A Note on Evaluation of Derivatives with Collaterals*

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Abstract: This is a revised version of our published paper Han et al. (2014). In this paper, valuation of a derivative, which is traded between default-free counterparties, partially collateralized in another specific (foreign) currency and funding in a third specific currency is studied. Replication pricing approach together with PDE and martingale solutions are presented. Our findings show that the current marking-to-market value of such a derivative consists of three components: A) the price of the perfectly collateralized derivative (a.k.a. the price by discounting in the collateral rate of the trading/payoff currency), B) the value adjustment due to the collateral rate difference between the collateral currency and the trading currency, and C) the value adjustment resulting from the uncollateralized portion of the derivative exposure. These results generalize previous works on discounting for fully collateralized derivatives and on funding value adjustments for partially collateralized or uncollateralized derivatives. It is also shown that, different from the conventional risk-neutral pricing, the derivative price is represented as an expectation in a measure where the underlying assets grow in their individual funding rates, and this measure may be reduced to the conventional risk-neutral measure in a special case.

Keywords: derivative pricing, collateralization, funding and discounting, funding value adjustments, derivative hedging

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1 Introduction

The impact of collateralization to valuation of Over-The-Counter (OTC) derivatives is well recognized and observed in the market, in particular when the borrowing rate of the derivative desk is significantly higher than the return rate of the collateral (a.k.a. *collateral rate*) designated in its Credit Support Annex (CSA) since the 2007-2008 credit and liquidity crunch. The conventional LIBOR-OIS[1] spread is usually regarded as an indicator of such a gap. This impact has been extensively investigated in practice and in theory (e.g., Johannes and Sundaresan (2007), Piterbarg (2011), Fujii et al. (2010b)). As a consequence, the approach to discounting projected cashflows with the collateral rate, a.k.a. *collateral rate discounting* or *CSA discounting*, is addressed. Collateral rate discounting for a derivative with its trading in a single currency, however, implies several model assumptions Piterbarg (2011), Fujii et al. (2010b), including:

1. Full collateralization, i.e., the posted collateral amount equals the Marking-to-Market (MtM) of the derivative;

2. Bilateral collateralization with the same collateral rate for both counterparties, i.e., each counterparty posts collateral when the derivative has a negative MtM (out of the money) from its view and receives the same collateral rate;

3. Continuous settlement, i.e., the collateral adjustment is settled immediately when MtM changes;

4. Domestic collateralization, i.e., collateral is in the same trading currency as that of the derivative payoff;

5. Cash-equivalent collateral, i.e., the posted collateral must have the highest quality and be "risk-free".

Intuitively, under above assumptions, a derivative may be regarded as "secured" and the Counterparty Credit Risk (CCR) becomes negligible. In this paper, a collateralized derivative with all above assumptions being fulfilled is referred as *perfectly collateralized*[2], whereas the term *full collateralization* refers to the relaxation of perfect collateralization in a specific collateral currency being allowed different from the trading currency. As shown in Fujii et al. (2010b,a), Fujii and Takahashi (2011), Castagna (2012), Piterbarg (2012), the value of such a derivative depends on the collateral currency even in the fully collateralized case. We here restrict ourselves within the case of a specific foreign collateral currency for any derivative, so the embedded Cheapest-To-Deliver (CTD) option of collateral posting for some CSAs that allow more than one collateral currency is out of the scope of this paper. A derivative with its payoff in a single trading currency is called *domestic collateralized* if it is (possibly partially) collateralized in its trading currency, and *foreign collateralized* if the specific collateral currency is different from the trading currency.

It is also worth noting that an implicit assumption widely made for pricing is that the unsecured borrowing rate and unsecured lending rate of the derivative desk are the same. This assumption might be regarded as counter-intuitive. However, the derivative desk has to borrow cash from its funding source (e.g., treasury desk) to start trading, so it is usually in debt regarding cash positions and needs to pay its borrowing rate as well. With any incoming cashflow, the derivative desk tends to reduce its borrowing positions of cash, if it can not lend the cash with a higher rate. Therefore, it is safe to make such an assumption, and our results could be extend to a relaxed case of different borrowing and lending rates. In our framework, the borrowing/lending rate is referred as the (unsecured) *funding rate* of a counterparty, and the spread between its unsecured funding rate and the collateral rate determined in CSA is named *funding spread*.

[1] Overnight index swap.

[2] Cleared derivatives, i.e., derivatives traded in clearing houses, are usually regarded as perfectly collateralized. Therefore, in this paper, we do not specify cleared derivatives but simply categorize them as perfectly collateralized.

The collateral settled in a daily basis is the most common practice, in particular, in consistent with requirements by clearing houses (e.g., LCH). Therefore, in many cases the collateral rate is defined to be the overnight index rate of the collateral currency in accordance with the settlement frequency. In such a case, the collateral rate discounting is equivalent to the overnight rate discounting, a.k.a. *OIS discounting*. In addition, eligible collateral assets may not be limited to cash, and government bonds in the collateral currency with minimum sovereign risk are frequently agreed for collateral. It also occurs that risky assets are posted as collateral with certain hair-cut defined in non-standard CSAs. Again, in our theoretical framework it is assumed that collateral is posted only in cash.

In addition to collateralization in a foreign currency, partial collateralization and unsecured funding in a third currency, which may be different from the trading currency and/or the collateral currency, are also considered in this paper.[3] Therefore, presented results generalize many previous works, and value adjustments due to the collateral currency and funding cost/benefit incurred by partial collateralization[4] are both included. Similar to Piterbarg (2011), Han et al. (2014), Antonov et al. (2013), counterparties of a derivative are both assumed to be default-free, and extension of our results to defaultable counterparties will be a topic of our future research.

1.1 Related Works

The theoretical foundation of valuation for derivatives partially collateralized in the domestic currency is developed in the seminal work Piterbarg (2011) by a replication and PDE approach, where both counterparties are assumed to be default-free. As special cases, approaches of collateral rate discounting and funding rate discounting are presented for the perfectly collateralized case and uncollateralized case, respectively. Furthermore, the funding value adjustment (FVA) due to partial collateralization is also implied in Piterbarg (2011). A small gap in the theory in Piterbarg (2011) is pointed out and filled in Brigo et al. (2012), and is acknowledged in the Correction Note at the end of Piterbarg (2012), while results in Piterbarg (2011) are valid. Alternatively, two different valuation approaches by expectation for perfectly collateralized derivatives are proposed in Fujii et al. (2010b) to obtain collateral rate discounting results as well as the application in interest rate curve building. These works are further extended to fully collateralized case in a foreign collateral currency Fujii et al. (2010b,a), Fujii and Takahashi (2011) to build the multiple discounting framework. A survey on multiple discounting and curve frameworks can be found at Ametrano and Bianchetti (2013). It is also worth noting that valuation methodologies in Piterbarg (2011) and in Fujii et al. (2010b,a), Fujii and Takahashi (2011) may be under different measures. Such a difference is addressed in Han et al. (2014) and this paper, as well as the link between them.

Prior to this paper, in Han et al. (2014) the valuation methodologies for derivatives partially collateralized in a foreign currency is developed, where some similar results to this work are reported. However, in this paper the market risk-free rates for relevant currencies are introduced, and collateral shortfall of the derivative can be funded in a third currency. A similar theoretical framework of FVA is proposed in Antonov et al. (2013) together with some approximation methods to calculate FVA effectively, while only domestic collateralization is studied in Antonov et al. (2013).

For uncollateralized derivatives traded between defaultable counterparties, comprehensive valuation methodologies are studied in Burgard and Kjaer (2011a,b) by replication and in Pallavicini et al. (2011, 2012), Brigo and Pallavicini (2014) by expectation, to include both bilateral credit value adjustment (CVA)[5] and funding cost.[6] Furthermore, generalized and canonical forms of CVA/DVA/FVA models are studied in Bielecki and Rutkowski (2013), and more valuation adjustments including replacement costs

[3] That is, assumptions 1 and 4 of perfect collateralization are both relaxed.

[4] Intuitively, this cost or benefit happens when a perfectly collateralized derivative is employed to hedge a partially collateralized derivative to match cashflows. The extra posted or received collateral for the hedging position may be borrowed or lent with a rate higher than the collateral rate, resulting in such a cost or a benefit.

[5] A.k.a. CVA and DVA (debt/debit/default value adjustment).

[6] Impact due to some other trading operations decided by real contracts is also addressed in Pallavicini et al. (2011, 2012), Brigo and Pallavicini (2014).

are investigated in Crépey (2012a,b). However, the impact of collateral currency is not covered in these works. The replication approach is further applied in Castagna (2011, 2012) to capture the impact of collateral and its currency for partially collateralized derivatives and results reported are similar to part of results in this paper, where this adjustment is termed as liquidity value adjustment (LVA) in those works. On the other hand, a collateral rate adjustment (CRA) is proposed on top of OIS discounting results in Hull and White (2013b,a), Hull (2014) for perfectly collateralized derivatives with collateral rate different from OIS rate.

Recent research in Albanese and Andersen (2014) and Albanese et al. (2014) proposes new accounting rules to internally transfer the FVA at enterprise level into bondholders such that FVA is not taken into account in pricing quote to clients. Similar idea is also proposed in Burgard and Kjaer (2011a) to eliminate FVA via funding the derivative by issuance of senior bonds and purchase of subordinated bonds. However, this topic is beyond the scope of this paper, and we shall always restrict ourselves within pricing derivative at trade level.

A comprehensive list of literature on new discounting theory due to collateralization as well as CVA, DVA and FVA can be found in Kenyon and Stamm (2012), Brigo et al. (2013), while we here restrict ourselves within the framework without CCR as in Piterbarg (2011), Antonov et al. (2013), and give only a few previous works directly related to our work in above.

1.2 Our Contribution

We study a derivative with a given payoff in a single trading currency, which is the same as the denominated currency of its underlyings, partially collateralized in a specific foreign currency[7] traded between two default-free counterparties and unsecured funded in a third currency if needed. To calculate its present value (PV) with respect to the impact of collateral, two types of solutions are employed.

In our approach, following ideas in Piterbarg (2011) and with the similar analysis on self-financing condition to Brigo et al. (2012) and Han et al. (2014), a self-financing portfolio is constructed including the underlying assets of the derivative and cash positions with various funding sources and return rates to replicate the value of the derivative, which might be regarded as a generalization of the Black-Scholes-Merton's framework as well. Applying Feynman-Kac formula, a PDE solution is formulated which yields our main results on the value of such a derivative. The second type of solution follows martingale approach.

The current MtM value of such a derivative can be further decomposed into three components: the price by discounting the derivative payoff with the return rate as if it was collateralized in the trading currency[8], a value adjustment due to the mismatch of collateral rates[9] of the trading (domestic) currency and the collateral (foreign) currency, and the value adjustment resulting from the uncollateralized portion of the derivative value (i.e., collateral shortfall) which is further partitioned into two parts due to the mismatch of the MtM value of the derivative and due to the mismatch (shortfall) of the collateral. Several special cases for either domestic collateral or fully collateralization are discussed, and consistent results to those in Piterbarg (2011), Fujii et al. (2010b,a), Fujii and Takahashi (2011) are reported.

It is worth noting that the term of FVA in this paper and in Piterbarg (2011), Bielecki and Rutkowski (2013), Han et al. (2014) refers to the discrepancy of price of a non-market standard derivative (e.g., partial collateralized in a foreign currency while collateral shortfalls funded in another currency) and that of the market standard derivative (e.g., cleared or fully collateralized in trading currency). This endogenous FVA is implied in pricing formula and reflects the extra funding cost and benefit to hedge

[7]In this paper, the haircut of foreign currency collateral is not taken into account, while it is straightforward to include this haircut if the corresponding schedule is simple (e.g., as defined in Basel Committee on Banking Supervision (2013)).

[8]This may be different from the actual collateral currency defined in the CSA as the derivative could be foreign collateralized. This component is in fact the collateral rate discounting result, and as a special case, the OIS discounting result if the assumed domestic collateral rate is its overnight index rate.

[9]As the difference between the currency basis adjusted unsecured funding rate of the derivative desk and the return rate of the collateral defined in CSA.

the derivative. On the other hand, in many FVA models, an exogenous FVA term is additional to the pricing formula in an "add-on" form, which is different from ours in principle.

The remainder of this paper is organized as follows. The model setup and main results are given in Section 2. The valuation methodology by replication is presented in Section 3 together with its pricing solutions by PDE and martingale in Section 4. These theoretical results are further discussed in Section 5. Section 6 presents a numerical example, and Section 7 concludes this paper.

2 Model Setup and Main Results

In a trading currency d market, let us consider a derivative which matures at $T > 0$ with a well-defined payoff of $V_{\mathsf{d}}(T)$ in d-currency[10]. This derivative is collateralized in a specified (collateral) currency c with currency exchange rate $X_t^{\mathsf{d}/\mathsf{c}}$ at time $t \geqslant 0$, which is expressed as the number of units in d per one unit in c.[11] The (cash-equivalent) collateral amount C_t^{c} in c-currency at time $t \in [0, T)$ against the derivative is assumed to be dependent of the CSA definition and the value of the derivative at t, and may differ from the derivative value denominated in d-currency in general (partially collateralized) cases. Let us introduce a (funding) currency f, which is used for unsecured funding, and the exchange rate $X_t^{\mathsf{d}/\mathsf{f}}$. In general, the three currencies (d, c, f) can be different.

Assume that the derivative is defined on a set of underlying assets whose prices $\boldsymbol{S}_t := (S_t^{(1)}, \cdots, S_t^{(n)})^\top \in \mathbb{R}^n$[12] are denominated in d-currency, where $n \geqslant 1$ is an integer. Denote by r^{d}, r^{c} and r^{f} the d-, c- and f-currency risk-free short rates, respectively. In addition, let $r^{\mathrm{F},\mathsf{u}}$ be the u-currency unsecured funding rate, and $r^{\mathrm{C},\mathsf{v}}$ be the v-currency collateral short rate, where $\mathsf{u} \in \{\mathsf{d}, \mathsf{f}\}$ and $\mathsf{v} \in \{\mathsf{d}, \mathsf{c}\}$. The d-*currency basis adjusted interest rate* is defined as $r^{\mathrm{Y},\mathsf{w}/\mathsf{d}} = r^{\mathrm{Y},\mathsf{w}} + r^{\mathsf{d}} - r^{\mathsf{w}}$ where $(\mathrm{Y}, \mathsf{w}) \in \{(\mathrm{F}, \mathsf{u}), (\mathrm{C}, \mathsf{v})\}$ can be either unsecured u-funding rate or v-collateral rate. Finally, $\lambda^{(\mathsf{u},\mathsf{v})/\mathsf{d}}$ is the spread between the d-currency basis adjusted unsecured u-funding rate and the d-currency basis adjusted v-collateral rate. We will discuss in more details on $r^{\mathrm{Y},\mathsf{w}/\mathsf{d}}$ and $\lambda^{(\mathsf{u},\mathsf{v})/\mathsf{d}}$ in Section 3.

In this paper, we focus on the derivative with a single payoff in d-currency which is possibly partially c-collateralized and possibly f-funded. Counterparties trading this derivative are assumed to be default-free. At any time $t \in [0, T)$, the time t-value of the derivative as well as its special cases are denoted as in Table 2.1, where there is no collateral currency for uncollateralized case and no unsecured funding currency for fully collateralized case.

Table 2.1: Value of the derivative and special cases

Value	Collateral Currency	Funding Currency	Financial Meaning
$V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)$	c	f	partially c-collateralized and f-funded
$V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t)$	c	d	partially c-collateralized and domestic funded
$V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t)$	d	f	partially domestic collateralized and f-funded
$V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t)$	d	d	partially domestic collateralized and funded
$V_{\mathsf{d}}^{0,\mathsf{f}}(t)$	n/a	f	uncollateralized and f-funded
$V_{\mathsf{d}}^{0,\mathsf{d}}(t)$	n/a	d	uncollateralized and domestic funded
$\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)$	c	n/a	fully c-collateralized
$\bar{V}_{\mathsf{d}}^{\mathsf{d}}(t)$	d	n/a	fully domestic collateralized (perfectly collateralized)

[10] Assume no intermediate cashflow of the derivative within the time interval (t, T).

