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ELECTION GOALS AND THE ALLOCATION OF CAMPAIGN RESOURCES[1]

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This paper compares the equilibrium behavior and outcomes in a model of two-party competition for legislative seats, under two different assumptions about the parties' goals: (i) parties maximize the expected number of seats won, and (ii) parties maximize the probability of winning a majority of the seats. The two goals may lead to qualitatively different behavior, and studying the differences yields insights into the relationship between the goals, and the role of assymetries between the parties.

KEYWORDS: Formal political models, game theory.

1. INTRODUCTION

IN LEGISLATIVE ELECTIONS with many distinct but simultaneous contests for individual seats, an important aspect of the campaign strategies of large actors, such as political parties and wealthy interest groups, is the allocation of scarce resources across the various legislative districts. This is true particularly for political systems with single-member districts and first-past-the-post electoral rules, in which it is necessary to win "local majorities," that is, majorities within district boundaries, to win seats in the legislature.[2] This type of allocation problem is also an important feature of some other types of elections, such as U.S. presidential elections, where the fact that each states' electors vote as a block make it necessary to win majorities within states. In this paper, I study the strategic allocation of campaign resources when two parties compete for legislative seats (or electoral votes). Treating the problem as a two-person game, I characterize the equilibrium strategies and outcomes under different assumptions about the parties' goals. Also, I pay special attention to situations in which the parties are not perfectly symmetric players, but, say, one party has some advantage over the other.

Political campaign expenditures are perhaps best viewed as a type of advertising, in which spending for a candidate increases the probability that the candidate wins the office he or she is seeking. The problem of allocating campaign resources may then be analyzed using game-theoretic advertising models such as those studied by Friedman (1958) and Karlin (1959), at least for the case of two

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[2] This is not to say that the distribution of campaign spending is unimportant in other systems, such as those with some form of proportional representation. Indeed, especially if there are significant regional differences in the relative strength of the various political parties, efficient allocation of campaign resources may be extremely important for winning seats.

competing parties. An excellent example is the work by Brams and Davis (1973, 1974), who extend one of these early models to study the effects of the electoral college on the distribution of spending across states in U.S. presidential campaigns.[3] They derive an interesting result, the "3/2's rule," which implies that candidates will, in equilibrium, devote a disproportionate share of their total campaign resources to larger states, relative to these states' electoral votes (and populations).[4]

As Brams and Davis note, their model includes several assumptions which require further investigation. I will mention two, as they are the subject of this paper. First, Brams and Davis consider only the perfectly symmetric case in which both candidates have the same total resources, and the "electoral production function" is the same for both candidates and is the same in all states. This is probably unrealistic given the considerable regional differences in the relative strength of the two major parties, as measured, say, by the relative number of voters with strong ties to each party (even if these ties are weaker now than in the past). Second, they assume that the goal of each candidate is to maximize his or her expected electoral vote. An alternative goal is to maximize the probability of winning the election (i.e., of winning a majority of the electoral votes). As the authors note, these goals need not lead to the same behavior or outcome: "The goal of maximizing one's expected electoral vote may or may not be equivalent to maximizing one's probability of winning the election ... We do not think that the implications of these two goals will be seriously contradictory in most cases, at least in two-candidate races, but this question needs to be explored further."[5]

Aranson, Hinich, and Ordeshook (1978) have investigated this latter issue in a somewhat more general framework, and show that if the game is symmetric enough, then maximizing the expected number of seats (or electoral votes) and maximizing the probability of winning a majority of the seats are equivalent goals in the sense that the same Nash equilibrium strategies are the same under each. The symmetry required is rather strong, however: at the equilibrium, the probability distribution of seats won by each party must be symmetric about its mean, and the expected number of seats won by each party must be the same.[6] As I show here, when the game is not so symmetric, and in particular, when one party has an advantage over the other, the two goals may yield rather different equilibria.

[3] Other examples are papers by Sankoff and Mellos (1972) and Young (1978). They study variations of the Colonel Blotto game, in which the party or candidate who spends the most resources in a district or state, by whatever amount, wins that district or state with certainty. As a number of writers have pointed out, a problem with these games is that they have no pure strategy equilibria except when one player has vastly more resources than the other, and it is difficult to interpret the mixed strategy equilibria in the context of political campaigns.

[4] See also the critique by Colantoni, Levesque, and Ordeshook (1975a), which led to a rather lively debate in Brams and Davis (1975) and Colantoni, Levesque, and Ordeshook (1975b).

[5] See Kramer (1966) for an insightful early discussion of the possible differences.

[6] Alternatively, they show that the two goals are equivalent if the variance of the probability distribution of seats won by each party is independent of the parties' strategies, but this condition is also quite strong.

The model I use is similar to one of those in Friedman, but I extend it to allow the effect of campaign spending to vary across districts for the two parties. I assume first that the goal of each party is to maximize the expected number of seats it wins. In this case, the campaigns in each district can be treated as independent rank-order tournaments, and, applying the results of Rosen (1986), it is straightforward to show that the equilibrium level of spending in a district is higher, the more competitive is the race in that district (i.e., the closer to $\frac{1}{2}$ is the equilibrium probability that either party wins the district). If the goal of each party is to maximize the probability of winning a majority of the seats, however, this is no longer true. The difference is that, when parties are concerned about winning a majority of the seats, there is an additional factor that affects the marginal product of spending in a district, namely, the probability that the seat is pivotal. Investigating in detail the case in which all districts are either relatively "safe" for one or the other party or "marginal," (these terms are defined precisely below) I show that if one party has more safe districts than the other, and the races in all these safe districts are equally (non)competitive, then, in equilibrium, more resources will be spent in the safe districts of the advantaged party than in the safe districts of the other party. Furthermore, even if the races in the safe districts of the disadvantaged party are closer, spending may be higher in the safe districts of the advantaged party. Only in the perfectly symmetric case, in which both parties have the same number of safe districts, is the level of spending always higher in closer races.

2. A SIMPLE MODEL OF CAMPAIGN SPENDING AS ADVERTISING

There is a set $N = \{1, \ldots, n\}$ of legislative districts, each of which selects by popular election one representative to serve in the legislature. Two political parties, call them X and Y, compete for these legislative seats by spending campaign resources to help their candidates win election. (Each party nominates at most one candidate to run in each district.) The election in each district is probabilistic, the probability that each party's candidate wins depending on the amount of resources that the parties spend in the district. Borrowing from the work by Friedman (1958), Brams and Davis (1973, 1974), and Rosen (1986), I use a particular functional form to describe the relationship between expenditures and election probabilities. Thus, letting x_i and y_i denote the amounts of resources spent in district i by X and Y respectively, I make the following assumption.

ASSUMPTION (A.1): *The probability that X's candidate wins the election in district i is*

$$\tilde{p}_i(x_i, y_i) = \frac{a_i h(x_i)}{a_i h(x_i) + (1 - a_i) h(y_i)},$$

where $a_i \in (0,1)$, *and h is twice continuously differentiable, with* $h(0) = 0$, *and* $h'(x) > 0$ *and* $h''(x) \leqslant 0$ *for all* $x \geqslant 0$.

The probability that Y's candidate wins the election is simply $1-\tilde{p}_i(x_i, y_i)$. This is exactly the functional form used by Rosen, and is somewhat more general than that used by Friedman and Brams and Davis.[7,8] Notice that if both parties spend the same amount of resources in district i (no matter what this amount is), the $\tilde{p}_i = a_i$. Thus, if $a_i > \frac{1}{2}$, then party X has a kind of "natural advantage" in district i, and if $a_i < \frac{1}{2}$, then Y has the advantage. This advantage may perhaps be due to incumbency, or to differences in the number of loyal party supporters in the district.

It will be useful below to note the following facts about $\tilde{p}_i$:

COMMENT 2.1: *Given the assumptions in* $(A.1)$,

$$\text{(i)} \qquad \frac{\partial \tilde{p}_i}{\partial x_i}(x_i, y_i) = \frac{h'(x_i)}{h(x_i)} \cdot \tilde{p}_i(x_i, y_i)[1-\tilde{p}_i(x_i, y_i)] \qquad \textit{for all } x_i > 0,$$

$$\text{(ii)} \qquad \frac{\partial \tilde{p}_i}{\partial y_i}(x_i, y_i) = -\frac{h'(y_i)}{h(y_i)} \cdot \tilde{p}_i(x_i, y_i)[1-\tilde{p}_i(x_i, y_i)] \qquad \textit{for all } y_i > 0,$$

$$\text{(iii)} \qquad \frac{\partial^2 \tilde{p}_i}{\partial x_i^2}(x_i, y_i) < 0 \qquad \textit{and} \qquad \frac{\partial^2 \tilde{p}_i}{\partial y_i^2}(x_i, y_i) < 0,$$

$$\textit{for all } (x_i, y_i) \textit{ with } x_i > 0 \textit{ and } y_i > 0.$$

Equations (i) and (ii) follow simply by differentiating $\tilde{p}_i$ and then substituting $\tilde{p}_i$ into the resulting partial derivatives; and (iii) follows by differentiating (2.1) twice and using the fact that, by assumption, h is weakly concave.

