EM316 - Numerical Methods for EEE

Problem Sheet 1

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Question 1

For the 10-bit binary word,

Sign	sign of exponential	exponential	mantissa
1	1	4	4

Converting 0.02832 into binary gives 0.0000011101
The number is stored as,

$$1.1101 \times 2^{-6}$$

0	1	0110	1101

Converting back to decimal this gives,

$$(0.0000011101)_2 = 0.02832$$

Thus Absolute error,

$$\begin{aligned} abs_{error} &= |0.02832 - 0.02832| \\ &= 0.00000 \end{aligned}$$

Relative Error,

$$\begin{aligned} R.E &= \frac{|0.02832 - 0.02832|}{0.02832} \\ &= 0.000 < 0.0625 \end{aligned}$$

Thus, the relative error is less than the machine epsilon

Question 2

a. Consider,

$$cos(0.01) = 0.9999500004$$

This has a significant 9-digit accuracy.

Consider,

$$1 - cos(0.01) = 0.0000499996$$

Which has only a significant 5-digit accuracy. Thus when subtracting two nearly numbers, accuracy reduces.

b. Applying Taylor Series,

$$\begin{aligned} cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + T(x) \\ T(x) &= \frac{x^8}{8!} cos(\xi) \end{aligned}$$

Thus,

$$\begin{aligned} f(x) &= \frac{1}{x^2}\left[1 - \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + T(x)\right]\right] \\ &= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^6}{6!} - \frac{1}{x^2}T(x) \\ &= \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^6}{6!} - \frac{x^6}{8!} cos(\xi) \end{aligned}$$

For $|x| \leq 0.1$,

$$\left|\frac{x^6}{8!} cos(\xi)\right| \leq \frac{10^{-6}}{8!} = 2.5 \times 10^{-11}$$

Hence, with this accuracy,

$$f(x) \approx \frac{1}{2!} - \frac{x^2}{4!} + \frac{x^4}{6!}$$

Question 3

a. Consider,

$$x^3 + 4x^2 - 10 = 0$$

$$x = \left[10 - 4x^2\right]^{1/3}, \qquad x = \left[\frac{10 - x^3}{4}\right]^{1/2}, \quad x = x^3 + 4x^2 + x - 10, \quad x = \sqrt{\frac{10}{4+x}}$$

Let $g_1(x) = \left[\frac{10-x^3}{4}\right]^{1/2}$

Consider,

$$\begin{aligned} g_1'(x) &= -\frac{3x^2}{4\sqrt{10 - x^3}} \\ g_1'(1) &= -\frac{3(1)^2}{4\sqrt{10 - (1)^3}} \\ &= -0.25 \end{aligned}$$

Thus, $|g_1'(1)| < 1$, hence $g_1(x)$ is a guaranteed convergence function.

Let $g_2(x) = \sqrt{\frac{10}{4+x}}$

Consider,

$$\begin{aligned} g_2'(x) &= -\frac{\sqrt{10}}{2(4+x)^{3/2}} \\ g_2'(1) &= -\frac{\sqrt{10}}{2(4+1)^{3/2}} \\ &= 0.1414 \end{aligned}$$

Thus, $|g_2(1)| < 1$. Hence $g_2(x)$ is a guaranteed convergence function.

b. Error e_n at n^{th} iteration can be written as,

$$e_n \leq \frac{\lambda^n}{1-\lambda}|x_1 - x_0|$$

Consider $g_1(x)$,

$$\lambda = 0.25$$
$$x_1 = \left[\frac{10 - (1.5)^3}{4}\right]^{1/2}$$
$$x_1 = 1.2869537$$
$$|x_1 - x_0| = 0.2130463$$

Thus,

$$\lambda^n \geq (1-\lambda)\frac{10^{-4}}{|x_1 - x_0|}$$
$$0.25^n \geq 0.75\frac{10^{-4}}{0.2130463}$$
$$0.25^n \geq 3.52036 \times 10^{-4}$$
$$n \geq \frac{\ln 3.52036 \times 10^{-4}}{\ln 0.25}$$
$$n \geq 5.73$$
$$i.e. \;\; n = 6$$

c.

n	$g_1(x)$	tolarence	$g_2(x)$	tolarance
1	1.286954	0.213046	1.348400	0.151600
2	1.402541	0.115587	1.367376	0.018977
3	1.345458	0.057082	1.364957	0.002419

d. Using Newton's Raphson method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$= x_n - \frac{x_n^3 + 4x^2 - 10}{3x_n^2 + 8x}$$

n	x_n	tolerance
1	1.373333	0.126667
2	1.365262	0.008071
3	1.365230	0.000032

Thus, there is a significant fast convergence when using Newton's Raphson method

Question 4

Using Newton's Method of Raphson,

$$f(x) = x^2 - a$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{x_n^2 - a}{2x_n}$$
$$x_{n+1} = \frac{x_n^2 + a}{2x_n}$$
$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{a}{x_n}\right]$$