[11] To simplify analysis, we only use the concept of the instantaneous currency exchange rate in this paper.

[12] $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$.

In all these cases, we always have following boundary conditions

$$\begin{cases} V_{\mathsf{d}}^{\alpha,\beta}(T) := V_{\mathsf{d}}^{\alpha,\beta}(T-) = V_{\mathsf{d}}(T) = \bar{V}_{\mathsf{d}}^{\xi}(T-) =: \bar{V}_{\mathsf{d}}^{\xi}(T)\,, \\ \quad \forall \alpha \in \{\mathsf{d},\mathsf{c},0\}\,,\ \ \forall \beta \in \{\mathsf{d},\mathsf{c},\mathsf{f}\}\,,\ \ \forall \xi \in \{\mathsf{d},\mathsf{c}\}\,. \end{cases} \tag{2.1}$$

Our main result of the derivative value for the generic case is

$$\begin{aligned} V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T) + \int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\mathrm{d}v\Big)\,\lambda_u^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\,X_u^{\mathsf{d}/\mathsf{c}}\,C_u^{\mathsf{c}}\,\mathrm{d}u\right] \quad (2.2)\\ &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right.\\ &\quad \left.-\int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v\Big)\Big([r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}} - r_u^{\mathrm{C},\mathsf{d}}]\,V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(u) - \lambda_u^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\,X_u^{\mathsf{d}/\mathsf{c}}\,C_u^{\mathsf{c}}\Big)\,\mathrm{d}u\right], \quad (2.3) \end{aligned}$$

where the pricing measure $\tilde{\mathcal{Q}}$ will be discussed in Section 3 as well as Subsection 4.3. In the other cases of partial collateralization, we have

$$\begin{aligned} V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right.\\ &\quad \left.-\int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v\Big)\Big([r_u^{\mathrm{F},\mathsf{d}} - r_u^{\mathrm{C},\mathsf{d}}]\,V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(u) - [r_u^{\mathrm{F},\mathsf{d}} - r_u^{\mathrm{C},\mathsf{c}/\mathsf{d}}]\,X_u^{\mathsf{d}/\mathsf{c}}\,C_u^{\mathsf{c}}\Big)\,\mathrm{d}u\right]; \quad (2.4)\\ V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right.\\ &\quad \left.-\int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v\Big)\,[r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}} - r_u^{\mathrm{C},\mathsf{d}}]\,\big(V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(u) - C_u^{\mathsf{d}}\big)\,\mathrm{d}u\right]; \quad (2.5)\\ V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right.\\ &\quad \left.-\int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v\Big)\,[r_u^{\mathrm{F},\mathsf{d}} - r_u^{\mathrm{C},\mathsf{d}}]\,\big(V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(u) - C_u^{\mathsf{d}}\big)\,\mathrm{d}u\right]. \quad (2.6) \end{aligned}$$

In the uncollateralized case $C^{\mathsf{c}} \equiv 0$ in (2.2), the following valuation results hold

$$\begin{aligned} V_{\mathsf{d}}^{0,\mathsf{f}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right]; \quad (2.7)\\ V_{\mathsf{d}}^{0,\mathsf{d}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{F},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right]. \quad (2.8) \end{aligned}$$

And finally, in the fully collateralized case $X^{\mathsf{d}/\mathsf{c}}\,C^{\mathsf{c}} \equiv \bar{V}_{\mathsf{d}}^{\mathsf{c}}$ in (2.4) or $C^{\mathsf{d}} \equiv \bar{V}_{\mathsf{d}}^{\mathsf{d}}$ (2.6), one can show that

$$\begin{aligned} \bar{V}_{\mathsf{d}}^{\mathsf{c}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right.\\ &\quad \left.-\int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C},\mathsf{d}}\,\mathrm{d}v\Big)\big(r_u^{\mathrm{C},\mathsf{c}/\mathsf{d}} - r_u^{\mathrm{C},\mathsf{d}}\big)\bar{V}_{\mathsf{d}}^{\mathsf{c}}(u)\,\mathrm{d}u\right]; \quad (2.9)\\ \bar{V}_{\mathsf{d}}^{\mathsf{d}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{C},\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right]. \quad (2.10) \end{aligned}$$

It is worth noting that many previous valuation results regarding funding and collateralization are covered by special cases above. For example, formula (2.8) gives the derivative value of traditional funding rate discounting[13] as in Piterbarg (2011), while (2.7) is used for traditional cross currency funding case. The derivative by collateral rate discounting is priced as in (2.10) as in Piterbarg (2011), Fujii et al. (2010b)[14], while the foreign collateral case is covered by (2.9) as in Fujii et al. (2010a), Fujii and Takahashi (2011), Piterbarg (2012). The domestic partially collateralized case in Piterbarg (2011) is provided in (2.6), and the foreign partially collateralized case in (2.4) has been studied in Han et al. (2014).

3 Replication

To replicate the derivative, we may consider a trading strategy which contains following components: underlying assets and their funding positions, the collateral account and an unsecured funding account. Let us elaborate. Denote

$$\boldsymbol{\theta}_t^{\mathrm{A}} := \left(\theta_t^{(1)}, \cdots, \theta_t^{(n)}\right)^\top \in \mathbb{R}^n$$

holding positions of underlying assets at the time $t \in [0, T]$. Then, we consider following types of cash positions:

- Underlying Asset Financing Account A:
  - Amount $\theta_t^{(i)} S_t^{(i)}$ is needed to finance long or short positions of the i-th underlying asset with a short rate $r_t^{(i)}$. Let us define $\boldsymbol{r}^{\mathrm{A}} := (r^{(1)}, \cdots, r^{(n)})^\top \in \mathbb{R}^n$.
  - Dividend cashflow is generated by the i-th underlying asset at the short rate $r_t^{\mathrm{D},i}$ which is paid to the owner of the asset,[15] for $i \in \{1, \ldots, n\}$. Let us define $\boldsymbol{r}_t^{\mathrm{D}} := \left(r_t^{\mathrm{D},1}, \cdots, r_t^{\mathrm{D},n}\right)^\top \in \mathbb{R}^n$.
- Derivative Collateral Account C: collateral amount C_t^{c} is posted at the time of t with a corresponding c-collateral short rate $r_t^{\mathrm{C,c}}$.
- Derivative Unsecured Financing Account F: amount $\left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}} C_t^{\mathsf{c}}\right) / X_t^{\mathsf{d/f}}$ in f-currency is to be financed with the unsecured f-funding short rate $r_t^{\mathrm{F,f}}$.

Similar to the discussion in Brigo et al. (2012), the first component of the trading strategy including the first two types of cash positions in the above trading strategy A is the portfolio of n (financing) contracts of the underlying assets. Let $\boldsymbol{v}^A$ and $\boldsymbol{g}^A$ be the price and the gain (or the yield Duffie (2003)) processes of A in d-currency. As the contracts are always instantaneously balanced, we have

$$\boldsymbol{v}_t^A = \mathbf{0} \in \mathbb{R}^n , \quad \forall t \in [0, T] , \tag{3.1}$$

while the gain process satisfies the following equation

$$\mathrm{d}\boldsymbol{g}_t^A = \mathrm{d}\boldsymbol{S}_t + \mathrm{diag}\left(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{A}}\right) \boldsymbol{S}_t \, \mathrm{d}t , \quad \forall t \in (0, T] , \tag{3.2}$$

where

$$\mathrm{diag}(\boldsymbol{a}) := \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{bmatrix} , \quad \forall \boldsymbol{a} = (a_1, \cdots, a_n)^\top \in \mathbb{R}^n .$$

[13] If the funding rate is assumed to be LIBOR rate, then this is the conventional LIBOR discounting case.

[14] When the collateral rate is designated as overnight index rate, this gives the price by so-called OIS discounting.

[15] If the financing contract defines that the dividend is paid to the seller, then the short rate of this contract should be $r_t^{(i)} - r_t^{\mathrm{D},i}$ according to the non-arbitrage arguments, and our results are still valid.

The second component C is the collateral account. Let v^C and g^C be the price and the gain processes of C in d-currency. As the collateral amount in c-currency is C^{c}, we have

$$v_t^C = \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}}\,, \ \ \forall t \in [0, T]\,, \tag{3.3}$$

and the gain process satisfies

$$\mathrm{d}g_t^C = r_t^{\mathrm{C,c}} \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}}\, \mathrm{d}t + \mathsf{C}_t^{\mathsf{c}}\, \mathrm{d}\mathsf{X}_t^{\mathsf{d/c}}\,, \ \ \forall t \in (0, T]\,. \tag{3.4}$$

The last component of the trading strategy F is the unsecured funding account in f-currency. Let v^F and g^F be the price and the gain processes of F in the d-currency. Similar to the second component, we have

$$v_t^F = V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}} = \left(\frac{1}{\mathsf{X}_t^{\mathsf{d/f}}}\, V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \frac{\mathsf{X}_t^{\mathsf{d/c}}}{\mathsf{X}_t^{\mathsf{d/f}}}\, \mathsf{C}_t^{\mathsf{c}} \right) \mathsf{X}_t^{\mathsf{d/f}}\ , \ \ \forall t \in [0, T]\,, \tag{3.5}$$

and

$$\mathrm{d}g_t^F = r_t^{\mathrm{F,f}} \left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}} \right) \mathrm{d}t + \left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}} \right) \frac{\mathrm{d}\mathsf{X}_t^{\mathsf{d/f}}}{\mathsf{X}_t^{\mathsf{d/f}}}\ , \ \ \forall t \in (0, T]\,. \tag{3.6}$$

Let us assume that there exists a function

$$\pi_{\mathsf{d}}^{\mathsf{c,f}} : (\boldsymbol{s}, c, x, x', t) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \pi^{\mathsf{d,c}}(\boldsymbol{s}, c, x, x', t) \in \mathbb{R}$$

such that the value of the aforementioned derivative can be written as

$$V_{\mathsf{d}}^{\mathsf{c,f}}(t) = \pi_{\mathsf{d}}^{\mathsf{c,f}}\big(\boldsymbol{S}_t,\, \mathsf{C}_t^{\mathsf{c}},\, \mathsf{X}_t^{\mathsf{d/c}},\, \mathsf{X}_t^{\mathsf{d/f}},\, t\big)\,, \ \ t \in [0, T]\,. \tag{3.7}$$

On the other hand, consider a strategy $\left({\boldsymbol{\theta}_t^{\mathrm{A}}}^\top, 1, 1 \right)^\top$ on (A, C, F). Clearly from (3.1), (3.3) and (3.5), the strategy gives a replication portfolio, denoted as Π_t, i.e., for $t \in [0, T]$,

$$\Pi_t = {\boldsymbol{\theta}_t^{\mathrm{A}}}^\top \cdot \boldsymbol{v}_t^A + 1 \cdot v_t^C + 1 \cdot v_t^F = V_{\mathsf{d}}^{\mathsf{c,f}}(t)\,, \tag{3.8}$$

and we further assume that the portfolio is of self-finance Duffie (2003).

Then, from (3.2), (3.4) and (3.6), we have

$$\begin{aligned} \mathrm{d}\Pi_t \ &= \ {\boldsymbol{\theta}_t^{\mathrm{A}}}^\top \cdot \left(\mathrm{d}\boldsymbol{S}_t + \mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{A}}\big) \boldsymbol{S}_t\, \mathrm{d}t \right) + 1 \cdot \left(r_t^{\mathrm{C,c}}\, \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}}\, \mathrm{d}t + \mathsf{C}_t^{\mathsf{c}}\, \mathrm{d}\mathsf{X}_t^{\mathsf{d/c}} \right) + \\ & \quad 1 \cdot \left(r_t^{\mathrm{F,f}} \left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}} \right) \mathrm{d}t + \left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathsf{X}_t^{\mathsf{d/c}}\, \mathsf{C}_t^{\mathsf{c}} \right) \frac{\mathrm{d}\mathsf{X}_t^{\mathsf{d/f}}}{\mathsf{X}_t^{\mathsf{d/f}}} \right)\,. \end{aligned} \tag{3.9}$$

Applying Ito's lemma to (3.7), we have

$$\begin{aligned} \mathrm{d}V_{\mathsf{d}}^{\mathsf{c,f}}(t) \ &= \ \mathrm{d}\pi_{\mathsf{d}}^{\mathsf{c,f}}\big(\boldsymbol{S}_t,\, \mathsf{C}_t^{\mathsf{c}},\, \mathsf{X}_t^{\mathsf{d/c}},\, \mathsf{X}_t^{\mathsf{d/f}},\, t\big) \\ &= \ \left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t} \right\} \mathrm{d}t + \left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial \boldsymbol{s}} \right\} \mathrm{d}\boldsymbol{S}_t + \left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial c} \right\} \mathrm{d}\mathsf{C}_t^{\mathsf{c}} + \left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x} \right\} \mathrm{d}\mathsf{X}_t^{\mathsf{d/c}} + \left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'} \right\} \mathrm{d}\mathsf{X}_t^{\mathsf{d/f}} + \\ & \quad \frac{1}{2} \sum_{\alpha, \beta \in \{s_1, \cdots, s_n, c, x, x'\}} \left\{ \frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial \alpha\, \partial \beta} \right\} \mathrm{d}[\zeta(\alpha), \zeta(\beta)]_t\,, \end{aligned} \tag{3.10}$$

where[16] $\boldsymbol{s} = (s_1, \cdots, s_n)^\top$, the mapping ζ is defined by

$$\zeta(s_i) = S^{(i)}\,, \ \ i \in \{1, \ldots, n\}\,, \ \ \zeta(c) = \mathsf{C}^{\mathsf{c}}\,, \ \ \zeta(x) = \mathsf{X}^{\mathsf{d/c}}\,, \ \ \zeta(x') = \mathsf{X}^{\mathsf{d/f}}\,,$$

---

[16] $\{\pi_{\mathsf{d}}^{\mathsf{c,f}}\}$ means $\pi_{\mathsf{d}}^{\mathsf{c,f}}\big(\boldsymbol{S}_t,\, \mathsf{C}_t^{\mathsf{c}},\, \mathsf{X}_t^{\mathsf{d/c}},\, \mathsf{X}_t^{\mathsf{d/f}};\, t\big)$. Similar meaning applies to $\left\{ \frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t} \right\}$, and etc.

and $[\cdot\,,\,\cdot]_t$ is a quadratic co-variation/variation process. From (3.8), we have

$$\mathrm{d}\Pi_t = \mathrm{d}V_{\mathsf{d}}^{\mathsf{c,f}}(t)\,,\quad t\in[0,T]\,. \tag{3.11}$$

Substituting (3.9) and (3.10) into (3.11), it follows that

$$\begin{aligned}
&\boldsymbol{\theta}_t^{\mathrm{A}\top}\,\mathrm{d}\boldsymbol{S}_t + C_t^{\mathsf{c}}\,\mathrm{d}X_t^{\mathsf{d/c}} + \left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right)\frac{\mathrm{d}X_t^{\mathsf{d/f}}}{X_t^{\mathsf{d/f}}} + \\
&\left[\boldsymbol{\theta}_t^{\mathrm{A}\top}\mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{A}}\big)\boldsymbol{S}_t + r_t^{\mathrm{C,c}}X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}} + r_t^{\mathrm{F,f}}\left(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right)\right]\mathrm{d}t \\
= \;&\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\boldsymbol{s}}\right\}\mathrm{d}\boldsymbol{S}_t + \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial c}\right\}\mathrm{d}C_t^{\mathsf{c}} + \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x}\right\}\mathrm{d}X_t^{\mathsf{d/c}} + \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'}\right\}\mathrm{d}X_t^{\mathsf{d/f}} + \\
&\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t}\right\}\mathrm{d}t + \frac{1}{2}\sum_{\alpha,\beta\in\{s_1,\cdots,s_n,c,x,x'\}}\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\alpha\,\partial\beta}\right\}\mathrm{d}[\zeta(\alpha),\zeta(\beta)]_t\,,\quad t\in[0,T]\,.
\end{aligned} \tag{3.12}$$

Based on (3.12), we impose

$$\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\boldsymbol{s}}\right\} = \boldsymbol{\theta}_t^{\mathrm{A}\top}\,,\quad \frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial c} = 0\,,\quad \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x}\right\} = C_t^{\mathsf{c}}\,,\quad \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'}\right\} = \frac{V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}}{X_t^{\mathsf{d/f}}}\,, \tag{3.13}$$

where the first equations indicates the Delta's of the derivative price with respect to underlying asset prices, the second equation implies that the derivative price does not explicitly depend on the collateral amount, the third and the fourth equations are consistent with intuition that the collateral account is the cash position in c-currency while the unsecured funding account is the cash position in f-currency.