3. MAXIMIZING THE EXPECTED NUMBER OF SEATS WON

I assume in this section that the goal of each party is to maximize the expected number of legislative seats it captures, less the resources it spends in trying to win these seats. That is, if $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_n)$ are the vectors of resource allocations, then the parties' payoffs are given by the following assumption:

ASSUMPTION (A.2):

$$u_X(\boldsymbol{x}, \boldsymbol{y}) = U \sum_{i=1}^{n} \tilde{p}_i(x_i, y_i) - c_X \sum_{i=1}^{n} x_i, \qquad \textit{and}$$

$$u_Y(\boldsymbol{x}, \boldsymbol{y}) = U \sum_{i=1}^{n} [1-\tilde{p}_i(x_i, y_i)] - c_Y \sum_{i=1}^{n} y_i,$$

$$\textit{where } U \geqslant c_X \textit{ and } U \geqslant c_Y.$$

[7] Also, special cases of this functional form appear in the rent-seeking literature. See, for example, Tullock (1980).

[8] A difference between the model here and that of Brams and Davis is that I make assumptions about the aggregate effect of campaign spending at the district level, while they make assumptions about the effect of spending on individual voters. (See their discussion of this point, and also the critique by Colantoni, Levesque, and Ordeshook (1975b).)

The parameters c_X and c_Y denote the "marginal cost" of raising and spending resources. They may be thought of as the disutility associated with fundraising effort. I have assumed constant marginal costs for simplicity, but the results hold for more general cost functions. Also, as discussed below, similar results hold if one assumes that each party maximizes its expected number of seats subject to a budget constraint on its total resources.

I assume that both parties have complete information about the technology and about each others' goals, and look for the pure-strategy Nash equilibria of the resulting two person, non-zero-sum game. (In what follows, I use "equilibrium," "Nash equilibrium," and "pure-strategy Nash equilibrium" interchangably.) A final assumption is that resource allocations must be nonnegative.

ASSUMPTION (A.3): *$x_i \geqslant 0$ and $y_i \geqslant 0$, for all i.*

Notice that the parties' objective functions are additively separable across districts. That is, the game really consists of n simultaneous but independent tournaments, and Rosen's analysis may be applied directly to characterize the equilibrium.

PROPOSITION 3.1: *In the game defined by* (A.1)–(A.3), *any pure-strategy Nash equilibrium* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *satisfies*

$$\frac{h(x_i^*)}{h'(x_i^*)} c_X = \frac{h(y_i^*)}{h'(y_i^*)} c_Y = U p_i^* [1 - p_i^*] \qquad \textit{for all } i,$$

where $p_i^* = \tilde{p}_i(x_i^*, y_i^*)$.

PROOF: Given $\boldsymbol{y}$, party X chooses $\boldsymbol{x}$ to maximize $u_X(\boldsymbol{x}, \boldsymbol{y})$, subject to the constraints that $x_i \geqslant 0$ for all i. The first-order conditions for this problem are

$$\frac{\partial \tilde{p}_i}{\partial x_i}(x_i, y_i) = \frac{c_X}{U} \qquad (i = 1, \dots, n).$$

Using (i) of Comment 2.1 and rearranging, this can be written as

$$\frac{h(x_i)}{h'(x_i)} c_X = U \tilde{p}_i(x_i, y_i)[1 - \tilde{p}_i(x_i, y_i)] \qquad (i = 1, \dots, n).$$

Similarly, given $\boldsymbol{x}$, party Y solves $\max_{\boldsymbol{y}} u_Y(\boldsymbol{x}, \boldsymbol{y})$, subject to $y_i \geqslant 0$ for all i, which yields the first-order conditions

$$\frac{h(y_i)}{h'(y_i)} c_Y = U \tilde{p}_i(x_i, y_i)[1 - \tilde{p}_i(x_i, y_i)] \qquad (i = 1, \dots, n).$$

Thus, any "interior" Nash equilibrium, at which the first-order conditions hold for both parties, satisfies the conditions of the proposition. And it is clear that there are no "corner" equilibria, in which some party spends zero resources in some district, because if one party spends zero resources in a district, then there is no solution to the other party's maximization problem. For example, if $y_i = 0$ for some i, then party X wants to choose x_i arbitrarily close to 0, but not equal to 0. Thus, any Nash equilibrium satisfies the condition of the proposition. *Q.E.D.*

Proposition 3.1 does not address the question of whether or not a Nash equilibrium exists, or is unique. In Propositions 3.2 and 3.3 below, I present conditions which guarantee existence and uniqueness. For the moment however, suppose that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is the unique equilibrium, and let $u_X^* = u_X(\boldsymbol{x}^*, \boldsymbol{y}^*)$ and $u_Y^* = u_Y(\boldsymbol{x}^*, \boldsymbol{y}^*)$ be the equilibrium payoffs. Then the model makes the following predictions about the two parties' spending and payoffs:

COMMENT 3.1: *Suppose the game described by $(A.1)$–$(A.3)$ has a unique pure-strategy Nash equilibrium, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. If $c_X = c_Y$, then (i) $x_i^* = y_i^*$, and hence $p_i^* = a_i$, for all i; (ii) $x_i^* > x_k^*$ if and only if $|p_i^* - \frac{1}{2}| < |p_k^* - \frac{1}{2}|$ (hence, if and only if $|a_i - \frac{1}{2}| < |a_k - \frac{1}{2}|$), for all i and k; (iii) if $\boldsymbol{a} = (a_1, \ldots, a_n)$ and $\boldsymbol{a}' = (a_1', \ldots, a_n')$ are two parameter vectors such that $|a_i - \frac{1}{2}| < |a_i' - \frac{1}{2}|$, $|a_k - \frac{1}{2}| < |a_k' - \frac{1}{2}|$, $a_j = a_j'$ for all j other than i and k, and $\sum_{i=1}^n a_i = \sum_{i=1}^n a_i'$, then the equilibrium payoffs of both parties are larger under $\boldsymbol{a}$ than under $\boldsymbol{a}'$.*

PROOF: If $c_X = c_Y$, then by Proposition 3.1,

$$\frac{h(x_i^*)}{h'(x_i^*)} = \frac{h(y_i^*)}{h'(y_i^*)} \qquad \text{for all } i.$$

Since h is nonnegative, strictly increasing, and weakly concave; h/h' is strictly increasing. Thus, $x_i^* = y_i^*$, proving (i). Proposition 3.1 also implies that

$$\frac{h(x_i^*)}{h'(x_i^*)} = \frac{p_i^*[1 - p_i^*]}{p_k^*[1 - p_k^*]} \cdot \frac{h(x_k^*)}{h'(x_k^*)} \qquad \text{for all } i \text{ and } k,$$

so, since h/h' is strictly increasing, $x_i^* > x_k^*$ if and only if $p_i^*[1 - p_i^*] > p_k^*[1 - p_k^*]$, which is true if and only if $|p_i^* - \frac{1}{2}| < |p_k^* - \frac{1}{2}|$. To prove (iii), note that the equilibrium payoff to party X is $u_X^* = U\sum_{i=1}^n a_i - c_X \sum_{i=1}^n x_i^*$, so, holding $\sum_{i=1}^n a_i$ fixed, u_X^* is larger, the smaller is total spending, $\sum_{i=1}^n x_i^*$. By (ii), total spending under $\boldsymbol{a}'$ is smaller than total spending under $\boldsymbol{a}$. The same argument holds for party Y. *Q.E.D.*

Thus, if both parties face the same marginal cost of raising and spending resources, then in each district, both parties spend the same amount of resources, regardless of the "natural advantage" one party might have, and thus, in each district, the actual odds of winning for the parties are equal to the "natural" odds.[9] Also, the level of spending is higher in districts where the race is closer.[10] Part (iii) says that, holding party X's overall advantage $(\Sigma_{i=1}^{n} a_i)$ fixed, both parties are better off if seats become "safer," i.e., if some of the a_i move closer to 0 or 1. Increasing "safeness" reduces the resources spent on campaigns.

If $c_X \neq c_Y$, so one party faces lower costs of raising campaign resources, then the results are slightly different:

COMMENT 3.2: *Suppose the game described by Assumptions* $(A.1)$–$(A.3)$ *has a unique pure-strategy Nash equilibrium,* $(\mathbf{x}^*, \mathbf{y}^*)$. *If, say,* $c_X < c_Y$, *then* (i) $x_i^* > y_i^*$, *and hence* $p_i^* > a_i$, *for all i;* (ii) $x_i^* > x_k^*$ *if and only if* $|p_i^* - \frac{1}{2}| < |p_k^* - \frac{1}{2}|$; (iii) *part* (iii) *of Comment* 3.1 *holds.*

The proof is analogous to that of Comment 3.1. Thus, the party with lower marginal costs spends more than the other party in every district, and thus the actual odds of winning, relative to the "natural" odds, favor the party with lower costs. As with equal costs, the level of spending by both parties is higher in districts where the race is closer. However, the level of spending is not monotonically related to the "natural" closeness of the race. For example, if $c_X < c_Y$, then over some interval $(a_0, \frac{1}{2})$, spending by both parties is *lower*, the closer a_i is to $\frac{1}{2}$. (Outside this interval, spending is monotonic in $|a_i - \frac{1}{2}|$.)

