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E/19/445

Assignement 2

Libraries

In [3]:
```
import numpy as np
import matplotlib.pyplot as plt
from sympy import *
from scipy import signal
from matplotlib import style
%matplotlib inline
```

In [54]:
```
class buddhi:
  def coswave(self):
    x=np.arange(-1.5,1.5,0.001)
    y=[]
    for val in x:
      if -0.5<val<0.5:
        y.append(np.cos(2*np.pi*val)+1)
      else:
        y.append(0)

    plt.ylim(-0.2,2.5)
    plt.xlim(-1.5,1.5)
    plt.axhline(y=0, color='black',linewidth = 0.5)
    plt.axvline(x=0, color='black',linewidth = 0.5)
    plt.plot(x,y)


  def repeatedcos(self):
    x=[]
    y=[]
    for i in range(-4,5,2):
      xx=np.arange(-1.5+i,1.5+i,0.001)
      for val in xx:
        x.append(val)
        if -0.5+i<val<0.5+i:
          y.append(np.cos(2*np.pi*val)+1)
        else:
          y.append(0)
    plt.ylim(-0.2,2.5)
    plt.xlim(-4,4)
    plt.axhline(y=0, color='black',linewidth = 0.5)
    plt.axvline(x=0, color='black',linewidth = 0.5)
    plt.plot(x,y)

  def square(self,shift,scale):
    y_values=[0,0,1,1,1,0,0]
    arr=[-2,-1,-1,0,1,1,2]
    x_values=[(t-shift)/scale for t in arr]
    plt.ylim(-0.2,2)
    plt.xlim(-3,3)
    plt.axhline(y=0, color='black',linewidth = 0.5)
    plt.axvline(x=0, color='black',linewidth = 0.5)
```

```
    plt.plot(x_values,y_values)

  def add2waves(self):
    y_values=[0,0,0,2,2,1,1,1,0]
    x_values=[-2,-1,0,0,1,1,1,2,2]
    plt.ylim(-0.2,2.5)
    plt.xlim(-3,3)
    plt.axhline(y=0, color='black',linewidth = 0.5)
    plt.axvline(x=0, color='black',linewidth = 0.5)
    plt.plot(x_values,y_values)

wave=buddhi()
```

Question 2.

$$g(t) = \begin{cases} 1 + cos(2\pi t) & for|t| \leq 0.5 \\ 0 & elsewhere \end{cases} \tag{1}$$

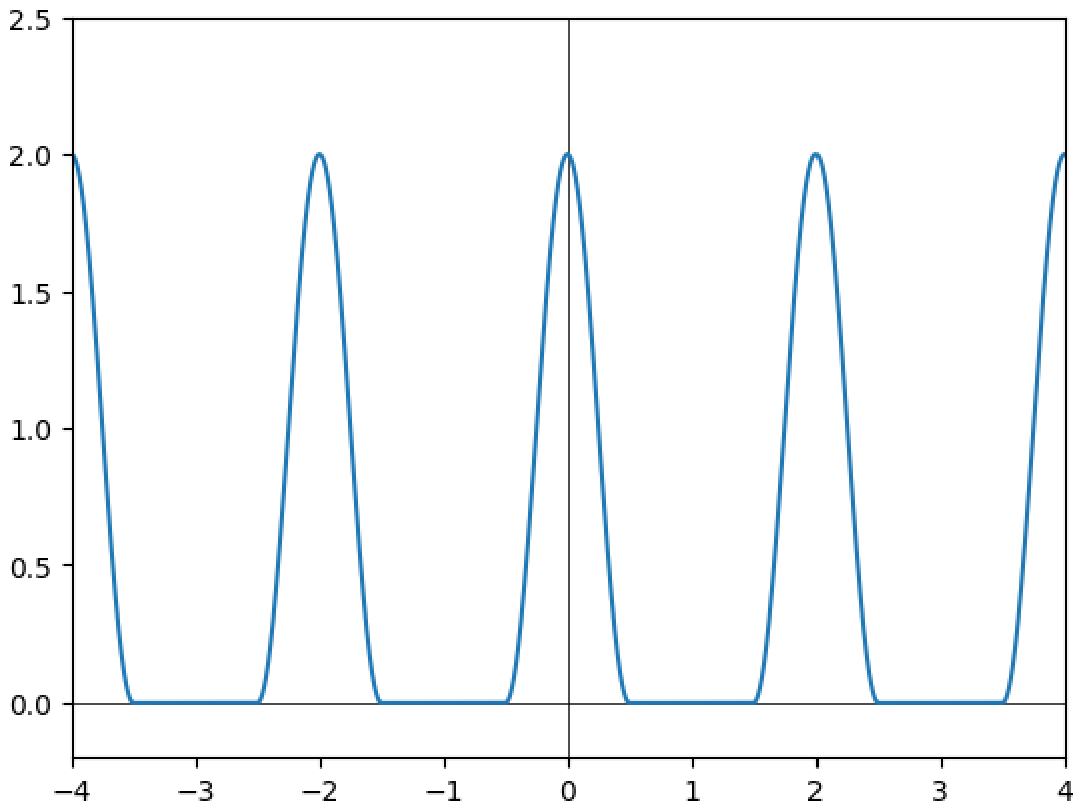
$$x(t) = \sum_{k=-\infty}^{\infty} g(t-2k) \tag{2}$$

