Probabilistic Inference and Learning
Lecture 07
Parametric Regression

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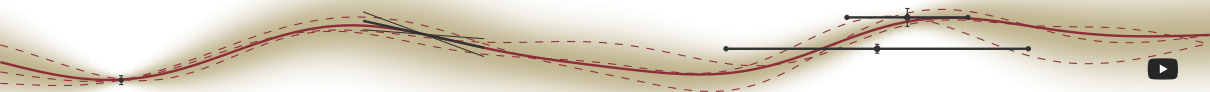

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Recap: Gaussian Distributions

Gaussian distributions provide the linear algebra of inference

- products of Gaussians are Gaussians $\quad C := (A^{-1} + B^{-1})^{-1} \quad c := C(A^{-1}a + B^{-1}b)$

$$\mathcal{N}(x; a, A)\mathcal{N}(x; b, B) = \mathcal{N}(x; c, C)\mathcal{N}(a; b, A + B)$$

- marginals of Gaussians are Gaussians

$$\int \mathcal{N}\left[\begin{bmatrix} x \\ y \end{bmatrix}; \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \begin{bmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{bmatrix}\right] dy = \mathcal{N}(x; \mu_x, \Sigma_{xx})$$

- (linear) conditionals of Gaussians are Gaussians

$$p(x \mid y) = \frac{p(x, y)}{p(y)} = \mathcal{N}\left(x; \mu_x + \Sigma_{xy}\Sigma_{yy}^{-1}(y - \mu_y), \Sigma_{xx} - \Sigma_{xy}\Sigma_{yy}^{-1}\Sigma_{yx}\right)$$

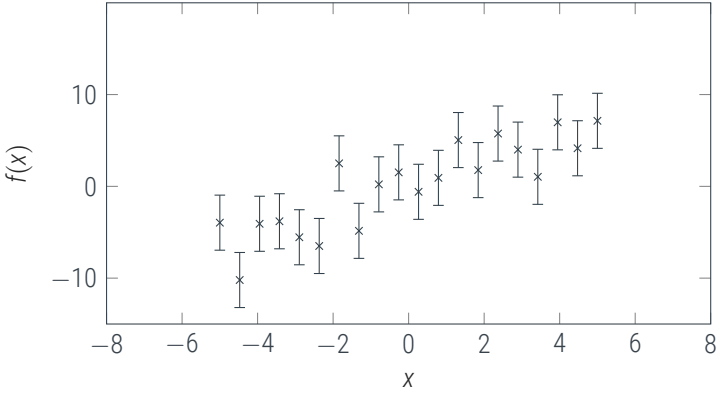
- linear projections of Gaussians are Gaussians

$$p(z) = \mathcal{N}(z; \mu, \Sigma) \quad \Rightarrow \quad p(Az) = \mathcal{N}(Az, A\mu, A\Sigma A^\intercal)$$

Bayesian inference becomes linear algebra $\qquad p(x) = \mathcal{N}(x; \mu, \Sigma) \qquad p(y \mid x) = \mathcal{N}(y; A^\intercal x + b, \Lambda)$

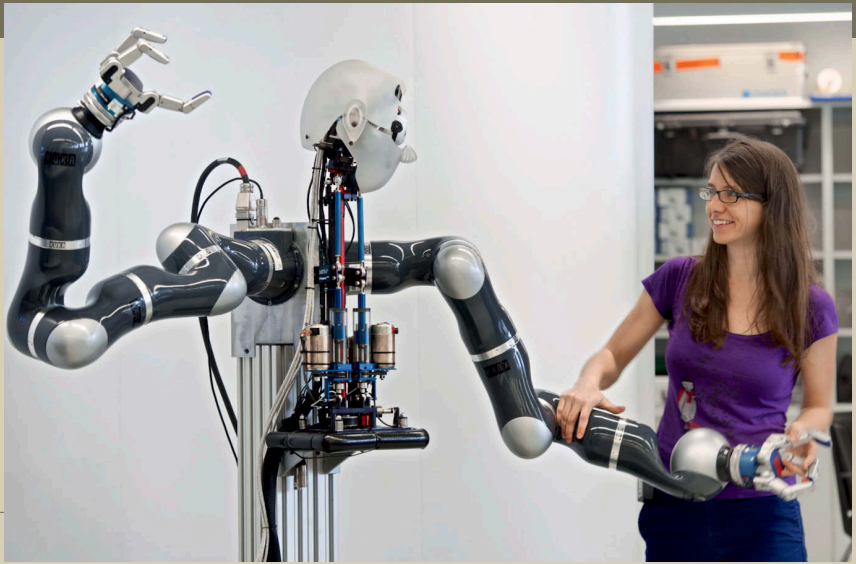
$$p(B^\intercal x + c \mid y) = \mathcal{N}[B^\intercal x + c; B^\intercal \mu + c + B^\intercal \Sigma A(A^\intercal \Sigma A + \Lambda)^{-1}(y - A^\intercal \mu - b), B^\intercal \Sigma B - B^\intercal \Sigma A(A^\intercal \Sigma A + \Lambda)^{-1} A^\intercal \Sigma B]$$

given: $\boldsymbol{y} \in \mathbb{R}^N, \quad p(\boldsymbol{y} \mid f) = \mathcal{N}(\boldsymbol{y}; \boldsymbol{f}(\boldsymbol{x}), \sigma^2 I_N)$. What is f?



Supervised Regression

A data set



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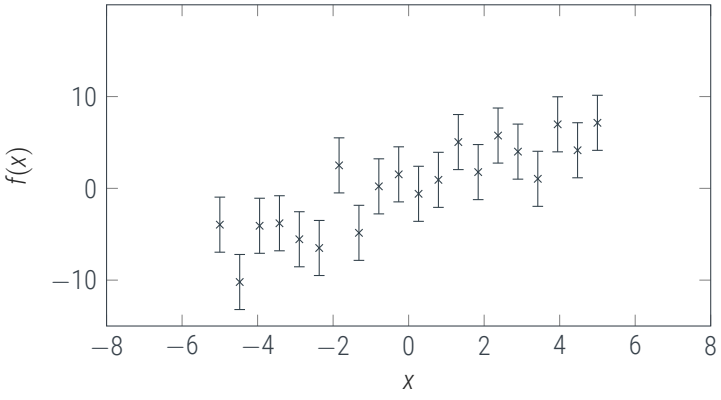
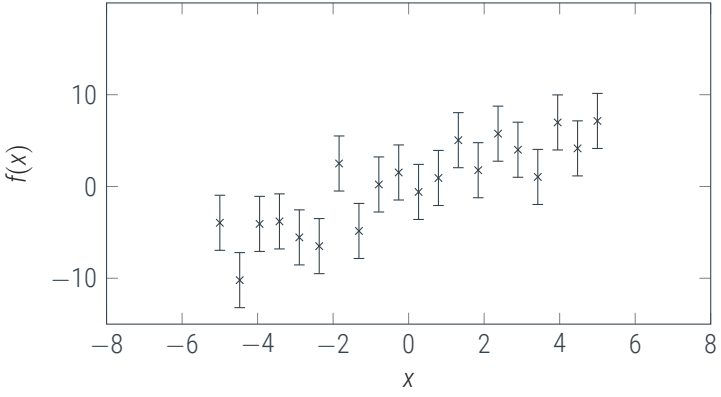
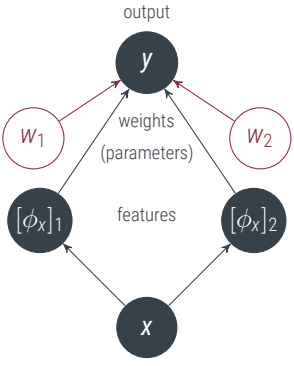