I should note also that the results are basically the same if one assumes that each party has a fixed total amount of resources to spend, and allocates these resources to maximize their expected number of seats, rather than a cost function. If both parties have the same total resources, then parts (i) and (ii) of Comment 3.1 hold, and if one party has more resources than the other, then parts (i) and (ii) of Comment 3.2 hold, where the party with more resources is in a position analogous to that of the party with lower marginal costs. The only

[9] This result depends on the specification of $\tilde{p}_i$, particularly on the assumption that the same function h applies to both parties, so that, given equal expenditures by the two parties, the marginal product of spending is the same for both.

[10] In fact, this result holds under more general assumptions about the "production" function $\tilde{p}_i$. For example, it holds if

$$\tilde{p}_i(x_i, y_i) = \frac{h_X(x_i)}{h_X(x_i) + h_Y(y_i)};$$

that is, the same function h need not apply to both parties. It does depend on the assumption that all districts are equally valuable. If some districts are worth more to a party than others, due, say, to differences in the leadership positions or seniority of the party's candidates in different districts (or, in the case of U.S. presidential elections, due to differences in the number of electoral votes across states), then the situation is more complicated, as parties also spend more, ceteris paribus, in the more valuable districts.

difference is that part (iii) does not apply, since if parties have a fixed amount of resources that they can devote only to campaigning (in the current election), then they are indifferent about the number of safe seats.

These observations are not new—one can find most of them in Rosen's article, for example—but I state them here because they are more interesting in the context of political elections than in most other kinds of tournaments. Contrary to the case of most tournament-like situations that come to mind, the "effort" exerted by each player in political campaigns, campaign expenditures, and the "relative strength" of the players (i.e., the players' relative odds of winning given equal effort), can be measured with some degree of accuracy. Hence, the predictions of Comments 3.1 and 3.2 are empirically testable. More importantly for the present paper, I wish to compare them with the predictions of the model when the parties wish to maximize the probability of winning a majority of the seats.

Before proceeding, however, I must return to the issue that was sidestepped above, regarding the existence and uniqueness of an equilibrium. When marginal costs are the same for both parties, the issue is easily resolved.

PROPOSITION 3.2: *In the game defined by Assumptions* (A.1)–(A.3), *if* $c_X = c_Y$ *then a unique pure-strategy Nash equilibrium exists.*

PROOF: As noted in Comment 3.1, if $c_X = c_Y$ then Proposition 3.1 implies that at any Nash equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, $x_i^* = y_i^*$ and thus $p_i^* = a_i$, for all i. Substituting these back into the first-order conditions for party X yields

$$\frac{h(x_i^*)}{h'(x_i^*)} = Ua_i(1-a_i)/c_X, \qquad \text{for all } i.$$

Let $g = h/h'$. Then $g(0) = 0$, since $h(0) = 0$ and $h'(0) > 0$. Also, $g'(x) = 1 - h(x)h''(x)/[h'(x)]^2 \geqslant 1$ for all x, since $h''(x) \leqslant 0$ (and $h(x) \geqslant 0$ and $h'(x) > 0$) for all x, so g_i is strictly increasing and continuous, and $\lim_{x \to +\infty} g(x) = +\infty$. Thus, for each i there is a unique value of x_i, call it x_i^{**}, such that $g(x_i^{**}) = Ua_i(1-a_i)/c_X$. Let $\boldsymbol{x}^{**} = (x_1^{**}, \ldots, x_n^{**})$. Then $(\boldsymbol{x}^{**}, \boldsymbol{x}^{**})$ is the only possible pure-strategy Nash equilibrium.

To see that $(\boldsymbol{x}^{**}, \boldsymbol{x}^{**})$ is in fact an equilibrium, note first that $x_i^{**} > 0$ for all i (since $a_i \in (0,1)$). Thus, given that Y plays $\boldsymbol{x}^{**}$, party X's maximization problem has a solution. Since X's payoff function is additively separable across districts, and $\boldsymbol{x}^{**}$ is the unique interior local extremum, the only possible solutions are $\boldsymbol{x}^{**}$, and "corner" points with $x_i^* = 0$ for some subset $M \subseteq N$ of districts and $x_i^* = x_i^{**}$ for the rest. Now, since $g(0) = 0$ and $g'(x) > 1$ for all x, $g(x) > x$ for all x ($g(x) = \int_0^x g'(z)\,dz > \int_0^x dz = x$). Thus $x_i^{**} < g_i(x_i^{**}) = Ua_i(1-a_i)/c_X < Ua_i/c_X$, so $Ua_i - c_X x_i^{**} > 0$ for all i. If $\boldsymbol{x}^*$ is a "corner" point, with $x_i^* = 0$ for all $i \in M \subseteq N$, and $x_i^* = x_i^{**}$ for all $i \notin M$, then $p_i(x_i, x_i^{**}) = 0$, so $u_X(\boldsymbol{x}^{**}, \boldsymbol{x}^{**}) - u_X(\boldsymbol{x}^*, \boldsymbol{x}^{**}) = \sum_{i \in M}[Ua_i(1-a_i) - c_X x_i^{**}] > 0$, and $\boldsymbol{x}^*$ is clearly worse for X. Thus, $\boldsymbol{x}^{**}$ is the uniquely optimal choice for X given that Y plays $\boldsymbol{x}^{**}$. The same argument holds for Y, so $(\boldsymbol{x}^{**}, \boldsymbol{x}^{**})$ is the unique Nash equilibrium. *Q.E.D.*

If costs are different for the parties, then the restrictions imposed by (A.1) are not enough to guarantee the existence of a unique Nash equilibrium. However, in the following case, a unique equilibrium does exist.

PROPOSITION 3.3: *In the game described by Assumptions* (A.1)–(A.3), *if* $h(x) = x^b$ *for all* x, *for some* $b \leqslant 1$, *then a unique pure-strategy Nash equilibrium exists.*

PROOF: If $h(x) = x^b$, then by Proposition 3.1, any equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ must satisfy $c_X x_i^* = c_Y y_i^* = bUp_i^*[1 - p_i^*]$, and thus $p_i^* = a_i/(a_i + (1 - a_i)(c_X/c_Y)^b)$, for all i. Substituting these back into the first-order conditions yields

$$x_i^* = \frac{bUa_i(1 - a_i)}{c_X\left[a_i + (1 - a_i)(c_X/c_Y)^b\right]} \qquad \text{and}$$

$$y_i^* = \frac{bUa_i(1 - a_i)}{c_Y\left[a_i + (1 - a_i)(c_X/c_Y)^b\right]}, \qquad \text{for all } i.$$

Letting $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ and $\boldsymbol{y}^* = (y_1^*, \ldots, y_n^*)$, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is the only possible Nash equilibrium.

As in the proof of Proposition 3.2, to show that $\boldsymbol{x}^*$ is optimal for party X given that Y plays $\boldsymbol{y}^*$, one need only compare X's payoff under strategies $\boldsymbol{x}$ with $x_i = 0$ for $i \in M$ and $x_i = x_i^*$ for $i \notin M$, where $M \subseteq N$ is some subset of districts. It is straightforward to check that $c_X x_i^* \leqslant Up_i^*$ for all i, and $p_i(x_i, y_i^*) = 0$ for all i with $x_i = 0$. So $u_X(\boldsymbol{x}^*, \boldsymbol{y}^*) - u_X(\boldsymbol{x}, \boldsymbol{y}^*) = \sum_{i \in M}[Up_i^* - c_X x_i^*] > 0$. Thus, $\boldsymbol{x}^*$ is uniquely optimal for X given that Y plays $\boldsymbol{y}^*$. A similar argument holds for Y, so $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is the unique Nash equilibrium. *Q.E.D.*

Finally, I should point out that a Nash equilibrium may exist even if the function h is not concave. For example, if $c_X = c_Y$ and $h(x) = x^b$ with $b \leqslant 2$, then a unique Nash equilibrium exists.

4. MAXIMIZING THE PROBABILITY OF GAINING CONTROL OF THE LEGISLATURE

I now assume that each party wishes to maximize, not the expected number of seats it wins in the legislature, but the probability that it wins a majority of the seats. Given that the legislature decides matters by majority rule, in a political environment with strong party discipline, the party with a majority of the seats effectively controls the legislature. In this case, maximizing the probability of winning a majority of the seats is perhaps the true goal of the parties, at least if both have a reasonable chance of achieving it. Thus, in this section I characterize the equilibrium strategies the parties would adopt given this goal, and compare them to the strategies the parties would adopt if they were maximizing the expected number of seats won. Interestingly, the strategies may be quite different, and in a systematic way, under the different goals.

