

A Cake-Cutting Solution to Gerrymandering

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Abstract

I propose a mechanism for redistricting inspired by cake-cutting mechanisms for fair division. The majority party proposes a partition of a state into districts. The minority party can accept it or undo any partisan disadvantage caused by irregular boundaries. Thus without imposing any requirement of regularity, the mechanism ensures that to the extent that irregular districts result from the process, the minority is never harmed by them.

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1 Introduction

The United States Constitution requires that a population Census be taken every 10 years and representation in the House of Representatives be reapportioned among the states according to their population. The U.S. Supreme Court in *Wesberry v Sanders* has interpreted this provision as requiring in addition that Congressional districts within a given state be as equal in population size as is practical. Whether because of changes in relative population across states or changes in population density within a given state, these requirements necessitate a redrawing of district boundaries in response to Census counts. The Constitution leaves it to the States to decide how such re-districting will be executed.

In practice this has led to *gerrymandering*: convoluted district maps that are drawn by the party in power and designed to maintain and extend their electoral advantage. Recent cases at the United States Supreme Court have raised the possibility that partisan re-districting could be judged unconstitutional should a manageable criterion for discerning such practices be forthcoming. However, the Justices are doubtful that a one-size-fits-all standard is possible. For example Justice Anthony Kennedy wrote the following in a concurring opinion in *Veith v Jubiliter* which upheld a redistricting plan challenged in Pennsylvania.

Because there are yet no agreed upon substantive principles of fairness in districting, we have no basis on which to define

clear, manageable, and politically neutral standards for measuring the particular burden a given partisan classification imposes on representational rights. Suitable standards for measuring this burden, however, are critical to our intervention. Absent sure guidance, the results from one gerrymandering case to the next would likely be disparate and inconsistent.

Indeed, states have a variety of irregular shapes, voters are distributed unevenly across them and their political divides do not predictably relate to geography or man-made boundaries. Even when such relationships exist today they will change in unpredictable ways in the future. Articulating a standard that could be expected to capture electoral fairness uniformly across such idiosyncrasies seems impossible.

1.1 You Cut I Choose

But the same challenges do not prevent the fair division of a piece of cake, which can be accomplished simply and without external arbitration regardless of the size and shape of the piece. Every family has independently discovered the “You cut, I choose” mechanism whereby one of two children, arbitrarily appointed the first-mover, offers a share the cake to the second who is free to reject the offer and take instead the proposer’s share. This game has a unique subgame-perfect equilibrium outcome in which each child enjoys an equal share of the cake and yet the rules of

this game make no reference to the dimensions or other qualitative and quantitative features of the particular slice to be divided. The mechanism allows for arbitrary and even irregular divisions. Indeed irregular divisions would be required if the given piece of cake came with its own irregularities. But by including a special form of veto power, the mechanism ensures that no child is ever *disadvantaged* by any irregular division.

I propose a redistricting game based on the same principles. The majority party moves first and proposes a partition of the state into districts. The minority party can either accept the proposal or veto it and enforce a modification of the proposed partition. The key ingredient of the mechanism is the definition of an allowable modification. The minority is not allowed to alter any regularly shaped district boundaries but is allowed to redraw boundaries that have irregular shapes.

Irregular districts proposed by the majority may or may not have been designed for political advantage. The mechanism empowers the minority to defend against gerrymander by redrawing any irregular districts provided she respects any larger, regular-shaped boundaries present in the proposed map. In this way, without *imposing* any requirement of regularity, the mechanism ensures that to the extent that irregular districts result from the process, the minority is never harmed by them.

In this paper I equate “regularity” with convexity. I thus formalize the feasible challenges for the minority party as follows. If the majority has proposed a re-districting plan P then we consider the finest convex coars-

ening of P . This is the partition of the state obtained by taking unions of any non-convex districts until all of the sets are convex. The resulting convex partition Q defines the regular boundaries that the minority must respect. The minority party can then redraw, i.e. refine, the partition into any convex, equal sized, districts within these boundaries. Formally, the minority party can either accept the original proposal P or select any re-districting map that is a convex refinement of Q .

To illustrate, consider [Figure 1](#) below. Suppose that the majority party proposes the gerrymandered map on the left. This map has two regular districts and two irregular districts. The finest convex coarsening yields the partition on the right, with the boundary between the yellow and pink districts removed. The mechanism empowers the minority party to redraw the removed boundary, thus eliminating any partisan advantage from the gerrymander. The minority party cannot alter any of the other boundaries.

