Appendix

1 Equilibrium Concepts

1.1 Scenario 1

The first possible scenario is

$$[(rrr)(dd\_)(d\_\_)]$$

R wins the first district and D wins the second district for sure. For the last district, we need to calculate the probilitity each player wins.
The probability d wins the third district is:

$$p_d = \frac{e_d^2 + 2e_r e_d}{(e_d + e_r)^2} \tag{1}$$

The expected payoff of a player d is:

$$E\pi_d = p_d v - e_d = \frac{e_d^2 + 2e_r e_d}{(e_d + e_r)^2} v - e_d \tag{2}$$

The standard modelling approach is to maximize (2) with respect to player d's effort, which yields the first-order condition:

$$\frac{2e_r^2}{(e_d + e_r)^3} v - 1 = 0 \tag{3}$$

Similarly, the probability r wins the third district is:

$$p_r = \frac{e_r^2}{(e_d + e_r)^2} \tag{4}$$

The expected payoff of r is:

$$E\pi_r = p_r v - e_r = \frac{e_r^2}{(e_d + e_r)^2} v - e_r \tag{5}$$

The standard modelling approach is to maximize (5) with respect to player r's effort, which yields the first-order condition:

$$\frac{2e_r e_d}{(e_d+e_r)^2}v - 1 = 0 \tag{6}$$

Assuming an interior solution, the Nash equilibrium outcome is attained by solving (3) and (6) simultaneously for e_d and e_r. This equilibrium is:

$$\frac{2e_r^2}{(e_d+e_r)^3} = \frac{2e_r e_d}{(e_d+e_r)^2} \tag{7}$$

$$e_d = e_r \tag{8}$$

Since e_d=e_r=e, the likelihood of d or r wins the third district is $\frac{3}{4}$ and $\frac{1}{4}$ respectively and the equilibrium level of effort is e_d=e_r=$\frac{1}{4}$v. The corresponding expected payoff of d and r are $\frac{1}{2}$v and 0 respectively.

1.2 Scenario 2

The second possible scenario is

$$[(rr\_)(dd\_)(dr\_)]$$

R wins the first district and D wins the second district for sure. We need to solve for the probilitity each player wins, but notice that both player has an advantage so it will be symmetric.
The probability d wins the third district is:

$$p_d = \frac{e_d}{(e_d+e_r)} \tag{9}$$

The expected payoff of a player d is:

$$E\pi_d = p_d v - e_d = \frac{e_d}{(e_d+e_r)}v - e_d \tag{10}$$

The standard modelling approach is to maximize (10) with respect to player d's effort, which yields the first-order condition:

$$\frac{e_r}{(e_d+e_r)^2}v - 1 = 0 \tag{11}$$

Assuming an interior solution, the Nash equilibrium outcome is attained by solving (11) simultaneously for e_d and e_r. This equilibrium is:

$$\frac{e_r}{(e_d+e_r)^2} = \frac{e_d}{(e_d+e_r)^2} \tag{12}$$

$$e_d = e_r \tag{13}$$

Since e_d=e_r=e, the likelihood of d or r wins the third district is $\frac{1}{2}$ and the equilibrium level of effort is e_d=e_r=$\frac{1}{4}$v. The corresponding expected payoff of d and r are $\frac{1}{4}$v.

1.3 Scenario 3

The third possible scenario is:

$$[(rrr)(ddd)(\_\_\_)]$$

R wins the first district and D wins the second district for sure. We need to solve for the probability each player wins, but no player has an advantage on the third district so it will be symmetric.
The probability d wins the third district is:

$$p_d = \frac{e_d^3 + 3e_d^2 e_r}{(e_d + e_r)^3} \tag{14}$$

The expected payoff of a player d is:

$$E\pi_d = p_d v - e_d = \frac{e_d^3 + 3e_d^2 e_r}{(e_d + e_r)^3} v - e_d \tag{15}$$

The standard modelling approach is to maximize (15) with respect to player d's effort, which yields the first-order condition:

$$\frac{6e_d e_r^2}{(e_d + e_r)^4} v - 1 = 0 \tag{16}$$

Assuming an interior solution, the Nash equilibrium outcome is attained by solving (16) simultaneously for e_d and e_r. This equilibrium is:

$$\frac{6e_d e_r^2}{(e_d + e_r)^4} = \frac{6e_d^2 e_r}{(e_d + e_r)^4} \tag{17}$$

$$e_d = e_r \tag{18}$$

Since e_d=e_r=e, the likelihood of d or r wins the third district is $\frac{1}{2}$ and the equilibrium level of effort is e_d=e_r=$\frac{3}{8}$v. The corresponding expected payoff of d and r are $\frac{1}{8}$v.

