Collateral Valuation and XVA Generation in DROP

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Collateralized Agreements and Derivatives Valuation

Background

1. Background: While economies without risk-free rates have been considered in the past (Black (1972)), typical derivatives pricing treatments have assumed the existence of such rates as a matter of course (e.g., Duffie (2001)).
2. Holy Grail of Curve Construction: Combining multiple curves, partial collateralization involving multiple currencies, with liquidity, counter party risk, funding, and credit risk factored in into a dynamic approach is treated in a variety of papers (Pallavicini and Tarenghi (2010), Fujii, Shimada, and Takahashi (2010a), Fujii, Shimada, and Takahashi (2010b), Fujii, Shimada, and Takahashi (2010c), Fujii and Takahashi (2011a, 2011b), Castagna (2012), Henrard (2013)).
3. Treatments of CVA/DVA: Partial collateralization results in non-zero counter-party risk, and these cases are covered in Burgard and Kjaer (2011a, 2011b), Brigo, Pallavicini, Buescu, and Liu (2012), Crpey (2012a), Crpey (2012b). Considerations regarding the risk of an "average" counter-party are treated in Morini (2009).

Introduction and Motivation

1. Counter party Credit Risk Free Asset: Closest to a counterparty credit-risk free asset is an asset that is fully collateralized on a continuous basis (ISDA (2009), ISDA (2011), Sawyer (2011), Piterbarg (2012)), i.e., the collateralized asset produces cash flows that are continuous with changes in both the derivative MTM and the collateral coupon. Macey (2011) and Piterbarg (2012) illustrate how to retain the traditional risk-neutral valuation in a collateralized context.

2. Collateralized Asset Process: At the inception of a fully collateralized trade, there is no cash exchange, i.e., the upfront payment amount is returned back as collateral. Further, in exchange for the continuous pay streams above, the trade can be cancelled at any time with zero net value for either side.
3. Price of a Collateralized Asset: The price of a collateralized asset is effectively the outstanding level of the collateral account, i.e., a collateralized transaction is an asset with a zero-drift price process and with the given cumulative dividend flows (Duffie (2001)).
4. Collateral Cash Flows: $V(t)$ is the asset price paid by A to B, and B posts this amount back as collateral. A now pays the contractual collateral coupon flow $c(t)$ back to B. In time unit Δt, the cash flow that is exchanged (i.e., paid to A) is $V(t + \Delta t) - V(t) - c(t)V(t)\Delta t$, i.e.,

$$\Delta\chi(t) = \Delta V(t) - c(t)V(t)\Delta t$$

Once this is exchanged, the transaction can terminate, and A can keep the collateral.

Two Collateralized Assets

1. Setup: Assume that each of the assets follows its corresponding real-world measures, but are exposed to the same risk factor ΔW, i.e.,

$$\Delta V_i(t) = \mu_i(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W$$

for

$$i = 1, 2$$

2. Hedge Portfolio: Say that the corresponding collateralized account for each of these assets has the dynamics

$$\Delta\chi_i(t) = \Delta V_i(t) - c(t)V_i(t)\Delta t$$

Construct a hedge portfolio using $-\sigma_1(t)V_1(t)$ of asset 2 and $+\sigma_2(t)V_2(t)$ of asset 1. The net change in the real world collateralized portfolio of these two assets is:

$$\Delta\chi_{12}(t) = \sigma_2(t)V_2(t)[\Delta V_1(t) - c(t)V_1(t)\Delta t] - \sigma_1(t)V_1(t)[\Delta V_2(t) - c(t)V_2(t)\Delta t]$$

$$\Delta\chi_{12}(t) = V_1(t)V_2(t)[\sigma_2(t)\{\mu_1(t) - c(t)\} - \sigma_1(t)\{\mu_2(t) - c(t)\}]\Delta t$$

3. <u>Application of the Collateral Rules</u>: The above amount is known at time t, and maybe exchanged at $t + \Delta t$, at zero additional cost to either party. Thus, the only way both can enter into this transaction is if the net cash flow is zero (this is the collateralized version of no arbitrage). This produces

$$\frac{\mu_1(t) - c(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c(t)}{\sigma_2(t)}$$

4. <u>Differences with Traditional Risk Neutral Pricing</u>: The main difference is: in the traditional risk-neutral pricing, the hedged portfolio grows at the "risk-free" rate. In collateralized pricing, the COLLATERALIZED + HEDGED portfolio grows at ZERO rate (i.e., does not grow at all) after incremental netting! Therefore the "risk-free" rate does not enter into this setting at all.
5. <u>Measure Change</u>: Create a new measure t where

$$\Delta W_Q = \Delta W + \frac{\mu_i(t) - c(t)}{\sigma_i(t)}$$

In this new measure, the individual assets grow as

$$\Delta V_i(t) = c(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W_Q$$

using which we estimate $V_i(t)$ as

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c(s)ds}V_i(T)\right]$$

As may be observed, measure Q looks like the traditional risk neutral measure.

6. Different Collateral Rates: The collateral rates $c_i(t)$ can be asset-specific within changing any of our principal conclusions, and $V_i(t)$ now becomes

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c_i(s)ds}V_i(T)\right]$$

Examples would be, say, a stock collateralized at its repo rate (or other funding rate), while the derivative would be collateralized at it collateral rate (e.g., Piterbarg (2010)).

7. Other Variants: Other collateralization variants include varying collateral processes, different counter-parties etc. Typically all these only end up varying the drift, thus you get

$$\frac{P_1(t,T)}{P_2(t,T)} = \frac{\mathbb{E}_t^{Q_1}\left[e^{-\int_t^T c_1(s)ds}V(T)\right]}{\mathbb{E}_t^{Q_2}\left[e^{-\int_t^T c_2(s)ds}V(T)\right]}$$

Of course, the collateralization drift can also be stochastic. This measure change from collateralization scheme #1 to collateralization scheme #2 induces a drift to the scheme #2 as

$$\mathbb{E}_t^Q\left[e^{-\int_t^T [c_2(s)-c_1(s)]ds}V(T)\right]$$

8. Many Collateralized Assets: Will quickly flip through this, as Piterbarg (2012) spells out the details. N-dimensional asset $\vec{V}$ possesses the real-world dynamics

$$\Delta\vec{V}(t) = \vec{\mu}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\Delta\vec{W}$$

A linearly combined weight set $\vec{w}$ of the hedge portfolio satisfies the constraint

$$\vec{w}^T\vec{\sigma} = 0$$

Using the collateral cash flow matching arguments presented above, we get

$$\vec{w}^T\left[\vec{\mu}^T\vec{V} - \vec{c}^T\vec{V}\right] = 0$$

Measure Change => As before, there exists a measure Q with the drift vector $\vec{c}$, one for each asset, such that an adjustment $\vec{\lambda}$ can be made to the real world measure making it

$$\Delta\vec{V}(t) = \vec{c}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\left[\Delta\vec{W} + \vec{\lambda}\Delta t\right]$$

such that $\Delta\vec{W} + \vec{\lambda}\Delta t$ can become drift-less. Once again, in this new measure, the individual assets follow

$$V_i(t) = \mathbb{E}_t^Q\left[e^{-\int_t^T c_i(s)ds}V_i(T)\right]$$

Setup of the Collateral Curve Dynamics

1. Short-Rate Collateral Curve: Piterbarg (2010) considers the risk-free curve for lending, a curve that corresponds to the safest available collateral (cash). The corresponding short rate is denoted by $r_C(t)$ – where C stands for CSA, since the assumption is that this is the agreed upon overnight rate paid among collateral dealers under the CSA.
2. HJM Parametrization of the Collateral Curve: It is convenient to parametrize term curves in terms of discount factors $P_C(t,T)$

$$0 \leq t \leq T < \infty$$

standard HJM theory applies with the following dynamics for the yield curve:

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T \Delta W_C(t)$$

where W_C is a d-dimensional Brownina motion under the risk-neutral measure P and σ_C is a vector-valued, d-dimensional stochastic process.

3. Asset-specific Funding/Repo Rate: Piterbarg (2010) considers derivative contracts on a particular asset where the price process is denoted

$$S(t); t \geq 0$$

The short-rate on funding secured by this asset is r_R (R for repo).

4. Rates for Unsecured Funding: Finally the short-rate for unsecured funding is denoted

$$r_C(t), t \geq 0$$

As a rule, it would be expected that

$$r_C(t) \leq r_R(t) \leq r_F(t)$$

5. Funding Spread as a Default Premium: The existence of non-zero short rate spreads between short-rates of different collateral can be cast in the language of credit risk, by introducing joint defaults between the bank and the various assets used as collateral for funding.
6. Default Intensity of the Bank: In particular, the funding spread

$$s_F(t) \triangleq r_F(t) - r_C(t)$$

can be thought of as the stochastic intensity of default of the bank. The dynamics of the intensity is pursued in Gregory (2009) and Burgard and Kjaer (2009), while Piterbarg (2009) postulates the dynamics of the funding curves directly instead.

Collateralized Black-Scholes Formulation

1. Dynamics of the Derivative Value: This section examines how the regular Black-Scholes pricing methodology changes in the presence of CSA. Let $S(t)$ be an asset that follows in real-world the dynamics

$$\frac{\Delta S(t)}{S(t)} = \mu_S(t)\Delta t + \sigma_S(t)\Delta W_S(t)$$

2. Full Change in the Derivative Value: Let $V(t,S)$ be a derivative on the asset. By Ito's lemma it follows that

$$\Delta V(t) = \mathcal{L}\big(V(t)\big)\Delta t + \mathcal{X}(t)\Delta V(t)$$

where $\mathcal{L}$ is the standard pricing operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2}{\partial S^2}$$

and $\mathcal{X}(t)$ is the option's delta

$$\mathcal{X}(t) = \frac{\partial V(t)}{\partial S}$$

3. Full/Partial Collateral Cash Account: Let $C(t)$ be the collateral, i.e., the cash held in the collateral account, at time t against the derivative. For flexibility this amount may be different from $V(t)$.
4. Replicating Portfolio for the Derivative Payoff: To replicate the derivative at time t we hold $\mathcal{X}(t)$ units of the asset and $\gamma(t)$ units of cash. The value of the replication portfolio, which we denote by $\Pi(t)$ is equal to

$$V(t) = \Pi(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

where

$$\gamma(t) = C(t) + [V(t) - C(t)] - \mathcal{X}(t)S(t)$$

5. Decomposition of the Cash Account: The cash amount $\gamma(t)$ is split among a number of accounts;
    a. Amount $C(t)$ is in collateral
    b. Amount $V(t) - C(t)$ needs to be borrowed/lent from the treasury desk
    c. Amount $\mathcal{X}(t)S(t)$ is borrowed to finance the purchase of $\mathcal{X}(t)$ assets. It is secured by the assets purchased.
    d. The assets pay dividend at the rate $r_D(t)$
6. Full Growth of the Cash Account: The growth of all the cash accounts is given by

$$g(t)\Delta t = [r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)\mathcal{X}(t)S(t)]\Delta t$$

7. Applying the Self-Financing Criterion: On the other hand, from

$$V(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

using the self-financing criterion one gets

$$g(t)\Delta t = \Delta V(t) - \mathcal{X}(t)\Delta S(t)$$

which becomes, by Ito's lemma

$$\Delta V(t) - \mathcal{X}(t)\Delta S(t) = \mathcal{L}\big(V(t)\big)\Delta t = \left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]\Delta t$$

8. Funding/Collateral Derivative Valuation PDE: Thus one obtains

$$\left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]$$
$$= r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)S(t)\frac{\partial V(t)}{\partial S}$$

which after re-arrangement results in

$$\frac{\partial V(t)}{\partial t} + [r_R(t) - r_D(t)]S(t)\frac{\partial V(t)}{\partial S} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}$$
$$= r_F(t)V(t) - C(t)[r_F(t) - r_C(t)]$$

9. Solution Using Feynman-Kac Integral: The solution may be obtained by essentially following the steps that lead to the Feynman-Kac formula (Karatzas and Shreve (1997)) and is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}V(T) + \int_t^T e^{-\int_t^u r_F(v)dv}\{r_F(u) - r_C(u)\}C(u)du\right]$$

in the measure in which the asset grows at the rate $r_R(t) - r_D(t)$, that is

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

10. The Right "Risk-Free" Rate: Note that if the probability space is rich enough, it can be taken to be the same risk-neutral measure P in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T \Delta W_C(t)$$

Thus this derivation validates the view of Barden (2009) (and Hull (2006)) that the repo rate $r_R(t)$ is the right "risk-free" rate to use when valuing derivatives on $S(t)$.