Assume **linear** function $f(x) = w_1 + w_2 x = \phi_x^\intercal \boldsymbol{w}$ with **features** $\phi(x) = \begin{bmatrix} 1 \\ x \end{bmatrix} =: \phi_x$

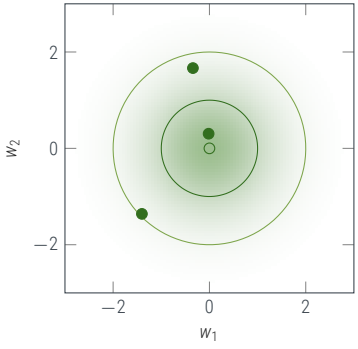




A linear **generative** model

if every variable is Gaussian and every relationship is linear, all marginals and conditionals are also Gaussian

$$f(x) = w_1 + w_2 x = \phi_x^\intercal w$$
$$p(w) = \mathcal{N}(w; \mu, \Sigma)$$



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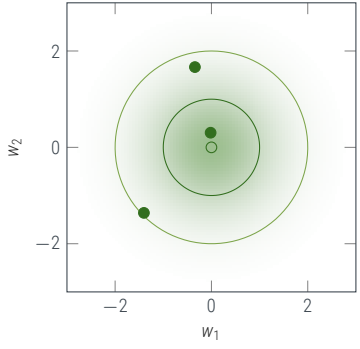

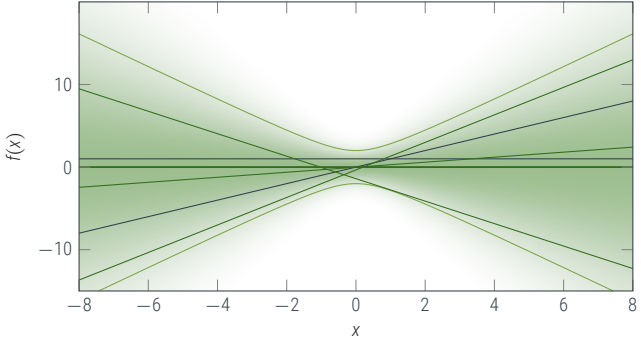

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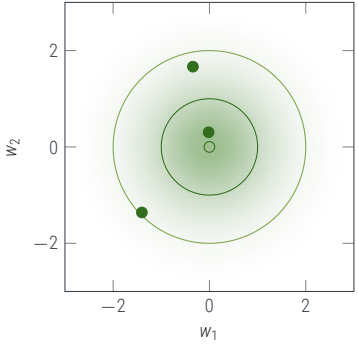
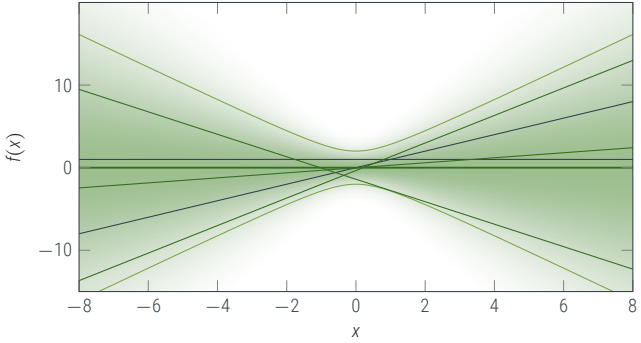


$$p(f) = \mathcal{N}(f; \phi_x^\intercal \mu, \phi_x^\intercal \Sigma \phi_x)$$



Notation

this will become exceedingly helpful later on

Dataset: $X := \begin{bmatrix} x_1 \\ \vdots \\ x_N \end{bmatrix} \in \mathbb{X}^N, \boldsymbol{y} := \begin{bmatrix} y_1 \\ \vdots \\ y_N \end{bmatrix} \in \mathbb{R}^N$. We will use the following very sloppy notation, sloppily

$$\phi_x := \phi(x) = \begin{bmatrix} 1 \\ x \end{bmatrix} \in \mathbb{R}^F \qquad \phi_X := \begin{bmatrix} \phi(x_1) & \phi(x_2) & \cdots & \phi(x_N) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_N \end{bmatrix} \in \mathbb{R}^{F \times N}$$

$$f_x := f(x) \in \mathbb{R} \qquad f_X := \phi_X^\intercal \boldsymbol{w} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) \\ \phi_1(x_2) & \phi_2(x_2) \\ \vdots & \vdots \\ \phi_1(x_N) & \phi_2(x_N) \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} \phi_{x_1}^\intercal \boldsymbol{w} \\ \phi_{x_2}^\intercal \boldsymbol{w} \\ \vdots \\ \phi_{x_N}^\intercal \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{bmatrix} \in \mathbb{R}^N$$

Think of f as an infinitely long vector, indexed by x:

$$\boldsymbol{v} \in \mathbb{R}^N, I \in \mathbb{N}^d \Rightarrow v_I := [v_{I_1}, \ldots, v_{I_d}] \in \mathbb{R} \quad \Longleftrightarrow \quad f \in \mathbb{R}^\infty, X \in \mathbb{R}^N \Rightarrow f_X := [f_{x_1}, \ldots, f_{x_N}] \in \mathbb{R}^N.$$

Gaussian Inference on a linear function

weight space / function space

$$\text{prior} \quad p(w) = \mathcal{N}(w; \mu, \Sigma) \quad \Rightarrow \quad p(f) = \mathcal{N}(f_X; \phi_X^\intercal \mu, \phi_X \Sigma \phi_X)$$

$$\text{likelihood} \quad p(y \mid w, \phi_X) = \mathcal{N}(y; \phi_X^\intercal w, \sigma^2 I) = \mathcal{N}(y; f_X, \sigma^2 I)$$