To make the problem tractable, I assume that the elections in the districts are all statistically independent.[11] That is, if $\boldsymbol{p} = (p_1, \ldots, p_n)$, where p_i is the probability that party X wins the election in district i, then I make the following assumption:

Assumption (A.4): *For any nonempty subset $C \subset N$ of districts, the probability that X wins all of the districts in C and loses all of the districts in C' (where $C' = N - C$ is the complement of C) is $f(C) = [\prod_{i \in C} p_i][\prod_{i \in C'}(1 - p_i)]$. Also, $f(\phi) = \prod_{i=1}^{n}(1 - p_i)$ and $f(N) = \prod_{i=1}^{n} p_i$.*

Then, for any subset $D \subseteq N$ of districts, the probability that party X wins exactly s seats out of those in D (and any number of the seats in D') is

$$q(s, \boldsymbol{p}; D) = \begin{cases} f(\phi), & s = 0, \\ \sum_{\{C \mid \#C = s, C \subset D\}} f(C), & 0 < s < \#D, \\ f(D), & s = \#D \end{cases}$$

($\#C$ denotes the cardinality of C). Assuming that there are an odd number of seats in the legislature, the probability that party X wins more than half of the seats is

$$q_M(\boldsymbol{p}) = \sum_{s=\frac{n+1}{2}}^{n} q(s, \boldsymbol{p}; N) = \sum_{s=\frac{n+1}{2}}^{n} \sum_{\{C \mid \#C = s\}} f(C).$$

It is useful for what follows to note that, for each i, q_M can be written as follows:

Comment 4.1:

$$q_M(\boldsymbol{p}) = \sum_{s=\frac{n+1}{2}}^{n-1} q(s, \boldsymbol{p}; N - \{i\}) + p_i \cdot q((n-1)/2, \boldsymbol{p}; N - \{i\}).$$

[11] The independence assumption is quite strong, as it rules out "spillovers" in spending across districts and, perhaps more importantly, uncertainty about national variables that may affect the electoral outcomes in all districts simultaneously, such as changes in aggregate output or foreign policy crises. Some preliminary work extending the model here to a case where the election outcomes are correlated across districts suggests that the basic insights of the present paper still apply; however, more work on this issue is needed.

PROOF:

$$
\begin{aligned}
q_M(\boldsymbol{p}) = \sum_{s=\frac{n+1}{2}}^{n-1} \Bigg[& \sum_{\{C \mid \#C=s,\, i\in C\}} p_i \Bigg[\prod_{j\in C-\{i\}} p_j\Bigg]\Bigg[\prod_{j\in C'}(1-p_j)\Bigg] \\
& + \sum_{\{C \mid \#C=s,\, i\notin C\}} (1-p_i)\Bigg[\prod_{j\in C} p_j\Bigg] \\
& \times \Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg]\Bigg] + \prod_{j=1}^{n} p_j \\
= \sum_{s=\frac{n+1}{2}}^{n-1} & \sum_{\{C \mid \#C=s,\, i\notin C\}} \Bigg[\prod_{j\in C} p_j\Bigg]\Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg] \\
+ p_i \cdot \Bigg[& \sum_{s=\frac{n+1}{2}}^{n-1} \sum_{\{C \mid \#C=s-1,\, i\notin C\}} \Bigg[\prod_{j\in C} p_j\Bigg]\Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg] \\
& + \prod_{j\neq i} p_j - \sum_{s=\frac{n+1}{2}}^{n-1} \sum_{\{C \mid \#C=s,\, i\notin C\}} \Bigg[\prod_{j\in C} p_j\Bigg] \\
& \times \Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg]\Bigg] \\
= \sum_{s=\frac{n+1}{2}}^{n-1} & \sum_{\{C \mid \#C=s,\, i\notin C\}} \Bigg[\prod_{j\in C} p_j\Bigg]\Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg] \\
+ p_i \cdot & \sum_{\left\{C \mid \#C=\frac{n-1}{2}\right\}} \Bigg[\prod_{j\in C} p_j\Bigg]\Bigg[\prod_{j\in C'-\{i\}}(1-p_j)\Bigg] \\
= \sum_{s=\frac{n+1}{2}}^{n-1} & q(s, \boldsymbol{p}; N-\{i\}) + p_i \cdot q((n-1)/2, \boldsymbol{p}; N-\{i\}).
\end{aligned}
$$

Q.E.D.

This makes sense, since X wins a majority of the seats if and only if either (i) X wins a majority out of the seats other than i (and X either wins or loses seat

i), or (ii) X wins seat i and wins exactly half of the remaining seats; and (i) and (ii) are mutually exclusive events. The first term in the expression above gives the probability that (i) occurs, and the second term gives the probability that (ii) occurs.

Note that the first term does not depend on p_i, nor does the term that multiplies p_i. Thus, q_M is linear in each p_i.

COMMENT 4.2: *$\partial q_M/\partial p_i$ is equal to the probability that party X (hence, party Y as well) wins exactly one half of the districts other than i.*

This is intuitive—party X only cares about winning the election in district i if the district is pivotal, that is, if winning or losing in district i would make the difference between winning or losing a majority in the legislature. Obviously this occurs only if X wins exactly half of the other seats. Thus, the higher the probability that X wins exactly half of the other seats, the more it cares about winning in district i.

This means, of course, that the marginal impact on q_M of increasing p_i depends on all of the other p_j. Thus, if a party wants to maximize the probability of winning a majority of the seats, its resource allocation problem is not separable across districts, as it is for the case where the party maximizes the expected number of seats won; rather, the party's optimal strategy in one district will generally depend critically on what is happening in the other districts.

If parties care about the probability of winning a majority then their payoffs are given by the following assumption:

ASSUMPTION (A.5):

$$v_X(\boldsymbol{x}, \boldsymbol{y}) = Vq_m(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) - c_X \sum_{i=1}^{n} x_i, \qquad \textit{and}$$

$$v_Y(\boldsymbol{x}, \boldsymbol{y}) = V[1 - q_m(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y}))] - c_Y \sum_{i=1}^{n} y_i,$$

where $\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y}) = (\tilde{p}_1(x_1, y_1), \dots, \tilde{p}_n(x_n, y_n))$.

Again, I assume that allocations must be nonnegative. Then, analogous to Proposition 3.1 above, the equilibrium to the game described in this section can be characterized as follows:

PROPOSITION 4.1: *In the game defined by* (A.1), (A.3), (A.4), *and* (A.5), *any pure-strategy Nash equilibrium* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *satisfies*

$$\frac{h(x_i^*)}{h'(x_i^*)} c_X = \frac{h(y_i^*)}{h'(y_i^*)} c_Y = V p_i^* [1 - p_i^*] \frac{\partial q_M}{\partial p_i}(\boldsymbol{p}^*) \qquad \textit{for all } i,$$

where $\boldsymbol{p}^* = (p_1^*, \dots, p_n^*) = (\tilde{p}_1(x_1^*, y_1^*), \dots, \tilde{p}_n(x_n^*, y_n^*))$.

PROOF: Given y, party X solves $\max_x v_X(\boldsymbol{x}, \boldsymbol{y})$, which yields the first-order conditions

$$\frac{\partial q_M}{\partial p_i}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \cdot \frac{\partial \tilde{p}_i}{\partial x_i}(x_i, y_i) = \frac{c_X}{V} \qquad (i = 1, \ldots, n).$$

Substituting from (i) of Comment 2.1 and rearranging, this can be written as

$$\frac{h(x_i)}{h'(x_i)} c_X = V\tilde{p}_i(x_i, y_i)[1 - \tilde{p}_i(x_i, y_i)] \frac{\partial q_M}{\partial p_i}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \qquad \text{for all } i.$$

Similarly, the first-order conditions for Y's maximization problem can be written as

$$\frac{h(y_i)}{h'(y_i)} c_Y = V\tilde{p}_i(x_i, y_i)[1 - \tilde{p}_i(x_i, y_i)] \frac{\partial q_M}{\partial p_i}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \qquad \text{for all } i.$$

Thus, for any "interior" equilibrium, at which the first-order conditions hold for both parties, the conditions of the proposition are satisfied.

Next, to see that there are no "corner" equilibria, with zero spending in some districts by one or both parties, suppose, for example, that $y_i = 0$ for some i. If $\partial q_M / \partial p_i > 0$, then there is no solution to X's maximization problem, since X would want to choose x_i arbitrarily close to, but not equal to, 0. On the other hand, if $\partial q_M / \partial p_i = 0$, then using Comment 4.2 it is straightforward to show that either (i) $p_j = 0$ for at least $(n+1)/2$ of the districts in $N - \{i\}$, or (ii) $p_j = 1$ for at least $(n+1)/2$ of the districts in $N - \{i\}$. If (i) holds then party Y is certain not to win a majority of the seats, which means that the equilibrium must have $y_j = 0$ for all j (else Y's payoff would be negative). But then X's maximization problem clearly has no solution, since X would want to spend an arbitrarily small but positive amount of resources in some subset C of districts with $\#C = (n+1)/2$, and zero elsewhere. If (ii) holds, then party X is sure not to win a majority, and the same argument applies with the roles of X and Y reversed. Thus, no "corner" equilibrium can exist. *Q.E.D.*

Notice that, as in the game where parties maximize the expected number of seats won, if $c_X = c_Y$ then the equilibrium satisfies $x_i^* = y_i^*$, and hence $p_i^* = a_i$, for all i; and if $c_X < c_Y$ then $x_i^* > y_i^*$ and $p_i^* > a_i$, for all i.

