

Introduction to singular perturbation methods

Nonlinear oscillations

This text is part of a set of lecture notes written by A. Aceves, N. Ercolani, C. Jones, J. Lega & J. Moloney, for a summer school held in Cork, Ireland, from 1994 to 1997. The links below will take you to online overviews of some of the concepts used here.

- Phase space, from Scholarpedia
- Stability of equilibria, from Scholarpedia
- Big-o and little-o notation, from MathWorld
- If you have MATLAB, PPLANE can be downloaded from the PPLANE page. Otherwise, you may use the online PPLANE JAVA applet.

1 Introduction

The simple pendulum is an excellent paradigm for studying the nonlinear behaviour of nonequilibrium systems. Moreover, a sequence of coupled pendula will provide a natural setting for introducing the Sine Gordon nonlinear partial differential equation (pde), an integrable nonlinear pde, from which another integrable pde, the Nonlinear Schrödinger (NLS) equation, can be derived as a small amplitude approximation. These provide a nice illustration of how the techniques of weakly nonlinear analysis are developed systematically.

Exact solutions are available to the simple pendulum problem for comparison. We will see that the nonlinear dependence of the frequency of oscillation of the pendulum on its amplitude of oscillation is a crucial signature of nonlinear behavior. In fact, this naturally suggests two weakly nonlinear paradigms: the anharmonic oscillator, a weakly nonlinear oscillator and the Van der Pol oscillator, an essentially linear oscillator with a nonlinear damping. These two models will illustrate the use of singular perturbation methods to derive uniformly valid perturbation corrections to the basic oscillator amplitude and frequency. The principal idea is to expand the oscillator amplitude in an asymptotic series, allowing for sufficient flexibility to avoid

unbounded (secular) growth of the correction term to the amplitude at each order in the perturbation expansion. We will first see how a regular perturbation expansion lacks this flexibility.

Singular perturbation expansions are extremely powerful analytic tools for studying a whole class of nonlinear problems. They will form the basis for deriving many of the hierarchy of soliton equations (the Nonlinear Schrödinger equation (NLS), in particular) and universal order parameter equations (of Complex Ginzburg-Landau type) valid near a bifurcation point in spatially extended systems. As these perturbation theories involve expansions in a small parameter, one might be left with the impression that they are of limited utility. Remarkably, in many instances the results prove accurate even for values of the parameter approaching $O(1)$.

The equation of motion for a simple pendulum is expressed in terms of the rate of change of the angle of the pendulum, measured with respect to its equilibrium position,

$$\begin{aligned}\ddot{\theta} + \omega^2 \sin \theta &= 0, \\ \omega &= \sqrt{\frac{g}{l}}.\end{aligned}$$

Multiplying by $\dot{\theta}$ gives

$$\begin{aligned}\dot{\theta}\ddot{\theta} + \omega^2 \dot{\theta} \sin \theta &= 0 \\ \frac{d}{d\theta} \left(\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta \right) &= 0,\end{aligned}$$

or

$$\frac{1}{2} \dot{\theta}^2 - \omega^2 \cos \theta = c, \text{ a constant,}$$

which is a statement of the conservation of energy. The phase portrait of the simple pendulum is the graph of $\dot{\theta}$ vs. θ for different values of the arbitrary constant c . This yields the one-parameter family of curves graphed in Figure 1.

The arrows indicate how θ is changing: $\dot{\theta} > 0 \Rightarrow \theta$ is increasing and, $\dot{\theta} < 0 \Rightarrow \theta$ is decreasing. Such a graph conveniently summarizes all possible states of motion of the pendulum.

The pendulum has the advantage that an exact solution can be obtained and expressed in terms of elliptic functions. This, however, is not very illuminating. For example the period of oscillation is given by:

$$Period = \frac{4}{\omega} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - m^2 \sin^2 \psi}}, m^2 = \sin^2 \frac{A}{2},$$

where A denotes the maximum amplitude of the motion. Note that the period (frequency) is amplitude dependent.