Collateralization and Funding Derivative Valuation

1. Incremental Change in the Derivative Value: By re-arranging the Feynman-Kac expression above for $V(t)$ one obtains another useful expression for the valuation of the derivative:

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

It can be seen that

$$\mathbb{E}_t[\Delta V(T)] = [r_F(t)V(t) - \{r_F(t) - r_C(t)\}C(t)]\Delta t = [r_F(t)V(t) - s_F(t)C(t)]\Delta t$$

2. Derivative Value Under Full Collateralization: Thus the rate of growth in the derivative security is the funding spread $s_F(t)$ applied to the collateral. In particular, if the collateral is equal to the value $V(t)$ then

$$\mathbb{E}_t[\Delta V(T)] = r_C(t)C(t)\Delta t$$

and therefore

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T)\right]$$

and the derivative value grows at the risk free (i.e., collateral) rate. The final value is the only payment that appears in the discounted expression as the other payments net out given the assumption of full collateralization. This is consistent with the drift in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T \Delta W_C(t)$$

as $P_C(t,T)$ corresponds to the deposits secured by cash collateral.

3. Derived Value Under Unsecured Trading: On the other hand if the collateral is zero then

$$\mathbb{E}_t[\Delta V(T)] = r_F(t)V(t)\Delta t$$

and the rate of growth is equal to the bank's unsecured funding rate, or, using the credit risk language, adjusted for the probability of the bank default.

4. Collateral and Funding Measure Numeraires: Therefore the case

$$C = V$$

could be handled using the measure that corresponds to the risk free bond

$$P_C(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as a numeraire, and likewise, the case

$$C = 0$$

corresponds to the risky bond

$$P_F(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

as a numeraire.

5. <u>Portfolio Effects of the Collateral Position</u>: When two dealers are trading with each other the collateral is applied to the overall value of the portfolio between them with the positive exposures on some trades offsetting the negative exposures on the other trades (so-called netting). Hence the valuation of individual trades should take into account the collateral position of the whole portfolio.
6. <u>Simplification of Full Collateralization/Trading</u>: Fortunately in the simples case of the collateral being a linear function of the exact value of the portfolio - the case that includes both the no-collateral case

$$C = 0$$

as well as the full collateral case

$$C = V$$

- the value of the portfolio is just the sum of the values of the individual trades (with the collateral attributed to the trades by the same linear function). This easily follows from the pricing formula linearity of C and V in

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

Collateral PDE Formulation

1. <u>PDE Collateralization Treatments</u>: Bjork (2009), Piterbarg (2010), Castagna (2011), Fujii and Takahashi (2011a, 2011b), Henrard (2012), Piterbarg (2012), Ametrano and Bianchetti (2013), and Han, He, and Zhang (2013) extend the no-arbitrage to the collateralization case.
2. <u>Review of Derivative PDE Using Replication</u>: The derivative that is replicated using n assets and a bond via

$$V = nS + B$$

undergoes the evolution through the self-financing formulation

$$\Delta V = n\Delta S + \Delta B$$

This is matched to the derivative change

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S$$

Equating the two, setting

$$\frac{\partial V}{\partial S} = n$$

to eliminate stochasticity, and noticing that

$$\Delta B = rB\Delta t$$

we get

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\frac{1}{r} = B$$

Using the expression for V, this may be re-composed as the Black-Scholes PDE from

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

(Harrison and Kreps (1979), Harrison and Pliska (1981), Harrison and Pliska (1983)).

3. Derivative Replication with Collateral Account: The replication strategy now involves the assets, the bank funding account, and the collateral account.

$$V = aS + B + C$$

where a and B are the number of assets and the bank funding notional account, respectively, and C is the collateral account. Under perfect collateralization

$$C \equiv V$$

Further

$$\Delta C = r_C V \Delta t$$

$$\Delta B = r_f B \Delta t$$

and

$$\Delta S = \mu_S S \Delta t + \sigma_S S \Delta W$$

4. Derivative Value Change: Applying the self-financing condition

$$\Delta V = a\Delta S + \Delta B + \Delta C$$

Using the perfect collateral condition we get

$$V = aS + B + V$$

which implies

$$B = -aS$$

Thus

$$\Delta V = a\Delta S + r_f B\Delta t + r_C V\Delta t$$

results in

$$\Delta V = a\Delta S - a r_f S\Delta t + r_C V\Delta t$$

We refer to the quantity

$$\Gamma\left(r_C, r_f\right) = -a r_f S\Delta t + r_C V\Delta t$$

as the cash account.

5. <u>The Collateralization PDE</u>:

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S = a\Delta S - a r_f S\Delta t + r_C V\Delta t$$

Setting

$$a = \frac{\partial V}{\partial S}$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S \frac{\partial V}{\partial S} = r_C V$$

Re-casting using the appropriate measure terminology, we get

$$V(S, t) = \mathbb{E}_t^{Q^f}[D_C(S, T)V(S, T)]$$

where

$$D_C(t, T) = e^{-\int_t^T r_C(u)du}$$

Forward Contract Valuation

1. Repo'd Zero Strike Call Option:
    a. Asset Delivery at Future Time => Possibly the simplest derivative contract on an asset is the promise to deliver this asset at a given future time T. The contract should be seen as a zero strike call option with expiry T. In the standard theory, of course, the value of the derivative is the same as the value of the asset itself (in the absence of dividends).
    b. Forwards Value and Derivative Value => The payout of the derivative is given by

    $$V(T) = S(T)$$

    and the value at time t, assuming no CSA, is given by

$$V_{ZSC}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]$$

On the other hand, if

$$r_D(t) \neq 0$$

then

$$S(t) = \mathbb{E}_t\left[e^{-\int_t^T r_R(u)du} S(T)\right]$$

as follows from

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

and clearly

$$V_{ZSC}(t) \neq S(t)$$

c. Repo Impact on the Value => The difference in the value between the derivative and the asset are now easily understood as the zero-strike call-option carries the credit risk of the bank, while the asset $S(t)$ does not. Or, in the language of funding, the asset $S(t)$ can be used to secure the funding – which is reflected in the corresponding repo rate applied – while $V_{ZSC}(t)$ cannot be used for such a purpose.

2. <u>No-CSA Forwards Valuation</u>:
    a. Forwards Contract without CSA => This section considers a forward contract on $S(t)$ where at a time t the bank agrees to deliver the asset at time T against a cash payment at time T.

b. No-CSA Forward Contract Definition => A no-CSA forward contract could be seen as a derivative with payout $S(T) - F_{NoCSA}(t,T)$ at a time T where $F_{NoCSA}(t,T)$ is the forward price at a time t for a delivery at T. As the forward contract is cost free, we have by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

that

$$0 = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - F_{NoCSA}(t,T)\}\right]$$

so we get

$$F_{NoCSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

c. Valuation of the No-CSA Forward => From the above expression for $F_{NoCSA}(t,T)$ define

$$P_F(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

Note that this is essentially a credit-risky bond issued by the bank. Thus the expression for $F_{NoCSA}(t,T)$ can be re-written as

$$F_{NoCSA}(t,T) = \widetilde{\mathbb{E}}_t^T[S(T)]$$

where the measure $\tilde{P}_T$ is defined by the numeraire $P_F(t,T)$ as

$$e^{-\int_0^t r_F(u)du} P_F(t,T) = \mathbb{E}_t\left[e^{-\int_0^T r_F(u)du}\right]$$

is a P-martingale. Thereby $F_{NoCSA}(t,T)$ is a $\tilde{P}$-martingale.

d. No-CSA Forward Probability Measure => Note that the value of the asset under no-CSA at time t is given by

$$\mathbb{E}_t[\Delta V(T)] = r_F(t)V(t)\Delta t$$

to be

$$V(t) = \mathbb{E}_t\left[e^{-\int_0^T r_F(u)du} V(T)\right] = P_F(t,T)\widetilde{\mathbb{E}}_t^T[V(T)]$$

so it could be calculated simply by taking the expected value of the payout in the risky T-forward measure.

3. Forwards Contract with CSA:
    a. Full CSA Forward Contract Definition => Now let us consider a forward contract covered by a CSA where we assume that the collateral posted C is always equal to the value of the contract.
    b. Valuation of the CSA-Based Forward => Let the forward price $F_{CSA}(t,T)$ be fixed at t; then the value from

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

is given by

$$0 = V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T)\right] = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right]$$

so we get

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

Comparing this with

$$F_{NoCSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

we see that

$$F_{NoCSA}(t,T) \neq F_{CSA}(t,T)$$

By the arguments similar to the no-CSA case we obtain

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

where P^T is the standard T-forward measure – that is a measure defined by

$$P_C(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as its numeraire.

c. CSA Based Forward Probability Measure => Note that the value of an asset under CSA at a time t with a payout $V(t)$ is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T)\right] = P_C(t,T) = \mathbb{E}_t^T[V(T)]$$

so it could be simply calculated by taking the expected value of the payout in the risk-free T-forward measure.

4. <u>Calculating CSA Convexity Adjustment</u>:
    a. The Funding Basis Spread Numeraire => This section focusses on the difference between the CSA and the non-CSA forward prices. It can be seen that

$$F_{NoCSA}(t,T) = \widetilde{\mathbb{E}}_t^T[S(T)] = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} e^{-\int_t^T \{r_F(u) - r_C(u)\}du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{P_C(t,T)}{P_F(t,T)} \mathbb{E}_t^T\left[e^{-\int_t^T s_F(u)du} S(T)\right] = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)} S(T)\right]$$

where

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

is a P^T-martingale, as

$$M(t,T) = \mathbb{E}_t^T\left[e^{-\int_t^T s_F(u)du}\right]$$

    b. CSA vs. no-CSA Convexity => It can be noted trivially that

$$\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right] = 1$$

so

$$F_{NoCSA}(t,T) - F_{CSA}(t,T) = \mathbb{E}_t^T\left[\left\{\frac{M(T,T)}{M(t,T)} - \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right]\right\}\{S(t,T) - F_{CSA}(t,T)\}\right]$$

$$= \frac{1}{M(t,T)} Covariance_t^T[M(T,T), F_{CSA}(T,T)]$$

c. Funding Spread Dynamical Model => To obtain the actual value of the adjustment the joint dynamics of $s_F(u)$ and $S(u), u \geq t$ needs to be postulated. A simple model presented later shows the results of these corrections.

5. <u>Futures vs. CSA Forward Contracts</u>:
   a. Futures vs. CSA Forward – Similarity => At first sight the forward contract under CSA looks like a futures contract on the asset. With the futures contract, the daily price difference gets credited/debited to the margin account. In the same way, as the forward prices move, a CSA forward contract also specifies that money exchanges hands.
   b. Futures vs. CSA Forward – Differences => There is, however, an important difference. Consider the value of the forward contract at

$$t' > t$$

a contract that was entered at time t, so

$$V(t) = 0$$

Then

$$V(t') = \mathbb{E}_{t'}\left[e^{-\int_{t'}^T r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right]$$

$$= \mathbb{E}_{t'}\left[e^{-\int_{t'}^T r_C(u)du} S(T)\right] - \mathbb{E}_{t'}\left[e^{-\int_{t'}^T r_C(u)du}\right] F_{CSA}(t,T)$$

From

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

one gets

$$V(t') - V(t) = \mathbb{E}_{t'}\left[e^{-\int_{t'}^T r_C(u)du}\right]\{F_{CSA}(t',T) - F_{CSA}(t,T)\}$$

so the difference between the contract values on t and t' that exchanges hands on t' is equal to the discounted T difference in the forward prices. For a futures contract the difference will not be discounted.

c. Futures vs. CSA Forward Convexity => Therefore the types of convexity seen in the futures contract are different from those seen in the CSA vs. non-CSA forward contracts, a conclusion different from those reached by Johannes and Sundaresan (2007).

