Autonomous Equations / Stability of Equilibrium Solutions

First order autonomous equations, Equilibrium solutions, Stability, Long-term behavior of solutions, direction fields, Population dynamics and logistic equations

Autonomous Equation: A differential equation where the independent variable does not explicitly appear in its expression. It has the general form of

$$y' = f(y).$$

Examples:

$$y' = e^{2y} - y^3$$

$$y' = y^3 - 4y$$

$$y' = y^4 - 81 + \sin y$$

Every autonomous ODE is a separable equation. Because, assuming that $f(y) \neq 0$,

$$\frac{dy}{dt} = f(y) \qquad \rightarrow \qquad \frac{dy}{f(y)} = dt \qquad \rightarrow \qquad \int \frac{dy}{f(y)} = \int dt \,.$$

Hence, we already know how to solve them. What we are interested now is to predict the behavior of an autonomous equation's solutions without solving it, by using its direction field. But what happens if the assumption that $f(y) \neq 0$ is false? We shall start by answering this very question.
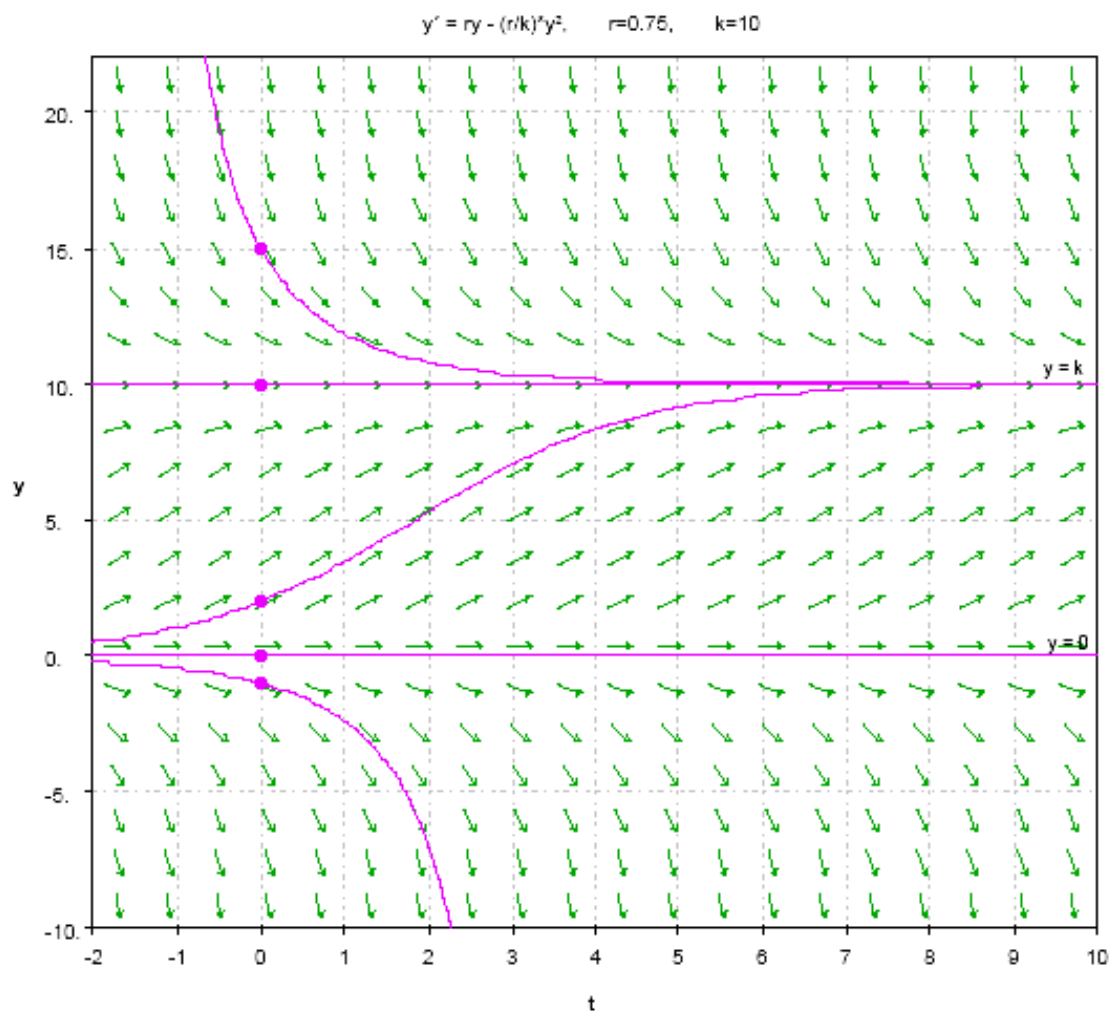
Equilibrium solutions

Equilibrium solutions (or critical points) occur whenever $y' = f(y) = 0$. That is, they are the roots of $f(y)$. Any root c of $f(y)$ yields a constant solution $y = c$. (*Exercise*: Verify that, if c is a root of $f(y)$, then $y = c$ is a solution of $y' = f(y)$.) Equilibrium solutions are constant functions that satisfy the equation, i.e., they are the constant solutions of the differential equation.