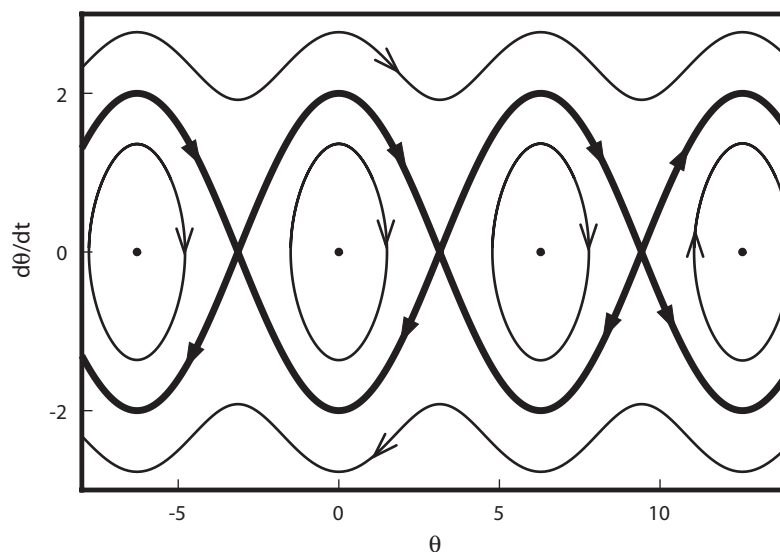


Figure 1: Phase portrait of the simple pendulum.

Exercise: Derive the above equation for the period of the pendulum.

For the phase portrait analysis, let $\theta = x, \dot{\theta} = y$. Then, the system reads

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega^2 \sin x.\end{aligned}$$

Critical (equilibrium) points occur when $(\dot{x}, \dot{y}) = (0, 0)$.

Exercise: Using the software package PPLANE, construct phase portraits for non-linear dynamical systems such as the simple pendulum.

We now review some important consequences of nonlinearity on the dynamics of oscillators.

- The frequency of oscillation depends on the amplitude of the swing.
- Tracing out separatrices takes infinite time.
- When externally driven, hysteresis may occur between different states of oscillation, thereby leading to the coexistence of multiple states.

We noted above that the pendulum equation cannot be solved in terms of elementary functions. For relatively small displacements from equilibrium one can expand $\sin \theta \simeq \theta - \theta^3/3! \dots$. The lowest truncation $\ddot{\theta} + \omega^2 \theta = 0$ yields the simple harmonic oscillator, a linear system. The solution of this differential equation is simply $\theta(t) = a \cos \omega t + b \sin \omega t$, with a and b arbitrary constants. At the next level

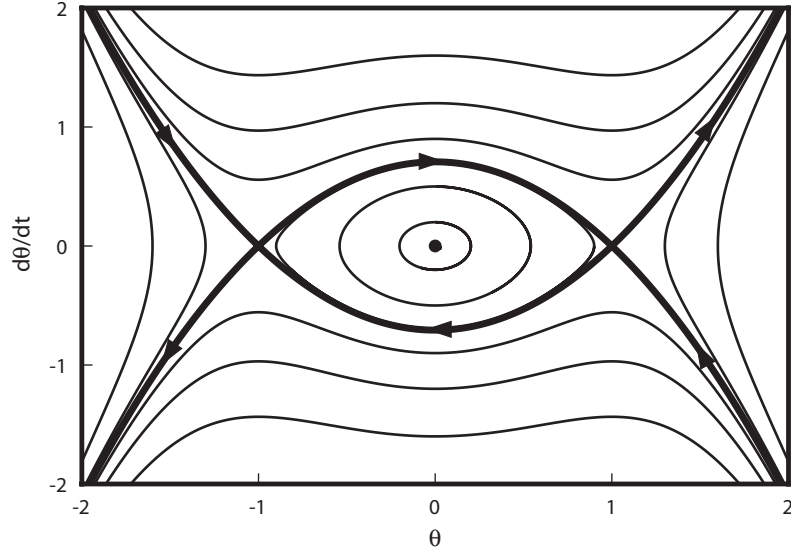


Figure 2: Phase portrait of the anharmonic oscillator ($\omega^2 = 1$, $\epsilon = 1$).

of truncation we obtain the anharmonic oscillator, $\ddot{\theta} + \omega^2\theta = \frac{\omega^2}{6}\theta^3$ which exhibits nonlinear behavior. If we think of the anharmonic term as a small (ϵ) correction to simple harmonic motion, we can replace $\frac{\omega^2}{6}$ by ϵ

$$\ddot{\theta} + \omega^2\theta = \epsilon\theta^3.$$

This is also the nonlinear spring model with a restoring force $F = -kx + k'x^3$ where θ now represents the displacement of the spring from equilibrium, $\omega^2 = k/m$ and $\epsilon = k'/m$.