Gaussian Inference on a linear function

weight space / function space

$$
\begin{aligned}
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\text{likelihood} \quad p(y \mid w, \phi_X) &= \mathcal{N}(y; \phi_X^\intercal w, \sigma^2 I) = \mathcal{N}(y; f_X, \sigma^2 I) \\
\text{posterior on } \boldsymbol{w} \quad p(w \mid \boldsymbol{y}, \phi_X) &= \mathcal{N}(w; \mu + \Sigma\phi_X(\phi_X^\intercal \Sigma \phi_X + \sigma^2 I)^{-1}(\boldsymbol{y} - \phi_X^\intercal \mu), \\
&\qquad \Sigma - \Sigma\phi_X(\phi_X^\intercal \Sigma \phi_X + \sigma^2 I)^{-1}\phi_X^\intercal \Sigma) \\
&= \mathcal{N}\Big(w; (\Sigma^{-1} + \sigma^{-2}\phi_X^\intercal \phi_X)^{-1} \left(\Sigma^{-1}\mu + \sigma^{-2}\phi_X \boldsymbol{y}\right), \\
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&\qquad \phi_x^\intercal \Sigma \phi_x - \phi_x^\intercal \Sigma \phi_X (\phi_X^\intercal \Sigma \phi_X + \sigma^2 I)^{-1}\phi_X^\intercal \Sigma \phi_x) \\
&\qquad \mathcal{N}\Big(f_x; \phi_x(\Sigma^{-1} + \sigma^{-2}\phi_X^\intercal \phi_X)^{-1}\left(\Sigma^{-1}\mu + \sigma^{-2}\phi_X \boldsymbol{y}\right), \\
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\end{aligned}
$$

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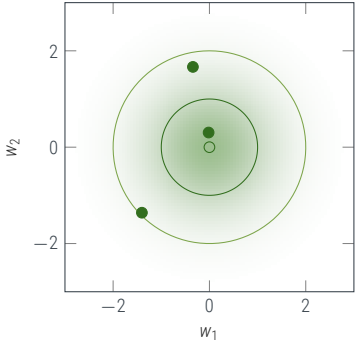

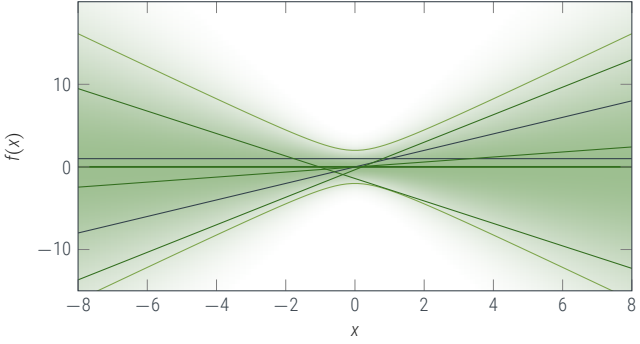

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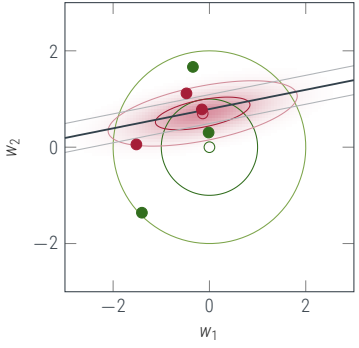

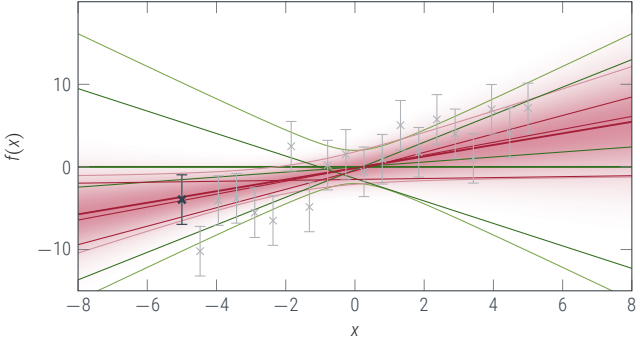

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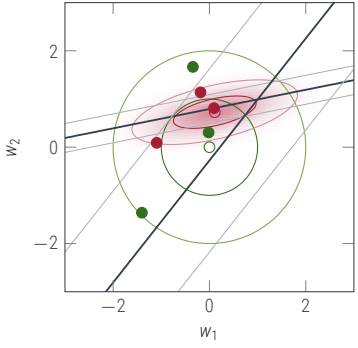

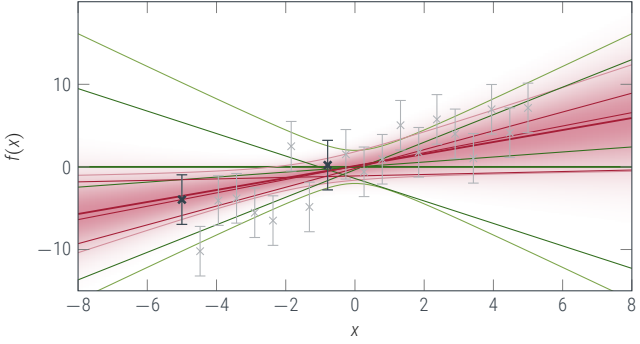

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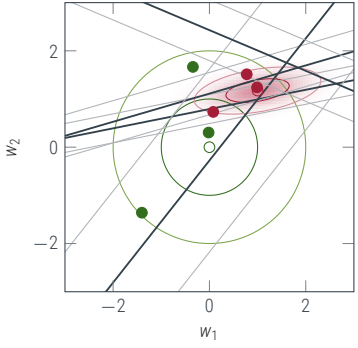

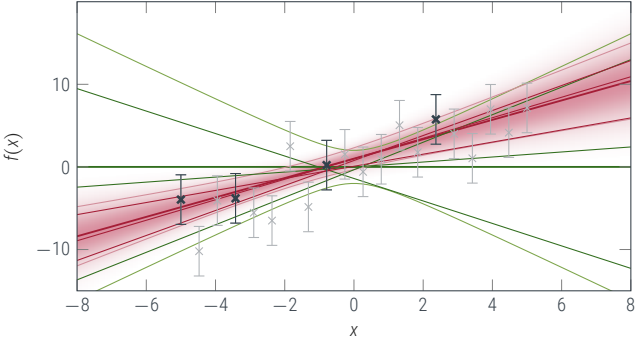

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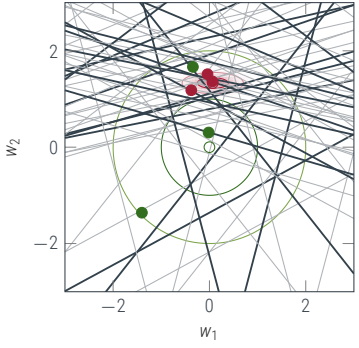

