2.5.4: Second Order System Natural Response – Part II

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Overview

In chapter 2.5.2, we developed the form of the solution of the differential equation governing second order systems. The form of the solution contained so-called complex exponentials; the background material relative to these signals was provided in chapter 2.5.3. We are thus now in a position to re-examine and interpret the solution presented in chapter 2.5.2.

In chapter 2.5.2, we noted that the form of the natural response of a second order system was strongly dependent upon the damping ratio, ζ. If the damping ratio was greater than one, all terms in the response decay exponentially, but if the damping ratio was between zero and one some terms in the response became complex exponentials – in chapter 2.5.3, we saw that this corresponded to an oscillating signal. Thus, depending upon the value of damping ratio, the response could decay exponentially or oscillate. In this chapter, we will quantify and formalize these results.

This chapter concludes with an extended example of a second order system natural response.

Before beginning this chapter, you should be able to:

- Define damping ratio and natural frequency from the coefficients of a second order differential equation (Chapter 2.5.1)
- Write the form of the natural response of a second order system (Chapter 2.5.2)
- State conditions on the damping ratio which results in the natural response consisting of decaying exponentials (Chapter 2.5.2)
- State conditions on the damping ratio which results in the natural response consisting of complex exponentials (Chapter 2.5.2)
- Use complex exponentials to represent sinusoidal signals (Chapter 2.5.3)

After completing this chapter, you should be able to:

- Classify *overdamped*, *underdamped*, and *critically damped* systems according to their damping ratio
- Identify the expected shape of the natural response of over-, under-, and critically damped systems

This chapter requires:

- N/A

In chapter 2.5.1, the differential equation governing the natural response of a second order system was written as

$$\frac{d^2 y_h(t)}{dt^2} + 2\varsigma\omega_n \frac{dy_h(t)}{dt} + \omega_n^2 y_h(t) = 0 \tag{1}$$

where *y(t)* is any system parameter of interest, ω_n is the *undamped natural frequency* and ζ is the *damping ratio*. The initial conditions are the value of the function *y(t)* at *t = 0* and the derivative of the function *y(t)* at *t = 0:*

$$\begin{aligned} &y(t=0) = y_0 \\ &\left.\frac{dy(t)}{dt}\right|_{t=0} = y'_0 \end{aligned} \tag{2}$$

In chapter 2.5.2, we wrote the solution to equation (1) in the form

$$y_h(t) = e^{-\zeta\omega_n t}\left[K_1 e^{\left(\omega_n\sqrt{\zeta^2-1}\right)t} + K_2 e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)t}\right] \tag{3}$$

where K_1 and K_2 are unknown coefficients which can be determined by application of the initial conditions provided in equation (2). The form of the solution of equation (3) will fall into one of three categories, depending on the value of damping ratio. The three possible cases are:

1. If $\zeta > 1$, all terms in the solution will be either growing or decaying exponentials and the solution will decay exponentially with time. If the damping ratio is large, this decay rate can be very slow. A system with $\zeta > 1$ is said to be *overdamped*.
2. If $\zeta < 1$, the terms $e^{\pm\left(\omega_n\sqrt{\zeta^2-1}\right)t}$ are *complex exponentials*. Thus, have terms in our solution which are exponentials raised to an imaginary power and the solution can oscillate. A system with $\zeta < 1$ is said to be *underdamped*.
3. $\zeta = 1$; the form of the solution in this case is approximately that of case 1 above, in that the solution will decay exponentially. However, in this case, the response decay rate will be faster than the response of any overdamped system with the same natural frequency. Systems with $\zeta = 1$ are said to be *critically damped*.

Details of the responses for each of the above three cases are provided in the subsections below.

1. <u>Overdamped system,:</u>

   For an overdamped system, $\zeta > 1$, and equation (3) becomes as shown in equation (4).

$$y_h(t) = e^{-\zeta\omega_n t}\left[\frac{\dot{y}_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n y_0}{2\omega_n\sqrt{\zeta^2 - 1}} e^{\omega_n t\sqrt{\zeta^2-1}} + \frac{\dot{y}_0 - \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n y_0}{2\omega_n\sqrt{\zeta^2 - 1}} e^{-\omega_n t\sqrt{\zeta^2-1}}\right] \qquad (4)$$

   In equation (4), the $e^{-\zeta\omega_n t}$ term is a decaying exponential with time constant $1/\zeta\omega_n$. The $e^{\omega_n t\sqrt{\zeta^2-1}}$ is a growing exponential with time constant $1/\omega_n\sqrt{\zeta^2-1}$. The $e^{-\omega_n t\sqrt{\zeta^2-1}}$ term is a decaying exponential, also with time constant $1/\omega_n\sqrt{\zeta^2-1}$. Thus, the overall system response is a sum of two decaying exponential signals, one which is proportional to $e^{-\zeta\omega_n t}\cdot e^{\omega_n t\sqrt{\zeta^2-1}}$ and the other which is proportional to $e^{-\zeta\omega_n t}\cdot e^{-\omega_n t\sqrt{\zeta^2-1}}$.