The phase portraits of anharmonic oscillators are sketched by first writing the above as a system $y = \dot{\theta}$, $x = \theta$

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= -\omega^2x + \epsilon x^3\end{aligned}$$

and then obtaining the critical points by setting $(\dot{x}, \dot{y}) = (0, 0)$. This yields the equilibria as solutions of $y = 0$; $x(\epsilon x^2 - \omega^2) = 0$. These are $y = 0$ and $x = 0, \pm \frac{\omega}{\sqrt{\epsilon}}$ ¹ (see Figure 2).

For a realistic physical description we typically need to add a frictional force proportional to the velocity. We now consider two types of damping, linear and nonlinear. The anharmonic oscillator with linear damping is simply

$$\ddot{x} + \delta\dot{x} + \omega^2x = \epsilon x^3$$

¹If $\epsilon < 0$ (hard spring) only the origin is a critical point. We observe the same qualitative behavior as the pendulum.

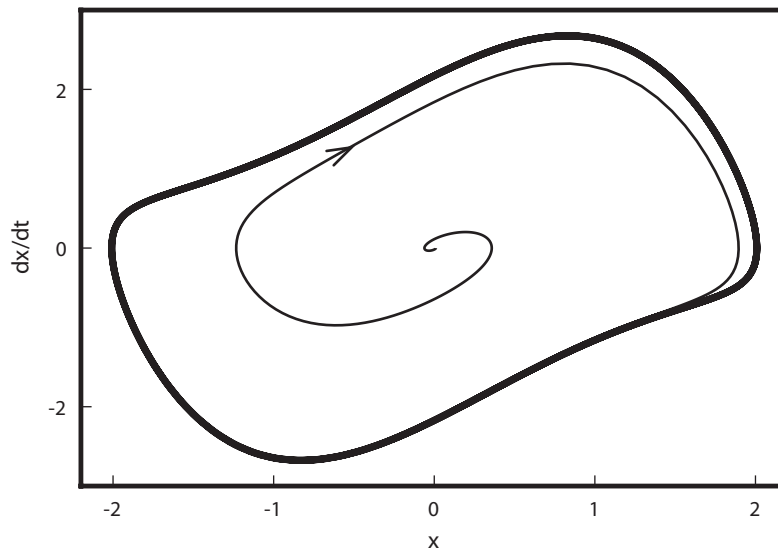


Figure 3: Phase portrait of the Van der Pol oscillator ($\omega^2 = 1$, $\epsilon = 1$).

This equation exhibits a simple stable (attracting) critical point and two unstable equilibria. This can be inferred geometrically from its associated phase portraits.

Exercise: Use PPLANE to draw the phase portraits of the damped anharmonic oscillator.

A linear oscillator with nonlinear damping is given by the Van der Pol equation

$$\ddot{x} + \omega^2 x + \epsilon(x^2 - 1)\dot{x} = 0$$

Here the damping coefficient depends on the amplitude $x(t)$ and displays a very interesting behavior depending on the magnitude of the displacement x . If $|x| < 1$, we obtain negative damping or growth, and, if $|x| > 1$, the damping is positive or the solution decays. We can immediately infer from this observation that the Van der Pol equation possesses a stable limit cycle solution (see Figure 3).

Exercise: Use PPLANE to construct a phase portrait of the Van der Pol oscillator for different magnitudes of the “small” parameter $\epsilon = 0.01, 0.1$, and 1.0 .

We will first proceed to analyze the Van der Pol oscillator for $\epsilon \ll 1$, using regular perturbation theory. This will allow us to highlight the shortcomings of this approach in an explicit manner and devise a better solution method.

2 Approximating the Limit Cycle of the Van der Pol Oscillator: Regular Perturbation Expansion

When $\epsilon = 0$, we recover the simple harmonic oscillator (SHO) which possesses a family of periodic solutions parameterized by ω . We base the perturbation expansion on this

as the leading order behavior. Our starting equation is

$$\ddot{x} + x + \epsilon(x^2 - 1)\dot{x} = 0 \quad \epsilon \ll 1 \quad (1)$$

For any ϵ , one can show that this equation possesses a limit cycle (periodic) solution and that any initial condition (besides the rest state $(\dot{x}, x) = (0, 0)$) will eventually lead to this limit cycle. As ϵ is a small parameter and $\epsilon = 0$ gives the SHO $\ddot{x} + x = 0$, we “perturb off of” the $\epsilon = 0$ solution.