Using given data,

$$T_0 = 2 \tag{3}$$

$$\omega_0 = \pi \tag{4}$$

In [28]:
```
wave.repeatedcos()
```

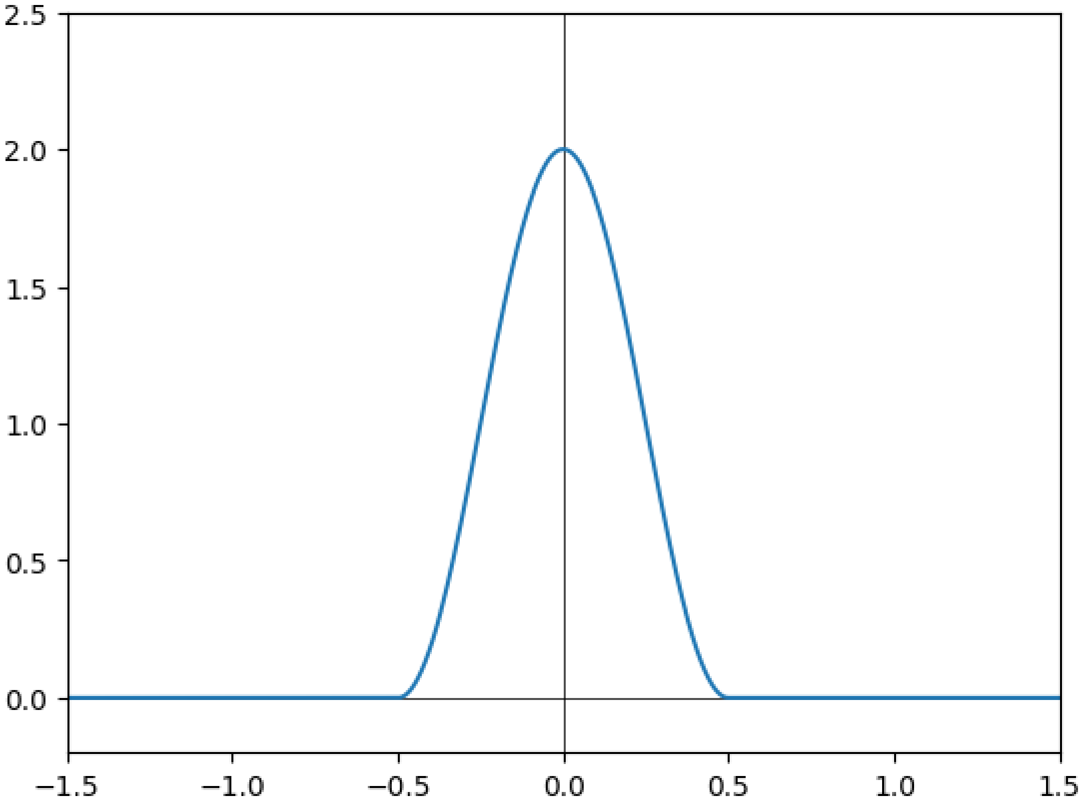


Since the signal is an even signal the Fourier Series of $x(t)$,

$$x(t) = a_0 + \sum_{k=0}^{\infty} 2a_k cos(k\omega_0 t) \tag{5}$$

where a_k is the Fourier Coeffient

In [19]:
```
wave.coswave()
```



$$
\begin{aligned}
a_k &= \frac{1}{T_0}\int_{T_0} x(t)cos(k\omega_0 t)dt && (6)\\
&= \frac{1}{T_0}\int_{-0.5}^{0.5} (1+cos(2\pi t))cos(k\omega_0 t)dt && (7)\\
&= \frac{1}{2}\int_{-0.5}^{0.5} (1+cos(2\pi t))cos(k\pi t)dt && (8)
\end{aligned}
$$