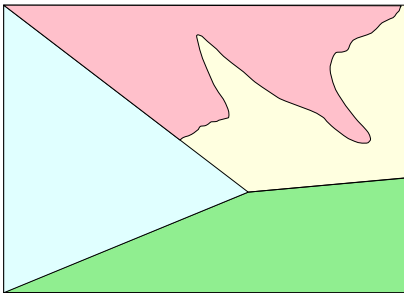


Figure (a) Gerrymandered proposal

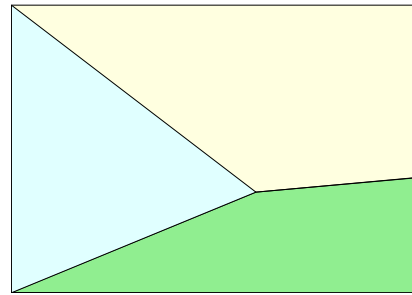


Figure (b) Finest convex coarsening

Figure 1: Illustration of the mechanism

Convexity plays a dual role in the analysis of this mechanism. First, convexity is an appealing property *per se* for an electoral district. It is a politically neutral criterion that captures regularity. But in addition to its role as a formalization of regularity, convexity is crucial for ensuring that the mechanism is well-behaved. I show that for any proposed redistricting map P , the set of feasible challenges $\mathcal{Q}(P)$ is compact and that the two parties' payoffs are continuous functions of $Q \in \mathcal{Q}(P)$. This guarantees that the minority party has a sequential best-reply and in turn that the majority party has an optimal strategy. These properties are not satisfied by arbitrary, non-convex partitions.¹

In a subgame-perfect equilibrium the minority party chooses its best-response to any map proposed by the majority party. Thus the minority party is no worse off than its most preferred feasible re-drawing. The majority party, foreseeing whether and how the minority party would redraw any proposed map, chooses its optimal proposal. Of course the majority party always has the option of offering any convex partition that appears in the range of the minority's equilibrium strategy. An already-convex proposal leaves no boundaries for the minority to re-draw. Thus, in equilibrium, the majority is no worse off than its most-preferred convex district map.

¹For example, Lebesgue measure is not a continuous function in the standard Hausdorff topology on the space of all closed sets. I model party vote shares as a measure on the state. Without a restriction like convexity these vote shares, and hence payoffs, will generally not be continuous functions on the strategy space.

In this paper I assume that each party has the goal of maximizing the number of districts won. This makes the redistricting mechanism a zero-sum game and therefore the inequalities in the preceding paragraph are in fact equalities.²

Of course in practice, other considerations justify district boundaries that violate convexity. For example, districts may align with county or municipal boundaries which themselves are irregularly shaped. Or there may be geographical features that naturally define distinct but irregular regions of a state. These considerations can be included in the mechanism, still preserving its tractability. As an illustration I show in [Appendix A](#) that for any exogenously given collection \mathcal{F} of subsets of the state (representing, say, unions of Counties), we can add to the set of “regular” district maps all partitions obtained by taking intersections of convex sets with sets in \mathcal{F} . [Figure 2](#) shows an example. The left panel depicts a state with two non-convex county boundaries. The right panel shows a partition of this state into four districts that have one irregular boundary inherited from the exogenous county lines.

More generally, convexity is just one way of defining regular district boundaries. The mechanism is flexible and can accomodate other definitions provided the requisite technical conditions are satisfied to guarantee existence of best-responses. Furthermore, for some realistic political ob-

²The same would be true if each party was maximizing the probability of winning a majority of districts.

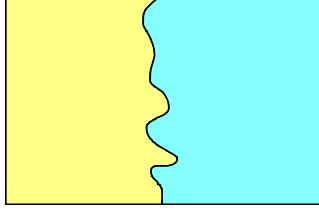


Figure (a) Exogenous county boundaries

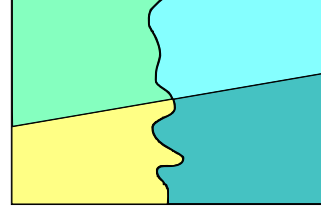


Figure (b) A partition that uses county lines

Figure 2: Incorporating exogenous boundaries

jectives the redistricting problem is not zero-sum. These extensions are discussed further in the concluding section [Section 6](#).