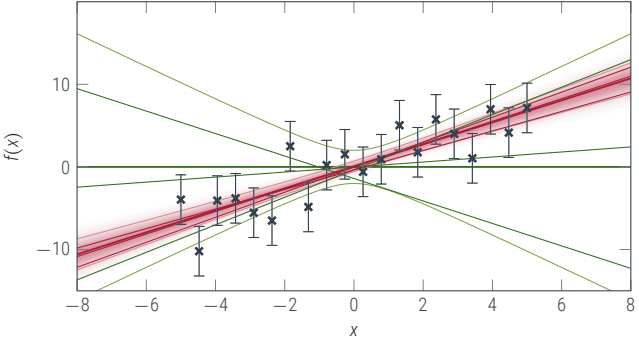

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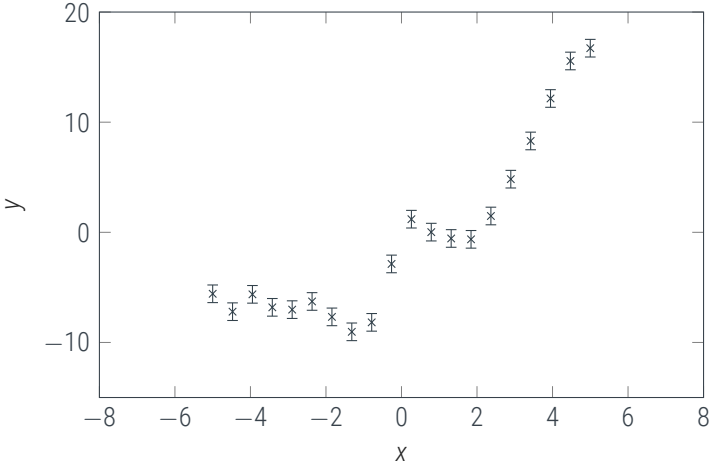
&\qquad (\Sigma^{-1} + \sigma^{-2}\phi_X^\intercal \phi_X)^{-1}\Big) \\
\text{posterior on } f \quad p(f_x \mid y, \phi_X) &= \mathcal{N}(f_x; \phi_x^\intercal \mu + \phi_x^\intercal \Sigma \phi_X (\phi_X^\intercal \Sigma \phi_X + \sigma^2 I)^{-1}(y - \phi_X^\intercal \mu), \\
&\qquad \phi_x^\intercal \Sigma \phi_x - \phi_x^\intercal \Sigma \phi_X (\phi_X^\intercal \Sigma \phi_X + \sigma^2 I)^{-1}\phi_X^\intercal \Sigma \phi_x) \\
&\qquad \mathcal{N}\Big(f_x; \phi_x(\Sigma^{-1} + \sigma^{-2}\phi_X^\intercal \phi_X)^{-1}\left(\Sigma^{-1}\mu + \sigma^{-2}\phi_X \boldsymbol{y}\right), \\
&\qquad \phi_x(\Sigma^{-1} + \sigma^{-2}\phi_X^\intercal \phi_X)^{-1}\phi_x^\intercal\Big)
\end{aligned}
$$

Code

Gaussian_Linear_Regression.ipynb

A more Realistic Dataset

General linear regression

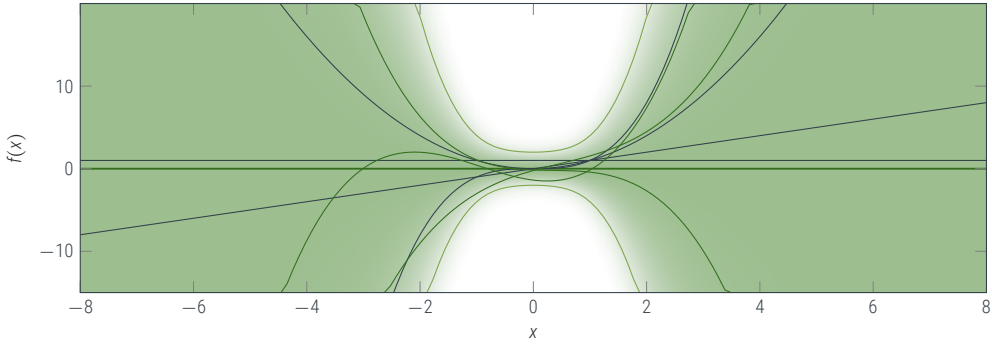


$$f(x) = w_1 + w_2 x = \phi_x^\intercal w$$

$$\phi_x := \begin{bmatrix} 1 \\ x \end{bmatrix}$$

Cubic Regression

$$f(x) = \phi(x)^\intercal w \qquad \phi(x) = \begin{bmatrix} 1 & x & x.^2 & x.^3 \end{bmatrix}^\intercal$$



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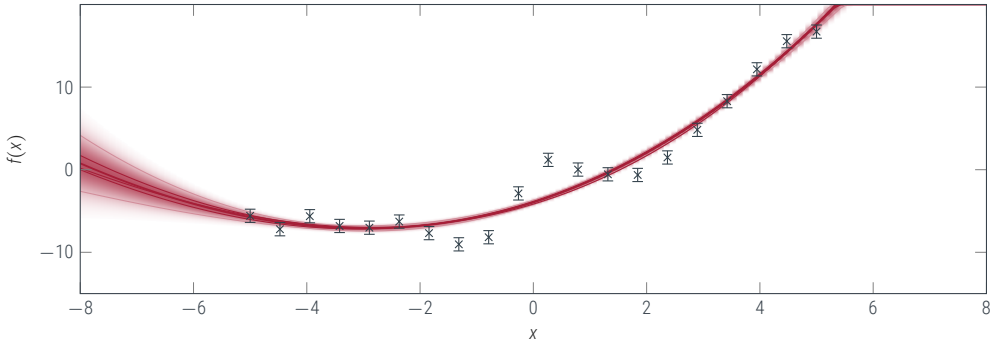
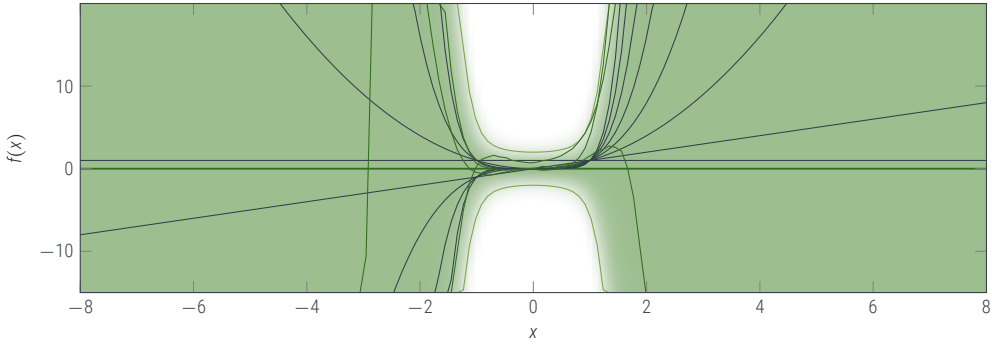