We first expand $x(t)$ as a power series in ϵ

$$x(t) = x_0(t) + \epsilon x_1(t) + \epsilon^2 x_2(t),$$

and solve recursively for the unknown functions $x_i(t)$. Substitution of the expansion into 1 gives

$$\begin{aligned} x_0'' + x_0 + \epsilon(x_1'' + x_1 + x_0'(x_0^2 - 1)) \\ + \epsilon^2(x_2'' + x_2 + x_1'(x_0^2 - 1) + 2x_0x_0'x_1) + O(\epsilon^3) = 0. \end{aligned}$$

Since we want a solution valid for all (small) values of ϵ , each coefficient of ϵ^n , $n = 0, 1, 2, \dots$ must be identically zero. This yields the following sequence of problems,

$$\begin{aligned} \epsilon^0 : x_0'' + x_0 &= 0 && - SHO \\ \epsilon^1 : x_1'' + x_1 &= x_0'(1 - x_0^2) \\ \epsilon^2 : x_2'' + x_2 &= x_1'(1 - x_0^2) - 2x_0x_0'x_1 \\ \vdots & & \vdots & \end{aligned}$$

Normally we need to specify the initial conditions $x(0), x'(0)$ in order to solve the problem. As the Van der Pol equation is autonomous (it has no explicit time dependence) we can choose the instant $t = 0$ to lie anywhere on the limit cycle and choose it, for convenience, such that $x'(0) = 0$.

This implies that $x_0'(0) = x_1'(0) = x_2'(0) = \dots = 0$.

Solving the ϵ^0 problem (which corresponds to the unperturbed equation), subject to this initial condition, we get

$$x_0(t) = B_0 \cos t,$$

where B_0 is to be determined. Using this solution in the ϵ^1 problem, we obtain

$$\begin{aligned} x_1'' + x_1 &= -B_0 \sin t (1 - B_0^2 \cos^2 t) \\ &= \left[\frac{B_0^3}{4} - B_0 \right] \sin t + \frac{B_0^3}{4} \sin 3t. \end{aligned}$$

The general solution is

$$x_1(t) = \left[\frac{B_0^3}{4} - B_0 \right] \left(-\frac{t}{2} \cos t \right) - \frac{B_0^3}{32} \sin 3t + B_1 \cos t + A_1 \sin t$$

Here A_1 and B_1 are arbitrary constants of integration. Note that the term $t \cos t$ causes the solution to grow without bound as $t \rightarrow \infty$. In fact, to this order,

$$\begin{aligned} x(t) &= B_0 \cos t + \left[\frac{B_0^3}{4} - B_0 \right] \left(-\frac{\epsilon t}{2} \cos t \right) \\ &\quad - \frac{B_0^3}{32} \epsilon \sin 3t + \epsilon B_1 \cos t + \epsilon A_1 \sin t \end{aligned}$$

Therefore, when t is $O(1/\epsilon)$ the perturbation expansion, which assumes that each term of the series is smaller than the preceding one, breaks down. This unbounded growth term leads to secular divergence of the solution. Since we are looking for a periodic solution, such secular terms must be removed. If we choose $B_0 = 2$, this term vanishes at this order at least.

We now apply the initial condition $x'_1(0) = 0$.

$$\begin{aligned} x'_1(t) &= -\frac{24}{32} \cos 3t - B_1 \sin t + A_1 \cos t \\ x'_1(0) &= -\frac{3}{4} + A_1 = 0 \quad \text{or} \quad A_1 = 3/4 \end{aligned}$$

Therefore,

$$x_1(t) = -\frac{1}{4} \sin 3t + \frac{3}{4} \sin t + B_1 \cos t.$$

At $O(\epsilon^2)$, we have the following equation,

$$\begin{aligned} x''_2 + x_2 &= \left(-\frac{3}{4} \cos 3t + \frac{3}{4} \cos t - B_1 \sin t \right) (1 - 4 \cos^2 t) \\ &\quad + 8 \cos t \sin t \left(-\frac{1}{4} \sin 3t + \frac{3}{4} \sin t + B_1 \cos t \right) \\ &= \frac{1}{4} \cos t + 2B_1 \sin t - \frac{3}{2} \cos 3t + 3B_1 \sin 3t + \frac{5}{4} \cos 5t \end{aligned}$$

Now the terms proportional to $\cos t$ and $\sin t$ give rise to secular growth terms in the solution $x_2(t)$ and we have no flexibility to remove them! Therefore the regular perturbation method fails.