2 Related Literature

There is a large formal literature on redistricting. One strand of the literature is concerned with devising measures of geographic regularity of districts. For example [Fryer Jr and Holden \(2011\)](#) propose a measure of the compactness of districts while [Chambers and Miller \(2010\)](#) and [Chambers and Miller \(2013\)](#) attempt to formally classify highly irregular districts. Normative measures based on fairness to parties and/or voters have also been studied. For example [McGhee \(2014\)](#) and [Stephanopoulos and McGhee \(2015\)](#) proposed a popular measure known as the efficiency gap which received attention by the Supreme Court in *Gill v Whitford*.³

³See [Chambers, Miller and Sobel \(2017\)](#) for a critique.

Coate and Knight (2007) analyze the social planner’s redistricting problem when the objective is utilitarian welfare of voters.

The strategy of redistricting is the subject of Friedman and Holden (2008*b*), Friedman and Holden (2008*a*), and Gul and Pesendorfer (2010). These papers study optimal gerrymandering in the dictatorial setting (i.e. no recourse for the minority party) and without geographical constraints.

Most closely related are Landau, Reid and Yershov (2009) and Pegden, Procaccia and Yu (2017) which also consider sequential redistricting games empowering both the majority and the minority. In Landau, Reid and Yershov (2009), an impartial third party divides the state into two regions and the two parties engaged in a structured negotiation over the right to exclusively redistrict each region. Pegden, Procaccia and Yu (2017) analyze a protocol in which the two parties take turns drawing districts until the state is fully partitioned. Neither has an explicit geographic model and both assume that voters can be arbitrarily assigned to districts. The technical results presented here could also be used to formally analyze equilibria in these and other protocols using an explicit model of voters distributed within a state.

3 Model

There is a state S that must be divided into K districts of equal population size. The state is populated by voters each of whom is affiliated with

one of two parties referred to as Left and Right. We model the state as a closed subset $S \subset \mathbf{R}^2$ of the plane, and we describe party representation by two measures μ_L and μ_R . For any (measurable) subset $E \subset S$, the measures $\mu_L(E)$ and $\mu_R(E)$ quantify the total mass of Left-affiliates and Right-affiliates respectively. We assume μ_L and μ_R are absolutely continuous with respect to Lebesgue measure and that the total population of any subset E is $\mu(E) = \mu_R(E) + \mu_L(E)$. Normalize the total population of the state $\mu(S)$ to equal K so that each district will have measure 1.

A redistricting plan is a partition of S . A partition is a family P of subsets $\{P_1, P_2, \dots, P_K\}$ satisfying

1. $P_j \subset S$ is closed,
2. $\mu(P_j \cap P_l) = 0$ for all j, l
3. $\bigcup_j P_j = S$.

Let \mathcal{T} be the collection of all partitions. A redistricting partition is a partition with K elements with equal population size $\mu(P_j) = 1$. Let \mathcal{P} be the collection of all district partitions.

Voting is noisy. If λ is the fraction of Left-affiliates in a given district then the probability that the Left party candidate wins the district is $\omega(\lambda)$ where $\omega : [0, 1] \rightarrow [0, 1]$ is an increasing continuous function satisfying $\omega(\lambda) = 1 - \omega(1 - \lambda)$. Each party's payoff is equal to the number of districts it wins.

4 Redistricting Game

It simplifies notation to assume S is a convex set.⁴ Let \mathcal{O} be the collection of partitions of S consisting of convex subsets, and \mathcal{Q} the collection of convex district partitions. A convex partition $P \in \mathcal{O}$ is a convex coarsening of P' if each element of P is a union of elements of P' . We also say that P' is a refinement of P . Every partition has a unique finest convex coarsening, i.e. the convex coarsening such that there is no other convex coarsening that refines it.

Say that a partition $Q \in \mathcal{P}$ is a feasible challenge to a partition P if $Q \in \mathcal{Q}$ is a convex refinement of the finest convex coarsening T of P and $Q_l \subset T(P_l)$ where $T(P_l)$ is the element of T that contains P_l .

The redistricting game is the following extensive-form game with perfect information. The Left party moves first and proposes a redistricting plan $P \in \mathcal{P}$. Next the Right party moves and either accepts P or selects a feasible challenge. Either way the game ends and the partition chosen by Right is enacted.