Septic Regression (?)

$$f(x) = \phi(x)^\intercal w \qquad \phi(x) = \begin{bmatrix} 1 & x & x.^2 & \cdots & x.^7 \end{bmatrix}^\intercal$$



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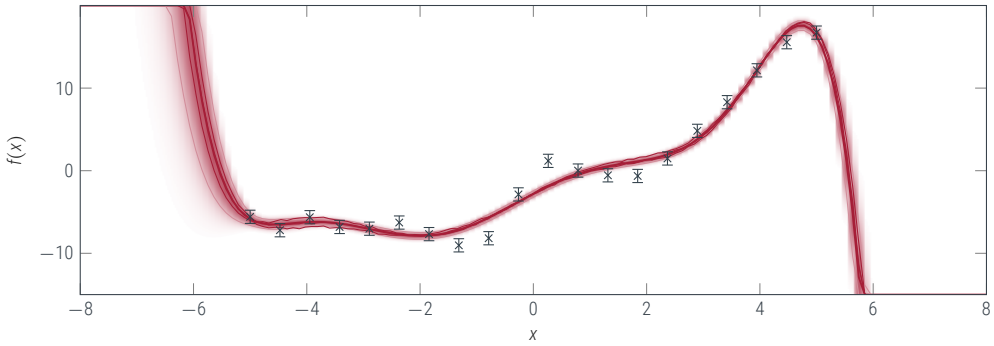
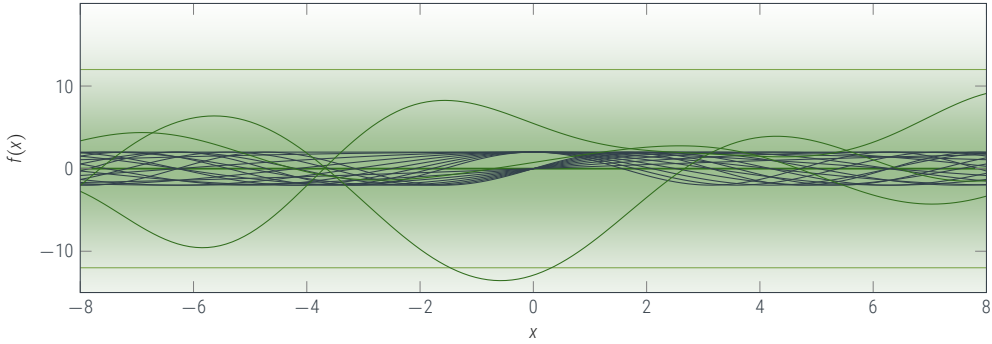


Fourier Regression

$$f(x) = \phi(x)^\intercal w \qquad \phi(x) = \begin{bmatrix}\cos(x) & \cos(2x) & \cos(3x) & \dots & \sin(x) & \sin(2x) & \dots\end{bmatrix}^\intercal$$



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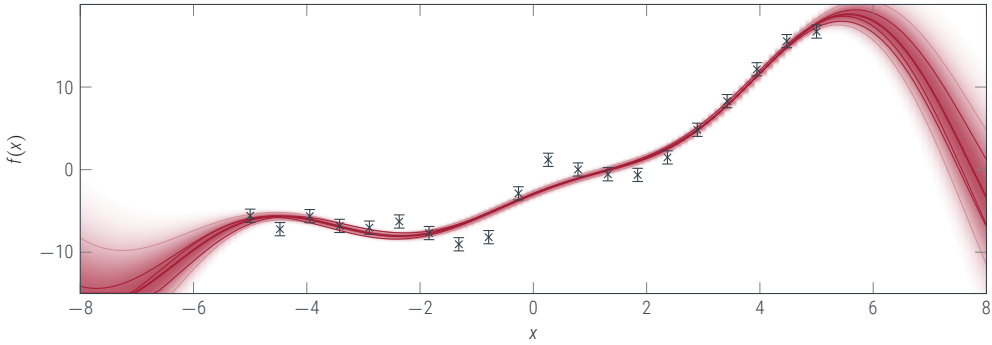
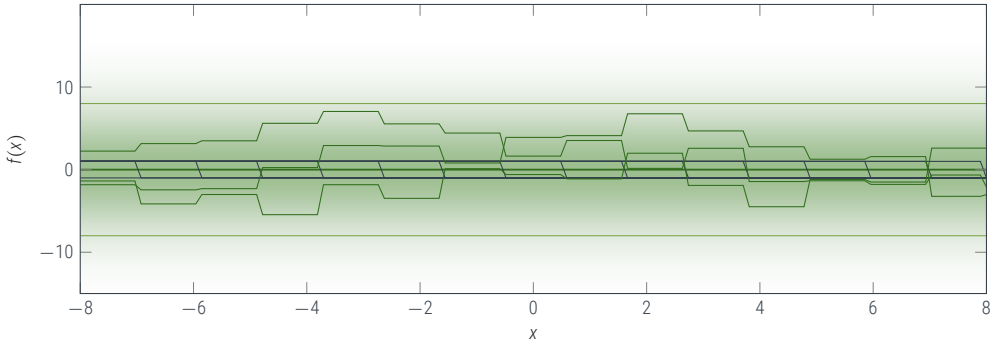


Pixel Regression

$$\phi(x) = -1 + 2\begin{bmatrix}\theta(x-8) & \theta(8-x) & \theta(x-7) & \theta(7-x) & \dots\end{bmatrix}^\intercal$$



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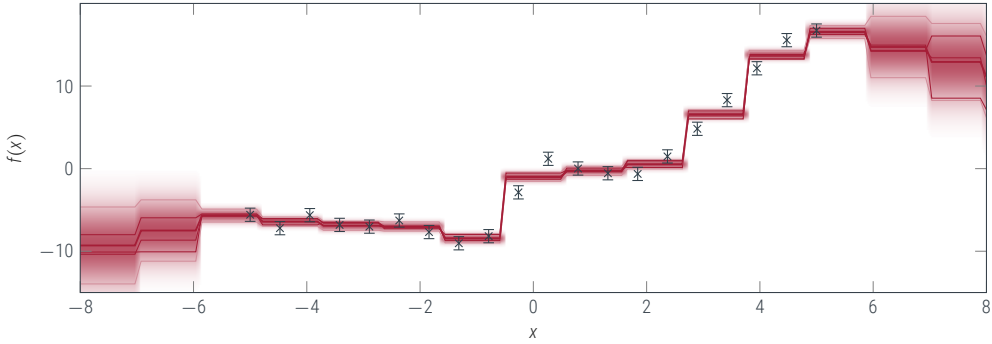
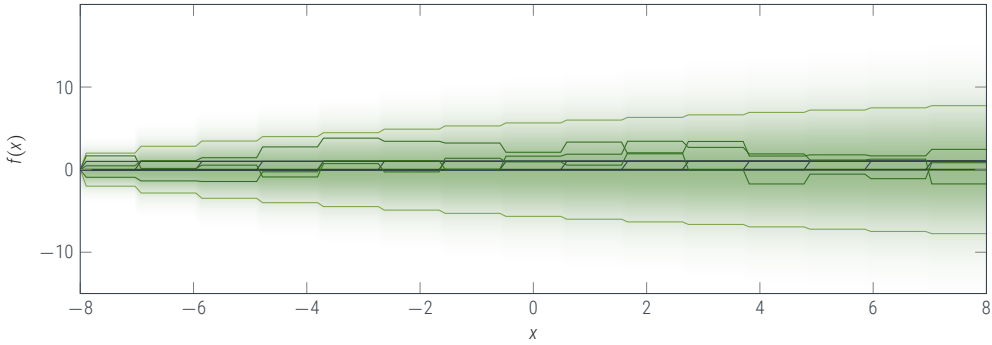


