Omega_T(\mathbb{Q})\\ \mathrm{mod}\,\Gamma_j}} \varphi(h_j^{-1}x)\cdot\frac{1}{2}\left\langle \hat{z}(g_1',x_1),\hat{z}(g_2',x_2)\right\rangle_\infty .
\end{aligned}
$$

This is precisely the claimed expression, via Corollary 11.10. Here we are again using the fact that, when $\det(T)\neq 0$, the stabilizer in Γ_j of $x\in\Omega_T(\mathbb{Q})$ is trivial.

In the case when d_1d_2 is a square, the nonsingular part (Definition 12.3) of this expression can be obtained by the same computation, but with the sum on x_1 and x_2 in the third line replaced by the sum over pairs x_1, x_2 for which $\det(T)\neq 0$. □

Note that the quantity

$$
\begin{aligned}
(12.26)\qquad \Big\langle \hat{Z}(g_1',d_1,\varphi_1),\ \hat{Z}(g_2',d_2,\varphi_2)\Big\rangle_\infty^{\mathrm{ns.}}&\\
:= vol(K)\Big\langle \hat{Z}(g_1',d_1,\varphi_1,K),\ \hat{Z}(g_2',d_2,\varphi_2,K)\Big\rangle_\infty^{\mathrm{ns.}}&
\end{aligned}
$$

occuring in Proposition 12.5 is independent of the choice of the compact open subgroup K, and so defines a pairing on $\widehat{CH}^1(X)_{\mathbb{C}}$. Also note that the cases where $d_i\leq 0$ are included.

Finally, we compare this archimedean height pairing to the archimedean part of a Fourier coefficient of our Eisenstein series. For our pair of weight functions φ_1 and $\varphi_2 \in S(V(\mathbb{A}_f))$, let $\varphi = \varphi_1 \otimes \varphi_2$, and let $\Phi(s)$ be the associated incoherent standard section, as in (7.12). Attached to $\Phi(s)$, we have the incoherent Eisenstein series $E(g, s, \Phi)$ on the metaplectic group of genus 2, and the restriction $F(g_1', g_2', \Phi)$ of its derivative at $s = 0$ to $G_{\mathbb{A}}' \times G_{\mathbb{A}}'$. For d_1, and $d_2 \in \mathbb{Q}$ we have the Fourier coefficient $F_{d_1,d_2}(g_1', g_2', \Phi)$ of $F(g_1', g_2', \Phi)$, as in (7.15). Finally, let $F_{d_1,d_2}(g_1', g_2', \Phi)_\infty$ be the archimedean part of this coefficient, as described in (ii) of Proposition 7.3, i.e., the part coming from the terms with $\det(T) \neq 0$ and $\mathrm{Diff}(\mathcal{C}, T) = \{\infty\}$. Note that, if $d_1 d_2$ is not a square, every T contributing to $F_{d_1,d_2}(g_1', g_2', \Phi)$ has $\det T \neq 0$, i.e., $F_{d_1,d_2}(g_1', g_2', \Phi)_{\mathrm{sing}} = 0$.

Combining Proposition 12.5, Corollary 11.10, and (ii) of Proposition 7.3, we obtain the main result of this paper.

THEOREM 12.6. *For g_1' and $g_2' \in G_{\mathbb{R}}'$, and for any d_1 and $d_2 \in \mathbb{Q}$,*

$$F_{d_1,d_2}(g_1', g_2', \Phi)_\infty = 2\pi^2 \cdot \left\langle \hat{Z}(g_1', d_1, \varphi_1), \hat{Z}(g_2', d_2, \varphi_2) \right\rangle_\infty^{\mathrm{ns.}},$$

where the pairing on the right-hand side is the nonsingular archimedean part of the height pairing (12.26) *on $\widehat{\mathrm{CH}}^1(X)_{\mathbb{C}}$. If $d_1 d_2$ is not a square in $\mathbb{Q}$, then the superscript "ns." can be dropped.*

The reader who prefers to work at a finite level, on X_K, say, can modify the right-hand side of this result, via (12.26).

Remark 12.7. Recall that the superscript "ns.", which denotes the part of the height coming from pairs x_1, x_2 with $\det T \neq 0$, as in Definition 12.3. Of course, if $d_1 d_2$ is not a square, then only such pairs occur.

Proof. By (ii) of Proposition 7.3, Corollary 11.10, and Proposition 12.5, we have

$$\begin{aligned} &F_{d_1,d_2}(g_1', g_2', \Phi)_\infty \\ &\quad = \frac{1}{2iC_\infty} \sum_T W_{T,\infty}'(\iota(g_1', g_2'), 0, \Phi_\infty^{\frac{3}{2}}) \cdot \mathrm{Rep}(T, \varphi, V) \\ &\quad = \pi^2 W_{d_1}^{\frac{3}{2}}(g_1')\, W_{d_2}^{\frac{3}{2}}(g_2') \sum_T Ht(T[a])_\infty \cdot \mathrm{Rep}(T, \varphi, V) \\ &\quad = 2\pi^2 \cdot \left\langle \hat{Z}(g_1', d_1, \varphi_1), \hat{Z}(g_2', d_2, \varphi_2) \right\rangle_\infty^{\mathrm{ns.}}, \end{aligned}$$

as claimed. □

13. *An invariance property of the archimedean height.* In this section we prove the rather surprising invariance property

$$Ht(T[k_\theta])_\infty = Ht(T)_\infty, \tag{13.1}$$

of Theorem 11.6. Although the proof amounts to an elaborate calculus exercise, it sheds some light on the nature of the archimedean height function $Ht(T)_\infty$.

We consider the function:

$$\text{(13.2)}\quad \begin{aligned}\frac{1}{2}Ht(T)_\infty &= -Ei\left(-4\pi\frac{|\det T|}{d_1}\right) - Ei\left(-4\pi\frac{|\det T|}{d_2}\right)\\ &\quad -\frac{1}{\pi}\int_{D^+}\frac{M(M+2m)}{R_1\cdot R_2}\cdot e^{-2\pi(R_1+R_2)}\cdot\frac{du\,dv}{v^2},\end{aligned}$$

in the notation of Lemma 11.5. Here the term $-Ei(-4\pi\frac{|\det T|}{d_i})$ is to be omitted if $d_i \le 0$. We must show that this quantity depends only on the eigenvalues of the matrix T, or equivalently, that it is invariant under the action of SO(2).

We begin by choosing convenient coordinates. By scaling all vectors suitably, we may as well assume that $\kappa = -1$. Write

$$\text{(13.3)}\qquad T = {}^t k_\theta \begin{pmatrix}\delta_+ & \\ & -\delta_-\end{pmatrix} k_\theta = \begin{pmatrix} d_1 & m \\ m & d_2\end{pmatrix},$$

with

$$\text{(13.4)}\qquad k_\theta = \begin{pmatrix} c & s \\ -s & c\end{pmatrix}$$

and $c = \cos(\theta)$, $s = \sin(\theta)$. Here δ_+ and δ_- are both positive. We note that

$$\text{(13.5)}\qquad \begin{aligned} d_1 &= c^2\delta_+ - s^2\delta_- \\ d_2 &= s^2\delta_+ - c^2\delta_- \\ m &= cs(\delta_+ + \delta_-),\end{aligned}$$

and that

$$\text{(13.6)}\qquad \frac{d}{d\theta}d_1 = -2m, \qquad \frac{d}{d\theta}d_2 = 2m, \qquad \text{and} \qquad \frac{d}{d\theta}m = d_1 - d_2.$$

Next, we let

$$\text{(13.7)}\qquad \begin{aligned} x_0 &= \sqrt{\delta_+}\cdot\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},\\ y_0 &= \sqrt{\delta_-}\cdot\begin{pmatrix} & 1 \\ 1 & \end{pmatrix},\end{aligned}$$

and consider the pair of vectors

$$\text{(13.8)}\qquad \begin{aligned} x = x_\theta &= c\cdot x_0 - s\cdot y_0 = \begin{pmatrix} & x_2 \\ x_3 & \end{pmatrix}\\ y = y_\theta &= s\cdot x_0 + c\cdot y_0 = \begin{pmatrix} & y_2 \\ y_3 & \end{pmatrix}\end{aligned}$$

which move with θ. Then

$$T = \frac{1}{2}\begin{pmatrix} (x,x) & (x,y) \\ (y,x) & (y,y) \end{pmatrix}, \tag{13.9}$$

and

$$d_1 = -x_2x_3, \qquad d_2 = -y_2y_3, \qquad 2m = -x_2y_3 - x_3y_2. \tag{13.10}$$

Note that

$$(x, x(\tau)) = -v^{-1}(x_3|\tau|^2 - x_2), \tag{13.11}$$

$$(y, x(\tau)) = -v^{-1}(y_3|\tau|^2 - y_2),$$

and that

$$(x, x(\tau))|_{\theta=0} = \sqrt{\delta_+}\cdot\frac{(|\tau|^2+1)}{v}, \tag{13.12}$$

and

$$(y, x(\tau))|_{\theta=0} = -\sqrt{\delta_-}\cdot\frac{(|\tau|^2-1)}{v}.$$

Observe that, if $d_1 > 0$, then

$$x = \sqrt{d_1}\cdot x(\tau_1) \tag{13.13}$$

where

$$\tau_1 = i\,v_1, \qquad v_1 = \frac{x_2}{\sqrt{d_1}} = \frac{c\sqrt{\delta_+} - s\sqrt{\delta_-}}{\sqrt{d_1}}, \tag{13.14}$$

and similarly, when $d_2 > 0$,

$$\tau_2 = i\,v_2, \qquad v_2 = \frac{y_2}{\sqrt{d_2}} = \frac{s\sqrt{\delta_+} + c\sqrt{\delta_-}}{\sqrt{d_2}}. \tag{13.15}$$

Finally, we set

$$\begin{aligned} R &= R(\theta,\tau) = \begin{pmatrix} R_1 & M \\ M & R_2 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} (x,x(\tau))^2 & (x,x(\tau))(y,x(\tau)) \\ (x,x(\tau))(y,x(\tau)) & (y,x(\tau))^2 \end{pmatrix} - 2T. \end{aligned} \tag{13.16}$$

Note that

$$R(\theta,\tau) = {}^t k_\theta \cdot R(0,\tau)\cdot k_\theta \tag{13.17}$$

and that

$$R_1(0,\tau) = \frac{1}{2}\delta_+\left(\frac{(|\tau|^2+1)^2}{v^2} - 4\right), \tag{13.18}$$

$$R_2(0,\tau) = \frac{1}{2}\delta_-\left(\frac{(|\tau|^2-1)^2}{v^2} + 4\right),$$

and

$$M(0,\tau) = \frac{1}{2}\sqrt{\delta_+\delta_-}\left(\frac{|\tau|^4-1}{v^2}\right). \tag{13.19}$$

Thus, the quantities $\mathrm{tr}(R) = R_1 + R_2$ and $\det R = R_1R_2 - M^2$ are independent of θ.

LEMMA 13.1.

$$R_1 + R_2 = -2(\delta_+ - \delta_-) + \delta_+ \cdot \frac{(|\tau|^2+1)^2}{2v^2} + \delta_- \cdot \frac{(|\tau|^2-1)^2}{2v^2},$$

and

$$\det R = 4\,|\det T\,|\,\frac{u^2}{v^2}.$$

Also, if $d_1 > 0$, *then* $R_1 = 0$ *if and only if* $\tau = \tau_1 = \tau_1(\theta)$, *and then*

$$R(\theta,\tau_1) = \begin{pmatrix} 0 & 0 \\ 0 & R_2(\theta,\tau_1) \end{pmatrix}.$$

We consider the family of mappings

$$\alpha_\theta\colon\ D^+ \longrightarrow \mathrm{Sym}_2(\mathbb{R}) \qquad \tau \mapsto R(\theta,\tau) \tag{13.20}$$

and

$$\alpha\colon\ \mathbb{R}\times D^+ \longrightarrow \mathrm{Sym}_2(\mathbb{R}) \qquad \{\theta,\tau\} \mapsto R(\theta,\tau), \tag{13.21}$$

which can be viewed as giving a homotopy between the various α_θ's.

