EM316 - Numerical Methods for EEE

Problem Sheet 2

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1

Converting to Augmented Matrix,

$$\left[\begin{array}{cc|c} 0.004 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right]$$

Using Partial Pivoting,

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0.004 & 1 & 1 \end{array}\right]$$

Applying $R_2 - R_1 \times 0.004 \rightarrow R_2$ and two digit rounding,

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0.996 & 0.992 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 0.99 \end{array}\right]$$

This gives,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.01 \\ 0.99 \end{bmatrix}$$

2

a

Converting to Augmented Matrix,

$$\left[\begin{array}{cc|c} 0.780 & 0.563 & 0.217 \\ 0.913 & 0.659 & 0.254 \end{array}\right]$$

Using Partial Pivoting,

$$\left[\begin{array}{cc|c} 0.913 & 0.659 & 0.254 \\ 0.780 & 0.563 & 0.217 \end{array}\right]$$

Applying $R_2 - R_1 \times \frac{0.780}{0.914} \to R_2$

$$\left[\begin{array}{cc|c} 0.913 & 0.659 & 0.254 \\ 0 & 0.001 & 0.001 \end{array}\right]$$

This gives,

$$\begin{aligned} x_2 &= 1.000 \\ x_1 &= \frac{0.254 - 0.659}{0.913} \\ &= -0.443 \end{aligned}$$

$$x^* = (-0.443 \quad 1.000)^T$$

b

Residual r,

$$\begin{aligned} r &= b - Ax^* \\ r &= \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix} - \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \begin{bmatrix} -0.443 \\ 1.000 \end{bmatrix} \\ r &= \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix} - \begin{bmatrix} 0.21746 \\ 0.25454 \end{bmatrix} \\ r &= \begin{bmatrix} -0.000460 \\ -0.000541 \end{bmatrix} \end{aligned}$$

c

error e,

$$\begin{aligned} e &= x - x^* \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -0.443 \\ 1.000 \end{bmatrix} \\ &= \begin{bmatrix} 1.443 \\ -2.000 \end{bmatrix} \end{aligned}$$

Thus, even though the residual value is very low error is much larger. So the residual does not provide a good measure to determine the accuracy in this case.

d

Consider $AA^T = H$,

$$AA^T = \begin{bmatrix} 0.780 & 0.563 \\ 0.913 & 0.659 \end{bmatrix} \begin{bmatrix} 0.780 & 0.913 \\ 0.563 & 0.659 \end{bmatrix}$$
$$H = \begin{bmatrix} 0.925369 & 1.083157 \\ 1.083157 & 1.26785 \end{bmatrix}$$

Consider $H - \lambda I$,

$$H - \lambda I = \begin{bmatrix} 0.925369 - \lambda & 1.083157 \\ 1.083157 & 1.26785 - \lambda \end{bmatrix}$$
$$det(H - \lambda I) = \lambda^2 - 2.193219\lambda + 1.000 \times 10^{-12} = 0$$

For Eigenvalues,

$$det(H - \lambda I) = 0$$
$$\lambda^2 - 2.193219\lambda + 1.000 \times 10^{-12} = 0$$
$$\lambda_1 = 4.56092595033819 \times 10^{-13} \qquad \lambda_1 = 2.19321899999954$$

Thus singular values of the coefficient matrix A are,

$$\sigma_1 = \sqrt{\lambda_1}$$
$$= 1.48095205864320$$
$$\sigma_2 = \sqrt{\lambda_2}$$
$$= 0.000000675346$$

e

For condition number $K(A)$,

$$K(A) = ||A||\ ||A^{-1}||$$
$$||A||_2 = max(\sigma_1, \sigma_2)$$
$$||A^{-1}|| = \frac{1}{min(\sigma_1, \sigma_2)}$$

Thus, 2- norm condition number,

$$\begin{aligned}
K(A) &= \frac{\sigma_1}{\sigma_2} \\
&= \frac{1.48095205864320}{0.000000675346} \\
&= 2.1921899 \times 10^6
\end{aligned}$$

Thus, the condition number is very large, hence the system is ill-conditioned.

f

For $||r||_2$,

$$\begin{aligned}
r &= \begin{bmatrix} -0.00046 \\ -0.000541 \end{bmatrix} \\
||r||_2 &= \sqrt{0.00046^2 + 0.000541^2} \\
&= 0.000710
\end{aligned}$$