Switch Regression

$$\phi(x) = \begin{bmatrix} \theta(x-8) & \theta(8-x) & \theta(x-7) & \theta(7-x) & \dots \end{bmatrix}^\intercal$$



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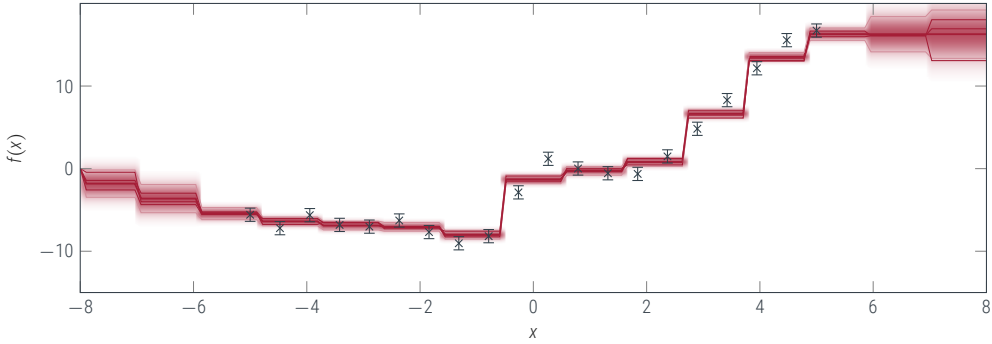
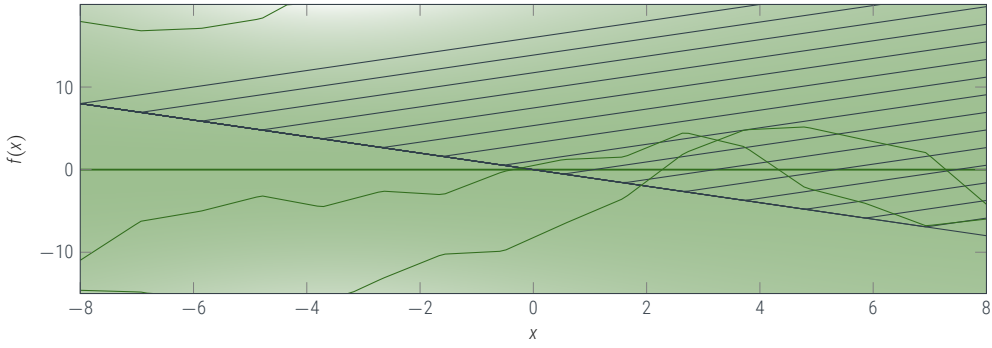


V Regression

$$\phi(x) = \begin{bmatrix} |x-8|-8 & |x-7|-7 & |x-6|-6 & \dots \end{bmatrix}^\intercal$$



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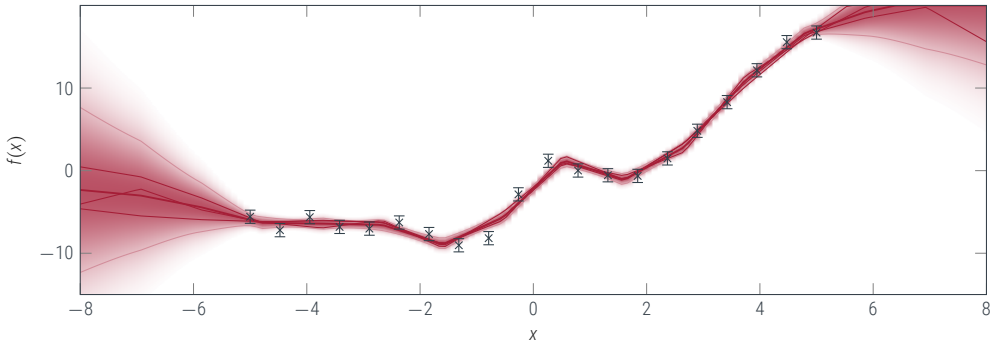
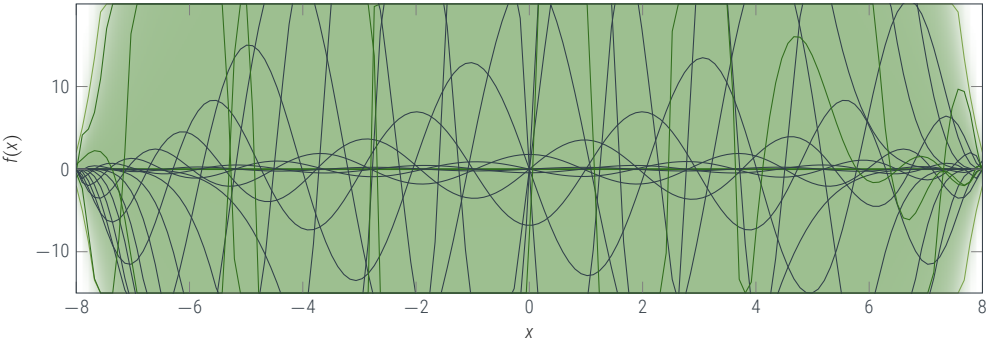


Legendre Regression

$$\phi(x) = \left[b^0 P_0(x), b^1 P_1(x), \ldots, b^{13} P_{13}(x)\right]^\intercal \qquad P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$$



