# Asset Liability Management in DROP

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# Asset Liability Matching Model

## Overview

1. Competing Goals of Individual Investors: An individual investor has two competing goals – principal protection and asset growth.
2. Model Addressing the Competing Goals: The objective of this chapter is to provide a model allowing an investor to precisely weight these factors and implementing a portfolio maximizing the potential of achieving their goals with a clear understanding of their downside risk.
3. Addressing the Income Generation Functionality: Income generation is a critical component for some investors. This factor is addressed at the end of the chapter.

## Example

1. Quantification of the Exposed Risk: The classic investor dilemma is to decide between principal protection and growth. The model in this chapter provides clarity to this choice by estimating how much the investors are risking in dollar terms as opposed to merely categorizing themselves as conservation, moderate, or aggressive.
2. Portfolios from Stock-Bond Mix: For example, a portfolio with 60% S&P and 40% Bloomberg Barclays Aggregate will have an expected return of 6% and 10% volatility. The initial investment needed to obtain $1,000 in 10 years, with a 6% expected return, is $560. The investor will know through a Monte Carlo simulation that there is a 25% likelihood that the final value in 10 years will be worth less than $830. The $1,000 target amount is not guaranteed, and the results will vary depending upon the market conditions.

3. Tweaking for Growth/Protection Balance: The investor can scale up or down between principal protection and asset growth, which will be measures by the dollar dispersion around the future liability.
4. Principal Protection/Growth Trade off: More growth means a lower initial investment, but a higher dollar dispersion. On the other hand, greater principal protection will reduce the potential loss over the investment horizon, but requires a higher initial investment.
5. Typical Extreme Growth/Protection Portfolios: For example, if the investor wants zero dispersion around the liability, the model will create a portfolio with a single Treasury Strip that matures on the date that the liability is due. If the investor wants to maximize their expected returns and minimize their initial investment, the model will create a portfolio with 100% equities.
6. Trade-off Based Estimation of Investment: The investor adjusting the model will incrementally determine the trade-off between the dispersion around the liability, and the initial investment required.
7. Construction of Optimal Component Portfolios: Once the investor determines their preferences, the portfolio will be created from an optimal blend of maturing assets – e.g., a single Treasury Strip held to maturity – and non-maturing assets – e.g., the Russell 1000 Index.
8. Controlling the Volatility of Returns: The vital component of the model is enabling the investor to incrementally adjust the volatility of the optimization, which will directly impact the Monte Carlo simulated expected dispersion around the liability.
9. Incorporating Liquidity and Component Default: The portfolio will have daily liquidity, but the mean-variance optimizer (MVO) that the risk of the maturing asset is the risk of the default, not its decreasing volatility over the period. For instance, insurance companies do not mark fixed income instruments' daily volatility on their balance sheet if it is held to maturity.
10. Impact of the Volatility of the Maturing Asset: The reason for this is that the volatility should not matter to an investor holding the security to maturity. A KMV model may be used to determine the risk of default. It is assumed here that a data set can be licensed that provides a KMV estimate for each bond issue, and that this estimate can be used in the MVO.

11. Portfolio Returns Monte Carlo Simulation: However, the Monte Carlo simulation needs to include the maturing asset actual declining volatility to inform the investor of the potential return variation over the entire investment period.
12. Double Convex Portfolio Returns Pattern: The expectation is that the Monte Carlo simulation will forecast a double convex expected return pattern giving the investor confidence that they will achieve their objective. A double convex pattern converges on both ends, but is thicker across the middle.

## Construction Objectives

1. The Asset-Liability Matching Model: Create a dynamic asset-liability matching model that makes it possible to incrementally increase of decrease the dollar dispersion of expected returns around a future liability. The dollar dispersion represents the amount of money an individual investor can expect to make or lose at the end of the investment period. The dollar denominated risk profile is more precise than the standard conservative, moderate, and aggressive description. The investor will change the dispersion around the future liability based on their tolerance for accepting losses.
2. Returns Portfolio Monte Carlo Simulation: Create a Monte Carlo simulation that tracks the expected returns of the portfolio. A statistical illustration of all potential return paths for an asset-liability matching model will mitigate negative performance anxiety and make assets stickier during downturns.
3. Optimizer to cover all Assets: Enable the optimizer to include both the maturing and the non-maturing assets. This capability will allow the investor to create a Monte-Carlo simulation with risk patters ranging from double convexity – e.g., treasury strips – to wide dollar dispersion around the liability target – e.g., 100% equities.
4. Impact Analysis on the Target Metrics: The capabilities listed above will alter the following quantities:
    a. Asset Matching Portfolio Holdings
    b. Expected Risk and Returns
    c. Initial investment required to match the Liability

    d. Monte Carlo simulation

  The investor adjusting the model will be able to see in real-time the impact on all these four factors.

## Details

1. Primary Components of the Model: There are three primary components to this model.
    a. The mean variance optimizer (MVO) to determine the portfolio holdings and expected returns over each level of risk.
    b. Monte Carlo simulation to project the expected returns *journey* over the investment period and the dispersion of expected returns around the liability.
    c. Dynamic functionality to discount the liability back to its present value using the expected return derived from changes in dispersion.
2. Parameters Governing Non-Maturing Assets: The MVO available assets can be sub-divided into two groups – maturing assets and non-maturing assets. The allocation between these two groups will vary as the risk levels change to reflect the dispersion around the target value. The non-maturing assets risk/return performance is based on historical performance. This data is expected to be included, along with the list of assets.
3. Parameters Governing the Maturing Assets: Maturing assets have verifiable results, but volatility that diminishes over time, given its decrease in duration. The only risk for the maturing asset that should be included in the MVO is the probability of default. A KMV model can calculate this likelihood. The maturing assets are assumed to be held to maturity, and daily market volatility is not factored into the long-term portfolio optimization.
4. Rationale behind the Portfolio Simulation: The Monte-Carlo projections need to reflect the bond volatility despite the effort to not overstate price instability within the MVO framework. There are two rationales behind this. First, the investor may have daily liquidity and will need to know the NAV. Second, a bond's volatility that decreases as it approaches maturity is favorable for a strategy looking to narrow its dispersion around its liability.

5. Investor’s Control Function: The investor will have the ability to increase and decrease the dispersion around the liability by increasing or decreasing the MVO volatility.
6. Investors’ Inputs:
    a. Future liability
    b. Date the liability is due
7. Model Outputs:
    a. Portfolio expected returns
    b. Monte Carlo simulation
    c. Portfolio holdings
    d. Initial investment
    e. Monthly cash flow
8. Available MVO Outputs: There are two categories of assets – non-maturing and maturing assets. The non-maturing assets include expected risk/returns based on historical data. The maturing assets include bond ETF’s with fixed terms between 1 and 10 years. Other maturing assets are fixed income categories where individual securities will be utilized in the portfolio. The yield curves for these securities will serve as proxies for individual securities that will be purchased at a later date and held to maturity unless the investor liquidates the portfolio.