Example: Logistic Equation of Population

$$y' = r\left(1 - \frac{y}{K}\right)y = ry - \frac{r}{K}y^2$$

Both r and K are positive constants. The solution y is the population size of some ecosystem, r is the *intrinsic growth rate*, and K is the *environmental carrying capacity*. The intrinsic growth rate is the natural rate of growth of the population provided that the availability of necessary resource (food, water, oxygen, etc) is limitless. The environmental carrying capacity (or simply, carrying capacity) is the maximum sustainable population size given the actual availability of resource.

Without solving this equation, we will examine the behavior of its solution. Its direction field is shown in the next figure.

$y' = ry - (r/k)'y'$,   r=0.75,   k=10

Notice that the long-term behavior of a particular solution is determined solely from the initial condition $y(t_0) = y_0$. The behavior can be categorized by the initial value y_0:

If $y_0 < 0$, then $y \rightarrow -\infty$ as $t \rightarrow \infty$.

If $y_0 = 0$, then $y = 0$, a constant/equilibrium solution.

If $0 < y_0 < K$, then $y \rightarrow K$ as $t \rightarrow \infty$.

If $y_0 = K$, then $y = K$, a constant/equilibrium solution.

If $y_0 > K$, then $y \rightarrow K$ as $t \rightarrow \infty$.

Comment: In a previous section (applications: air-resistance) you learned an easy way to find the limiting velocity without having to solve the differential equation. Now we can see that the limiting velocity is just the equilibrium solution of the motion equation (which is an autonomous equation). Hence it could be found by setting $v' = 0$ in the given differential equation and solve for v.

Stability of an equilibrium solution

The *stability* of an equilibrium solution is classified according to the behavior of the integral curves near it – they represent the graphs of particular solutions satisfying initial conditions whose initial values, y_0, differ only slightly from the equilibrium value.

> If the nearby integral curves all converge towards an equilibrium solution as t increases, then the equilibrium solution is said to be *stable*, or *asymptotically stable*. Such a solution has long-term behavior that is insensitive to slight (or sometimes large) variations in its initial condition.

> If the nearby integral curves all diverge away from an equilibrium solution as t increases, then the equilibrium solution is said to be *unstable*. Such a solution is extremely sensitive to even the slightest variations in its initial condition − as we can see in the previous example that the smallest deviation in initial value results in totally different behaviors (in both long- and short-terms).

Therefore, in the logistic equation example, the solution $y = 0$ is an unstable equilibrium solution, while $y = K$ is an (asymptotically) stable equilibrium solution.

An alternative graphical method: Plotting $y' = f(y)$ versus y. This is a graph that is easier to draw, but reveals just as much information as the direction field. It is rather similar to the *First Derivative Test*[*] for local extrema in calculus. On any interval (they are separated by equilibrium solutions / critical points, which are the horizontal-intercepts of the graph) where $f(y) > 0$, y will be increasing and we denote this fact by drawing a <u>rightward</u> arrow. (Because, y in this plot happens to be the horizontal axis; and its coordinates increase from left to right, from $-\infty$ to ∞.) Similarly, on any interval where $f(y) < 0$, y is decreasing. We shall denote this fact by drawing a <u>leftward</u> arrow. To summarize: $f(y) > 0$, y goes up, therefore, rightward arrow; $f(y) < 0$, y goes down, therefore, leftward arrow. The result can then be interpreted in the following way: Suppose $y = c$ is an equilibrium solution (i.e. $f(y) = 0$), then

> (i.) If $f(y) < 0$ on the left of c, and $f(y) > 0$ on the right of c, then the equilibrium solution $y = c$ is <u>unstable</u>. (Visually, the arrows on the two sides are moving away from c.)
>
> (ii.) If $f(y) > 0$ on the left of c, and $f(y) < 0$ on the right of c, then the equilibrium solution $y = c$ is <u>asymptotically stable</u>. (Visually, the arrows on the two sides are moving toward c.)