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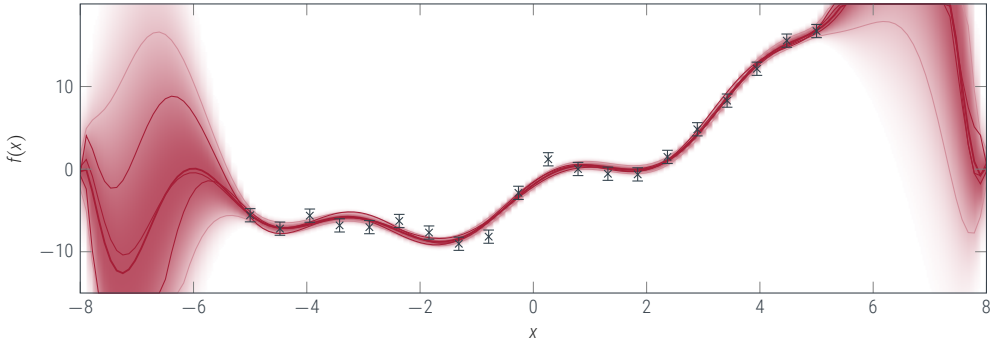
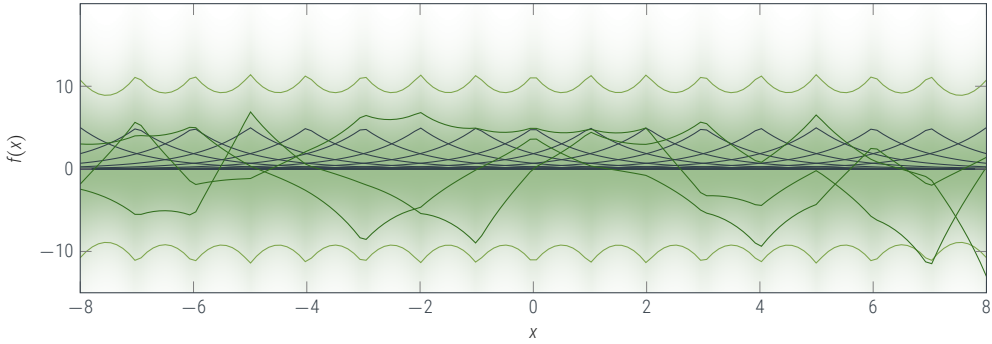


Eiffel Tower Regression

$$\phi(x) = \begin{bmatrix} e^{-|x-8|} & e^{-|x-7|} & e^{-|x-6|} & \dots \end{bmatrix}^\intercal$$



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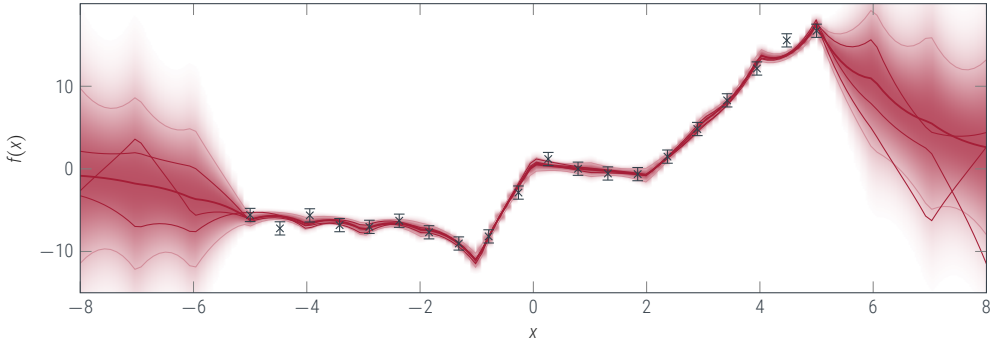
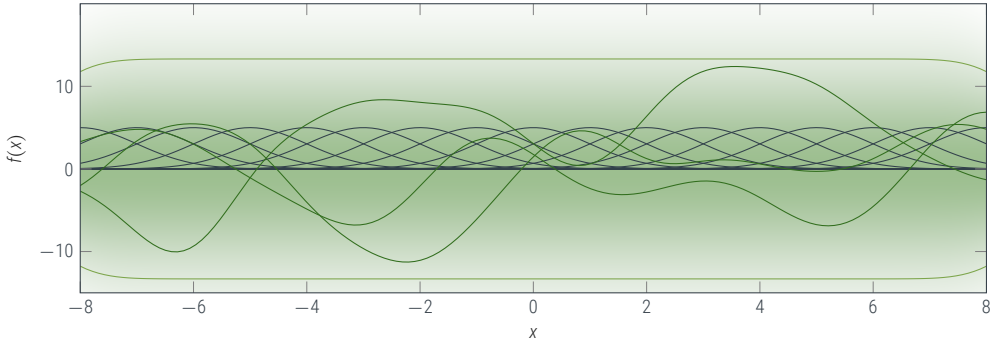


Bell Curve Regression

$$\phi(x) = \begin{bmatrix} e^{-\frac{1}{2}(x-8)^2} & e^{-\frac{1}{2}(x-7)^2} & e^{-\frac{1}{2}(x-6)^2} & \dots \end{bmatrix}^\intercal$$



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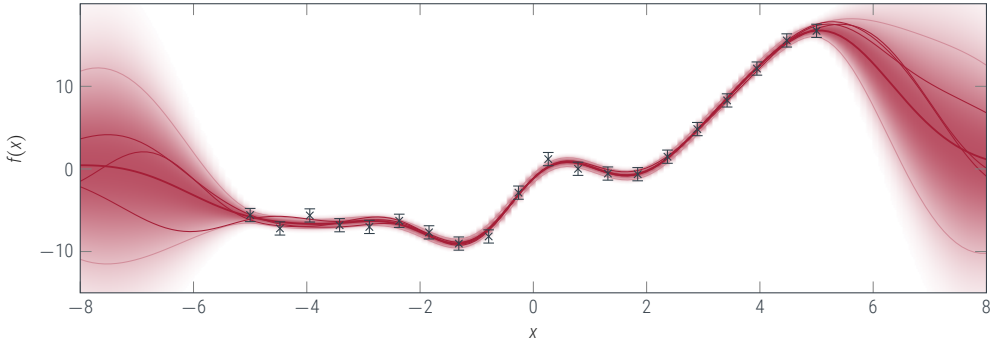
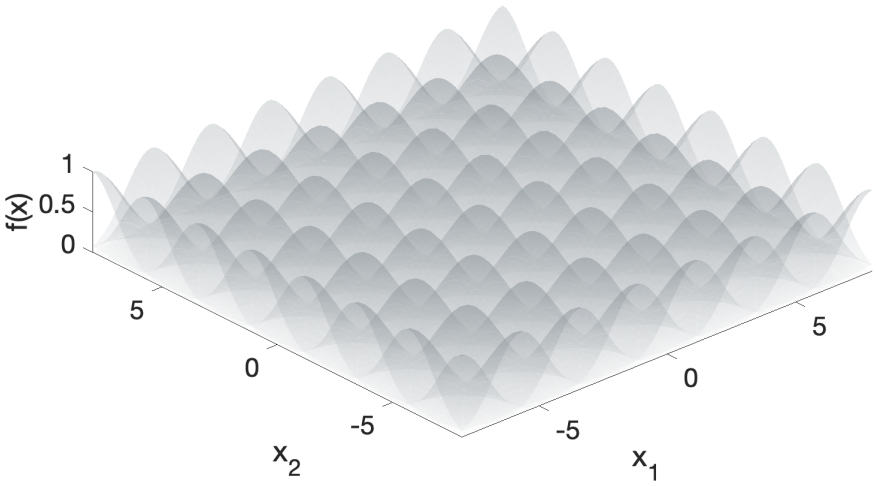