3 Poincaré-Lindstedt Perturbation Method

This approach permits removal of all secular terms at each order. The basic idea is as follows. We know from the simple pendulum that the period of oscillation depends on the amplitude of the motion (consider for instance the extreme case where the pendulum is balanced straight up and takes an infinite time to swing from this state

back to the same state). The new perturbation expansion allows the frequency to adapt to the nonlinearity by defining the “stretched time variable”

$$\tau = \omega t$$

where ω , the frequency of the response, is expanded in a power series in ϵ ,

$$\omega = 1 + k_1\epsilon + k_2\epsilon^2 + \dots,$$

where $\omega_0 = 1$ is the base harmonic oscillator frequency and the constants $k_i, i = 1, 2, \dots$ are to be determined. We will see that these constants introduce sufficient flexibility to remove secular growth terms at each order in the perturbation expansion.

With the above change of variable, Van der Pol’s equation becomes

$$\omega^2 \frac{d^2 x}{d\tau^2} + x + \epsilon(x^2 - 1)\omega \frac{dx}{d\tau} = 0.$$

Substituting the series expansions for $x(\tau)$ and ω , we obtain,

$$\begin{aligned} & (1 + 2k_1\epsilon + (k_1^2 + 2k_2)\epsilon^2 + 0(\epsilon^3)) \left(\frac{d^2}{d\tau^2}(x_0 + \epsilon x_1 + \epsilon^2 x_2 \dots) \right) \\ & + (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \dots) \\ & + \epsilon(x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2(x_1^2 + 2x_0 x_1) + \dots - 1)(1 + \epsilon k_1 + \epsilon^2 k_2 \dots) \\ & \frac{d}{d\tau}(x_0 + \epsilon x_1 + \epsilon^2 x_2 \dots) = 0. \end{aligned}$$

Regrouping into contributions at each order in ϵ , we get

$$\begin{aligned} O(\epsilon^0) : x_0'' + x_0 &= 0 \\ O(\epsilon^1) : x_1'' + x_1 &= x_0'(1 - x_0^2) - 2k_1 x_0'' \\ O(\epsilon^2) : x_2'' + x_2 &= x_1'(1 - x_0^2) - 2x_0 x_1 x_0' - 2k_1 x_1'' \\ &\quad - (2k_2 + k_1^2)x_0'' + k_1(1 - x_0^2)x_0' \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \end{aligned}$$

As before, the $O(\epsilon^0)$ problem has solution $x_0(\tau) = B_0 \cos \tau$. Removing secular terms at $O(\epsilon^1)$ requires again that $B_0 = 2$ and, in addition, $k_1 = 0$. The solution for $x_1(\tau)$ is as before,

$$x_1(\tau) = B_1 \cos \tau - \frac{1}{4} \sin 3\tau + \frac{3}{4} \sin \tau$$

Substituting for $x_0(\tau)$ and $x_1(\tau)$ into the RHS of the $O(\epsilon^2)$ equation yields,

$$x_2'' + x_2 = x_1'(1 - x_0^2) - 2x_0 x_1 x_0' - 2k_2 x_0'',$$

i.e.

$$\begin{aligned}
& x_2'' + x_2 \\
&= \frac{1}{4} \cos \tau + 2B_1 \sin \tau - \frac{3}{2} \cos 3\tau + 3B_1 \sin 3\tau + \frac{5}{4} \cos 5\tau + 4k_2 \cos \tau \\
&= (4k_2 + 1/4) \cos \tau + 2B_1 \sin \tau - \frac{3}{2} \cos 3\tau + 3B_1 \sin 3\tau + \frac{5}{4} \cos 5\tau.
\end{aligned}$$

Secular terms can now be removed by the choice $k_2 = -1/16$ and $B_1 = 0$.

