Problem definition: Determine a two-term expansion for the relationship between frequency and response amplitude for the following

$$m\ddot{x} + k_1 x + k_3 x^3 = mg \tag{1}$$

Solution approach: To calculate the stationary point of Equation (1), we need to set the time derivative equal to zero. By doing this we end up with a cubic polynomial as shown in Equation (2) that has a root x_0.

$$k_3 x_0^3 + k_1 x_0 - mg = 0 \tag{2}$$

x_0 is calculated as follows:

$$x_0 = \mathcal{A} - \frac{k_1}{3\mathcal{A}k_3} \quad , \quad \mathcal{A} = \left(\left(\frac{k_1^3}{27k_3^3} + \frac{m^2g^2}{4k_3^2}\right) + \frac{mg}{2k_3}\right)^{1/3} \tag{3}$$

We define a new variable, $x = x + x_0$, and substitute in Equation (1) to get the following:

$$\ddot{x} + \omega_0^2 x + \epsilon\alpha x^2 + \epsilon\beta x^3 = 0 \tag{4}$$

where

$$\omega_0^2 = \frac{k_1 + 3k_3x_0^2}{m}$$
$$\epsilon\alpha = \frac{3x_0k_3}{m}$$
$$\epsilon\beta = \frac{k_3}{m}$$

As can be seen in Equation (4), by shifting the coordinates to the equilibrium point, the constant term disappears. Next we define our differential operators as follow:

$$\frac{d}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2\left(D_1^2 + 2D_0D_2\right) \tag{5a}$$

$$\frac{d}{dt} = D_0 + \epsilon D_1 \tag{5b}$$

x is approximated using the following equation:

$$x(t) = x_0(T_0, T_1, T_2) + \epsilon x_1(T_0, T_1, T_2) \tag{6}$$

Substituting (5) and (6) in (4) and gathering the coefficients of ϵ^0 and ϵ^1 we get the following equations:

$$\epsilon^0: \quad D_0^2 x_0 + \omega_0^2 x_0 = 0 \tag{7a}$$

$$\epsilon^1: \quad D_0^2 x_1 + \omega_0^2 x_1 = -\left(2D_0D_1x_0 + \alpha x_0^2 + \beta x_0^3\right) \tag{7b}$$

For Equation (7a), we get the following solution. Here, we assumed that the amplitude of the solution, A, is only a function of T_1.

$$x_0\left(T_0, T_1\right) = Ae^{i\omega_0 T_0} + \bar{A}e^{-i\omega_0 T_0} \tag{8}$$

It should be noted that $\bar{A}$ is the complex conjugate of A. Substituting Equation (8) into Equation (7b) we get the following equation

$$D_0^2 x_1 + \omega_0^2 x_1 = -\alpha A\bar{A} - \alpha A^2 e^{2i\omega_0 T_0} - \beta A^3 3^{3i\omega_0 T_0} - \underbrace{\left[3\beta A^2 \bar{A} + 2i\omega_0 \frac{dA}{dT_1}\right] e^{i\omega_0 T_0}}_{\text{secular term}} + cc \tag{9}$$

where "cc" is the complex conjugate term. As can be seen above, the frequency of secular term is the same as the natural frequency of the system and it will cause resonance. For this not to happen, we set the coefficient of the secular term equal to zero. To solve the resulting differential equation for the secular term, we assume $A(T_1)$ as follow:

$$A(T_1) = a(T_1)e^{ib(T_1)} \Rightarrow \frac{dA}{dT_1} = A' = a'e^{ib} + iab'e^{ib} \tag{10}$$

Substituting Equation (10) in the secular term, we get the following system of equations for a and b:

$$\begin{cases} a'\omega_0 = 0 \\ 3\beta - 2ab'\omega_0 = 0 \end{cases} \Rightarrow \begin{cases} a = \text{constant} = \mathcal{C} \\ b = \dfrac{3\beta}{2\mathcal{C}\omega_0} T_1 \end{cases}$$

Therefore, $A(T_1)$ can be written as

$$A(T_1) = \mathcal{C} e^{\frac{3\beta}{2\mathcal{C}\omega_0} T_1 i} \tag{11}$$

The particular solution for Equation (9) is shown in the following equation. This can be derived using the frequency response function method.