For $||b||_2$,

$$\begin{aligned}
b &= \begin{bmatrix} 0.217 \\ 0.254 \end{bmatrix} \\
||b||_2 &= \sqrt{0.217^2 + 0.254^2} \\
&= 0.33407
\end{aligned}$$

Substituting to the relationship,

$$\begin{aligned}
\frac{||r||}{||b||K(A)} &\leq \frac{||x - x^*||}{||x||} \leq K(A)\frac{||r||}{||b||} \\
\frac{0.000710}{0.3341 \times 2.19 \times 10^6} &\leq \frac{||x - x^*||}{||x||} \leq 2.19 \times 10^6 \frac{0.000710}{0.3341} \\
9.694015 \times 10^{-10} &\leq \frac{||x - x^*||}{||x||} \leq 4658.649
\end{aligned}$$

g

For relative error,

$$\begin{aligned}
x - x^* &= \begin{bmatrix} 1.443 \\ -2 \end{bmatrix} \\
||x - x^*||_2 &= \sqrt{1.443^2 + 2^2} \\
&= 2.466221 \\
||x||_2 &= \sqrt{1^2 + 1^2} \\
&= 1.41421
\end{aligned}$$

This gives the relative error,

$$\begin{aligned}
\frac{||x - x^*||}{||x||} &= \frac{2.466221}{1.41421} \\
&= 1.743882
\end{aligned}$$

Since the estimations of relative error ranges over 13 orders of magnitude it is totally unreliable to rely on the residual.

4

a

Considering,

$$\begin{aligned}
K(A) &= ||A||\ ||A^{-1}|| \\
K(\alpha A) &= ||\alpha A||\ ||(\alpha A)^{-1}|| \\
K(\alpha A) &= ||\alpha||\ ||A||\ \frac{||A^{-1}||}{||\alpha||} \\
K(\alpha A) &= ||A||\ ||A^{-1}||
\end{aligned}$$

Thus,

$$K(\alpha A) = K(A)$$

b

Consider diagonal matrix A with elements $a_1, a_2, \ldots, a_n$

$$||A|| = max(a_i)$$
$$||A^{-1}|| = \frac{1}{min(a_i)}$$

Thus,

$$\begin{aligned} K(A) &= ||A||\ ||A^{-1}|| \\ &= \frac{max(a_i)}{min(a_i)} \end{aligned}$$

c

For a orthogonal matrix A,

$$AA^T = A^T A = I$$

Thus all the singular values of A are 1 and hence,

$$max(\sigma_i) = 1$$
$$min(\sigma_i) = 1$$

Thus,

$$\begin{aligned} K(A) &= ||A||\ ||A^{-1}|| \\ &= \frac{max(\sigma_i)}{\min(\sigma_i)} \\ &= 1 \end{aligned}$$

Since ratio of largest to smallest singular values is minimum, they are well-behaved in computations.

5

Given that,

$$Ax = b \tag{1}$$
$$A(x + \delta x) = b + \delta b \tag{2}$$

Using 1 and 2,

$$
\begin{aligned}
A\delta x &= \delta b \\
\delta x &= A^{-1}\delta b
\end{aligned}
$$

Taking norms of vectors and matrices,

$$
\begin{aligned}
||\delta x|| &= ||A^{-1}\delta b|| \\
||A^{-1}\delta b|| &\leq ||A^{-1}||\ ||\delta b|| \\
&\leq ||A^{-1}||\ ||b||\ \frac{||\delta b||}{||b||}
\end{aligned}
$$

Since $Ax = b$,

$$
\begin{aligned}
||\delta x|| \leq ||A^{-1}||\ ||b||\ \frac{||\delta b||}{||b||} &= ||A^{-1}||\ ||Ax||\ \frac{||\delta b||}{||b||} \\
||A^{-1}||\ ||Ax||\ \frac{||\delta b||}{||b||} &\leq ||A^{-1}||\ ||A||\ ||x||\ \frac{||\delta b||}{||b||}
\end{aligned}
$$

Thus, this gives,

$$
\begin{aligned}
||\delta x|| &\leq ||A^{-1}||\ ||A||\ ||x||\ \frac{||\delta b||}{||b||} \\
\frac{||\delta x||}{||x||} &\leq ||A^{-1}||\ ||A||\ \frac{||\delta b||}{||b||} \\
\frac{||\delta x||}{||x||} &\leq K(A)\frac{||\delta b||}{||b||}
\end{aligned}
$$