The frequency to $O(\epsilon^2)$ becomes

$$\omega = 1 - \frac{\epsilon^2}{16}.$$

This formula shows that nonlinearity reduces the frequency of oscillation and therefore increases the period. The amplitude of oscillation to $O(\epsilon^2)$ is given by

$$\begin{aligned}
x(\tau) &= x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) \\
&= 2 \cos \tau - \epsilon \frac{[\sin(3\tau) - 3 \sin \tau]}{4} - \epsilon^2 \frac{[5 \cos 5\tau - 18 \cos 3\tau + 12 \cos \tau]}{96}.
\end{aligned}$$

Exercise: Carry out the singular perturbation expansion to order $O(\epsilon^4)$ and show that the frequency and amplitude of oscillation are given, to this order, by:

$$\begin{aligned}
\omega &= 1 - \frac{\epsilon^2}{16} + \frac{17\epsilon^4}{3072} \\
x(t) &= 2 \cos \omega t - \epsilon \frac{[\sin 3\omega t - 3 \sin \omega t]}{4} \\
&\quad - \frac{\epsilon^2 [5 \cos 5\omega t - 18 \cos 3\omega t + 12 \cos \omega t]}{96} \\
&\quad + \epsilon^3 \frac{[28 \sin 7\omega t - 140 \sin 5\omega t + 189 \sin 3\omega t - 63 \sin \omega t]}{2304}
\end{aligned}$$

Exercise: Compare the analytically computed expression for $x(t)$ with a numerical solution of the original equation over a single oscillation for $\epsilon = 0.1, 0.5, 1.0$ and 2.0 and remark on the accuracy of the result.

4 Multiple Scales Expansion

While the Poincaré-Linstedt method gives an approximation to the limit cycle solution, it yields no information regarding its stability. The multiple (time) scales expansion yields evolution equations on slow time scales which give the dynamics in the vicinity of an equilibrium point or a limit cycle. Therefore the stability of the critical point or limit cycle is determined. The idea behind the multiple time scales expansion is to obtain a uniform approximation to a differential equation valid for

times $t \sim 0(\epsilon^{-n})$ by defining the ordered “slow” variables, $T_0 = t, T_1 = \epsilon t, T_2 = \epsilon^2 t, \dots, T_n = \epsilon^n t$. The latter are treated as independent variables in subsequent manipulations.² The approximation is uniform in the sense that, to the order obtained, it does not blow-up for all time although strictly speaking it approximates the true solution for times $t \sim 0(\epsilon^{-n})$.

4.1 Anharmonic Oscillator

The anharmonic oscillator provides a simple example of the implementation of the multiple time scale expansion procedure. This procedure will be extended in later chapters to include spatial degrees of freedom in deriving complex order parameter equations and the NLS equation. The anharmonic oscillator model is:

$$\ddot{x} + x - \epsilon x^3 = 0; \quad x(0) = a, \quad \dot{x}(0) = 0$$

Substituting the multiple scales expansion: $T_0 = t, \quad T_1 = \epsilon t, \quad T_2 = \epsilon^2 t, \dots$, and applying the chain rule for differentiation, yields

$$\begin{aligned} \frac{d}{dt} &= \sum_{n=0}^{\infty} \frac{\partial}{\partial T_n} \frac{\partial T_n}{\partial t} = \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \dots \\ \frac{d^2}{dt^2} &= \frac{\partial^2}{\partial T_0^2} + \epsilon \left[\frac{\partial^2}{\partial T_1 \partial T_0} + \frac{\partial^2}{\partial T_0 \partial T_1} \right] + \epsilon^2 \left[\frac{\partial^2}{\partial T_1^2} + \frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_2 \partial T_0} \right] \\ &\quad + 0(\epsilon^3). \end{aligned}$$

Expanding, as before, $x(t)$ in an asymptotic series in ϵ ,

$$x(t, \epsilon) = x_0 + \epsilon x_1 + \epsilon^2 x_2 \dots,$$

and separating into individual orders in ϵ , yields the sequence of problems,

$$\begin{aligned} O(\epsilon^0) : \frac{\partial^2 x_0}{\partial T_0^2} + x_0 &= 0 \\ O(\epsilon^1) : \frac{\partial^2 x_1}{\partial T_0^2} + x_1 &= -\frac{2\partial^2 x_0}{\partial T_1 \partial T_0} + x_0^3 \\ O(\epsilon^2) : \frac{\partial^2 x_2}{\partial T_0^2} + x_2 &= -\left[\frac{\partial^2}{\partial T_1^2} + \frac{2\partial^2}{\partial T_0 \partial T_2} \right] x_0 - \frac{2\partial^2}{\partial T_1 \partial T_0} x_1 + 3x_0^2 x_1 \\ &\vdots \quad \quad \quad \vdots \quad \quad \quad \vdots \end{aligned}$$