Let $u_R(P)$ and $u_L(P)$ denote the expected payoffs from a partition P . Note that these payoffs depend only on the party representations of each district.

⁴Appendix A describes how to adjust the definitions for non-convex S , leaving all of the results unchanged.

5 Outcome

Theorem 1. *An equilibrium exists. All equilibria are payoff equivalent to the outcome in which the majority party proposes its most favorable convex partition and the minority accepts.*

As in any two-stage bargaining problem, in equilibrium the second-mover chooses a best-response to any proposal by the first-mover and the first-mover's proposal is the one that induces her most-favorable best-response. The technical challenge is then two-fold. First we must show that the second-mover Right party has a best-response to any proposal and secondly we must show that the range of best-responses has an optimal element for the first-mover Left party.

I show below that for any partition P , the set of equal-population convex refinements is a compact space with continuous payoffs. This ensures existence of a best-response for the Right party. The same result implies that the Left party has a most favorable convex partition. Moreover the first-mover can secure a payoff no worse than its most favorable convex partition by simply offering it. Finally the zero-sum nature of the game is used to prove that there is no proposal which can induce a best-response leading to any higher payoff than this. Thus, offering her most favorable convex partition is a best-response for the first-mover establishing the existence and payoff equivalence results in the Theorem.

In particular, let $\mathcal{Q}(P)$ be the set of feasible challenges to a proposal

P . In response to any proposal $P \in \mathcal{P}$, the strategy σ_R of the Right party selects

$$\sigma_R(P) \in \operatorname{argmax}_{Q \in \{P\} \cup \mathcal{Q}(P)} u_R(Q).$$

We show that the argmax is always non-empty.

The Left party thus earns $\pi_L(P) = u_L(\sigma_R(P))$ from any proposal P .

We show that there exists a $P^* \in \mathcal{P}$ such that

$$\pi_L(P^*) = \sup_{P \in \mathcal{P}} \pi_L(P).$$

The Left party proposes (any such) P^* .

Note that the mechanism provides a first-mover advantage to the party in the majority while providing protection to the minority. Indeed while the minority party is no worse off than the least favorable convex partition, the majority party effectively has the advantage of choosing among these.

6 Conclusion

Convexity plays two roles in the mechanism. It is a way of describing regular boundaries and it ensures the existence of equilibrium. Weaker conditions however could suffice for both. A goal for future research is to characterize well-behaved subspaces of the space of all partitions. Convex partitions comprise just one such subspace.

Achieving this would have an additional benefit. The present formu-

lation uses the zero-sum nature of the game in establishing existence of a best-response for the first mover. The results on convexity are not enough because the first mover is not constrained to offer a convex partition. Finding a weak but still tractable restriction on the space of first-mover offers would allow us to use the same mechanism for non-zero sum objectives of which there are many natural examples. If each party's goal is to maximize the probability of winning a super-majority of districts, for example, then the game is not zero sum.

These extensions are the subject of ongoing work.

7 Proofs

Let $d(x, x')$ be the Euclidean distance between points $x, x' \in S$. The Hausdorff distance between any two closed, non-empty subsets $F \subset S$ and $E \subset S$ is

$$h(F, E) = \max \left\{ \max_{x \in F} \min_{x' \in E} d(x, x'), \max_{x' \in E} \min_{x \in F} d(x, x') \right\}$$

The space \mathcal{F} of closed, non-empty, subsets of S endowed with the Hausdorff distance is a compact metric space. The limit of a convergent sequence F^t from \mathcal{F} is equal to the topological \limsup of F^t , namely the set of points x such that for every neighborhood V of x , there are infinitely many t such that $F^t \cap V \neq \emptyset$. The following immediate implications will

be used in the results that follow.

1. If $F^t \rightarrow F$ and $x \in F$ then there exists a sequence $x^t \in F^t$ converging to x .
2. If $F^t \rightarrow F$ and for all t , $F^t \subset E \in \mathcal{F}$, then also $F \subset E$.
3. When they exist, $\lim [F^t \cap E^t] \subset [\lim F^t] \cap [\lim E^t]$

Proposition 1. *Let $\mathcal{C} \subset \mathcal{F}$ be the subspace of convex subsets.*

1. \mathcal{C} is compact.
2. For any measure μ that is absolutely continuous with respect to Lebesgue measure, the map

$$\mu : \mathcal{C} \rightarrow \mathbf{R}$$

is continuous.