We define the following 2-form on $\mathrm{Sym}_2(\mathbb{R})$:

$$\Omega = \frac{1}{4\pi}\,e^{-2\pi(R_1+R_2)}\frac{M}{R_1\cdot R_2}\cdot\frac{dR_1\wedge dR_2}{\sqrt{\det(R)}}. \tag{13.22}$$

Note that Ω is singular where $R_1\cdot R_2\cdot\det(R) = 0$.

PROPOSITION 13.2. (i) *For a fixed value of* θ,

$$\alpha_\theta^*(\Omega) = -\frac{1}{\pi}\,e^{-2\pi(R_1+R_2)}\cdot\frac{M(M+2m)}{R_1\cdot R_2}\cdot\frac{du\wedge dv}{v^2}.$$

(ii) *On* $\mathrm{Sym}_2(\mathbb{R})$,

$$d\Omega = \frac{1}{4\pi}\, e^{-2\pi(R_1+R_2)} \cdot \frac{dR_1 \wedge dR_2 \wedge dM}{\det(R)^{\frac{3}{2}}}.$$

(iii) *On* $\mathbb{R} \times D^+$,

$$\alpha^*(d\Omega) = \frac{1}{2\pi} e^{-2\pi(R_1+R_2)} \cdot \frac{\delta_+ + \delta_-}{\sqrt{|\det(T)|}} \cdot \frac{v^2}{u^2} \cdot \left(\frac{|\tau|^4 - 1}{v^2}\right) \cdot \frac{du \wedge dv \wedge d\theta}{v^2}.$$

Note that there is no dependence on θ *here.*

Proof. First observe that

$$(13.23) \qquad \frac{d}{d\theta} R_1 = -2M, \qquad \frac{d}{d\theta} R_2 = 2M, \qquad \text{and} \qquad \frac{d}{d\theta} M = R_1 - R_2.$$

Thus, on $\mathbb{R} \times D$,

$$(13.24) \qquad \begin{aligned} dR_1 &= (x, x(\tau))(x, dx(\tau)) - 4M\, d\theta \\ dR_2 &= (y, x(\tau))(y, dx(\tau)) + 4M\, d\theta \\ dM &= \frac{1}{2}(x, x(\tau))(y, dx(\tau)) \\ &\quad + \frac{1}{2}(y, x(\tau))(x, dx(\tau)) - 2(R_1 - R_2)\, d\theta. \end{aligned}$$

and so, on a level surface of θ,

$$(13.25) \qquad dR_1 \wedge dR_2 = (x, x(\tau))\,(y, x(\tau))\,(x, dx(\tau)) \wedge (y, dx(\tau)).$$

For fixed θ:

$$(13.26) \quad (x, dx(\tau)) = d(x, x(\tau)) = -v^{-1}(x, x(\tau))\, dv - v^{-1} x_3(2u\, du + 2v\, dv),$$

and so, after a short calculation:

$$(13.27) \quad \begin{aligned} (x, dx(\tau)) \wedge (y, dx(\tau)) &= -2(x_2 y_3 - x_3 y_2) \cdot \frac{u}{v} \cdot \frac{du \wedge dv}{v^2} \\ &= -4\sqrt{\delta_+ \delta_-} \cdot \frac{u}{v} \cdot \frac{du \wedge dv}{v^2}. \end{aligned}$$

Thus, on a level surface of θ,

$$(13.28) \qquad \begin{aligned} dR_1 \wedge dR_2 &= -8(M + 2m) \cdot \sqrt{\delta_+ \delta_-} \cdot \frac{u}{v} \cdot \frac{du \wedge dv}{v^2} \\ &= -4(M + 2m) \cdot \sqrt{\det(R)} \cdot \frac{du \wedge dv}{v^2}. \end{aligned}$$

This yields (i), while (ii) is immediate. □

LEMMA 13.3.

$$\alpha^*(dR_1 \wedge dR_2 \wedge dM) = 8\sqrt{|\det(T)|}\,(\delta_+ + \delta_-) \left(\frac{|\tau|^4 - 1}{v^2}\right) \sqrt{\det(R)} \cdot \frac{du \wedge dv \wedge d\theta}{v^2}.$$

Proof. A direct calculation. □

For convenience, we write $Ht(\theta) = Ht(T)_\infty$, and, from (i) of Proposition 13.2, we obtain

$$\frac{1}{2}Ht(\theta) = -Ei(-2\pi R_2(\theta,\tau_1)) - Ei(-2\pi R_1(\theta,\tau_2)) + \int_{D^+} \alpha^*_\theta(\Omega). \tag{13.29}$$

Note that, as functions of τ, the quantities R_1, R_2 and M are all invariant under $u \mapsto -u$. Thus

$$\int_{D^+} \alpha^*_\theta(\Omega) = 2\int_Q \alpha^*_\theta(\Omega), \tag{13.30}$$

where $Q = \{\tau \mid u > 0,\ v > 0\}$.

The singularities of $\alpha^*_\theta(\Omega)$ lie on the axis $u = 0$, and hence on the boundary of Q. We want to apply Stokes' Theorem, away from these singularities. To do this, for $\varepsilon > 0$ (small), let

$$Q_\varepsilon = \left\{\tau = re^{i\mu} \mid r > 0,\ 0 < \mu \le \frac{\pi}{2} - \varepsilon\right\}, \tag{13.31}$$

and, for any fixed θ_0, consider the 'slab' $S_\varepsilon = [\theta_0, \theta] \times Q_\varepsilon$. We then have

$$\int_{S_\varepsilon} \alpha^*(d\Omega) = \int_{Q_\varepsilon} \alpha^*_\theta(\Omega) - \int_{Q_\varepsilon} \alpha^*_{\theta_0}(\Omega) + \int_{B_\varepsilon} \alpha^*(\Omega), \tag{13.32}$$

where

$$B_\varepsilon = [\theta_0, \theta] \times \{\tau = re^{\frac{\pi}{2}-\varepsilon}\}. \tag{13.33}$$

Here we have used the fact that the factor $e^{-2\pi(R_1+R_2)}$ decays very rapidly as τ goes to the boundary of D^+.

LEMMA 13.4.
$$\int_{S_\varepsilon} \alpha^*(d\Omega) = 0.$$

Proof. Consider the mapping $\tau \mapsto \bar{\tau}^{-1}$. This preserves the region Q_ε. Since the quantity $(|\tau|^4 - 1)/v^2$ is odd for this involution, while the rest of the integrand is invariant,

$$\int_{Q_\varepsilon} e^{-2\pi(R_1+R_2)} \cdot \frac{v^2}{u^2} \cdot \left(\frac{|\tau|^4 - 1}{v^2}\right) \frac{du\,dv}{v^2} = 0. \tag{13.34}$$

This proves the lemma. □

By (13.32) and Lemma 13.4,

$$\int_Q \alpha^*_\theta(\Omega) - \int_Q \alpha^*_{\theta_0}(\Omega) = -\lim_{\varepsilon\to 0} \int_{B_\varepsilon} \alpha^*(\Omega). \tag{13.35}$$

It turns out that the term on the right side of this expression can be computed explicitly! First, we need to consider the restriction of the 2-form $\alpha^*(\Omega)$ to the 'wall' B_ε.

LEMMA 13.5. *On B_ε, $u = r\sin(\varepsilon)$, $v = r\cos(\varepsilon)$,*

$$\det(R) = 4|\det(T)|\,\tan^2(\varepsilon),$$

$$dR_1 \wedge dR_2 = \frac{2}{\cos^2(\varepsilon)} \cdot (\delta_+ + \delta_-) \cdot M\,(r^2 - r^{-2}) \cdot \frac{dr}{r} \wedge d\theta,$$

and hence

$$\begin{aligned}\alpha^*(\Omega) \quad &= \quad \frac{1}{4\pi}\frac{\delta_+ + \delta_-}{\sqrt{|\det(T)|}} \cdot \frac{1}{\cos(\varepsilon)\sin(\varepsilon)} \cdot e^{-2\pi(R_1+R_2)} \\ &\quad \times \left(1 - \frac{\det(R)}{R_1R_2}\right)(r^2 - r^{-2}) \cdot \frac{dr}{r} \wedge d\theta.\end{aligned}$$

Proof. Since

$$dR_1 = \frac{\partial R_1}{\partial r} \cdot dr - 2M \cdot d\theta \tag{13.36}$$

and

$$dR_2 = \frac{\partial R_2}{\partial r} \cdot dr + 2M \cdot d\theta,$$

we have

$$dR_1 \wedge dR_2 = 2M \cdot \frac{\partial}{\partial r}\{R_1 + R_2\} \cdot dr \wedge d\theta. \tag{13.37}$$

Since, on B_ε,

$$R_1+R_2 = -2(\delta_+-\delta_-)+\frac{1}{2\cos^2(\varepsilon)}\delta_+(r+r^{-1})^2+\frac{1}{2\cos^2(\varepsilon)}\delta_-(r-r^{-1})^2, \tag{13.38}$$

we have

$$dR_1 \wedge dR_2 = \frac{2}{\cos^2(\varepsilon)} \cdot (\delta_+ + \delta_-) \cdot M(r^2 - r^{-2}) \cdot \frac{dr}{r} \wedge d\theta, \tag{13.39}$$

as claimed. □

We now consider the integral:

$$\begin{aligned}-\int_{B_\varepsilon} \alpha^*(\Omega) \quad &= \quad \frac{1}{4\pi}\frac{\delta_+ + \delta_-}{\sqrt{|\det(T)|}} \cdot \frac{1}{\cos(\varepsilon)\sin(\varepsilon)} \\ &\quad \times \int_{\theta_0}^{\theta}\int_0^\infty e^{-2\pi(R_1+R_2)}\left(1 - \frac{\det(R)}{R_1R_2}\right)(r^2 - r^{-2})\,\frac{dr}{r}\,d\theta.\end{aligned} \tag{13.40}$$

Note that

$$\int_0^\infty e^{-2\pi(R_1+R_2)}(r^2 - r^{-2})\,\frac{dr}{r} = 0, \tag{13.41}$$

since the quantity $R_1 + R_2$ is invariant under $r \mapsto r^{-1}$. Thus, recalling Lemma 13.1, we have simply:

$$(13.42) \quad -\int_{B_\varepsilon} \alpha^*(\Omega) = -\frac{1}{\pi}(\delta_+ + \delta_-)\sqrt{|\det(T)|} \cdot \frac{\sin(\varepsilon)}{\cos^3(\varepsilon)} \times \int_{\theta_0}^{\theta}\int_0^\infty e^{-2\pi(R_1+R_2)} \frac{1}{R_1R_2}(r^2 - r^{-2})\,\frac{dr}{r}\,d\theta.$$

We need to consider the limit of this expression as $\varepsilon \to 0$. It is not difficult to check that this limit can be interchanged with the integral with respect to θ. Hence, for fixed θ, we consider the quantity

$$(13.43) \quad \lim_{\varepsilon\to 0}\left(\sin(\varepsilon) \cdot \int_0^\infty e^{-2\pi(R_1+R_2)} \frac{1}{R_1R_2}(r^2 - r^{-2})\,\frac{dr}{r}\right).$$

It will turn out that the only contributions to this limit arise from the singularities at the points τ_1, i.e., where $R_1 = 0$ in case $d_1 > 0$, and τ_2, i.e., where $R_2 = 0$ in case $d_2 > 0$.