1.4 Scenario 4

The fourth possible scenario is:

$$[(rd\_)(rd\_)(rd\_)]$$

No player has an advantage in any district.
The probability d wins all three districts or at least two districts is:

$$p_d = p_1p_2p_3 + p_1p_2(1-p_3) + p_1(1-p_2)p_3 + (1-p_1)p_2p_3 \tag{19}$$

For simplification, define

$$\theta(\cdot) = (e_{d,1}+e_{r,1})(e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3})$$
$$\text{and}$$
$$\phi(\cdot) = e_{d,1}e_{d,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3} + e_{d,1}e_{r,2}e_{d,3} + e_{r,1}e_{d,2}e_{d,3}$$

Now, let's state the expected profit of player d in this particular case.

$$E\pi_d = \frac{\phi(e_{d,1},e_{d,2},e_{d,3};e_{r,1},e_{r,2},e_{r,3})}{\theta(e_{d,1},e_{d,2},e_{d,3};e_{r,1},e_{r,2},e_{r,3})}v - (e_{d,1}+e_{d,2}+e_{d,3}) \tag{20}$$

Player d seeks to maximize their expected profit function over their choice variables $e_{d,1}, e_{d,2}, e_{d,3}$ so we naturally find first order conditions:

$$[e_{d,1}] : \frac{\theta(\cdot)\phi_{e_{d,1}}(\cdot) + \phi(\cdot)\theta_{e_{d,1}}(\cdot)}{(\theta(\cdot))^2}v = 1 \tag{21}$$

$$[e_{d,2}] : \frac{\theta(\cdot)\phi_{e_{d,2}}(\cdot) + \phi(\cdot)\theta_{e_{d,2}}(\cdot)}{(\theta(\cdot))^2}v = 1 \tag{22}$$

$$[e_{d,3}] : \frac{\theta(\cdot)\phi_{e_{d,3}}(\cdot) + \phi(\cdot)\theta_{e_{d,3}}(\cdot)}{(\theta(\cdot))^2}v = 1 \tag{23}$$

Now player d would like to figure out their best response to whatever their opponent might do given their choice of other efforts. So, assuming. In other words, they would like a function $e_{d,1}$ of parameters $e_{d,2}, e_{d,3}, e_{r,1}, e_{r,2}, e_{r,3}$. This means there is some rearranging to be done.

Due to the symmetric nature of the problem at hand, focusing on equation (21) will provide insight into the other first order conditions. From (21) we have

$$\Big(\theta(\cdot)\phi_{e_{d,1}}(\cdot) + \phi(\cdot)\theta_{e_{d,1}}(\cdot)\Big)v = \Big(\theta(\cdot)\Big)^2$$

which is

$$\Big[\theta(\cdot)(e_{d,2}e_{d,3} + e_{d,2}e_{r,3} + e_{r,2}e_{d,3}) - \phi(\cdot)\Big((e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3})\Big)\Big]v = \Big(\theta(\cdot)\Big)^2$$

and making use of the definition of $\theta(\cdot)$ we have

$$\Big[\theta(\cdot)(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-\frac{\phi(\cdot)}{(e_{d,1}+e_{r,1})}\theta(\cdot)\Big]v=\Big(\theta(\cdot)\Big)^2$$

which, dividing by $\theta(\cdot)$, yields

$$\Big[(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-\frac{\phi(\cdot)}{(e_{d,1}+e_{r,1})}\Big]v=\theta(\cdot)$$

and multiplying by $(e_{d,1}+e_{r,1})$ gives us

$$\Big[(e_{d,1}+e_{r,1})(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-\phi(\cdot)\Big]v=\theta(\cdot)(e_{d,1}+e_{r,1}),$$

an equation that is actually more useful than it appears. Consider only the left hand side. Making use of the definition of $\phi(\cdot)$ we have

$$\Big[(e_{d,1}+e_{r,1})(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-e_{d,1}e_{d,2}e_{d,3}-e_{d,1}e_{d,2}e_{r,3}-e_{d,1}e_{r,2}e_{d,3}-e_{r,1}e_{d,2}e_{d,3}\Big]v$$

which can be rewritten as

$$\Big[(e_{d,1}+e_{r,1})(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-e_{d,1}(e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3})-e_{r,1}e_{d,2}e_{d,3}\Big]v$$

and defining $\psi=e_{d,2}e_{d,3}+e_{d,2}e_{r,3}+e_{r,2}e_{d,3}$ we have reduced the previous equality to

$$\Big[(e_{d,1}+e_{r,1})\psi-e_{d,1}\psi-e_{r,1}e_{d,2}e_{d,3}\Big]v=(e_{d,1}+e_{r,1})^2(e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3}).$$