The following lemma is used in the proof of [Proposition 1](#) and subsequent results.

Lemma 1. *If C'' is a closed subset of the interior of $C \in \mathcal{C}$, then there exists $\varepsilon > 0$ such that $C'' \subset C'$ for every convex $C' \in B(C, \varepsilon)$.*

Proof. For each point $x \in C''$ there are three distinct elements x_1, x_2, x_3 of C such that x is in the interior of the convex hull of $\{x_1, x_2, x_3\}$. Let N_x be a neighborhood of x that is included in the interior of the convex hull of $\{x_1, x_2, x_3\}$.

By continuity there exist neighborhoods $N_{x_1}, N_{x_2}, N_{x_3}$ of x_1, x_2 , and x_3 respectively such that $N_x \subset \text{conhull} \{y_1, y_2, y_3\}$ for any selection $y_l \in N_{x_l}$. The family $\{N_x : x \in C''\}$ is an open covering of the compact set C'' and hence it has a finite subcovering. Let A be the finite set of associated points x , and $\{N_{x_l} : x \in A; l = 1, 2, 3\}$ be the associated finite family of neighborhoods.

Let $\varepsilon > 0$ be small enough so that $B(x_l, \varepsilon) \subset N_{x_l}$ for every $x \in A$ and $l = 1, 2, 3$. If $C' \in B(C, \varepsilon)$ then

$$\sup_{x \in C} \inf_{x' \in C'} d(x, x') < \varepsilon.$$

In particular $C' \cap N_{x_l} \neq \emptyset$ for all $x \in A$ and $l = 1, 2, 3$. Let y_{x_l} be an element of the intersection. We thus have

$$C'' \subset \bigcup_{x \in A} N_x \subset \text{conhull} \{y_{x_l} : x \in A, l = 1, 2, 3\} \subset C'$$

since the latter is convex. □

Proof of Proposition 1. The space of closed subsets of S is compact in the Hausdorff metric so any sequence C_n from \mathcal{C} has a convergent subsequence. Let C be the limit. To show that \mathcal{C} is compact it is enough to show that C is convex. Take any two points x, x' belonging to C and suppose $y = \tau x + (1 - \tau)x'$ for some $\tau \in [0, 1]$. There exist sequences $x_n \in C_n$ and $x'_n \in C_n$ converging to x and x' respectively. Since C_n is convex, the point

$\tau x_n + (1 - \tau)x'_n$ belongs to C_n and the sequence of such points converges to y by continuity.

Next we show that $\mu : \mathcal{C} \rightarrow \mathbf{R}$ is both upper- and lower-semicontinuous. Pick $\delta > 0$, and let C be any element of \mathcal{C} . The first goal is to show that there exists $\varepsilon > 0$ such that every element C' of the ball $B(C, \varepsilon)$ has measure less than $\mu(C) + \delta$. Take ε small enough so that the set

$$C^\varepsilon = \bigcup_{x \in C} B(x, \varepsilon)$$

has measure $\mu(C^\varepsilon) < \mu(C) + \delta$. This can be done because μ is absolutely continuous with respect to Lebesgue measure. Now if $C' \in B(C, \varepsilon)$ then

$$\sup_{x \in C'} \inf_{x' \in C} d(x, x') < \varepsilon$$

i.e. $C' \subset C^\varepsilon$ and therefore $\mu(C') < \mu(C^\varepsilon) < \mu(C) + \delta$.

The next goal is to show that there exists $\varepsilon > 0$ such that every element C' of the ball $B(C, \varepsilon)$ has measure greater than $\mu(C) - \delta$. Pick a closed subset C'' of the interior of C such that $\mu(C'') > \mu(C) - \delta$. This is possible because μ is absolutely continuous with respect to Lebesgue measure. By [Lemma 1](#) $C'' \subset C'$ and hence $\mu(C') > \mu(C'') > \mu(C) - \delta$. \square

An immediate implication of [Proposition 1](#) is that the subspace of \mathcal{C} consisting of unit measure sets is compact. Let \mathcal{F}^K be the K -fold Cartesian product of the space \mathcal{F} of S with itself. It is compact in the product

topology. For any $P \in \mathcal{P}$ the set $\mathcal{Q}(P)$ is a subspace.