If $d_j = d_j(\theta) > 0$, let $\tau_j = iv_j$ be as above, so that $x = \sqrt{d_1} \cdot x(\tau_1)$, (resp. $y = \sqrt{d_2} \cdot x(\tau_2)$), and

$$(13.44) \quad \begin{aligned} R_j &= \frac{1}{2}d_j\left((x(\tau_j), x(\tau))^2 - 4\right) \\ &= \frac{1}{2}d_j\left(\frac{|\tau-\tau_j|^2}{vv_j}\right)\cdot\left(\frac{|\tau-\tau_j|^2}{vv_j}+4\right) \\ &= (r^2 - 2rv_j\cos(\varepsilon) + v_j^2)\cdot\frac{d_j}{2vv_j}\cdot\left(\frac{|\tau-\tau_j|^2}{vv_j}+4\right) \\ &= \left((r - v_j\cos(\varepsilon))^2 + v_j^2\sin^2(\varepsilon)\right)\cdot\frac{d_j}{2vv_j}\cdot\left(\frac{|\tau-\tau_j|^2}{vv_j}+4\right). \end{aligned}$$

We break up the integral in (13.43) into integrals along the pieces:

$$(13.45) \quad I_1 = \begin{cases} (v_1\cos(\varepsilon) - \nu_1, v_1 - \cos(\varepsilon) + \nu_1) & \text{if } d_1 > 0 \\ \phi & \text{if } d_1 \le 0, \end{cases}$$

$$(13.46) \quad I_2 = \begin{cases} (v_2\cos(\varepsilon) - \nu_2, v_2 - \cos(\varepsilon) + \nu_2) & \text{if } d_2 > 0 \\ \phi & \text{if } d_2 \le 0, \end{cases}$$

and $C = (0,\infty) - I_1 - I_2$, the complement.

PROPOSITION 13.6. *Let* $\nu_1 = \nu_2 = \sqrt{\varepsilon}$. *Then:*

$$\lim_{\varepsilon\to 0}\left(\sin(\varepsilon)\cdot\int_C e^{-2\pi(R_1+R_2)}\frac{1}{R_1R_2}(r^2 - r^{-2})\,\frac{dr}{r}\right) = 0.$$

If $d_1 > 0$, *then*

$$\lim_{\varepsilon\to 0}\left(\sin(\varepsilon)\cdot\int_{I_1} e^{-2\pi(R_1+R_2)}\frac{1}{R_1R_2}(r^2-r^{-2})\,\frac{dr}{r}\right) = -2\pi\cdot\frac{e^{-2\pi R_2(\tau_1)}}{R_2(\tau_1)}\cdot\frac{\sqrt{|\det(T)|}}{d_1^2}\cdot cs.$$

If $d_2 > 0$, *then*

$$\lim_{\varepsilon\to 0}\left(\sin(\varepsilon)\cdot\int_{I_2} e^{-2\pi(R_1+R_2)}\frac{1}{R_1R_2}(r^2-r^{-2})\,\frac{dr}{r}\right) = 2\pi\cdot\frac{e^{-2\pi R_1(\tau_2)}}{R_1(\tau_2)}\cdot\frac{\sqrt{|\det(T)|}}{d_2^2}\cdot cs.$$

Here $cs = \cos(\theta)\sin(\theta)$.

Note that, in the definition of I_1 and I_2, any monotone decreasing function $\nu_1 = \nu_2 = \nu(\varepsilon)$ of ε for which $\lim_{\varepsilon\to 0}\nu(\varepsilon) = 0$ but $\lim_{\varepsilon\to 0}\frac{\nu(\varepsilon)}{\sin(\varepsilon)} = \infty$ would do in place of $\sqrt{\varepsilon}$.

Proof. First consider the integral over I_1, which we write as

$$\int_{I_1} F_1(r)\cdot\Big((r-v_1\cos(\varepsilon))^2 + v_1^2\sin^2(\varepsilon)\Big)^{-1}\,dr \tag{13.47}$$

where the function $F_1(r)$ also depends on ε and is continuous in regions under consideration. This integral is equal to

$$\begin{aligned}\int_{-\nu_1}^{\nu_1} F_1(w+v_1\cos(\varepsilon))\cdot(w^2+v_1^2\sin^2(\varepsilon))^{-1}\,dw \\ = \frac{1}{v_1\sin(\varepsilon)}\cdot\int_{-\frac{\nu_1}{v_1\sin(\varepsilon)}}^{\frac{\nu_1}{v_1\sin(\varepsilon)}} F_1(v_1\sin(\varepsilon)w+v_1\cos(\varepsilon))\cdot\frac{dw}{1+w^2}.\end{aligned}\tag{13.48}$$

We multiply this expression by $\sin(\varepsilon)$ and take the limit as ε goes to 0. For any fixed ε, the argument of the function F_1 runs over the interval $I_1 = I_1(\varepsilon)$, which contracts to the point v_1 as ε goes to 0. Thus, in the limit, we obtain

$$v_1^{-1}F_1(v_1)\cdot\int_{-\infty}^{\infty}\frac{dw}{1+w^2} = \pi\cdot v_1^{-1}F_1(v_1). \tag{13.49}$$

Finally, consider the contribution of C. If $[a,b]$ is any interval with constant endpoints contained in C, we have

$$\lim_{\varepsilon\to 0}\left(\int_{[a,b]} e^{-2\pi(R_1+R_2)}\frac{1}{R_1R_2}(r^2-r^{-2})\,\frac{dr}{r}\right) < \infty, \tag{13.50}$$

so that such intervals make no contribution after we multiply by $\sin(\varepsilon)$. Choosing a suitable (fixed) reference point $a > v_1$, we consider, for example,

$$(13.51)\quad \sin(\varepsilon)\cdot\int_{\nu_1+v_1\cos(\varepsilon)}^{a} F_1(r)\cdot\Big((r-v_1\cos(\varepsilon))^2+v_1^2\sin^2(\varepsilon)\Big)^{-1}dr$$
$$=\frac{1}{v_1}\cdot\int_{\frac{\nu_1}{v_1\sin(\varepsilon)}}^{\frac{a-v_1\cos(\varepsilon)}{v_1\sin(\varepsilon)}} F_1\left(v_1\sin(\varepsilon)w+v_1\cos(\varepsilon)\right)\cdot\frac{dw}{w^2+1},$$

as before. Now the argument of F_1 runs over the interval (ν_1, a) and therefore remains bounded, uniformly in ε. Thus, the whole expression is bounded by a fixed constant, independent of ε, times

$$(13.52)\qquad \int_{\frac{\nu_1}{v_1\sin(\varepsilon)}}^{\frac{a-v_1\cos(\varepsilon)}{v_1\sin(\varepsilon)}}\frac{dw}{w^2}=\frac{v_1\sin(\varepsilon)}{\nu_1}-\frac{v_1\sin(\varepsilon)}{a-v_1\cos(\varepsilon)},$$

and this expression goes to 0 as ε goes to 0, by our choice of $\nu_1(\varepsilon)$. The other 'moving endpoints' of C are handled in the same way. □

Returning to our main calculation and substituting the results of Proposition 13.6, we now have:

$$(13.53)\quad -\lim_{\varepsilon\to0}\left(\int_{B_\varepsilon}\alpha^*(\Omega)\right)=\int_{\theta_0}^{\theta}\Bigg(2\cdot\frac{e^{-2\pi R_2(\tau_1)}}{R_2(\tau_1)}\cdot\frac{|\det(T)|}{d_1^2}\cdot cs(\delta_++\delta_-)$$
$$-2\cdot\frac{e^{-2\pi R_1(\tau_2)}}{R_1(\tau_2)}\cdot\frac{|\det(T)|}{d_2^2}\cdot cs(\delta_++\delta_-)\Bigg)\,d\theta.$$

Here, as always, the first (resp. second) term occurs only when $d_1 > 0$ (resp. $d_2 > 0$). Note that $m = cs(\delta_+ + \delta_-)$. Since

$$(13.54)\qquad R_2(\theta,\tau_1)=2\frac{|\det(T)|}{d_1},$$

and

$$(13.55)\qquad R_1(\theta,\tau_2)=2\frac{|\det(T)|}{d_2},$$

we have

$$(13.56)\qquad \frac{d}{d\theta}\{R_2(\theta,\tau_1)\}=4|\det(T)|\cdot\frac{m}{d_1^2}$$

and, similarly,

$$(13.57)\qquad \frac{d}{d\theta}\{R_1(\theta,\tau_2)\}=-4|\det(T)|\cdot\frac{m}{d_2^2}.$$

Thus the right side of (13.53) becomes

$$(13.58)\qquad \frac{1}{2}\int_{R_2(\theta_0,\tau_1)}^{R_2(\theta,\tau_1)}\frac{e^{-2\pi z}}{z}\,dz+\frac{1}{2}\int_{R_1(\theta_0,\tau_2)}^{R_1(\theta,\tau_2)}\frac{e^{-2\pi z}}{z}\,dz$$
$$=\frac{1}{2}Ei(-2\pi R_2(\theta,\tau_1))+\frac{1}{2}Ei(-2\pi R_1(\theta,\tau_2))$$
$$-\frac{1}{2}Ei(-2\pi R_2(\theta_0,\tau_1))-\frac{1}{2}Ei(-2\pi R_1(\theta_0,\tau_2)).$$

COROLLARY 13.7. *For any θ and θ_0,*

$$-Ei(-2\pi R_2(\theta,\tau_1)) - Ei(-2\pi R_1(\theta,\tau_2)) + \int_{D^+} \alpha_\theta^*(\Omega)$$
$$= -Ei(-2\pi R_2(\theta_0,\tau_1)) - Ei(-2\pi R_1(\theta_0,\tau_2)) + \int_{D^+} \alpha_{\theta_0}^*(\Omega).$$

This is the (long sought!) independence of θ.

14. *Special cycles via moduli.* In this section, which is the outcome of discussions with M. Rapoport, we turn to the contribution of the finite primes to the height pairing of our cycles. First, we review the modular interpretation of the Shimura curves X_K. In particular, following Zink [51], we will fix a prime p and define a moduli problem which is 'p-integral', and whose solution provides a model of our Shimura curve over $\mathbb{Z}_{(p)}$. We also give a 'p-integral', modular definition of the special cycles. We then sketch a computation of the intersection pairing of two such cycles under the assumption that they do not meet in the generic fiber and that $p \nmid \kappa D(B)$. This computation proceeds in two stages. First, we use the well known description of the geometric points of the special fiber [37], [38] and determine the points of intersection of our cycles. This method allows us to work with an arbitrary level away from p. Then the local multiplicity of a point of intersection was determined by Keating [18], based on his earlier work with Gross [10], by using the deformation theory of p-divisible groups and their endomorphisms.

We retain the notation of Section 10, and we let $\mathcal{O}_B$ be a maximal order in B. Also recall that $\chi(x) = (x,\kappa)_{\mathbb{A}}$ where $\kappa < 0$ is a square-free integer.

Following Zink [51], we consider the following moduli problem. Define a functor from schemes over $\mathbb{Q}$ to sets which assigns to a scheme T over $\mathbb{Q}$ the set of isomorphism classes of triples $(A, \iota, \bar{\eta})$, where:

(i) A is an abelian scheme over T, up to isogeny,

(ii) ι: $\mathcal{O}_B \hookrightarrow \mathrm{End}^0(A) := \mathrm{End}(A) \otimes \mathbb{Q}$ is an imbedding, and

(iii) $\bar{\eta}$: $\hat{V}(A) \xrightarrow{\sim} B(\mathbb{A}_f)$ is an equivalence class of $\mathcal{O}_B$ equivariant isomorphisms, for the action of $\mathcal{O}_B$ on $B(\mathbb{A}_f)$ by left multiplication, taken modulo the right action of K (i.e., a K level structure).

Here, recall that

$$\hat{V}(A) = \left(\prod_\ell T_\ell(A)\right) \otimes \mathbb{Q} \tag{14.1}$$

where $T_\ell(A)$ is the Tate module of A.