For $k = 0$,

$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-0.5}^{0.5} (1+cos(2\pi t))cos(0)dt && (9)\\
&= \frac{1}{2}\int_{-0.5}^{0.5} (1+cos(2\pi t))dt && (10)\\
&= \frac{1}{2}[t+\frac{sin(2\pi t)}{2\pi}]|_{-0.5}^{0.5} && (11)\\
&= \frac{1}{2} && (12)
\end{aligned}
$$

For $k = 1$,

$$
\begin{aligned}
a_1 &= \frac{1}{2}\int_{-0.5}^{0.5}(1+cos(2\pi t))cos(\pi t)dt && (13)\\
&= \frac{1}{2}\int_{-0.5}^{0.5} cos(\pi t)dt + \frac{1}{2}\int_{-0.5}^{0.5} cos(2\pi t)cos(\pi t)dt && (14)\\
&= \frac{1}{2}[sin\frac{\pi}{2} + sin\frac{\pi}{2}] + \frac{1}{4}\int_{-0.5}^{0.5}[cos(3\pi t)+cos(\pi t)]dt && (15)\\
&= 1 + \frac{1}{4}\int_{-0.5}^{0.5} cos(3\pi t)dt + \int_{-0.5}^{0.5} cos(\pi t)dt && (16)\\
&= 1 + \frac{1}{2}sin(\frac{3\pi}{2}) + \frac{1}{2}sin(\frac{\pi}{2}) && (17)\\
&= 1 && (18)
\end{aligned}
$$