Equation (3.13) suggests that, instead of (3.7), we should have

$$V_{\mathsf{d}}^{\mathsf{c,f}}(t) = \pi_{\mathsf{d}}^{\mathsf{c,f}}\Big|_{(\boldsymbol{s},x,x',t)=\left(\boldsymbol{S}_t,\,X_t^{\mathsf{d/c}},\,X_t^{\mathsf{d/f}},\,t\right)}\,,\quad t\in[0,T]\,. \tag{3.14}$$

Hence (3.12) can be re-written as[17]

$$\begin{aligned}
&\left[\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\boldsymbol{s}}\right\}\mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{D}} - \boldsymbol{r}_t^{\mathrm{A}}\big)\boldsymbol{S}_t + r_t^{\mathrm{F,f}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) + \left(r_t^{\mathrm{C,c}} - r_t^{\mathrm{F,f}}\right)X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t \\
= \;&\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t}\right\}\mathrm{d}t + \frac{1}{2}\sum_{\alpha,\beta\in\{s_1,\cdots,s_n,x,x'\}}\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\alpha\,\partial\beta}\right\}\mathrm{d}[\zeta(\alpha),\zeta(\beta)]_t\,,\quad t\in[0,T]\,,
\end{aligned}$$

or

$$\begin{aligned}
&\left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t}\right\}\mathrm{d}t + \left\{\frac{\partial\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\boldsymbol{s}}\right\}\mathrm{diag}\big(\boldsymbol{r}_t^{\mathrm{A}} - \boldsymbol{r}_t^{\mathrm{D}}\big)\boldsymbol{S}_t\,\mathrm{d}t\; + \\
&\frac{1}{2}\left[\sum_{i,j=1}^{n}\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial s_j}\right\}\mathrm{d}[S^{(i)},S^{(j)}]_t + 2\sum_{i=1}^{n}\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial x}\right\}\mathrm{d}[S^{(i)},X^{\mathsf{d/c}}]_t + 2\sum_{i=1}^{n}\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial x'}\right\}\mathrm{d}[S^{(i)},X^{\mathsf{d/f}}]_t\; +\right. \\
&\left.\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial^2 x}\right\}\mathrm{d}[X^{\mathsf{d/c}},X^{\mathsf{d/c}}]_t + 2\left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x\partial x'}\right\}\mathrm{d}[X^{\mathsf{d/c}},X^{\mathsf{d/f}}]_t + \left\{\frac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial^2 x'}\right\}\mathrm{d}[X^{\mathsf{d/f}},X^{\mathsf{d/f}}]_t\right] \\
= \;&\left[r_t^{\mathrm{F,f}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) + \left(r_t^{\mathrm{C,c}} - r_t^{\mathrm{F,f}}\right)X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t\,,\quad t\in[0,T]\,.
\end{aligned} \tag{3.15}$$

Let us introduce dynamics for the asset price $\boldsymbol{S}_t$ and the FX rate $\boldsymbol{X}_t = (X_t^{\mathsf{d/c}}, X_t^{\mathsf{d/f}})^\top$. Let $\boldsymbol{\mu}^{\mathrm{A}}$ and $\boldsymbol{\sigma}^{\mathrm{A}}$ be $\mathbb{R}^n$-valued and $\mathbb{R}_+^n$-valued processes for underlying assets, respectively, while $\boldsymbol{\mu}^{\mathrm{X}} = (\mu,\mu')^\top$ and

[17] From the second equation in (3.13), we have $\dfrac{\partial^2\pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial\alpha\partial c} = 0$, $\forall\alpha\in\{s_1,\cdots,s_n,c,x,x'\}$.

$\boldsymbol{\sigma}^{\mathsf{X}} = (\sigma, \sigma')^\top$ be $\mathbb{R}^2$-valued and $\mathbb{R}^2_+$-valued processes for the FX rate, respectively. Assume that, under a given measure, $\boldsymbol{S}_t$, $\mathrm{X}_t^{\mathsf{d/c}}$ and $\mathrm{X}_t^{\mathsf{d/f}}$ satisfy the following dynamics

$$\mathrm{d}\begin{pmatrix}\boldsymbol{S}_t\\ \boldsymbol{X}_t\end{pmatrix} = \begin{pmatrix}\boldsymbol{\mu}^{\mathrm{A}}\\ \boldsymbol{\mu}^{\mathrm{X}}\end{pmatrix}\mathrm{d}t + \mathrm{diag}\begin{pmatrix}\boldsymbol{\sigma}^{\mathrm{A}}\\ \boldsymbol{\sigma}^{\mathrm{X}}\end{pmatrix}\mathrm{d}\begin{pmatrix}\boldsymbol{W}_t^{\mathrm{A}}\\ \boldsymbol{W}_t^{\mathrm{X}}\end{pmatrix}, \tag{3.16}$$

where $(\boldsymbol{W}^{\mathrm{A}\top}, \boldsymbol{W}^{\mathrm{X}\top})^\top$ is some $\mathbb{R}^n \times \mathbb{R}^2$-valued correlated Wiener process with

$$\mathrm{d}\left[\begin{pmatrix}\boldsymbol{W}^{\mathrm{A}}\\ \boldsymbol{W}^{\mathrm{X}}\end{pmatrix}, \left(\boldsymbol{W}^{\mathrm{A}\top}, \boldsymbol{W}^{\mathrm{X}\top}\right)\right]_t = \rho\,\mathrm{d}t\,,\quad \rho := \begin{pmatrix}\rho^{\mathrm{A}} & \rho^{\mathrm{AX}}\\ \rho^{\mathrm{AX}\top} & \rho^{\mathrm{X}}\end{pmatrix}, \tag{3.17}$$

and $[\rho]_{(n+2)\times(n+2)}$ is a given correlation matrix. From (3.17), we also have

$$\mathrm{d}\left[\begin{pmatrix}\boldsymbol{S}\\ \boldsymbol{X}\end{pmatrix}, \left(\boldsymbol{S}^\top, \boldsymbol{X}^\top\right)\right]_t = \begin{pmatrix}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}})\rho^{\mathrm{A}}\,\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & \mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}})\rho^{\mathrm{AX}}\,\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{X}})\\ \mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{X}})\rho^{\mathrm{AX}\top}\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{A}}) & \mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{X}})\rho^{\mathrm{X}}\,\mathrm{diag}(\boldsymbol{\sigma}^{\mathrm{X}})\end{pmatrix}\mathrm{d}t\,. \tag{3.18}$$

Then one may find some measure, denoted as $\tilde{\mathcal{Q}}$, such that, under $\tilde{\mathcal{Q}}$, the dynamics (3.16) can be written as

$$\mathrm{d}\begin{pmatrix}\boldsymbol{S}_t\\ \boldsymbol{X}_t\end{pmatrix} = \begin{pmatrix}\tilde{\boldsymbol{\mu}}^{\mathrm{A}}\\ \tilde{\boldsymbol{\mu}}^{\mathrm{X}}\end{pmatrix}\mathrm{d}t + \mathrm{diag}\begin{pmatrix}\boldsymbol{\sigma}^{\mathrm{A}}\\ \boldsymbol{\sigma}^{\mathrm{X}}\end{pmatrix}\mathrm{d}\begin{pmatrix}\tilde{\boldsymbol{W}}_t^{\mathrm{A}}\\ \tilde{\boldsymbol{W}}_t^{\mathrm{X}}\end{pmatrix},\quad \begin{pmatrix}\tilde{\boldsymbol{\mu}}^{\mathrm{A}}\\ \tilde{\boldsymbol{\mu}}^{\mathrm{X}}\end{pmatrix} := \mathrm{diag}\begin{pmatrix}\boldsymbol{r}_t^{\mathrm{A}} - \boldsymbol{r}_t^{\mathrm{D}}\\ \boldsymbol{\delta}_t\end{pmatrix}\begin{pmatrix}\boldsymbol{S}_t\\ \boldsymbol{X}_t\end{pmatrix} \tag{3.19}$$

where $(\tilde{\boldsymbol{W}}^{\mathrm{A}\top}, \tilde{\boldsymbol{W}}^{\mathrm{X}\top})^\top$ is some $\mathbb{R}^n \times \mathbb{R}^2$-valued ρ-correlated Wiener process under $\tilde{\mathcal{Q}}$[18], and

$$\boldsymbol{\delta}_t := \left(\delta_t^{\mathsf{d,c}},\ \delta_t^{\mathsf{d,f}}\right)^{\mathsf{T}} := \left(r_t^{\mathsf{d}} - r_t^{\mathsf{c}},\ r_t^{\mathsf{d}} - r_t^{\mathsf{f}}\right)^{\mathsf{T}}, \tag{3.20}$$

which are currency basis spreads or risk-free rate spreads. With the consideration of (3.18), (3.19) and (3.20) together with the third and fourth equations in (3.13), we re-visit (3.15), which can be now further re-written as

$$\begin{aligned}
&\left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial t}\right\}\mathrm{d}t\ +\\
&\left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial \boldsymbol{s}}\right\}\mathrm{diag}(\boldsymbol{r}_t^{\mathrm{A}} - \boldsymbol{r}_t^{\mathrm{D}})\boldsymbol{S}_t\,\mathrm{d}t + \left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x}\right\}\delta_t^{\mathsf{d,c}}\,\mathrm{X}_t^{\mathsf{d/c}}\,\mathrm{d}t + \left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'}\right\}\delta_t^{\mathsf{d,f}}\,\mathrm{X}_t^{\mathsf{d/f}}\,\mathrm{d}t\ +\\
&\frac{1}{2}\left[\sum_{i,j=1}^{n}\left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial s_j}\right\}\sigma_i^{\mathrm{A}}\rho_{i,j}^{\mathrm{X}}\sigma_j^{\mathrm{A}} + 2\sum_{i=1}^{n}\left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial x}\right\}\sigma_i^{\mathrm{A}}\rho_{i,1}^{\mathrm{AX}}\sigma + 2\sum_{i=1}^{n}\left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial s_i\,\partial x'}\right\}\sigma_i^{\mathrm{A}}\rho_{i,2}^{\mathrm{AX}}\sigma'\ +\right.\\
&\qquad\left.\left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x^2}\right\}\sigma^2 + 2\left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x\partial x'}\right\}\sigma\rho_{1,2}^{\mathrm{X}}\sigma' + \left\{\frac{\partial^2 \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'^2}\right\}\sigma'^2\right]\mathrm{d}t\\
=\ &\left[r_t^{\mathsf{F,f}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) + \left(r_t^{\mathsf{C,c}} - r_t^{\mathsf{F,f}}\right)\mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t + \left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x}\right\}\delta_t^{\mathsf{d,c}}\,\mathrm{X}_t^{\mathsf{d/c}}\,\mathrm{d}t + \left\{\frac{\partial \pi_{\mathsf{d}}^{\mathsf{c,f}}}{\partial x'}\right\}\delta_t^{\mathsf{d,f}}\,\mathrm{X}_t^{\mathsf{d/f}}\,\mathrm{d}t\\
=\ &\left[r_t^{\mathsf{F,f}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) + \left(r_t^{\mathsf{C,c}} - r_t^{\mathsf{F,f}}\right)\mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t + \delta_t^{\mathsf{d,c}}\,\mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\,\mathrm{d}t + \delta_t^{\mathsf{d,f}}\left[V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t\\
=\ &\left[(r_t^{\mathsf{F,f}} + \delta_t^{\mathsf{d,f}})\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \left([r_t^{\mathsf{F,f}} + \delta_t^{\mathsf{d,f}}] - [r_t^{\mathsf{C,c}} + \delta_t^{\mathsf{d,c}}]\right)\mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t\\
=\ &\left[r_t^{\mathsf{F,f/d}}\left\{\pi_{\mathsf{d}}^{\mathsf{c,f}}\right\} - \lambda_t^{(\mathsf{f,c})/\mathsf{d}}\,\mathrm{X}_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\right]\mathrm{d}t\,,\quad t \in [0,T]\,,
\end{aligned} \tag{3.21}$$

[18] Actually, we may first get

$$\boldsymbol{\eta} := \left(\mathrm{diag}\begin{pmatrix}\boldsymbol{\sigma}^{\mathrm{A}}\\ \boldsymbol{\sigma}^{\mathrm{X}}\end{pmatrix}\right)^{-1}\begin{pmatrix}\boldsymbol{\mu}^{\mathrm{A}} - \mathrm{diag}(\boldsymbol{r}^{\mathrm{A}} - \boldsymbol{r}^{\mathrm{D}})\boldsymbol{S}\\ \boldsymbol{\mu}^{\mathrm{X}} - \mathrm{diag}(\boldsymbol{\delta})\boldsymbol{X}\end{pmatrix}.$$

We further assume that $\boldsymbol{\eta}$ satisfies some regular conditions such that the measure $\tilde{\mathcal{Q}}$ can be obtained by the Girsanov transformation with the kernel of $\boldsymbol{\eta}$ and

$$\mathrm{d}\begin{pmatrix}\tilde{\boldsymbol{W}}^{\mathrm{A}}\\ \tilde{\boldsymbol{W}}^{\mathrm{X}}\end{pmatrix} = \mathrm{d}\begin{pmatrix}\boldsymbol{W}^{\mathrm{A}}\\ \boldsymbol{W}^{\mathrm{X}}\end{pmatrix} + \boldsymbol{\eta}\,\mathrm{d}t\,.$$

where, for $\forall \mathsf{u} \in \{\mathsf{d}, \mathsf{f}\}$ and $\forall \mathsf{v} \in \{\mathsf{d}, \mathsf{c}\}$,

$$r_t^{\mathrm{F},\mathsf{u}/\mathsf{d}} := r_t^{\mathrm{F},\mathsf{u}} + \delta_t^{\mathsf{d},\mathsf{u}}\ , \quad r_t^{\mathrm{C},\mathsf{v}/\mathsf{d}} := r_t^{\mathrm{C},\mathsf{v}} + \delta_t^{\mathsf{d},\mathsf{v}}\ , \quad \delta_t^{\mathsf{d},\mathsf{d}} := 0\ , \quad \lambda_t^{(\mathsf{u},\mathsf{v})/\mathsf{d}} := r_t^{\mathrm{F},\mathsf{u}/\mathsf{d}} - r_t^{\mathrm{C},\mathsf{v}/\mathsf{d}}\ . \tag{3.22}$$

in which $r^{\mathrm{F},\mathsf{u}/\mathsf{d}}$ is the currency basis (d/u)-adjusted unsecured u-funding (short) rate, $r^{\mathrm{C},\mathsf{v}/\mathsf{d}}$ is the currency basis (d/v)-adjusted v-collateral (short) rate and $\lambda^{(\mathsf{u},\mathsf{v})/\mathsf{d}}$ is the spread between the currency basis adjusted unsecured u-funding rate and the currency basis adjusted v-collateral rate. In short, we may also call the spread λ the (currency) basis adjusted funding spread. Clearly, for any $\mathsf{u}, \mathsf{v} \in \{\mathsf{d}, \mathsf{c}, \mathsf{f}\}$, we simply have $r^{\mathrm{F},\mathsf{u}/\mathsf{u}} = r^{\mathrm{F},\mathsf{u}}$, $r^{\mathrm{C},\mathsf{v}/\mathsf{v}} = r^{\mathrm{C},\mathsf{v}}$ and $\lambda^{(\mathsf{u},\mathsf{u})/\mathsf{v}} = r^{\mathrm{F},\mathsf{u}} - r^{\mathrm{C},\mathsf{u}} =: \lambda^{\mathsf{u}}$ which is the u-funding spread. The last equation also shows that if the (unsecured) funding and the collateral are in same currency, then the currency basis adjustment has actually no effect on the funding spread.