This moduli problem is represented by a quasi-projective scheme X_K over $\mathbb{Q}$ which is smooth if K is sufficiently small and is projective if $D(B) \neq 1$. The Riemann surface $X_K(\mathbb{C})$ of Section 10 can be identified as the set of complex points of the Shimura curve X_K, hence the notation.

For $d \in \mathbb{Q}^{\times}_{>0}$, let $\mathrm{k}_{\kappa d} = \mathbb{Q}(\sqrt{\kappa d})$, and let

$$\Omega_d = \left\{ x \in V \mid -\kappa\nu(x) = \frac{1}{2}(x,x) = d \right\}, \tag{14.2}$$

as in Section 10. For a K-orbit $\underline{x} \subset \Omega_d(\mathbb{A}_f)$, consider the moduli problem which assigns to a scheme T over $\mathbb{Q}$ the set of isomorphism classes of collections $(A, \iota, \bar{\eta}, y)$, where $(A, \iota, \bar{\eta})$ is as before and where

(iv) $y \in \mathrm{End}^0(A, \iota)$ with $y^2 = d/\kappa$. Moreover, for $\eta \in \bar{\eta}$, the O_B endomorphism of $B(\mathbb{A}_f)$ corresponding, under η, to the endomorphism induced by y on $\hat{V}(A)$ is given by right multiplication by an element $y_\eta \in \underline{x} \subset \Omega_d(\mathbb{A}_f)$.

Note that the condition $y_\eta \in \underline{x}$ of (iv) is independent of the choice of $\eta \in \bar{\eta}$.

These additional conditions give a modular definition of $Z(d, \underline{x}; K)$, a 0-cycle in X_K. For a function $\varphi \in S(V(\mathbb{A}_f))^K$, write

$$\mathrm{supp}(\varphi) \cap \Omega_d(\mathbb{A}_f) = \coprod_r \underline{x}_r, \tag{14.3}$$

where the $\underline{x}_r$'s are K-orbits. Then set

$$Z(d, \varphi; K) = \sum_r \varphi(\underline{x}_r)\, Z(d, \underline{x}_r; K). \tag{14.4}$$

It is easily checked that the set of complex points $Z(d, \varphi; K)(\mathbb{C}) \subset X_K(\mathbb{C})$ is precisely the 0-cycle defined in Section 10.

Fix a prime p, and suppose that the compact open subgroup K in $H(\mathbb{A}_f)$ has the form $K = K_p K^p$, where K_p is a maximal compact subgroup of $H(\mathbb{Q}_p)$ and $K^p \subset H(\mathbb{A}_f^p)$ is arbitrary. In fact, to eliminate various finite stabilizers, we assume that K^p is *neat*. This means that the image of $K^p \subset H(\mathbb{A}_f^p)$ in $H(\mathbb{A}_f^p)/Z(\mathbb{A}_f^p)$ is torsion free. When $p \nmid D(B)$, assume that $O_B \otimes \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p)$, and that $K_p \simeq \mathrm{GL}_2(\mathbb{Z}_p)$, under our fixed identification $B_p \simeq M_2(\mathbb{Q}_p)$.

As in [51, p. 17], we consider the following moduli problem. Define a functor from schemes over $\mathbb{Z}_{(p)}$ to sets which to a scheme T over $\mathbb{Z}_{(p)}$ assigns the set of isomorphism classes of triples $(A, \iota, \bar{\eta})$, where:

(i) A is an abelian scheme over T, up to prime to p isogeny,

(ii) ι: $O_B \hookrightarrow \mathrm{End}(A) \otimes \mathbb{Z}_{(p)}$ is an imbedding satisfying the 'special' condition [51, p. 17],

(iii) $\bar{\eta}^p$: $\hat{V}^p(A) \xrightarrow{\sim} B(\mathbb{A}_f^p)$ is an equivalence class of O_B equivariant isomorphisms, taken modulo the right action of K^p (i.e., a K^p level structure).

Here

$$\hat{V}^p(A) = \left(\prod_{\ell \neq p} T_\ell(A)\right) \otimes \mathbb{Q} \tag{14.5}$$

where $T_\ell(A)$ is the Tate module of A.

For sufficiently small K^p, this functor is representable by a scheme $\mathcal{X} = \mathcal{X}_{K^p} = \mathcal{X}_K$ which is quasi-projective over $\mathbb{Z}_{(p)}$, [51, p. 28]. It is not difficult to check, say following [2], [52], or [37], that

$$\mathcal{X}_K \times_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq X_K, \tag{14.6}$$

so that $\mathcal{X}_K$ gives a model of X_K over $\mathbb{Z}_{(p)}$. If $p \nmid D(B)$ and if K^p is sufficiently small, this model is smooth over $\mathbb{Z}_{(p)}$. For any p, the model $\mathcal{X}_K$ also has the following important property:

PROPOSITION 14.1. (Serre-Tate property, [3], [6], [51]). *Suppose that K^p is sufficiently small. For any geometric point $x \in \mathcal{X}_K(\bar{\mathbb{F}}_p)$, let (A,ι) be the corresponding abelian variety over $\bar{\mathbb{F}}_p$ with O_B action. Let $\hat{O}_x$ be the strict completion of the local ring at x of the strucure sheaf of $\mathcal{X}$. Then* $\mathrm{Spf}(\hat{O}_x)$ *is the universal deformation space of the p-divisible group $(A(p),\iota)$ together with with its O_B action (formal O_B module).*

We next want to define p-integral versions of the special cycles. We assume that K^p is sufficiently small. For a K^p orbit $\underline{x}^p \subset \Omega_d(\mathbb{A}_f^p)$, consider the set valued functor on schemes over $\mathbb{Z}_{(p)}$ which assigns to T the set of isomorphism classes of collections $(A, \iota, \bar{\eta}, y)$, where the first three elements are as before and where $y \in \mathrm{End}(A, \iota) \otimes \mathbb{Z}_{(p)}$ is an endomorphism such that:

(iv) $y^2 = d/\kappa$ and under any isomorphism $\eta^p \in \bar{\eta}^p$, y corresponds to an element of $\underline{x}^p$.

Note that the condition that y lie in $\underline{x}^p$ does not depend on the choice of $\eta^p \in \bar{\eta}^p$. The extra structure given by the endomorphism y cuts out a subscheme $\mathcal{Z}(d, \underline{x}^p; K^p)$ in the moduli scheme $\mathcal{X}_K$. Finally, if $\varphi^p \in S(V(\mathbb{A}_f^p))$ is a K^p-invariant function, we set

$$\mathcal{Z}(d, \varphi^p; K^p) = \sum_{\underline{x}_r^p} \varphi^p(\underline{x}_r^p) \cdot \mathcal{Z}(d, \underline{x}_r^p, K^p), \tag{14.7}$$

where $\underline{x}_r^p$ runs over the K^p orbits in $\mathrm{supp}(\varphi^p) \cap \Omega_d(\mathbb{A}_f^p)$.

The following result is not difficult to check.

LEMMA 14.2. *Suppose that $\varphi = \varphi_p \cdot \varphi^p \in S(V(\mathbb{A}_f))^K$, with φ_p equal to the characteristic function of the set $V(\mathbb{Z}_p)$. Then*

$$\mathcal{Z}(d,\varphi^p;K^p) \times_{\mathbb{Z}_{(p)}} \mathbb{Q} \simeq Z(d,\varphi;K),$$

as schemes over $\mathbb{Q}$. *In particular, the divisor* $\mathcal{Z}(d,\varphi^p;K^p)$ *on the curve* $\mathcal{X}_K$ *over* $\mathbb{Z}_{(p)}$ *extends the* 0*-cycle* $Z(d,\varphi;K)$ *on the generic fiber* X_K.

Note that the cycle $Z(d,\varphi;K)$ can have points in charararacteristic 0 only when $\Omega_d(\mathbb{Q})$ is nonempty. In turn, $\Omega_d(\mathbb{Q})$ is nonempty if and only if $k_{\kappa d} = \mathbb{Q}(\sqrt{\kappa d})$ splits B. Similarly, for $\mathcal{Z}(d,\varphi^p;K^p)$ to have points in characteristic p, we must have $\Omega_d(\mathbb{A}_f^p)$ nonempty. This is because the endomorphism y on $\hat{V}^p(A) \simeq B(\mathbb{A}_f^p)$ must be given by a right multiplication by an element of $B(\mathbb{A}_f^p)$. Suppose that $\Omega_d(\mathbb{A}_f^p)$ is nonempty. Then, since B is indefinite, the only obstruction to $k_{\kappa d}$ splitting B occurs when B_p is division, i.e., $p \mid D(B)$, and p splits in $k_{\kappa d}$. In that case the cycle $\mathcal{Z}(d,\varphi^p;K^p)$ is supported in the fiber at p.

Remark 14.3. When $p \mid D(B)$ and p splits in $k_{\kappa d}$, the cycles $\mathcal{Z}(d,\varphi^p;K^p)$, supported in the fiber at p, should be the analogues of the $\hat{Z}(g',d,\varphi)$'s of Definition 12.3 in the case $d<0$, i.e., in the case when the prime ∞ splits in $k_{\kappa d}$!

We now assume that K^p is sufficiently small and consider the intersection pairing of these divisors in $\mathcal{X}_K$, which is a regular scheme over $\mathbb{Z}_{(p)}$. Choose data d_1, φ_1^p and d_2, φ_2^p, and suppose that $\Omega_{d_1}(\mathbb{Q})$ and $\Omega_{d_2}(\mathbb{Q})$ are nonempty. Assume, for the moment, that $d_1 d_2 \notin \mathbb{Q}^{\times,2}$. Then, $\mathcal{Z}(d_1,\varphi_1^p;K^p)$ and $\mathcal{Z}(d_2,\varphi_2^p;K^p)$ intersect only in the fiber at p and their intersection is proper.

We consider the height pairing

$$\langle \mathcal{Z}(d_1,\varphi_1^p;K^p), \mathcal{Z}(d_2,\varphi_2^p;K^p)\rangle_p . \tag{14.8}$$

As in [10] and [18], this height pairing is a sum over the points $x \in \mathcal{X}_K(\bar{\mathbb{F}}_p)$. At each point, we sum over the pairs of branches $(\mathcal{Z}_1,\mathcal{Z}_2)$ of the cycles through x. The contribution at x of a pair of branches is then

$$\mathrm{vol}(K)\cdot \log |\mathcal{O}_x/(f_1,f_2)| , \tag{14.9}$$

where f_1 and f_2 are the local defining equations of the divisors $\mathcal{Z}_1$ and $\mathcal{Z}_2$, and $\mathcal{O}_x$ is the local ring of $\mathcal{X}_K$ at x. Here, as in the archimedean case, the factor $\mathrm{vol}(K)$ is introduced to make the pairing independent of $K = K_p K^p$ (sufficiently small).

From now on, we assume that $p \nmid D(B)$. As just noted, the condition $p \nmid D(B)$ implies that $\Omega_d(\mathbb{Q})$ is nonempty, whenever $\Omega_d(\mathbb{A}_f^p)$ is nonempty. Thus all components of our cycles are horizontal.

In this case, the isogeny classes of (A,ι)'s corresponding to points in $\mathcal{X}_K(\bar{\mathbb{F}}_p)$ are indexed by imaginary quadratic fields $E/\mathbb{Q}$ in which p *splits*, together with the supersingular isogeny class. If (A,ι) is not supersingular, then $\mathrm{End}^0(A,\iota) \simeq E$, while, in the supersingular case, $\mathrm{End}^0(A,\iota) \simeq B'$, where B'

is the quaternion algebra over $\mathbb{Q}$ whose invariants are the same as those of B, except at p and ∞. This algebra was denoted by $B^{(p)}$ in Section 7, cf. (7.6). If the cycle $\mathcal{Z}(d, \varphi^p; K^p)$ passes through a point in some isogeny class, then the endomorphism y gives an embedding $\mathrm{k}_{\kappa d} \hookrightarrow \mathrm{End}^0(A, \iota)$. Hence, in the nonsupersingular case, $E \simeq \mathrm{k}_{\kappa d}$, so that p must split in $\mathrm{k}_{\kappa d}$. In the supersingular case, $\mathrm{k}_{\kappa d} \hookrightarrow B'$, so that p cannot split in $\mathrm{k}_{\kappa d}$. Thus, for a fixed d, our cycles can have points in only one isogeny class in the special fiber $\mathcal{X}_K(\bar{\mathbb{F}}_p)$. In particular, if $d_1 d_2$ is not a square in $\mathbb{Q}^\times$, then the corresponding cycles cannot meet in the special fiber if p splits in one of the fields $\mathrm{k}_{\kappa d_1}$ or $\mathrm{k}_{\kappa d_2}$.