European Style Options

1. CSA vs. non-CSA Pricing:
    a. Basic European Option Pricing Setup => Consider a European style option on $S(T)$ with a strike K. Depending on the presence of absence of CSA we get two prices:

$$V_{CSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - K\}^+\right]$$

and

$$V_{NoCSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - K\}^+\right]$$

where for the CSA case we assume that the collateral posted C is always equal to the option value V_{CSA}.

b. CSA vs. non-CSA Numeraires => By the same measure change arguments as in the previous sections we get

$$V_{CSA}(t) = P_C(t, T)\mathbb{E}_t^T[\{S(T) - K\}^+]$$

and

$$V_{NoCSA}(t) = P_F(t, T)\tilde{\mathbb{E}}_t^T[\{S(T) - K\}^+]$$

c. CSA vs. non-CSA Raw Moments => The difference between the measures $\tilde{P}_t^T$ and P_t^T not only manifests itself in the mean of $S(T)$ – as already established – but also reveals itself in the characteristics of the distribution of $S(\cdot)$ such as its variance and higher moments.

2. <u>Distribution Impact of Convexity Adjustment</u>:
   a. No-CSA European Option Price => To see ow a change of measure affects the distribution of $S(\cdot)$, using

$$F_{NoCSA}(t, T) = \mathbb{E}_t^T\left[\frac{M(T, T)}{M(t, T)} S(t, T)\right]$$

one has

$$V_{NoCSA}(t) = \mathbb{E}_t^T\left[\frac{M(T, T)}{M(t, T)} \{S(T) - K\}^+\right]$$

where $M(t, T)$ is defined as

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

b. Conditional on $S(T)$ Option Price => From this, by conditioning on $S(T)$ one obtains

$$V_{NoCSA}(t) = P_F(t,T)\mathbb{E}_t^T\left[\alpha\big(t,T,S(T)\big)\{S(T)-K\}^+\right]$$

where the deterministic function $\alpha(t,T,x)$ is given by

$$\alpha(t,T,x) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}|S(T)=x\right]$$

c. Linearization of the Conditional Funding Basis => Using the approach of Antonov and Arneguy (2009), Piterbarg (2010) approximates the function $\alpha(t,T,x)$ by a function that is linear in x;

$$\alpha(t,T,x) = \alpha_0(t,T) + \alpha_1(t,T)x$$

and obtains α_0 and α_1 by minimizing the squared differences while using the fact that

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

and

$$\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right] = 1$$

as

$$\alpha_1(t,T) = \frac{\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right] - F_{CSA}(t,T)}{Variance_t^T[S(T)]}$$

and

$$\alpha_0(t,T) = 1 - \alpha_1(t,T)F_{CSA}(t,T)$$

d. Slop of the Conditional Funding Basis Distribution => Recognizing the term $\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right] - F_{CSA}(t,T)$ as the convexity adjustment of the forward between the n-CSA and the CSA versions of $F(t,T)$ one can write

$$\alpha_1(t,T) = \frac{F_{NoCSA}(t,T) - F_{CSA}(t,T)}{Variance_t^T[S(T)]}$$

e. Collateral vs. Funding Measure Relation => Differentiating

$$V_{NoCSA}(t) = P_F(t,T)\mathbb{E}_t^T\left[\alpha\big(t,T,S(T)\big)\{S(T)-K\}^+\right]$$

twice with respect to K one obtains the probability density functions (PDFs) of $S(T)$ under the two measures as

$$\tilde{P}_t^T(S(T) \in [K, K+\Delta K]) = [\alpha_0(t,T) + \alpha_1(t,T)K]P_t^T(S(T) \in [K, K+\Delta K])$$

So the PDF of $S(T)$ under the no-CSA measure is obtained by the density of $S(T)$ under the CSA measure by multiplying it with a linear function. It is not hard to see that the main impact of such a transformation is on the slope of the volatility smile of $S(\cdot)$.

3. Stochastic Funding Model: This section considers a simple stochastic funding model to estimate the impact of collateral rules on forwards and options. Consider an asset that follows a log-normal process

$$\frac{\Delta S(t)}{S(t)} \cong \mathcal{O}(\Delta t) + \sigma_S \Delta W_S(t)$$

and a funding spread that follows a simple one-factor Gaussian model of interest rates

$$\Delta s_F(t) = -\aleph_F[\theta - s_F(t)]\Delta t + \sigma_F \Delta W_F(t)$$

with

$$\langle \Delta W_F(t) \Delta W_S(t) \rangle = \rho \Delta t$$

where ρ is the correlation between the asset price and the funding spread.

a. Evolution of F_{CSA} under P => It is also assumed for simplicity that $r_C(t)$ and $r_R(t)$ are deterministic, and

$$r_D(t) = 0$$

Then

$$F_{CSA}(t, T) = \mathbb{E}_t^T[S(T)]$$

and

$$\frac{\Delta F_{CSA}(t, T)}{F_{CSA}(t, T)} = \sigma_S \Delta W_S(t)$$

with $W_S(t)$ being a Brownian motion under the risk-neutral measure P.

b. Evolution of the Credit-Risky Numeraire => On the other hand

$$\frac{\Delta P_F(t)}{P_F(t)} = \mathcal{O}(\Delta t) - \sigma_F b(T-t)\Delta W_F(t)$$

where

$$b(T-t) = \frac{1 - e^{-\aleph_F(T-t)}}{\aleph_F}$$

c. Evolution of the Funding Spread => As $M(t,T)$ is a martingale under P (since $r_C(t)$ is deterministic, the measures P and P_T coincide) one has from

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

that

$$\frac{\Delta M(t,T)}{M(t,T)} = -\sigma_F b(T-t)\Delta W_F(t)$$

d. Evolution of the Convexity Adjustment => Both $M(t,T)$ and $F_{CSA}(t,T)$ are martingales under P so it follows that

$$\frac{\Delta[M(t,T)F_{CSA}(t,T)]}{M(t,T)F_{CSA}(t,T)} = \mathcal{O}(\Delta t) + \rho\sigma_S\sigma_F b(T-t)\Delta W_F(t)$$

Using the fact that

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = \mathbb{E}\left[\frac{M(T,T)}{M(0,T)}\{F_{CSA}(T,T) - F_{CSA}(0,T)\}\right]$$

one gets

$$F_{NoCSA}(0,T) = F_{CSA}(0,T)e^{-\int_0^T \rho\sigma_S\sigma_F b(T-t)dt} = F_{CSA}(0,T)e^{-\rho\sigma_S\sigma_F\frac{T-b(t)}{\aleph_F}}$$

and in the case

$$\aleph_F = 0$$

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = F_{CSA}(0,T)\left[e^{-\frac{1}{2}\rho\sigma_S\sigma_F T^2} - 1\right]$$

e. Tenor Dependence of Convexity Correction => Note that the adjustment grows roughly as T^2. A similar formula was obtained by Barden (2009) using a model in which the funding spread was functionally linked to the value of the asset.

Cross Currency Model

1. LCH.ClearNet Collateral Rules: Single currency trades (currently mostly swaps) are collateralized in their own currencies, but multi-currency trades (e.g., cross currency swaps) are typically collateralized in USD.
2. Building Blocks: The building blocks typically are a) Domestic-Currency Collateralized Domestic Zero Coupon Bonds, b) Foreign-Currency Collateralized Foreign Zero Coupon Bonds, c) Collateralized FX Contracts. In practice, the former (the collateralized zeros) may not trade, whereas collateralized FX contracts typically do.
3. Foreign Bonds Collateralized in Domestic Currency: Consider a foreign zero-coupon bond collateralized with domestic collateral. The price of this zero coupon bond in foreign currency is $P_{f,d}(t)$. If $X(t)$ is the forex rate (i.e., the number of domestic units per foreign unit), the collateral account cash flow growth is

$$\Delta\chi_i(t) = \Delta\left[P_{f,d}(t)X(t)\right] - c_d(t)\left[P_{f,d}(t)X(t)\right]\Delta t$$

where $c_d(t)$ is the domestic collateral rate.

4. Collateralization of the FX: If $r_{d,f}(t)$ is the rate agreed on a domestic loan collateralized by foreign collateral, then the FX collateral account cash flow growth is

$$\Delta\chi_X(t) = \Delta X(t) - X(t)\Delta t$$

The contention by Piterbarg (2012) is that there is no relation between the collateralization rates $r_{d,f}(t)$, $c_d(t)$, and $c_f(t)$.

5. Collateralization Using Domestic Collateral:

$$\begin{bmatrix} \dfrac{\Delta X(t)}{X(t)} \\ \dfrac{\Delta P_{d,d}(t)}{P_{d,d}(t)} \\ \dfrac{\Delta P_{f,d}(t)}{P_{f,d}(t)} \end{bmatrix} = \begin{bmatrix} r_{f,d}(t) \\ c_d(t) \\ c_f(t) \end{bmatrix}\Delta t + \sigma_d \Delta W_d$$

Thus, under the domestic collateralization risk-neutral measure Q_d we have the following:

$$X(t) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T r_{d,f}(s)ds} X(T)\right]$$

$$P_{d,d}(t,T) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\right]$$

$$P_{f,d}(t,T) = \frac{1}{X(t)}\mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds} X(T)\right]$$

6. Collateralization Using Foreign Collateral:

$$\begin{bmatrix} \dfrac{\Delta\left[\dfrac{1}{X(t)}\right]}{\left[\dfrac{1}{X(t)}\right]} \\ \dfrac{\Delta P_{f,f}(t)}{P_{f,f}(t)} \\ \dfrac{\Delta P_{d,f}(t)}{P_{d,f}(t)} \end{bmatrix} = \begin{bmatrix} -r_{d,f}(t) \\ c_f(t) \\ c_d(t) \end{bmatrix} \Delta t + \sigma_f \Delta W_f$$

Thus, under the foreign collateralization risk-neutral measure Q_f we have the following:

$$\frac{1}{X(t)} = \mathbb{E}_t^{Q_f}\left[e^{\int_t^T r_{d,f}(s)ds} \frac{1}{X(T)}\right]$$

$$P_{f,f}(t,T) = \mathbb{E}_t^{Q_f}\left[e^{-\int_t^T c_f(s)ds}\right]$$

$$P_{d,f}(t,T) = X(t)\mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds} \frac{1}{X(T)}\right]$$

7. $P_{d,f}(t,T)$ and $P_{f,d}(t,T)$ Numeraires: These are effectively the cross-currency, oppositely collateralized numeraires, i.e., one unit of domestic/foreign currency collateralized using the corresponding foreign/domestic collateral. Thus these numeraires, as such, can form the basis for cross-currency discount curves employed in cross-currency swaps. Further, while these building blocks are primarily only discounting oriented – securities with forward/floater leg may also require a quanto adjustment to be applied.
8. Cross Currency Model Parameters: All the model parameters and the process dynamical parameters in the set of equations above can be independently observed.
9. "Implied" Cross Currency Risk Free Rate: The measure change from Q_d to Q_f under the Q_d measure is captured by the Q_d martingale

$$M(t)=\frac{\partial Q_f}{\partial Q_d}=e^{-\int_t^T r_{d,f}(s)ds}\frac{X(t)}{X(0)}$$

Thus, the corresponding growth rate $r_{d,f}(t)$ also helps clarify the references to the "cross-currency risk-free Rates" (e.g., Fujii and Takahashi (2011a, 2011b)) – viz., they are instantaneous FX collateralization rate using the foreign collateral.