4 Pricing

4.1 Valuation by PDE Solutions

Now from (3.21), we may conclude that if the derivative price $V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}$ has the form of (3.14), then the function $\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}$ is a solution of the following PDE

$$\mathcal{D}.\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}} = r^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}} - \lambda^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\,x\,C^{\mathsf{c}}\ , \tag{4.1}$$

with a terminal condition for $\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(\boldsymbol{s}, x, x', T)$ which is given by the derivative matured payoff, i.e.,

$$\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(\boldsymbol{s}, x, x', T)\Big|_{(\boldsymbol{s},x,x')=(\boldsymbol{S}_T, X_T^{\mathsf{d}/\mathsf{c}}, X_T^{\mathsf{d}/\mathsf{f}})} = V_{\mathsf{d}}(T)\ , \tag{4.2}$$

where

$$\left\{\begin{array}{rcl} \mathcal{D} & := & \dfrac{\partial\,\cdot}{\partial t} + \dfrac{\partial\,\cdot}{\partial \boldsymbol{s}}\,\mathrm{diag}\big(\boldsymbol{r}^{\mathrm{A}} - \boldsymbol{r}^{\mathrm{D}}\big)\boldsymbol{s} + \dfrac{\partial\,\cdot}{\partial x}\,\delta^{\mathsf{d},\mathsf{c}}\,x + \dfrac{\partial\,\cdot}{\partial x'}\,\delta^{\mathsf{d},\mathsf{f}}\,x' + \\ & & \dfrac{1}{2}\Bigg[\displaystyle\sum_{i,j=1}^{n} \sigma_i^{\mathrm{A}}\rho_{i,j}^{\mathrm{A}}\sigma_j^{\mathrm{A}}\,\frac{\partial^2\,\cdot}{\partial s_i\,\partial s_j} + 2\sum_{i=1}^{n}\sigma_i^{\mathrm{A}}\rho_{i,1}^{\mathrm{AX}}\sigma\,\frac{\partial^2\,\cdot}{\partial s_i\,\partial x} + 2\sum_{i=1}^{n}\sigma_i^{\mathrm{A}}\rho_{i,2}^{\mathrm{AX}}\sigma'\,\frac{\partial^2\,\cdot}{\partial s_i\,\partial x'} + \\ & & \quad \sigma^2\,\dfrac{\partial^2\,\cdot}{\partial x^2} + 2\,\sigma\,\rho_{1,2}^{\mathrm{X}}\,\sigma'\,\dfrac{\partial^2\,\cdot}{\partial x\partial x'} + \sigma'^2\,\dfrac{\partial^2\,\cdot}{\partial x'^2}\Bigg]\ , \end{array}\right. \tag{4.3}$$

which is called the Dynkin or Kolmogorov backward operator. We also assume that $\boldsymbol{\sigma}^{\mathrm{A}}$, $\boldsymbol{\sigma}^{\mathrm{X}}$, ρ, $\delta^{\mathsf{d},\mathsf{c}}$, $\delta^{\mathsf{d},\mathsf{f}}$, $\boldsymbol{r}^{\mathrm{D}}$, $\boldsymbol{r}^{\mathrm{A}}$, $r^{\mathrm{F},\mathsf{f}/\mathsf{d}}$, $\lambda^{(\mathsf{f},\mathsf{c})/\mathsf{d}}$, and C^{c} are all functions of $(\boldsymbol{S}, X^{\mathsf{d}/\mathsf{c}}, X^{\mathsf{d}/\mathsf{f}}, t)$. According to Feynman-Kac formula (e.g., Theorem 5.7.6 of Karatzas and Shreve (1997) or Appendix E of Duffie (2003)), the following theorem holds about the solution to (4.1)-(4.2) as summarized in (2.2):

Theorem 4.1. *With regular conditions for (4.1)-(4.2), its unique solution with sub-exponential growth admits the following stochastic representation:*

$$V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t) = \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\Big(-\int_t^T r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T) + \int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{F},\mathsf{f}/\mathsf{d}}\,\mathrm{d}v\Big)\,\lambda_u^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\,X_u^{\mathsf{d}/\mathsf{c}}\,C_u^{\mathsf{c}}\,\mathrm{d}u\right]\ , \tag{4.4}$$

where $\tilde{\mathcal{Q}}$ is the measure introduced in (3.19).

In the uncollateralized case $C^{\mathsf{c}} \equiv 0$, the derivative values in (2.7) and (2.8) can be obtained from (4.4).

As similarly pointed in Piterbarg (2011) and summarized in (2.3), we may express (4.4) in the following way:[19]

[19] If we claim that the solution $\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}$ to (4.1)-(4.2) exists, then it also satisfies (4.5), which may be obtained by re-arranging the right hand side of (4.1) to be

$$r^{\mathrm{C},\mathsf{d}}\,\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}} - \Big[-\Big(r^{\mathrm{F},\mathsf{f}/\mathsf{d}} - r^{\mathrm{C},\mathsf{d}}\Big)\,\pi_{\mathsf{d}}^{\mathsf{c},\mathsf{f}} + \lambda^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\,x\,C^{\mathsf{c}}\Big]\ ,$$

and then applying Theorem 5.7.6 of Karatzas and Shreve (1997) again. We provide another rigorous proof in Appendix A.

Theorem 4.2. *The solution (4.4) has another equivalent form:*

$$\begin{aligned} V_{\mathsf{d}}^{\mathsf{c,f}}(t) &= \mathrm{E}_t^{\tilde{\mathcal{Q}}}\Bigg[\exp\Big(-\int_t^T r_u^{\mathrm{C,d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T) \\ &\quad - \int_t^T \exp\Big(-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v\Big)\,\Big([r_u^{\mathrm{F,f/d}} - r_u^{\mathrm{C,d}}]\,V_{\mathsf{d}}^{\mathsf{c,f}}(u) - \lambda_u^{(\mathsf{f,c})/\mathsf{d}}\,X_u^{\mathsf{d/c}}\,C_u^{\mathsf{c}}\Big)\,\mathrm{d}u\Bigg]\,. \end{aligned} \tag{4.5}$$

Again, when either the collateral currency or funding currency are same as the trading currency, the pricing results (2.4), (2.5) and (2.6) for these special cases immediately follow, which further yield (2.9) and (2.10) in the special case of full collateralization.

4.2 Valuation by Martingales

From equations (3.11) and by substituting the dynamics (3.19) into (3.9), we have, under the measure $\tilde{\mathcal{Q}}$,

$$\left\{\begin{aligned} \mathrm{d}V_{\mathsf{d}}^{\mathsf{c,f}}(t) &= \Big[r_t^{\mathrm{C,c}}\,X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}} + \delta_t^{\mathsf{d,c}}\,X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}} + r_t^{\mathrm{F,f}}\Big(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\Big) + \\ &\qquad \delta_t^{\mathsf{d,f}}\,\Big(V_{\mathsf{d}}^{\mathsf{c,f}}(t) - X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\Big)\Big]\,\mathrm{d}t + (\cdots)\,\mathrm{d}\begin{pmatrix}\tilde{W}_t^{\mathrm{A}}\\ \tilde{W}_t^{\mathrm{X}}\end{pmatrix} \\ \text{(or)}\quad &=: \Big(r_t^{\mathrm{F,f/d}}V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \lambda_t^{(\mathsf{f,c})/\mathsf{d}}\,X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\Big)\,\mathrm{d}t + \Sigma_t^{\mathrm{drv}}\,\mathrm{d}W_t^{\mathrm{drv}}\,, \end{aligned}\right. \tag{4.6}$$

where Σ^{drv} is the diffusion term of the derivative price $V_{\mathsf{d}}^{\mathsf{c,f}}$ and W^{drv} is some $\tilde{\mathcal{Q}}$-Wiener process. By the discussion in Chapter 5 of Duffie (2003), the conditional expected rate of change of the derivative value at time t becomes[20]

$$\lim_{\tau\to t}\frac{\mathrm{d}}{\mathrm{d}\tau_+}\mathrm{E}_t^{\tilde{\mathcal{Q}}}\Big[V_{\mathsf{d}}^{\mathsf{c,f}}(\tau)\Big] = \Big(r_t^{\mathrm{F,f/d}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \lambda_t^{(\mathsf{f,c})/\mathsf{d}}\,X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\Big)\,,$$

or by using the associated abuse of notations, we may write

$$\mathrm{E}_t^{\tilde{\mathcal{Q}}}\Big[\mathrm{d}V_{\mathsf{d}}^{\mathsf{c,f}}(t)\Big] = \Big(r_t^{\mathrm{F,f/d}}\,V_{\mathsf{d}}^{\mathsf{c,f}}(t) - \lambda_t^{(\mathsf{f,c})/\mathsf{d}}\,X_t^{\mathsf{d/c}}\,C_t^{\mathsf{c}}\Big)\,\mathrm{d}t\,.$$

Thus, the growth rate of the derivative (under the measure $\tilde{\mathcal{Q}}$) is the currency basis (d/f)-adjusted unsecured f-funding rate $r^{\mathrm{F,f/d}}$ applied to its value less the currency basis adjusted funding spread $\lambda^{(\mathsf{f,c})/\mathsf{d}}$ applied to the d-currency equivalent collateral.

Let us consider three special cases, in which two of them are "boundary" cases. The first "boundary" case is that the derivative is uncollateralized, i.e., $C_t^{\mathsf{c}} \equiv 0$. Then, from (4.6), we have

$$\mathrm{E}_t^{\tilde{\mathcal{Q}}}\Big[\mathrm{d}V_{\mathsf{d}}^{0,\mathsf{f}}(t)\Big] = r_t^{\mathrm{F,f/d}}V_{\mathsf{d}}^{0,\mathsf{f}}(t)\,\mathrm{d}t\,,$$

i.e., $\exp\Big(-\int_0^t r_u^{\mathrm{F,f/d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}^{0,\mathsf{f}}(t)$ is a $\tilde{\mathcal{Q}}$-martingale. Thus, the time t MtM value of the uncollateralized derivative can be simply written as

$$V_{\mathsf{d}}^{0,\mathsf{f}}(t) = \mathrm{E}_t^{\tilde{\mathcal{Q}}}\Big[\exp\Big(-\int_t^T r_u^{\mathrm{F,f/d}}\,\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\Big]\,, \tag{4.7}$$

[20] "$\mathrm{d}/\,\mathrm{d}\tau_+$" is the right derivative at τ.

which is consistent with the traditional funding rate discounting Piterbarg (2011)[21]. In the other "boundary" case, the derivative is fully collateralized in c-currency. Then (4.6) gives

$$\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\mathrm{d}\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)\right] = \left(r_t^{\mathrm{F},\mathsf{f}/\mathsf{d}} - \lambda_t^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\right)\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)\,\mathrm{d}t = r_t^{\mathrm{C},\mathsf{c}/\mathsf{d}}\,\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)\,\mathrm{d}t\,,$$

which implies $\exp\left(-\int_0^t r_u^{\mathrm{C},\mathsf{c}/\mathsf{d}}\,\mathrm{d}u\right)\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)$ is a $\tilde{\mathcal{Q}}$-martingale. Therefore, the time t MtM value of the fully foreign collateralized derivative becomes

$$\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t) = \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\left(-\int_t^T r_u^{\mathrm{C},\mathsf{c}/\mathsf{d}}\,\mathrm{d}u\right)V_{\mathsf{d}}(T)\right]\,, \tag{4.8}$$

which is equivalent to those in Fujii et al. (2010b), Fujii and Takahashi (2011). In the two "boundary" cases, we clearly see that the present value of the derivative is the expectation of its matured payoff with an appropriate "discounting". In other words, the time t value of the derivative is indifferent to a path towards $V_{\mathsf{d}}(T)$.

Furthermore, for a very special single currency case[22], instead of regarding d and c as currencies, let us assume that the superscript "d" stands for a "standard" collateralization such that the collateral rate $r^{\mathrm{C},\mathsf{d}}$ is the overnight index rate of the trading/collateral currency, and that the superscript "c" for a "non-standard" collateralization such that its collateral rate $r^{\mathrm{C},\mathsf{OI}}$ is different from the overnight index rate $r^{\mathrm{C},\mathsf{d}}$. Clearly, in this case, "currency basis" adjusted c-collateralized rate $r^{\mathrm{C},\mathsf{OI}/\mathsf{d}}$ is identical to the non-standard collateralized rate $r^{\mathrm{C},\mathsf{OI}}$. Therefore, from (4.8), to value this fully OI-collateralized derivative with the collateral rate different from $r^{\mathrm{C},\mathsf{d}}$, it also holds that

$$\bar{V}_{\mathsf{d}}^{\mathsf{OI}} = \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\left(-\int_t^T r_u^{\mathrm{C},\mathsf{OI}}\,\mathrm{d}u\right)V_{\mathsf{d}}(T)\right]\,, \tag{4.9}$$

i.e., discounting with the "non-standard" collateral rate $r^{\mathrm{C},\mathsf{OI}}$, which seems like the collateral rate adjustment (CRA) proposed in Hull and White (2013b,a), Hull (2014) when collateral rate differs from overnight index rate[23]. From our analysis above, it would be more appropriate to make this adjustment on the "currency basis" adjusted collateral rate rather than simply on the collateral rate.

In the last case, we introduce a *collateral-ratio* process γ such that

$$\gamma_t\,V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t) := X_t^{\mathsf{d}/\mathsf{c}}\,C_t^{\mathsf{c}}\,,\ \ \forall t\in[0,T]\,, \tag{4.10}$$

which is a generalization of that in Castagna (2011, 2012). Then, we similarly have

$$\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\mathrm{d}V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)\right] = \left(r_t^{\mathrm{F},\mathsf{f}/\mathsf{d}} - \gamma_t\,\lambda_t^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\right)V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)\,\mathrm{d}t = \left[(1-\gamma_t)\,r_t^{\mathrm{F},\mathsf{f}/\mathsf{d}} + \gamma_t\,r_t^{\mathrm{C},\mathsf{c}/\mathsf{d}}\right]V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)\,\mathrm{d}t\,, \tag{4.11}$$

which also implies $\exp\left(-\int_0^t \left[r_u^{\mathrm{F},\mathsf{f}/\mathsf{d}} - \gamma_u\,\lambda_u^{(\mathsf{f},\mathsf{c})/\mathsf{d}}\right]\mathrm{d}u\right)V_{\mathsf{d}}^{\mathsf{f},\mathsf{c}}(t)$ is a $\tilde{\mathcal{Q}}$-martingale, and hence the time t MtM value of the partially c-collateralized and f-funded derivative becomes

$$V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t) = \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\exp\left(-\int_t^T y_{\mathsf{d}}^{\mathsf{f},\mathsf{c}}(u;\gamma)\,\mathrm{d}u\right)V_{\mathsf{d}}(T)\right]\,, \tag{4.12}$$

[21] Conventional "LIBOR discounting" when the funding rate is assumed LIBOR rate. This result may also be directly obtained by (4.4).

[22] Where both payoff and collateral are in the same currency.