Remember, a leftward arrow means y is decreasing as t increases. It corresponds to downward-sloping arrows on the direction field. While a rightward arrow means y is increasing as t increases. It corresponds to upward-sloping arrows on the direction field.

---

[*] All the steps are really the same, only the interpretation of the result differs. A result that would indicate a local minimum now means that the equilibrium solution/critical point is unstable; while that of a local maximum result now means an asymptotically stable equilibrium solution.

As an example, let us apply this alternate method on the same logistic equation seen previously: $y' = ry - (r/K)y^2$, $r = 0.75$, $K = 10$.

The y'-versus-y plot is shown below.
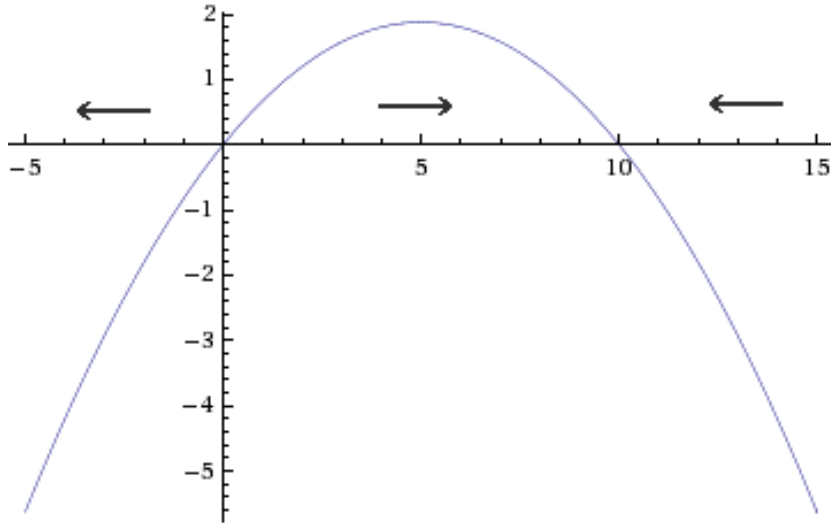
As can be seen, the equilibrium solutions $y = 0$ and $y = K = 10$ are the two horizontal-intercepts (confusingly, they are the y-intercepts, since the y-axis is the horizontal axis). The arrows are moving apart from $y = 0$. It is, therefore, an unstable equilibrium solution. On the other hand, the arrows from both sides converge toward $y = K$. Therefore, it is an (asymptotically) stable equilibrium solution.

*Example*: Logistic Equation with (Extinction) Threshold

$$y' = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y$$

Where r, T, and K are positive constants: $0 < T < K$.
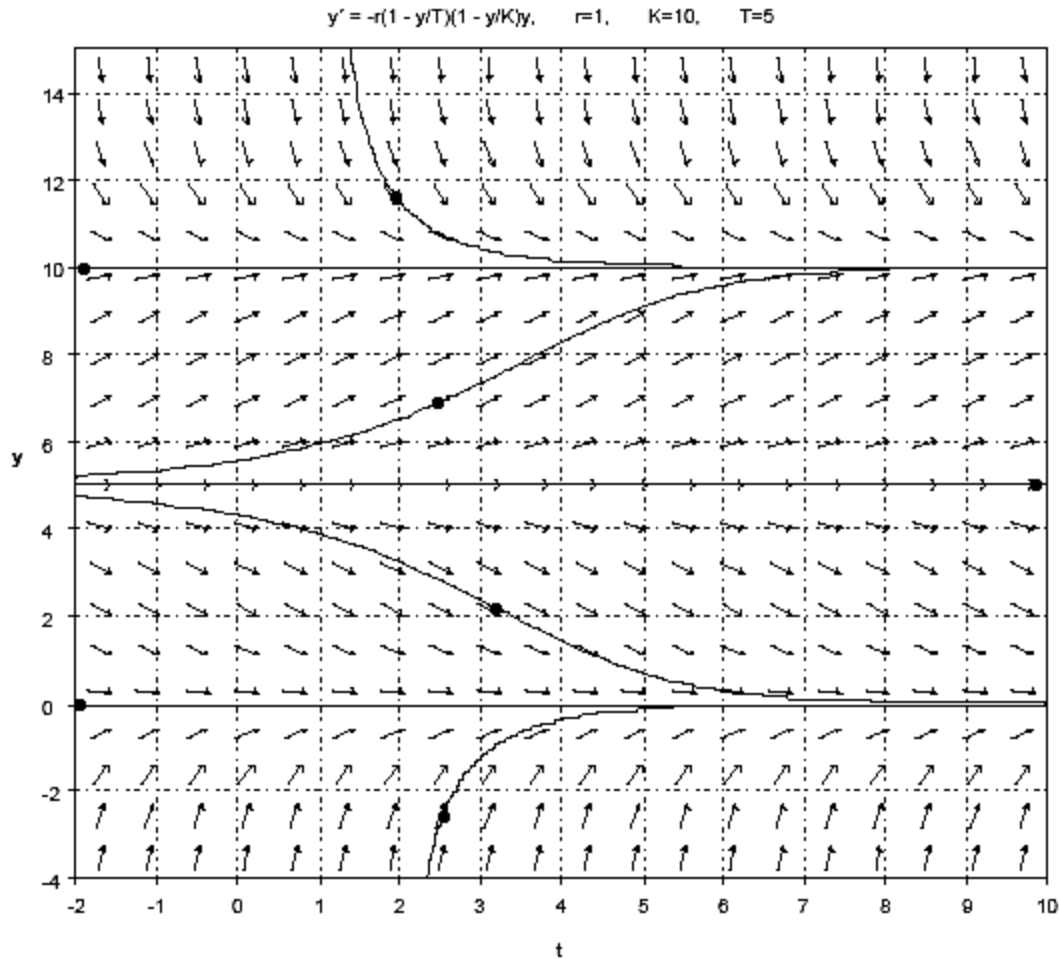
The values r and K still have the same interpretations, T is the extinction threshold level below which the species is endangered and eventually become extinct. As seen above, the equation has (asymptotically) stable equilibrium solutions $y = 0$ and $y = K$. There is an unstable equilibrium solution $y = T$.

