EE352 - AUTOMATIC CONTROL

Week 2 Activity 2

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Consider,

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + ... + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + b_{m-2}\frac{d^{m-2}u(t)}{dt^{m-2}} + ... + b_1\frac{du(t)}{dt} + b_0 u(t)$$

Assuming all zero initial conditions by taking Laplace Transformations from both sides,

$$(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0)Y(s)$$
$$= (b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + ... + b_1 s + b_0)U(s)$$

This yields,

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0}$$

This can also be represented as,

$$\frac{W(s)}{U(s)}\frac{Y(s)}{W(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + b_{m-2}s^{m-2} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0}$$

Where,

$$\frac{W(s)}{U(s)} = \frac{1}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0}$$
$$(s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + ... + a_1 s + a_0)W(s) = U(s)$$

Converting back to the time domain this gives,

$$\overset{n}{\dot{w}} = -a_{n-1}\overset{n-1}{\dot{w}} - a_{n-2}\overset{n-2}{\dot{w}} - ... - a_1\dot{w} - a_0 w + u$$

Similarly, we can show that,

$$y = b_m \overset{m}{\dot{w}} + b_{m-1} \overset{m-1}{\dot{w}} + b_{m-2} \overset{m-2}{\dot{w}} + ... + b_1 \dot{w} + b_0 w$$

Let's select the states as,

$$\begin{aligned} x_1 &= w \\ x_2 &= \dot{w} \\ \vdots \quad & \quad \vdots \\ x_{n-2} &= \overset{n-2}{\dot{w}} \\ x_{n-1} &= \overset{n-1}{\dot{w}} \end{aligned}$$

Hence, the state equations are,

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ \vdots \quad & \quad \vdots \\ \dot{x}_{n-2} &= x_{n-1} \\ \dot{x}_{n-1} = \overset{n}{\dot{w}} &= -a_{n-1} \overset{n-1}{\dot{w}} - a_{n-2} \overset{n-2}{\dot{w}} - ... - a_1 \dot{w} - a_0 w + u \end{aligned}$$

This yields the state space system,

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \vdots \\ \dot{x}_{n-2} \\ \dot{x}_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-2} \\ x_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u$$

For $y(t)$,

$$y = \begin{bmatrix} b_0 & b_1 & b_2 & \dots & b_{m-1} & b_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{m-1} \\ x_m \end{bmatrix}$$