[23] However, the concrete form of CRA is not provided in Hull and White (2013b,a), Hull (2014) so we are unable to make accurate comparison.

where

$$y_{\mathsf{d}}^{\mathsf{f},\mathsf{c}}(u;\gamma) := (1-\gamma_u)\, r_u^{\mathsf{F},\mathsf{f}/\mathsf{d}} + \gamma_u\, r_u^{\mathsf{C},\mathsf{c}/\mathsf{d}}\;, \tag{4.13}$$

which can be considered as the adjusted "*effective yield*" that is $(1-\gamma,\, \gamma)$-weighted “average” of the currency basis adjusted unsecured f-funded rate and the currency basis adjusted c-collateralized rate. One, however, should not be misled by the expression of (4.12). Since the collateral ratio process (4.10) may depend on the value process $V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}$, hence in general the expectation (4.12) may be subject to a distribution of value paths in $\{V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t) \,:\, t \in (t,T]\}$.

4.3 Risk Neutral Pricing

In Section 3, we have discussed that to hold the i-th underlying asset positions, either long or short, the amount of $\theta^{(i)}S^{(i)}$ is needed to be financed. If the underlying is eligible as collateral, then the finance rate $r^{(i)}$ can be the asset repo short rate. Otherwise, an unsecured funding rate can be the finance rate. It is seen that the measure $\tilde{\mathcal{Q}}$ is obtained by the Girsanov transformation with an appropriate kernel $\boldsymbol{\eta}$ such that the resulting dynamics of $(\boldsymbol{S},\boldsymbol{X})$ has the drift term given in (3.19). Clearly, $\tilde{\mathcal{Q}}$ may not be an equivalent martingale measure (EMM) corresponding to a given numéraire. Actually, this is not even needed by the valuation approaches in Sections 3, 4.1 and 4.2. It is also known that, if $\tilde{\mathcal{Q}}$ is *not* an EMM for a numéraire, it becomes inconvenient to do measure change in valuation applications.

However, we may consider an idealized situation in the original portfolio A in Section 3. We assume that, regardless the underlying assets are eligible as collateral or not, one has to always use d-market risk free rate r^{d} to finance the asset positions. In this case, we have $\boldsymbol{r}^{\mathsf{A}} = r^{\mathsf{d}}\mathbf{1}_n$ where $\mathbf{1}_n = (1,\cdots,1)^\top \in \mathbb{R}^n$. Then one may use the d-risk neutral measure $\mathcal{Q}^{\mathsf{d}}$, where the d-money market account is taken as numéraire

$$B_t^{\mathsf{d}} := \exp\Big(\int_0^t r_u^{\mathsf{d}}\,\mathrm{d}u\Big)\;,\;\; t \geqslant 0\,, \tag{4.14}$$

such that under the d-risk neutral measure, the drift term of $(\boldsymbol{S},\boldsymbol{X})$ becomes

$$\mathrm{diag}\left[\left((\boldsymbol{r}_t^{\mathsf{d}}\mathbf{1}_n - \boldsymbol{r}_t^{\mathrm{D}})^\top,\, \boldsymbol{\delta}_t^\top\right)\right]\begin{pmatrix}\boldsymbol{S}_t\\ \boldsymbol{X}_t\end{pmatrix}\;. \tag{4.15}$$

(See also Chapter 6 in Duffie (2003) and Chapter 7 in Hunt and Kennedy (2000).) It is now clear to see that, in this case, one may use $\mathcal{Q}^{\mathsf{d}}$ to replace $\tilde{\mathcal{Q}}$ in Section 3 and 4. In applications, particularly due to frequent measure change requirements, the d-risk neutral measure $\mathcal{Q}^{\mathsf{d}}$ is definitely preferred than $\tilde{\mathcal{Q}}$. In the rest of the paper, we simply use $\mathcal{Q}$ to represent either $\tilde{\mathcal{Q}}$ or $\mathcal{Q}^{\mathsf{d}}$ accordingly.

5 Funding Value Adjustments

In this subsection, it is further assumed that collateral yields in both d-currency and c-currency are their corresponding overnight index rates, unless specified otherwise.[24] It is worth noting that in the MtM value (4.5) for a partially (foreign) c-collateralized derivative, the first term, which is also (2.10), is the pricing result for the corresponding perfectly collateralized derivative as in Piterbarg (2011), Fujii et al. (2010b).[25] The second term, however, is a value adjustment of the price of perfectly collateralized derivative to incorporate the impact of funding mismatching due to imperfection of collateralization, or in the

[24] This assumption may be dropped easily by adding another FVA term as the difference between the current MtM value of the fully d-collateralized derivative with a collateral rate different from the overnight index rate and that of the OIS discounting.

[25] That is the OIS discounting result. In the current market, it is well accepted that most liquid derivatives can be regarded as perfectly collateralized with overnight index rate as the collateral rate. In particular this is the case for derivatives traded in clearing houses or with standard CSA. As a consequence, it is safe to assume that most market quotes are such prices by OIS discounting.

conventional term, FVA, which is defined as follows:

$$\begin{aligned}\mathrm{FVA}(t) &:= \left[V_{\mathsf{d}}^{\mathsf{c,f}}(4.5) - \bar{V}_{\mathsf{d}}^{\mathsf{d}}(2.10)\right] \\ &= -\mathrm{E}_t^{\mathcal{Q}}\left[\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\left[(r_u^{\mathsf{F,f/d}} - r_u^{\mathsf{C,d}})\,V_{\mathsf{d}}^{\mathsf{c,f}}(u) - \lambda_u^{(\mathsf{f,c})/\mathsf{d}}\,X_u^{\mathsf{d/c}}\,C_u^{\mathsf{c}}\right]\mathrm{d}u\right]. \end{aligned} \tag{5.1}$$

Clearly, the term FVA is naturally defined as the difference of the value of partially collateralized derivative less that of the perfectly collateralized derivative. This is consistent with the concepts of FVA in literature, e.g., Burgard and Kjaer (2011a,b), Pallavicini et al. (2011).[26] From the viewpoint of the derivative desk, it has to hedge a partially collateralized derivative with the most liquid instrument in the market, i.e., the corresponding perfectly collateralized derivative, to fulfill the cashflow liability. Hence, any difference between the value $V_{\mathsf{d}}^{\mathsf{c,f}}$ and the price $\bar{V}_{\mathsf{d}}^{\mathsf{d}}$ incurs extra funding cost or benefit for the derivative desk.

The FVA term (5.1) can be further decomposed into following two major components:

$$\mathrm{FVA}(t) = \mathrm{FVA}_1(t) + \mathrm{FVA}_2(t)\,, \tag{5.2}$$

where

$$\left\{\begin{array}{lcl}\mathrm{FVA}_1(t) &:=& [\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)(2.9) - \bar{V}_{\mathsf{d}}^{\mathsf{d}}(t)(2.10)]\,, \\ \mathrm{FVA}_2(t) &:=& [V_{\mathsf{d}}^{\mathsf{c,f}}(t)(4.5) - \bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)(2.9)]\,. \end{array}\right. \tag{5.3}$$

We see that FVA_1 means the difference between the value of the fully (foreign) c-collateralized derivative and that of the perfectly collateralized derivative, while FVA_2 represents the difference between the value of the partially c-collateralized derivative and that of the fully c-collateralized derivative. In below these two FVA terms are thoroughly studied.

First, substituting (2.9) and (2.10) into the equation of FVA_1 in (5.3) yields

$$\mathrm{FVA}_1(t) = \mathrm{E}_t^{\mathcal{Q}}\left[-\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\left[r_u^{\mathsf{C,c/d}} - r_u^{\mathsf{C,d}}\right]\bar{V}_{\mathsf{d}}^{\mathsf{c}}(u)\,\mathrm{d}u\right]\,, \tag{5.4}$$

which is due to the mismatch of cost to fulfill collateral requirements in the fully collateralized cases.[27] If the currency basis adjusted c-collateral rate happens to be equal to the d-collateral rate, i.e., $r^{\mathsf{C,c/d}} = r^{\mathsf{C,d}}$, then FVA_1 vanishes even if the trading and collateral are in different currencies. From hedging point of view, when the derivative desk hedge a fully foreign collateralized derivative with its corresponding perfectly collateralized derivative, it receives collateral (or posts collateral, respective) in c-currency if it is in the money (or out of the money, respectively) for the original derivative, and posts collateral (or receives collateral, respectively) in d-currency for the hedging position, with the corresponding collateral yield $r^{\mathsf{C,c/d}}$ or $r^{\mathsf{C,d}}$. The desk's collateral commitment and its different collateral funding cost for d- and c-currencies leads to the value adjustment in (5.4), which can be a cost or a benefit depending on the difference of the collateral yields of the two currencies,[28] even though the collateral of the original derivative is always equivalent to its MtM at any time.

Based on (4.8), an equivalent form of the MtM value of a fully foreign collateralized derivative reads:

$$\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t) = \mathrm{E}_t^{\mathcal{Q}}\left[\exp\Big(-\int_t^T \left[r_u^{\mathsf{C,c}} + \delta_u^{\mathsf{d,c}}\right]\mathrm{d}u\Big)\,V_{\mathsf{d}}(T)\right]\,, \tag{5.5}$$

[26] Or the combination of FVA and LVA discussed in Castagna (2011, 2012).

[27] Notice that it is not perfect collateralization because the collateral currency is not the same as the derivative trading currency.

[28] If the collateral currency can be chosen from a set of different currencies, the derivative desk would choose the one in its most favour to reduce its funding cost when it is out of the money. This leads to the embedded cheapest-to-deliver (CTD) option in collateral management. On the other hand, when it is in the money, it seems to sell such a CTD option to its counterparty. This optionality is not within the scope of this paper.

which implies that the fully (foreign) c-collateralized derivative would be valued by discounting its matured payoff with a synthetic discount curve with short rate $r^{\mathsf{C,c}} + \delta^{\mathsf{d,c}}$. This approach leads to the *multiple discounting* framework depending on collateral currency which is widely employed in industry, e.g., Fujii et al. (2010a), Fujii and Takahashi (2011). In this way, impact of FVA_1 can be replaced by choosing an appropriate (synthetic) discounting curve, though extra attention should be paid on the correlation between derivative payoff and the short rates/spreads in general cases.

Second, Substituting (4.5) and (2.9) into the second equation of FVA_2 in (5.3) gives

$$\begin{cases} \mathrm{FVA}_2(t) &=& \mathrm{E}_t^{\mathcal{Q}}\left[-\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\big[r_u^{\mathsf{F,f/d}} - r_u^{\mathsf{C,d}}\big]\Big(V_{\mathsf{d}}^{\mathsf{c,f}}(u) - \bar{V}_{\mathsf{d}}^{\mathsf{c}}(u)\Big)\,\mathrm{d}u\right] \\ & & + \,\mathrm{E}_t^{\mathcal{Q}}\left[-\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\lambda_u^{(\mathsf{f,c})/\mathsf{d}}\Big(\bar{V}_{\mathsf{d}}^{\mathsf{c}}(u) - X_u^{\mathsf{d/c}}\,C_u^{\mathsf{c}}\Big)\,\mathrm{d}u\right] \\ &=:& \mathrm{FVA}_2^{\mathrm{MtM}}(t) + \mathrm{FVA}_2^{\mathrm{Collateral}}(t). \end{cases} \tag{5.6}$$

FVA_2, as the funding value adjustment due to partial collateralization, can be further decomposed into two parts: $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ due to mismatch between MtM value of the partially collateralized derivative and that of the corresponding fully collateralized derivative, as well as $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ due to mismatch (shortfall) of collateral amount between them. From hedging point of view, the derivative desk hedges a partially collateralized derivative with its corresponding fully (foreign) collateralized derivative. Since here the two derivatives have different MtMs $V_{\mathsf{d}}^{\mathsf{c,f}}(u)$ and $\bar{V}_{\mathsf{d}}^{\mathsf{c}}(u)$, respectively, the derivative desk's accounting profit or loss $V_{\mathsf{d}}^{\mathsf{c,f}} - \bar{V}_{\mathsf{d}}^{\mathsf{c}}$ of the portfolio consisting of the original derivative and its hedging position denominated in d-currency on its book, but this MtM profit or loss amount is not realized. Otherwise, the derivative desk would have used this realized profit in cash to earn $r_u^{\mathsf{F,f/d}}$ return (or have borrowed in $r_u^{\mathsf{F,f/d}}$ to cover the realized loss, respectively) if it is in the money (or out of the money, respectively) for this portfolio, and would have paid back (or have received, respectively) the accrued interest at next MtM calculation date,[29] with the return rate $r_u^{\mathsf{C,d}}$ due to the cash position and the calculation frequency equivalent to the collateral position. However, proceeds of this unrealized profit or loss cannot be booked, thus $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ occurs. On the other hand, the derivative desk only receives (or posts, respectively) C_u^{c} amount of collateral in c-currency, but has to post (or receive, respectively) $\bar{C}_u^{\mathsf{c}} := \bar{V}_{\mathsf{d}}^{\mathsf{c,f}}/X_u^{\mathsf{d/c}}$ amount of collateral in c-currency, with dividend yield $\lambda_u^{(\mathsf{f,c})/\mathsf{d}}$. Then $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ for this part follows. Again, $\mathrm{FVA}_2(t)$ could be cost or benefit depending on the MtM dynamics.

In the special case that the collateral is posted in d-currency as well, FVA_1 simply vanishes from (5.4). In addition, though both $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ and $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ exist, the identity $\lambda^{(\mathsf{f,d})/\mathsf{d}} = r^{\mathsf{F,f/d}} - r^{\mathsf{C,d}}$ in (5.6) leads to the cancellation of the term with $\bar{V}_{\mathsf{d}}^{\mathsf{d}}$, and we have the following effectively FVA amount:

$$\begin{aligned} \mathrm{FVA} &= [V_{\mathsf{d}}^{\mathsf{d,f}}(t)(2.5) - \bar{V}_{\mathsf{d}}^{\mathsf{d}}(t)(2.10)] \\ &= \mathrm{E}_t^{\mathcal{Q}}\left[-\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\big[r_u^{\mathsf{F,f/d}} - r_u^{\mathsf{C,d}}\big]\Big(V_{\mathsf{d}}^{\mathsf{f,d}}(u) - C_u^{\mathsf{d}}\Big)\,\mathrm{d}u\right], \end{aligned} \tag{5.7}$$

which is consistent with the results in Burgard and Kjaer (2011a,b), Pallavicini et al. (2011), Castagna (2011). These identical yield rates in $\mathrm{FVA}_2^{\mathrm{MtM}}(t)$ and $\mathrm{FVA}_2^{\mathrm{collateral}}(t)$ frequently mislead people simply thinking of the credit exposure being directly used for FVA calculation, like in (5.7).

In a summary, at any future time $u \in (t, T)$, let us consider the whole portfolio of a derivative and its collateral. We notice that any component of this portfolio causes funding/collateral adjustment if it cannot be hedged/funded/replicated by the portfolio of the corresponding perfectly collateralized hedging position with its collateral. Therefore, a term *funding exposure* is coined here for such a component with

[29] This is also the next collateral calculation/settlement date.

a single funding spread. As a result, the generic FVA can be formed as follows:

$$\mathrm{FVA}(t) = \mathrm{E}_t^{Q}\left[-\int_t^T \exp\Big(-\int_t^u r_v^{\mathsf{C,d}}\,\mathrm{d}v\Big)\left(\mathbf{\Lambda}_u^\top \cdot \boldsymbol{V}_u^{\mathrm{F}}\right)\mathrm{d}u\right], \tag{5.8}$$

where $\boldsymbol{V}_u^{\mathrm{F}} := (V_u^{\mathrm{F},1}, \ldots, V_u^{\mathrm{F},m})^\top$ and each $V_u^{\mathrm{F},j}$ is a funding exposure, for $j \in \{1, \ldots, m\}$, and $\mathbf{\Lambda}_u = (\lambda_u^{(1)}, \ldots, \lambda_u^{(m)})^\top$ is a vector of corresponding funding spread yields. Then in the partially (foreign) collateralized case, in (5.8), $m = 3$,

$$\left\{\begin{array}{lcl} \boldsymbol{V}_u^{\mathrm{F}} & = & \left(\bar{V}_{\mathsf{d}}^{\mathsf{c}}(u),\ V_{\mathsf{d}}^{\mathsf{c,f}}(u) - \bar{V}_{\mathsf{d}}^{\mathsf{c}}(u),\ \bar{V}_{\mathsf{d}}^{\mathsf{c}}(u) - X_u^{\mathsf{d/c}} C_u^{\mathsf{c}}\right)^\top, \\ \mathbf{\Lambda}_u & = & \left(r_u^{\mathsf{C,c/d}} - r_u^{\mathsf{C,d}},\ r_u^{\mathsf{F,f/d}} - r_u^{\mathsf{C,d}},\ \lambda_u^{(\mathsf{f,c})/\mathsf{d}}\right)^\top. \end{array}\right. \tag{5.9}$$

Give the above generic form (5.8) of FVA as well as the fact that modelling the funding spread of the derivative desk is at least as difficult as counterparty's default process, it implies that the complexity of FVA calculation is not less than that of CVA calculation.[30] Also notice that in the fully (foreign) collateralized case, though the credit exposure is zero, FVA_1 still exists while CVA vanishes. It is only in the very special case of (domestic) d-collateralization, there is only one type of funding exposure, which happens to be equivalent to the credit exposure, and the FVA calculation may be similar to CVA calculation.

