Stability of Numerical Methods

We will consider two types of stability for numerical methods:

zero-stability Growth of round-off errors.

a-stability Growth of rapidly decaying modes of the solution

To study zero-stability, we consider the test problem

$$y' = 0\,, \quad y(0) = a\,.$$

The exact solution is a constant and we would like the numerical approximation to result in a constant solution. If we consider a 1-step method applied to this test problem, we find

$$y_{n+1} = y_n$$

If we assume a solution of the form $y_i = \zeta^i$, we get

$$\zeta^n(\zeta - 1) = 0$$

We have one root of $\zeta = 1$, which we must have for the method to be consistent, the other roots are all 0. As long as all the other root satisfy $\zeta_i \leq 1$, the method is zero stable.

We will now apply this method to a multi-step method. Consider the third order Backwards Difference Formula:

$$y_{n+3} - \frac{18}{11}y_{n+2} + \frac{9}{11}y_{n+1} - \frac{2}{11}y_n = \frac{6}{11}hf(t_{n+3}, y_{n+3})\,.$$

If we apply our this method to our test problem, we get

$$y_{n+3} - \frac{18}{11}y_{n+2} + \frac{9}{11}y_{n+1} - \frac{2}{11}y_n = 0\,.$$

Now we let $y_i = \zeta^i$, then we have

$$\zeta^3 - \frac{18}{11}\zeta^2 + \frac{9}{11}\zeta - \frac{2}{11} = 0\,.$$

The three roots are

$$\zeta_1 = 1\,, \quad \zeta_2 = \frac{7 + \sqrt{39}i}{22}\,, \quad \zeta_2 = \frac{7 - \sqrt{39}i}{22}\,.$$

Here $|\zeta_{1,2}| = \frac{\sqrt{22}}{11} \sim 0.426 < 1$, so this method is stable.

Here is another linear multi-step scheme considered in the book.

$$y_{n+1} + \frac{3}{2}y_n - 3y_{n-1} + \frac{1}{2}y_{n-2} = 3hf(t_n, y_n)$$

This is a 3rd order consistent method. If we check for zero-stability, we will get the polynomial

$$\zeta^3 + \frac{3}{2}\zeta^2 - 3\zeta + \frac{1}{2} = 0\,.$$

The roots are given by

$$\zeta_1 = 0\,, \quad \zeta_2 = \frac{-5 + \sqrt{33}}{4}\,, \quad \zeta_3 = \frac{-5 - \sqrt{33}}{4}\,.$$

In this case, $|\zeta_3| \sim 2.686 > 1$, so this method isn't zero-stable. This method can't be used for any problems.

Absolute Stability

The other form of stability that we will consider is absolute stability. We consider the test problem,

$$y' = \lambda y\,.$$

We would like the approximate solutions to decay when $\mathrm{Re}(\lambda) < 0$ We define the region of the complex $h\lambda$ plane where the approximate solution satisfies $|y_{i+1}| < |y_i|$. Here h is the step-size.

For example we will find the region for the mid-point method,

$$y_{i+1} = y_i + hf(t_i + \frac{h}{2}, y_i + \frac{h}{2} f(t_i, y_i)) .$$

We apply this method to our test problem,

$$\begin{aligned} y_{i+1} &= y_i + h\lambda \left(y_i + \frac{h}{2} \lambda y_i \right) , \\ y_{i+1} &= y_i \left(1 + h\lambda + \frac{(h\lambda)^2}{2} \right) . \end{aligned}$$

The region of absolute stability is then defined by

$$\left| 1 + z + \frac{z^2}{2} \right| \leq 1 .$$

To find the region, we plot the solutions to $1 + z + \frac{z^2}{2} = e^{i\theta}$ as θ ranges from 0 to 2π. We can plot this in Matlab using the roots command. Look up the help page for it. The session is given below.

```
octave:8> t=linspace(0,2*pi);
octave:9> for j=1:100
> p=[1/2 1 1-exp(i*t(j))];
> plot(roots(p),'x');
> hold on
> end
octave:10> print("rk2stab.eps")
octave:11> diary
```

Note that we must be careful with the variable `i`. If we were to use it as an index, we could no longer use it as an imaginary number. The resulting graph is in Figure . Just to try something new, lets find the region for the standard

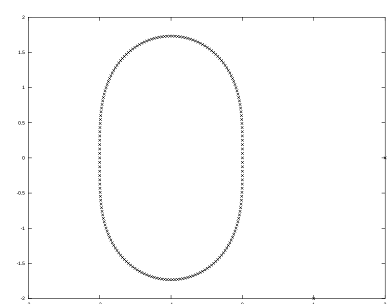

Figure 1: The region of absolute stability for the mid-point method

4th order method with the table, $\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$ If we apply this method to our test problem carefully, we get

$$y_{n+1} = y_n \left(1 + h\lambda + (h\lambda)^2 + \frac{(h\lambda)^3}{2} + \frac{(h\lambda)^4}{4} \right) .$$