Thus, we assume that $\mathrm{k}_{\kappa d_1}$ and $\mathrm{k}_{\kappa d_2}$ both split B', and we must consider the intersections of our cycles in the supersingular set $\mathcal{X}_K(\bar{\mathbb{F}}_p)^{ss.}$. Recall that,

$$\mathcal{X}_K(\bar{\mathbb{F}}_p)^{ss.} \simeq H'(\mathbb{Q}) \backslash \left(X_p \times H(\mathbb{A}_f^p)/K^p \right) \simeq H'(\mathbb{Q}) \backslash H'(\mathbb{A}_f)/K', \tag{14.10}$$

where $H' = (B')^\times$ and $K' = K^p K'_p$. This is described in [38, p. 77]. The set X_p is a certain set of lattices in the (contravariant, rational) Dieudonné module $D'A$, for any A in the isogeny class. In our simple case, the group $H'(\mathbb{Q}_p) \simeq (B'_p)^\times$ acts transitively on X_p, and the base point Λ_p^0 can be taken so that its stabilizer is $(\mathcal{O}'_p)^\times := K'_p$. (In [38], our $H'(\mathbb{Q}_p)$ is denoted by $\bar{G}(\mathbb{Q}_p)$.) Note that the set of endomorphisms α of the Dieudonné module such that $\alpha \Lambda_p^0 \subset \Lambda_p^0$ is just $\mathcal{O}'_p$.

Suppose that $(A, \iota, \bar{\eta}^p)$ corresponds to a point $x \in \mathcal{X}_K(\bar{\mathbb{F}}_p)^{ss.}$ and hence to a double coset $H'(\mathbb{Q}) h K'$. Then $y \in \mathrm{End}^0(A, \iota) \simeq B'$ with $y^2 = d/\kappa$ can be identified with an element $y \in \Omega'_d(\mathbb{Q}) \subset V'(\mathbb{Q}) \subset B'$, since the condition on y^2 implies that $tr(y) = 0$. The collection $(A, \iota, \bar{\eta}^p, y)$ then defines a point of $\mathcal{Z}(d, \underline{x}; K^p)(\bar{\mathbb{F}}_p)$ precisely when

(a) $y \cdot \Lambda_p \subset \Lambda_p$, i.e.,

$$h_p^{-1} \cdot y = h_p^{-1} y h_p \in V'(\mathbb{Z}_p),$$

and

(b)

$$h_0^{-1} \cdot y = h_0^{-1} y h_0 \in \underline{x},$$

where $h = h_0 h_p$, with $h_0 \in H(\mathbb{A}_f^p)$.

Here V' is the space of trace zero elements in B'.

Thus, we obtain:

LEMMA 14.4. *Let $\varphi'_p \in S(V'(\mathbb{Q}_p))$ be the characteristic function of the set $V'(\mathbb{Z}_p)$, and let $\varphi' = \varphi'_p \varphi^p$. Then, for any $y \in \Omega'_d(\mathbb{Q})$, the collection $(A,\iota,\bar{\eta}^p,y)$ defines a point of $\mathcal{Z}(d,\varphi^p;K^p)(\bar{\mathbb{F}}_p)$ of weight $\varphi'(h^{-1}y)$, where $(A,\iota,\bar{\eta}^p)$ corresponds to the double coset $H'(\mathbb{Q})hK'$ under the bijection* (14.10).

Here it should be recalled that we are assuming that K^p is *neat*. Since B' is definite, for any $h \in H'(\mathbb{A}_f)$, $H'(\mathbb{Q}) \cap hK'h^{-1}$ has finite image in $H'(\mathbb{A}_f^p)/Z(\mathbb{A}_f^p)$. Thus, for K^p neat, the groups $H'(\mathbb{Q}) \cap hK'h^{-1}$ lie in the center $Z(\mathbb{Q})$ and hence act trivially on $V'(\mathbb{Q}) \supset \Omega'_d(\mathbb{Q})$.

Returning to our pair of cycles, and writing

$$H'(\mathbb{A}_f) = \coprod_j H'(\mathbb{Q}) h_j K', \tag{14.11}$$

we find the following.

PROPOSITION 14.5. *Assume that $d_1 d_2 \notin \mathbb{Q}^2$. Then*

$$\langle \mathcal{Z}(d_1,\varphi_1^p;K^p),\mathcal{Z}(d_2,\varphi_2^p;K^p)\rangle_p = \mathrm{vol}(K) \cdot \sum_j \sum_{y_1 \in \Omega'_{d_1}(\mathbb{Q})} \sum_{y_2 \in \Omega'_{d_2}(\mathbb{Q})} \varphi'_1(h_j^{-1}y_1)\varphi'_2(h_j^{-1}y_2)\, Ht(y_1,y_2)_{p,j},$$

where $Ht(y_1,y_2)_{p,j}$ is the intersection multiplicity, at the point of $\mathcal{X}_K(\bar{\mathbb{F}}_p)^{\mathrm{ss.}}$ corresponding to h_j, of the branches of the cycles determined by y_1 and y_2.

Now we want to invoke the Serre-Tate property to compute $Ht(y_1, y_2)_{p,j}$. Here we follow the discussion of [18] and [10].

Let $(A, \iota, \bar{\eta}^p)$ correspond to h_j. Since $p \nmid D(B)$, we have an action of $\mathcal{O}_B \otimes \mathbb{Z}_p \simeq M_2(\mathbb{Z}_p)$ on the p-divisible (formal) group $A(p)$ of A, so that this formal group is a product $A(p) \simeq A_0(p) \times A_0(p)$, where $A_0(p)$ is a p-divisible formal group of dimension 1 and height 2 over $\bar{\mathbb{F}}_p$. Note that $\mathrm{End}(A_0(p)) \simeq \mathcal{O}'_p$. The deformations of $(A(p), \iota)$ are the same as those of $A_0(p)$. Thus, the universal deformation space of $(A(p), \iota)$ is

$$\hat{\mathcal{U}} = \mathrm{Spf}\, W_p[[u]], \tag{14.12}$$

where $W_p = W(\bar{\mathbb{F}}_p)$ is the ring of Witt vectors of $\bar{\mathbb{F}}_p$. By the Serre-Tate property,

$$\hat{\mathcal{O}}_x \simeq W_p[[u]], \tag{14.13}$$

where $\mathcal{O}_x$ is the local ring of $\mathcal{X}_K$ at x and $\hat{\mathcal{O}}_x$ is its strict localization.

Following Keating, [18, p. 15], for any subset $S \subset \mathrm{End}(A_0(p))$, let $\hat{\mathcal{U}}(S)$ be the largest formal subscheme of $\hat{\mathcal{U}}$ along which the endomorphisms in S deform. Then $\hat{\mathcal{U}}(S) = \mathrm{Spf}\, R(S)$ for some quotient $R(S)$ of $W_p[[u]]$. If the algebra $R(S)$ has finite length as a W_p module, we write

$$\delta_p(S) := \mathrm{length}_{W_p}(R(S)). \tag{14.14}$$

The following fundamental result is proved in [10] and [18]:

PROPOSITION 14.6. (Keating) *Let y_1 and $y_2 \in \mathrm{End}(A_0(p))$ be endomorphisms of $A_0(p)$, viewed as elements of $\mathcal{O}'_p \subset B'_p$. Let*

$$T = \frac{1}{2}\begin{pmatrix} (y_1,y_1) & (y_1,y_2) \\ (y_2,y_1) & (y_2,y_2) \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q}_p)$$

be the matrix of inner products of y_1 and y_2 with respect to the norm form $(x,y) = -\kappa\,\mathrm{tr}(xy^{\iota})$ on B'_p. If $p \neq 2$, $T \in \kappa\,\mathrm{Sym}_2(\mathbb{Z}_p)$ and

$$\kappa^{-1\,t}\alpha T\alpha = \kappa^{-1}T[\alpha] = \begin{pmatrix} \varepsilon_1 p^a & \\ & \varepsilon_2 p^b \end{pmatrix},$$

for some $\alpha \in \mathrm{GL}_2(\mathbb{Z}_p)$, and with integers a and b satisfying $0 \le a \le b$. If $p = 2$, integer invariants a and b, with $0 \le a \le b$, are defined as in [10]. *Assume that* $\det(T) \neq 0$. *Then $R(y_1,y_2)$ has finite length as a W_p module, and that length is given by*

$$\delta_p(y_1,y_2) = \begin{cases} \sum_{j=0}^{\frac{a-1}{2}} (a+b-4i)\ p^j & \text{if } a \text{ is odd}, \\ \sum_{j=0}^{\frac{a}{2}-1} (a+b-4j)\, p^j + \frac{1}{2}(b-a+1)p^{\frac{a}{2}} & \text{if } a \text{ is even}. \end{cases}$$

In particular, $\delta_p(y_1,y_2) = \delta_p(T)$ depends only on the $\mathrm{GL}_2(\mathbb{Z}_p)$ equivalence class of the matrix T.

Remark 14.7. (i) The $\mathrm{GL}_2(\mathbb{Z}_p)$ invariance of $\delta_p(T)$ might be thought of as the analogue of the θ invariance of $Ht(T)_\infty$ (Theorem 11.6) in the archimedean case.

(ii) As in [18], Proposition 14.6 can be obtained by specializing Proposition 5.4 of [10] to the case $G = A_0(p)$, $\hat{f}_1 = id$, $\hat{f}_2 = y_1$ and $\hat{f}_3 = y_2$. Then, the formula for $\alpha_p(Q)$ in [10, Prop. 5.4], with $a_1 = 0$, $a_2 = a$ and $a_3 = b$ reduces to that for $\delta_p(y_1, y_2)$ in Proposition 14.6.

By the Serre-Tate property, this result yields:

COROLLARY 14.8. *Suppose that $y_1 \in \Omega'_{d_1}(\mathbb{Q})$ and $y_2 \in \Omega'_{d_2}(\mathbb{Q})$, are such that the matrix of inner products*

$$T = \frac{1}{2}\begin{pmatrix} (y_1,y_1) & (y_1,y_2) \\ (y_2,y_1) & (y_2,y_2) \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Q}_p)$$

has $\det(T) \neq 0$. *Assume, moreover, that $\varphi'_1(h_j^{-1}y_1)$ and $\varphi'_2(h_j^{-1}y_2)$ are nonzero for some j. Then, for all j,*

$$Ht(y_1,y_2)_{p,j} = Ht(y_1,y_2)_p = Ht(T)_p = \delta_p(T)\cdot\log(p).$$

Combining this result with Proposition 14.5, we obtain:

THEOREM 14.9. *Assume that d_1d_2 is not a square. Then*

$$\begin{aligned}\langle \mathcal{Z}(d_1,\varphi_1^p;K^p),\mathcal{Z}(d_2,\varphi_2^p;K^p)\rangle_p &= \sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Q}_p)\\ T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}}} \delta_p(T)\cdot\log(p)\cdot\mathrm{vol}(K)\cdot\sum_j \sum_{\substack{y\in V^{(p)}(\mathbb{Q})^2\\ \frac{1}{2}(y,y)=T}} \varphi^{(p)}(h_j^{-1}y)\\ &= \frac{\mathrm{vol}(K)}{\mathrm{vol}(K^{(p)})}\sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Q}_p)\\ T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}}} Ht(T)_p\cdot \mathrm{Rep}(T,\varphi^{(p)},V^{(p)}),\end{aligned}$$

where $Ht(T)_p = \log(p)\cdot\delta_p(T)$ is as in the Proposition 14.6 *above. Here*

$$\varphi^{(p)} = \varphi_1'\otimes\varphi_2' = (\varphi_p'\otimes\varphi_1^p)\otimes(\varphi_p'\otimes\varphi_2^p)\in S(V^{(p)}(\mathbb{A}_f)^2),$$

$\varphi_p'\in S(V^{(p)}(\mathbb{Q}_p))$ as in Lemma 14.4, *$K^{(p)} = K_p'K^p$, and* $\mathrm{Rep}(T,\varphi^{(p)},V^{(p)})$ *is as in* (7.30).