6 Numerical Examples

In this section, we illustrate quantitative impacts of the collateralization ratio and the collateral and funding currency on derivative pricing and on the discount factor.

6.1 Example Settings

We consider a simple example of European options on a single non-dividend stock, whose dynamics is assumed to be uncorrelated with interest rate processes. Table 6.1 lists details of these options.

Table 6.1: Details of sample options

Parameter	Value
Spot price (S_t)	\$100.00
Strike (K)	\$100.00
Volatility (σ)	30.00%
Time to maturity ($\tau \triangleq T - t$)	1 year
Type	call/put

We assume all interest rates are constant. Four cases are considered in our examples, where interest rates in each case are listed in Table 6.2 and 6.3. Case 1, 2 and 3 are used to examine the impact on European option prices, where Case 1 and Case 2 differ in the underlying funding rate, and Case 1 and Case 3 differ in currency basis spreads. Case 4 is designed to check the impact on the discount factor.

[30] Notice that here the credit exposure is simply $V_u^{\mathsf{d,f}} - X_u C_u^{\mathsf{f}}$ if the counterparties were defaultable. This argument also works for the case with wrong way risk, as in that case the dependence between the desk's funding spread and the exposure has to be taken into account.

Table 6.2: Sample currency basis spreads $\delta^{\mathsf{d},\cdot}$, collateral rates $r^{\mathrm{C},\cdot}$ and funding rates $r^{\mathrm{F},\cdot}$

Currency	Case 1			Case 2			Case 3			Case 4		
	$\delta^{\mathsf{d},\cdot}$	$r^{\mathrm{C},\cdot}$	$r^{\mathrm{F},\cdot}$	$\delta^{\mathsf{d},\cdot}$	$r^{\mathrm{C},\cdot}$	$r^{\mathrm{F},\cdot}$	$\delta^{\mathsf{d},\cdot}$	$r^{\mathrm{C},\cdot}$	$r^{\mathrm{F},\cdot}$	$\delta^{\mathsf{d},\cdot}$	$r^{\mathrm{C},\cdot}$	$r^{\mathrm{F},\cdot}$
d	0.00%	1.80%	2.30%	0.00%	1.80%	2.30%	0.00%	1.80%	2.30%	0.00%	1.80%	3.00%
c	-0.10%	2.00%	n/a	-0.10%	2.00%	n/a	0.40%	2.00%	n/a	0.25%	0.60%	n/a
f	0.15%	n/a	1.90%	0.15%	n/a	1.90%	-0.20%	n/a	1.90%	-0.25%	n/a	1.30%

Table 6.3: Sample underlying funding rates r^{A}

Case 1	Case 2	Case 3	Case 4
$r^{\mathsf{repo}} = 1.70\%$	$r^{\mathrm{F},\mathsf{d}} = 2.30\%$	$r^{\mathsf{repo}} = 1.70\%$	$r^{\mathsf{repo}} = 1.70\%$

For each case, we change collateralization ratios from $\gamma = 0\%$ (uncollateralized) to $\gamma = 110\%$ (over-collateralized)[31], including the fully collateralized case $\gamma = 100\%$. As mentioned in Table 2.1, we are interested in prices of options under following collateral/funding assumptions:

- $V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)$: c-collateralized and f-funded;
- $V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t)$: c-collateralized and domestic funded;
- $V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t)$: domestic collateralized and f-funded;
- $V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t)$: domestic collateralized and domestic funded.

Notice that, with introduction of the collateralization ratio γ, following special cases in Table 2.1 are covered as well:

- $V_{\mathsf{d}}^{0,\mathsf{f}}(t)$: uncollateralized and f-funded, which is a special case of $V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)$ and $V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t)$ by setting $\gamma = 0$;
- $V_{\mathsf{d}}^{0,\mathsf{d}}(t)$: uncollateralized and domestic funded, which is a special case of $V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t)$ and $V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t)$ by setting $\gamma = 0$;
- $\bar{V}_{\mathsf{d}}^{\mathsf{c}}(t)$: fully c-collateralized, which is a special case of $V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)$ and $V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t)$ by setting $\gamma = 1$;
- $\bar{V}_{\mathsf{d}}^{\mathsf{d}}(t)$: fully d-collateralized (perfectly collateralized), which is a special case of $V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t)$ and $V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t)$ by setting $\gamma = 1$.

6.2 Pricing Formula

Based on settings described in the previous subsection, payoffs of sample options are

$$V_{\mathsf{d}}(T) = (\varsigma (S_T - K))^+ ,$$

where ς is the call/put indicator:

$$\varsigma = \begin{cases} 1, & \text{call,} \\ -1, & \text{put.} \end{cases}$$

Since all interest rates are constant, from (4.12), (4.13) and (3.22), we have

$$V_{\mathsf{d}}^{\mathsf{u},\mathsf{v}}(t) = e^{-y_{\mathsf{d}}^{\mathsf{u},\mathsf{v}}(\gamma)\tau} \mathbb{E}_t^{\mathcal{Q}} \left[(\varsigma (S_T - K))^+ \right] , \tag{6.1}$$

[31] In practice, cleared trades are actually over-collateralized due to extra costs to maintain initial margin, default funds and other fees. However, in that case the collateralization ratio γ is path dependent. Here we only provide an impact study for pricing over-collateralized derivatives.

where $\mathsf{u} \in \{\mathsf{f}, \mathsf{d}\}$, $\mathsf{v} \in \{\mathsf{c}, \mathsf{d}\}$, and

$$
\begin{aligned}
y_{\mathsf{d}}^{\mathsf{u},\mathsf{v}}(\gamma) &= (1-\gamma) r^{\mathrm{F},\mathsf{u}/\mathsf{d}} + \gamma r^{\mathrm{C},\mathsf{v}/\mathsf{d}}, && (6.2)\\
r^{\mathrm{F},\mathsf{u}/\mathsf{d}} &= r^{\mathrm{F},\mathsf{u}} + \delta^{\mathsf{d},\mathsf{u}}, && (6.3)\\
r^{\mathrm{C},\mathsf{v}/\mathsf{d}} &= r^{\mathrm{C},\mathsf{v}} + \delta^{\mathsf{d},\mathsf{v}}. && (6.4)
\end{aligned}
$$

The expectation in (6.1) are calculated with the assumption that S_t follows the dynamics as in (3.19), which gives

$$\mathbb{E}_t^{\mathcal{Q}}\left[\left(\varsigma\left(S_T - K\right)\right)^+\right] = \varsigma\left(e^{r^{\mathrm{A}}\tau} S_t \Phi\left(\varsigma d_+\right) - K\Phi\left(\varsigma d_-\right)\right),$$

where $\Phi(\cdot)$ is the cumulative density function of the standard normal random variable, and

$$d_{\pm} = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{S_t}{K} + \left(r^{\mathrm{A}} \pm \frac{\sigma^2}{2}\right)\tau\right).$$

6.3 Numerical Results

6.3.1 Impact on European option prices

For Case 1, 2 and 3, we illustrate European option prices, FVAs, and FVA components in Figure 6.1, and these results, together with the percentages of total FVA with respect to prices by collateral rate discounting, i.e., those of perfectly collateralized cases, are presented in Table B.1-B.3 and Table C.1-C.12 as well. Pricing results of perfectly collateralized cases are marked in italic bold font, which are based on collateral rate discounting[32] as in (2.10). They are regarded as market prices of corresponding derivatives and used as benchmarks in this example. When $\gamma = 0$, it is reduced to uncollateralized case, and pricing results are based on funding rate discounting (2.7) or (2.8) so collateral currency has no impact as shown in the first row of Table B.1-B.3.

In Case 1 and 2, European option prices increase with increase of collateralization ratio under all collateral/funding assumptions. The reason is as follows. Equation (6.2) shows that, the effective discounting yield, $y_{\mathsf{d}}^{\mathsf{u},\mathsf{v}}(\gamma)$, is an affine combination of the currency basis-adjusted u-funding rate, $r^{\mathrm{F},\mathsf{u}/\mathsf{d}}$, and the currency basis-adjusted v-collateral rate, $r^{\mathrm{C},\mathsf{v}/\mathsf{d}}$. Therefore, in the case where the currency basis-adjusted u-funding rate is greater (smaller) than the currency basis-adjusted v-collateral rate, as γ increases, the effective discounting yield decreases (increases), hence the price increases (decreases). As shown in Table 6.4, currency basis adjusted funding spreads, $\lambda^{(\mathsf{u},\mathsf{v})/\mathsf{d}} = r^{\mathrm{F},\mathsf{u}/\mathsf{d}} - r^{\mathrm{C},\mathsf{v}/\mathsf{d}}$ are positive under all collateral/funding assumptions in Case 1 and 2, hence European option prices increases with the collateralization ratio. Intuitively, from the perspective of the option seller, he has to post more collateral to cover (one-way) exposures, which leads to higher costs for the seller to be included into the prices. From the option buyer's standpoint, he receives collateral of the amount $\gamma V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(u)$ for $u \in (t, T)$, based on which he pays collateral rate $r^{\mathrm{C},\mathsf{v}/\mathsf{d}}$ and earns funding rate $r^{\mathrm{F},\mathsf{u}/\mathsf{d}}$. Hence the net amount of intermediate cashflows is $\gamma\lambda^{(\mathsf{u},\mathsf{v})/\mathsf{d}} V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(u)$. In the case of positive currency basis adjusted funding spread, the option buyer receives more intermediate cashflows when the collateralization ratio increases, therefore, European option prices must increase with increase of the collateralization ratio.

[32] Or OIS discounting if the collateral rate is the corresponding overnight index rate.

Figure 6.1: European option prices, FVAs and FVA components

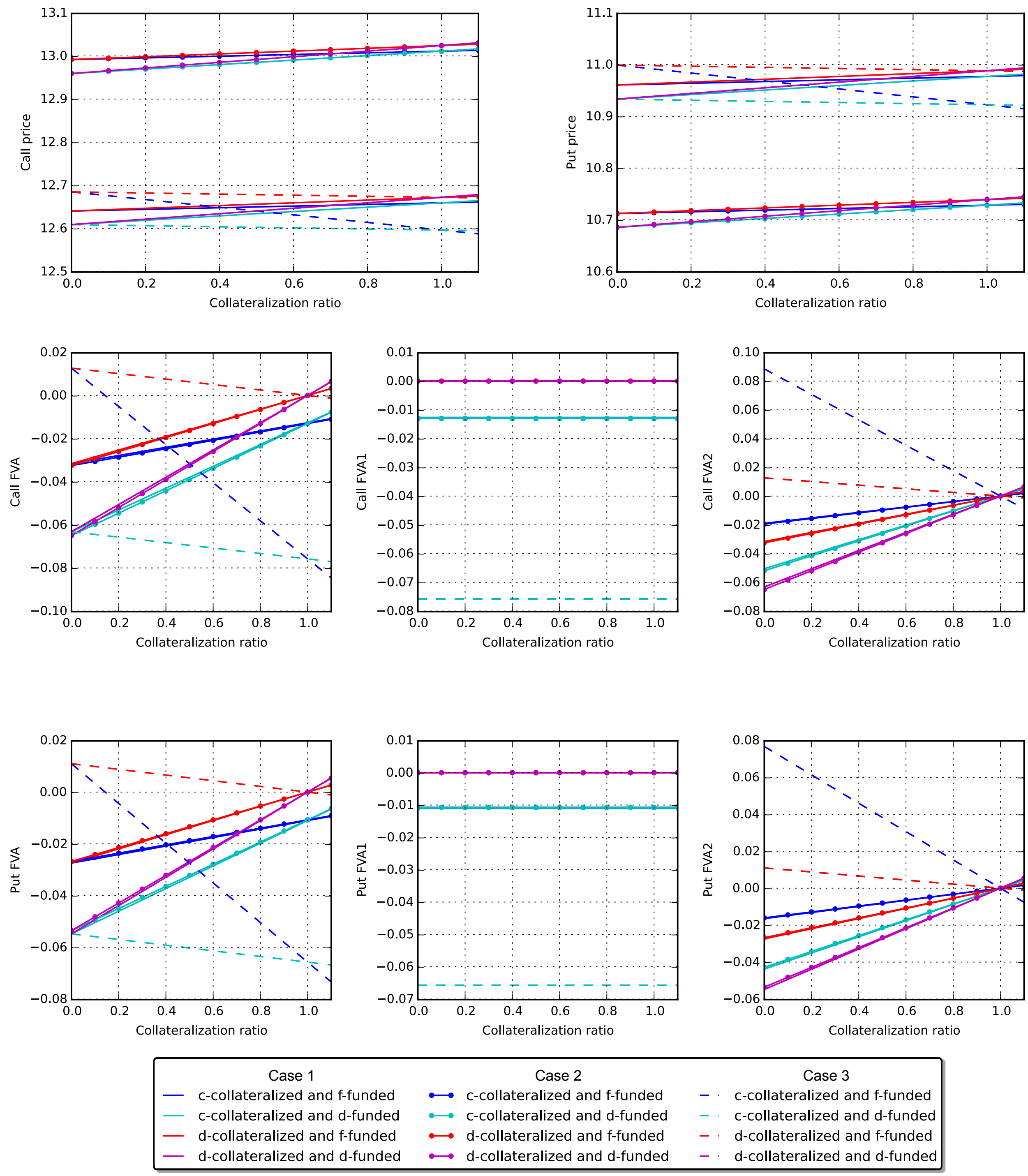


Table 6.4: Currency basis adjusted funding spreads

Collateral/funding	Symbol	Value			
		Case 1	Case 2	Case 3	Case 4
c-collateralized and f-funded	$\lambda^{(\mathsf{c},\mathsf{f})/\mathsf{d}}$	0.15%	0.15%	-0.70%	0.20%
c-collateralized and d-funded	$\lambda^{(\mathsf{c},\mathsf{d})/\mathsf{d}}$	0.40%	0.40%	-0.10%	2.15%
d-collateralized and f-funded	$\lambda^{(\mathsf{d},\mathsf{f})/\mathsf{d}}$	0.25%	0.25%	-0.10%	-0.75%
d-collateralized and d-funded	$\lambda^{(\mathsf{d},\mathsf{d})/\mathsf{d}}$	0.50%	0.50%	0.50%	1.20%

In Case 3, European option prices decrease with increase of collateralization ratio if they are c-collateralized and f-funded, c-collateralized and d-funded or d-collateralized and f-funded, since currency basis adjusted funding spreads under these collateral/funding assumptions are negative. Notice that, from Table 6.2, for the same currency, the collateral rate is less than the funding rate, which is consistent with market observation. However, the currency basis adjusted collateral rate may be greater than the currency basis adjusted funding rate due to certain combination of different risk-free rates, which makes European option prices decrease with increase of collateralization ratio.