To insure that a Nash equilibrium in fact exists, more restrictions are needed on the h function. Finding very general restrictions that work appears to be a difficult problem, because q_M is not concave in $\boldsymbol{p}$, and thus, even if $\tilde{p}_1, \ldots, \tilde{p}_n$ are concave, v_X may not be concave in $\boldsymbol{x}$ (and similarly for v_Y). Thus, strategies on the boundary of the feasible sets, with $x_i = 0$ or $y_i = 0$ for some subset of the districts, may often yield a higher payoff than the interior local maximum for one or the other party. In the following case, however, it is straightforward to show that a unique equilibrium exists:

PROPOSITION 4.2: *In the game defined by* (A.1), (A.3), (A.4), *and* (A.5), *if* $h(x) = x^b$ *for all* x, *with* $b \leqslant 1/n$, *then a unique pure-strategy Nash equilibrium exists.*

PROOF: If $h(x) = x^b$, then by Proposition 4.1, any equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ must satisfy

$$c_X x_i^* = c_Y y_i^* = bVp_i^*[1-p^*]\frac{\partial q_M}{\partial p_i}(\boldsymbol{p}^*),$$

and thus

$$p_i^* = \frac{a_i}{a_i + (1-a_i)(c_X/c_Y)^b},$$

for all i. Substituting these back into the first-order conditions, it is clear that there is only one possible equilibrium $(\boldsymbol{x}^*, \boldsymbol{y}^*)$, and this equilibrium has $x_i^* > 0$ and $y_i^* > 0$ for all i.

The condition $b \leqslant 1/n$ guarantees that, for each $\boldsymbol{y}$, v_X is concave in $\boldsymbol{x}$, and hence $\boldsymbol{x}^*$ is the uniquely optimal strategy for X given $\boldsymbol{y}^*$. A sketch of the proof is as follows: v_X is concave in $\boldsymbol{x}$ at $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ if

$$\psi_{ij}(\boldsymbol{x}, \boldsymbol{y}, \alpha_i, \alpha_j) = \alpha_i^2\frac{\partial^2 v_X}{\partial x_i^2}(\boldsymbol{x}, \boldsymbol{y}) + 2(n-1)\alpha_i\alpha_j\frac{\partial^2 v_X}{\partial x_i\,\partial x_j}(\boldsymbol{x}, \boldsymbol{y}) + \alpha_j^2\frac{\partial^2 v_X}{\partial x_j^2}(\boldsymbol{x}, \boldsymbol{y}) \leqslant 0$$

for all real numbers α_i and α_j, for all i and j, $i \neq j$ (this is a sufficient condition for concavity, but not necessary). Differentiating v_X twice and discarding terms that are zero,

$$\frac{\partial^2 v_X}{\partial x_i^2}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial q_M}{\partial p_i}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \cdot \frac{\partial^2 \tilde{p}_i}{\partial x_i^2}(x_i, y_i)$$

$$= \frac{\partial q_M}{\partial p_i}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \cdot p_i[1-p_i][b(b-1)/x_i^2 - 2p_i b^2/x_i^2]$$

for all i, and

$$\frac{\partial^2 v_X}{\partial x_i\,\partial x_j}(\boldsymbol{x}, \boldsymbol{y}) = \frac{\partial^2 q_M}{\partial p_i\,\partial p_j}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \cdot \frac{\partial \tilde{p}_i}{\partial x_i}(x_i, y_i) \cdot \frac{\partial \tilde{p}_j}{\partial x_j}(x_j, y_j)$$

$$= \frac{\partial^2 q_M}{\partial p_i\,\partial p_j}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \cdot p_i p_j[1-p_i][1-p_j][b^2/x_i x_j],$$

for all i and j, $\quad i \neq j$.

Now, for any $\boldsymbol{p}$, and any $j \neq i$,

$$\begin{aligned}\frac{\partial q_M}{\partial p_i}(\boldsymbol{p}) &= q((n-1)/2, \boldsymbol{p}; N-\{i\}) \\ &= p_j q((n-3)/2, \boldsymbol{p}; N-\{i, j\}) \\ &\quad + (1-p_j) q((n-1)/2, \boldsymbol{p}; N-\{i, j\}) \\ &\geqslant p_j(1-p_j) \cdot |q((n-3)/2, \boldsymbol{p}; N-\{i, j\}) \\ &\qquad - q((n-1)/2, \boldsymbol{p}; N-\{i, j\})| \\ &= p_j(1-p_j) \cdot \left| \frac{\partial^2 q_M}{\partial p_i \, \partial p_j}(\boldsymbol{p}) \right|.\end{aligned}$$

Substituting all of these results into ψ_{ij} above, using the condition $b \leqslant 1/n$, and reducing yields

$$\begin{aligned}\psi_{ij}(\boldsymbol{x}, \boldsymbol{y}, \alpha_i, \alpha_j) \leqslant -&\left| \frac{\partial^2 q_M}{\partial p_i \, \partial p_j}(\tilde{\boldsymbol{p}}(\boldsymbol{x}, \boldsymbol{y})) \right| \cdot p_i p_j [1-p_i][1-p_j] \\ &\cdot b^2 \left[(\alpha_i/x_i - \alpha_j/x_j)^2 + 2\left(p_i/x_i^2 - p_j/x_j^2 \right) \right] \leqslant 0\end{aligned}$$

as desired.

Similarly, $b \leqslant 1/n$ guarantees that v_Y is concave in $\boldsymbol{y}$ for all $\boldsymbol{x}$. Thus $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is a Nash equilibrium. *Q.E.D.*

The conditions of the proposition above are sufficient, but not necessary, to guarantee that the game has a unique equilibrium. One could search for more general sufficient conditions, but, as this is not essential for the purposes of this paper, I leave it as an exercise for the future. For now, I will simply assume that a unique equilibrium exists.

The relative spending in any two districts i and k is then given by the first-order conditions:

COMMENT 4.3: *Suppose the game defined by Assumptions* (A.1), (A.3), (A.4), *and* (A.5) *has a unique pure-strategy equilibrium*, $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. *Then* $x_i^* > x_k^*$ *if and only if*

$$\frac{p_k^*(1-p_k^*)}{p_i^*(1-p_i^*)} \cdot \frac{q((n-1)/2, \boldsymbol{p}; N-\{i\})}{q((n-1)/2, \boldsymbol{p}; N-\{k\})} > 1.$$

This follows directly from Proposition 4.1 and the fact that h/h' is strictly increasing. The first ratio in the expression above is simply a measure of the relative closeness of the election in the two districts. Recall that if each party's goal is to maximize the expected number of seats it wins, then this is the only factor that matters, so parties spend more resources in a district, the closer is the race in the district. Here, however, there is an additional factor to consider, namely the probability that a seat is pivotal; this is captured in the second ratio. Thus, whenever the relative probability of being pivotal differs significantly from 1 for some pair of districts, the relative spending in the two districts will be significantly different, depending on whether the goal of the parties is to maximize the expected number of seats won, or the probability of winning a majority of the seats.

It is difficult in general to determine which of two districts is more likely to be pivotal. However, in the following case it can be determined: Suppose that for every district in which party Y is more likely to win in equilibrium, there is another district in which party X is more likely to win, and in which the probability that party X wins is at least as great as party Y's probability of winning in the first district. That is, suppose that for all k such that $p_k^* \leqslant \frac{1}{2}$, there exists an i such that $p_i^* \geqslant 1 - p_k^*$. (For example, there might be 5 districts, with $p_1^* = .2$, $p_2^* = .4$, $p_3^* = .5$, $p_4^* = .7$, and $p_5^* = .9$.) This is equivalent to assuming that $n-1$ of the districts can be paired in such a way that for each pair (i, k), $p_i^* + p_k^* \geqslant 1$, with the district that is "leftover" (suppose it is district n) satisfying $p_n^* \geqslant \frac{1}{2}$. Then the following holds:

PROPOSITION 4.3: *Let p_i be the probability that party X wins the election in district i, $i = 1, \ldots, n$, and suppose that $p_n > \frac{1}{2}$, and that $p_1, \ldots, p_{n-1}$ can be paired in such a way that, for each pair (p_i, p_k), $p_i + p_k \geqslant 1$. Then, for each pair (p_i, p_k), if $p_i > p_k$ then district i is more likely to be pivotal than district k, that is, $q((n-1)/2, \boldsymbol{p}; N - \{i\}) > q((n-1)/2, \boldsymbol{p}; N - \{k\})$.*

PROOF: See Appendix.