## Non-Maturing Assets

| **Ticker** | **ETF Description** | **Volatility (Annualized)** | **Arithmetic Mean (Annualized)** | **Geometric Mean (Annualized)** | |
|---|---|---|---|---|---|
| TIP | iShares Bond TIPS | 5.69% | 4.11% | 3.95% | Jan 2004 – Jul 2019 |
| BMOIX | iShares Blackrock Aggregate Bond Index | 2.88% | 2.60% | 2.55% | Apr 2011 – Jul 2019 |
| EMB | iShares JP Morgan Emerging Bond Index | 11.50% | 6.82% | 6.11% | Jan 2008 – Jul 2019 |
| IWB | iShares Russell 1000 Index ETF | 14.63% | 7.80% | 6.65% | Jun 2000 – Jul 2019 |

| | | | | | |
|---|---|---|---|---|---|
| MASKX | iShares Russell 2000 Small-Cap Index Fund | 19.81% | 9,40$ | 7.27% | May 1997 – Jul 2019 |
| IXUS | iShares MSCI Total International Stock ETF | 11.86% | 5.00% | 4.28% | Nov 2012 – Jul 2019 |
| BIRDX | iShares Developed Real Estate CL K | 11.67% | 7.48% | 6.78% | Sep 2015 – Jul 2019 |

## Maturing ETF Assets

1. Invesco and Blackrock FI ETF: Invesco BulletShares and Blackrock iBond are fixed-term exchange-traded funds (ETFs) that provide defined maturity exposure. Each BulletShares of iBond ETF comprises a diversified portfolio of individual bonds that mature or are anticipated to be called in a specific year.
2. Operating Mechanism for BulletShares ETFs: BulletShares corporate bond ETFs begin moving to cash in the final six months of the maturity year. BulletShares high-yield corporate bonds and emerging markets ETFs start to move to cash in the final 12 months. Cash form these securities is re-invested in T-bills. The fund stops trading on the maturity date of the designated year. At this point, the ETF will de-list from the exchange and make a final distribution to the shareholders, similar to the principal re-payment of an individual bond at maturity.
3. Invesco BulletShares Corporate Bond ETFs:

| **Ticker** | **Maturity Date** | **Yield To Maturity** | **Yield To Worst** | **Effective Duration (years)** | **30-Day SEC Yield** | **Distribution Rate** | **Number of Holdings** |
|---|---|---|---|---|---|---|---|
| BSCK | 12/31/20 | 2.32% | 2.24% | 0.82 | 2.26% | 2.57% | 373 |
| BSCL | 12/31/21 | 2.38% | 2.24% | 1.69 | 2.26% | 2.75% | 399 |
| BSCM | 12/31/22 | 2.35% | 2.32% | 2.64 | 2.32% | 2.91% | 380 |
| BSCN | 12/31/23 | 2.41% | 2.38% | 3.43 | 2.39% | 3.06% | 318 |
| BSCO | 12/31/24 | 2.56% | 2.53% | 4.28 | 2.53% | 3.19% | 240 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| BSCP | 12/31/25 | 2.69% | 2.66% | 5.11 | 2.64% | 3.34% | 236 |
| BSCQ | 12/31/26 | 2.78% | 2.77% | 5.94 | 2.76% | 3.29% | 249 |
| BSCR | 12/31/27 | 2.91% | 2.89% | 6.66 | 2.84% | 3.43% | 214 |
| BSCS | 12/31/28 | 2.99% | 2.97% | 7.22 | 2.90% | 3.47% | 147 |

4. Invesco BulletShares High Yield Corporate Bond ETFs:

| **Ticker** | **Maturity Date** | **Yield To Maturity** | **Yield To Worst** | **Effective Duration (years)** | **30-Day SEC Yield** | **Distribution Rate** | **Number of Holdings** |
|---|---|---|---|---|---|---|---|
| BSJK | 12/31/20 | 4.56% | 3.36% | 0.64 | 3.89% | 4.17% | 79 |
| BSJL | 12/31/21 | 5.45% | 4.80% | 1.52 | 4.63% | 5.19% | 125 |
| BSJM | 12/31/22 | 5.90% | 4.97% | 1.72 | 4.82% | 5.33% | 177 |
| BSJN | 12/31/23 | 6.26% | 5.49% | 2.18 | 4.93% | 5.65% | 207 |
| BSJO | 12/31/24 | 5.76% | 4.96% | 2.43 | 4.90% | 5.43% | 190 |
| BSJP | 12/31/25 | 6.05% | 5.74% | 3.27 | 5.45% | 5.82% | 243 |
| BSJQ | 12/31/26 | 6.02% | 5.73% | 2.75 | 5.54% | 5.63% | 152 |

5. Invesco BulletShares Emerging Market Debt:

| **Ticker** | **Maturity Date** | **Yield To Maturity** | **Yield To Worst** | **Effective Duration (years)** | **30-Day SEC Yield** | **Distribution Rate** | **Number of Holdings** |
|---|---|---|---|---|---|---|---|
| BSAE | 12/31/21 | 2.95% | 2.95% | 1.72 | 2.65% | N/A | 50 |
| BSBE | 12/31/22 | 3.32% | 3.31% | 2.58 | 2.89% | N/A | 55 |
| BSCE | 12/31/23 | 3.48% | 3.46% | 3.31 | 2.96% | N/A | 48 |
| BSDE | 12/31/24 | 3.69% | 3.63% | 3.97 | 3.27% | N/A | 48 |

6. Blackrock iBond Corporate Term ETFs:

| **Ticker** | **Maturity Date** | **Yield To Maturity** | **Weighted Avg. Mat (years)** | **Effective Duration (years)** |
|---|---|---|---|---|
| IBDL | 12/15/20 | 2.32 | 0.86 | 0.84 |
| IBDM | 12/15/21 | 2.31 | 1.77 | 1.70 |
| IBDN | 12/15/22 | 2.35 | 2.74 | 2.62 |
| IBDO | 12/15/23 | 2.44 | 3.67 | 3.44 |
| IBDP | 12/15/24 | 2.58 | 4.68 | 4.29 |
| IBDQ | 12/15/25 | 2.71 | 5.62 | 5.08 |
| IBDR | 12/15/26 | 2.82 | 6.63 | 5.91 |
| IBDS | 12/15/27 | 2.95 | 7.58 | 6.63 |
| IBDT | 12/15/28 | 3.02 | 8.58 | 7.25 |

7. Blackrock iBond Term Muni Bond ETFs:

| **Ticker** | **Maturity Date** | **Yield To Maturity** | **Weighted Avg. Mat (years)** | **Effective Duration (years)** |
|---|---|---|---|---|
| IBMI | 9/1/20 | 0.97 | 0.92 | 0.91 |
| IBMJ | 9/1/21 | 0.99 | 1.92 | 1.83 |
| IBMK | 9/1/22 | 1.01 | 2.93 | 2.73 |
| IBML | 9/1/23 | 1.03 | 3.88 | 3.55 |
| IBMM | 9/1/24 | 1.06 | 4.90 | 4.41 |
| IBMN | 9/1/25 | 1.12 | 5.89 | 5.21 |
| IBMO | 9/1/26 | 1.75 | 6.64 | 5.82 |
| IBMP | 9/1/27 | 1.89 | 7.50 | 6.49 |
| IBMQ | 9/1/28 | 1.90 | 8.22 | 7.11 |

## Single Name Maturing Assets

1. Long-Term Liability Matching Instruments: The objective is to empower the investors to create a liability matching portfolio that extends out as far as 30 years with intra-day liquidity. The fixed-term ETFs enable a portfolio to be implemented up to 10 years quiet easily. Beyond 10 years, it is uncertain at this time how to make purchases of single-name debt instruments, i.e., long corporate debt, intra-day, which will likely be in small amounts, i.e. $50. It is assumed here that the issue will be addressable.
2. Maturity Matching Yield Curve Instruments: What follow are the single-name maturity asset categories that the optimizer would have access to initially when determining the portfolio. The most-straightforward approach is to use the yield curve for each category and designate the term that reflects the maturity date matching the liability. The idea is that the yield curve can serve as a substitute for a future single bond purchase.
3. Instruments:
    a. US Treasury Strips => 1 – 30 years Yield Curve
    b. US Corporate Investment Grade => 1 – 20 years Yield Curve
    c. US High Yield => 1 – 20 years Yield Curve
    d. Emerging Market Debt => 1 – 20 years Yield Curve

## Running Income Prioritization

1. Motivation for Running Income Prioritization: Income is the third factor investors consider beyond principal protection and asset growth. For some investors, maximizing monthly cash flows takes priority over choosing between capital preservation and asset growth.
2. Running Income Prioritization Preference Scheme: This preference requires:
    a. An option for investors to prioritize monthly income
    b. Categorization of assets by their cash flow potential as well as expected returns and risk
    c. Restricting MVO to assets with highest potential income

d. The investor can still scale up between capital preservation and growth. The only difference is that the MVO will prioritize expected cash flow over expected returns. It is expected that this will lower the returns throughout the different risk levels.

