Problem definition: Determine the decay envelope using the method of multiple scales.

$$\ddot{x} + \omega_0^2 x = \epsilon\left(\dot{x} - \dot{x}^3\right) \tag{1}$$

We define our differential operators as follow:

$$\frac{d}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2\left(D_1^2 + 2D_0 D_2\right) \tag{2a}$$

$$\frac{d}{dt} = D_0 + \epsilon D_1 \tag{2b}$$

We approximate x as follows:

$$x(t) = x_0(T_0, T_1, T_2) + \epsilon x_1(T_0, T_1, T_2) \tag{3}$$

Substituting Equations (2) and (3) into (1) and grouping terms with respect to ϵ we get the following system of equations:

$$\epsilon^0: \quad D_0^2 x_0 + \omega_0^2 x_0 = 0 \tag{4a}$$

$$\epsilon^1: \quad D_0^2 x_1 + \omega_0^2 x_1 = -\left(D_0^3 x_0^3 + 2D_0 D_1 x_0 - D_0 x_0\right) \tag{4b}$$

We assume the solution of Equation (4a) as follows:

$$x_0 = A(T_1)\exp\left(i\omega_0 T_0\right) + cc \tag{5}$$

Substituting this equation into Equation (4b) we get:

$$D_0^2 x_1 + \omega_0^2 x_1 = \left[A(T_1) - \frac{dA(T_1)}{dT_1}\right] 2i\exp\left(iT_0\right) \tag{6}$$

To remove the secular terms, the coefficient of the exponential needs to be equal to zero. By assuming $A(T_1)$ in the form of $a(T_1)\exp(ib(T_1))$ we get:

$$a = \exp\left(T_1\right)$$
$$b' = 0 \Rightarrow b = \mathcal{C}$$

The particular solution of Equation (4b) can be written as:

$$x_1 = A(T_1)\exp\left(i\omega_0 T_0\right) \tag{8}$$

Finally by substituting Equations (5) and (8) into (3), the solution to (1) can be written as:

$$x(t) = e^{T_1 + i\mathcal{C}}\exp\left(i\omega_0 T_0\right) + \epsilon\left[e^{T_1 + i\mathcal{C}}\exp\left(i\omega_0 T_0\right)\right] \tag{9}$$

The decay envelop is:

$$(1+\epsilon)\, e^{T_1+i\mathcal{C}} \tag{10}$$

This means that the amplitude of the vibrations increases. This is true due to negative damping coefficient in Equation (1).