Note that our assumption that K^p is neat implies that the groups Γ_j of (7.30) are all trivial and $|\Gamma_{j,x}| = 1$. Thus (7.30) simplifies in our present case.

Similarly, with no assumption about d_1d_2, we have the quantity:

$$(14.15)\quad \begin{aligned}&\langle \mathcal{Z}(d_1,\varphi_1^p;K^p),\mathcal{Z}(d_2,\varphi_2^p;K^p)\rangle_p^{\mathrm{ns.}}\\ &\quad := \frac{\mathrm{vol}(K)}{\mathrm{vol}(K^{(p)})}\sum_{\substack{T\in \mathrm{Sym}_2(\mathbb{Q}_p)\\ T=\begin{pmatrix} d_1 & m\\ m & d_2\end{pmatrix}\\ \det(T)\neq 0}} Ht(T)_p\cdot \mathrm{Rep}(T,\varphi^{(p)},V^{(p)}),\end{aligned}$$

which is the 'transverse part' of the intersection pairing. It coincides with the previous expression in the case when d_1d_2 is not a square.

The following fact is easily checked, and can also be found in [49, p. 49] .

LEMMA 14.10.

$$\frac{\mathrm{vol}(K)}{\mathrm{vol}(K^{(p)})} = \frac{\mathrm{vol}(K_p)}{\mathrm{vol}(K_p')} = (p-1).$$

Comparing these results to those of Proposition 7.3 (i) and Corollary 8.7, we obtain:

THEOREM 14.11. *Assume that $p\nmid \kappa D(B)$ and that K and φ_1 and φ_2 are as above. Then, for d_1 and $d_2\in\mathbb{Q}_{>0}^\times$, and for g_1' and $g_2'\in G_{\mathbb{R}}'$,*

$$\begin{aligned}F_{d_1,d_2}(g_1',g_2',\Phi)_p &= 2\pi^2\cdot W_{d_1}^{\frac{3}{2}}(g_1')\,W_{d_2}^{\frac{3}{2}}(g_2')\cdot\langle \mathcal{Z}(d_1,\varphi_1^p;K^p),\mathcal{Z}(d_2,\varphi_2^p;K^p)\rangle_p^{\mathrm{ns.}}\\ &:= 2\pi^2\cdot\left\langle \hat{Z}(g_1',d_1,\varphi_1),\hat{Z}(g_2',d_2,\varphi_2)\right\rangle_p^{\mathrm{ns.}}.\end{aligned}$$

The last expression here is the nonsingular nonarchimedean part of the height pairing on $\widehat{CH}^1(X)_{\mathbb{C}}$, [9], *of the weighted cycles of Definition* 12.3. *If* d_1d_2 *is not a square in* $\mathbb{Q}^{\times}$, *then the superscript* "ns." *can be dropped.*

Together, Theorem 14.11 and Theorem 12.6 give the two cases of the main result of this paper.

15. *The global pairing.* We now collect our local results and consider the global height pairing. For simplicity, in this section we assume that $D(B) > 1$.

Fix K sufficiently small, as before, and let $X = X_K$ be the canonical model of our Shimura curve over $\mathbb{Q}$. Let $N = D(B)N'$, where N' is the product of the primes $p \nmid D(B)$ at which K_p is not $\mathrm{GL}_2(\mathbb{Z}_p)$. Thus, X has good reduction for all $p \nmid N$. The assumption that K is sufficiently small also implies that X has positive genus. Let $\mathcal{Y}$ be a minimal regular model of X over $\mathrm{Spec}(\mathbb{Z})$; this is unique up to isomorphism.

For $p \nmid N$, let $\mathcal{X}^{(p)}$ be the modular model of X over $\mathbb{Z}_{(p)}$ defined in Section 14. Then, since $\mathcal{X}^{(p)}$ is smooth over $\mathbb{Z}_{(p)}$,

$$\mathcal{Y} \times_{\mathbb{Z}} \mathbb{Z}_{(p)} \simeq \mathcal{X}^{(p)}. \tag{15.1}$$

For data d_1, φ_1, and d_2, φ_2, let $Z_1 = Z(d_1, \varphi_1; K)$ and $Z_2 = Z(d_2, \varphi_2; K)$ be the 0-cycles in X, and let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be their closures in $\mathcal{Y}$. Also, let $\mathcal{Z}_1^{(p)}$ and $\mathcal{Z}_2^{(p)}$ be the corresponding modular cycles in $\mathcal{X}^{(p)}$, as defined in Section 14. Then

$$\mathfrak{Z}_1 \times_{\mathbb{Z}} \mathbb{Z}_{(p)} \simeq \mathcal{Z}_1^{(p)} \qquad \text{and} \qquad \mathfrak{Z}_2 \times_{\mathbb{Z}} \mathbb{Z}_{(p)} \simeq \mathcal{Z}_2^{(p)}. \tag{15.2}$$

For g_1' and $g_2' \in G_{\mathbb{R}}'$, let $\hat{\mathfrak{Z}}_1(g_1')$ and $\hat{\mathfrak{Z}}_2(g_2') \in \widehat{\mathrm{CH}}^1(\mathcal{Y}) \otimes \mathbb{C}$ be the elements of the arithmetic Chow group of $\mathcal{Y}$, as in Definition 12.3. The global Gillet-Soulé height pairing ([8])

$$\left\langle \hat{\mathfrak{Z}}_1(g_1'), \hat{\mathfrak{Z}}_2(g_2') \right\rangle = \sum_{p \leq \infty} \left\langle \hat{\mathfrak{Z}}_1(g_1'), \hat{\mathfrak{Z}}_2(g_2') \right\rangle_p \tag{15.3}$$

is then defined. Moreover, for $p < \infty$ with $p \nmid N$, we have

$$\left\langle \hat{\mathfrak{Z}}_1(g_1'), \hat{\mathfrak{Z}}_2(g_2') \right\rangle_p = W_{d_1}^{\frac{3}{2}}(g_1') W_{d_2}^{\frac{3}{2}}(g_2') \cdot \left\langle \mathcal{Z}_1^{(p)}, \mathcal{Z}_2^{(p)} \right\rangle_p, \tag{15.4}$$

while, for $p = \infty$ the height pairing is given by Proposition 12.5 and Theorem 12.6.

Thus we obtain the following global result:

THEOREM 15.1. *Assume that d_1d_2 is not a square. If p is a prime of good reduction (including ∞) and if χ_p is unramified, then*

$$\left\langle \hat{\mathfrak{Z}}_1(g_1'), \hat{\mathfrak{Z}}_2(g_2') \right\rangle_p = \frac{1}{2\pi^2} \cdot F_{d_1,d_2}(g_1', g_2', \Phi)_p.$$

In particular, if the cycles $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ only meet in the fibers of good reduction, i.e., those for $p \nmid N$. Then

$$\left\langle \hat{\mathfrak{Z}}_1(g_1'),\hat{\mathfrak{Z}}_2(g_2')\right\rangle = \frac{1}{2\pi^2}\cdot F_{d_1,d_2}(g_1',g_2',\Phi).$$

Remark 15.2. If we drop the condition on d_1d_2, then analogous statements hold for the nonsingular part of the height pairing.

16. Generalizations In this section, we indicate how one might hope to generalize the results of this paper to higher dimensional situations. Our notation will be that of Sections 2–6. In particular, F will be a totally real number field of degree d over $\mathbb{Q}$ and $G_{\mathbb{A}}$ will be as in (2.2), for arbitrary n.

Let V be a nondegenerate quadratic space over F with

$$\text{(16.1)} \qquad \operatorname{sig}(V) = ((n-1,2),(n+1,0),\ldots,(n+1,0)).$$

Let $\mathcal{C}$ be the incoherent collection with $\mathcal{C}_v \simeq V_v$ for each finite place of F and with

$$\text{(16.2)} \qquad \operatorname{sig}(\mathcal{C}) = ((n+1,0),(n+1,0),\ldots,(n+1,0)).$$

Fix a decomposition $n = n_1 + n_2$, with n_1 and $n_2 \geq 1$, and choose factorizable weight functions $\varphi_1 \in S(V(\mathbb{A}_f)^{n_1})$ and $\varphi_2 \in S(V(\mathbb{A}_f)^{n_2})$. Let $\varphi = \varphi_1 \otimes \varphi_2 \in S(V(\mathbb{A}_f)^n)$, and let $\Phi_f(s) \in I_{n,f}(s,\chi)$ be the factorizable standard section associated to φ, i.e., with $\lambda(\varphi) = \Phi_f(0)$, in the global analogue of (1.11)–(1.13). Let

$$\text{(16.3)} \qquad \Phi_\infty(s) = \otimes_{j=1}^{d}\Phi_{\infty_j}^{\frac{n+1}{2}}(s),$$

where $\Phi_{\infty_j}^{\frac{n+1}{2}}(s) \in I_{n,\infty_j}(s,\chi_{\infty_j})$ is the standard section of weight $\frac{n+1}{2}$. Here we note that χ_∞ is determined by our choice of $\operatorname{sig}(V)$. Then $\Phi(s) = \Phi_\infty(s) \otimes \Phi_f(s)$ is an incoherent standard and factorizable section with $\Phi(0) \in \Pi_n(\mathcal{C})$. Let $E(g,s,\Phi)$ be the associated incoherent Eisenstein series, and let $F(g_1',g_2',\Phi)$ be the pullback of $E'(g,0,\Phi)$ to $G_{1,\mathbb{A}}' \times G_{2,\mathbb{A}}'$, as defined in (6.11)–(6.14). For $g_1' \in G_{1,\mathbb{R}}'$ and $g_2' \in G_{2,\mathbb{R}}'$, with $G_{i,\mathbb{R}}'$ given by (6.13), the Fourier coefficient $F_{d_1,d_2}(g_1',g_2',\Phi)$ of (6.15) has a structure, given by Corollary 6.2, which is analogous to that studied in this paper in the special case $n=2$ and $n_1 = n_2 = 1$. Of course, the general case will be much more complicated. In particular, it will not be so simple to eliminate the singular contribution $F_{d_1,d_2}(g_1',g_2',\Phi)_{\text{sing}}$.

On the geometric side, let $H = \operatorname{G}\operatorname{Spin}(V)$. By our assumption on the signature of V, to any compact open subgroup $K \subset H(\mathbb{A}_f)$, there is a Shimura variety X_K of dimension $n-1$, whose canonical model is defined over F (which we view as a subfield of $\mathbb{R}$ via the embedding ∞_1), and such that

$$\text{(16.4)} \qquad X_K(\mathbb{C}) \simeq H(F)\backslash D \times H(\mathbb{A}_f)/K.$$

For simplicity, we now assume that $d > 1$, so that X_K is projective. Given totally positive matrices $d_1 \in \mathrm{Sym}_{n_1}(F)$ and $d_2 \in \mathrm{Sym}_{n_2}(F)$, and weight functions φ_1 and φ_2, as above, there are weighted algebraic cycles $Z(d_1, \varphi_1; K)$ and $Z(d_2, \varphi_2; K)$ of codimensions n_1 and n_2 in X_K respectively. These are defined in [23], are rational over F, and have many nice properties. Note that, in our present situation, we have $n_1 + n_2 = n = \dim X_K + 1$.