# Fokker Planck Equation

## Overview

1. Description of the Fokker-Planck Equation: The Fokker-Planck equation is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag and random forces, as in Brownian motion (Wikipedia (2019)). The equation can be generalized to other variables as well (Kadanoff (2000)).
2. Kolmogorov Forward and Backward Equations: It is named after Adrian Fokker and Max Planck (Fokker (1914) and Planck (1917)) and is also known as the Kolmogorov forward equation after Andrey Kolmogorov, who independently discovered the concept in 1931 (Kolmogorov (1931)).
3. Particle Position PDF - Smoluchowski Equation: When applied to particle position distributions, it is better known as the *Smoluchowski equation* – after Marion Smoluchowski – and in this context it is equivalent to the convection-diffusion equation, i.e., the Smoluchowski equation is the Fokker-Planck equation for the probability density function of the particle positions of the Brownian particles (Dhont (1996)).
4. Liouville Equation/Kramers-Moyal Expansion: The case with zero diffusion is known in statistical mechanics as the Liouville equation. The Fokker Planck equation is obtained from the master equation through the Kramers-Moyal expansion.
5. Unified Derivation for Classical/Quantum Systems: The first consistent microscopic derivation of the Fokker-Planck equation in the single unified scheme of classical and quantum mechanics was given by Nikolay Bogoliubov and Nikolay Krylov (Bogoliubov and Krylov (1939), Bogoliubov and Sankevich (1994)).

## One Dimension

In one spatial dimension, for an Ito process driven by a standard Weiner process $W(t)$ and described by the stochastic differential equation (SDE)

$$\Delta X(t) = \mu(X(t), t)\Delta t + \sigma(X(t), t)W(t)$$

with drift $\mu(X(t), t)$ and diffusion coefficient

$$D(X(t), t) = \frac{1}{2}\sigma^2(X(t), t)$$

the Fokker-Planck equation for the probability density $p(X(t), t)$ of the random variable $X(t)$ is

$$\frac{\partial p(X(t), t)}{\partial t} = -\frac{\partial}{\partial X(t)}[\mu(X(t), t)p(X(t), t)] + \frac{\partial^2}{\partial X^2(t)}[D(X(t), t)p(X(t), t)]$$

## Link Between the Ito SDE and the Fokker-Planck Equation

1. Infinitesimal Generator of Expected Probabilities: The following uses

$$\sigma = \sqrt{2D}$$

This section uses the approach laid out in Ottinger (1996). The infinitesimal generator $\mathcal{L}$ is defined as

$$\mathcal{L}p(X(t),t) = \underset{\Delta t \to 0}{Limit} \frac{1}{\Delta t}\{\mathbb{E}[p(X(t+\Delta t), t+\Delta t) \mid X(t), t] - p(X(t), t)\}$$

2. <u>Use of the Transition Probability</u>: The *transition probability* $P(X(t), t \mid X'(t'), t')$ defined as the probability of going from $(X'(t'), t')$ to $(X(t), t)$ is used here; the expectation is then written as

$$\mathbb{E}[p(X(t+\Delta t), t+\Delta t) \mid X(t), t] = \int p(y,t)P(y, t+\Delta t \mid X(t), t)dy$$

3. <u>Evolution from s Fixed Starting Variate $(X'(t'), t')$</u>: The above expression is replaced by the definition of $\mathcal{L}$ multiplied by $P(X(t), t \mid X'(t'), t')$ and integrated over $X(t)$:

$$\int p(y,t) \int P(y, t+\Delta t \mid X(t), t)P(X(t), t \mid X'(t'), t')dX(t)\, dy \\ - \int p(X(t), t)P(X(t), t \mid X'(t'), t')dX(t)$$

4. <u>Explicit Introduction of the Time Derivative</u>: It may be noted that

$$\int P(y, t + \Delta t \mid X(t), t) P(X(t), t \mid X'(t'), t') dX(t) = P(y, t + \Delta t \mid X'(t'), t')$$

which is the Chapman-Kolmogorov theorem. Changing the dummy variable from $y$ to $X(t)$ one gets

$$\int p(X(t), t) \underset{\Delta t \to 0}{Limit} \frac{1}{\Delta t} [P(X(t + \Delta t), t + \Delta t \mid X'(t'), t') - P(X(t), t \mid X'(t'), t')] dX(t)$$

which is a time derivative.

5. Arriving at the Kolmogorov Equation: Finally, one arrives at

$$\int [\mathcal{L} p(X(t), t)]\, P(X(t), t \mid X'(t'), t') dX(t) = \int p(X(t), t) \frac{\partial P(X(t), t \mid X'(t'), t')}{\partial t} dX(t)$$

6. Fokker Planck Kolmogorov Forward Equation: If we instead use the adjoint operator of $\mathcal{L}$, $\mathcal{L}^{\dagger}$ defined such that

$$\int [\mathcal{L} p(X(t), t)] P(X(t), t \mid X'(t'), t') dX(t) = \int p(X(t), t) [\mathcal{L}^{\dagger} P(X(t), t) \mid X'(t'), t']\, dX(t)$$

one then arrives at the Kolmogorov Forward Equation, or the Fokker-Planck equation, which, simplifying the notation

$$p(X(t), t) = P(X(t), t \mid X'(t'), t')$$

in its differential form reads

$$\mathcal{L}^{\dagger} p(X(t),t) = \frac{\partial p(X(t),t)}{\partial t}$$

7. <u>Incorporating the Operator Form for $\mathcal{L}$</u>: Remains the issue of defining $\mathcal{L}$ explicitly. This can be done by taking the expectation from the integral form of the Ito's lemma:

$$\mathbb{E}[p(X(t),t)] = p(X_0,t_0) + \mathbb{E}\left[\int_0^t \left(\frac{\partial}{\partial t} + \mu\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}\right) p(X'(t'),t')dt'\right]$$

The part that depended on $W(t)$ vanished because of the martingale property.

8. <u>Fokker Planck Equation Differential Form</u>: Then, for a particle subject to an Ito equation, using

$$\mathcal{L} = \mu\frac{\partial}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial x^2}$$

it can be easily calculated, using integration by parts, to get

$$\mathcal{L}^{\dagger} = \frac{\partial}{\partial x}(\mu\,\cdot) + \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\sigma^2\,\cdot\right)$$

which brings us to the Fokker-Planck equation

$$\frac{\partial p(X(t),t)}{\partial t} = -\frac{\partial}{\partial X(t)}[\mu(X(t),t)p(X(t),t)] + \frac{\partial^2}{\partial X^2(t)}\left[\frac{1}{2}\sigma^2(X(t),t)p(X(t),t)\right]$$

9. Generating Prior Distributions - Feynman-Kac: While the Fokker-Planck equations is used with problems where the initial distribution is known, if the problem is to know the distribution at previous times, the Feynman-Kac formula can be used, which is a consequence of the Kolmogorov backward equation.
10. Ito Process in the Stratonovich Convention: The stochastic process defined above in the Ito sense can be re-written within the Stratonovich convention as a Stratonovich SDE:

$$\left[\mu(X(t),t) - \frac{1}{2}\frac{\partial}{\partial X(t)}D(X(t),t)\right]\Delta t + \sqrt{2D(X(t),t)} \circ \Delta W(t)$$

    This includes an added noise-induced drift term due to diffusion gradient effects if the noise is state-dependent. This convention is more often used in physical applications. Indeed, it is well-known that any solution to the Stratonovich SDE is a solution to the Ito SDE.
11. Case of Classical Brownian Motion: The zero-drift equation with constant diffusion can be considered as a model of classical Brownian motion:

$$\frac{\partial p(X(t),t)}{\partial t} = D_0 \frac{\partial^2 p(X(t),t)}{\partial X^2(t)}$$

12. Fixed Boundary Spectrum of Solutions: The model has discrete spectrum of solutions if the condition of fixed boundaries is added for

$$0 \leq x \leq L$$

$$p(0,t) = p(L,t) = 0$$

$$p(x,0) = p_0(x)$$

13. Coordinate Space Velocity Uncertainty Relation: It has been shown (Kamenshchikov (2014)) that in this case that the analytical spectrum of solutions allows one to derive the local uncertainty relation for the coordinate-velocity phase volume:

$$\Delta x \Delta v \geq D_0$$

Here $D_0$ is a minimal value of a corresponding diffusion spectrum $D_j$ while $\Delta x$ and $\Delta v$ represent the uncertainty of coordinate-velocity definition.