$$x_1(T_1) = -\frac{\alpha A\bar{A}}{\omega_0^2} - \frac{\alpha A^2}{\omega_0^2 - 4\omega_0^2} e^{2i\omega_0 T_0} - \frac{\beta A^3}{\omega_0^2 - 9\omega_0^2} e^{3i\omega_0 T_0} + cc \tag{12}$$

Therefore, by substituting Equations (8), (11), and (12) in Equation (6) the solution of (4) can be written as follows

$$x = \mathcal{C} e^{i\frac{3\beta}{2\mathcal{C}\omega_0} T_1} e^{i\omega_0 T_0} - \epsilon \frac{\mathcal{C}^2}{\omega_0^2} + \epsilon \frac{\alpha \mathcal{C}^2}{3\omega_0^2} e^{i\frac{3\beta}{\mathcal{C}\omega_0} T_1} e^{2i\omega_0 T_0} + \epsilon \frac{\beta \mathcal{C}^3}{8\omega_0^2} e^{i\frac{9\beta}{2\mathcal{C}\omega_0} T_1} e^{3i\omega_0 T_0} + cc \tag{13}$$

This can further be simplified to

$$x = \mathcal{C} e^{i\left(\frac{3\beta}{2\mathcal{C}\omega_0} T_1 + \omega_0 T_0\right)} - \epsilon \frac{\mathcal{C}^2}{\omega_0^2} + \epsilon \frac{\alpha \mathcal{C}^2}{3\omega_0^2} e^{i\left(\frac{3\beta}{\mathcal{C}\omega_0} T_1 + 2\omega_0 T_0\right)} + \epsilon \frac{\beta \mathcal{C}^3}{8\omega_0^2} e^{i\left(\frac{9\beta}{2\mathcal{C}\omega_0} T_1 + 3\omega_0 T_0\right)} + cc \tag{14}$$

where

$$T_0 = t$$
$$T_1 = \epsilon t$$

The oscillatory terms are the harmonics of the natural frequency of the linear system (ω_0) with is expected for systems like this. The amplitude of the response, is the coefficient of exponentials.

Problem definition: Consider

$$\dot{x} = x^3 + \delta x^2 - \mu x \tag{15}$$

Determine the fixed points of the system and study the biforcations in the (x, μ) plane for non-zero values of δ

Solution approach: The stationary points are calculated by setting $\dot{x}$ equal to zero.

$$2x = 0 \tag{16a}$$

$$x = \frac{-\delta \pm \sqrt{\delta^2 + 4\mu}}{2} \tag{16b}$$

The Jacobian for Equation (15) is calculated as

$$D_x F = 3x^2 + 2\delta x - \mu \tag{17}$$

where $F = x^3 + \delta x^2 - \mu x$. The eigenvalue is written as

$$\lambda = 3x^2 + 2\delta x - \mu \tag{18}$$

The derivative of the forcing function to the control parameter, μ, is calculated as:

$$F_\mu = -x \tag{19}$$

For $(x, \mu) = (0, 0)$ we have the following conditions:

$$\begin{cases} F(0,0) = 0 \\ D_x F \text{ has zero eigenvalue} \end{cases} \tag{20}$$