The same result can, of course, be obtained by looking at the y'-versus-y plot (in this example, $T = 5$ and $K = 10$):
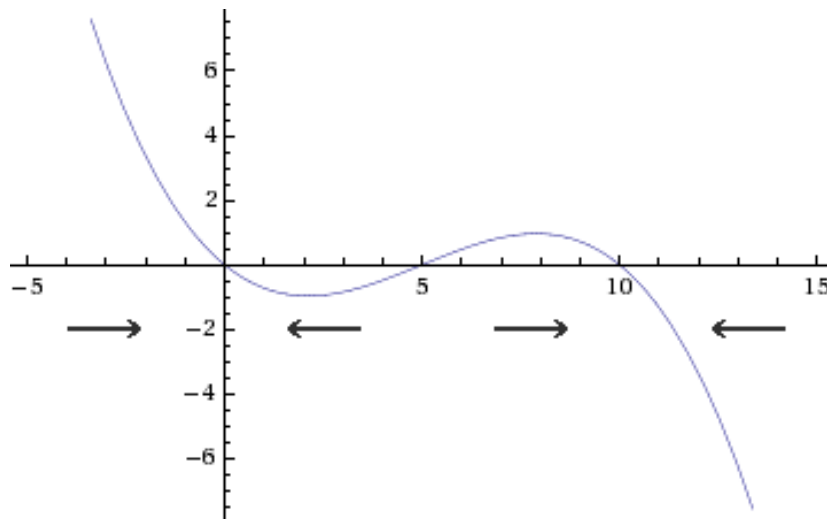
We see that $y = 0$ and $y = K$ are (asymptotically) stable, and $y = T$ is unstable.

Once again, the long-term behavior can be determined just by the initial value y_0:

>   If $y_0 < 0$, then $y \to 0$ as $t \to \infty$.
>   If $y_0 = 0$, then $y = 0$, a constant/equilibrium solution.
>   If $0 < y_0 < T$, then $y \to 0$ as $t \to \infty$.
>   If $y_0 = T$, then $y = T$, a constant/equilibrium solution.
>   If $T < y_0 < K$, then $y \to K$ as $t \to \infty$.
>   If $y_0 = K$, then $y = K$, a constant/equilibrium solution.
>   If $y_0 > K$, then $y \to K$ as $t \to \infty$.

Semistable equilibrium solution

A third type of equilibrium solutions exist. It exhibits a half-and-half behavior. It is demonstrated in the next example.