## Higher Dimensions

1. Multi-dimensional Fokker-Planck Equations: More generally, if

$$\Delta \boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t), t)\Delta t + \boldsymbol{\sigma}(\boldsymbol{X}(t), t)\Delta \boldsymbol{W}(t)$$

where $\boldsymbol{X}(t)$ and $\boldsymbol{\mu}(\boldsymbol{X}(t), t)$ are $N$-dimensional random vectors, $\boldsymbol{\sigma}(\boldsymbol{X}(t), t)$ is an $N \times M$ matrix, and $\boldsymbol{W}(t)$ is an $M$-dimensional standard Wiener process, the probability density $p(\boldsymbol{X}(t), t)$ for $\boldsymbol{X}(t)$ satisfies the Fokker-Planck equation

$$\frac{\partial p(\boldsymbol{X}(t),t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial X_i(t)} [\mu_i(\boldsymbol{X}(t),t)p(\boldsymbol{X}(t),t)] + \sum_{i=0}^{N}\sum_{j=0}^{N} \frac{\partial^2}{\partial X_i(t)\partial X_j(t)} \left[D_{ij}(\boldsymbol{X}(t),t)p(\boldsymbol{X}(t),t)\right]$$

with the drift vector

$$\boldsymbol{\mu} = \{\mu_1, \cdots, \mu_N\}$$

and the diffusion vector

$$\boldsymbol{D} = \frac{1}{2}\boldsymbol{\sigma}\boldsymbol{\sigma}^{\boldsymbol{T}}$$

i.e.,

$$D_{ik}(\boldsymbol{X}(t),t) = \frac{1}{2}\sum_{j=1}^{M} \sigma_{ij}(\boldsymbol{X}(t),t)\sigma_{jk}(\boldsymbol{X}(t),t)$$

2. Use of the Stratonovich Convention: If instead of an Ito SDE, a Stratonovich SDE is considered,

$$\Delta\boldsymbol{X}(t) = \boldsymbol{\mu}(\boldsymbol{X}(t),t)\Delta t + \boldsymbol{\sigma}(\boldsymbol{X}(t),t) \circ \Delta\boldsymbol{W}(t)$$

the Fokker-Planck equation will read (Ottinger (1996))

$$\frac{\partial p(\boldsymbol{X}(t),t)}{\partial t} = -\sum_{i=1}^{N} \frac{\partial}{\partial X_i(t)} [\mu_i(\boldsymbol{X}(t),t) p(\boldsymbol{X}(t),t)] + \sum_{k=0}^{N} \sum_{i=0}^{N} \frac{\partial}{\partial X_i(t)} \left\{ \sigma_{ik}(\boldsymbol{X}(t),t) \sum_{j=1}^{N} \frac{\partial}{\partial X_j(t)} \left[ \sigma_{jk}(\boldsymbol{X}(t),t) p(\boldsymbol{X}(t),t) \right] \right\}$$

## Wiener Process

A standard scalar Wiener process is generated by the stochastic differential equation

$$\Delta X(t) = \Delta W(t)$$

Here the drift term is zero and the diffusion coefficient is $\frac{1}{2}$. Thus, the corresponding Fokker-Planck equation is

$$\frac{\partial p(X(t),t)}{\partial t} = \frac{1}{2} \frac{\partial^2 p(X(t),t)}{\partial X^{\mathbf{2}}(t)}$$

which is the simplest form of a diffusion equation. If the initial condition is

$$p(x,t) = \delta(x)$$

the solution is

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t^2}}$$

## Ornstein-Uhlenbeck Process

The Ornstein-Uhlenbeck process is a process defined as

$$\Delta X(t) = -aX(t)\Delta t + \sigma \Delta W(t)$$

with

$$a > 0$$

The corresponding Fokker-Planck equation is

$$\frac{\partial p(X(t),t)}{\partial t} = a \frac{\partial}{\partial X(t)} [X(t)p(X(t),t)] + \frac{1}{2} \sigma^2 \frac{\partial^2 p(X(t),t)}{\partial X^{\mathbf{2}}(t)}$$

The stationary solution corresponding to

$$\frac{\partial p(X(t),t)}{\partial t} = 0$$

is

$$p_{STEADY-STATE}(x) = \sqrt{\frac{a}{\pi\sigma^2}}e^{-\frac{ax^2}{\sigma^2}}$$

## Plasma Physics

1. Boltzmann Equation for Particle Species Distribution: In plasma physics, the distribution for a particle species $s$, $p_S(\boldsymbol{x}, \boldsymbol{v}, t)$. takes the place of the probability density function. The corresponding Boltzmann equation is given by

$$\frac{\partial p_S}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} p_S + \frac{Z_S e}{m_S}(\boldsymbol{E} + \boldsymbol{v} \times p_S) \cdot \boldsymbol{\nabla}_v p_S = -\frac{\partial}{\partial v_i}(p_S\langle\Delta v_i\rangle) + \frac{1}{2}\frac{\partial^2}{\partial v_i \partial v_j}\left(p_S\langle\Delta v_i \Delta v_j\rangle\right)$$

   where the third term includes the particle acceleration due to the Lorentz force and the Fokker-Planck term on the right-hand side represents the effect of particle collisions.

2. Incorporating Cross Species Particle Collision: The quantities $\langle\Delta v_i\rangle$ and $\langle\Delta v_i \Delta v_j\rangle$ are the average change in velocity a particular type $s$ experiences due to collisions with all other particle species in unit time. Expressions for these quantities are given in Rosenbluth (1957). If the collisions are ignored, the Boltzmann equation reduces to the Vlasov equation.

## Computational Considerations

Brownian motion follows the Langevin equation, which can be solved for many different stochastic forcings, with the results being averaged using, for example, the Monte-Carlo method, or the canonical ensemble in molecular dynamics. However, instead of this computationally intensive approach, one can use the Fokker-Planck equation and consider the probability $p_S(\boldsymbol{v}, t)\Delta\boldsymbol{v}$ of a particle having a velocity in the interval $(\boldsymbol{v}, \boldsymbol{v} + \Delta\boldsymbol{v})$ when it starts its motion with $\boldsymbol{v_0}$ at time $t$.

## Solution

1. Analytical Solutions of Fokker-Planck: Being a partial differential equation, the Fokker-Planck equation can only be solved in special cases. A formal analogy of the Fokker-Planck equation with the Schrodinger's equation allows the use of advance operator techniques from quantum mechanics for its solution in a number of cases.
2. Simplification under Over-Damped Dynamics: Furthermore, in the case of over-damped dynamics when the Fokker-Planck equation contains second partial derivatives with respect to all variables, the equation can be written in a master form that may be solved easily numerically (Holubec, Kroy, and Steffenoni (2019)).
3. Solutions under Other Special Situations: In many situations, one is only interested in the steady-state probability distribution $p_{STEADYSTATE}(X(t), t)$ which can be found from

$$\frac{\partial p(X(t), t)}{\partial t} = 0$$

The computation of the mean passage times and the splitting probabilities can be reduced to the solution of an ordinary differential equation that is intimately related to the Fokker-Planck equation.

## Particular Cases with Known Solution and Inversion

1. Option Smile Local Volatility Calibration: In mathematical finance for volatility smile modeling of local volatility, one has the problem of deriving a diffusion coefficient $\boldsymbol{\sigma}(\boldsymbol{X}(t), t)$ consistent with the probability density obtained from market quotes. The problem is therefore an inversion of the Fokker-Planck equation. Given the probability density $f(\boldsymbol{X}(t), t)$ of the option underlying $\boldsymbol{X}(t)$ deduced from the option market, one aims to find the local volatility $\boldsymbol{\sigma}(\boldsymbol{X}(t), t)$ consistent with $f(\boldsymbol{X}(t), t)$.
2. Parametric and Non-parametric Solution Approaches: This inverse problem has been solved in general by Dupire (1994, 1997) with a non-parametric solution. Brigo and Mercurio (2002) and Brigo, Mercurio, and Sartorelli (2003) propose a solution in parametric form via a particular local volatility $\boldsymbol{\sigma}(\boldsymbol{X}(t), t)$ consistent with the solution of the Fokker-Planck equation given by a mixture model. More information is available in Fengler (2008), Gatheral (2008), and Musiela and Rutkowski (2008).