10. Forward Forex Contract Collateralized with Domestic Collateral: This contract pays

$$X(T)-K$$

in the domestic currency, and is collateralized using the domestic collateral. Thus

$$\mathbb{E}_t^{Q_{f,d}}[X(T)-K]=X(t)P_{f,d}(t,T)-KP_{d,d}(t,T)$$

Therefore, the par strike K for this contract is

$$K=\frac{X(t)P_{f,d}(t,T)}{P_{d,d}(t,T)}$$

11. Forward Forex Contract Collateralized with Foreign Collateral: This contract pays $1-\frac{K}{X(T)}$ in the foreign currency, and is collateralized using the foreign collateral. Thus

$$\mathbb{E}_t^{Q_f}\left[1-\frac{K}{X(T)}\right]=P_{f,f}(t,T)-K\frac{P_{d,f}(t,T)}{X(T)}$$

Therefore, the par strike K for this contract is

$$K=\frac{X(T)P_{f,f}(t,T)}{P_{d,f}(t,T)}$$

12. Same Currency Collateralization:

$$V_{d,d}(t,T) = \mathbb{E}_t^{Q_{d,d}{}^T}[X(T) - K] = P_{d,d}(t,T)[X(T) - K]$$

and

$$V_{f,f}(t,T) = \mathbb{E}_t^{Q_{f,f}{}^T}\left[1 - \frac{K}{X(T)}\right] = \left[1 - \frac{K}{X(T)}\right]P_{f,f}(t,T)$$

No rocket science, really, with simple forwards. Question, however, is that whether $X(T)$ would ever be domestically collateralized, and that whether $\frac{1}{X(T)}$ would ever be collateralized in foreign currency. Same currency collateralization is uncommon presumably for these reasons.

13. Market Quotes for Collateralized Forex Forwards: Strictly speaking, all Forex Forwards should always be collateralized using either foreign or domestic collateral. Thus, the Forward Prices should be different depending on the collateralization currency. However, this DOES NOT appear to be the market practice, as the quotes are independent of collateral.

Collateral Choice Model

1. Setup: Here, an American style path-dependent collateral is chosen at every incremental step by opting for the collateral among the choices available that maximizes the incremental collateral cash flow.
2. Motivation: Collateralization at the domestic collateral accrual rate is c_d. On switching over to the foreign collateral, the rate becomes $c_f + r_{d,f}$. Thus at each time step we want to maximize the incremental collateral cash flow

$$\max(c_d, c_f + r_{d,f}) = c_d + \max(c_f + r_{d,f} - c_d, 0)$$

We begin by setting

$$q_{d,f} = c_f + r_{d,f} - c_d$$

3. Dynamics of $Q_{d,f}$: Consider the dynamics of

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

This entity has a drift

$$q_{d,f} = c_f + r_{d,f} - c_d$$

First of all, the dynamics of $c_d(t)$, $r_{d,f}(t)$, and $c_f(t)$ may be worked out using one of several typically accepted practices – e.g., the HJM-type dynamics, or an even more simplified Hull-White type dynamics.

- Using

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

  it is fairly straightforward to show that

$$\frac{\Delta Q_{d,f}}{Q_{d,f}} = q_{d,f}(t)\Delta t + \sigma_q(t)\Delta W_q$$

  where

$$\sigma_q(t)\Delta W_q = \sigma_f(t)\Delta W_f + \sigma_X(t)\Delta W_X - \sigma_d(t)\Delta W_d$$

4. Piterbarg (2012) Expression for $q_{d,f}(t)$: Piterbarg (2012) employs a combination of HJM machinery as listed above and additional techniques outlined in Andersen and Piterbarg (2010) to obtain $q_{d,f}(t)$.
5. Collateral Choice - Deterministic $q_{d,f}(t)$: If $q_{d,f}(t)$ is deterministic, there will be no optionality involved; however, depending upon the sign of $q_{d,f}(t)$, there will be a collateral switch at each time increment. Piterbarg (2012) demonstrates this in his framework by turning the volatility explicitly down to zero.
6. Deterministic and Incremental Curve Decay Collateral: If the collateral discounting path choice can be proxied using a "curve roll up" phenomenon, the collateral choice discount factor becomes

$$P_{d,CC}(t_0, t_n) = \prod_{i=1}^{n} \min\left(\{P_{d,j}(t_{i-1}, t_i)\}_{j=1}^{r}\right)$$

where

$$j = 1, \cdots, r$$

are the r possible collateral choices, $j = 0$ is the domestic collateral curve, $P_{d,j}(t_{i-1}, t_i)$ is the discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the foreign collateral j, and $P_{d,CC}(t_{i-1}, t_i)$ is the collateral choice discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the most appropriate incremental collateral. Note that this discount curve is artificial and deterministic.

- Advantages of using deterministic collateral choices => All the advantages stem from the computational simplicity. They are:
    - More than one collateral currency may be used, thus optimizing over the multiple collateral choices (USD, GBP, EUR, JPY, etc.)
    - Empirical Curve Representations using splining techniques may be usable

7. Valuing the Collateral Choice Option: The value we seek is of the form

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max\left(q_{d,f}(s),0\right)ds}V(T)\right]$$

where $V(T)$ is the terminal payoff at the time instant T. It may be a fixed amount (i.e., the fixed swap rate) or a variable amount (the floating swap coupon).

- Closed Form => Typically

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max\left(q_{d,f}(s),0\right)ds}\right]$$

has to be computed using Monte-Carlo or a PDE, therefore we seek an alternative fast analytic approximation. By Jensen's inequality, Piterbarg (2012) noticed that

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max\left(q_{d,f}(s),0\right)ds}\right] \geq e^{-\int_t^T \max\left(q_{d,f}(s),0\right)ds}$$

This approximation may be used to compute the fixed leg value for the swap above. For the floater leg, the term $V(T)$ may be pushed outside to a separated expectation to get

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}[V(T)]\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max\left(q_{d,f}(s),0\right)ds}\right]$$

Piterbarg (2012) performs a full set of comparison to demonstrate that these approximations behave favorably with the Monte-Carlo under several situations.

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Cross Asset Random Number Generator

Introduction

1. Centralized Random Number Generation Methodology: This chapter describes the methodology and the implementation of the Centralized Random Generator, and its usage in DRIP's XVA framework.
2. Market Risk Factor Random Numbers: The Central Random Generator is the central price for establishing a cross-asset framework in DRIP's XVA framework. In the market generation step of the XVA framework, each market risk factor is simulated using the centralized random numbers.

Centralized Random Number Generator

1. Centralized Random Number Generation Module: The centralized random number generation module is responsible for producing the incremental shocks of major market risk factors between consecutive CVA simulation dates, taking into account the correlations between these risk factors.
2. Factor/Time/Path Random Increments: Each factor includes a sufficient number of time steps, paths, and components if needed (principal components or multiple factors). The module outputs standard normal random numbers to represent the shocks (increments) at each time step for each path and each component.

3. Cross Sectional Factor Correlations Incorporation: For each individual risk factor, random factors are mutually independent across time steps, paths, and components. For distinct market risk factors, the random numbers incorporate cross-sectional correlations between them.

Data Structures

1. The Random Sequence Component ID: For any risk factor, each random number produced is labeled by a component ID (tenor ID), a time step index, and a path index. These components are typically strings, such as "PC1" (the first principal component), or "3M" (the 3M tenor).
2. The Random Sequence Date Index: A time step index is an ordinal number to denote the order of the CVA dates, starting from 0. The CVA dates are also attached to the random number matrix.
3. The Random Sequence Path Index: The path index is also an ordinal number used to label the paths in the simulation, starting from 0.

Factor Model for Correlation Handling

1. Two-Tier Risk Factor Model: To model the correlation structure of the risk factors, a two-tier factor-risk model is used. The risk factors are classified into two groups – the primary (driving) market risk factors and the secondary risk factors – which are determined by their importance in terms of their market risk impact as well as the bank's exposure.

2. Modeling Primary/Secondary Risk Factors: The correlation between the primary market risk factors are modeled using a full correlation matrix. The secondary market risk factors are modeled as dependent factors on the primary market risk factor so that their correlations are derived from the latter.

Variance Reduction

1. Generating the Antithetic Variable Sequence: The random number generator is capable of sampling the antithetic variable sequence. It is a common practice to use antithetic variables in Monte Carlo as a variance reduction technique. In the output of random numbers with antithetic variables, the $(2p+1)^{th}$ path is the negative of the $(2p)^{th}$ path

$$p = 0, 1, 2, \cdots$$

2. Variance Reduction with Quadratic Resampling: In some instances, it is desired that the standard deviation of the generated random numbers to be exactly equal to one, to reduce the Monte Carlo errors. This can be achieved by quadratic resampling.
3. Quadratic Resampling - The Two Steps: For a sequence of random numbers $r_0, r_1, \cdots, r_{T-1}$ the quadratic resampling follows the two step procedure outlined below.
4. Sub-dividing the sequence into Blocks: First, the random numbers sequence is divide into blocks of size N where N is typically large, but no greater than the length of the random number sequence T. If T is not a multiple of N, then the last block will be incomplete, i.e., it contains less than N numbers, and it will be excluded from the resampling.
5. Mean-Centering across the Block: Without loss of generality, it is assumed that the sequence is generated with antithetic sampling, i.e.,

$$r_{2p+1} = -r_{2p}$$

and N is even, so that for each complete block, the average is zero. Otherwise a mean-centering adjustment can be made to make it zero.

6. Variance Scaling inside the Block: Then inside each complete block $r_{iN}, r_{iN+1}, \cdots, r_{(i+1)N-1}$

$$0 \le i \le \left[\frac{T}{N}\right] - 1$$

let

$$s_i = \sqrt{\frac{1}{N}\sum_{j=0}^{N-1} {r_{iN+j}}^2}$$

denote the standard deviation.

7. Unbiased Estimate of the Standard Deviation: Another definition of the standard deviation is

$$s_i = \sqrt{\frac{1}{N-1}\sum_{j=0}^{N-1} {r_{iN+j}}^2}$$

These two definitions yield little difference when N is large.

8. The Quadratically Resampled Random Sequence: The quadratically resampled random numbers then are

$$\tilde{r}_{iN+j} = \frac{r_{iN+j}}{s_i}$$

$$j = 0, \cdots, N-1$$

It is straightforward to verify that taking a sequence of length $\{\tilde{r}_0, \cdots, \tilde{r}_{kN-1}\}$, it standard deviation is 1, where k is a positive integer.

Implementation – The Scope of the Risk Factors

1. The Primary Market Risk Factors: The risk factors considered in the random number generator are aligned with the asset classes, including the bench-mark interest rates (forward curves), foreign exchange rates, equity indexes and stock prices, commodity futures prices, CDS indexes, inflation rates, etc. and they are consistent with the risk drivers in typical productions of risk reports in each asset class.
2. Pre-generation of Random Numbers: To hasten the CVA generation, given the large number of risk factors, the random numbers for the risk factors are pre-generated before the CVA market generation step is initiated. I particular, random numbers can be pre-generated and stored in a system infrastructure and can be accessed upon request.
3. On-Demand Random Number Generation: For those risk factors that do not have these pre-generated random numbers, random numbers can be generated at the run time when the CVA market generation is initiated.
4. Rates Primary Market Risk Factor: Forward rate curves for the major currencies are considered the risk factors, which includes the ones listed below.
5. 3M LIBOR Forward Rate States: Forward rate curves for 3M LIBOR for USD, EUR, GBP, JPY, and CHF are the primary risk factors.
6. xM LIBOR Forward Rate States: Forward rate curves for LIBOR at other tenors.
7. Non-G5 DM IBOR Forwards: Forward rate curves for other major currencies that are not listed in LIBOR (such as AUD, CAD, etc.)
8. FX Primary Market Risk Factor: The spot FX rate for G10 currencies EUR, JPY, GBP, CHF, AUD, NZD, CAD, SEK, and NOK. For example, in Calypso, the foreign exchange rates all

use USD as the base currency. In addition, other currencies that are available in Calypso are included, including CNY, DKK, HKD, etc.

9. Equity Primary Market Risk Factor: The risk factors in the equity space is primarily S&P 500, since over 80% of the gross equity deltas for typical equity derivatives books is in .SPX.
10. Equity Secondary Market Risk Factor: Other equity indices and individual stocks, such as AAPL, BRK.B, MET etc. are assigned as secondary (dependent) risk factors.
11. Commodity Primary Market Risk Factor: The risk factors are commodities futures curves such as WTI, BRT, and CGOLD. Each commodities futures type will have at least one futures curve being assigned as the primary risk factor – the rest can be assigned as secondary risk factors. For important basis curves (e.g., natural gas at major locations), they will also be assigned as the primary risk factor.
12. Credit Primary Market Risk Factor: Credit default swap indexes, such as CDX.NA.IG, CDX.NA.HY, iTRAXX Europe, etc. are used as primary risk factors.
13. Inflation Primary Market Risk Factor: CPI-U (consumer price index for all urban consumers) is included as the primary market factor for inflation risk. CPI-U is the reference index for TIPS and other inflation swaps.