To demonstrate the proposition's usefulness, I now apply it to the following interesting special case. There are three types of districts, (i) districts that are "safe" for party X, in which party X has a substantial natural advantage, (ii) districts that are safe for party Y, in which party Y has a natural advantage, and (iii) "marginal" districts, in which neither party has a natural advantage. Formally, I make the following assumption:

ASSUMPTION (A.6): *$N = C_X \cup C_Y \cup C_0$, where $C_X = \{i | a_i = a_X\}$, $C_Y = \{i | a_i = a_Y\}$, and $C_0 = \{i | a_i = \frac{1}{2}\}$, with $a_X \in (\frac{1}{2}, 1)$ and $a_Y \in (0, \frac{1}{2})$. Let $n_X = \#C_X$, $n_Y \#C_Y$, and $n_0 = \#C_0$.*

If $a_X = 1 - a_Y$, so that the safe districts of each party are "equally safe," then $a_X(1 - a_X) = a_Y(1 - a_Y)$. Suppose the parties face the same marginal costs (i.e., $c_X = c_Y$). Then, if each party attempts to maximize the expected number of seats it wins, the level of spending in all safe districts (of both parties) will be the same. However, if parties attempt to maximize the probability of winning a majority of the seats, this will not be true. Rather, spending will be higher in the safe districts of the party which has more safe districts, that is, in the safe districts of the party with an overall advantage. Only in a perfectly symmetric game, in which the parties have the same number of safe districts, will the level of spending be the same in all safe districts. The reason for this is simply that, if one party has more safe districts than the other, then its safe districts are more likely to be pivotal than the safe districts of the other party.

COROLLARY 4.1: *Consider the game described by* (A.1), (A.3), (A.4), (A.5), *and* (A.6), *and suppose that* $c_X = c_Y$ *and* $a_X = 1 - a_Y$. *If* i *and* k *are any two districts with* $a_i = a_X$ *and* $a_k = a_Y$ (*i.e., district* i *is safe for party* X, *and district* k *is safe for party* Y), *then* $x_i^* > x_k^*$ *if and only if* $n_X > n_Y$, *and* $x_i^* = x_k^*$ *if and only if* $n_X = n_Y$.

PROOF: Since $p_i^* = a_i$ for all $i \in N$ in equilibrium, Comment 4.3 implies that $x_i^* > x_k^*$ if and only if

$$\frac{a_i(1-a_i)}{a_k(1-a_k)} \cdot \frac{q((n-1)/2, \mathbf{p}; N - \{i\})}{q((n-1)/2, \mathbf{p}; N - \{k\})} > 1.$$

If district i is safe for party X and district k is safe for party Y, then $a_i = 1 - a_k$, so $x_i^* > x_k^*$ iff

$$q((n-1)/2, \mathbf{p}; N - \{i\}) > q((n-1)/2, \mathbf{p}; N - \{k\}),$$

and $x_i^* = x_k^*$ iff

$$q((n-1)/2, \mathbf{p}; N - \{i\}) = q((n-1)/2, \mathbf{p}; N - \{k\}).$$

If $n_X > n_Y$, then

$$q((n-1)/2, \mathbf{p}; N - \{i\}) > q((n-1)/2, \mathbf{p}; N - \{k\}),$$

by the last part of Proposition 4.3 (after pairing $n - 1$ of the districts so as to satisfy the hypothesis of the proposition, the "leftover" district, j, will satisfy $a_j = a_X > \frac{1}{2}$). If $n_X = n_Y$, then

$$q((n-1)/2, \mathbf{p}; N - \{i\}) = q((n-1)/2, \mathbf{p}; N - \{k\})$$

is obvious by symmetry. *Q.E.D.*

The difference in spending in the safe districts of the two parties may be substantial, even if the number of districts is large, as Table I below demonstrates. With a large number of districts, the probability that any given district is pivotal is very small; however, it is the *ratio* of these probabilities that matters in determining relative spending between any two districts, and these ratios may be quite large. Column 3 of the table gives the relative probability of being pivotal for districts that are safe for X and safe for Y (party X has the overall advantage in all cases shown). If, say, $h(x) = x^b$ for all x, then this relative probability is equal to the ratio of spending in the two types of districts. Thus, for example, if 30 of the 65 districts are safe for party X and 20 are safe for party Y, then spending will be nearly 50% higher in the districts that are safe for X than in those that are safe for Y. Note that the difference in spending in the safe seats of

TABLE I

$n = 65,\ a_X = .9,\ a_Y = .1,\ a_O = .5,\ a_i = a_X,\ a_k = a_Y$			
n_X	n_Y	$\dfrac{h'(x_i)/h(x_i)}{h'(x_k)/h(x_k)}$	$q_M(\boldsymbol{a})$
30	15	1.681	.976
30	16	1.642	.969
30	17	1.601	.960
30	18	1.561	.949
30	19	1.519	.935
30	20	1.476	.918
30	21	1.433	.898
30	22	1.388	.873
30	23	1.343	.844
30	24	1.296	.810
30	25	1.249	.770
30	26	1.201	.725
30	27	1.152	.675
30	28	1.102	.621
30	29	1.051	.562
30	30	1.000	.500

the two parties is greater, the greater the advantage that one party has over the other.

Whenever the race in one district is closer than that in another, but the second district is more likely to be pivotal, the two effects captured in the expression in Comment 4.3 are conflicting. In many cases, the latter effect dominates the former, with the result that *more* resources are spent in the race that is *less* close. For example, suppose that there are three types of districts, but that the safe districts of one party are "safer" than the safe districts of the other party. Table II shows such a case, in which the safe districts of party X are safer than those of party Y, with $a_X = .9$ and $a_Y = .15$ (and $a_0 = .5$, as before). Since $(a_X(1-a_X))/(a_Y(1-a_Y)) = 12/17 < 1$, if the parties maximize the expected number of seats won, then spending will be higher in the safe districts of party Y than in those of party X. On the other hand, as indicated by column 3 of Table II, if the parties maximize their probability of winning a majority of the seats, and if fewer than 24 of the 65 districts are safe for party Y (and 30 are safe for party X), then more resources will be spent in the safe districts of party X.

Another interesting corollary of Proposition 4.3 is that, if the conditions of the proposition are met, then increasing the safety of some districts, keeping the expected number of districts won by each party constant, increases the probability that the party with the overall advantage wins a majority of the districts.

COROLLARY 4.2: *Let* $\boldsymbol{p}^0 = (p_1^0, \ldots, p_n^0)$, *and suppose that* $p_n^0 > \frac{1}{2}$, *and that* $p_1^0, \ldots, p_{n-1}^0$ *can be paired in such a way that, for each pair* (p_i^0, p_k^0), $p_i^0 + p_k^0 \geqslant 1$. *Suppose that* (p_i^0, p_k^0) *is one such pair, with* $p_i^0 > p_k^0$, *and let* $\boldsymbol{p}^1 = (p_1^1, \ldots, p_n^1)$ *satisfy* $p_i^1 > p_i^0$, $p_k^1 = p_k^0 - (p_i^1 - p_i^0) < p_k^0$, *and* $p_j^1 = p_j^0$ *for all* $j \neq i, k$. *Then* $q_M(\boldsymbol{p}^1) > q_M(\boldsymbol{p}^0)$.

TABLE II

$n = 65$, $a_X = .9$, $a_Y = .15$, $a_O = .5$, $a_i = a_X$, $a_k = a_Y$			
n_X	n_Y	$\dfrac{h'(x_i)/h(x_i)}{h'(x_k)/h(x_k)}$	$q_M(a)$
30	15	1.192	.985
30	16	1.169	.981
30	17	1.146	.976
30	18	1.123	.970
30	19	1.099	.962
30	20	1.075	.953
30	21	1.050	.941
30	22	1.026	.928
30	23	1.001	.912
30	24	.975	.893
30	25	.950	.870
30	26	.923	.845
30	27	.896	.816
30	28	.870	.783
30	29	.842	.746
30	30	.815	.746
30	31	.787	.661
30	32	.760	.614
30	33	.732	.564

PROOF: Without loss of generality, suppose that $i = 1$ and $k = 2$. For each $p_1 \in [p_1^0, p_1^1]$, let $\boldsymbol{p}(p_1) = (p_1, p_2^0 - (p_1 - p_1^0), p_3^0, \dots, p_n^0)$, so $\boldsymbol{p}^0 = \boldsymbol{p}(p_1^0)$ and $\boldsymbol{p}^1 = \boldsymbol{p}(p_1^1)$. Then, since q_M is continuously differentiable,

$$q_M(\boldsymbol{p}^1) = q_M(\boldsymbol{p}^0) + \int_{p_1^0}^{p_1^1} \left[\frac{\partial q_M}{\partial p_1}(\boldsymbol{p}(p_1)) - \frac{\partial q_M}{\partial p_2}(\boldsymbol{p}(p_1)) \right] dp_1.$$

By Proposition 4.3,

$$\frac{\partial q_M}{\partial p_1}(\boldsymbol{p}(p_1)) > \frac{\partial q_M}{\partial p_2}(\boldsymbol{p}(p_1))$$

for all $p_1 \in (p_1^0, p_1^1]$ (recall that $\partial q_M(\boldsymbol{p})/\partial p_1$ is the probability that district i is pivotal, given $\boldsymbol{p}$). Thus, $q_M(\boldsymbol{p}^1) > q_M(\boldsymbol{p}^0)$. *Q.E.D.*

Thus, again considering the case where there are three types of seats, if one party has more safe districts than the other, then increasing the number of safe districts won by each party, keeping the expected number of districts won by each party constant, increases the probability that the party with more safe districts wins a majority of the districts. Such a shift also causes the total spending by both parties to fall, since more resources are spent in marginal districts than in either party's safe districts. Thus, the party with an overall advantage strictly prefers any change (e.g., redistricting) that creates an equal number of new safe districts for each party. The party with fewer safe seats may or may not prefer the change, however, depending on the relative magnitude of the decrease in spending and the decrease in the probability that it wins a majority of the seats.