Example: $y' = y^3 - 2y^2$

    The equilibrium solutions are $y = 0$ and 2. As can be seen below, $y = 2$ is an unstable equilibrium solution. The interesting thing here, however, is the equilibrium solution $y = 0$ (which corresponding a double-root of $f(y)$.
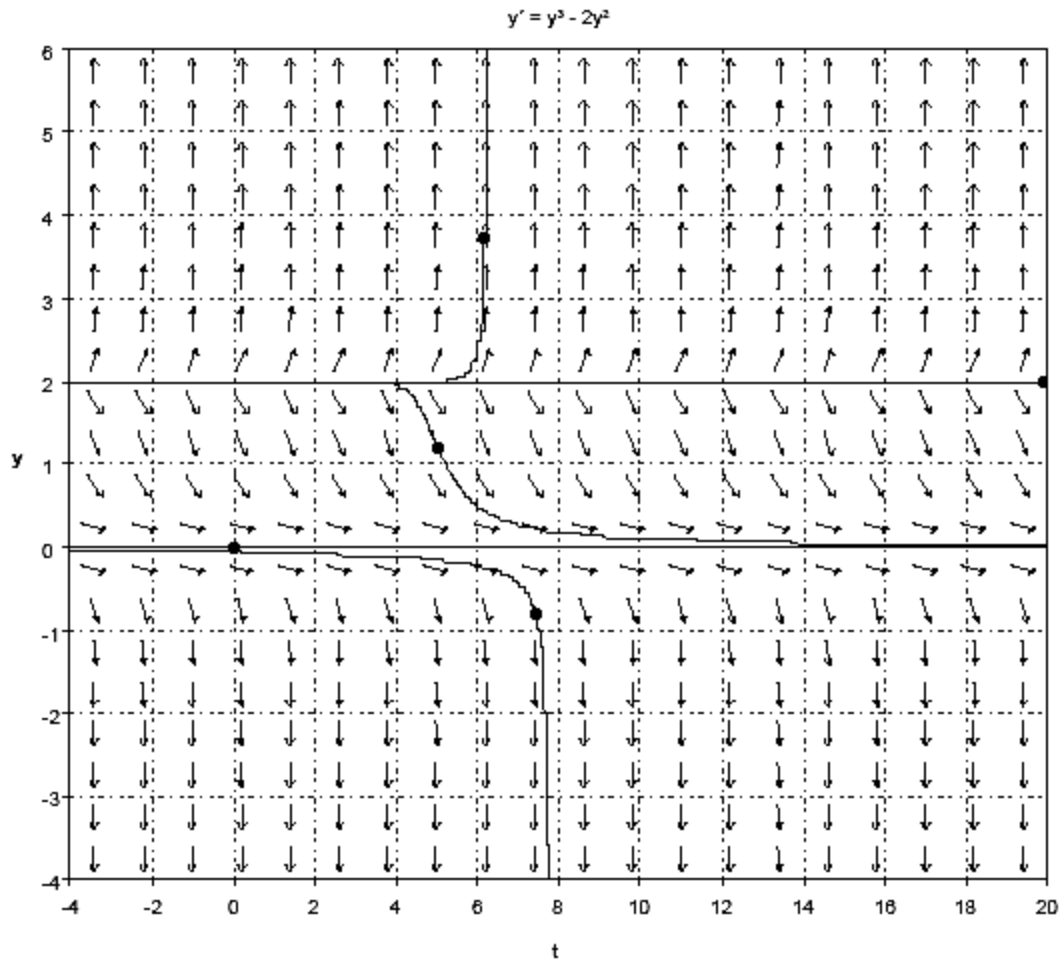
Notice the behavior of the integral curves near the equilibrium solution $y = 0$. The integral curves just above it are converging to it, like it is a stable equilibrium solution, but all the integral curves below it are moving away and diverging to $-\infty$, a behavior associated with an unstable equilibrium solution. A behavior such like this defines a *semistable* equilibrium solution.

An equilibrium solution is semistable if y' has the same sign on both adjacent intervals. (In our analogy with the First Derivative Test, if the result would indicate that a critical point is neither a local maximum nor a minimum, then it now means we have a semistable equilibrium solution.

(iii.) If $f(y) > 0$ on both sides of c, or $f(y) < 0$ on both sides of c, then the equilibrium solution $y = c$ is semistable. (Visually, the arrows on one side are moving toward c, while on the other side they are moving away from c.)

Comment: As we can see, it is actually not necessary to graph anything in order to determine stability. The only thing we need to make the determination is the sign of y' on the interval immediately to either side of an equilibrium solution (a.k.a. critical point), then just apply the above-mentioned rules. The steps are otherwise identical to the first derivative test: breaking the number line into intervals using critical points, evaluate y' at an arbitrary point within each interval, finally make determination based on the signs of y'. This is our version of the *first derivative test* for classifying stability of equilibrium solutions of an autonomous equation. (The graphing methods require more work but also will provide more information – unnecessary for our purpose here – such as the instantaneous rate of change of a particular solution at any point.)