Before proceeding to solve this sequence of problems, we introduce some notational simplification which will prove convenient throughout the text. Notice that the left

² In many instances, when dealing with differential operators defined in terms of these slow variables, care must be taken to preserve the order of operation as these may not commute.

side of each equality, at each order, involves the linear differential operator $\left[\frac{\partial^2}{\partial T_0^2} + 1\right]$ applied to the unknown function x_i . This only involves the original (fast) time scale and it is convenient to introduce the compact notation

$$Lx_i = \left[\frac{\partial^2}{\partial T_0^2} + 1\right] x_i.$$

This form will be used in what follows.

The $O(\epsilon^0)$ problem is simply the linear harmonic oscillator, which is solved easily,

$$x_0 = A(T_1, T_2)e^{iT_0} + A^*(T_1, T_2)e^{-iT_0} + c.c.,$$

where the use of complex notation will facilitate algebraic manipulations later. Here, the complex “constant” A is only constant with respect to the fastest timescale T_0 and will in general depend on all slower timescales $T_i, i = 1, 2, \dots$. The explicit time dependence of A on these slower timescales will appear at progressively higher orders in ϵ .

Substituting into the RHS of the $O(\epsilon^1)$ problem, we get

$$Lx_1 = -2i(e^{iT_0}\frac{\partial A}{\partial T_1} - e^{-iT_0}\frac{\partial A^*}{\partial T_1}) + 3|A|^2 Ae^{iT_0} + A^3 e^{3iT_0} + A^{*3} e^{-3iT_0} + c.c.$$

Removal of secular terms imposes the constraint:

$$2i\frac{\partial A}{\partial T_1} = 3|A|^2 A,$$

or

$$A(T_1) = e^{-\frac{3i}{2}|A|^2 T_1} A(0),$$

which follows immediately from the fact that $|A|^2$ is a constant of the motion, i.e $\frac{\partial |A|^2}{\partial T_1} = 0$. This implies that

$$x_0 = e^{it - \frac{3i}{2}|A|^2 \epsilon t} A(0) + c.c.$$

and

$$\dot{x}_0 = i[1 - \frac{3}{2}|A|^2 \epsilon] e^{it - \frac{3i}{2}|A|^2 \epsilon t} A(0) + c.c.$$

Finally, using the initial conditions on $x(t)$

$$\begin{aligned} x_0(0) &= A(0) + A^*(0) = a \\ \dot{x}_0(0) &= i[1 - \frac{3}{2}|A(0)|^2 \epsilon] A(0) - i[1 - \frac{3}{2}|A(0)|^2 \epsilon] A^*(0) = 0, \end{aligned}$$

we obtain

$$A(0) = A^*(0) = a/2.$$

Therefore,

$$x_0(t) = a \cos[1 - \frac{3a^2}{8} \epsilon]t.$$

Observe that truncating the expansion to this order gives a correction to the frequency of oscillation and the motion remains bounded between ± 1 for all time.

The constraint on A above appears as an evolution equation and determines its explicit dependence on T_1 . The dependence of A on the slower scale T_2 is determined in a similar fashion at order ϵ^2 . We won't pursue this further here. Conditions of this form are called *solvability conditions* and they play a central role in deriving complex order parameter equations of CGL and NLS type later.

4.2 Van der Pol Oscillator

Returning to the Van der Pol oscillator model analyzed in the previous section via the Poincaré-Lindstedt method, we now apply the multiple time scales procedure to this equation,

$$\ddot{x} + \epsilon(x^2 - 1)\dot{x} + x = 0; \quad x(0) = a_0, \quad \dot{x}(0) = 0,$$

to obtain

$$\begin{aligned} & [\frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_1 \partial T_0} + \epsilon^2 (\frac{\partial^2}{\partial T_1^2} + \frac{2\partial^2}{\partial T_2 \partial T_0}) \cdots] (x_0 + \epsilon x_1 + \epsilon^2 x_2 \cdots) \\ & + \epsilon \{ [(x_0^2 + 2\epsilon x_0 x_1 + \epsilon^2 (x_1^2 + 2x_0 x_2)) - 1] (\frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} \cdots) \\ & (x_0 + \epsilon x_1 + \epsilon^2 x_2 \cdots) \} + (x_0 + \epsilon x_1 + \epsilon^2 x_2 \cdots) = 0. \end{aligned}$$

