FYS 3120: Classical Mechanics and Electrodynamics
Spring semester 2014

A note on "Small Oscillations"

Consider a system described by a Lagrangian $L(q, \dot{q})$, for simplicity with only one generalized coordinate q. Assume that we know that q_0 is an equilibrium point, which means that $q = q_0$ is stationary solution of the equation of motion. If this is a *stable* equilibrium point, we know that if the the system is given initial values where the deviations from equilibrium, $q - q_0$ and $\dot{q}$, are sufficiently small, the system will oscillate about q_0

The purpose of this note is to discuss how a systematic expansion in small quantities, either of the Lagrangian or the equation of motion, will lead to a harmonic oscillator equation for the deviation from equilibrium, from which the oscillation frequency can be extracted.

Let us therefore consider as situation where the deviation from equilibrium, $\rho = q - q_0$, is at all time a small quantity. This can be obtained by making the initial values $\rho(0)$ and $\dot{\rho}(o)$ sufficiently small. Note that by making the initial values of these variables small, thereby giving a small amount of energy available for oscillations, this will make not only ρ, but also $\dot{\rho}$ and $\ddot{\rho}$, to be small for all t.

Let us consider the equation of motion of the system written in the form

$$\ddot{q} = f(q, \dot{q}) \tag{1}$$

where the function $f(q, \dot{q})$ is determined by the Lagrangian. Expressed in the deviation ρ from equilibrium, the equation takes the form

$$\ddot{\rho} = f(q_0 + \rho, \dot{\rho}) \tag{2}$$

For small deviations a power expansion in ρ and $\dot{\rho}$ is meaningful, and by regulating the initial values of these, the importance of the higher order terms can be reduced to the extent that only the first order terms are relevant

$$\ddot{\rho} = f(q_0, 0) + \rho \frac{\partial f}{\partial \rho}(q_0, 0) + \dot{\rho}\frac{\partial f}{\partial \dot{\rho}}(q_0, 0) \tag{3}$$

Since $q = q_0$ is an equilibrium point, and $\rho = 0$ therefore is a solution of the equation of motion, the constant term must vanish, $f(q_0, 0) = 0$. Eq.(3), is therefore a linear, homogenous differential equation in $\rho(t)$, of the form

$$\ddot{\rho} + a\dot{\rho} + b\rho = 0 \tag{4}$$

with a and b as constants determined by the partial derivatives of f.

This equation has oscillationary solutions, $\rho(t) = e^{i\omega t}$, provided the secular equation

$$-\omega^2 + ia\omega + b = 0 \tag{5}$$

is satisfied with real values for ω. The linearized equation thus gives a consistency condition for the point $q = q_0$ being a stable equilibrium, and if that is satisfied, it determines the angular frequency of small oscillations about the point.

As a simple example, we take a particle moving in a potential. The Lagrangian is

$$L = \frac{1}{2}m\dot{x}^2 - V(x) \tag{6}$$

with the corresponding equation of motion

$$\ddot{x} = -\frac{1}{m}V'(x) \tag{7}$$

with V' as the derivative of V with respect to x. Assume $x = x_0$ is a point of stable equilibrium, with $\eta = x - x_0$ as the deviation from equilibrium. Expanding the potential about x_0 gives the linearized equation

$$\ddot{\eta} + \frac{1}{m}V''(x_0)\eta = 0 \tag{8}$$

where we have used that $V'(x_0) = 0$ at the equilibrium point. The secular equation now gives $\omega^2 = V''(x_0)/m$, which has real solutions provided $V''(x_0)$ is positive. This is indeed the condition for the equilibrium to be stable.

As a second example let us consider the radial equation in Problem 4.3. When the angular variable θ has been eliminated it takes the form

$$(1 + \lambda^2 r^2)\ddot{r} + \lambda^2 r\dot{r}^2 - \frac{l^2}{m^2 r^3} + g\lambda r = 0 \tag{9}$$

where l is a constant of motion, identified as the z component of the angular momentum. This one-dimensional problem has an equilibrium solution $r = r_0$, with

$$r_0^2 = \frac{l}{m\sqrt{g\lambda}} \tag{10}$$

We introduce the deviation from equilibrium, $r = r_0 + \rho$, with $\dot{r} = \dot{\rho}$ and $\ddot{r} = \ddot{\rho}$. Introduced in the equation of motion this gives

$$(1 + \lambda^2 (r_0 + \rho)^2)\ddot{\rho} + \lambda^2 (r_0 + \rho)\dot{\rho}^2 - \frac{l^2}{m^2 (r_0 + \rho)^3}\rho + g(\lambda r_0 + \rho) = 0 \tag{11}$$

For sufficiently small deviations from equilibrium we may use the linearized form of the equation, which means that only first order terms in ρ, $\dot{\rho}$ and $\ddot{\rho}$ are kept when the expressions above are expanded in these variables. This gives the equation

$$(1 + \lambda^2 r_0^2)\ddot{\rho} + (3\frac{l^2}{m^2 r_0^4} + g\lambda)\rho = 0 \tag{12}$$

This can be simplified to

$$\ddot{\rho} + \Omega^2 \rho = 0 \tag{13}$$

with

$$\Omega = 2\sqrt{\frac{g\lambda}{1 + \lambda^2 r_0^2}} \tag{14}$$

as the angular frequency of the radial oscillations.

If this solution is used in the expressions for all variables r, z and θ in Problem 4.3, with $z = \frac{1}{2}\lambda r^2$ and $\dot{\theta} = l/mr^2$, this gives for these coordinates, to first order in the deviation ρ,

$$
\begin{aligned}
r(t) &= r_0 + \rho_0 \cos \Omega t \\
z(t) &= \frac{1}{2}\lambda r_0^2 + \lambda r_0 \rho_0 \cos \Omega t \\
\theta(t) &= \frac{l}{mr_0^2}t - 2\sqrt{gl}\,\frac{\rho_0}{\Omega r_0}\sin \Omega t
\end{aligned}
\tag{15}
$$

where we have chosen the initial conditions

$$
\rho(0) = \rho_0, \quad \dot{\rho}(0) = 0, \quad \theta(0) = 0
\tag{16}
$$

In the discussion given above we have focussed on linearizating the equation of motion (Lagrange's equation) with respect to the small variables. We could, however, have made the "small oscillation approximation" already in the the expression for the Lagrangian. However expanding L around a point of equilibrium, would then give no contribution to first order in the small terms, and expansion to second order would therefore be necessary. This we can understand from the form of Lagrange's equation, which depends only on the derivatives of L with respect to q and $\dot{q}$. Second order terms in L will therefore contribute with first order terms in Lagrange's equation.