EM316 - Numerical Methods for EEE

Problem Sheet 3

W.M.B.S.K.Wijenayake (E/19/445)

5/12/2023

Question 1

Given that,

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \quad X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

a

Iteration 1,

$$AX_0 = \lambda X_1$$
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \tag{1}$$

Iteration 2,

$$AX_1 = \lambda X_2$$
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} \tag{2}$$

Iteration 3,

$$AX_2 = \lambda X_3$$
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ -0.929 \end{bmatrix} \tag{3}$$

For an approximate eigenvector of A,

$$X = \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$$

Thus, the approximate dominant eigenvalue,

$$\lambda_{max} = 2.8$$

b

Considering the final result,

$$X_3 = \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}$$

Approximate Dominant eigenvalue using Rayleigh Quotient,

$$\begin{aligned}
\lambda &= \frac{X^T A X}{X^T X} \\
&= \frac{\begin{bmatrix} 1 & -0.929 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}}{\begin{bmatrix} 1 & -0.929 \end{bmatrix} \begin{bmatrix} 1 \\ -0.929 \end{bmatrix}} \\
&= \frac{\begin{bmatrix} 1 & -0.929 \end{bmatrix} \begin{bmatrix} 2.929 \\ -2.858 \end{bmatrix}}{1.863041} \\
&= \frac{5.584082}{1.863041} \\
&= 2.9973
\end{aligned}$$

c

For exact values of Eigenvectors and Eigenvalues

$$\begin{aligned}
det(A - \lambda I) &= 0 \\
(2 - \lambda)^2 - 1 &= 0 \\
\lambda_1 = 3, &\quad \lambda_2 = 1
\end{aligned}$$

For the Dominant Eigenvector,

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This gives,

$$\begin{aligned}
2x_1 - x_2 &= 3x_1 \\
-1x_1 + 2x_2 &= 3x_2 \\
x_1 &= -x_2
\end{aligned}$$

Thus the eigenvector,

$$X = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

d

For the error ϵ,

$$\begin{aligned} \epsilon &= \frac{|3 - 2.8|}{3} \times 100\% \\ &= 6.667\% \end{aligned}$$

e

For the inverse matrix of A,

$$\begin{aligned} A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \end{aligned}$$

Let's take $X_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Iteration 1,

$$\begin{gathered} A^{-1} X_0 = \mu X_1 \\ \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} \end{gathered} \tag{1}$$

Iteration 2,

$$\begin{gathered} A^{-1} X_1 = \mu X_2 \\ \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2.5 \\ 2 \end{bmatrix} = \frac{2.5}{3} \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} \end{gathered} \tag{2}$$

Iteration 3,

$$\begin{gathered} A^{-1} X_2 = \mu X_3 \\ \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2.8 \\ 2.6 \end{bmatrix} = \frac{2.8}{3} \begin{bmatrix} 1 \\ 0.929 \end{bmatrix} \end{gathered} \tag{3}$$

This gives approximate smallest eigenvalue,

$$\begin{aligned}\lambda_{min} &= \frac{1}{\mu} \\ &= \frac{3}{2.8} \\ &= 1.071\end{aligned}$$

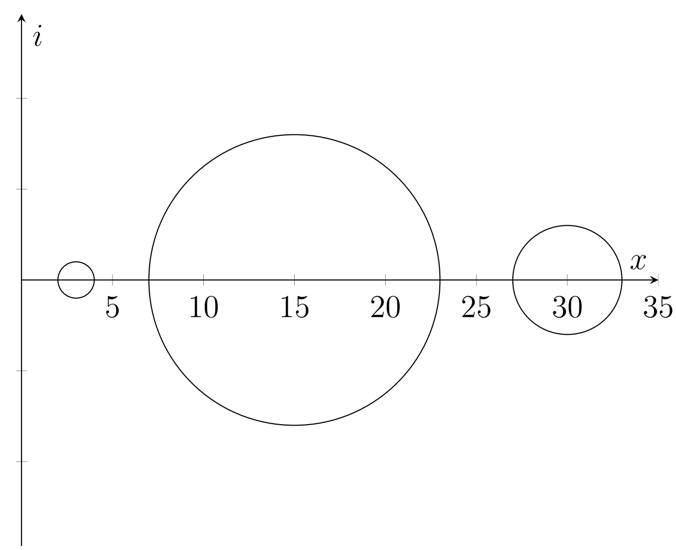
Question 2

Given that,

$$A = \begin{bmatrix} 30 & 1 & 2 \\ 4 & 15 & -4 \\ -1 & 0 & 3 \end{bmatrix} \quad X_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