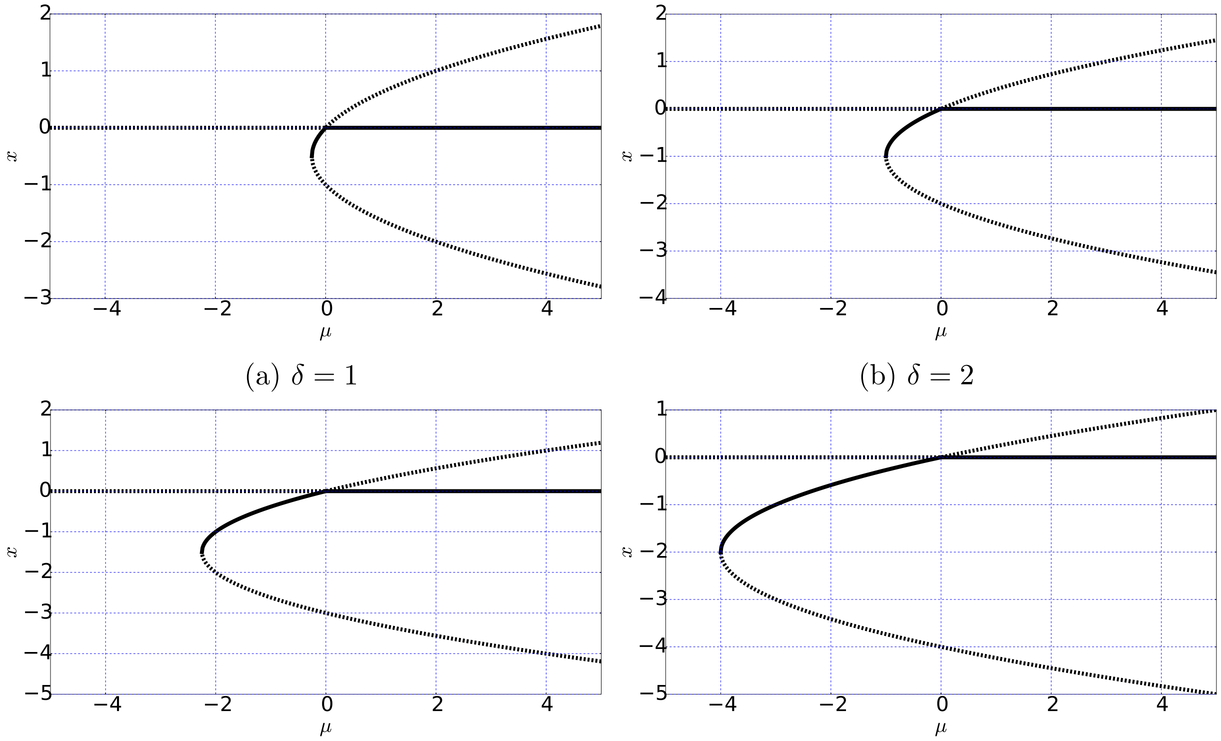
Therefore, we have satisfied the necessary conditions for the bifurcations. Since F_μ is in the range of $D_x F$ at $(0, 0)$, we have a pitch fork bifurcation.

The bifurcation diagrams for various values of δ are shown in Figure 1. In these plots, the dashed line represents the *unstable branch* and the solid line represents the *stable branch*. To calculate the turning point of the bifurcation diagram, we need to calculate the derivative of control variable, μ, with respect to x and set it equal to zero.

$$x = \frac{-\delta \pm \sqrt{\delta^2 + 4\mu}}{2} \Rightarrow \frac{d\mu}{dx} = \delta + 2x \tag{21}$$

Therefore, the turning point is at $(-\frac{\delta}{2}, -\frac{\delta^2}{4})$. This can also be seen in Figure 1.

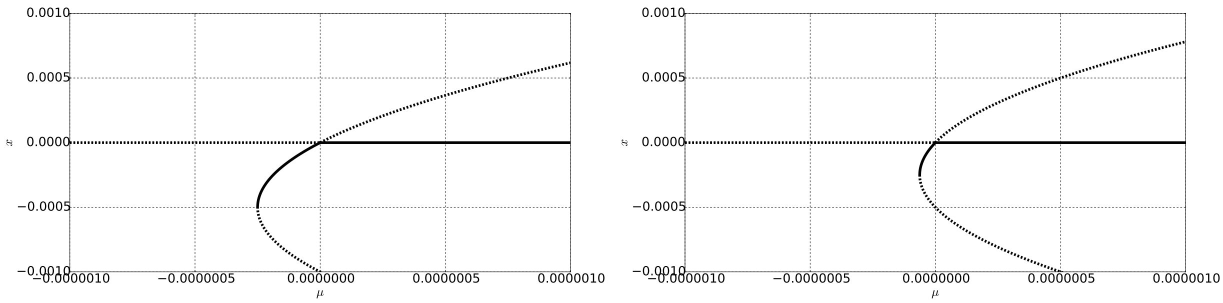
For transcritical bifurcation, the number of stationary points for $\mu < 0$ and $\mu > 0$ remains the same (here 3) however, their characteristics change. Stable stationary points become unstable and unstable ones become stable. For small values of μ, the pitchfork bifurcation at $(0, 0)$ can be considered a transcritical bifurcation for this reason. The bifurcation plots for small values of x and δ are shown in Figure 2.



(a) $\delta = 1$ (b) $\delta = 2$

(c) $\delta = 3$ (d) $\delta = 4$

Figure 1: Bifurcation diagram for different positive values for δ. The dashed line is the *unstable branch* and the solid line is the *stable branch*.



(a) $\delta = 0.001$ (b) $\delta = 0.0005$

Figure 2: Bifurcation diagram for small values of x and δ. The dashed line is the *unstable branch* and the solid line is the *stable branch*.