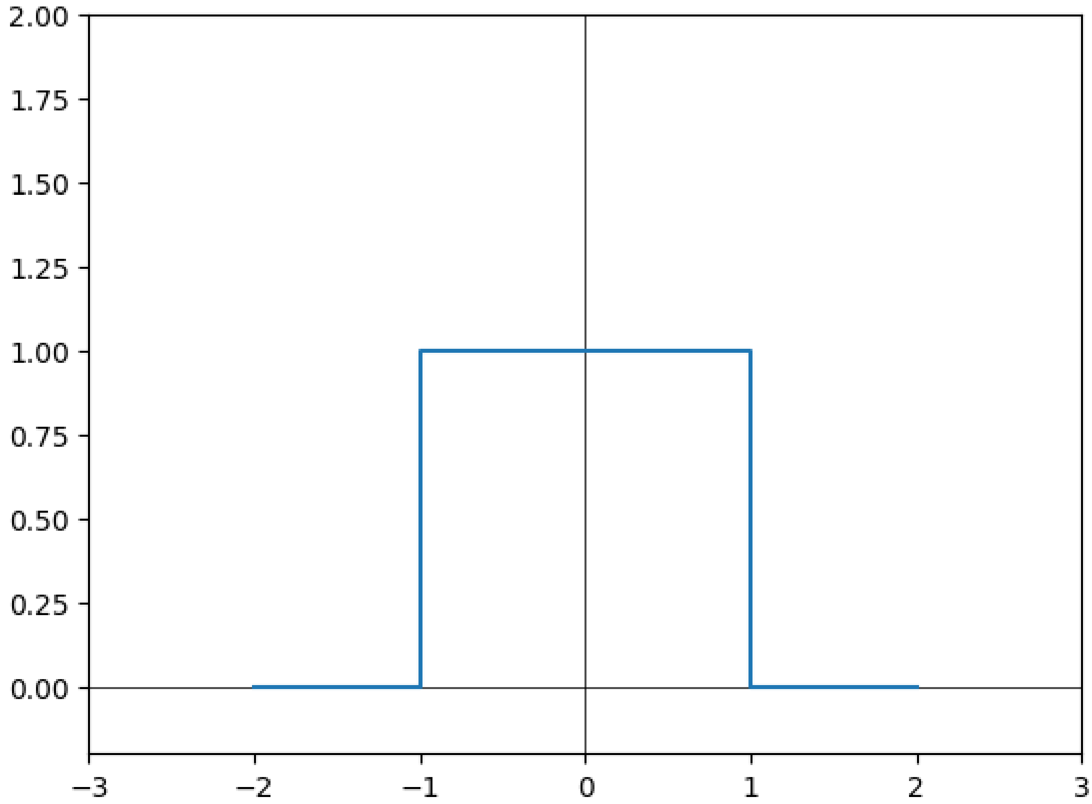
For $k = 2$,

$$
\begin{aligned}
a_2 &= \frac{1}{2}\int_{-0.5}^{0.5}(1+cos(2\pi t))cos(2\pi t)dt && (19)\\
&= \frac{1}{2}\int_{-0.5}^{0.5} cos(2\pi t)dt + \frac{1}{2}\int_{-0.5}^{0.5} cos^2(2\pi t)dt && (20)\\
&= \frac{1}{4}\int_{-0.5}^{0.5}[1+cos(4\pi t)]dt && (21)\\
&= \frac{1}{4} && (22)
\end{aligned}
$$

Question 4

$$
x(t) = \begin{cases} 1 & for \quad |t| < T_1 \\ 0 & for \quad T_1 < |t| < T/2 \end{cases} \tag{23}
$$

In [42]:
```
wave.square(0,1)
```



a. For X_0,

$$\begin{aligned} X_0 &= \frac{1}{T}\int_T x(t)dt && (24)\\ &= \frac{1}{T}\int_{-T_1}^{T_1} dt && (25)\\ &= \frac{2T_1}{T} && (26) \end{aligned}$$

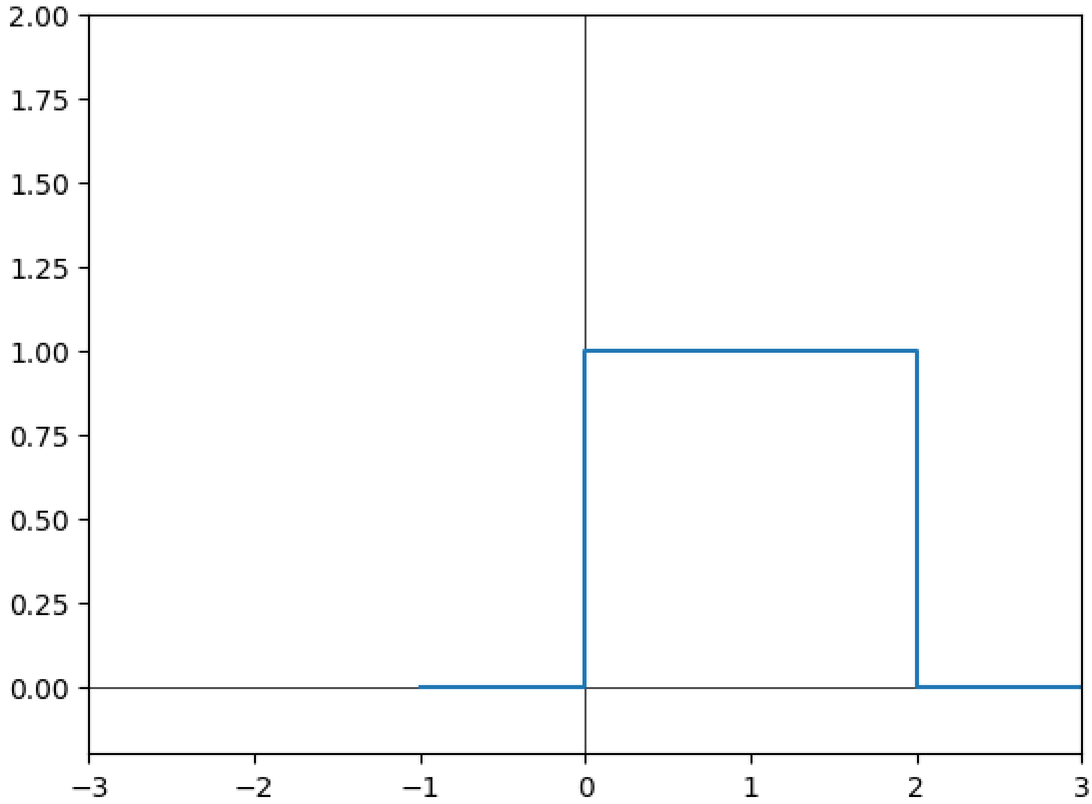
For X_k,

$$\begin{aligned} X_k &= \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt && (27)\\ &= \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt && (28)\\ &= \frac{1}{\pi k}[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}] && (29)\\ &= \frac{1}{\pi k} sin(k\omega_0 T_1) && (30) \end{aligned}$$

b.