Computationally, stability classification tells us the sensitivity (or lack thereof) to slight variations in initial condition of an equilibrium solution. An unstable equilibrium solution is very sensitive to deviations in the initial condition. Even the slightest change in the initial value will result in a very different asymptotical behavior of the particular solution. An asymptotically stable equilibrium solution, on the other hand, is quite tolerant of small changes in the initial value − a slight variation of the initial value will still result in a particular solution with the same kind of long-term behavior. A semistable equilibrium solution is quite insensitive to slight variation in the initial value in one direction (toward the converging, or the stable, side). But it is extremely sensitive to a change of the initial value in the other direction (toward the diverging, or the unstable, side).

Exercises A-2.1:

1 − 8 Find and classify all equilibrium solutions of each equation below.
1. $y' = 100y - y^3$

2. $y' = y^3 - 4y$

3. $y' = y(y - 1)(y - 2)(y - 3)$

4. $y' = \sin y$

5. $y' = \cos^2(\pi y / 2)$

6. $y' = 1 - e^y$

7. $y' = (3y^2 - 2y - 1)e^{-2y}$

8. $y' = y(y - 1)^2(3 - y)(y - 5)^2$

9. For each of problems 1 through 8, determine the value to which y will approach as t increases if (a) $y_0 = -1$, and (b) $y_0 = \pi$.

10. Consider the air-resistance equation from an earlier example, $100v' = 10000 - 4v^2$. (i) Find and classify its equilibrium solutions. (ii) Given $y(t_0) = 0$, determine the range of $y(t)$. (iii) Given $y(8) = -60$, determine the range of $y(t)$.

11. Verify the fact that every first order linear ODE with constant coefficients only is also an autonomous equation (and, therefore, is also a separable equation).

12. Give an example of an autonomous equation having no (real-valued) equilibrium solution.

13. Give an example of an autonomous equation having exactly n equilibrium solutions ($n \geq 1$).

Answers A-2.1:

1. $y = 0$ (unstable), $y = \pm 10$ (asymptotically stable)
2. $y = 0$ (asymptotically stable), $y = \pm 2$ (unstable)
3. $y = 0$ and $y = 2$ (asymptotically stable), $y = 1$ and $y = 3$ (unstable)
4. $y = 0, \pm 2\pi, \pm 4\pi, \ldots$ (unstable), $y = \pm\pi, \pm 3\pi, \pm 5\pi, \ldots$ (asymptotically stable)
5. $y = \pm 1, \pm 3, \pm 5, \ldots$ (all are semistable)
6. $y = 0$ (asymptotically stable)
7. $y = -1/3$ (asymptotically stable), $y = 1$ (unstable)
8. $y = 0$ (unstable), $y = 1$ and $y = 5$ (semistable), $y = 3$ (asymptotically stable)
9. (1) $-10, 10$; (2) $0, \infty$; (3) $0, \infty$; (4) $-\pi, \pi$; (5) $-1, 5$; (6) $0, 0$;
    (7) $-1/3, \infty$; (8) $-\infty, 3$.
10. (i) $y = -50$ (unstable) and $y = 50$ (asymptotically stable); (ii) $(-50, 50)$;
    (iii) $(-\infty, -50)$
11. For any constants α and β, $y' + \alpha y = \beta$ can be rewritten as $y' = \beta - \alpha y$,
which is autonomous (and separable).
12. One example (there are infinitely many) is $y' = e^y$.
13. One of many examples is $y' = (y - 1)(y - 2)(y - 3)\ldots(y - n)$.

Exact Equations

An *exact equation* is a first order differential equation that can be written in the form

$$M(x,y) + N(x,y) \, y' = 0,$$

provided that there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y).$$

Note 1: Often the equation is written in the alternate form of

$$M(x,y) \, dx + N(x,y) \, dy = 0.$$

> *Theorem* (Verification of *exactness*): An equation of the form
>
> $$M(x,y) + N(x,y) \, y' = 0$$
>
> is an exact equation if and only if
>
> $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Note 2: If $M(x)$ is a function of x only, and $N(y)$ is a function of y only, then trivially $\dfrac{\partial M}{\partial y} = 0 = \dfrac{\partial N}{\partial x}$. Therefore, every separable equation,

$$M(x) + N(y)y' = 0,$$

can always be written, in its standard form, as an exact equation.