## Fokker-Planck Equation and Path Integral

1. Fokker-Planck Path Integral Equivalence: Every Fokker-Planck equation is equivalent to a path integral. The path integral formulation is an excellent starting point for the application of the field theory models (Justin-Zinn (1996)). This is used, for instance, in critical dynamics.

2. Derivation of the Path Integral: The derivation of the path integral is possible in a similar way as in quantum mechanics. The derivation for a Fokker-Planck equation with one variable $X(t)$ is as follows. Start by inserting a delta function and then integrating by parts:

$$\frac{\partial p(X'(t),t)}{\partial t} = -\frac{\partial}{\partial X'(t)}[\mu(X'(t),t)p(X'(t),t)] + \frac{\partial^2}{\partial X'^2(t)}[D(X'(t),t)p(X'(t),t)]$$

$$= \int_{-\infty}^{+\infty} dX(t)\left[\mu(X(t),t)\frac{\partial}{\partial X(t)} + D(X(t),t)\frac{\partial^2}{\partial X^2(t)}\right]\delta\big(X'(t)$$

$$- X(t)\big)p(X(t),t)$$

3. Time Integral of the Delta Function: The $X(t)$ derivatives here act only on the $\delta$ function and not on $p(X(t),t)$. Integrating over a time interval $\epsilon$ results in

$$p(X'(t),t+\epsilon)$$

$$= \int_{-\infty}^{+\infty} dX(t)\left\{\left[1+\epsilon\left\{\mu(X(t),t)\frac{\partial}{\partial X(t)} + D(X(t),t)\frac{\partial^2}{\partial X^2(t)}\right\}\right]\delta\big(X'(t)\right.$$

$$\left. - X(t)\big)\right\}p(X(t),t) + \mathcal{O}(\epsilon^2)$$

4. Fourier Transform of Fokker Planck: Inserting the Fourier integral

$$\delta\big(X'(t)-X(t)\big) = \int_{-i\infty}^{+i\infty} \frac{d\tilde{x}}{2\pi i}e^{\tilde{x}\left(X'(t)-X(t)\right)}$$

for the $\delta$ function results in

$$p(X'(t),t+\epsilon)$$

$$= \int_{-\infty}^{+\infty} dX(t) \int_{-i\infty}^{+i\infty} \frac{d\tilde{x}}{2\pi i}[1$$

$$+ \epsilon\{\tilde{x}\mu(X(t),t) + \tilde{x}^2 D(X(t),t)\}]e^{\tilde{x}\left(X'(t)-X(t)\right)} p(X(t),t) + \mathcal{O}(\epsilon^2)$$

$$= \int_{-\infty}^{+\infty} dX(t) \int_{-i\infty}^{+i\infty} \frac{d\tilde{x}}{2\pi i} e^{\epsilon\left\{-\tilde{x}\frac{X'(t)-X(t)}{\epsilon}+\tilde{x}\mu(X(t),t)+\tilde{x}^2 D(X(t),t)\right\}} p(X(t),t) + \mathcal{O}(\epsilon^2)$$

5. Expression for Path Integral Action: The above equation expresses $p(X'(t),t+\epsilon)$ as a functional of $p(X(t),t)$. Iterating $\frac{X'(t)-X(t)}{\epsilon}$ times and taking the limit

$$\epsilon \rightarrow 0$$

gives a path integral with action

$$S = \int dt \left\{\tilde{x}\mu(X(t),t) + \tilde{x}^2 D(X(t),t) - \tilde{x}\frac{\partial X(t)}{\partial t}\right\}$$

The variable $\tilde{x}$ conjugate to $x$ are called the *response variables* (Janssen (1976)).

6. Fokker Planck vs Path Integral Comparison: Although formally equivalent, different problems may be solved more easily in the Fokker Planck equation or the path integral formulation. The equilibrium distribution, for example, may be obtained more directly from the Fokker-Planck equation.

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# Ornstein Uhlenbeck Process

## Overview

1. Definition of Ornstein-Uhlenbeck Process: The *Ornstein-Uhlenbeck* process is a stochastic process with applications in financial mathematics and physical sciences. It original application in physics was as a model for the velocity of a massive Browning particle under the influence of friction, also called a *Damped Random Walk* (Uhlenbeck and Ornstein (1930), MacLeod, Ivezic, Kochanek, Kozlowski, Kelly, Bullock, Kimball, Sesar, Westman, Brooks, Gibson, Becker, and De Vries (2010), Wikipedia (2019)).
2. Mean-Reverting Stationary Gauss-Markov: The process is a stationary Gaussian-Markov process, which means that it is a Gaussian process, it is a Markov process, and it is temporally homogenous. The Ornstein-Uhlenbeck process is the only non-trivial process that satisfies these three conditions, up to allowing linear transformations for time and space variables (Doob (1942)). Over time, the process tends to drift towards its long-term mean; such a process is called *mean-reverting*.
3. Mean-Reverting Modified Weiner Process: The process can be considered to be a modification to the walk random process in continuous time, or Weiner process, in which the properties of the process have been changed such that there is a tendency of the walk to move back to a central location, with a greater attraction when the process is further away from the center. The Ornstein-Uhlenbeck process can also be considered to be a continuous-time analogue of the discrete time AR (1) process.

## Definition

1. Stochastic Difference Form for the Process: The Ornstein-Uhlenbeck process is defined by the following stochastic difference equation:

$$\Delta x(t) = -\theta x(t)\Delta t + \sigma \Delta W(t)$$

where

$$\theta > 0$$

and

$$\sigma > 0$$

are parameters and $W(t)$ denotes the Weiner process (Gard (1988), Gardiner (2009)).

2. Vasicek Extension to Ornstein-Uhlenbeck: An additional drift term is sometimes added:

$$\Delta x(t) = \theta[\mu - x(t)]\Delta t + \sigma \Delta W(t)$$

where $\mu$ is a constant. In financial mathematics, this is also known as the Vasicek model (Bjork (2009)).

## Fokker Planck Equation Representation

1. Evolution of the Ornstein-Uhlenbeck PDE: The Ornstein-Uhlenbeck process can also be described in terms of a probability density function $P(x, t)$ which specifies the probability of finding the process in the state $x$ at a time $t$ (Risken and Till (1996)).
2. Ornstein-Uhlenbeck Fokker-Planck Equation: $P(x, t)$ satisfies

$$\frac{\partial P(x, t)}{\partial t} = \theta \frac{\partial}{\partial x}[xP(x, t)] + D\frac{\partial^2 P(x, t)}{\partial x^2}$$

where

$$D = \frac{1}{2}\sigma^2$$

This is a linear parabolic differential equation that may be solved by a variety of techniques.

3. Expression for the Transition Probability: The transition probability $P(x, t \mid x_0, t_0)$ is Gaussian with mean $x_0 e^{-\theta(t-t_0)}$ and variance $\frac{D}{\theta}\left[1 - e^{-2\theta(t-t_0)}\right]$

$$P(x, t \mid x_0, t_0) = \sqrt{\frac{\theta}{2\pi D\left[1 - e^{-2\theta(t-t_0)}\right]}}\, e^{-\frac{\theta}{2D}\frac{\left[x - x_0 e^{-2\theta(t-t_0)}\right]^2}{1 - e^{-2\theta(t-t_0)}}}$$

This gives the probability of state $x$ occurring at time $t$ given the initial state $x_0$ at time

$$t_0 < t$$

Equivalently, $P(x, t \mid x_0, t_0)$ is the solution of the Fokker-Planck equation with the initial condition

$$P(x, t_0) = \delta(x - x_0)$$

## Mathematical Properties

1. Ornstein-Uhlenbeck Process Expectation/Covariance: Assuming $x_0$ is a constant, the mean is

$$\mathbb{E}[x(t)] = x_0 e^{-\theta t} + \mu\left[1 - e^{-\theta t}\right]$$

and the time covariance is

$$\mathbb{V}[x(s), x(t)] = \frac{\sigma^2}{2\theta}\left[e^{-\theta|t-s|} - e^{-\theta|t+s|}\right]$$

2. Bounded Variance Stationary Probability Distribution: The Ornstein-Uhlenbeck process is an example of a Gaussian process that has a bounded variance and admits a stationary probability distribution, in contrast to the Wiener process; the difference between the two is in their drift term. For the Wiener process the drift is constant, whereas for the Ornstein-Uhlenbeck process it is dependent on the current value of the state variable; if the current value of the process is less than the long-term mean, the drift will be positive; if the current value of the process is greater than the long-term mean, the drift will be negative. In other words, the mean acts as the equilibrium level for the process. This gives it the informative name *mean-reverting*.