For the similar reason, in Case 1, 2 and 3, if the collateralization ratio is fixed, European option prices in Figure 6.1 show that $V_{\mathsf{d}}^{\mathsf{d},\mathsf{f}}(t) \geq V_{\mathsf{d}}^{\mathsf{c},\mathsf{f}}(t)$ and $V_{\mathsf{d}}^{\mathsf{d},\mathsf{d}}(t) \geq V_{\mathsf{d}}^{\mathsf{c},\mathsf{d}}(t)$ since the currency basis adjusted funding spread is positive in all cases and $\lambda^{(\mathsf{d},\mathsf{f})/\mathsf{d}} > \lambda^{(\mathsf{c},\mathsf{f})/\mathsf{d}}$ and $\lambda^{(\mathsf{d},\mathsf{d})/\mathsf{d}} > \lambda^{(\mathsf{c},\mathsf{d})/\mathsf{d}}$.

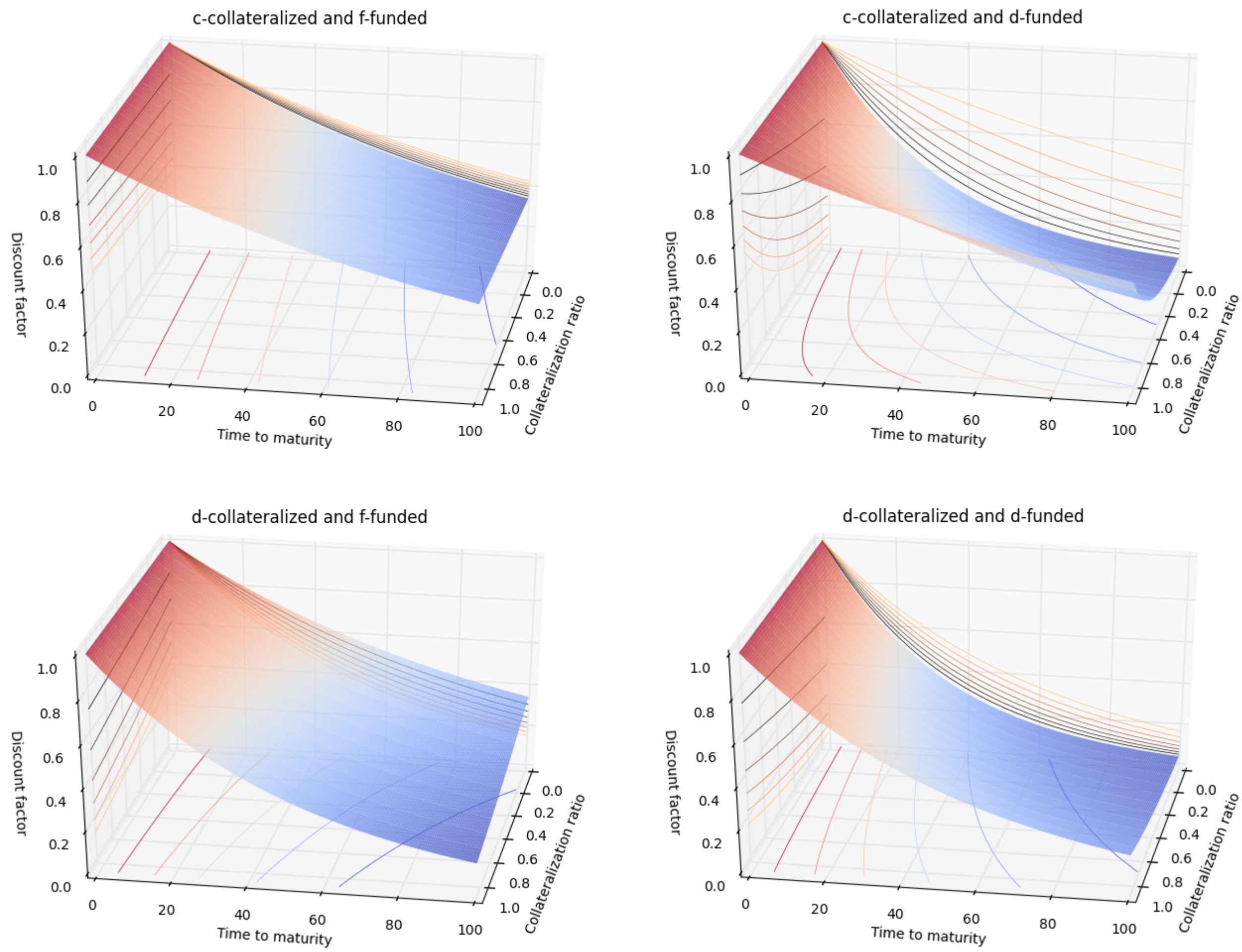
Comparing Case 1 and Case 2, which only differ in the funding rate of the underlying stock, it can be observed that, under all collateral/funding assumptions, call prices in Case 1 are smaller than those in Case 2, while put prices in Case 1 are greater than those in Case 2. Notice that the funding rate of the underlying stock in Case 1 is less than that in Case 2. This shows that call (put) prices increase (decrease) with the underlying funding rate, which is consistent with the relationship between call/put prices and risk-free rates in Black-Scholes model.

Figure 6.1 and Table C.1-C.12 also show the visible impact of collateralization to pricing results. It is observed that FVA_1 results are constant regardless of collateralization ratio, because according to its definition (5.3) it is defined as the difference between prices of the foreign fully collateralized derivative and that of the perfectly collateralized derivative.

6.3.2 Impact on Discount Factor

We conclude this section by presenting the impact of the collateralization ratio on the discount factor. Under different collateral/funding assumptions, we change the collateralization ratio from 0 to 1.1 and the time to maturity from 0 year to 100 years, and calculate the discount factor $e^{-y_{\mathsf{d}}^{\mathsf{u},\mathsf{v}}(\gamma)\tau}$ using interest rates in Case 4. Numerical results are illustrated in Figure 6.2. Due to different values of currency basis adjusted funding spreads under different collateral/funding assumptions, different shapes of discount factor surface are observed.

Figure 6.2: Discount factor surface



7 Conclusion

Derivatives partially collateralized and funded in foreign currencies are studied in this paper, the valuation methodology by replication, together with their PDE solutions and martingale approaches are presented, and corresponding FVA terms are discussed.

Extension of our work in this paper to various applications to swaps and swaptions are of our particular interest. Furthermore, extension to the case of defaultable counterparties is also our interest. It is anticipated that CVA and bilateral FVA will be included, and the double counting between funding benefit adjustment and DVA will be naturally avoided.

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A A Direct Proof of Theorem 4.2

Proof. To show (4.5), first from (4.4), we have

$$V_{\mathsf{d}}^{\mathsf{c,f}}(u) = \mathrm{E}_u^{\tilde{\mathcal{Q}}}\left[e^{-\int_u^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, V_{\mathsf{d}}(T) + \int_u^T e^{-\int_u^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}}\,\mathrm{d}\xi\right], \quad \forall u \in [t,T].$$

Let $\lambda^{\mathrm{C,f/d}} := r^{\mathrm{F,f/d}} - r^{\mathrm{C,d}}$. With the Law of Total Expectation and Fubini's Theorem, from the second term of (4.5), it holds that

$$\begin{aligned}
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\, V_{\mathsf{d}}^{\mathsf{c,f}}(u)\,\mathrm{d}u\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\, \mathrm{E}_u^{\tilde{\mathcal{Q}}}\left[e^{-\int_u^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, V_{\mathsf{d}}(T)\right]\,\mathrm{d}u\right] + \\
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\, \mathrm{E}_u^{\tilde{\mathcal{Q}}}\left[\int_u^T e^{-\int_u^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}}\,\mathrm{d}\xi\right] du\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v} e^{-\int_u^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\, V_{\mathsf{d}}(T)\,\mathrm{d}u\right] + \\
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T \int_u^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v} e^{-\int_u^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}} \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}}\,\mathrm{d}\xi\,\mathrm{d}u\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, V_{\mathsf{d}}(T) \int_t^T e^{\int_t^u \lambda_v^{\mathrm{C,f/d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\,\mathrm{d}u\right] + \\
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}} \left(\int_t^{\xi} e^{\int_t^u \lambda_v^{\mathrm{C,f/d}}\,\mathrm{d}v}\, \lambda_u^{\mathrm{C,f/d}}\,\mathrm{d}u\right) \mathrm{d}\xi\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, V_{\mathsf{d}}(T) \left(e^{\int_t^u \lambda_v^{\mathrm{C,f/d}}\,\mathrm{d}v}\right)\Big|_{u=t}^{u=T}\right] + \\
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}} \left(e^{\int_t^u \lambda_v^{\mathrm{C,f/d}}\,\mathrm{d}v}\right)\Big|_{u=t}^{u=\xi}\,\mathrm{d}\xi\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\left(e^{-\int_t^T r_v^{\mathrm{C,d}}\,\mathrm{d}v} - e^{-\int_t^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\right) V_{\mathsf{d}}(T)\right] + \\
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T \left(e^{-\int_t^{\xi} r_v^{\mathrm{C,d}}\,\mathrm{d}v} - e^{-\int_t^{\xi} r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\right) \lambda_{\xi}^{(\mathsf{f,c})/\mathsf{d}} X_{\xi}^{\mathsf{d/c}} C_{\xi}^{\mathsf{c}}\,\mathrm{d}\xi\right].
\end{aligned}$$

Replacing notation ξ by u in the second term and substituting the above equality to the right-hand side of (4.5) yield

$$\begin{aligned}
&\mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r_v^{\mathrm{C,d}} dv}\, V_{\mathsf{d}}(T)\right] - \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[\int_t^T e^{-\int_t^u r_v^{\mathrm{C,d}}\,\mathrm{d}v}\left[\lambda_u^{\mathrm{C,f/d}}\, V_{\mathsf{d}}^{\mathsf{c,f}}(u) - \lambda_u^{(\mathsf{f,c})/\mathsf{d}}\, X_u^{\mathsf{d/c}}\, C_u^{\mathsf{c}}\right] \mathrm{d}u\right] \\
= \;& \mathrm{E}_t^{\tilde{\mathcal{Q}}}\left[e^{-\int_t^T r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, V_{\mathsf{d}}(T) + \int_t^T e^{-\int_t^u r_v^{\mathrm{F,f/d}}\,\mathrm{d}v}\, \lambda_u^{(\mathsf{f,c})/\mathsf{d}}\, X_u^{\mathsf{d/c}}\, C_u^{\mathsf{c}}\,\mathrm{d}u\right] \;=\; V_{\mathsf{d}}^{\mathsf{c,f}}(t),
\end{aligned}$$

which completes the proof of (4.5). $\square$

B Numerical Results: European Option Price

Table B.1: Numerical results: European option prices in Case 1

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	$12.6404	$10.9606	$12.6088	$10.9333	$12.6404	$10.9606	$12.6088	$10.9333
0.1	$12.6423	$10.9623	$12.6139	$10.9376	$12.6435	$10.9634	$12.6151	$10.9387
0.2	$12.6442	$10.9639	$12.6189	$10.9420	$12.6467	$10.9661	$12.6214	$10.9442
0.3	$12.6461	$10.9656	$12.6240	$10.9464	$12.6499	$10.9689	$12.6277	$10.9497
0.4	$12.6480	$10.9672	$12.6290	$10.9508	$12.6530	$10.9716	$12.6341	$10.9552
0.5	$12.6499	$10.9689	$12.6341	$10.9552	$12.6562	$10.9743	$12.6404	$10.9606
0.6	$12.6518	$10.9705	$12.6391	$10.9595	$12.6594	$10.9771	$12.6467	$10.9661
0.7	$12.6537	$10.9721	$12.6442	$10.9639	$12.6625	$10.9798	$12.6530	$10.9716
0.8	$12.6556	$10.9738	$12.6492	$10.9683	$12.6657	$10.9826	$12.6594	$10.9771
0.9	$12.6575	$10.9754	$12.6543	$10.9727	$12.6688	$10.9853	$12.6657	$10.9826
1.0	$12.6594	$10.9771	$12.6594	$10.9771	$12.6720	$10.9881	***$12.6720***	***$10.9881***
1.1	$12.6613	$10.9787	$12.6644	$10.9815	$12.6752	$10.9908	$12.6784	$10.9936

Table B.2: Numerical results: European option prices in Case 2

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	$12.9917	$10.7122	$12.9592	$10.6855	$12.9917	$10.7122	$12.9592	$10.6855
0.1	$12.9936	$10.7138	$12.9644	$10.6897	$12.9949	$10.7149	$12.9657	$10.6908
0.2	$12.9956	$10.7154	$12.9696	$10.6940	$12.9982	$10.7176	$12.9722	$10.6962
0.3	$12.9975	$10.7170	$12.9748	$10.6983	$13.0014	$10.7203	$12.9787	$10.7015
0.4	$12.9995	$10.7186	$12.9800	$10.7026	$13.0047	$10.7229	$12.9852	$10.7069
0.5	$13.0014	$10.7203	$12.9852	$10.7069	$13.0079	$10.7256	$12.9917	$10.7122
0.6	$13.0034	$10.7219	$12.9904	$10.7111	$13.0112	$10.7283	$12.9982	$10.7176
0.7	$13.0053	$10.7235	$12.9956	$10.7154	$13.0144	$10.7310	$13.0047	$10.7229
0.8	$13.0073	$10.7251	$13.0008	$10.7197	$13.0177	$10.7337	$13.0112	$10.7283
0.9	$13.0092	$10.7267	$13.0060	$10.7240	$13.0209	$10.7363	$13.0177	$10.7337
1.0	$13.0112	$10.7283	$13.0112	$10.7283	$13.0242	$10.7390	***$13.0242***	***$10.7390***
1.1	$13.0131	$10.7299	$13.0164	$10.7326	$13.0274	$10.7417	$13.0307	$10.7444

Table B.3: Numerical results: European option prices in Case 3

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	$12.6847	$10.9991	$12.6088	$10.9333	$12.6847	$10.9991	$12.6088	$10.9333
0.1	$12.6758	$10.9914	$12.6076	$10.9322	$12.6834	$10.9980	$12.6151	$10.9387
0.2	$12.6669	$10.9837	$12.6063	$10.9311	$12.6822	$10.9969	$12.6214	$10.9442
0.3	$12.6581	$10.9760	$12.6050	$10.9300	$12.6809	$10.9958	$12.6277	$10.9497
0.4	$12.6492	$10.9683	$12.6038	$10.9289	$12.6796	$10.9947	$12.6341	$10.9552
0.5	$12.6404	$10.9606	$12.6025	$10.9278	$12.6784	$10.9936	$12.6404	$10.9606
0.6	$12.6315	$10.9530	$12.6013	$10.9267	$12.6771	$10.9925	$12.6467	$10.9661
0.7	$12.6227	$10.9453	$12.6000	$10.9256	$12.6758	$10.9914	$12.6530	$10.9716
0.8	$12.6139	$10.9376	$12.5987	$10.9245	$12.6746	$10.9903	$12.6594	$10.9771
0.9	$12.6050	$10.9300	$12.5975	$10.9234	$12.6733	$10.9892	$12.6657	$10.9826
1.0	$12.5962	$10.9223	$12.5962	$10.9223	$12.6720	$10.9881	***$12.6720***	***$10.9881***
1.1	$12.5874	$10.9147	$12.5950	$10.9212	$12.6708	$10.9870	$12.6784	$10.9936