$$y(t) = \begin{cases} 1 & for \quad 0 \leq \ t < T_1 \\ 0 & for \quad 2 \leq \ t < 4 \end{cases} \qquad (31)$$

In [43]:
```
wave.square(-1,1)
```



Lets take $T_1 = 1$ and $T = 4$,

Thus,

$$\omega_0 = \frac{\pi}{2} \tag{32}$$

This shows,

$$y(t) = x(t-1) \tag{33}$$
$$x(t) \longrightarrow X_k \tag{34}$$
$$y(t) \longrightarrow X_k e^{-jk\omega_0} \tag{35}$$

Thus,

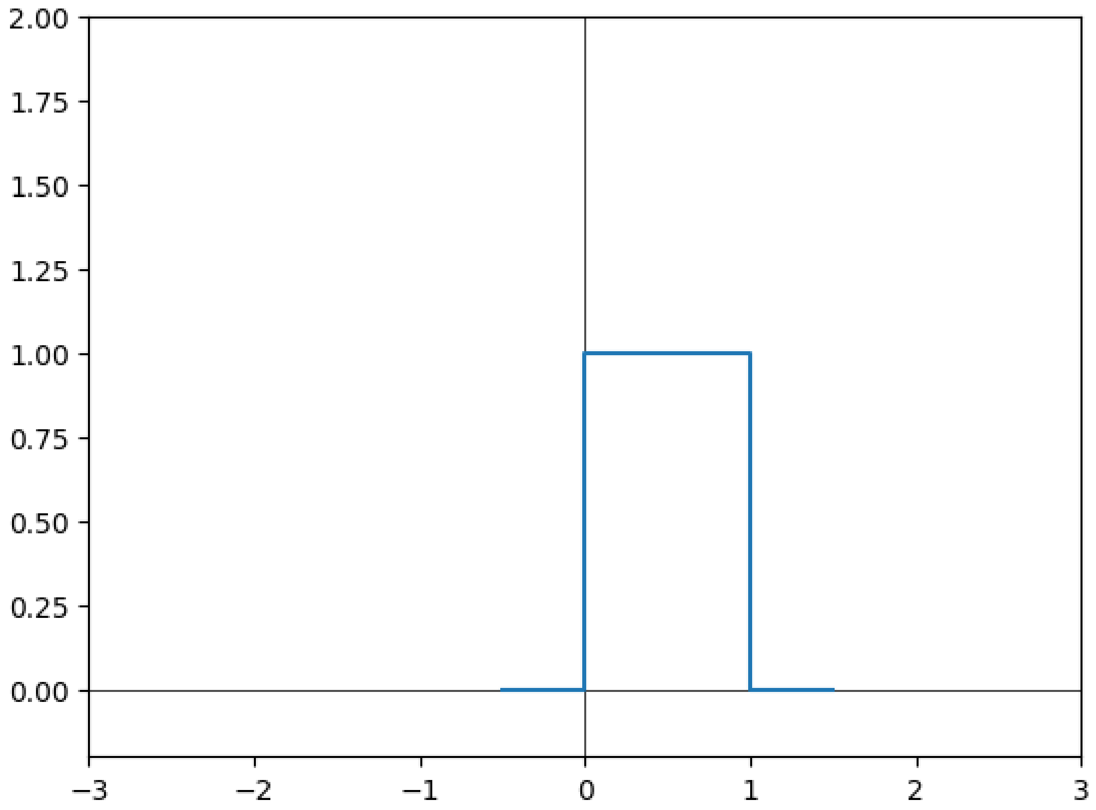
$$Y_k = \frac{1}{\pi k} sin(k\omega_0 T_1) e^{-jk\omega_0} \tag{36}$$
$$= \frac{1}{\pi k} sin(\frac{k\pi}{2}) e^{\frac{-jk\pi}{2}} \tag{37}$$
$$\tag{38}$$
$$Y_0 = \frac{1}{2} \tag{39}$$