## Properties of Sample Paths

1. Scaled, Time Transformed Wiener Process: A temporally homogenous Ornstein-Uhlenbeck process can be represented as a scaled, time-transformed Wiener process:

$$x(t) = \frac{\sigma}{\sqrt{2\theta}} e^{-\theta t} W\left(e^{2\theta t}\right)$$

where $W(t)$ is the standard Wiener process (Doob (1942)). Equivalently, using the change of variables

$$s = e^{2\theta t}$$

this process becomes

$$W(s) = \frac{\sqrt{2\theta}}{\sigma} s^{\frac{1}{2}} x\left(\frac{\log s}{2\theta}\right)$$

$$s > 0$$

2. Inference from Known Brownian Properties: Using the above mapping, one can translate the known properties of $W(t)$ into the corresponding statements for $x(t)$. For example, the law of iterated logarithm for $W(t)$ becomes (Doob (1942))

$$\begin{matrix} Limit \\ t \to 0 \end{matrix} \sup \frac{x(t)}{\sqrt{\frac{\sigma^2}{\theta} \log t}} = 1$$

with probability 1.

## Formal Solution

1. Variation of Parameters – Differences Form: The stochastic differential equation for $x(t)$ can be formally solved by variation of parameters (Gardner (2009)). Writing

$$f(x(t), t) = x(t)e^{\theta t}$$

one gets

$$\Delta f(x(t), t) = \theta x(t)e^{\theta t}\Delta t + e^{\theta t}\Delta x(t) = e^{\theta t}\theta\mu\Delta t + \sigma e^{\theta t}\Delta W(t)$$

2. Variation of Parameters - Integral Form: Integrating from 0 to $t$ one gets

$$x(t)e^{\theta t} = x_0 + \int_0^t \theta x(s)e^{\theta s}ds + \int_0^t \sigma e^{\theta s}dW(s)$$

3. Ornstein Uhlenbeck Process Leading Moments: From this representation, the first moment, i.e., the mean, is shown to be

$$x(t) = x_0 e^{-\theta t} + \mu\left[1 - e^{-\theta t}\right]$$

assuming $x_0$ is a constant. Moreover, the Ito isometry can be used to calculate the covariance function by

$$\begin{aligned}\mathbb{V}[x(s), x(t)] &= \mathbb{E}\left[x(s) - \mathbb{E}[x(s)]\right] \cdot \mathbb{E}\left[x(t) - \mathbb{E}[x(t)]\right] \\ &= \mathbb{E}\left[\int_0^s \sigma e^{\theta(u-s)} dW(u) \cdot \int_0^t \sigma e^{\theta(v-t)} dW(v)\right] \\ &= \sigma^2 e^{-\theta|t+s|} \cdot \mathbb{E}\left[\int_0^s \sigma e^{\theta u} dW(u) \cdot \int_0^t \sigma e^{\theta v} dW(v)\right] \\ &= \frac{\sigma^2}{2\theta} e^{-\theta|t+s|}\left[e^{2\theta \min(s,t)} - 1\right] = \frac{\sigma^2}{2\theta}\left[e^{-\theta|t-s|} - e^{-\theta|t+s|}\right]\end{aligned}$$

## Numerical Sampling

By using discretely sampled data at time intervals of width $t$, the maximum likelihood estimators for the parameters of the Ornstein-Uhlenbeck process are asymptotically normal to their true values (Ait Sahalia (2002)). More precisely,

$$\begin{aligned}&\sqrt{n}\left[\begin{pmatrix}\hat{\theta}_n \\ \hat{\mu}_n \\ \hat{\sigma}_n\end{pmatrix} - \begin{pmatrix}\theta \\ \mu \\ \sigma\end{pmatrix}\right] \\ &\to \mathcal{N}\left(\begin{bmatrix}0 \\ 0 \\ 0\end{bmatrix}, \begin{bmatrix}\dfrac{e^{2t\theta} - 1}{t^2} & 0 & \dfrac{\sigma^2\left\{e^{2t\theta} - 1 - 2t\theta\right\}}{t^2\theta} \\ 0 & \dfrac{\sigma^2\left\{e^{t\theta} + 1\right\}}{2\{e^{t\theta} - 1\}\theta} & 0 \\ \dfrac{\sigma^2\left\{e^{2t\theta} - 1 - 2t\theta\right\}}{t^2\theta} & 0 & \dfrac{\sigma^4\left\{\left(e^{2t\theta} - 1\right)^2 + 2t^2\theta^2\left(e^{2t\theta} + 1\right) + 4t\theta\left(e^{2t\theta} - 1\right)\right\}}{t^2\{e^{2t\theta} - 1\}\theta^2}\end{bmatrix}\right)\end{aligned}$$

## Scaling Limit Interpretation

The Ornstein-Uhlenbeck process can be interpreted as a scaling limit of a discrete process in the same way that Brownian motion is a scaling limit of random walks. Consider an urn containing $n$ blue and yellow balls. At each step a ball is chosen at random and replaced by a ball of the opposite color. Let $X_n$ be the number of blue balls after $n$ steps. Then $\frac{X_n - \frac{n}{2}}{\sqrt{n}}$ converges to an Ornstein-Uhlenbeck process as $n$ tends to infinity.

## Application in Physical Sciences

1. <u>Dynamics of a Noisy Relaxation Process</u>: The Ornstein-Uhlenbeck process is a prototype of a noisy relaxation process. Consider for example a Hookean spring with spring constant $k$, whose dynamics is highly over-damped with friction coefficient $\gamma$.
2. <u>Spring Oscillation under Thermal Fluctuation</u>: In the presence of thermal fluctuations with temperature $T$, the length $x(t)$ of the spring will fluctuate stochastically around the spring rest-length $x_0$; its stochastic dynamic is described by an Ornstein-Uhlenbeck process with

$$\theta = \frac{k}{\gamma}$$

$$\mu = x_0$$

$$\sigma = \sqrt{\frac{2k_BT}{\gamma}}$$

where $\sigma$ is derived from the Stokes-Einstein equation

$$D = \frac{\sigma^2}{2} = \frac{k_BT}{\gamma}$$

for effective diffusion constant.

3. Langevin Equation Form of the Process: In physical sciences, the stochastic differential equation of an Ornstein-Uhlenbeck process is rewritten as a Langevin equation

$$\dot{x}(t) = -\frac{k}{\gamma}[x(t) - x_0] + \xi(t)$$

where $\xi(t)$ is the white Gaussian noise with

$$\mathbb{E}[\xi(t_1)\xi(t_2)] = \frac{2k_BT}{\gamma}\delta(t_1 - t_2)$$

4. Correlations among the Spring Fluctuations: Fluctuations are correlated as

$$\mathbb{E}[(x(t_0) - x_0)(x(t_0 + t) - x_0)] = \frac{k_BT}{k}\delta(t_1 - t_2)e^{-\frac{|t|}{\tau}}$$

with a correlation time

$$\tau = \frac{\gamma}{k}$$

5. Spring Energy under Equi-partition Theorem: At equilibrium, the spring stores an average energy

$$\mathbb{E}[E] = k\frac{\mathbb{E}[(x - x_0)^2]}{2} = \frac{k_BT}{2}$$

in accordance with the equi-partition theorem.