We can further reduce this through cancelation of the ψ terms, leaving us with

$$\Big[e_{r,1}\psi-e_{r,1}e_{d,2}e_{d,3}\Big]v=(e_{d,1}+e_{r,1})^2(e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3}).$$

Again, consider the left hand side. After making use of the definition of ψ we have

$$\Big[e_{r,1}e_{d,2}e_{d,3}+e_{r,1}e_{d,2}e_{r,3}+e_{r,1}e_{r,2}e_{d,3}-e_{r,1}e_{d,2}e_{d,3}\Big]v$$

where the outer terms inside the brackets cancel such that our full equality now reduces through the process

$$\Big[e_{r,1}e_{d,2}e_{r,3}+e_{r,1}e_{r,2}e_{d,3}\Big]v=(e_{d,1}+e_{r,1})^2(e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3})$$

$$\frac{e_{r,1}e_{d,2}e_{r,3}+e_{r,1}e_{r,2}e_{d,3}}{(e_{d,2}+e_{r,2})(e_{d,3}+e_{r,3})}v=(e_{d,1}+e_{r,1})^2$$

to the relatively concise best response function

$$e_{d,1} = \sqrt{\frac{e_{r,1}e_{d,2}e_{r,3} + e_{r,1}e_{r,2}e_{d,3}}{(e_{d,2} + e_{r,2})(e_{d,3} + e_{r,3})}v} - e_{r,1} \tag{24}$$

Now, by symmetry we know

$$e_{r,1} = \sqrt{\frac{e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}}{(e_{r,2} + e_{d,2})(e_{r,3} + e_{d,3})}v} - e_{d,1}$$

or alternatively

$$e_{d,1} = \sqrt{\frac{e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}}{(e_{r,2} + e_{d,2})(e_{r,3} + e_{d,3})}v} - e_{r,1}$$

which means

$$\sqrt{\frac{e_{r,1}e_{d,2}e_{r,3} + e_{r,1}e_{r,2}e_{d,3}}{(e_{d,2} + e_{r,2})(e_{d,3} + e_{r,3})}v} - e_{r,1} = \sqrt{\frac{e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}}{(e_{r,2} + e_{d,2})(e_{r,3} + e_{d,3})}v} - e_{r,1}$$

$$\sqrt{\frac{e_{r,1}e_{d,2}e_{r,3} + e_{r,1}e_{r,2}e_{d,3}}{(e_{d,2} + e_{r,2})(e_{d,3} + e_{r,3})}v} = \sqrt{\frac{e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}}{(e_{r,2} + e_{d,2})(e_{r,3} + e_{d,3})}v}$$

$$\frac{e_{r,1}e_{d,2}e_{r,3} + e_{r,1}e_{r,2}e_{d,3}}{(e_{d,2} + e_{r,2})(e_{d,3} + e_{r,3})}v = \frac{e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}}{(e_{r,2} + e_{d,2})(e_{r,3} + e_{d,3})}v$$

$$e_{r,1}e_{d,2}e_{r,3} + e_{r,1}e_{r,2}e_{d,3} = e_{d,1}e_{r,2}e_{d,3} + e_{d,1}e_{d,2}e_{r,3}$$

$$e_{r,1}(e_{d,2}e_{r,3} + e_{r,2}e_{d,3}) = e_{d,1}(e_{r,2}e_{d,3} + e_{d,2}e_{r,3})$$

$$e_{r,1} = e_{d,1}$$

and, without loss of generality

$$e_{r,i} = e_{d,i} \text{ for } i \in \{1, 2, 3\}.$$

Applying this property to equation (24) we now see

$$e_{d,1} = \sqrt{\frac{e_{d,1}e_{d,2}e_{d,3} + e_{d,1}e_{d,2}e_{d,3}}{(e_{d,2} + e_{d,2})(e_{d,3} + e_{d,3})}v} - e_{d,1}$$

$$2e_{d,1} = \sqrt{\frac{2e_{d,1}e_{d,2}e_{d,3}}{4e_{d,2}e_{d,3}}v}$$

$$4(e_{d,1})^2 = \frac{e_{d,1}}{2}v$$

$$e_{d,1} = \frac{1}{8}v$$

And thus, with out loss of generality, we have

$$e_{j,i} = \frac{1}{8}v \text{ for } j \in \{d, r\} \text{ and } i \in \{1, 2, 3\}.$$

Since $e_{j,i} = \frac{1}{8}v$ and since $p_d = \frac{1}{2}$ player i now has expected profit

$$E\pi_i = \frac{1}{2}v - \frac{3}{8}v$$

$$E\pi_i = \frac{1}{8}v$$

which is similar to the previous scenario.