PROBLEM 16.1. *Are there Green forms of logarithmic type such that $F_{d_1,d_2}(g_1',g_2',\Phi)$ is a constant multiple of the height pairing*

$$\left\langle\ \hat{\mathfrak{Z}}(g_1',d_1,\varphi_1;K),\ \hat{\mathfrak{Z}}(g_2',d_2,\varphi_2;K)\ \right\rangle,$$

where

$$\hat{\mathfrak{Z}}(g_i',d_i,\varphi_i;K) = \left(\ W_{d_i}^{\frac{n+1}{2}}(g_i') \cdot \mathfrak{Z}(d_i,\varphi_i;K),\ \Xi(g_i',d_i,\varphi_i;K)\ \right) \in \widehat{CH}^{n_i}(\mathcal{Y}_K),$$

where $\mathcal{Y}_K$ is a suitable regular model of X_K over $\mathcal{O}_F$, and $\mathfrak{Z}(d_i,\varphi_i;K)$ is an extension of the cycle $Z(d_i,\varphi_i;K)$ to this model?

Of course, one could isolate nonsingular parts associated to the various places of F and ask for partial results along the lines of those we have proved in this paper in our special case. For example, the study of nonarchimedean places does not require that we know the Green forms. It seems to me that we are very far from being able to answer such a question at present, but it may be possible to obtain partial results supporting this picture.

For example, for $F = \mathbb{Q}$, if we consider instead the decomposition $W = W_1+W_2+W_3$ in the case $n = 3$ and $n_1 = n_2 = n_3 = 1$, then we obtain functions $F_{d_1,d_2,d_3}(g_1', g_2', g_3', \Phi)_p$. The cycles in question are Hirzebruch-Zagier curves on quaternion Hilbert modular surfaces, and, for good primes p, the analogues of the arguments of our Sections 7–8 and 14 together with the results of Gross and Keating [10] and of Kitaoka [19] two yield the analogue of Theorem 14.11. This is (part of) the type of result envisioned in the last section of [12]. Details will appear elsewhere.

It should be noted that are analogous incoherent Eisenstein series and algebraic cycles defined for unitary groups of signature $((n-1, 1), (n, 0), \ldots, (n, 0))$. One can formulate the same problem in that case.

Appendix. Interpolation of local representation densities

In this section, we consider the nonarchimedean case and recall the relation between $W_{T,v}(g, s, \Phi_v)$ of (1.17) and the classical local representation densities. We then give an interpretation of the derivative $W_{T,v}'(g, 0, \Phi_v)$ in the same language.

We fix a nonarchimedean place v and temporarily suppress it from the notation, so that F is a nonarchimedean local field of characteristic zero with ring of integers $\mathcal{O}$, maximal ideal $\mathcal{P}$ and uniformizer ϖ. Let ψ be a fixed additive character of F, trivial on $\mathcal{O}$ and nontrivial on $\mathcal{P}^{-1}$. Recall that, for our fixed character χ, the local induced representation decomposes as:

$$I_n(0,\chi) = R_n(V_\chi^+) \oplus R_n(V_\chi^-) \tag{A.1}$$

where $\dim V_\chi^+ = \dim V_\chi^- = n+1$. We take a standard local section $\Phi(s)$ such that $\Phi(0) \in R_n(V)$ for $V = V_\chi^\pm$. There is then a function $\varphi \in S(V^n)$ such that

$$\Phi(g,s) = (\omega_V(g)\varphi)(0)\,|a(g)|^s, \tag{A.2}$$

where $\omega_V(g)$ is the action of $g \in \mathrm{Mp}(W)$ in $S(V^n)$ via the Weil representation. Recall, [22], that, for $g \in \mathrm{Sp}(W)$, and $\underline{g} = (g,1) \in \mathrm{Mp}(W) \simeq \mathrm{Sp}(W) \times \mathbb{C}^1$ under our fixed Rao isomorphism, this action is defined, by

$$\omega_V(\underline{g}) = \eta_V(g)\, r_V(g) \tag{A.3}$$

where $r_V(g)$ is the pullback to $\mathrm{Sp}(W)$ (via the imbedding ι_V: $\mathrm{Mp}(W) \to \mathrm{Mp}(V \otimes W)$, this last group having the 'Leray' cocycle) of the 'standard' (projective) action of $\mathrm{Sp}(V \otimes W)$ on $S(V^n)$. Also,

$$\eta_V(g) = (x(g), \det V)_F\, \gamma_F(\det V, \psi)^{-j}\, \varepsilon(V)^j\, \gamma_F(x(g),\psi)^{-m}\, \gamma_F(\psi)^{-mj}, \tag{A.4}$$

is the factor from [22], which reduces the pullback of the Leray cocycle to the m^{th} power of the Rao cocycle. Here $m = \dim_F V$, and $j = j(g)$ is the index of the Bruhat cell associated to g. For more details, see [22]. When m is odd, this defines a genuine representation of $\mathrm{Mp}(W) \simeq \mathrm{Sp}(W) \times \mathbb{C}^1$.

Now for any $r \in \mathbb{Z}_{\geq 0}$, define

$$U_r = V + V_{r,r}, \tag{A.5}$$

where $V_{r,r}$ is a hyperbolic plane of dimension $2r$, and let ω_{U_r} denote the corresponding Weil representation of $\mathrm{Mp}(W)$ on

$$S(U_r^n) \simeq S(V^n) \otimes S(V_{r,r}^n). \tag{A.6}$$

Remark A.1. Here, the action of the element $(1,t)$, for $t \in \mathbb{C}^1$, in the Weil representation of the even dimensional quadratic space $V_{r,r}$ is taken to be *trivial* (cf. (1.8)).

LEMMA A.2. $\omega_{U_r} = \omega_V \otimes \omega_{V_{r,r}}$.

Proof. It is known that the representation r_{U_r} has this property [43], so it suffices to show that the 'normalizing' factors behave correctly, i.e., that $\eta_{U_r} = \eta_V \cdot \eta_{V_{r,r}}$. We omit the calculation. □

Let $\varphi_r^0 \in S(V_{r,r}^n)$ be the characteristic function of the lattice $M_{2r,n}(\mathcal{O})$ (where we have chosen a basis for $V_{r,r}$ for which the quadratic form has matrix

$\frac{1}{2}\begin{pmatrix} 0 & 1_r \\ 1_r & 0 \end{pmatrix})$). This function is K-invariant, where K is the inverse image in $\mathrm{Mp}(W)$ of the maximal compact subgroup $\mathrm{Sp}_n(\mathcal{O})$ in $\mathrm{Sp}_n(F)$, and satisfies

$$\left(\omega_{V_{r,r}}(g)\varphi_r^0\right)(0) = |a(g)|^r. \tag{A.7}$$

LEMMA A.3. *Set $\varphi^{[r]} = \varphi \otimes \varphi_r^0 \in S(U_r^n)$. Then, for any $g \in \mathrm{Mp}(W)$,*

$$\Phi(g,r) = \left(\omega_{U_r}(g)\varphi^{[r]}\right)(0).$$

In short, the function $\Phi(g,s)$ 'interpolates' the functions $\left(\omega_{U_r}(g)\varphi^{[r]}\right)(0)$ as s takes on integer values $r \geq 0$.

Proof. Using Lemma A.2, (A.7), and (A.2),

$$\begin{aligned}\left(\omega_{U_r}(g)\varphi^{[r]}\right)(0) &= \left(\omega_V(g)\varphi\right)(0)\cdot\left(\omega_{V_{r,r}}(g)\varphi_r^0\right)(0) \\ &= \left(\omega_V(g)\varphi\right)(0)\cdot|a(g)|^r \\ &= \Phi(g,r). \end{aligned}\tag{A.8}$$

□

In the range $r > \frac{n+1}{2}$ (which insures convergence of the integral in question), we use Lemma A.3 to obtain:

PROPOSITION A.4. *With the notation just introduced and for any $r \in \mathbb{Z}$, with $r > \frac{n+1}{2}$,*

$$W_T(e,r,\Phi) = \gamma(V)\cdot\alpha(T,U_r;\varphi^{[r]}),$$

where

$$\alpha(T,U_r;\varphi^{[r]}) = \lim_{t\to\infty} q^{t\cdot\frac{n(n+1)}{2}} \int_{\substack{x\in U_r^n \\ \frac{1}{2}(x,x)-T\in\varpi^t \mathrm{Sym}_n(\mathcal{O})^*}} \varphi^{[r]}(x)\,dx,$$

is the weighted (by $\varphi^{[r]} \in S(U_r^n)$) representation density of T by the quadratic space $U_r = V + V_{r,r}$. Here the constant $\gamma(V)$ is given by

$$\gamma(V) = \gamma_F(\det V,\psi)^{-n}\varepsilon_F(V)^n\gamma_F(\psi)^{-n(n+1)},$$

and hence depends only on $\det V$, $\varepsilon(V)$, and n.

Proof. If $r > \frac{n+1}{2}$, we can simply substitute the expression for $\Phi(g,r)$ given by Lemma A.3 into the integral:

$$\begin{aligned} &W_T(e,r,\Phi) \\ &= \int_{\mathrm{Sym}_n(F)} \Phi(w^{-1}n(b),r)\psi(-\operatorname{tr}(Tb))\,db \\ &= \int_{\mathrm{Sym}_n(F)} \left(\omega_{U_r}(w^{-1}n(b))\varphi^{[r]}\right)(0)\psi(-\operatorname{tr}(Tb))\,db \\ &= \gamma(V)\cdot\int_{\mathrm{Sym}_n(F)}\int_{U_r^n} \psi\left(\frac{1}{2}\operatorname{tr}(b(x,x))\right)\varphi^{[r]}(x)\,dx\;\psi(-\operatorname{tr}(Tb))\,db \end{aligned}\tag{A.9}$$

$$
\begin{aligned}
&= \gamma(V) \\
&\quad \cdot \lim_{t\to\infty} \int_{\varpi^{-t}\operatorname{Sym}_n(\mathcal{O})} \int_{U_r^n} \psi\left(\frac{1}{2}\operatorname{tr}(b(x,x))\right) \varphi^{[r]}(x)\,dx\ \psi(-\operatorname{tr}(Tb))\,db \\
&= \gamma(V)\cdot \lim_{t\to\infty} q^{t\frac{n(n+1)}{2}} \int_{\substack{x\in U_r^n \\ \frac{1}{2}(x,x)-T\in\varpi^t\operatorname{Sym}_n(\mathcal{O})^*}} \varphi^{[r]}(x)\,dx \\
&:= \gamma(V)\cdot\alpha(T,U_r;\varphi^{[r]}).
\end{aligned}
$$

Here, via (A.3) and (A.4),

$$
\begin{aligned}
\text{(A.10)}\qquad \gamma(V) &= \eta_V(w^{-1}) \\
&= \gamma_F(\det V,\psi)^{-n}\varepsilon_F(V)^n\gamma_F(\psi)^{-n(n+1)}. \qquad \square
\end{aligned}
$$

For a moment, to obtain a more classical formula, we slightly shift our notation. In general, suppose that $U \simeq F^m$ with quadratic form $\frac{1}{2}(x,x) = S[x] = {}^t xSx$, for $S \in \operatorname{Sym}_m(F)$. For a function $\varphi \in S(U^n)$ and for $T \in \operatorname{Sym}_n(F)$, we set

$$
\text{(A.11)}\qquad \alpha(S,T;\varphi) := \lim_{t\to\infty} q^{t\cdot\frac{n(n+1)}{2}} \int_{\substack{x\in U^n \\ S[x]-T\in\operatorname{Sym}_n(\mathcal{O})^*}} \varphi(x)\,dx,
$$

and view this as a weighted representation density. The following homogeneity property follows immmediately from the definition.