   The term $e^{-\zeta\omega_n t}\cdot e^{\omega_n t\sqrt{\zeta^2-1}}$ is the product of two exponentials: one which grows with time, and the other which decays with time. The decaying exponential time constant, $1/\zeta\omega_n$, is smaller than the growing exponential time constant, $1/\omega_n\sqrt{\zeta^2-1}$. Thus, the product of the two will decay with time, though the decay rate may be very slow. (Note that in the limit as $\zeta \to \infty$, $1/\zeta\omega_n \approx 1/\omega_n\sqrt{\zeta^2-1}$, the two time constants are nearly identical, and this term becomes constant with time.)

   The term $e^{-\zeta\omega_n t}\cdot e^{-\omega_n t\sqrt{\zeta^2-1}}$ is the product of two decaying exponentials; this term will, in general, decay quickly relative to the $e^{-\zeta\omega_n t}\cdot e^{\omega_n t\sqrt{\zeta^2-1}}$ term.

   An example of the response of an overdamped system is shown in Figure 1, for various values of damping ratio. The two system time constants are readily observable in this example. Note that as the damping ratio increases, the overall time required for the system response to decay to zero increases. The response of overdamped systems cannot oscillate, however, the response can change sign <u>once</u> (e.g. the function is allowed one zero-crossing).

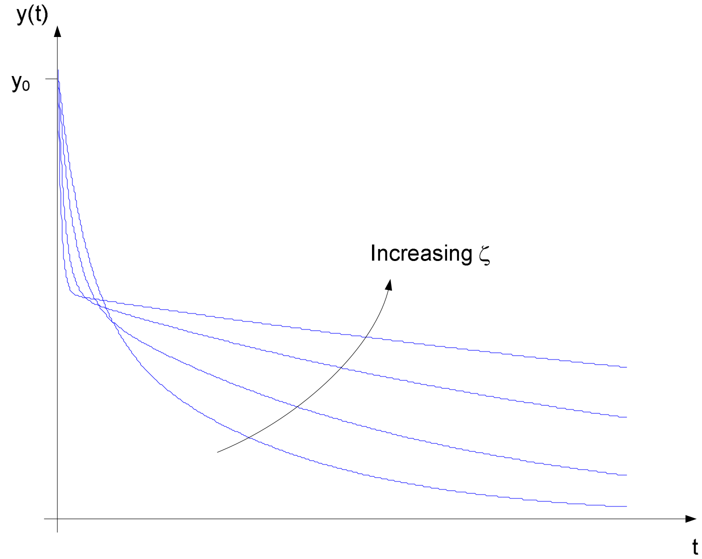


Figure 1. Overdamped system response.

2. Underdamped system:

For an underdamped system, $\zeta < 1$, and equation (3) becomes as shown in equation (5).

$$y_h(t) = e^{-\zeta\omega_n t}\left[\frac{\dot{y}_0 + \zeta\omega_n y_0}{\omega_n\sqrt{1-\zeta^2}} \sin\left(\omega_n t\sqrt{1-\zeta^2}\right) + y_0 \cos\left(\omega_n t\sqrt{1-\zeta^2}\right)\right] \tag{5}$$

The solution is a decaying sinusoid. The decay rate is set by the term $e^{-\zeta\omega_n t}$, while the oscillation frequency of the sinusoid is $\omega_n\sqrt{1-\zeta^2}$. The oscillation frequency seen in the natural response is thus not identically the natural frequency of the system; it is also influenced by the damping ratio. This leads to the definition of the *damped natural frequency*:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{6}$$

Oscillations seen in the system response will have radian frequency ω_d; thus, the period of the oscillations is $\frac{2\pi}{\omega_d}$.