Implementation – The Correlation Matrix

1. Correlations between Primary Market Factors: The correlation coefficients between the primary market risk factors are obtained from multiple sources as shown in the table below, e.g., the correlations between rates and FX spot are obtained from Calypso.
2. Source of DRIP Correlation Coefficients:

	Rates	FX	Commodity	Equity	Credit	Inflation
Rates	Calypso	Calypso	Historical	Historical	Historical	Historical

FX	Calypso	Calypso	Historical	Historical	Historical	Historical
Commodity	Historical	Historical	Perimeter Historical	Historical	Historical	Historical
Equity	Historical	Historical	Historical	Historical	Historical	Historical
Credit	Historical	Historical	Historical	Historical	Historical	Historical
Inflation	Historical	Historical	Historical	Historical	Historical	Historical

3. Correlations across Major Commodities Futures: Correlations between the prompt contracts of the major commodities futures are obtained from Perimeter (market implied correlation from European basket options on commodity pairs), while all other correlation coefficients are estimated from historical data.
4. Log-Normal Historical Returns Correlation: Pearson correlation with multiple years of historical data based on lognormal returns are computed. Correlation matrix can typically be updated on a monthly basis (or as needed).
5. Validity of Pooled Correlation Matrices: As some of the correlation coefficients are obtained from Calypso/Perimeter, and others are calculated from historical data, when they are pooled, the validity of the correlation matrices (e.g., positive definiteness, as well as numerical singularity) needs to be checked.
6. Conversion to Nearest Correlation Matrix: To handle numerical singularity, Jackel's approach is used; it finds a similar matrix that is positive definite (replacing non-positive eigenvalues with a small positive value).
7. Centralized Random Number Generation Customization:

Parameter	**Description**
OutputRequest	Outputs desired

Name	CRNG Name
ScenarioName	Scenario Name
TimeSteps	Array of Time Steps
NumPaths	Number of Paths
useQuadResample	Whether Quadratic resampling is used
isAntithetic	Whether Antithetic Paths are Simulated
PrimaryFactors	Primary Market Factors
Correlation	Correlation between the Primary Market Factors
SecondaryFactors	
rfRequestedFactors	Risk Factors to Generate Random Numbers for
Storage	Storage Type for the Output
Seed	Seed for the Random Number Generator
URI	URI for the Storage
BlockSize	Block Size used in Quadratic Resampling
BumpCorrFactors	
BumpCorrAmount	
BumpBetaFactors	
BumpBetaAmount	Bump Size for the Beta Coefficients
MarketDate	Optional, Run Date for the Random Numbers

Testing

1. Characterizing Monte Carlo Sequence Errors: By design the cross-correlations between the time steps are zero, and the cross-correlations between the risk factors are input correlations. Consider the Monte-Carlo error; the actual correlations will only be close to the theoretical values.
2. Risk Factors Cross Correlation Errors: The results for an initial test are contained in the table below. For cross-correlation between the different risk factors, the simulation error is about 1%. For cross-correlation between the steps the simulation error is about 3%.
3. Risk Factor Serial Correlation Errors: The first column is the input correlation coefficients between these risk factors and the USD 3M LIBOR. The second column contains the correlation coefficients of the generated random numbers. Column 3 contains the differences. Column 4 contains the average step-wise correlation for the first 5 steps. The calculation is based on 1,000 paths and 5 time steps.

Risk Factor	Input Correlation with USD_LIBOR_3M	Simulated Correlation	Difference	Average Cross-Step Correlation
EUR_LIBOR_3M	16.1%	15.8%	-0.3%	-1.8%
JPY_LIBOR_3M	24.5%	26.1%	1.6%	-3.3%
GBP_LIBOR_3M	25.9%	25.7%	-0.2%	-0.8%
CHF_LIBOR_3M	35.2%	36.2%	1.0%	-0.05%
EUR FX	16.6%	15.5%	-1.0%	-0.07%
JPY FX	0.3%	-0.3%	-0.6%	1.5%

GBP FX	11.4%	12.4%	1.0%	0.06%
CHF FX	3.8%	2.8%	-1.0%	-1.6%

4. Historical vs. Model Correlation Comparison: For equities, only one primary risk factor (SPX) is used in the factor model. The table above gives the results for comparing the factor model correlations and the true historical correlations for AAPL and .STOXX50E. As can be seen the differences are not significant.

	Lognormal 1Y	**Lognormal 5Y**	**Normal 1Y**	**Normal 5Y**
Historical Correlation	30.29%	27.48%	30.95%	26.38%
Factor Model Correlation	31.37%	33.76%	31.45%	32.34%
Difference (Factor Minus Historical)	1.09%	6.28%	0.51%	5.96%

Random Number Generators

1. Pseudo-Random Number Sequence Generators: The random numbers used in computer programs are pseudo-random, which means that they are generated in a predictable fashion using algorithms. These algorithms can generally create long runs of numbers with good properties, but eventually the sequence repeats.

2. Seed induced Pseudo-Random Sequence: The series of values generated by such algorithms is generally determined by a fixed number called a seed. If one gives the program the same seed twice, it produces the same sequence of random numbers. As these numbers are by no means "completely random", they are sometimes referred to as pseudo-random numbers.
3. RNG as a Finite State Machine: A random number generator (RNG) can be described as a state machine; each time you ask for a random number, the state changes, so that the random number is different the next time you ask. In fact, since the RNG runs on a computer, which is finite, it must be a finite state machine.
4. Components of the RNG State: Formally, and RNG is defined by three components.
    a. An initial state s_0
    b. A transition function S on the states such that

    $$s_{k+1} = S(s_k)$$

    for all

    $$k \geq 0$$

    c. An output function V on the states such that for all

    $$k \geq 0$$

    $V(s_k)$ is the random number at invocation k of the RNG.
5. Algorithm #1 - Linear Congruential Operator: One of the most common RNG's is the linear congruential generator, which uses recurrence based on modular integer arithmetic

    $$s_{k+1} = (as_k + b)\ mod\ M$$

    With appropriately selected a and b the sequence can achieve the maximum period M.

6. Algorithm #1 - L'Ecuyer's Recursive Generator: By combining multiple recursive sequences, a generator can have a large state space with good randomness properties, such as the L'Ecuyer's multiple recursive generator MRG32k3a

$$y_{1,n} = \left(a_{12}y_{1,n-2} + a_{13}y_{1,n-3}\right) \, mod \, m_1$$

$$y_{2,n} = \left(a_{21}y_{2,n-1} + a_{23}y_{2,n-3}\right) \, mod \, m_2$$

$$s_n = y_{1,n} + y_{2,n}$$

for all

$$n \geq 3$$

where

$$a_{12} = 1403580$$

$$a_{13} = -810728$$

$$m_1 = 2^{32} - 209$$

$$a_{21} = 527612$$

$$a_{23} = -1370589$$

$$m_2 = 2^{32} - 22853$$

This generator has a period length of approximately 2^{191} ($\approx 10^{57}$)

7. Algorithm #2 - Shift Register Generator: Another important class of RNG is the shift register generator which takes the form

$$x_{n+k} = \sum_{i=0}^{k-1} a_i x_{n+i} (mod\ 2)$$

where the x_n's and the a_i's are either 0 or 1.

8. Algorithm #2 - The Mersenne Twister: The maximum period of $2^k - 1$ can be achieved using as few as two non-zero values of a_i. This leads to a very fast RNG. The famous Mersenne Twister (MT19937) belongs to this class.

Multi-Stream RNG's

1. Eliminating the Seed-Path Overlap: In Monte Carlo multiple independent random number sequences are usually required. A naïve way of getting multiple independent random number seeds is by setting different seeds. It is simple to implement, but there is no guarantee that truly independent sequences are obtained, as multiple sequences may overlap.
2. Multi-Stream Seed Path Overlap: An ideal solution is to use multi-stream random number generator. This is achieved by splitting the random number sequence into multiple, uncorrelated streams.
3. Seed Path Overlap Elimination Schemes: The most commonly used methods are the skip-ahead method and the leap-frog method. Suppose that the threads are numbered from 0 to $N - 1$ and the elements in the original sequence are numbered from 0 to $T - 1$.
4. Simple Skip Ahead Overlap Elimination: Start thread t

$$0 \leq t \leq N - 1$$

at spot $T \times \frac{t}{N}$ in the sequence, and let each thread step through its length $\left[\frac{T}{N}\right]$ sub-sequence.

5. Leapfrog Path Overlap Elimination: Start thread t at spot t in the sub-sequence, and let each thread skip N elements at a time in the sequence.
6. Hybrid Seed Path Overlap Elimination: A skip-ahead is performed at the thread block level, and within one block each thread does a leapfrog generation.
7. Parallelizability of the Multi-Stream Generators: The key feature of the multi-stream RNG's is that each stream can be generated independently of the others. This makes them CUDA friendly and ready to be used in parallel algorithms.

Core CVA/DVA Model

Abstract and Synopsys

1. Credit Risky CVA/DVA Adjustment: This chapter provides a general methodology for computing the Credit Valuation Adjustment and the Debt Valuation Adjustment for credit risk. It is assumed that either party to the contract can default and that the counter party defaults are independent of one another and the interest rate moves.
2. HW Monte Carlo Rates Models: In general the valuation is through the Hull White for the interest rate movements for the interest rate products. In certain circumstances analytical calculations can be used to improve the speed and the accuracy of the calculation. For cross-currency swap portfolios, Monte Carlo based simulation model is presented.

Introduction

1. CVA vs. Counter Party Risk: CVA (Credit valuation adjustment) is defined as the adjustment to the risk free portfolio value due to the counter party risk. Assuming that the counter party is default free, CVA is the same as the counter party credit risk defined in the literature.
2. The Bank Debt Value Adjustment: In reality the default probability of the bank is non-zero. On the other hand, the possibility of the bank's default leads to another adjustment to the risk free portfolio value. This is called the Debt Valuation Adjustment (DVA).

3. Combined CVA and DVA Adjustments: Viewed from the counter party, this is their CVA due to the bank's default. The total adjustment then is the sum of the CVA and the DVA.
4. Valuation Adjustment Mathematical Framework: The chapter is organized as follows. The next section addresses the general CVA/DVA valuation in a mathematical framework.
5. Independence between Rates and Credit: It is shown that under the assumption of independence between the defaults and the interest rate movements, the valuation of CVA/DVA can be separated into an interest rate part and a credit part.
6. Default Probabilities and Recovery Rates: Default probabilities are inferred from hazard curves. The recovery rate map is used when the recovery rate depends on the exposure, and is discussed in the corresponding section.
7. CVA/DVA for IR Portfolios: The next two sections discuss the details of the computation of the CVA/DVA for portfolios of interest rate products and portfolio specific calibrations.
8. CVA/DVA from Components/Portfolios: Single fixed-for-floating swap CVA/DVA can be valued as a linear combination of swaptions weighted by default probabilities. For portfolios of swaps and option underlyings the Hull White model is utilized to compute the interest rate risk.
9. Impact of the Posted Collateral: Under the bilateral margin agreement, both the bank and the counter party need to post collateral if the uncollateralized amount exceeds a certain threshold. The impact of the margin agreements and the margin periods of risk on the CVA/DVA valuation is discussed in their own section.
10. CVA/DVA for Muni FPA: A separate section presents an approximation method for capturing the CVA/DVA of the Muni FPA product by constructing an equivalent LIBOR swap for the FPA pricing.
11. Cross Currency Swap CVA/DVA: For cross-currency swap portfolios, since there are more risk factors to model, a Monte Carlo based simulation to model the interest rates in each currency and the FX dynamics has been built, and is discussed in the penultimate section. The final section concludes this chapter.