6

a

Given system,

$$
\begin{bmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}
$$

Let's take L, U, D as,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -7 \end{bmatrix}$$

For Jacobi Matrix B,

$$\begin{aligned} Ax &= b \\ (L+U+D)x &= b \\ Dx &= -(L+U)x + b \\ x &= -D^{-1}(L+U)x + D^{-1}b \\ x &= Bx + D^{-1}b \end{aligned}$$

Thus,

$$B = \begin{bmatrix} \frac{-1}{5} & 0 & 0 \\ 0 & \frac{-1}{9} & 0 \\ 0 & 0 & \frac{1}{7} \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ -3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{5} & \frac{-3}{5} \\ \frac{1}{3} & 0 & \frac{-1}{9} \\ \frac{2}{7} & \frac{-1}{7} & 0 \end{bmatrix}$$

For Gauss Seidel Matrix G,

$$\begin{aligned} Ax &= b \\ (L+U+D)x &= b \\ (D+L)x &= -Ux + b \\ x &= -(D+L)^{-1}Ux + (D+L)^{-1}b \\ x &= Gx + (D+L)^{-1}b \end{aligned}$$

Thus,

$$G = \begin{bmatrix} 5 & 0 & 0 \\ -3 & 9 & 0 \\ 2 & -1 & -7 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{15} & \frac{1}{9} & 0 \\ \frac{1}{21} & \frac{-1}{63} & \frac{-1}{7} \end{bmatrix} \begin{bmatrix} 0 & 2 & -3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{2}{5} & \frac{-3}{5} \\ 0 & \frac{2}{15} & \frac{-14}{45} \\ 0 & \frac{2}{21} & \frac{-8}{63} \end{bmatrix}$$

b

Using B we can obtain the recurrence equation as,

$$x_1^{k+1} = \frac{1}{5}(-1 + 2x_2^k - 3x_3^k)$$
$$x_2^{k+1} = \frac{1}{9}(2 + 3x_1^k - x_3^k)$$
$$x_3^{k+1} = \frac{-1}{7}(3 - 2x_1^k + x_2^k)$$

k	x_1	Δx_1	x_2	Δx_2	x_3	Δx_3
1	-0.200	-0.200	0.222	0.222	-0.429	-0.429
2	0.146	0.346	0.203	-0.019	-0.517	-0.089
3	0.192	0.046	0.328	0.125	-0.416	0.102
4	0.181	-0.011	0.332	0.004	-0.421	-0.005
5	0.185	0.004	0.329	-0.003	-0.424	-0.004
6	0.186	0.001	0.331	0.002	-0.423	0.002
7	0.186	-0.000	0.331	0.000	-0.423	0.000

c

Considering G,

$$(D+L)^{-1}b = \begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{1}{15} & \frac{1}{9} & 0 \\ \frac{1}{21} & \frac{-1}{63} & \frac{-1}{7} \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-1}{5} \\ \frac{7}{45} \\ \frac{-32}{63} \end{bmatrix}$$

This gives the recurrence relationship as,

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{2}{5} & \frac{-3}{5} \\ 0 & \frac{2}{15} & \frac{-14}{45} \\ 0 & \frac{2}{21} & \frac{-8}{63} \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix} + \begin{bmatrix} \frac{-1}{5} \\ \frac{7}{45} \\ \frac{-32}{63} \end{bmatrix}$$

k	x_1	Δx_1	x_2	Δx_2	x_3	Δx_3
1	-0.200	-0.200	0.156	0.156	-0.508	-0.508
2	0.167	0.367	0.334	0.179	-0.429	0.079
3	0.191	0.024	0.333	-0.001	-0.422	0.007
4	0.186	-0.005	0.331	-0.002	-0.423	-0.001
5	0.186	-0.000	0.331	-0.000	-0.423	-0.000

7

a

$$\begin{bmatrix} 5 & 2 & 1 \\ 4 & 11 & 15 \\ 7 & 8 & 16 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 20 \\ 31 \end{bmatrix}$$

Taking L, U, D as,

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 7 & 8 & 0 \end{bmatrix} \quad U = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 16 \end{bmatrix}$$