## Application in Financial Mathematics

The Ornstein-Uhlenbeck process is one of the several approaches used to model – with modifications – interest rates, currency exchange rates, and commodity prices stochastically. The parameter $\mu$ represents the equilibrium of the mean value supported by fundamentals; $\sigma$ the degree of volatility around it caused by market moves, and $\theta$ the rate at which these moves dissipate and the variable reverts towards the mean. One application of the process is a strategy known as pairs trading (Rampertshammer (2007), Skiena (2008), Leung and Li (2015)).

## Application in Evolutionary Biology

The Ornstein-Uhlenbeck process has been proposed as an improvement over a Brownian motion model for modeling change in organismal phenotypes over time (Martins (1994)). A Brownian motion model implies that a phenotype can move without a limit, whereas for most phenotypes natural selection imposes a cost for moving too far in either direction.

## Generalizations

1. Extension to Background Levy Process: It is possible to extend the Ornstein-Uhlenbeck processes to process where the background driving process is a Levy process instead of a simple Brownian motion.
2. Chan-Karolyi-Longstaff-Sanders Enhancement: In addition, in finance, stochastic processes are used where the volatility increase for larger values of the variable. In particular, the CKLS process (Chan, Karolyi, Longstaff, and Sanders (1992)), where the volatility term is replaced by $\sigma x^{\varpi} \Delta W(t)$, can be solved in closed form for

$$\varpi = 1$$

as well as for

$$\varpi = 0$$

which corresponds to the conventional Ornstein-Uhlenbeck process. Another special case is

$$\varpi = \frac{1}{2}$$

which corresponds to the Cox-Ingersoll-Ross model (CIR model).

## Higher Dimensions

1. Multi-dimensional Ornstein-Uhlenbeck Process: A multi-dimensional version of the Ornstein-Uhlenbeck process, denoted by the $N$ dimensional vector $\boldsymbol{x}(t)$, can be defined from

   $$\Delta \boldsymbol{x}(t) = -\boldsymbol{\beta} \boldsymbol{x}(t) \Delta t + \boldsymbol{\sigma} \Delta \boldsymbol{W}(t)$$

   where $\boldsymbol{W}(t)$ is an $N$ dimensional Weiner process, $\boldsymbol{\beta}$ and $\boldsymbol{\sigma}$ are constant $N \times N$ matrices (Gardiner (2009)).
2. Multi-dimensional Ornstein-Uhlenbeck Solution: The solution is

   $$\boldsymbol{x}(t) = e^{-\boldsymbol{\beta} t} \boldsymbol{x}_0 + \int_0^t e^{-\boldsymbol{\beta}(t-s)} d\boldsymbol{W}(s)$$

   and the mean is

   $$\mathbb{E}[\boldsymbol{x}(t)] = e^{-\boldsymbol{\beta} t} \mathbb{E}[\boldsymbol{x}_0]$$

   Note that these expressions make use of matrix exponential.

3. Multi-dimensional Fokker Planck Equation: The process can also be described in terms of the probability density function $P(\boldsymbol{x}, t)$ which satisfies the Fokker-Planck equation (Gardiner (2009))

$$\frac{\partial P(\boldsymbol{x}, t)}{\partial t} = \sum_{i,j} \beta_{ij} \frac{\partial}{\partial \boldsymbol{x_i}} \left[ \boldsymbol{x_j} P(\boldsymbol{x}, t) \right] + \sum_{i,j} D_{ij} \frac{\partial^2 P(\boldsymbol{x}, t)}{\partial \boldsymbol{x_i} \partial \boldsymbol{x_j}}$$

   where the matrix $\boldsymbol{D}$ with components $D_{ij}$ is defined by

$$\boldsymbol{D} = \frac{\boldsymbol{\sigma}\boldsymbol{\sigma}^T}{2}$$

4. $P(\boldsymbol{x}, t)$ as a Linearly Transformed Gaussian: As is the case for on-dimension, the process is a linear transformation of Gaussian random variables, and therefore itself must be a Gaussian. Because of this, the transition probability $P(\boldsymbol{x}, t \mid \boldsymbol{x_0}, t_0)$ is a Gaussian which can be written down explicitly.
5. Explicit Stationary Solution for $P(\boldsymbol{x}, t)$: If the real part of the eigenvalues of $\boldsymbol{\beta}$ are larger than zero, a stationary solution $P_{STATIONARY}(\boldsymbol{x})$ exists, and is given by

$$P_{STATIONARY}(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{N}{2}}} \sqrt{\frac{1}{Det\ \boldsymbol{\Omega}}}\, e^{-\frac{1}{2}\boldsymbol{x^T}\boldsymbol{\Omega}^{-1}\boldsymbol{x}}$$

   where the matrix $\boldsymbol{\Omega}$ is determined from

$$\boldsymbol{\beta\Omega} + \boldsymbol{\Omega\beta^T} = \boldsymbol{D}$$

   (Risken and Till (1996))

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# Vasicek Model

## Overview

1. <u>One-Factor Interest Rate Model</u>: The *Vasicek model* describes the evolution of interest rates. It is a type of one-factor short-rate model as it describes the interest rate movements as driven by only one source of market risk (Wikipedia (2019)).
2. <u>Application to Other Market Factors</u>: The model is also used in the valuation of interest rate derivatives, as well as been adapted to credit markets. It was introduced by Vasicek (1977), and may also be viewed as a stochastic investment model.

## Model Description

1. <u>Dynamics of the Vasicek Model</u>: The model specifies that the instantaneous interest rate follows the stochastic difference equation

   $$\Delta r(t) = a[b - r(t)]\Delta t + \sigma \Delta W(t)$$

   where $W(t)$ is the Wiener process under the risk-neutral framework modeling the market-risk factor, in that it models the continuous inflow of randomness into the system.
2. <u>Details of the Model Parameters</u>: The standard deviation parameter $\sigma$ determines the volatility of the interest rate, and in a way characterizes the amplitude of the instantaneous random inflow. The parameters $a$, $b$, and $\sigma$, together with the initial condition $x_0$, completely determine the dynamics, and are characterized as follows, assuming $a$ to be non-negative:
    a. $b$ => *Long-Term Mean Level* All future trajectories of $r$ will revolve around a mean level $b$ in the long term.

    b. $a$ => *Speed of Reversion* – $a$ characterizes the velocity at which such trajectories will regroup around $b$ in time.
    c. $\sigma$ => *Instantaneous Volatility* – measures instant-by-instant the amplitude of randomness entering the system. Higher $\sigma$ implies more randomness.

3. <u>Expression for Long-Term Variance</u>: In addition, the *Long-Term Variance* $\frac{\sigma^2}{2a}$ is also of interest. All future trajectories of $r$ will regroup around the long-term mean with such variance after a long time.
4. <u>Opposing Nature of $a$ and $\sigma$</u>: $a$ and $\sigma$ tend to oppose each other. Increasing $\sigma$ increases the amount of randomness entering the system, but at the same time, increasing $a$ amounts to increasing the speed at which the system will statistically stabilize around the long-term mean $b$ with a corridor of variance given by $a$. This is clear from the long-term variance $\frac{\sigma^2}{2a}$ which increases with $\sigma$ but decreases with $a$.
5. <u>Cointelation - Stochastic Long-Term Mean</u>: This model is a type of Ornstein-Uhlenbeck stochastic process. Making the long term stochastic to another SDE is a simplified version of the Cointelation SDE (Mahdavi-Damghani (2014)).

## Analysis of the Vasicek Model

1. <u>Mean-reverting Nature of Interest Rates</u>: Vasicek model was the first one to capture mean-reversion, as essential characteristic of interest rates that sets it apart from other financial quantities. This is because very high levels would hamper economic activity, prompting a decrease in the interest rates. As a result, interest rates move in a limited range, showing a tendency to revert to a long-run value.
2. <u>Mean-reversion Produced by Drift</u>: The drift factor $a[b - r(t)]$ represents the expected instantaneous change in the interest rate at time $t$. The parameter $b$ represents the long-term equilibrium value towards which the interest rate reverts. Indeed, in the absence of shocks, when

$$\Delta W(t) = 0$$

the interest rate remains constant at

$$r(t) = b$$

The parameter $a$, governing the speed to the adjustment, needs to be positive to ensure the stability around the long-term value. For example, when $r(t)$ is below $b$, the drift term $a[b - r(t)]$ becomes positive for positive $a$, generating a tendency for the interest rates to move upwards towards equilibrium.