General Framework for CVA/DVA Pricing

1. Unilateral Counter Party Credit Risk: CVA/DVA model can be viewed as an extension to the counter party credit risk. Counter party credit risk valuation assumes that the bank is default free, and this leads to 0 DVA value, and the counter party credit risk is equivalent to the CVA.
2. Bilateral Counter Party Credit Risk: To properly adjust portfolio values, it is necessary to take the bank's credit condition into account. Therefore it needs to be assumed that the probability of bank's default is non-zero.
3. Separation of Rates/Credit Components: However, it is assumed that there is no correlation between defaults and interest rate movements. Such an assumption essentially separates the credit risk from the interest rate risk, and allows avoiding the simulation of both parts.
4. General Description of the Framework: The general description of the problem is as follows. Let V denote the default free contract value and τ the default time.
5. Default of the Counter Party: At default there are two scenarios. The first is that the counter party defaults. If the contract value is positive to the bank, a loss equal to $(1 - R_{CPTY})V^+$ is incurred, where R_{CPTY} is the counter party's recovery value.
6. Default of the Bank Entity: The second scenario is that the bank defaults. If the remaining contract value is negative, the counter party receives $R_{BANK}V^-$
7. Adjustment to the Default Free Value: Putting all these possibilities together, the expression for the adjustment of the default-free contract value when the default occurs between t and $t + \Delta t$ is given by

$$\begin{aligned} & Adjustment(t \leq \tau < t + \Delta t) \\ & \qquad = [(1 - R_{CPTY})V^+] \cdot \mathrm{I}_{t \leq \tau_{CPTY} < t + \Delta t} - [(1 - R_{BANK})V^-] \cdot \mathrm{I}_{t \leq \tau_{BANK} < t + \Delta t} \\ & \qquad = CVA(t \leq \tau < t + \Delta t) + DVA(t \leq \tau < t + \Delta t) \end{aligned}$$

$$V^+ = \max(0, V)$$

$$V^- = \max(0, -V)$$

where I_X is the indicator function that takes 1 when condition X is met and 0 otherwise.

8. <u>CVA and DVA Component Terms</u>: The two terms above are the CVA and the DVA. This formulation can be compared to the corresponding one in Duffie and Huang (1996) treatment on swap rate with credit quality adjustment.
9. <u>Total Bank/Counter Party Adjustment</u>: After taking expectation of the above expression and integrating over all possible default times, the total adjustment is

$$CVA + DVA = \int_0^T \left\{(1 - R_{CPTY})\mathbb{E}[V^+] \cdot \frac{\partial F_{CPTY}(t)}{\partial u}\right\} - \int_0^T \left\{(1 - R_{BANK})\mathbb{E}[V^-] \cdot \frac{\partial F_{BANK}(t)}{\partial u}\right\}$$

where

$$F_{CPTY}(u) = \mathbb{P}(\tau_{CPTY} \le u)$$

is the marginal distribution of the counter party's defaults, and F_{BANK} is the bank's.

10. <u>The Expected Positive/Negative Exposures</u>: The expectation $\mathbb{E}[V^+]$ is short-hand for

$$\mathbb{E}[V^\pm] = \mathcal{N}(0)\mathbb{E}_{\mathcal{N}}\left[\frac{V^\pm}{\mathcal{N}(\tau)}\right]$$

where $\mathcal{N}(\cdot)$ is the appropriate numeraire. For example, when the underlying is a swap, the swap measure with the numeraire $DV01(T)$ is the most convenient. In general the measure is used by the model used to determine the value of the option on the underlying portfolio.

11. <u>Discretization of the CVA/DVA Integrals</u>: The above integral can be approximated by a discrete sum:

$$CVA + DVA \approx \sum_n \{(1 - R_{CPTY}) \cdot \mathbb{E}[V^+(t_n)]\} \cdot \mathbb{P}(t_n \leq \tau_{CPTY} < t_n + \Delta t_n) - \sum_n \{(1 - R_{BANK}) \cdot \mathbb{E}[V^-(t_n)]\} \cdot \mathbb{P}(t_n \leq \tau_{BANK} < t_n + \Delta t_n)$$

as was done with the counter party credit risk evaluation. The expectations $\mathbb{E}[V^{\pm}]$ are just call/put options on the underlying portfolio values. Their valuation will be discussed later.

12. The Incremental Probability of Default: The distribution of defaults can be inferred from a hazard curve easily. The "discount factors" calculated from the hazard curves are just the survival probabilities. Thus the probability of default between t_n and $t_n + \Delta t_n$ is given by

$$\mathbb{P}(t_n \leq \tau_{BANK} < t_n + \Delta t_n) = DF_{Hazard}(t_n) - DF_{Hazard}(t_n + \Delta t_n)$$

13. Calls/Puts on the Underlying Portfolio: Calculation of the probabilities using the equation above is straightforward. The difficult part in the CVA/DVA valuation is the calculation of the call/put option values on the underlying portfolio.
14. Valuation of Forward Basket Options: As explained in the counter party credit valuation literature, this step is equivalent to valuing forward basket options, which requires the forward volatilities of the term structure. Generally this is model dependent.
15. Hull White Basket Evolution Model: For CVA/DVA on a portfolio of instruments, the Hull White model is utilized to value options on a basket of options. The details are presented in the next two sections.
16. Single Swap CVA/DVA Estimation: The next section discusses CVA/DVA on a single swap. This is the case where the credit risk is directly proportional to the swaption value, and therefore can be prices by a volatility surface directly.

Recovery Rate Map

1. Threshold Dependent Recovery Rate Map: For the recovery rate the user can also use a recovery rate map – a two column array in which the first column showing the threshold above which a recovery rate – shown in the second column – is going to be realized when the counter party/bank defaults.
2. Piecewise Constant Recovery Map:

$Threshold1$	$RecoveryRate1$
$Threshold2$	$RecoveryRate2$
$Threshold3$	$RecoveryRate3$

The first column should be increasing in the order of threshold with

$$Threshold1 = 0.0$$

The recovery rate in the second column should be between 0.0 and 1.0 For the portion of the exposure between $Threshold1$ and $Threshold2$ $RecoveryRate1$ is used, and for the portion of the exposure between $Threshold2$ and $Threshold3$ the $RecoveryRate2$ is used, and so on. For any portion of the exposure above the last threshold, the last recovery rate will be used.

3. MC Grid Based CVA/DVA: If a recovery rate map is available, the grid or the MC simulation is used to compute the CVA/DVA.

CVA/DVA Model for Interest Rates Products

1. Aggregation across the Basket Components: When there is more than one instrument in the portfolio, the expectation becomes

$$\mathbb{E}\left[\left(\sum V_i\right)^{\pm}\right] = \mathbb{E}\left[\mathbb{E}\left[\left(\sum V_i\right)^{\pm} |\mathcal{F}_\tau\right]\right]$$

   Instrument values are aggregated first before their positive/negative values are taken. The valuation of loss given default then becomes a basket option.

2. Forward Starting Options and Swaps: When there are only swaps in the portfolio, it is an option on forward swap cash flows. But when there are swaptions/caps/floors, it becomes an option on forward starting options. In this section, these issues are addressed in the setting of a Hull White model.
3. CVA/DVA for Swaps/Options: The next section discusses the simplest case for an interest rate product, a single fixed-for-floating swap. The remaining cases discuss the CVA/DVA for a portfolio of interest rate products in general.

CVA/DVA Valuation of a Single Swap

1. Expected Exposure for Single Swap: As seen earlier the expectation $\mathbb{E}[V^{\pm}]$ is shorthand for

$$\mathbb{E}[V^{\pm}] = \mathcal{N}(0)\mathbb{E}\left[\mathbb{E}\left[\frac{V^{\pm}}{\mathcal{N}} |\mathcal{F}_\tau\right] |\mathcal{F}_0\right]$$

For a swap, this becomes

$$V^{\pm} = |S(\tau) - K|^{\pm} DV01(\tau)$$

where $S(\tau)$ is the swap rate and K the fixed rate, assuming the swap is a payer swap.

2. Application of the Tower Rule: The tower rule of iterated expectation then shows that

$$\mathbb{E}[V^{\pm}] = DV01(\tau)\mathbb{E}[\{S(\tau) - K\}^{\pm}|\mathcal{F}_0]$$

which is the swaption value.

3. Series of Forward Starting Options: Then the expression

$$CVA + DVA \approx \sum_n \{(1 - R_{CPTY}) \cdot \mathbb{E}[V^{+}(t_n)]\} \cdot \mathbb{P}(t_n \leq \tau_{CPTY} < t_n + \Delta t_n) - \sum_n \{(1 - R_{BANK}) \cdot \mathbb{E}[V^{-}(t_n)]\} \cdot \mathbb{P}(t_n \leq \tau_{BANK} < t_n + \Delta t_n)$$

becomes a weighted sum of the swaption values

$$CVA + DVA \approx \sum_n (1 - R_{CPTY}) \cdot DV01 \cdot \mathbb{E}[\{S(\tau) - K\}^{+}] \cdot \mathbb{P}(t_n \leq \tau_{CPTY} < t_n + \Delta t_n) - \sum_n (1 - R_{BANK}) \cdot DV01 \cdot \mathbb{E}[\{S(\tau) - K\}^{-}] \cdot \mathbb{P}(t_n \leq \tau_{BANK} < t_n + \Delta t_n)$$

4. Expiry of the Underlying Swaption: The swaption expiries are given by t_n. Swaptions can be priced by volatility surfaces directly by DRIP's generic swaption pricer. The pricer covers vanilla swaps and non-standard swaps (amortizing, in-arrears, compounding, and averaging).

5. Equivalent Cash Flow Based Pricer: To make the computation more efficient, the CVA/DVA pricer calls a version of the swaption pricer based on equivalent cash flows. The time between the valuation date and the last cash flows date of the swap is discretized first in order to apply the expression for $CVA + DVA$ above. The dates t_n can be specified by the user or they will be a union of swap accrual dates from both legs.

Approximate Computation of the Convexity Based Adjustment for Contracts based on Averaging Index

1. Index Averaging Based Payout: For the purposes of CVA/DVA valuation of contracts based on Averaging Indices such as PRIME 1D index, the following approximation procedure it utilized when computing convexity adjustments. In a typical situation for an averaging index trade, each payment is made on multiple rates – for example the PRIME 1D index is averaged daily and paid monthly.
2. Numeraire Mismatch Induced Convexity Correction: Each one of these rates has a small convexity correction, which is a smooth function of the time difference between reset rate pf the index and the payment date.
3. Convexity Correction causes Computational Degradation: Exact computation of the convexity correction is relatively slow, leading to significant decrease in performance when pricing trades that are based on averaging indices. Though important for valuation purposes, convexity correction can be computed approximately in the case of CVA/DVA valuation.
4. Approximation Schemes for Convexity Correction: One of the simplest approximations is based on computing the convexity correction for only one reset rate in the middle of the period and then applying this value to all the reset rates for the current payment period. In the case where some of the rates in the current payment period have been reset already, the convexity correction is computed for the reset date in the middle of the period.

5. Performance Gain from the Approximation: From a practical point of view this approach provides a significant performance improvement and does not lead to noticeable CVA/DVA differences. The method is used only when computing CVA/DVA and is not used for regular NPV computations.

Hull White Model Calibration

1. Hull White Global Volatility Calibration: For the purpose of CVA/DVA valuation, the Hull White model with global calibration is used. The global calibration of the Hull White model calibrates the Hull White volatility to the standard ATM swaptions constructed with the expiries and the tenors from the volatility surface.
2. Hull White Portfolio Volatility Calibration: The portfolio specific calibration, which is consistent with the underlying securities in the portfolio, is to calibrate the Hull White volatility to the basket option on specific portfolio, which is described a few sections below in this chapter.
3. Challenges with the Portfolio Volatility Calibration: In general, globally calibrated Hull-White should be used for all the portfolios. Portfolio-specific calibration is not recommended, since there will be cases where the portfolio specific calibrations are not stable, and it results in unstable delta and gamma reports (up shifts and down shifts) for the Desk and the Market Risk. A typical case of unstable portfolio calibration is when the portfolios are extremely ITM or OTM.
4. Portfolio Specific Volatility Surface Calibration: Since the portfolio specific calibration is a complex process in general, involving the credit-date selection, basket option pricing, and Hull-White volatility bootstrapping, an entire section is dedicated to that approach.
5. Base vs. Stressed Case Calibration: The Hull White model is calibrated using the given market data – interest rate curves and volatility surfaces – either on the base or the stressed scenario. If the market data is in a base scenario as in daily pricing, the unstressed interest

rate curves and the volatility surfaces are used in the Hull-White calibration; if the data has been stressed, the stressed interest rate curves and the volatility surfaces are used in the Hull-White model calibration.