The solution of an exact equation

Suppose a function $\psi(x,y)$ exists such that $\dfrac{\partial \psi}{\partial x} = M(x,y)$ and

$\dfrac{\partial \psi}{\partial y} = N(x,y)$. Let y be an implicit function of x as defined by the differential equation

$$M(x,y) + N(x,y)\,y' = 0. \qquad\qquad (1)$$

Then, by the Chain Rule of partial differentiation,

$$\frac{d}{dx}\,\psi(x, y(x)) = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = M(x,y) + N(x,y)y' \; .$$

As a result, equation (1) becomes

$$\boxed{\frac{d}{dx}\,\psi(x, y(x)) = 0} \; .$$

Therefore, we could, in theory at least, find the (implicit) general solution by integrating both sides, with respect to x, to obtain

$$\boxed{\psi(x,y) = C.}$$

Note 3: In practice $\psi(x,y)$ could only be found after two partial integration steps: Integrate $M\,(= \psi_x)$ respect to x, which would recover every term of ψ that contains at least one x; and also integrate $N\,(= \psi_y)$ with respect to y, which would recover every term of ψ that contains at least one y. Together, we can then recover every non-constant term of ψ.

Note 4: In the context of multi-variable calculus, the solution of an exact equation gives a certain *level curve* of the function $z = \psi(x,y)$.

Example: Solve the equation
$$(y^4 - 2) + 4xy^3 y' = 0$$

First identify that $M(x,y) = y^4 - 2$, and $N(x,y) = 4xy^3$.

Then make sure that it is indeed an exact equation:
$$\frac{\partial M}{\partial y} = 4y^3 \qquad \text{and} \qquad \frac{\partial N}{\partial x} = 4y^3$$

Finally find $\psi(x,y)$ using partial integrations. First, we integrate M with respect to x. Then integrate N with respect to y.

$$\psi(x,y) = \int M(x,y)\,dx = \int (y^4 - 2)\,dx = xy^4 - 2x + C_1(y) ,$$

$$\psi(x,y) = \int N(x,y)\,dy = \int 4xy^3\,dy = xy^4 + C_2(x) .$$

Combining the result, we see that $\psi(x,y)$ must have 2 non-constant terms: xy^4 and $-2x$. That is, the (implicit) general solution is:
$xy^4 - 2x = C$.

Now suppose there is the initial condition $y(-1) = 2$. To find the (implicit) particular solution, all we need to do is to substitute $x = -1$ and $y = 2$ into the general solution. We then get $C = -14$.

Therefore, the particular solution is $\quad xy^4 - 2x = -14$.

Example: Solve the initial value problem

$$(y\cos(xy) + \frac{y}{x} + 2x)\,dx + (x\cos(xy) + \ln x + e^y)\,dy = 0, \quad y(1) = 0.$$

First, we see that $M(x,y) = y\cos(xy) + \frac{y}{x} + 2x$ and

$N(x,y) = x\cos(xy) + \ln x + e^y$.

Verifying:

$$\frac{\partial M}{\partial y} = -xy\sin(xy) + \cos(xy) + \frac{1}{x} = \frac{\partial N}{\partial x} = -xy\sin(xy) + \cos(xy) + \frac{1}{x}$$

Integrate to find the general solution:

$$\psi(x,y) = \int \left(y\cos(xy) + \frac{y}{x} + 2x \right) dx = \sin(xy) + y\ln x + x^2 + C_1(y),$$

as well,

$$\psi(x,y) = \int \left(x\cos(xy) + \ln x + e^y \right) dy = \sin(xy) + y\ln x + e^y + C_2(x).$$

Hence, $\quad \sin xy + y\ln x + e^y + x^2 = C.$

Apply the initial condition: $x = 1$ and $y = 0$:

$$C = \sin 0 + 0\ln(1) + e^0 + 1 = 2$$

The particular solution is then $\quad \sin xy + y\ln x + e^y + x^2 = 2.$

Example: Write an exact equation that has general solution
$$x^3 e^y + x^4 y^4 - 6y = C.$$

We are given that the solution of the exact differential equation is

$$\psi(x,y) = x^3 e^y + x^4 y^4 - 6\,y = C.$$

The required equation will be, then, simply

$$M(x,y) + N(x,y)\,y' = 0,$$

such that $\dfrac{\partial \psi}{\partial x} = M(x,y)$ and $\dfrac{\partial \psi}{\partial y} = N(x,y)$.