5. DISCUSSION AND FURTHER POSSIBILITIES

A number of scholars, such as Jacobson and Kernel (1984), have asserted that the optimal distribution of campaign resources involves spending more resources in closer races. The results here support this assertion, provided that the goal of parties is to maximize the expected number of seats they win. If the goal of parties is to maximize the probability that they win a majority of the seats, however, then the situation is more complicated, and this conclusion must be revised somewhat.

The available empirical evidence indicates that campaign spending is, in fact, generally higher in closer races. For example, Jacobson (1980) and Bronars (1986) find this for U.S. Congressional elections. Of course, while these findings are roughly consistent with the predictions of simple, two-player, resource allocation games such as that presented here, the realities of Congressional campaign finance are probably much more complex, since campaigns are so candidate-oriented, incumbency is so important, and political parties supply only a small fraction of the total campaign resources.[12] Perhaps more relevant, Colantoni, Levesque, and Ordeshook (1975a, 1975b) find that U.S. presidential candidates spend more resources, as measured by campaign visits, in states where the outcome is closer (controlling for the number of electoral votes in each state). Unfortunately, it is not possible to use the results presented in these studies directly to test the more subtle prediction that spending in the "safe" districts (or states) of the party (or Presidential candidate) with an overall advantage will be higher than those of its opponent.

Finally, I should mention two extensions of the work here that may prove interesting. Jacobson (1980) finds that, in Congressional races, the marginal product of spending for incumbent candidates is lower than that for challengers. This is not inconsistent with the assumption here that marginal products are equal given equal spending, since he also finds that incumbents spend considerably more than challengers. However, he argues (as others argue), that one of the important advantages of incumbency is that it gives the incumbent a large amount of free publicity, and access to resources that can be used for campaigning. An interesting alternative specification of $\tilde{p}_i$, which captures some of the flavor of this argument, is the following:

$$\tilde{p}_i(x_i, y_i) = \frac{a_i h(x_I + x_i)}{a_i h(x_I + x_i) + (1 - a_i) h(y_i)}.$$

[12] In 1982, about 1.8 percent of the total spent by or on behalf of Democratic House candidates, and 10.3 percent of that spent by or on behalf of Republican candidates, was supplied by the respective national party committees. The figures are somewhat higher for Senate campaigns—about 4.4 percent for Democrats and 15.6 percent for Republicans. For good treatments of Congressional campaign finance, see Jacobson (1980) and Welch (1981).

Here, party X's candidate is the incumbent in district i, and x_I is part of the "incumbency advantage," a chunk of free, nondisposable advertising. If Y's candidate were the incumbent, then $\tilde{p}_i$ would be altered accordingly. This specification can also be used to deal with the problem of spending by actors other than the political parties (although it treats these actors as nonstrategic, their spending being fixed).

Lastly, if one party has very little chance of winning a majority of the seats, then it may view this goal, however desirable, as unrealistic, and attempt instead simply to win as many seats as possible in the current election, "waiting for another day" to try to win control of the legislature. The other party, however, may want to be as certain as possible that it wins a majority today. Thus, it would be interesting to investigate the case in which the party with the overall advantage maximizes the probability of winning a majority of the seats, while the other party maximizes the expected number of seats that it wins. In particular, it would be interesting to know the conditions under which the disadvantaged party adopts a strategy which appears to be "riskier" than that of its opponent, and how the distribution of outcomes is affected by the difference in the parties' goals.

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APPENDIX

Here, I prove Proposition 4.3. Let $\boldsymbol{p} = (p_1, \dots, p_n)$, where p_i is the probability that party X wins in district i. Then the probability that district i is pivotal is

$$q((n-1)/2, \boldsymbol{p}; N - \{i\}) = p_k \cdot q((n-3)/2, \boldsymbol{p}; N - \{i,k\}) + (1-p_k) \cdot q((n-1)/2, \boldsymbol{p}; N - \{i,k\}).$$

Similarly, the probability that district k is pivotal is

$$q((n-1)/2, \boldsymbol{p}; N - \{k\}) = p_i \cdot q((n-3)/2, \boldsymbol{p}; N - \{i,k\}) + (1-p_i) \cdot q((n-1)/2, \boldsymbol{p}; N - \{i,k\}).$$

Thus,

$$q((n-1)/2, \boldsymbol{p}; N - \{i\}) - q((n-1)/2, \boldsymbol{p}; N - \{k\}) = (p_i - p_k) \cdot [q((n-1)/2, \boldsymbol{p}; N - \{i,k\}) - q((n-3)/2, \boldsymbol{p}; N - \{i,k\})].$$

If $p_i > p_k$, this is positive iff

$$q((n-1)/2, \boldsymbol{p}; N - \{i,k\}) > q((n-3)/2, \boldsymbol{p}; N - \{i,k\}).$$

Let w denote the total number of seats won by party X out of the districts in $N - \{i,k\}$. Since the elections in the districts are assumed to be independent, the probability generating function for w is $g(t) = \prod_{j \in N - \{i,k\}} (1 - p_j + p_j t)$. If $\sum_{r=0}^{n-2} e_r t^r$ is the expansion of $g(t)$, then $q(r, \boldsymbol{p}; N - \{i,k\}) = c_r$

for all $r = 0, \ldots, n-2$, so

$$q\big((n-1)/2, \boldsymbol{p}; N - \{i,k\}\big) - q\big((n-3)/2, \boldsymbol{p}; N - \{i,k\}\big) = e_{(n-1)/2} - e_{(n-3)/2}.$$

To show that $e_{(n-1)/2} > e_{(n-3)/2}$ provided that the conditions in the proposition hold, I first prove a pair of lemmas.

LEMMA 1: *Let $b_1 \in [\frac{1}{2}, 1)$ and $b_2 \in (0,1)$, with $b_1 + b_2 \geqslant 1$, and let $c_0 + c_1 t + c_2 t^2 = (1 - b_1 + b_1 t) \cdot (1 - b_2 + b_2 t)$. Then $c_0 < c_1$, and $c_0 \leqslant c_2$, with $c_0 = c_2$ iff $b_1 + b_2 = 1$.*

PROOF: Evidently, $c_0 = (1-b_1)(1-b_2)$, $c_1 = b_1(1-b_2) + b_2(1-b_1)$, and $c_2 = b_1 b_2$. Thus, $c_1 \geqslant (1-b_1)(1-b_2) + b_2(1-b_1)$, since $b_1 \geqslant \frac{1}{2}$, so $c_1 > c_0$. Also, $c_2 - c_0 = b_1 + b_2 - 1$, which is nonnegative by assumption, and clearly $c_2 = c_0$ iff $b_1 + b_2 = 1$. *Q.E.D.*

LEMMA 2: *Let $b_0, \ldots, b_R$ be positive numbers such that $b_{r-1} < b_r$ and $b_r \leqslant b_{R-r}$ for all $r \leqslant R/2$. Let $c_0 < c_1$ and $c_0 \leqslant c_2$, and let $d_0, \ldots, d_{R+2}$ satisfy $\sum_{r=0}^{R+2} d_r t^r = [c_0 + c_1 t + c_2 t^2][\sum_{r=0}^{R} b_r t^r]$. Then $d_{r-1} < d_r$ and $d_r \leqslant d_{R-r}$ for all $r \leqslant (R+2)/2$. Furthermore, if $c_0 < c_2$ then $d_r < d_{R-r}$ for all $r < (R+2)/2$.*

PROOF: Evidently, $d_0 = c_0 b_0$, $d_1 = c_0 b_1 + c_1 b_0$, $d_r = c_0 b_r + c_1 b_{r-1} + c_2 b_{r-2}$ for $r = 2, \ldots, R$, $d_{R+1} = c_1 b_R + c_2 b_{R-1}$, and $d_R = c_2 b_R$. Thus, for all $r \leqslant R/2$, $d_{r-1} < d_r$ follows immediately from the assumption that $b_{r-1} < b_r$ for all such r. If R is even, then

$$d_{R/2+1} = c_0 b_{R/2+1} + c_1 b_{R/2} + c_2 b_{R/2-1} \geqslant c_0 b_{R/2-1} + c_1 b_{R/2} + c_2 b_{R/2-1},$$

so

$$d_{R/2+1} - d_{R/2} \geqslant (c_1 - c_0)\big(b_{R/2} - b_{R/2-1}\big) + c_2\big(b_{R/2-1} - b_{R/2-2}\big) > 0.$$

Similarly, if R is odd, then

$$\begin{aligned} d_{(R+1)/2} &= c_0 b_{(R+1)/2} + c_1 b_{(R-1)/2} + c_2 b_{(R-3)/2} \\ &\geqslant c_0 b_{(R-1)/2} + c_1 b_{(R-1)/2} + c_2 b_{(R-3)/2} \\ &> c_0 b_{(R-1)/2} + c_1 b_{(R-3)/2} + c_2 b_{(R-5)/2} = d_{(R-1)/2}. \end{aligned}$$

Thus, in either case, $d_{r-1} < d_r$ for all $r \leqslant (R+2)/2$.