6. CVA/DVA Hull White Grid: The mean reversion parameter in the Hull White model is not calibrated under either the base or the stressed scenarios. Once the Hull White model has been calibrated in either the base or the stressed scenario, the Hull White grid is built for pricing as described for pricing as described in the DRIP Fixed Income Specification, particularly for pricing CVA/DVA as described in a later section.

CVA/DVA of Portfolio of Swaps, Caps, Floors, and Swaptions

1. Credit Exposure Data Set Generation: First of all, the general set of dates needs to be generated, which would cover all important dates for each instrument in the portfolio. For caps, swaps, and floors, these are just accrual dates; for swaptions notifications dates are also included.
2. Maximum Spacing Data Set Parameter: To improve the accuracy of the calculation additional data sets can be added. This can be done by setting a maximum step size; additional dates are added accordingly making sure that the grid dates are not too far (but not too close either, for efficiency purposes).
3. Minimum Spacing Data Set Parameter: An extra input parameter called the $MinimumSpacing$ is introduced to control the credit dates. If

$$MinimumSpacing > 1$$

the following four credit dates are inserted; $ValDate + 1$, $ValDate + 3$, $ValDate + 7$, and $ValDate + 14$; then those credit dates with differences larger or equal to the $MinimumSpacing$ are retained.

4. Purpose behind the Minimum Spacing Parameter: $MinimumSpacing$ is an important feature to eliminate more credit dates when a portfolio contains a large number of underlyings and generates several credit dates.
5. Recommended Minimum/Maximum Spacing Size: DRIP recommends using 7 or 14 when this feature is turned on to make sure that the $MinimumSpacing$ is less than the step size to avoid eliminating too many important dates. For larger counterparties with more than or equal to 100 underlying trades, global grid dates are recommended.
6. Standardized Global Exposure Grid Tenors: The global grid dates are generated from the valuation date and the following set of tenors: 0D, 1D, 3D, 1W, 2W, 1M, 2M, 3M, 4M, 5M, 6M, 9M, 12M, 15M, 18M, 21M, 24M, 27M, 30M, 33M, 36M, 39M, 42M, 45M, 48M, 51M, 54M, 57M, 5Y, 6Y 7Y, 8Y, 9Y, 10Y, 11Y, 12Y, 13Y, 14Y, 15Y, 16Y, 17Y, 18Y, 19Y, 20Y 21Y, 22Y, 23Y, 24Y, 25Y, 26Y, 27Y, 28Y, 29Y, 30Y, 31Y, 32Y, 33Y, 34Y, 35Y, 36Y, 37Y, 38Y, 39Y, 40Y.
7. Decreasing Density of the Credit Dates: The dates are more dense at the beginning. The step sizes gradually change from daily, weekly, biweekly, monthly, quarterly, and finally annual.
8. Harmonization across the Dealer Trades: Since there are a large number of underlying trades from the dealers, DRIP has configured the pricing environment to use the global dates and to use global Hull White model calibration for the dealers.
9. Global Exposures and Date Construction: Once all the important dates are aggregated, the generic grid based on these dates and the calibrated Hull White model are constructed.
10. Instrument/Date Equivalent Cash Flows: Equivalent cash flows are created for every important date for every instrument in the portfolio to be able to compute the value of every instrument at every node.
11. Positive/Negative Exposure Amount Aggregation: These computations are implemented using an efficient backward induction algorithm preserving the values of every instrument at every grid node. Also, at every node, one can apply $(\cdot)^+$ or $(\cdot)^-$ operators to compute the expected values for the positive and the negative parts for every grid date (time t_n).
12. HW Grid/Direct Surface Comparisons: For caps/floors, NPV's can be calculated from both the volatility surface with the cap pricer directly and from the HW grid. Comparison of the

NPV for each cap/floor causes the introduction of a multiplicative adjuster to adjust the value from the HW grid to match that of the underlying pricer.

13. Applying the Basis Mismatch Multiplier: The ratio of the NPV for a cap pricer to that from the gird is multiplied to the equivalent cash flow sets. The updated equivalent cash flow sets are used for the CVA/DVA calculation. This is similar to the *Out-of-Model-Adjustment* described in Andersen and Piterbarg (2010), in particular on the section on *Adjusters*.
14. Physically Exercised Bermudan Swaption #1: The pricing for a Bermudan swaption in the Hull White grid is described in the DRIP Fixed Income Specification. For a swaption with a physical type of exercise, the grid values of the combined portfolios of the swaptions (before exercise) and the underlying swap (after exercise) cannot be computed be computed using just a backward induction because at the time the exercise boundary is realized, it is impossible to go back and determine the expected value of the underlying swap, whose existence depends on this exercise boundary.
15. Physically Exercised Bermudan Swaption #2: In this case, therefore, one has to determine the exercise boundary using backward induction and then determine the conditional exercise probability of the underlying swap at every node using forward induction.
16. Physically Exercised Bermudan Swaption #3: The values of the combined portfolios above on the grid are computed as a sum of the swaption values and the swap values weighted by their corresponding existence probabilities.
17. Physically Exercised Bermudan Swaption #4: Let t be the grid date, j be the node index on the grid date t, $V_{SWAP}(t, j)$ and $V_{SWAPTION}(t, j)$ be the swap value and the swaption value at the node (t, j), respectively, with P_E being the probability that the swaption is already exercised conditional to reaching that node.
18. Physically Exercised Bermudan Swaption #5: When the value of the combined portfolio is

$$V(t, j) = P_E(t, j)V_{SWAP}(t, j) + [1 - P_E(t, j)]V_{SWAPTION}(t, j)$$

The conditional probability is calculated as the ratio of two unconditional probabilities, i.e.,

$$P_E(t,j) = \frac{P_{Ex}(t,j)}{P_{At}(t,j)}$$

where $P_{Ex}(t,j)$ is the unconditional probability to get to this node with the exercised option, and $P_{At}(t,j)$ is the unconditional probability to get to this node.

19. Discretized Total Credit Exposure Aggregation: The total aggregated credit exposure is then calculated as follows:

$$CVA + DVA \approx \sum_n \mathbb{E}\left[\left\{1 - R_{CPTY}\left(V^+(t_{n+1})\right)\right\} \cdot V^+(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{CPTY} < t_n + \Delta t_n) - \sum_n \mathbb{E}\left[\left\{1 - R_{BANK}\left(V^-(t_{n+1})\right)\right\} \cdot V^-(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{BANK} < t_n + \Delta t_n)$$

20. Recovery Map Date Node Choice: Notice that the option values on t_{n+1} (end of period) is used instead – in contrast to

$$CVA + DVA \approx \sum_n (1 - R_{CPTY}) \cdot DV01 \cdot \mathbb{E}[\{S(t_n) - K\}^+] \cdot \mathbb{P}(t_n \le \tau_{CPTY} < t_n + \Delta t_n) - \sum_n (1 - R_{BANK}) \cdot DV01 \cdot \mathbb{E}[\{S(t_n) - K\}^-] \cdot \mathbb{P}(t_n \le \tau_{BANK} < t_n + \Delta t_n)$$

21. Left vs. Right Integral Approximation: This choice is made to capture the different tenors in the swap portfolio. The difference between

$$CVA + DVA \approx \sum_n (1 - R_{CPTY}) \cdot DV01 \cdot \mathbb{E}[\{S(t_n) - K\}^+] \cdot \mathbb{P}(t_n \le \tau_{CPTY} < t_n + \Delta t_n) - \sum_n (1 - R_{BANK}) \cdot DV01 \cdot \mathbb{E}[\{S(t_n) - K\}^-] \cdot \mathbb{P}(t_n \le \tau_{BANK} < t_n + \Delta t_n)$$

and

$$CVA + DVA \approx \sum_n \mathbb{E}\left[\left\{1 - R_{CPTY}\left(V^+(t_{n+1})\right)\right\} \cdot V^+(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{CPTY} < t_n + \Delta t_n) - \sum_n \mathbb{E}\left[\left\{1 - R_{BANK}\left(V^-(t_{n+1})\right)\right\} \cdot V^-(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{BANK} < t_n + \Delta t_n)$$

is actually the same as the left vs. the right approximations to the integral.

22. First Period WHOLE vs STUB: Cash flows are generated in the HW model assuming that the stub type is STUB. In CVA/DVA valuation it is desirable to have the WHOLE stub type, especially for the first period of the swap because the rate is already set. For default dates before the end of the first period, the first cash flows are generated using the STUB stub type.
23. Accounting for the Anterior Accrual Period: An additional single cash flow is generated at the accrual end date of the first period with the amount being the difference between the WHOLE and the STUB swap NPV's. Since the first period rate is already set, the first period's cash flows are deterministic, and thus this approach is the same as generating cash flows using the WHOLE stub type.
24. Handling of the Later Period Flows: For default dates that are after the first period accrual end date, however, the STUB setting is used to generate cash flows. It is expected that the majority of the difference between WHOLE and STUB comes from the first period whose payment is already known. This logic of equivalent cash flow for the WHOLE stub type has been implemented in IR CVA model and the cross-currency swap CVA model.

Risk Participation Swap (RPS) Valuation

1. Definition of the Risk Participation Swap: The NPV of the risk participation swap is the unilateral credit value adjustment to the risk free underlying swap due to the counter party credit risk.
2. Unilateral CVA Bank Credit Indifference: The risk participation swap is priced the same way as that of the underlying swap, except that a zero hazard curve for the bank (along with a 100% recovery rate) is used.

CVA for Swaps with Prepay Risk

1. Prepayment Cash Flow Based Collateral: This is a portfolio of swaps with principal instruments (PI) with a special from the collateral backing the swap payments. Thus the reserve calculation has to account for the default risk and the prepay risk.
2. External Computation of Prepayment Risk: Often the risk of prepay is calculated externally from spreadsheets and then modeled into the system (e.g., Calypso) using hazard curves. Thus these PI trades can be incorporated into the IR CVA model.

Negative Rates Distribution

1. Forward vs. Spot Negative Rates: There are two sources of negative rates in the CVA model. The first is the negative forward rate from the interest rate curve and the second is the negative short rate in the HW interest rate model.

2. Negative Rate from Stresses Scenarios: For an interest rate curve there could be a set of time segments in which the forward rates are negative, especially for the cases where the curves are downward shifted (in parallel or buckets) in stress scenarios.
3. Optional Flooring of the Negative Rates: There is a mechanism in DRIP to optionally floor the forward rates when shifting the curve such that the forward rates are non-negative. The CVA model will take shifted interest rates curves as inputs.
4. Negative Short Rates in HW: The second is the negative short rates in the HW interest rate model. Since the HW model is a normal interest rate model it supports negative short rates.
5. Short Rate HW Volatility Impact: Negative short rates can be observed even if the forward rates in the interest rates curves are non-negative. The short rate dynamics are generated through the HW tree; the higher the HW volatility is, the wider the rate distribution is.
6. Statistics on Realized Short Rates: On each node on each credit date on the HW tree, both the probability of being in that node and the corresponding short rate are known. Using this rates distribution, the statistics on negative short rate at each node can be generated.
7. *Thin* Short Rate Statistical Measures: The distribution of short rates is available in the model output. The same set of credit dates as in the CVA/DVA profile have been used to show the percentage of negative short rates on each credit date. Also output are the mean, the standard deviation, the minimum, and the maximum of the short rates, and the mean and the standard deviation of the negative short rates.