a

$$D_i = \left(a_{i,i}, \sum_{j \neq i} |a_{i,j}| \right) \quad D_1 = (30, 3) \;\; D_2 = (15, 8) \;\; D_3 = (3, 1)$$



b

Iteration 1,

$$AX_0 = \lambda X_1$$

$$\begin{bmatrix} 30 & 1 & 2 \\ 4 & 15 & -4 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -28 \\ -8 \\ 4 \end{bmatrix} = 28 \begin{bmatrix} -1 \\ -0.286 \\ 0.143 \end{bmatrix} \tag{1}$$

Iteration 2,

$$AX_1 = \lambda X_2$$

$$\begin{bmatrix} 30 & 1 & 2 \\ 4 & 15 & -4 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -0.286 \\ 0.143 \end{bmatrix} = \begin{bmatrix} -30.00 \\ -8.862 \\ 1.429 \end{bmatrix} = 30 \begin{bmatrix} -1 \\ -0.2954 \\ 0.048 \end{bmatrix} \tag{2}$$

Iteration 3,

$$AX_2 = \lambda X_3$$

$$\begin{bmatrix} 30 & 1 & 2 \\ 4 & 15 & -4 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ -0.295 \\ 0.048 \end{bmatrix} = \begin{bmatrix} -30.199 \\ -8.617 \\ 1.144 \end{bmatrix} = 30.199 \begin{bmatrix} -1 \\ -0.285 \\ 0.037 \end{bmatrix} \tag{3}$$

c

For the dominant eigenvalue, choosing the scalar as 30,

$$A - 30I = \begin{bmatrix} 0 & 1 & 2 \\ 4 & -15 & -4 \\ -1 & 0 & -27 \end{bmatrix}$$

$$(A - 30I)^{-1} = \frac{1}{82} \begin{bmatrix} 405 & 27 & 26 \\ 112 & 2 & 8 \\ -15 & -1 & -4 \end{bmatrix}$$

Selecting $X_0 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

Iteration 1,

$$(A - 30I)^{-1}X_0 = \lambda X_1$$
$$\frac{1}{82}\begin{bmatrix} 405 & 27 & 26 \\ 112 & 2 & 8 \\ -15 & -1 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{82}\begin{bmatrix} -379 \\ -104 \\ 11 \end{bmatrix} = 4.622\begin{bmatrix} -1 \\ -0.274 \\ 0.029 \end{bmatrix} \tag{1}$$

Iteration 2,

$$(A - 30I)^{-1}X_1 = \lambda X_2$$
$$\frac{1}{82}\begin{bmatrix} 405 & 27 & 26 \\ 112 & 2 & 8 \\ -15 & -1 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ -0.274 \\ 0.029 \end{bmatrix} = \frac{1}{82}\begin{bmatrix} -404.246 \\ -112.316 \\ 15.158 \end{bmatrix} = 4.929\begin{bmatrix} -1 \\ -0.278 \\ 0.037 \end{bmatrix} \tag{2}$$

Iteration 3,

$$(A - 30I)^{-1}X_2 = \lambda X_3$$
$$\frac{1}{82}\begin{bmatrix} 405 & 27 & 26 \\ 112 & 2 & 8 \\ -15 & -1 & -4 \end{bmatrix}\begin{bmatrix} -1 \\ -0.278 \\ 0.037 \end{bmatrix} = \frac{1}{82}\begin{bmatrix} -404.788 \\ -112.260 \\ 15.13 \end{bmatrix} = 4.936\begin{bmatrix} -1 \\ -0.277 \\ 0.037 \end{bmatrix} \tag{3}$$