Since

$$\frac{\partial \psi}{\partial x} = 3x^2 e^y + 4x^3 y^4, \qquad \text{and}$$

$$\frac{\partial \psi}{\partial y} = x^3 e^y + 4x^4 y^3 - 6.$$

Therefore, the exact equation is:

$$(3x^2 e^y + 4x^3 y^4) + (x^3 e^y + 4x^4 y^3 - 6)\,y' = 0.$$

Summary: Exact Equations

$$M(x,y) + N(x,y) y' = 0$$

Where there exists a function $\psi(x,y)$ such that

$$\frac{\partial \psi}{\partial x} = M(x,y) \quad \text{and} \quad \frac{\partial \psi}{\partial y} = N(x,y).$$

1. Verification of exactness: it is an exact equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

2. The general solution is simply

$$\psi(x,y) = C.$$

Where the function $\psi(x,y)$ can be found by combining the result of the two integrals (write down each distinct term only once, even if it appears in both integrals):

$$\psi(x, y) = \int M(x, y)\, dx, \quad \text{and}$$

$$\psi(x, y) = \int N(x, y)\, dy.$$

Exercises A-2.2:

1 − 2 Write an exact equation that has the given solution. Then verify that the equation you have found is exact.

1. It has the general solution $x^2 \tan y + x^3 - y^2 - 3x^4 y^2 = C$.

2. It has a particular solution $2xy - \ln xy + 5y = 9$.

3 − 10 For each equation below, verify its exactness then solve the equation.

3. $2x + 2x \cos(x^2) + 2y\, y' = 0$

4. $4x^3 y^4 - \dfrac{2x}{y} - 2x + (4x^4 y^3 + \dfrac{x^2}{y^2} + 5)y' = 0$

5. $(2x - 2y) + (2y - 2x)y' = 0$, $y(10) = -5$

6. $(3x^2 y + y^3 + 4 - y\, e^{xy}) + (x^3 + 3xy^2 - x\, e^{xy})y' = 0$, $y(2) = 0$

7. $(5 - 2y^2 e^{2x}) + (-5 - 2y\, e^{2x})y' = 0$, $y(0) = -4$

8. $(\dfrac{\sin x}{y^2} + \dfrac{2x}{y}) + (\dfrac{2\cos x}{y^3} - \dfrac{x^2}{y^2})y' = 0$, $y(0) = 1$

9. $(\dfrac{2xy}{x^4 + 1} + \dfrac{1}{y^2}) + (\arctan(x^2) - \dfrac{2x}{y^3})y' = 0$, $y(1) = 2$

10. $-\sin(x)\sin(2y) + y\cos(x) + (2\cos(x)\cos(2y) + \sin(x))\, y' = 0$, $y(\pi/2) = \pi$

11. Rewrite the equation into an exact equation, verify its exactness, and then solve the initial value problem.

$$y' = \dfrac{-e^y}{xe^y - \sin(y)}, \qquad y(1) = 0.$$

12 − 14 Find the value(s) of λ such that the equation below is an exact equation. Then solve the equation.

12. $(2\lambda x^5 y^3 - \dfrac{1}{x^2}) + (3x^6 y^2 - \lambda)y' = 0$

13. $(\lambda y \sec^2(2xy) - \lambda xy^2) + (2x \sec^2(2xy) - \lambda x^2 y) y' = 0$

14. $(10y^4 - 6xy + 6x^2 \sin(x^3)) + (40xy^3 - 3x^2 + \lambda \cos(x^3)) y' = 0$

Answers A-2.2:

1. $(2x \tan y + 3x^2 - 12x^3 y^2) + (x^2 \sec^2 y - 2y - 6x^4 y) y' = 0$

2. $(2y - \dfrac{1}{x}) + (2x - \dfrac{1}{y} + 5) y' = 0$

3. $x^2 + y^2 + \sin(x^2) = C$

4. $x^4 y^4 - \dfrac{x^2}{y} - x^2 + 5y = C$

5. $x^2 - 2xy + y^2 = 225$

6. $x^3 y + xy^3 + 4x - e^{xy} = 7$

7. $5x - 5y - y^2 e^{2x} = 4$

8. $\dfrac{-\cos x}{y^2} + \dfrac{x^2}{y} = -1$

9. $y \arctan(x^2) + \dfrac{x}{y^2} = \dfrac{2\pi + 1}{4}$

10. $\cos(x)\sin(2y) + y\sin(x) = \pi$

11. The equation is $e^y + (x e^y - \sin y) y' = 0$; $\qquad x e^y + \cos y = 2$

12. $\lambda = 3$; $\quad x^6 y^3 + x^{-1} - 3y = C$

13. $\lambda = 2$; $\quad \tan(2xy) - x^2 y^2 = C$

14. $\lambda = 0$; $\quad 10xy^4 - 3x^2 y - 2\cos(x^3) = C$