Example responses for underdamped systems are shown in Figure 2; the responses shown are all for the same natural frequency and initial conditions – only the damping ratio varies. Note that smaller damping ratios result in slower decay rates for the response, oscillations persist for longer and are more pronounced for smaller damping ratios.

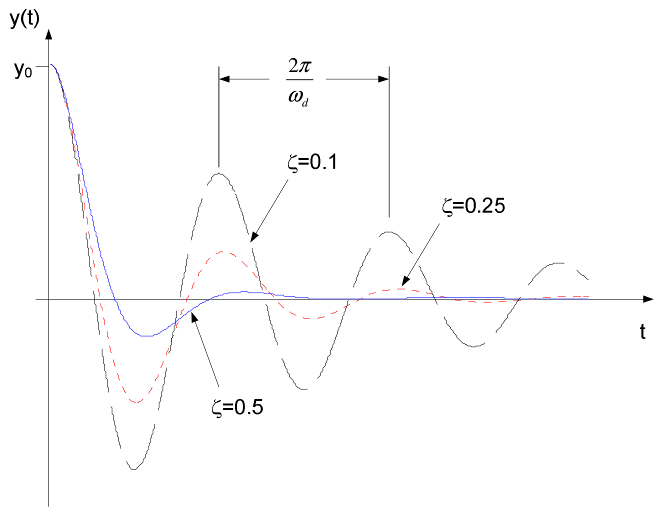


Figure 2. Underdamped system response.

3. Critically damped system

   For a critically damped system, $\zeta = 1$, and equation (3) becomes as shown in equation (7).

$$y_h(t) = e^{-\zeta\omega_n t}\left[y_0 + \left(\dot{y}_0 + \omega_n y_0 t\right)\right] \quad (7)$$

   The critically damped system response does not oscillate although, as with the overdamped case, one zero crossing of the function is allowed. The importance of the critically damped system response is that, for a particular natural frequency, it has the shortest decay time <u>without oscillation</u> of any system.

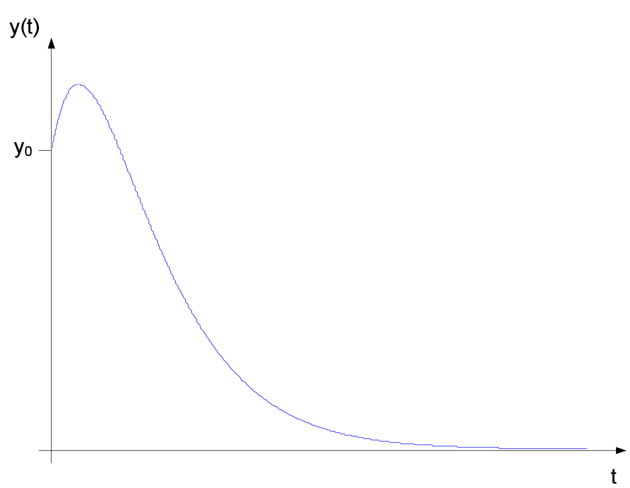
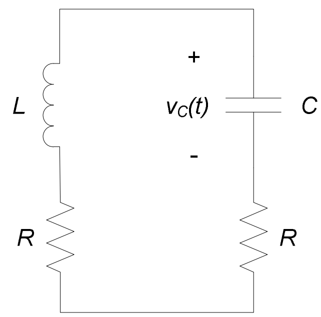


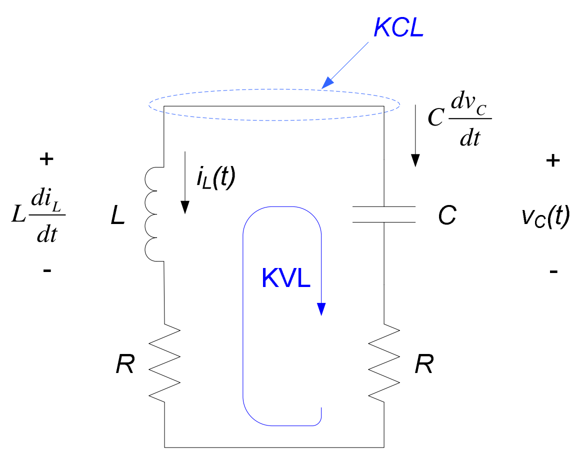
Figure 3. Critically damped system response.