Thus, the eigenvalue,

$$\begin{aligned} \lambda_{max} &= \frac{1}{\mu} + \sigma \\ &= \frac{1}{4.936} + 30 \\ &= 30.203 \end{aligned}$$

It is very clear that the shifted inverse power method converges to the actual eigen value faster than the power method.

d

For the inverse matrix of A,

$$A^{-1} = \frac{1}{1372}\begin{bmatrix} 45 & -3 & -34 \\ -8 & 92 & 128 \\ 15 & -1 & 446 \end{bmatrix}$$

Iteration 1,

$$A^{-1}X_0 = \mu X_1$$

$$\frac{1}{1372}\begin{bmatrix} 45 & -3 & -34 \\ -8 & 92 & 128 \\ 15 & -1 & 446 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{1372}\begin{bmatrix} -79 \\ 136 \\ 431 \end{bmatrix} = 0.314\begin{bmatrix} -0.183 \\ 0.315 \\ 1 \end{bmatrix} \tag{1}$$

Iteration 2,

$$A^{-1}X_1 = \mu X_2$$

$$\frac{1}{1372}\begin{bmatrix} 45 & -3 & -34 \\ -8 & 92 & 128 \\ 15 & -1 & 446 \end{bmatrix}\begin{bmatrix} -0.183 \\ 0.315 \\ 1 \end{bmatrix} = \frac{1}{1372}\begin{bmatrix} -43.18 \\ 158.444 \\ 442.94 \end{bmatrix} = 0.32\begin{bmatrix} -0.097 \\ 0.358 \\ 1 \end{bmatrix} \tag{2}$$

Iteration 3,

$$A^{-1}X_2 = \mu X_3$$

$$\frac{1}{1372}\begin{bmatrix} 45 & -3 & -34 \\ -8 & 92 & 128 \\ 15 & -1 & 446 \end{bmatrix}\begin{bmatrix} -0.097 \\ 0.358 \\ 1 \end{bmatrix} = \frac{1}{1372}\begin{bmatrix} -39.44 \\ 161.71 \\ 444.19 \end{bmatrix} = 0.323\begin{bmatrix} -0.088 \\ 0.364 \\ 1 \end{bmatrix} \tag{3}$$

Thus the smallest Eigenvalue approximation,

$$\begin{aligned} \lambda_{min} &= \frac{1}{\mu} \\ &= \frac{1}{0.323} \\ &= 3.0959 \end{aligned}$$

Question 3

Given that,

$$A = \begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix}$$

a

Eigenvalues,

$$\lambda_1 = 3 \quad \lambda_2 = -6 \quad \lambda_3 = 0$$

Corresponding eigen vectors,

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

b

Taking $X_0 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$

Iteration 1,

$$AX_0 = \lambda X_1$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \\ 8 \end{bmatrix} = -8 \begin{bmatrix} 0.5 \\ -0.25 \\ -1 \end{bmatrix} \tag{1}$$

Iteration 2,

$$AX_1 = \lambda X_2$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0.5 \\ -0.25 \\ -1 \end{bmatrix} = \begin{bmatrix} -5.25 \\ -0.75 \\ 3.75 \end{bmatrix} = -5.25 \begin{bmatrix} 1 \\ 0.143 \\ -0.714 \end{bmatrix} \tag{2}$$

Iteration 3,

$$AX_2 = \lambda X_3$$

$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.143 \\ -0.714 \end{bmatrix} = \begin{bmatrix} -4.713 \\ 0.429 \\ 5.571 \end{bmatrix} = -5.571 \begin{bmatrix} 0.846 \\ 0.076 \\ -1 \end{bmatrix} \tag{3}$$

Iteration 4,

$$AX_3 = \lambda X_4$$
$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 0.846 \\ 0.076 \\ -1 \end{bmatrix} = \begin{bmatrix} -5.616 \\ -0.078 \\ 5.46 \end{bmatrix} = -5.616 \begin{bmatrix} 1 \\ 0.014 \\ -0.972 \end{bmatrix} \tag{4}$$