Multiple Inputs

Input domain $\mathbb{X}$ can be anything



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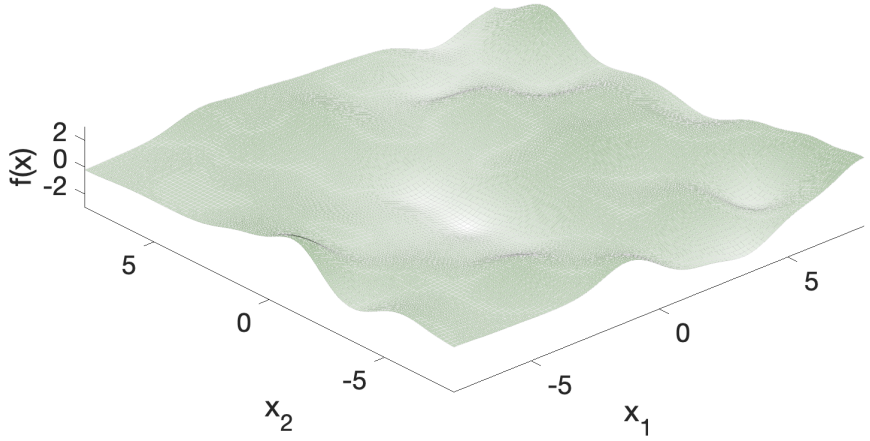
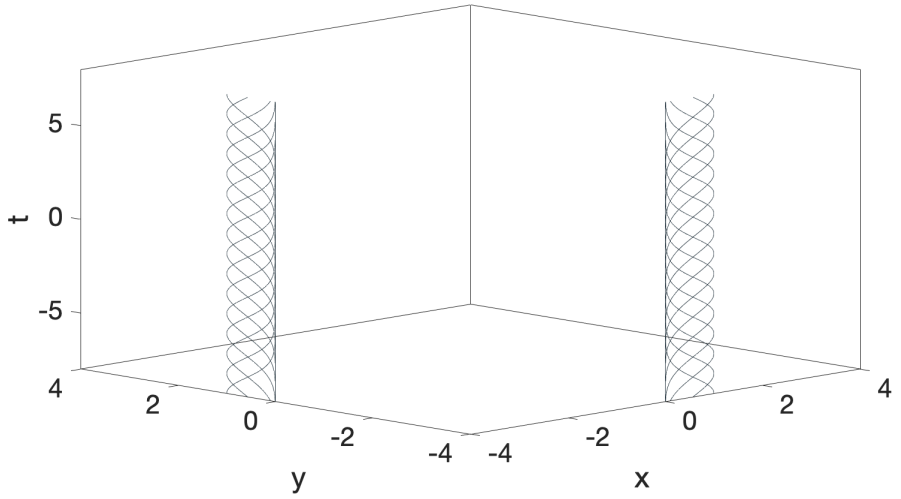


Multiple Outputs

The output domain can be anything isomorphic to $\mathbb{R}^N$



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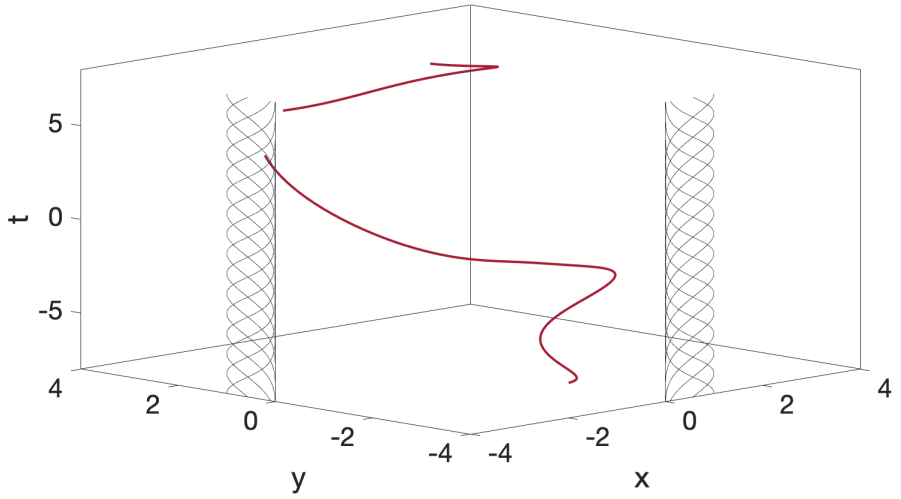

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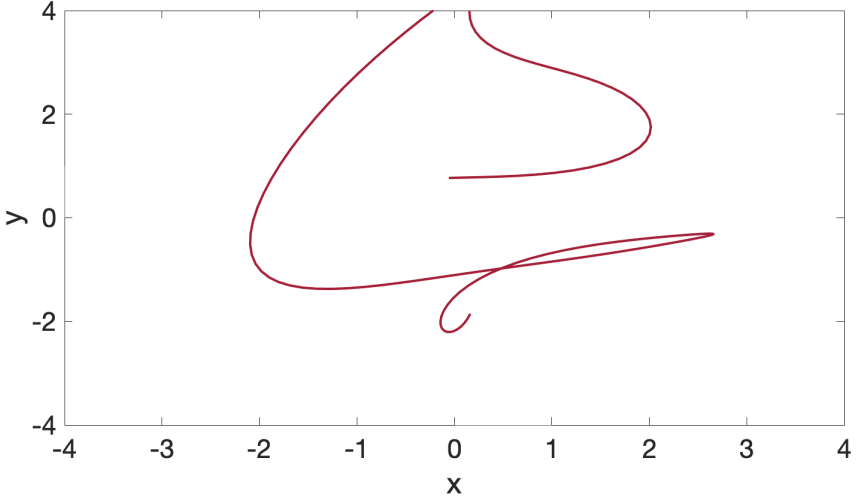

Summary:

- Gaussian distributions can be used to learn functions
- Analytical inference is possible using general linear models

$$f(x) = \phi(x)^\intercal \boldsymbol{w} = \phi_x^\intercal \boldsymbol{w}$$

- Then the posterior on both w and f is Gaussian
- The choice of features $\phi : \mathbb{X} \to \mathbb{R}$ is essentially unconstrained