Let $a = 10$,

$$x_{n+1} = \frac{1}{2}\left[x_n + \frac{10}{x_n}\right]$$

Iteration	x	abs(error)
1	3.0000000000	0.162277660168
2	3.1666666667	0.004389006498
3	3.1622807018	0.000003041586
4	3.1622776602	0.000000000001

Question 7

a. From Newton's Method of Raphson,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{x_n^3 - 0.165x_n^2 - 3.993 \times 10^{-4}}{3x_n^2 - 0.33x_n}$$
$$x_{n+1} = \frac{2x_n^3 - 0.165x_n^2 + 3.993 \times 10^{-4}}{3x_n^2 - 0.33x_n}$$

Choose x_0 as 0.2,

$$x_1 = \frac{2(0.2)^3 - 0.165(0.2)^2 + 3.993 \times 10^{-4}}{3(0.2)^2 - 0.33(0.2)}$$
$$x_1 = 0.1814685185185$$
$$x_2 = \frac{2(0.1814685185185)^3 - 0.165(0.1814685185185)^2 + 3.993 \times 10^{-4}}{3(0.1814685185185)^2 - 0.33(0.1814685185185)}$$
$$x_2 = 0.177792606316553$$
$$x_3 = \frac{2(0.17779260631)^3 - 0.165(0.17779260631)^2 + 3.993 \times 10^{-4}}{3(0.17779260631)^2 - 0.33(0.17779260631)}$$
$$x_3 = 0.1776521994207141$$

n	x_n	error	significant digits
1	0.1814685185	0.0038163191	1
2	0.1777926063	0.0001404069	3
3	0.1776521994	0.0000000000	10

Question 8

Using Newton's Raphson Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_{n+1} = x_n - \frac{\left[0.05 - \frac{x}{20}\right] cos10x + \frac{1}{200} sin10x - 0.03}{-10sin10x\left[0.05 - \frac{x}{20}\right]}$$

Let's take $x_0 = 2$,

n	x	tolerance
1	2.010421	0.010421
2	2.006641	0.003780
3	2.007794	0.001153
4	2.007418	0.000376
5	2.007538	0.000120
6	2.007499	0.000039

Question 9

$$q(t) = q_0 e^{-Rt/2L} \cos\left(t\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right)$$

a. Consider,

$$f(r) = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}$$

For $f(r)$ to be real,

$$\frac{1}{LC} \geq \left(\frac{R}{2L}\right)^2$$
$$\frac{1}{5 \times 10^{-4}} \geq \left(\frac{R}{2 \times 5}\right)^2$$
$$R^2 \leq 20 \times 10^4$$
$$|R| \leq 447.21\Omega$$

Thus, an appropriate range for $\tilde{R}$ would be,

$$0 \leq \tilde{R} \leq 400$$

b. at $t = 0.05s$,

$$0.01q_0 = q_0 e^{-R0.05/10} cos\left(0.05\sqrt{2000 - 0.01R^2}\right)$$
$$0.01 = e^{-0.005R} cos\left(0.05\sqrt{2000 - 0.01R^2}\right)$$

Let $f(R) = 0.01 - e^{-0.005R} cos\left(0.05\sqrt{2000 - 0.01R^2}\right)$

Using the method of bisection,

n	R	$f(R)$
1	200.000000	0.163092
2	300.000000	0.029503
3	350.000000	-0.020915
4	325.000000	0.003155

Using the Newton's Method of Raphson,

$$f(R) = 0.01 - e^{-0.005R}cos\left(0.05\sqrt{2000 - 0.01R^2}\right)$$

$$f'(R) = 0.005e^{-0.005R}cos\left(0.05\sqrt{2000 - 0.01R^2}\right) -$$
$$e^{-0.005R}sin\left(0.05\sqrt{2000 - 0.01R^2}\right)0.05\frac{0.02R}{\sqrt{2000 - 0.01R^2}}$$

Consider,

$$R_{n+1} = R_n - \frac{f(R_n)}{f'(R_n)}$$

n	R_n	$f(R_n)$
1	366.986497	-0.063122
2	348.611594	-0.036088
3	338.784970	-0.019634
4	333.639055	-0.010384

Considering the first 4 iterations, the bisect method has a faster convergence than Newton's method of Raphson

c.