Iteration 5,

$$AX_4 = \lambda X_5$$
$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.014 \\ -0.972 \end{bmatrix} = \begin{bmatrix} -5.874 \\ -0.042 \\ 5.958 \end{bmatrix} = -5.958 \begin{bmatrix} 0.986 \\ 0.007 \\ -1 \end{bmatrix} \tag{5}$$

Taking $X_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Iteration 1,

$$AX_0 = \lambda X_1$$
$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{1}$$

Iteration 2,

$$AX_1 = \lambda X_2$$
$$\begin{bmatrix} -2 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{2}$$

Since $X_2 = X_1$, over the next iterations it will produce the same answer with eigen value equals to 3, and the eigenvector in this scenario converges to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Question 4

Given that,

$$f(x) = \begin{bmatrix} x_1^2 - x_1 + x_2 - 1 = 0 \\ x_1^2 - 2x_2^2 - x_2 = 0 \end{bmatrix}$$

For the Jacobian Matrix J,

$$\begin{aligned} J &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 - 1 & 1 \\ 2x_1 & -4x_2 - 1 \end{bmatrix} \end{aligned}$$

Iteration 1,

$$X_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{1}$$

$$J(X_0) = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \tag{2}$$

$$f(X_0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{3}$$

Using Newtons Method,

$$J(X_0)\delta X_{1,0} = -f(X_0)$$

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \delta X_{1,0} = -\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This gives,

$$X_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \tag{4}$$

$$J(X_1) = \begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} \tag{5}$$

$$f(X_1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Using Newton's Method,

$$J(X_1)\delta X_{2,1} = -f(X_1)$$
$$\begin{bmatrix} -3 & 1 \\ -2 & -1 \end{bmatrix} \delta X_{2,1} = -\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Using Gaussian Elimination,

$$\left[\begin{array}{cc|c} -3 & 1 & -1 \\ -2 & -1 & -1 \end{array}\right]_{R2-R3\times\frac{2}{3}}$$
$$\left[\begin{array}{cc|c} -3 & 1 & -1 \\ 0 & -\frac{5}{3} & -\frac{1}{3} \end{array}\right]_{R2\times\frac{-3}{5}}$$
$$\left[\begin{array}{cc|c} -3 & 1 & -1 \\ 0 & 1 & \frac{1}{5} \end{array}\right]_{R_1-R_2}$$
$$\left[\begin{array}{cc|c} -3 & 0 & \frac{-6}{5} \\ 0 & 1 & \frac{1}{5} \end{array}\right]_{R_1\times\frac{1}{3}}$$
$$\left[\begin{array}{cc|c} 1 & 0 & \frac{2}{5} \\ 0 & 1 & \frac{1}{5} \end{array}\right]$$

This gives,

$$X_2 = \begin{bmatrix} -\frac{3}{5} \\ \frac{1}{5} \end{bmatrix} \tag{7}$$
$$J(X_2) = \begin{bmatrix} -\frac{11}{5} & 1 \\ -\frac{12}{5} & -\frac{9}{5} \end{bmatrix} \tag{8}$$
$$f(X_2) = \begin{bmatrix} \frac{4}{25} \\ \frac{2}{25} \end{bmatrix} \tag{9}$$

Using Newton's Method,

$$J(X_1)\delta X_{3,2} = -f(X_2)$$
$$\begin{bmatrix} -\frac{11}{5} & 1 \\ -\frac{12}{5} & -\frac{9}{5} \end{bmatrix} \delta X_{3,2} = \begin{bmatrix} -\frac{4}{25} \\ -\frac{2}{25} \end{bmatrix}$$

Using Gaussian Elimination,

$$\left[\begin{array}{cc|c} -\frac{11}{5} & 1 & -\frac{4}{25} \\ -\frac{12}{5} & -\frac{9}{5} & -\frac{2}{25} \end{array}\right]$$
$$\left[\begin{array}{cc|c} 1 & 0 & \frac{46}{795} \\ 0 & 1 & -\frac{26}{795} \end{array}\right]$$

This gives,

$$X_3 = \begin{bmatrix} -0.658 \\ 0.232 \end{bmatrix}$$