Hence the Fourier Series of $y(t)$ can be written as,

$$y(t) = Y_0 + \sum_{\forall k \neq 0} Y_k e^{jk\omega_0 t} \tag{40}$$
$$= \frac{1}{2} + \sum_{\forall k \neq 0} \frac{1}{\pi k} sin(\frac{k\pi}{2}) e^{\frac{-jk\pi}{2}} e^{\frac{jk\pi t}{2}} \tag{41}$$
$$= \frac{1}{2} + \sum_{\forall k \neq 0} \frac{1}{\pi k} sin(\frac{k\pi}{2}) e^{\frac{jk\pi(t-1)}{2}} \tag{42}$$

d. $g(t) = y(2t)$

In [44]:
```
wave.square(-1,2)
```



By Fourier Series property, \ New fundamental frequency $= 2 \times \frac{\pi}{2} = \pi$

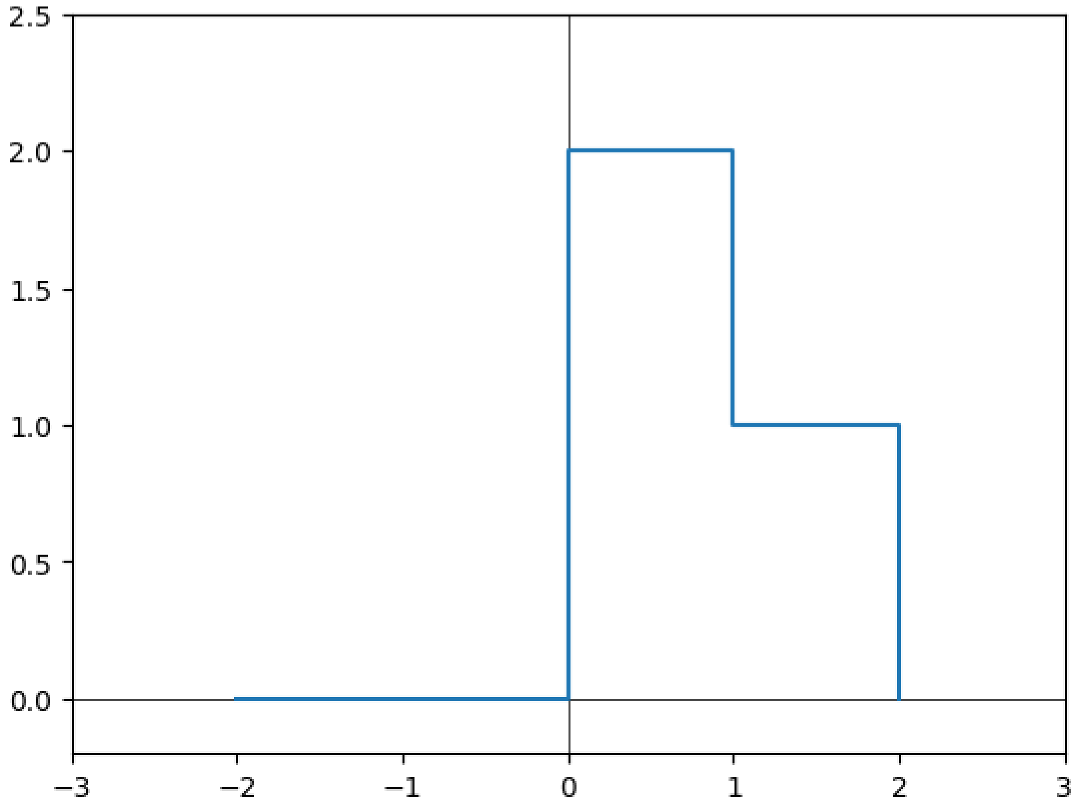
Fourier Series coefficients do not change due to scaling,

Thus,

$$G_k = Y_k \tag{43}$$
$$= \frac{1}{\pi k} sin(\frac{k\pi}{2})e^{\frac{-jk\pi}{2}} \tag{44}$$

e. $m(t) = y(t) + g(t)$

In [55]:
```
wave.add2waves()
```



since the fundermental frequencies are different we can't just add them but,

$$g(t) = \sum_{\forall k} G_k e^{jk\pi t} \tag{45}$$

$$= \sum_{\forall k} G_k e^{j(2k)\frac{\pi}{2}t} \tag{46}$$

Considering obtained formulea it has values when only k is even.