Lemma 2. *For any $P \in \mathcal{P}$, the subspace $\mathcal{Q}(P)$ is compact.*

Proof. It is enough to show that $\mathcal{Q}(P)$ is closed. If $Q^s \rightarrow Q \in \mathcal{F}^K$ then each $Q_j^s \rightarrow Q_j$. To show that Q belongs to $\mathcal{Q}(P)$ first note that [Proposition 1](#) implies that each Q_j is convex with unit population. Next, every Q^s is a refinement of the finest convex coarsening T of P and in particular $Q_j^s \subset T(P_j)$. It follows that $Q_j \subset T(P_j)$ and therefore that Q is also a refinement of T .

The proof is concluded by showing that $\mu(Q_j \cap Q_i) = 0$. Suppose instead $\mu(Q_j \cap Q_i) > 0$. Then there is a closed subset C'' of the interior of $Q_j \cap Q_i$ such that $\mu(C'') > 0$. Since $Q_j^s \rightarrow Q_j$ and $Q_i^s \rightarrow Q_i$, for any $\varepsilon > 0$ we have $Q_j^s \in B(Q_j, \varepsilon)$ and $Q_i^s \in B(Q_i, \varepsilon)$ for sufficiently large s . Since C'' is a closed subset of the interior of both Q_j and Q_i , by [Lemma 1](#) we can pick ε small enough so that $C'' \subset Q_j^s \cap Q_i^s$, implying $\mu(Q_j^s \cap Q_i^s) \geq \mu(C'') > 0$, a contradiction. \square

There exist partitions whose finest convex coarsening is the trivial partition consisting of the single set S . Therefore [Lemma 2](#) also implies that the full subspace \mathcal{O} of convex district partitions is compact.

Lemma 3. *For any $P \in \mathcal{P}$ the expected payoff functions $u_R(P)$ and $u_L(P)$ are continuous on $\mathcal{Q}(P)$*

Proof. The payoffs are continuous in the representation shares of each district. By [Proposition 1](#) the representation shares are continuous functions

on $\mathcal{Q}(P)$. □

Proof of Theorem 1. We characterize the set of all subgame-perfect equilibria of the game. Let Σ_R^* be the set of strategies σ_R such that

$$\sigma_R(P) \in \operatorname{argmax}_{P' \in \{P\} \cup \mathcal{Q}(P)} u_R(P')$$

for every $P \in \mathcal{T}$. Note that the argmax and hence the set Σ_R^* are not empty by Lemma 2 and Lemma 3. Define

$$\Pi_L^* = \{u_L(Q) : Q \in \mathcal{Q}(P) \text{ for some } P \in \mathcal{P}\}.$$

and for any $\sigma_R \in \Sigma_R^*$

$$\Sigma_L^*(\sigma_R) = \{P : u_L(\sigma_R(P)) \geq \sup \Pi_L^*\}.$$

The set $\Sigma_L^*(\sigma_R)$ is non-empty because by Lemma 2 and Lemma 3 there exists $Q \in \mathcal{Q}(P)$ such that $u_L(Q) = \sup \Pi_L^*$ and note that $\mathcal{Q}(Q) = \{Q\}$ and hence $\sigma_R(Q) = Q$.

A profile (σ_L^*, σ_R^*) is a subgame perfect equilibrium if and only if $\sigma_R^* \in \Sigma_R^*$ and $\sigma_L^* \in \Sigma_L^*(\sigma_R^*)$. To prove this, first note that by definition Σ_R^* is the entire set of sequentially rational strategies for Right. We conclude the proof by showing that $\Sigma_L^*(\sigma_R)$ is the entire set of best-replies for Left to any $\sigma_R \in \Sigma_R^*$.

First, since $\Sigma_L^*(\sigma_R)$ is non-empty, any best-response to σ_R must be an

element. Indeed every element is a best-response as we now show. Since $\sigma_R^* \in \Sigma_R^*$, for any $P \in \mathcal{P}$,

$$u_R(\sigma_R^*(P)) \geq \max_{Q \in \mathcal{Q}(P)} u_R(Q)$$

and therefore since $\sigma_L^* \in \Sigma_L^*(\sigma_R)$ and $u_L(P) = K - u_R(P)$,

$$u_L(\sigma_R^*(\sigma_L^*)) \geq \sup \Pi_L^* \geq u_L(\operatorname{argmax}_{Q \in \mathcal{Q}(P)} u_R(Q)) \geq u_L(\sigma_R^*(P)).$$

so that σ_L^* is a best response. □

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A Appendix

If S is not convex we can treat it as a convex set without loss of generality. Define a convex subset of S to be any subset $C \subset S$ such that $[\text{conhull } C] \cap S = C$. This is equivalent to saying that any convex combination of points in C is either inside C or entirely outside of S .