Example: For the circuit shown below:

(a) Write the differential equation for $v_C(t)$.
(b) If $L = 1H$, $R = 200\Omega$, and $C = 1\times10^{-6}F$, find the undamped natural frequency, the damping ratio, and the damped natural frequency.
(c) For the conditions in part (b), is the system underdamped, overdamped, or critically damped?
(d) For the values of L and C in part (b), determine the value of R that makes the system critically damped.
(e) If $v_C(0) = 1V$ and $i_L(0) = 0.01A$, what are the appropriate initial conditions to solve the differential equation determined in part (a)?



(a) As usual, we define the voltage across the capacitor and the current through the inductor as our variables and write KVL and KCL in terms of these variables. The figure below shows these variables, along with the associated currents through capacitors and voltages across inductors.



KCL at the indicated node results in:

$$i_L(t) + C\frac{dv_C(t)}{dt} = 0 \tag{1}$$

KVL around the indicated loop provides

$$L\frac{di_L(t)}{dt} + 2Ri_L(t) = v_C(t) \tag{2}$$

The above two equations can be combined to obtain an equation for vC(t). To do this, we use the first equation to obtain:

$$i_L(t) = -C\frac{dv_C(t)}{dt} \tag{3}$$

Differentiating equation (3) provides:

$$\frac{di_L(t)}{dt} = -C\frac{d^2v_C(t)}{dt^2} \tag{4}$$

Substituting equations (3) and (4) into equation (2) results in:

$$-LC\frac{d^2v_C(t)}{dt^2} - 2RC\frac{dv_C(t)}{dt} = v_C(t)$$

Dividing the above by LC and grouping terms gives our final result

$$\frac{d^2v_C(t)}{dt^2} + \frac{2R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = 0 \tag{5}$$

(b) Equation (5) is of the form

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = 0 \tag{6}$$

Equating coefficients in equations (5) and (6) and substituting L = 1H, R = 200Ω, and C = 1×10^{-6}F results in:

$$2\zeta\omega_n = \frac{2R}{L} = 400 \tag{7}$$

$$\omega_n^2 = \frac{1}{LC} = 1\times10^6 \tag{8}$$

Solving equation (8) for the natural frequency results in $\omega_n = 1000$ rad/sec. Substituting this result into equation (7) and solving for the damping ratio gives $\zeta = 0.2$. The damped natural frequency is

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 979.8 \text{ rad/sec.}$$

(c) The damping ratio determined in part (b) is ζ = 0.2; since this is less than one, the system is underdamped.

(d) In order for the system to be critically damped, the damping ratio ζ = 1. From equation (7) with ζ = 1, we obtain:

$$2\zeta\omega_n = \frac{2R}{L} \Rightarrow 2(1)(1000) = \frac{2R}{1H} \Rightarrow R = 1000\Omega$$

(e) Initial conditions on $v_C(t)$ are $v_C(0)$ and $\left.\frac{dv_C(t)}{dt}\right|_{t=0}$. We are given $v_C(0)$ = 1V in the problem statement, but we need to determine $\left.\frac{dv_C(t)}{dt}\right|_{t=0}$; the current through the inductor, $i_L(0)$ can be used to determine this. The current through the inductor is related to the capacitor voltage via equation (3) above:

$$i_L(t) = -C\frac{dv_C(t)}{dt}$$

so at time t = 0,

$$i_L(0) = -(1\times10^{-6}F)\left.\frac{dv_C(t)}{dt}\right|_{t=0} = 0.01A$$

Solving for $\left.\frac{dv_C(t)}{dt}\right|_{t=0}$,

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = \frac{1}{C}i_L(0) = -\frac{0.01A}{1\times10^{-6}F} = -10{,}000 \text{ V/sec}$$

We conclude this example with plots of the system response for underdamped, critically damped, and overdamped conditions.

Figure 1 shows the response of the circuit described by the differential equation determined in part (a) above, for the circuit parameters provided in part (b), to the initial conditions of part (e). Thus, the governing differential equation is

$$\frac{d^2v_C(t)}{dt^2} + \frac{2R}{L}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) = 0$$

with L = 1H, R = 200Ω, and C = 1×10^{-6}F, the differential equation becomes:

$$\frac{d^2 v_C(t)}{dt^2} + 400\frac{dv_C(t)}{dt} + 1\times10^6 v_C(t) = 0$$

the initial conditions are, from part (e):

$v_C(0)$ = 1V

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = -10{,}000 \text{ V/sec}$$

Using MATLAB to evaluate the differential equation results in Figure 1. Figure 1 agrees with our expectations based on the calculations of part (b). In part (b), we determined that the damping ratio ζ = 0.2, so that the system is underdamped – Figure 1 exhibits the oscillations (multiple zero axis crossings) that we would expect from an underdamped system. Likewise, we determined in part (b) that the damped natural frequency of the system is approximately 980 rad/sec. The period of the oscillations we would expect to see in the response is therefore:

$$T = \frac{2\pi}{\omega_d} = 0.0064 \text{ seconds.}$$

This value is consistent with the period of the oscillations seen in Figure 1.