Next, $d_0 \leqslant d_{R+2}$ since $c_0 \leqslant c_2$ and $b_0 \leqslant b_R$, and $d_1 \leqslant d_{R+1}$ since $c_0 \leqslant c_2$, $b_0 \leqslant b_R$, and $b_1 \leqslant b_{R-1}$. For $r \geqslant 2$, $r < R/2$,

$$d_{R+2-r} = c_0 b_{R+2-r} + c_1 b_{R+1-r} + c_2 b_{R-r} \geqslant c_0 b_{r-2} + c_1 b_{r-1} + c_2 b_r,$$

so

$$d_{R+2-r} - d_r = (c_2 - c_0)(b_r - b_{r-2}) \geqslant 0$$

for all such r. If R is even, then

$$d_{R/2+2} = c_0 b_{R/2+2} + c_1 b_{R/2+1} + c_0 b_{R/2} \geqslant c_0 b_{R/2-2} + c_1 b_{R/2-1} + c_0 b_{R/2},$$

so

$$d_{R/2+2} - d_{R/2} \geqslant (c_2 - c_0)\big(b_{R/2} - b_{R/2-2}\big) \geqslant 0.$$

If R is odd, then

$$d_{(R+3)/2} = c_0 b_{(R+3)/2} + c_1 b_{(R+1)/2} + c_2 b_{(R-1)/2} \geqslant c_0 b_{(R-3)/2} + c_1 b_{(R+1)/2} + c_2 b_{(R-1)/2},$$

so

$$\begin{aligned} d_{(R+3)/2} - d_{(R+1)/2} &\geqslant (c_1 - c_0) b_{(R+1)/2} + (c_2 - c_1) b_{(R-1)/2} + (c_0 - c_2) b_{(R-3)/2} \\ &\geqslant (c_1 - c_0) b_{(R-1)/2} + (c_2 - c_1) b_{(R-1)/2} + (c_0 - c_2) b_{(R-3)/2} \\ &= (c_2 - c_0)\big(b_{(R-1)/2} - b_{(R-3)/2}\big) \geqslant 0. \end{aligned}$$

Thus, in either case, $d_r \leqslant d_{R+2-r}$ for all $r \leqslant (R+2)/2$. Finally, if $c_0 < c_2$, then, repeating the steps above, $d_r < d_{R+2-r}$ for all $r \leqslant (R+2)/2$. *Q.E.D.*

I now conclude the proof of Proposition 4.1. Suppose $p_n > \frac{1}{2}$, and that the $n-3$ districts in $N - \{i, k, n\}$ can be paired such that, for each pair (j, l), $p_j + p_l \geqslant 1$. Denote these $(n-3)/2$ pairs by $(p_j, p_{1(j)})$, $j \in C$, where $C \subseteq N - \{i, k, n\}$ with $\#C = (n-3)/2$. Then the probability generating function g can be written as

$$g(t) = \left[\prod_{j \in C} (1 - p_j + p_j t)(1 - p_{1(j)} + p_{1(j)} t)\right][1 - p_n + p_n t].$$

For each $j \in C$, let

$$c_{j0} + c_{j1} t + c_{j2} t^2 = (1 - p_j + p_j t)(1 - p_{1(j)} + p_{1(j)} t),$$

and let

$$\sum_{r=0}^{n-3} d_r t^r = \prod_{j \in C} (1 - p_j + p_j t)(1 - p_{1(j)} + p_{1(j)} t) = \prod_{j \in C} \left(c_{j0} + c_{j1} t + c_{j2} t^2\right).$$

By Lemma 1, $c_{j0} < c_{j1}$ and $c_{j0} \leqslant c_{j2}$ for all j, with $c_{j0} = c_{j2}$ iff $p_j + p_{1(j)} = 1$. Thus, applying Lemma 2 repeatedly, $d_0, \ldots, d_{n-3}$ satisfy $d_{r-1} < d_r$ and $d_r \leqslant d_{n-3-r}$ for all $r \leqslant (n-3)/2$.

Now, let

$$\sum_{r=0}^{n-2} e_r t^r = g(t) = \left[\sum_{r=0}^{n-3} d_r t^r\right][1 - p_n + p_n t].$$

Then

$$e_{(n-3)/2} = (1 - p_n) d_{(n-3)/2} + p_n d_{(n-5)/2}, \qquad \text{and}$$

$$e_{(n-1)/2} = (1 - p_n) d_{(n-1)/2} + p_n d_{(n-3)/2} \geqslant (1 - p_n) d_{(n-5)/2} + p_n d_{(n-3)/2},$$

so

$$e_{(n-1)/2} - e_{(n-3)/2} \geqslant (2p_n - 1)\left(d_{(n-3)/2} - d_{(n-5)/2}\right) > 0. \qquad Q.E.D.$$

REFERENCES

Aranson, P. H., M. J. Hinich, and P. C. Ordeshook (1978): "Election Goals and Strategies: Equivalent and Nonequivalent Candidate Objectives," *American Political Science Review*, 68, 135–152.

Brams, S. J., and M. D. Davis (1973): "Resource-Allocation Models in Presidential Campaigning: Implications for Democratic Representation," *Annals of the New York Academy of Sciences*, 219, 105–123.

——— (1974): "The 3/2's Rule in Presidential Campaigning," *American Political Science Review*, 68, 113–134.

——— (1975): "Comment on 'Campaign Resource Allocations under the Electoral College'," *American Political Science Review*, 69, 115–116.

Bronars, S. G. (1986): "Strategic Behavior in Tournaments," mimeo, 1986.

Colantoni, C. S., T. J. Levesque, and P. C. Ordeshook (1975a): "Campaign Resource Allocations Under the Electoral College," *American Political Science Review*, 69, 141–154.

——— (1975b): "Rejoinder to 'Comment' by S. J. Brams and M. D. Davis," *American Political Science Review*, 60, 157–161.

Friedman, L. (1958): "Game-Theory Models in the Allocation of Advertising Expenditures," *Operations Research*, 6, 699–709.

Hinich, M. J., and P. C. Ordeshook (1970): "Plurality Maximization vs. Vote Maximization: A Spatial Analysis with Variable Participation," *American Political Science Review*, 64, 772–791.

Jacobson, G. C. (1980): *Money in Congressional Elections*. New Haven: Yale University Press.

——— (1984): "Money in the 1980 and 1982 Congressional Elections," in *Money and Politics*, ed. by

M. J. Malbin. Washington, D.C.: American Enterprise Institute for Public Policy Research.

JACOBSON, G. C., AND S. KERNELL (1984): "Party Organization and the Efficient Distribution of Congressional Campaign Resources: Republicans and Democrats in 1982," mimeo.

KARLIN, S. (1959): *Mathematical Methods and Theory of Games, Programming and Economics: The Theory of Infinite Games*, Vol. 2. Reading, Mass: Addison-Wesley.

KRAMER, G. H. (1966): A Decision-Theoretic Analysis of a Problem in Political Campaigning," in *Mathematical Applications in Political Science*, Vol. II, ed. by J. L. Berndt. Dallas: Southern Methodist University Press.

ROSEN, S. (1986): "Prizes and Incentives in Elimination Tournaments," *American Economic Review*, 76, 701–709.

SANKOFF, D., AND K. MELLOS (1972): "The Swing Ratio and Game Theory," *American Political Science Review*, 66, 551–554.

TULLOCK, G. (1980): "Efficient Rent-Seeking," in *Towards a Theory of the Rent-Seeking Society*, ed. by J. Buchanan, R. Tollison, and G. Tullock. College Station: Texas A & M University Press.

WELCH, W. P. (1981): "Money and Votes: A Simultaneous Equation Model," *Public Choice*, 36, 209–234.

YOUNG, H. P. (1978): "The Allocation of Funds in Lobbying and Campaigning," *Behavioral Science*, 23, 21–31.