A convex partition of S is defined to be a convex partition of $\text{conhull } S$. If we set $\mu_L(S \setminus \text{conhull } S) = \mu_R(S \setminus \text{conhull } S) = \mu(S \setminus \text{conhull } S)$ then any

partition $\{P_1, P_2, \dots, P_K\}$ of $\text{conhull } S$ satisfying [item 1](#), [item 2](#), and [item 3](#) will satisfy the same conditions for the collection of sets $P_j \cap S$.

A.1 Adding County or Other Natural Boundaries

A closed subset E of S is regular if $\overline{E^\circ} = E$. Let \mathcal{F} be a finite collection of closed, regular subsets of S . Consider the following family of admissible district boundaries.

$$\mathcal{A} = \{E : E = \overline{[C \cap F]^\circ} \text{ for } C \in \mathcal{C} \text{ and } F \in \mathcal{F}\}$$

We establish the analog of [Proposition 1](#) for the class of sets \mathcal{A} . The remaining arguments are straightforward adaptations of previous results and are omitted.

Proposition 2. 1. \mathcal{A} is compact.

2. For any measure μ that is absolutely continuous with respect to Lebesgue measure, the map

$$\mu : \mathcal{A} \rightarrow \mathbf{R}$$

is continuous.

[Proposition 1](#) is proven using a series of lemmas.

Lemma 4. Suppose C^t is a sequence of closed convex sets converging to C and $F \subset S$ is closed. If the sequence $\overline{[C^t \cap F]^\circ}$ converges then its limit is $\overline{[C \cap F]^\circ}$.

Proof. Let A be the limit of the sequence $\overline{[C^t \cap F]^\circ}$. We first show that $\overline{[C \cap F]^\circ} \subset A$.

Let $x \in [C \cap F]^\circ$. Then the closed ball $\overline{B(x, \varepsilon)}$ is a subset of $[C \cap F]^\circ$ for sufficiently small $\varepsilon > 0$. By [Lemma 1](#), $\overline{B(x, \varepsilon)} \subset C^t$ for all t sufficiently large and hence $\overline{B(x, \varepsilon)} \subset [C^t \cap F]^\circ$. Thus $x \in \overline{[C^t \cap F]^\circ}$ for all sufficiently large t implying $x \in A$.

To prove the reverse inclusion, first let $x \in A^\circ$ and let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subset A$. We will show by contradiction that $B(x, \varepsilon) \subset C \cap F$. Suppose instead there exists $y \in B(x, \varepsilon) \cap \neg[C \cap F]$. Define $D = \lim_{t \rightarrow \infty} [C^t \cap F]$ (passing to a subsequence if necessary). By [item 3](#) $D \subset C \cap F$, hence $y \in B(x, \varepsilon) \cap \neg D$.

Since $B(x, \varepsilon) \cap \neg D$ is open there is $\varepsilon' > 0$ such that $B(y, \varepsilon') \subset B(x, \varepsilon) \cap \neg D \subset A \cap \neg D$. Since $y \in A$ there exists a sequence $y^t \rightarrow y$ with $y^t \in \overline{[C^t \cap F]^\circ} \subset C^t \cap F$. And in particular $y^t \in B(y, \varepsilon')$.

Therefore

$$\begin{aligned} h(C^t \cap F, D) &\geq \sup_{z \in C^t \cap F} \inf_{w \in D} d(z, w) \\ &\geq \inf_{w \in D} d(y^t, w) \\ &> \varepsilon', \end{aligned}$$

which is a contradiction since $C^t \cap F \rightarrow D$.

Therefore $B(x, \varepsilon) \subset C \cap F$ so that $x \in [C \cap F]^\circ$ and we have shown that

$A^\circ \subset [C \cap F]^\circ$. We conclude

$$A \subset \overline{A^\circ} \subset \overline{[C \cap F]^\circ}.$$

□

Lemma 5. *If $A = \overline{[C \cap F]^\circ} \in \mathcal{A}$ then*

$$\mu(A) = \mu(C \cap F)$$

for any μ that is absolutely continuous with respect to Lebesgue measure.