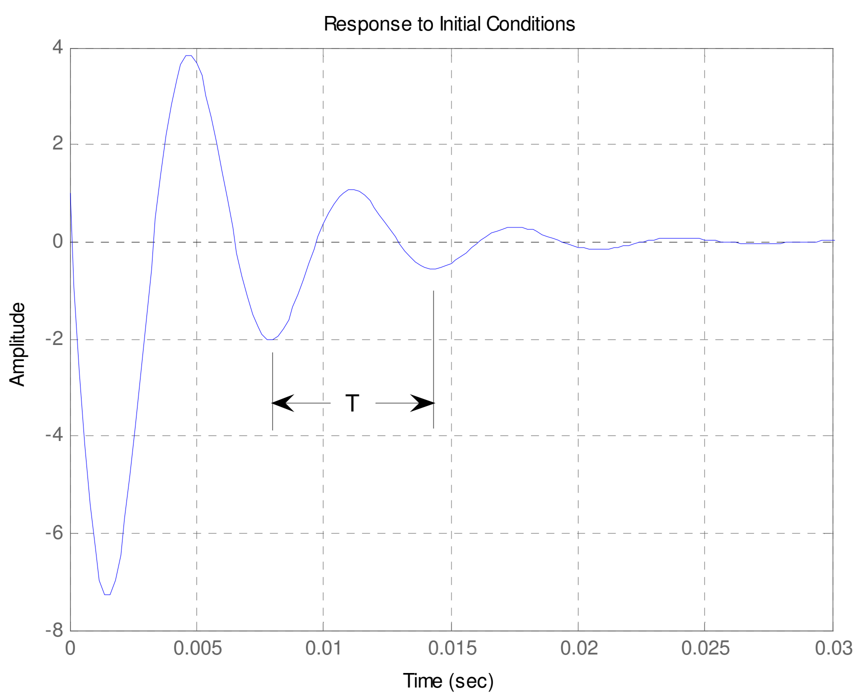


Figure 1. Underdamped response to initial conditions.

In part (d) above, we determined that the value of R resulting in a critically damped system is R = 1000Ω. Re-evaluating the above governing differential equation with this value for R results in

$$\frac{d^2 v_C(t)}{dt^2} + 2000\frac{dv_C(t)}{dt} + 1\times10^6 v_C(t) = 0$$

the initial conditions are as in the above example:

$v_C(0)$ = 1V

$$\left.\frac{dv_C(t)}{dt}\right|_{t=0} = -10{,}000 \text{ V/sec}$$

The resulting response is shown in Figure 2. This plot also matches our expectations, though we have fewer quantitative results against which to compare it. The response does not oscillate (the response does have one and only one zero crossing, which is allowable for a critically damped or overdamped system. The response also appears to be composed of exponential signals, which is consistent with our expectations.