Separating again at each order in ϵ , we get the sequence of problems

$$\begin{aligned} O(\epsilon^0) : Lx_0 &= 0 \\ O(\epsilon^1) : Lx_1 &= -2 \frac{\partial^2 x_0}{\partial T_1 \partial T_0} - (x_0^2 - 1) \frac{\partial x_0}{\partial T_0} \\ O(\epsilon^2) : Lx_2 &= -2 \frac{\partial^2 x_1}{\partial T_1 \partial T_0} - \left(\frac{\partial^2}{\partial T_1^2} + \frac{2\partial^2}{\partial T_2 \partial T_0} \right) x_0 \\ &\quad - (x_0^2 - 1) \frac{\partial x_0}{\partial T_1} - (x_0^2 - 1) \frac{\partial x_1}{\partial T_0} - 2x_0 x_1 \frac{\partial x_0}{\partial T_0} \end{aligned}$$

The $O(\epsilon^0)$ solution is

$$x_0 = A(T_1, T_2) e^{iT_0} + A^*(T_1, T_2) e^{-iT_0}.$$

Substituting this expression for x_0 into the x_1 equation, we get

$$\begin{aligned} Lx_1 &= -2i \left(\frac{\partial A}{\partial T_1} e^{iT_0} - \frac{\partial A^*}{\partial T_1} e^{-iT_0} \right) \\ &\quad - i \{ (|A|^2 A - A) e^{iT_0} - (|A|^2 A^* - A^*) e^{-iT_0} + A^3 e^{3iT_0} - A^{*3} e^{-3iT_0} \}. \end{aligned}$$

To remove secular terms at this order ($O(\epsilon^1)$), the condition

$$2 \frac{\partial A}{\partial T_1} = A - |A|^2 A$$

must be satisfied. Solving explicitly for A by setting $A = \frac{R(T_1)e^{i\theta(T_1)}}{2}$, we obtain separate equations for the amplitude and phase of A ,

$$\begin{aligned} \frac{\partial R}{\partial T_1} &= \frac{R}{2} - \frac{R^3}{8} \\ \frac{\partial \theta}{\partial T_1} &= 0, \end{aligned}$$

which implies that $\theta(T_1) = \text{const}$, and

$$R(T_1) = \frac{2R(0)e^{T_1/2}}{\sqrt{(e^{T_1} - 1)R(0)^2 + 4}},$$

where we have chosen $\theta = 0$. The initial conditions $x(0) = a_0$, and $\dot{x}(0) = 0$ imply that $R(0) = a_0$. Therefore

$$x_0 = \frac{a_0 e^{\epsilon t/2} e^{it}}{\sqrt{(e^{\epsilon t} - 1)a_0^2 + 4}} + c.c. = \frac{2}{\sqrt{1 + (4/a_0^2 - 1)e^{-\epsilon t}}} \cdot \cos t.$$

Observe that the expansion truncated to this order shows that the limit cycle of the Van der Pol oscillator is stable, as nearby initial conditions are attracted to this solution exponentially. However, we obtain a rather poor approximation to the limit cycle itself. If we proceed to the next order we recover the frequency correction of the Poincaré Lindstedt method. This is left as an exercise.

Exercise: Solve the ϵ^2 -problem above using the x_0 and x_1 solutions and show that the expansion of $x(t)$ to second order in ϵ is:

$$x(t) = a \cos \left((1 + b\epsilon^2)t + \phi_0 \right) - \epsilon \frac{1}{32} a^3 \sin \left(3[(1 + b\epsilon^2)t + \phi_0] \right) + O(\epsilon^2).$$

where $a = 2/\sqrt{1 + (\frac{4}{a_0^2} - 1)e^{-\epsilon t}}$ and $b = \frac{-1}{256}(7a^4 - 32a^2 + 32)$. Notice that since as $t \rightarrow +\infty$, $a \rightarrow 2$ and $b \rightarrow \frac{-1}{16}$, the frequency correction is identical to that obtained by the Poincaré-Lindstedt method.