3. Shortcoming of the Vasicek Model: The main disadvantage is that, under the Vasicek model, it is theoretically possible for the interest rate to become negative, an undesirable feature under the pre-crisis assumptions. This shortcoming is fixed in the Cox-Ingersoll-Ross model, the exponential Vasicek model, the Black-Derman-Toy model, and the Black-Karasinski model, among many others. The Vasicek model was further extended in the Hull-White model. The Vasicek model is also a canonical example of the affine term structure model, along with the Cox-Ingersoll-Ross model.

## Asymptotic Mean and Variance

1. Stochastic Integral Expression for $r(t)$: The stochastic difference equation above may be integrated to obtain

$$r(t) = r_0 e^{-at} + b[1 - e^{-at}] + \sigma e^{-at} \int_0^t e^{as} dW(s)$$

2. $r(t)$ Process Mean and Variance: Using similar techniques as applied to the Ornstein-Uhlenbeck stochastic process, one gets the $r(t)$ normally distributed with mean

$$\mathbb{E}[r(t)] = r_0 e^{-at} + b[1 - e^{-at}]$$

and variance

$$\mathbb{V}[r(t)] = \frac{\sigma^2}{2a}[1 - e^{-2at}]$$

3. Long-Term Mean and Variance: Consequently, one gets

$$\underset{t \to \infty}{Limit}\ \mathbb{E}[r(t)] = b$$

and

$$\underset{t \to \infty}{Limit}\ \mathbb{V}[r(t)] = \frac{\sigma^2}{2a}$$

## References

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# Cox-Ingersoll-Ross Model

## Overview

1. <u>One-Factor Short-Rate Model</u>: The *Cox-Ingersoll-Ross model* describes the evolution of the interest rates. It is a type of a *one-factor model* – short-rate model – as it describes the interest rate movements as described by only one source of market risk (Wikipedia (2019)).
2. <u>Extension to the Vasicek Model</u>: The model can be used in the valuation of the interest rate derivatives. It was introduced by Cox, Ingersoll, and Ross (1985) as an extension of the Vasicek model.

## The Model

1. <u>Dynamics of the CIR Process</u>: The CIR model specifies that the instantaneous interest rate $r(t)$ follows the stochastic differential equations, also named the CIR process –

   $$\Delta r(t) = a[b - r(t)]\Delta t + \sigma\sqrt{r(t)}\Delta W(t)$$

   where $W(t)$ is Wiener process that models the random market risk factor, and $a$, $b$, and $\sigma$ are parameters. The parameter $a$ corresponds to the speed of adjustment to the mean $b$, and $\sigma$ to the volatility. The drift term $a[b - r(t)]$ is exactly the same as in the Vasicek model, and ensures the mean-reversion of the interest rate towards the long-run value $b$, with the speed of adjustment governed by the strictly positive parameters $a$.
2. <u>Criteria for Avoiding Negative Rates</u>: The standard deviation factor $\sigma\sqrt{r(t)}$ avoids the possibility of negative interest rates for all positive values of $a$ and $b$. The interest rate of zero is also precluded if the condition

$$2ab \geq \sigma^2$$

is met. More generally, when the rate $r(t)$ is close to zero, the standard deviation $\sigma\sqrt{r(t)}$ also becomes very small, which dampens the effect of the random shock on the rate. Consequently, when the rate gets close to zero, the evolution becomes dominated by the drift factor, which pushes the rate upwards towards equilibrium.

3. Ergodic/Stationary CIR Process Probability Distribution: This process can also be defined as a sum of squared Ornstein-Uhlenbeck process. The CIR is an ergodic process, and possesses a stationary distribution. The same process is used in the Heston model to model stochastic volatility.

## Distribution

1. Distribution of the Future Values: The distribution of the future values of a CIR process can be computed in closed form:

$$r(t+T) = \frac{{\chi_{NC}}^2}{2c}$$

where

$$c = \frac{2a}{[1 - e^{-at}]\sigma^2}$$

and ${\chi_{NC}}^2$ is a non chi-squared distribution with $\frac{4ab}{\sigma^2}$ degrees of freedom and the non-centrality parameter $2cr(t)e^{-at}$.

2. Gamma Function Based Stationary Distribution: Owing to mean reversion, as time becomes large, the distribution of $r_\infty$ will approach a gamma distribution with the probability density

$$f(r_\infty; a, b, \sigma) = \frac{\beta^\alpha}{\Gamma(\alpha)} {r_\infty}^{\alpha-1} e^{-\beta r_\infty}$$

where

$$\beta = \frac{2a}{\sigma^2}$$

and

$$\alpha = \frac{2ab}{\sigma^2}$$

## Properties

1. Mean Reversion
2. Level Dependent Volatility $\sigma\sqrt{r(t)}$
3. For a given positive $r_0$ the process will never touch zero if

   $$2ab \geq \sigma^2$$

   Otherwise it can occasionally touch the zero point.
4. $$\mathbb{E}[r(t) \mid r_0] = r_0 e^{-at} + b[1 - e^{-at}]$$

   Therefore, the long-term mean is $b$
5. $$\mathbb{V}[r(t) \mid r_0] = r_0 \frac{\sigma^2}{2a} [e^{-at} - e^{-2at}] + \frac{b\sigma^2}{2a} [1 - e^{-at}]^2$$

## Calibration

1. Ordinary Least Squares => The continuous SDE can be discretized as

$$r(t+\Delta\mathrm{t}) - r(t) = a[b - r(t)]\Delta t + \sigma\sqrt{r(t)\Delta t}\varepsilon(t)$$

   which is equivalent to

$$\frac{r(t+\Delta\mathrm{t}) - r(t)}{\sqrt{r(t)}} = \frac{ab\Delta t}{\sqrt{r(t)}} - a\sqrt{r(t)}\Delta t + \sigma\sqrt{\Delta t}\varepsilon(t)$$

   provided that $\varepsilon(t)$ is i.i.d. $\mathcal{N}(0,1)$. The above equation can then be used for a linear regression.

2. Other techniques such as martingale estimation and maximum likelihood may also be used.

## Simulation

Stochastic simulation of the IR process can be achieved using two variants – discretized and exact.

## Bond Pricing

Under no-arbitrage assumptions, a bond may be priced using this interest rate process. The bond price is exponential affine in the interest rate:

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)}$$

where

$$A(t,T) = \left\{ \frac{2he^{\frac{(a+h)(T-t)}{2}}}{2h + (a+h)[e^{h(T-t)} - 1]} \right\}^{\frac{2ab}{\sigma^2}}$$

$$B(t,T) = \frac{2\left[e^{h(T-t)} - 1\right]}{2h + (a+h)[e^{h(T-t)} - 1]}$$

$$h = \sqrt{a^2 + 2\sigma^2}$$

It may be easily seen that

$$\underset{t \to T}{Limit}\, A(t,T) \to 1$$

and

$$\underset{t \to T}{Limit}\, B(t,T) \to 0$$

## Extensions

1. <u>Time-Dependent Mean/Volatility Functions</u>: Time-varying functions replacing coefficients can be introduced in the model in order to make it more consistent with the pre-assigned term-structure of interest rates and possibly volatilities. The most general approach is in Maghsoodi (1996). A more tractable approach is in Brigo and Mercurio (2001) where an external time-dependent shift is added to the model for consistency with an input term structure of rates.
2. <u>Extension Using Stochastic Mean/Volatility</u>: A significant extension to the CIR model in the case of stochastic mean and stochastic volatility is given by Chen (1996) and is known as the

Chen model. The CIR process is a special case of basic affine jump diffusion, which still permits a closed-form expression for bond prices.

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