For Jacobi Matrix B,

$$\begin{aligned}
Ax &= b \\
(L+U+D)x &= b \\
Dx &= -(L+U)x + b \\
x &= -D^{-1}(L+U)x + D^{-1}b \\
x &= Bx + D^{-1}b
\end{aligned}$$

Thus,

$$\begin{aligned}
B &= \begin{bmatrix} \frac{-1}{5} & 0 & 0 \\ 0 & \frac{-1}{11} & 0 \\ 0 & 0 & \frac{-1}{16} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 4 & 0 & 15 \\ 7 & 8 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{-2}{5} & \frac{-1}{5} \\ \frac{-4}{11} & 0 & \frac{-15}{11} \\ \frac{-7}{16} & \frac{-8}{16} & 0 \end{bmatrix}
\end{aligned}$$

For Gauss Seidel Matrix G,

$$\begin{aligned}
(L+U+D)x &= b \\
(L+D)x &= -Ux + b \\
x &= -(L+D)^{-1}Ux + (L+D)^{-1}b \\
x &= Gx + (L+D)^{-1}b
\end{aligned}$$

Thus,

$$\begin{aligned}
G &= -\begin{bmatrix} 5 & 0 & 0 \\ 4 & 11 & 0 \\ 7 & 8 & 16 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix} \\
&= -\begin{bmatrix} \frac{1}{5} & 0 & 0 \\ \frac{-4}{11} & \frac{1}{11} & 0 \\ \frac{-9}{176} & \frac{-8}{176} & \frac{1}{16} \end{bmatrix} \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{-2}{5} & \frac{-1}{5} \\ 0 & \frac{8}{11} & -1 \\ 0 & \frac{18}{176} & \frac{129}{176} \end{bmatrix}
\end{aligned}$$

b

Considering the rows of the coefficient matrix A,

$$\begin{aligned} 1: &\quad |5| > |2| + |1| \\ 2: &\quad |11| < |15| + |4| \end{aligned}$$

Thus, in the second row the diagonal element is less than the sum of other elements in that row. Hence the matrix A is not strictly diagonally dominant and the convergence is not guaranteed.

c

Considering error bound,

$$||x - x^k||_\infty \leq \frac{||B||_\infty^k}{(1 - ||B||_\infty)} ||x^1 - x^0||_\infty$$

Considering $(1 - ||B||_\infty) = \frac{-8}{11}$, gives a negative value hence the error is not bounded thus we cant say an exact k value where the error is less than 10^{-4}

d

Using Jacobi Method,

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-2}{5} & \frac{-1}{5} \\ \frac{-4}{11} & 0 & \frac{-15}{11} \\ \frac{-7}{16} & \frac{-8}{16} & 0 \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix} + \begin{bmatrix} \frac{8}{5} \\ \frac{20}{11} \\ \frac{31}{16} \end{bmatrix}$$

k	x_1	Δx_1	x_2	Δx_2	x_3	Δx_3
1	1.3000	0.8000	0.9545	0.4545	1.4688	0.9688
2	0.9244	-0.3756	-0.6574	-1.6119	0.8915	-0.5773

Using Gauss Siedel Method,

$$(L+D)^{-1}b = \begin{bmatrix} \frac{-1}{5} & 0 & 0 \\ \frac{4}{11} & \frac{-1}{11} & 0 \\ \frac{9}{176} & \frac{8}{176} & \frac{-1}{16} \end{bmatrix} \begin{bmatrix} 8 \\ 20 \\ 31 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{-8}{5} \\ \frac{12}{11} \\ \frac{-109}{176} \end{bmatrix}$$

This gives,

$$\begin{bmatrix} x_1^{k+1} \\ x_2^{k+1} \\ x_3^{k+1} \end{bmatrix} = \begin{bmatrix} 0 & \frac{-2}{5} & \frac{-1}{5} \\ 0 & \frac{8}{11} & -1 \\ 0 & \frac{18}{176} & \frac{129}{176} \end{bmatrix} \begin{bmatrix} x_1^k \\ x_2^k \\ x_3^k \end{bmatrix} + \begin{bmatrix} \frac{-8}{5} \\ \frac{12}{11} \\ \frac{-109}{176} \end{bmatrix}$$

k	x_1	Δx_1	x_2	Δx_2	x_3	Δx_3
1	1.3000	0.8000	0.6636	0.1636	1.0369	0.5369
2	1.1272	-0.1728	-0.0057	-0.6693	1.4472	0.4103