Proof. Because $\overline{[C \cap F]^\circ} \subset C \cap F$ we have

$$\mu(A) = \mu(C \cap F) - \mu([C \cap F] \setminus A)$$

We will show that $[C \cap F] \setminus A$ is a subset of the boundary of C . Since the boundary of any convex subset has Lebesgue measure zero, this will conclude the proof.

Suppose $x \in C^\circ \cap F$. For all sufficiently small $\varepsilon > 0$, $B(x, \varepsilon) \subset C^\circ$. Moreover since F is regular we have $x \in F = \overline{F^\circ}$ and thus $B(x, \varepsilon) \cap F^\circ \neq \emptyset$. Indeed $B(x, \varepsilon) \cap [C^\circ \cap F^\circ] \neq \emptyset$ for all sufficiently small $\varepsilon > 0$. We conclude $x \in A$.

Thus $C^\circ \cap F \subset A$. We now have

$$\begin{aligned} [C \cap F] \setminus A &\subset [C \cap F] \setminus [C^\circ \cap F] \\ &= [C \cap F] \setminus C^\circ \\ &\subset C \setminus C^\circ. \end{aligned}$$

□

Lemma 6. *Let C^t be a sequence from \mathcal{C} and let $F \in \mathcal{F}$. Suppose $[C^t \cap F] \rightarrow E$ and $C^t \rightarrow C$. Then $\mu(E) = \mu(C \cap F)$ for any μ that is absolutely continuous with respect to Lebesgue measure.*

Proof. By [item 3](#) we have $E \subset C \cap F$. We will establish the Lemma by proving that $\mu([C \cap F] \setminus E) = 0$.

The first step is to show that $C^\circ \cap F \subset E$. Let $x \in C^\circ \cap F$. Take any neighborhood V of x . Pick $\varepsilon > 0$ small enough so that $B(x, \varepsilon) \subset C^\circ \cap V$. Since F is regular, $B(x, \varepsilon) \cap F^\circ \neq \emptyset$, in particular $B(x, \varepsilon) \cap [C^\circ \cap F^\circ] \neq \emptyset$.

Pick $y \in B(x, \varepsilon) \cap F^\circ$ and a ball $B(y, \varepsilon') \subset B(x, \varepsilon) \cap F^\circ \subset V \cap F^\circ$. We claim that $C^t \cap B(y, \varepsilon') \neq \emptyset$ for infinitely many t . If not then $\inf_{x' \in C^t} d(y, x') > \varepsilon'$ and hence $h(C^t, C) > \varepsilon'$ for all but finitely many t , contradicting that $C^t \rightarrow C$. Thus

$$\emptyset \neq C^t \cap B(y, \varepsilon') \subset [C^t \cap F] \cap V$$

for infinitely many t , and since V was an arbitrary neighborhood of x we conclude $x \in E$.

We now have

$$\begin{aligned}
[C \cap F] \setminus E &\subset [C \cap F] \setminus [C^\circ \cap F] \\
&= [C \cap F] \setminus C^\circ \\
&\subset C \setminus C^\circ.
\end{aligned}$$

Since the boundary of a convex set has Lebesgue measure zero this concludes the proof. \square

Proof of Proposition 2. To prove compactness, since \mathcal{A} is a subspace of the compact space of closed subsets of S , it is enough to show that \mathcal{A} is closed. This is immediate from Lemma 4: if \mathcal{A}^t is a convergent sequence of elements of \mathcal{A} , then by Lemma 4 its limit belongs to \mathcal{A} .

To prove continuity, let A^t be a sequence from \mathcal{A} and $A^t \rightarrow A$. Since \mathcal{F} is finite and \mathcal{C} is compact we may pass to a subsequence with $A^t = [C^t \cap F]$ for $C^t \rightarrow C \in \mathcal{C}$ and $F \in \mathcal{F}$. Applying Lemma 4, Lemma 5 and Lemma 6 we obtain

$$\mu(A^t) = \mu\left(\overline{[C^t \cap F]^\circ}\right) = \mu(C^t \cap F) \rightarrow \mu(C \cap F) = \mu\left(\overline{[C \cap F]^\circ}\right) = \mu(A).$$

\square