LEMMA A.5. *If $a \in F^\times$, then*

$$
\alpha(aS,aT;\varphi) = |a|^{-\frac{n(n+1)}{2}}\cdot\alpha(S,T;\varphi).
$$

The terminology 'weighted representation density', is explained by the following example. Suppose that we choose an $\mathcal{O}$ lattice $L \subset V$ on which $(\ ,\ )$ is integral, and choose an $\mathcal{O}$ basis for L. If φ is the characteristic function of L^n, then $\varphi^{[r]}$ is the characteristic function of $M_{n+1+2r,n}(\mathcal{O})$.

PROPOSITION A.6. *Assume that $r \geq 0$ and that φ is the characteristic function of L^n for an $\mathcal{O}$ lattice in V. Then*

$$
\alpha(T,U_r;\varphi^{[r]}) = |\det S_0|^{\frac{n}{2}}\cdot\alpha(S_r,T),
$$

where

$$
\begin{aligned}
\alpha(S_r,T) &= \lim_{t\to\infty} q^{tn(\frac{n+1}{2}-(n+1+2r))}|2|^{n(n+1+2r)} \\
&\quad \#\{\, x\in M_{n+1+2r,n}((\mathcal{O}/\varpi^t\mathcal{O})) \mid S_r[x]-T\in\varpi^t\operatorname{Sym}_n(\mathcal{O})^*\ \}
\end{aligned}
$$

is the usual representation density [21], *where S_r is the matrix for the quadratic form on U_r with respect to the fixed basis of $L+\mathcal{O}^{2r}$. In particular,*

$$
W_T(e,r,\Phi) = |\det S_0|^{\frac{n}{2}}\cdot\gamma(V)\cdot\alpha(S_r,T).
$$

By [19], $\alpha(S_r,T)$ *is a rational function of* $X = q^{-r}$. *Hence,*

$$W'_T(e,0,\Phi) = -\log q \cdot |\det S_0|^{\frac{n}{2}} \cdot \gamma(V) \cdot \frac{\partial}{\partial X}\{\alpha(S_r,T)\}|_{X=1}.$$

Proof. In this situation we have

$$W_T(e,r,\Phi) = \gamma(V) \cdot \lim_{t\to\infty} q^{t\cdot\frac{n(n+1)}{2}} \int_{\substack{x\in M_{n+1+2r,n}(\mathcal{O}) \\ \frac{1}{2}(x,x)-T\in\varpi^t \operatorname{Sym}_n(\mathcal{O})^*}} dx. \tag{A.12}$$

Note that the measure dx is self dual with respect to the bilinear pairing defined by S_r. In particular,

$$\operatorname{vol}(M_{n+1+2r,n}(\mathcal{O})) = |\det S_0|^{\frac{n}{2}}.$$

Moreover, the domain of integration here is a union of cosets for $2\varpi^t M_{n+1+2r,n}(\mathcal{O})$, so that we get

$$\begin{aligned}(\text{A.13})\quad |\det S_0|^{\frac{n}{2}} \cdot \lim_{t\to\infty} q^{tn(\frac{n+1}{2}-(n+1+2r))}|2|^{n(n+1+2r)} \\ \#\{x\in M_{n+1+2r,n}((\mathcal{O}/\varpi^t\mathcal{O})) \mid S_r[x]-T\in\varpi^t\operatorname{Sym}_n(\mathcal{O})^*\}.\end{aligned}$$

Up to the factor $|\det S_0|^{\frac{n}{2}}$, this is just the classical definition of the density. □

It is known, [19], that the function $W_T(e,s,\Phi)$ is a polynomial in q^{-s}, and hence is determined by its values at r's with $r > \frac{n+1}{2}$. Thus, whenever an explicit formula for the representation density $\alpha(T,U_r;\varphi^{[r]})$, $r > \frac{n+1}{2}$, is known, we may substitute $r=0$ to obtain a formula for $W_T(e,0,\Phi)$, and we can differentiate to obtain a formula for $W'_T(e,0,\Phi)$, as was done in Section 8.

University of Maryland, College Park, MD
E-mail address: ssk@math.umd.edu

References

[1] S. Böcherer, Über die Funktionalgleichung automorpher L-Funktionen zur Siegelschen Modulgruppe, J. reine angew. Math. **362** (1985), 146–168.

[2] J.-F. Boutot and H. Carayol, Uniformisation p-adiques des courbes de Shimura: les théormès de Cerednik et de Drinfel'd, in: *Courbes Modulaires et Courbes de Shimura*, Astérisques **196–197** (1991), 45–158.

[3] H. Carayol, Sur la mauvaise réduction des courbes de Shimura, Comp. Math. **59** (1986), 151–230.

[4] W. Casselman, The Hasse-Weil ζ-function of some moduli varieties of dimension greater than one, in: *Automorphic forms, representations and L-functions*, Proc. Sympos. Pure Math., XXXIII, Part 2, AMS, Providence, VI, 1979, 141–163.

[5] H. Darmon, Heegner points, Heegner cycles, and congruences, in: *Elliptic Curves and Related Topics*, CRM Proc. and Lect. Notes **4**, H. Kisilevsky and R. Murty, eds., AMS, Providence, RI, 1994, 45–59.

[6] V. G. Drinfel'd, Coverings of p-adic symmetric regions, Funct. Anal. Appl., **10** (1977), 29–40.

[7] P. GARRETT, Pullbacks of Eisenstein series; Applications, in: *Automorphic Forms of Several Variables*, Taniguchi symposium, Katata, 1983, Birkhäuser, Boston, MA, 1984, 115–159.

[8] H. GILLET and C. SOULÉ, Arithmetic intersection theory, Publ. Math. IHES, **72** (1990), 93–174.

[9] B. H. GROSS, Local heights on curves in: *Arithmetic Geometry*, Cornell and Silverman, eds., Springer-Verlag, New York, 1986, 327–339.

[10] B. H. GROSS and K. Keating, On the intersection of modular correspondences, Invent. math. **112** (1993), 225–245.

[11] B. H. GROSS, W. KOHNEN, and D. ZAGIER, Heegner points and derivatives of L-series II, Math. Ann. **278** (1987), 497–562.

[12] B. H. GROSS and S. KUDLA, Heights and the central critical values of triple product L-functions, Comp. Math. **81** (1992), 143–209.

[13] B. H. GROSS, S. KUDLA, and D. ZAGIER, unpublished correspondence, 1990–92.

[14] B. H. GROSS and D. ZAGIER, Heegner points and the derivatives of L-series, Invent. math. **84** (1986), 225–320.

[15] M. HARRIS and S. Kudla, The central critical value of a triple product L-function, Annals of Math. **133** (1991), 605–672.

[16] M. KAREL, Values of certain Whittaker functions on a p-adic group, Illinois J. Math. **26** (1982), 552–575.

[17] G. KAUFHOLD, Dirichletsche Reihe mit Funktionalgleichung in der Theorie der Modulfunktion 2, Grades, Math. Ann. **137** (1959), 454–476.

[18] K. KEATING, Intersection numbers of Heegner divisors on Shimura curves, preprint, 1994.

[19] Y. KITAOKA, A note on local densities of quadratic forms, Nagoya Math. J. **92** (1983), 145–152.

[20] ———, Fourier coefficients of Eisenstein series of degree 3, Proc. Japan Akad. **60** (1984), 259–261.

[21] ———, *Arithmetic of Quadratic Forms*, Cambridge Tracts in Mathematics **106**, Cambridge Univ. Press, Cambridge, 1993.

[22] S. KUDLA, Splitting metaplectic covers of dual reductive pairs, Israel J. Math. **87** (1994), 361–401.

[23] ———, Algebraic cycles on Shimura varieties of orthogonal type, Duke Math. J. **86** (1997), 39–78.

[24] S. KUDLA and J. MILLSON, The theta correspondence and harmonic forms I, Math. Ann. **274** (1986), 353–378.

[25] ———, The theta correspondence and harmonic forms II, Math. Ann. **277** (1987), 267–314.

[26] ———, Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables, IHES Publ. Math. **71** (1990), 121–172.

[27] S. KUDLA and S. RALLIS, On the Weil-Siegel formula, J. reine angew. Math., **387** (1988), 1–68.

[28] ———, Degenerate principal series and invariant distributions, Israel J. Math. **69** (1990), 25–45.

[29] ———, Ramified degenerate principal series, Israel J. Math. **78** (1992), 209–256.

[30] ———, A regularized Siegel-Weil formula: The first term identity, Annals of Math. **140** (1994), 1–80.

[31] S. KUDLA, S. RALLIS, and D. SOUDRY, On the degree 5 L-function for Sp(2), Invent. math. **107** (1992), 483–541.

[32] J.-P. LABESSE, Fonction zeta locale et fonction L de formes automorphes, in: *Variétés de Shimura et Fonctions L*, Publ. Math. de l'Université Paris VII, 1979, 83–130.

[33] R. P. LANGLANDS, On the zeta functions of some simple Shimura varieties, Canad. J. Math. **31** (1979), 1121–1216.

[34] ———, On the notion of an automorphic form, *Proc. Symp. Pure Math.* **33** (1979), 203–207.
[35] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publ., Inc. New York, 1972.
[36] B. Myers, Local representation densities of non-unimodular quadratic forms, Thesis, University of Maryland, 1994.
[37] J. Milne, Points on Shimura varieties mod p, in: *Proc. Symp. Pure Math.* **33**, Part 2, AMS, Providence, RI, 1979, 165–184.
[38] ———, Étude d'une classe d'isogénie, in: *Variétés de Shimura et Fonctions L*, Pub. Math. de l'Université Paris VII, 1979, 73–81.
[39] I. Piatetski-Shapiro and S. Rallis, L-functions for the classical groups, Lecture Notes in Math. **1254**, Springer-Verlag, New York, 1987, 1–52.
[40] S. Rallis, On the Howe duality conjecture, Comp. Math. **51** (1984), 333–399.
[41] ———, L-functions and the oscillator representation, Lecture Notes in Math. **1245**, Springer-Verlag, New York, 1987.
[42] ———, Poles of standard L-functions, in: *Proc. of the Inter. Congress of Mathematicians, Kyoto*, Springer-Verlag, New York, 1990, 833–845.
[43] R. Ranga Rao, On some explicit formulas in the theory of Weil representation, Pacific J. Math. **157** (1993), 335–371.
[44] J.-P. Serre, *A Course in Arithmetic*, Springer-Verlag, New York, 1973.
[45] G. Shimura, Confluent hypergeometric functions on tube domains, Math. Ann. **260** (1982), 269–302.
[46] C. Soulé, D. Abramovich, J.-F. Burnol, and J. Kramer, *Lectures on Arakelov Geometry*, Cambridge Studies in Advanced Mathematics **33** Cambridge Univ. Press, Cambridge, 1992.
[47] W. J. Sweet, The metaplectic case of the Weil-Siegel formula, Thesis, Univ. of Maryland, 1990.
[48] ———, Functional equations of p-adic zeta integrals and representations of the metaplectic group, preprint, 1995.
[49] M.-F. Vigneras, Arithmétique des algèbres de quaternions, Lecture Notes in Math. **800**, Springer-Verlag, New York, 1980.
[50] N. Wallach, Lie algebra cohomology and holomorphic continuation of generalized Jacquet integrals, in: *Representations of Lie Groups*, K. Okamoto and K. Oshima, eds., (Adv. Stud. Pure Math. **14**, 123–151), Amsterdam, North Holland, 1988.
[51] T. Zink, Über die schlechte reduktion einiger Shimuramannifaltigkeiten, Comp. Math. **45** (1981), 15–107.
[52] ———, Isogenieklassen von Punkten von Shimuramannigfaltigkeiten mit Werten in einem endlichen Körper, Math. Nach. **112** (1983), 103–124.

(Received February 16, 1996)