Thus,

$$M_k = \begin{cases} Y_k + G_k & for \quad even \quad k \\ Y_k & elsewhere \end{cases} \tag{47}$$

Question 5

a. For $x'(t)$,

$$x'(t) = \begin{cases} 0 & t < -\frac{1}{2} \\ 1 & -\frac{1}{2} \leq t \leq \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases} \tag{48}$$

Fourier Transform of $x'(t)$,

$$\mathcal{F}x'(t) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{49}$$

$$X'(jw) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t}dt \tag{50}$$

$$= \frac{e^{j\frac{w}{2}} - e^{-j\frac{w}{2}}}{jw} \tag{51}$$

$$= 2\frac{sin(\frac{w}{2})}{w} \tag{52}$$

b. Thus the Fourier Transform of $x(t)$,

$$\mathcal{F}x(t) = \frac{X'(jw)}{jw} \tag{53}$$

$$X(jw) = -2j\frac{sin(\frac{w}{2})}{w^2} \tag{54}$$

c. Fourier Transform of $y'(t)$,

$$\mathcal{F}y'(t) = \int_{-\infty}^{\infty} y(t)e^{-j\omega t}dt \tag{55}$$

$$Y'(jw) = \int_{-\frac{1}{2}}^{\frac{1}{2}^-} e^{-j\omega t}dt - \frac{1}{2}\int_{\frac{1}{2}^-}^{\frac{1}{2}^+} \delta(t - \frac{1}{2})e^{-j\omega t}dt \tag{56}$$

$$= \frac{e^{j\omega} - e^{-j\omega}}{jw} - \frac{1}{2}e^{-jw/2} \tag{57}$$

$$= 2\frac{sinw}{w} - \frac{1}{2}e^{-jw/2} \tag{58}$$

d. Thus the Fourier Transform of $y(t)$,

$$\mathcal{F}y(t) = \frac{Y'(jw)}{jw} \tag{59}$$

$$Y(jw) = -2j\frac{sin(\frac{w}{2})}{w^2} + \frac{j}{2}e^{-jw/2} \tag{60}$$

e.

$$\mathcal{F}g(t) = \mathcal{F}x(t) - \frac{1}{2}\mathcal{F}(1) \tag{61}$$

$$G(jw) = X(jw) - \frac{1}{2}2\pi\delta(w) \tag{62}$$

$$= -2j\frac{sin(\frac{w}{2})}{w^2} - \pi\delta(w) \tag{63}$$

Question 6

a.

$$\mathcal{F}x(t) = X(jw) \tag{64}$$
$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{65}$$

By diffrentiating with respect to ω,

$$\frac{d}{d\omega}X(jw) = \int_{-\infty}^{\infty} x(t)\frac{d}{d\omega}e^{-j\omega t}dt \tag{66}$$
$$= \int_{-\infty}^{\infty} x(t)(-jt)e^{-j\omega t}dt \tag{67}$$
$$\frac{d}{d\omega}X(jw) = \mathcal{F}x(t)(-jt) \tag{68}$$
$$\mathcal{F}^{-1}[\frac{d}{d\omega}X(jw)] = (-jt)x(t) \tag{69}$$

b.

$$X(jw) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{70}$$
$$= \int_{-\infty}^{\infty} e^{-|t|}e^{-j\omega t}dt \tag{71}$$
$$= \int_{-\infty}^{0} e^{t}e^{-j\omega t}dt + \int_{0}^{\infty} e^{-t}e^{-j\omega t}dt \tag{72}$$
$$= \frac{1-0}{1-jw} - \frac{0-1}{1+jw} \tag{73}$$
$$= \frac{2}{1+w^2} \tag{74}$$

c.

$$\frac{d}{d\omega}X(jw) = \mathcal{F}x(t)(-jt) \tag{75}$$
$$\mathcal{F}[e^{-|t|}(-jt)] = \frac{d}{d\omega}\frac{2}{1+w^2} \tag{76}$$
$$= \frac{-4w}{(1+w^2)^2} \tag{77}$$
$$-j\mathcal{F}[e^{-|t|}(-jt)] = -j\frac{-4w}{(1+w^2)^2} \tag{78}$$
$$\mathcal{F}[te^{-|t|}] = \frac{4jw}{(1+w^2)^2} \tag{79}$$

d.

$$\mathcal{F}\frac{4jt}{(1+t^2)^2} = 2\pi(-we^{-|w|}) \tag{80}$$
$$\mathcal{F}\frac{4t}{(1+t^2)^2} = -2j\pi(-we^{-|w|}) \tag{81}$$
$$= 2j\pi we^{-|w|} \tag{82}$$