So we want to find the roots of the polynomial

$$1 + z + z^2 + \frac{1}{2} z^3 + \frac{1}{4} z^4 = e^{i\theta}$$

as θ ranges from 0 to 2π. Here is the Matlab session:

```
octave:2> t=linspace(0,2*pi);
octave:3> for j=1:100
> p=[1/4 1/2 1 1 1-exp(i*t(j))];
> plot(roots(p),'x');
> hold on
> end
octave:4> print("rk4stab.eps")
octave:5> diary
```

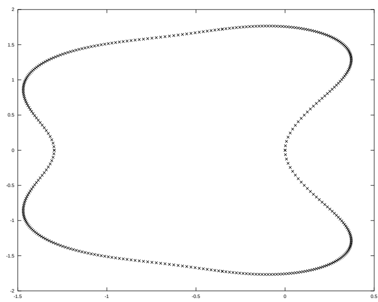

Figure 2: The region of absolute stability for the standard 4th order Runge-Kutta method

The process for determining the region of absolute stability for a multi-step scheme is a little different. For example we consider Addams-Bashford 2.

$$y_{i+1} = y_i + \frac{3h}{2} f_i - \frac{h}{2} f_{i-1}$$

We apply this method to our test problem,

$$y_{i+1} = y_i + \frac{3}{2} h\lambda y_i - \frac{h}{2} h\lambda y_{i-1}$$

We seek solutions to this relation with $y_i = \zeta^i$ with $\zeta = e^{i\theta}$.

$$\zeta^{i+1} = \zeta^i + \frac{3}{2} h\lambda \zeta^i - \frac{1}{2} h\lambda \zeta^{i-1}$$

$$\zeta^2 = \zeta + \frac{3}{2} \zeta h\lambda - \frac{1}{2} h\lambda$$

$$h\lambda = \frac{\zeta^2 - \zeta}{\frac{3}{2}\zeta - \frac{1}{2}}$$

Now we just plot $h\lambda$ as θ goes from 0 to 2π.

```
octave:2> t=linspace(0,2*pi);
octave:3> z=(exp(2*i.*t)-exp(i.*t))./(1.5*exp(i.*t)-.5);
octave:4> plot(z,'x')
octave:5> print("abstab.eps")
octave:6> diary
```

As I have discussed implicit methods have much better stability characteristics then explicit methods. This often makes them the better choice even though implicit methods require us to solve a non-linear system. Consider the 3rd order BDF method,

$$y_{n+3} - \frac{18}{11} y_{n+2} + \frac{9}{11} y_{n+1} - \frac{2}{11} y_n = \frac{6}{11} h f_{n+3}$$

We apply this method to our problem and solve for $h\lambda$. I will set $y_n = \zeta^n$ as well.

$$\zeta^3 - \frac{18}{11}\zeta^2 + \frac{9}{11}\zeta - \frac{2}{11} = \frac{6}{11} h\lambda \zeta^3 ,$$

$$h\lambda = \frac{\zeta^3 - \frac{18}{11}\zeta^2 + \frac{9}{11}\zeta - \frac{2}{11}}{\zeta^3}$$

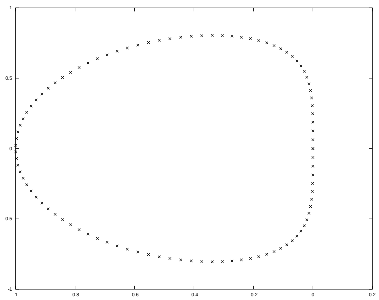

Figure 3: The region of absolute stability for the Addams Bashford 2nd order method.

We can now use Matlab to plot the stability region.

```
octave:2> t=linspace(0,2*pi);
octave:3> z=(exp(3.*i.*t)-18/11.*exp(2.*i.*t)+9/11.*exp(i.*t)-2/11)./exp(3.*i.*t);
octave:4> plot(z,'x')
octave:5> print("bdf3stab.eps");
octave:6> diary
```

Here is the resulting figure. The region of absolute stability is outside of the shape.

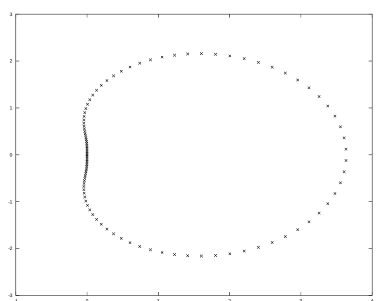

Figure 4: The region of absolute stability for the BDF 3 method is the outside of the enclosed shape. This method is almost A-stable. The two small lobes cross slightly into the left half plane.