C Numerical Results: FVA and FVA Components

Table C.1: Total FVA in Case 1

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-$0.0316	-$0.0274	-$0.0632	-$0.0548	-$0.0316	-$0.0274	-$0.0632	-$0.0548
0.1	-$0.0297	-$0.0258	-$0.0582	-$0.0504	-$0.0285	-$0.0247	-$0.0569	-$0.0493
0.2	-$0.0278	-$0.0241	-$0.0531	-$0.0461	-$0.0253	-$0.0220	-$0.0506	-$0.0439
0.3	-$0.0260	-$0.0225	-$0.0481	-$0.0417	-$0.0222	-$0.0192	-$0.0443	-$0.0384
0.4	-$0.0241	-$0.0209	-$0.0430	-$0.0373	-$0.0190	-$0.0165	-$0.0380	-$0.0329
0.5	-$0.0222	-$0.0192	-$0.0380	-$0.0329	-$0.0158	-$0.0137	-$0.0316	-$0.0274
0.6	-$0.0203	-$0.0176	-$0.0329	-$0.0285	-$0.0127	-$0.0110	-$0.0253	-$0.0220
0.7	-$0.0184	-$0.0159	-$0.0278	-$0.0241	-$0.0095	-$0.0082	-$0.0190	-$0.0165
0.8	-$0.0165	-$0.0143	-$0.0228	-$0.0198	-$0.0063	-$0.0055	-$0.0127	-$0.0110
0.9	-$0.0146	-$0.0126	-$0.0177	-$0.0154	-$0.0032	-$0.0027	-$0.0063	-$0.0055
1.0	-$0.0127	-$0.0110	-$0.0127	-$0.0110	$0.0000	$0.0000	$0.0000	$0.0000
1.1	-$0.0108	-$0.0093	-$0.0076	-$0.0066	$0.0032	$0.0027	$0.0063	$0.0055

Table C.2: FVA_1 in Case 1

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.1	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.2	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.3	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.4	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.5	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.6	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.7	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.8	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
0.9	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
1.0	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–
1.1	-$0.0127	-$0.0110	-$0.0127	-$0.0110	–	–	–	–

Table C.3: FVA_2 in Case 1

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-\$0.0190	-\$0.0165	-\$0.0505	-\$0.0438	-\$0.0316	-\$0.0274	-\$0.0632	-\$0.0548
0.1	-\$0.0171	-\$0.0148	-\$0.0455	-\$0.0394	-\$0.0285	-\$0.0247	-\$0.0569	-\$0.0493
0.2	-\$0.0152	-\$0.0132	-\$0.0404	-\$0.0351	-\$0.0253	-\$0.0220	-\$0.0506	-\$0.0439
0.3	-\$0.0133	-\$0.0115	-\$0.0354	-\$0.0307	-\$0.0222	-\$0.0192	-\$0.0443	-\$0.0384
0.4	-\$0.0114	-\$0.0099	-\$0.0303	-\$0.0263	-\$0.0190	-\$0.0165	-\$0.0380	-\$0.0329
0.5	-\$0.0095	-\$0.0082	-\$0.0253	-\$0.0219	-\$0.0158	-\$0.0137	-\$0.0316	-\$0.0274
0.6	-\$0.0076	-\$0.0066	-\$0.0202	-\$0.0175	-\$0.0127	-\$0.0110	-\$0.0253	-\$0.0220
0.7	-\$0.0057	-\$0.0049	-\$0.0152	-\$0.0132	-\$0.0095	-\$0.0082	-\$0.0190	-\$0.0165
0.8	-\$0.0038	-\$0.0033	-\$0.0101	-\$0.0088	-\$0.0063	-\$0.0055	-\$0.0127	-\$0.0110
0.9	-\$0.0019	-\$0.0016	-\$0.0051	-\$0.0044	-\$0.0032	-\$0.0027	-\$0.0063	-\$0.0055
1.0	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000
1.1	\$0.0019	\$0.0016	\$0.0051	\$0.0044	\$0.0032	\$0.0027	\$0.0063	\$0.0055

Table C.4: Percentages of FVA with respect to collateral rate discounting results in Case 1

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-0.25%	-0.25%	-0.50%	-0.50%	-0.25%	-0.25%	-0.50%	-0.50%
0.1	-0.24%	-0.24%	-0.46%	-0.46%	-0.23%	-0.23%	-0.45%	-0.45%
0.2	-0.22%	-0.22%	-0.42%	-0.42%	-0.20%	-0.20%	-0.40%	-0.40%
0.3	-0.21%	-0.21%	-0.38%	-0.38%	-0.18%	-0.18%	-0.35%	-0.35%
0.4	-0.19%	-0.19%	-0.34%	-0.34%	-0.15%	-0.15%	-0.30%	-0.30%
0.5	-0.18%	-0.18%	-0.30%	-0.30%	-0.13%	-0.13%	-0.25%	-0.25%
0.6	-0.16%	-0.16%	-0.26%	-0.26%	-0.10%	-0.10%	-0.20%	-0.20%
0.7	-0.15%	-0.15%	-0.22%	-0.22%	-0.08%	-0.08%	-0.15%	-0.15%
0.8	-0.13%	-0.13%	-0.18%	-0.18%	-0.05%	-0.05%	-0.10%	-0.10%
0.9	-0.12%	-0.12%	-0.14%	-0.14%	-0.03%	-0.03%	-0.05%	-0.05%
1.0	-0.10%	-0.10%	-0.10%	-0.10%	0.00%	0.00%	0.00%	0.00%
1.1	-0.09%	-0.09%	-0.06%	-0.06%	0.02%	0.02%	0.05%	0.05%

Table C.5: Total FVA in Case 2

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-\$0.0325	-\$0.0268	-\$0.0650	-\$0.0536	-\$0.0325	-\$0.0268	-\$0.0650	-\$0.0536
0.1	-\$0.0306	-\$0.0252	-\$0.0598	-\$0.0493	-\$0.0293	-\$0.0241	-\$0.0585	-\$0.0482
0.2	-\$0.0286	-\$0.0236	-\$0.0546	-\$0.0450	-\$0.0260	-\$0.0215	-\$0.0520	-\$0.0429
0.3	-\$0.0267	-\$0.0220	-\$0.0494	-\$0.0407	-\$0.0228	-\$0.0188	-\$0.0455	-\$0.0375
0.4	-\$0.0247	-\$0.0204	-\$0.0442	-\$0.0365	-\$0.0195	-\$0.0161	-\$0.0390	-\$0.0322
0.5	-\$0.0228	-\$0.0188	-\$0.0390	-\$0.0322	-\$0.0163	-\$0.0134	-\$0.0325	-\$0.0268
0.6	-\$0.0208	-\$0.0172	-\$0.0338	-\$0.0279	-\$0.0130	-\$0.0107	-\$0.0260	-\$0.0215
0.7	-\$0.0189	-\$0.0156	-\$0.0286	-\$0.0236	-\$0.0098	-\$0.0081	-\$0.0195	-\$0.0161
0.8	-\$0.0169	-\$0.0140	-\$0.0234	-\$0.0193	-\$0.0065	-\$0.0054	-\$0.0130	-\$0.0107
0.9	-\$0.0150	-\$0.0123	-\$0.0182	-\$0.0150	-\$0.0033	-\$0.0027	-\$0.0065	-\$0.0054
1.0	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	\$0.0000	\$0.0000	\$0.0000	\$0.0000
1.1	-\$0.0111	-\$0.0091	-\$0.0078	-\$0.0064	\$0.0033	\$0.0027	\$0.0065	\$0.0054

Table C.6: FVA_1 in Case 2

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.1	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.2	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.3	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.4	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.5	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.6	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.7	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.8	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
0.9	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
1.0	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–
1.1	-\$0.0130	-\$0.0107	-\$0.0130	-\$0.0107	–	–	–	–

Table C.7: FVA_2 in Case 2

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-$0.0195	-$0.0161	-$0.0519	-$0.0428	-$0.0325	-$0.0268	-$0.0650	-$0.0536
0.1	-$0.0176	-$0.0145	-$0.0468	-$0.0386	-$0.0293	-$0.0241	-$0.0585	-$0.0482
0.2	-$0.0156	-$0.0129	-$0.0416	-$0.0343	-$0.0260	-$0.0215	-$0.0520	-$0.0429
0.3	-$0.0137	-$0.0113	-$0.0364	-$0.0300	-$0.0228	-$0.0188	-$0.0455	-$0.0375
0.4	-$0.0117	-$0.0097	-$0.0312	-$0.0257	-$0.0195	-$0.0161	-$0.0390	-$0.0322
0.5	-$0.0098	-$0.0080	-$0.0260	-$0.0214	-$0.0163	-$0.0134	-$0.0325	-$0.0268
0.6	-$0.0078	-$0.0064	-$0.0208	-$0.0172	-$0.0130	-$0.0107	-$0.0260	-$0.0215
0.7	-$0.0059	-$0.0048	-$0.0156	-$0.0129	-$0.0098	-$0.0081	-$0.0195	-$0.0161
0.8	-$0.0039	-$0.0032	-$0.0104	-$0.0086	-$0.0065	-$0.0054	-$0.0130	-$0.0107
0.9	-$0.0020	-$0.0016	-$0.0052	-$0.0043	-$0.0033	-$0.0027	-$0.0065	-$0.0054
1.0	$0.0000	$0.0000	$0.0000	$0.0000	$0.0000	$0.0000	$0.0000	$0.0000
1.1	$0.0020	$0.0016	$0.0052	$0.0043	$0.0033	$0.0027	$0.0065	$0.0054

Table C.8: Percentages of FVA with respect to collateral rate discounting results in Case 2

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-0.25%	-0.25%	-0.50%	-0.50%	-0.25%	-0.25%	-0.50%	-0.50%
0.1	-0.24%	-0.24%	-0.46%	-0.46%	-0.23%	-0.23%	-0.45%	-0.45%
0.2	-0.22%	-0.22%	-0.42%	-0.42%	-0.20%	-0.20%	-0.40%	-0.40%
0.3	-0.21%	-0.21%	-0.38%	-0.38%	-0.18%	-0.18%	-0.35%	-0.35%
0.4	-0.19%	-0.19%	-0.34%	-0.34%	-0.15%	-0.15%	-0.30%	-0.30%
0.5	-0.18%	-0.18%	-0.30%	-0.30%	-0.13%	-0.13%	-0.25%	-0.25%
0.6	-0.16%	-0.16%	-0.26%	-0.26%	-0.10%	-0.10%	-0.20%	-0.20%
0.7	-0.15%	-0.15%	-0.22%	-0.22%	-0.08%	-0.08%	-0.15%	-0.15%
0.8	-0.13%	-0.13%	-0.18%	-0.18%	-0.05%	-0.05%	-0.10%	-0.10%
0.9	-0.12%	-0.12%	-0.14%	-0.14%	-0.03%	-0.03%	-0.05%	-0.05%
1.0	-0.10%	-0.10%	-0.10%	-0.10%	0.00%	0.00%	0.00%	0.00%
1.1	-0.09%	-0.09%	-0.06%	-0.06%	0.02%	0.02%	0.05%	0.05%

Table C.9: Total FVA in Case 3

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	\$0.0127	\$0.0110	-\$0.0632	-\$0.0548	\$0.0127	\$0.0110	-\$0.0632	-\$0.0548
0.1	\$0.0038	\$0.0033	-\$0.0645	-\$0.0559	\$0.0114	\$0.0099	-\$0.0569	-\$0.0493
0.2	-\$0.0051	-\$0.0044	-\$0.0657	-\$0.0570	\$0.0101	\$0.0088	-\$0.0506	-\$0.0439
0.3	-\$0.0139	-\$0.0121	-\$0.0670	-\$0.0581	\$0.0089	\$0.0077	-\$0.0443	-\$0.0384
0.4	-\$0.0228	-\$0.0198	-\$0.0682	-\$0.0592	\$0.0076	\$0.0066	-\$0.0380	-\$0.0329
0.5	-\$0.0316	-\$0.0274	-\$0.0695	-\$0.0603	\$0.0063	\$0.0055	-\$0.0316	-\$0.0274
0.6	-\$0.0405	-\$0.0351	-\$0.0708	-\$0.0614	\$0.0051	\$0.0044	-\$0.0253	-\$0.0220
0.7	-\$0.0493	-\$0.0428	-\$0.0720	-\$0.0625	\$0.0038	\$0.0033	-\$0.0190	-\$0.0165
0.8	-\$0.0582	-\$0.0504	-\$0.0733	-\$0.0635	\$0.0025	\$0.0022	-\$0.0127	-\$0.0110
0.9	-\$0.0670	-\$0.0581	-\$0.0745	-\$0.0646	\$0.0013	\$0.0011	-\$0.0063	-\$0.0055
1.0	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	\$0.0000	\$0.0000	\$0.0000	\$0.0000
1.1	-\$0.0846	-\$0.0734	-\$0.0771	-\$0.0668	-\$0.0013	-\$0.0011	\$0.0063	\$0.0055

Table C.10: FVA_1 in Case 3

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.1	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.2	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.3	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.4	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.5	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.6	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.7	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.8	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
0.9	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
1.0	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–
1.1	-\$0.0758	-\$0.0657	-\$0.0758	-\$0.0657	–	–	–	–

Table C.11: FVA_2 in Case 3

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	\$0.0885	\$0.0767	\$0.0126	\$0.0109	\$0.0127	\$0.0110	-\$0.0632	-\$0.0548
0.1	\$0.0796	\$0.0690	\$0.0113	\$0.0098	\$0.0114	\$0.0099	-\$0.0569	-\$0.0493
0.2	\$0.0707	\$0.0613	\$0.0101	\$0.0087	\$0.0101	\$0.0088	-\$0.0506	-\$0.0439
0.3	\$0.0619	\$0.0537	\$0.0088	\$0.0076	\$0.0089	\$0.0077	-\$0.0443	-\$0.0384
0.4	\$0.0530	\$0.0460	\$0.0076	\$0.0066	\$0.0076	\$0.0066	-\$0.0380	-\$0.0329
0.5	\$0.0442	\$0.0383	\$0.0063	\$0.0055	\$0.0063	\$0.0055	-\$0.0316	-\$0.0274
0.6	\$0.0353	\$0.0306	\$0.0050	\$0.0044	\$0.0051	\$0.0044	-\$0.0253	-\$0.0220
0.7	\$0.0265	\$0.0230	\$0.0038	\$0.0033	\$0.0038	\$0.0033	-\$0.0190	-\$0.0165
0.8	\$0.0176	\$0.0153	\$0.0025	\$0.0022	\$0.0025	\$0.0022	-\$0.0127	-\$0.0110
0.9	\$0.0088	\$0.0076	\$0.0013	\$0.0011	\$0.0013	\$0.0011	-\$0.0063	-\$0.0055
1.0	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000	\$0.0000
1.1	-\$0.0088	-\$0.0076	-\$0.0013	-\$0.0011	-\$0.0013	-\$0.0011	\$0.0063	\$0.0055

Table C.12: Percentages of FVA with respect to collateral rate discounting results in Case 3

Collateralization ratio	c-collateralized and f-funded		c-collateralized and d-funded		d-collateralized and f-funded		d-collateralized and d-funded	
	call	put	call	put	call	put	call	put
0.0	0.10%	0.10%	-0.50%	-0.50%	0.10%	0.10%	-0.50%	-0.50%
0.1	0.03%	0.03%	-0.51%	-0.51%	0.09%	0.09%	-0.45%	-0.45%
0.2	-0.04%	-0.04%	-0.52%	-0.52%	0.08%	0.08%	-0.40%	-0.40%
0.3	-0.11%	-0.11%	-0.53%	-0.53%	0.07%	0.07%	-0.35%	-0.35%
0.4	-0.18%	-0.18%	-0.54%	-0.54%	0.06%	0.06%	-0.30%	-0.30%
0.5	-0.25%	-0.25%	-0.55%	-0.55%	0.05%	0.05%	-0.25%	-0.25%
0.6	-0.32%	-0.32%	-0.56%	-0.56%	0.04%	0.04%	-0.20%	-0.20%
0.7	-0.39%	-0.39%	-0.57%	-0.57%	0.03%	0.03%	-0.15%	-0.15%
0.8	-0.46%	-0.46%	-0.58%	-0.58%	0.02%	0.02%	-0.10%	-0.10%
0.9	-0.53%	-0.53%	-0.59%	-0.59%	0.01%	0.01%	-0.05%	-0.05%
1.0	-0.60%	-0.60%	-0.60%	-0.60%	0.00%	0.00%	0.00%	0.00%
1.1	-0.67%	-0.67%	-0.61%	-0.61%	-0.01%	-0.01%	0.05%	0.05%