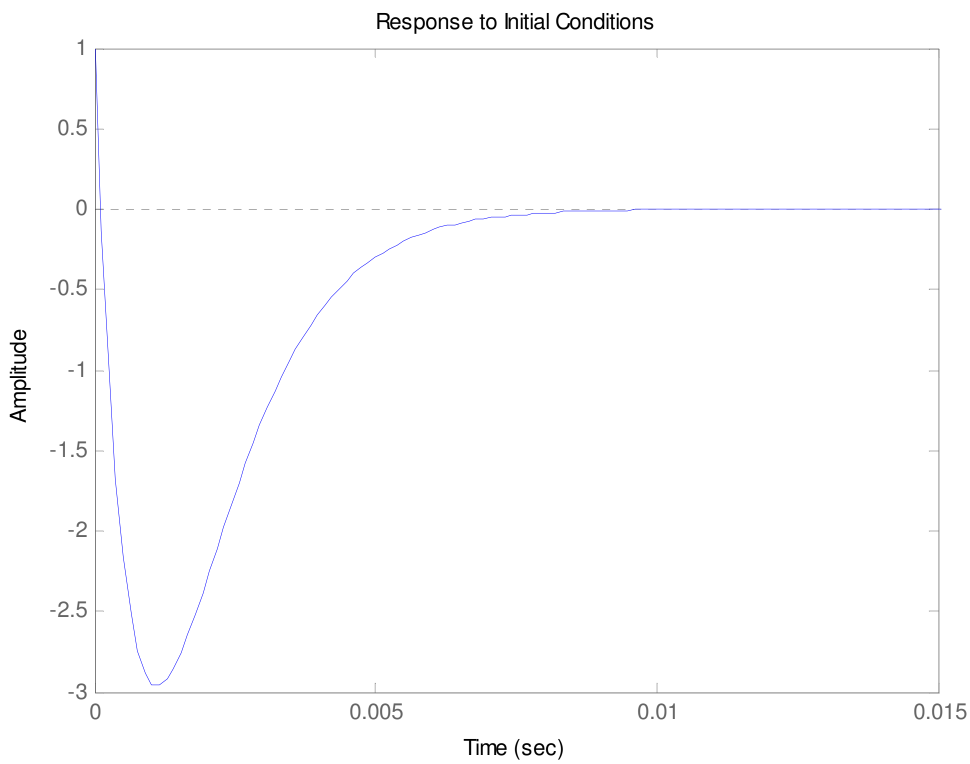


Figure 2. Critically damped system response.

In order to obtain a better understanding of critically damped vs. overdamped systems, we increase R to 3000Ω. The resulting damping ratio is $\zeta = 3$; increasing R above the critically damped value will result in an overdamped system since the damping ratio is proportional to R. We will expect the response shape to be somewhat like that shown in Figure 2 (it will still be composed of decaying exponential functions) but the overdamped system should decay more slowly. The system response shown in Figure 3. This response agrees with our qualitative expectations – the response does not oscillate, and the decay time is longer than that shown in Figure 2.

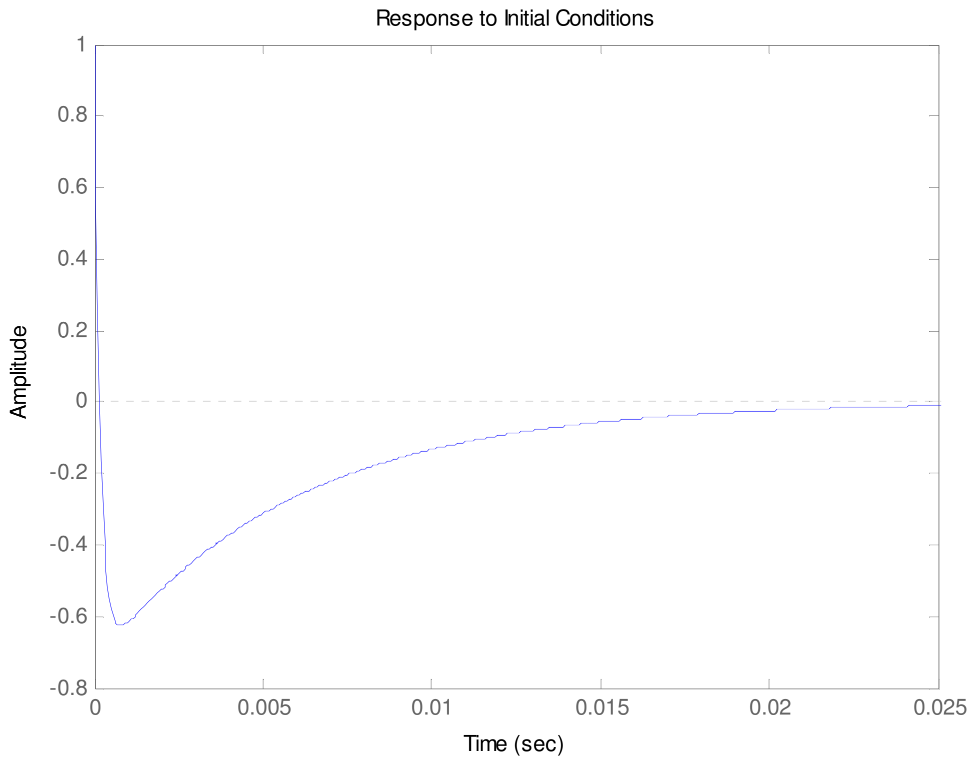


Figure 3. Overdamped system response.