Portfolio Specific Calibration of CVA/DVA for Interest Rates Products

1. Custom Portfolio HW Volatility Surface: As discussed earlier, DRIP recommends that the global calibration be used for all the portfolios in the "Hull White Model Calibration" section in this chapter. For the completeness of the document for an alternative calibration model, this section discusses the details of portfolio calibration in general for portfolios with swaps, caps/floors, and swaptions as underlyings.

2. Volatility Surface from Calibration Instruments: In calibrating the Hull White model for a portfolio of interest rates products, one obtains the term structure of volatilities in the HW model that is consistent with the portfolio of underlying securities.
3. Calibration against Co-terminal Swaptions: The calibration method follows an approach that is similar to the calibration of the Hull-White model when pricing individual Bermudan and cancelable swap contracts; this method is described in DRIP Fixed Income Analytics, and builds the model to match the prices of the corresponding European co-terminal swaptions. The most important difference is that the whole portfolio is considered instead of a single instrument when one values the CVA/DVA of the portfolio.

Portfolio Construction for the Calibration

1. Credit Exposure Dates Construction Methodology: The portfolio is calibrated to a set of credit dates. The credit dates are constructed using the procedure described below.
2. Dense Followed by Sparse Grids: Currently the set of credit dates is constructed starting with the valuation date according to the following schedule; quarterly dates are used for the first six months, semi-annual dates are used for the following five years, and annual dates are used up to the maturity of the longest deal in the portfolio.
3. Exposure Date Specific Calibration Portfolio: Thus every portfolio has a credit date corresponding to the standard tenors. For each credit date a calibration portfolio is constructed.
4. Choice of the Calibration Securities: For each individual underlying security, the current credit date is compared with both the tear-up date of the security and the last cash flow date. If the credit date is after the tear up date or after the last cash flow date, the instrument is not included in the calibration portfolio, because there is no credit exposure for that instrument on the given credit date. For a basket of swaps, caps, and swaptions, the following scheme is used to construct the swap portfolio.

5. Synthetic Swap in Calibration Portfolio: For a swap, a new synthetic swap with cash flows after the credit date is created, and is included in the calibration portfolio.
6. Synthetic Cap/Floor/Straddle Calibration: For a swap, a new instrument with the caplets remaining after the credit date is created. The new synthetic cap is added to the calibration portfolio. The same approach is applied to floors, straddles, etc.
7. Synthetic European Swaption Calibration #1: For European swaptions, if the credit date is before the exercise date, the European swaption is included in the calibration portfolio; if the credit date is after the exercise date, there are two situations.
8. Synthetic European Swaption Calibration #2: If the swaption is cash settled, it is not included in the portfolio because there is no credit exposure after the cash payment date. If the swaption is physically settled, a synthetic swap is created from the underlying swap of the swaption with cash flows after the credit date with the notional adjusted by the ratio of the DV01 of the European swaption over the DV01 of the swap. This adjustment is meant to match the DV01 for the parallel curve shift after the credit date.
9. Bermudan/American Swaption Calibration #1: For Bermudan and American swaptions, a similar approach is used – if there are exercise dates after the credit date, the next exercise date is located from the exercise schedule, and the European swaptions that are co-terminal with the next exercise date after the credit date are created.
10. Bermudan/American Swaption Calibration #2: If the credit date is after the last exercise date, a synthetic swap is created from the underlying swap of the swaption with the cash flows after the credit date, with the notionals adjusted by the ratio of DV01 of the last co-terminal swaption over the DV01 of the underlying swap.
11. Bermudan/American Swaption Calibration #3: In this approach, each American or Bermudan swaption is replicated using an European swaption, in the same of calibrating the HW model for pricing an American or an European swaption, in which the HW volatility is calibrated by boot-strapping for the next co-terminal swaption step-by-step.

Portfolio NPV and NPV Option Valuation

1. Using the Equivalent Cash Flows: After the calibration portfolios have been created for the credit dates, one has a set of calibration portfolios, where t is the credit date. For each asset in the portfolio for a given credit date, the asset is pried using its equivalent cash flows.
2. Swap/Swaption/Cap/Floor PVs: For a swap its cash flow NPV is directly calculated For a European swaption, the same approach as that in pricing generic European swaption using its equivalent cash flows is used (see the DRIP Fixed Income Specification for details). For a cap or floor, each caplet is priced using a generic European caplet pricer using its equivalent cash flows.
3. Decomposition of the Swaption Portfolio: At the same time, a portfolio option pricer that uses the equivalent cash flows of each asset in the portfolio in the same way as that in the generic European swaption pricer is implemented, where the portfolio of underlying securities is represented as a weighted sum of standard swaptions. The exact procedure for computing the weights is described elsewhere.
4. Portfolio Option Equivalent Cash Flows: After pricing the portfolio option NPV, the equivalent cash flows for the portfolio are obtained from the equivalent cash flows of the corresponding standard swaptions used in the construction of the weighted sum of standard options.
5. Generic European Swap Option Pricer: The DRIP Fixed Income Specification contains the details of the generic European swaption pricer and the equivalent cash flow. Therefore one has a sequence of data set on the calibration dates $\{NPV(P_i{}^+), CF_i, t_i\}$ which are then passed into the HW calibration procedure.

Bootstrapping the Volatility Term Structure

1. Positive Exposure Volatility Surface Calibration: After the values of $NPV\left(P_i^{+}\right)$ and the corresponding cash flows are calculated, the next step is to bootstrap the term structure of volatilities for the HW model.
2. Implying the Segment Volatility: Doing so one generally follows the procedure outlined in the DRIP Fixed Income Specification, where the values of the volatility in each segment is selected to match $NPV\left(P_i^{+}\right)$ for the corresponding credit date using the equivalent cash flows for this credit date.
3. Use of the Gaussian Quadrature Approach: The only significant difference from the method described in DRIP Fixed Income Specification is that Gaussian quadrature approach is applied when computing the value of the option on the portfolio of zero coupon bonds. This step is different from the Jamshidian approach described there and is used when pricing Bermudan and Cancelable swap contracts in the HW model framework.
4. Complexity of the Equivalent Cash Flows: The reason for this modification is that in many cases the equivalent cash flows for the portfolio of the underlying securities can be very complex and irregular, thus resulting in situations where one is not able to construct the interest rate curve, or not able to convert the option on a portfolio of zero coupon bonds into a portfolio of options on zero coupon bonds.
5. Integration over the Short Rate Distribution: Therefore the algorithm for computing the cash flow option has been improved using the quadrature approach. The integration is over the short rate. Because of the structure of the Hull-White model, the Gauss-Hermite scheme with 100 point selection of the standard normal distribution is used. A good reference on Gaussian quadrature is *Integration of Functions* in Press, Flannery, Teukolsky, and Vetterling (2002).

Margin Period of Risk for CVA/DVA

1. Bank/Counter Party Collateral Thresholds: Under bilateral margin agreements, both the bank and the counter party need to post collateral if the uncollateralized amount exceeds a certain limit; the counter party posts collateral when the bank's exposure to the counter party exceeds the counter party's threshold, while the bank posts collateral when the counter party's exposure to the bank exceeds the bank's threshold.
2. Margin Period of Risk - Definition: Due to the operational complexity of the collateral delivery and the margin calls, there is a time lag from the last margin call to the counter party's official default time, which is known as the margin period of risk (MPoR). Therefore the collateral available at a specific date is determined by the uncollateralized at the latest time when the margin call could be made.
3. Contractual Details of the Margin Process: The margin period of risk depends on the contractual margin call frequency and the liquidity of the portfolio, but a period of two weeks in calendar days is assumed usually for CVA/DVA calculation. More detailed description of the collateral dynamics under the Credit Support Annex (CSA) can be found in Pykhtin (2009) and Gregory (2012).
4. Bank/Counter Party Parameters #1: This section focuses on how to handle margin period of risk in CVA/DVA calculation. Let the threshold, the minimum transfer amount (MTA), and the independent amount (IA) for the counterparty be

$$H_{CPTY} \geq 0$$

$$MTA_{CPTY} \geq 0$$

and

$$IA_{CPTY} \geq 0$$

respectively.
5. Bank/Counter Party Parameters #2: The threshold, minimum transfer amount, and the independent amount for the bank are

$$H_{BANK} \geq 0$$

$$MTA_{BANK} \geq 0$$

and IA_{BANK} respectively.

6. Impact of Rounding on Collateralization: The impact of rounding on collateralization is not considered, since it has a small effect on the impact of the CVA/DVA calculation, as shown in Gregory (2012).
7. Margin Setting Sample Parameters #1: To illustrate a collateral call, Gregory (2012) gives an example:

$$H_{CPTY} = 1.0m$$

$$MTA_{CPTY} \geq 0.1m$$

$$IA_{CPTY} \geq 0$$

When a portfolio's MTM is

$$V_{MTM} \geq 1.7m$$

it exceeds the effective threshold

$$H_{CPTY} + MTA_{CPTY} - IA_{CPTY} = 1.1m$$

8. Margin Setting Sample Parameter #2: This triggers a collateral call, and the amount of collateral call is

$$CollateralCall = V_{MTM} - H_{CPTY} = 1.7mm - 1.0m = 0.7m$$

Note that this amount is the collateral above the threshold, namely the amount of threshold that is not collateralized.

9. Incorporation of the Minimum Transfer Amount: As described in Pykhtin (2009), in order to avoid modeling the complexity of the minimum transfer amount, it has been added to the threshold as an effective threshold.
10. Collateral Value at the Credit Date: Let $V(t)$ be the uncollateralized portfolio value at time t and

$$\Delta t = 2W$$

in calendar days be the margin period of risk, then the collateral value at time t is

$$C(t) = \begin{cases} V(t-\Delta t) - H_{CPTY} - MTA_{CPTY} + IA_{CPTY} & V(t-\Delta t) > +H_{CPTY} + MTA_{CPTY} - IA_{CPTY} \\ V(t-\Delta t) + H_{BANK} + MTA_{BANK} - IA_{BANK} & V(t-\Delta t) < -H_{BANK} - MTA_{BANK} - IA_{BANK} \\ 0 & Otherwise \end{cases}$$

11. Value of the Collateralized Portfolio #1: The collateralized portfolio value is

$$V_C(t) = V(t) - C(t) + CF(t-\Delta t, t)$$

where $CF(t-\Delta t, t)$ is the cash flow within the default window $(t-\Delta t, t]$.The payments on t are included and the payments at $t-\Delta t$ are excluded from $CF(t-\Delta t, t)$.

12. Value of the Collateralized Portfolio #2: By arranging the terms, the collateralized portfolio becomes

$$V_C(t) = \begin{cases} \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} + CF(t-\Delta t, t) & V(t) > \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} \\ \Delta V(t) - H_{BANK} - MTA_{BANK} + IA_{BANK} + CF(t-\Delta t, t) & V(t) < \Delta V(t) - H_{BANK} - MTA_{BANK} - IA_{BA} \\ \Delta V(t) + CF(t-\Delta t, t) & Otherwise \end{cases}$$

where

$$\Delta V(t) = V(t) - V(t - \Delta t)$$

is the difference of the uncollateralized portfolio values inside the margin period of risk.

13. Margin Period Look-back Window: In order to calculate the margin period risk, a full Monte Carlo framework is needed, with each credit date t having a two week look-back window $t - \Delta t$ to find the distribution of the portfolio value at time $t - \Delta t$ given the portfolio value at time t. This is handled in both the CVA for the interest rates products and the CVA for the cross-currency swap portfolio below.
14. HW Enhancement for MPoR: The CVA/DVA model has been enhanced to fully deal with the margin period o risk dynamically. For a portfolio of interest rates products, the Hull-White interest rate model is used in the CVA/DVA pricing.
15. Look back Window Portfolio Exposure: For each grid point at time t, the portfolio value $V(t)$ is calculated, and then a look-back of 14 calendar days from this period is applied. All the grid points i $i \in I$ that connect to the grid point at time t are located, and their corresponding portfolio values $V_i(t - \Delta t)$ and $P_i(t - \Delta t)$ are identified.
16. Approximation Approaches for the MPoR Calculation: There are different approaches/approximations to use these values in MPoR calculations; a boolean flag $UseExactHWGridForMPoR$ has been implemented in the pricing parameters to differentiate between these approximations.
17. Positive/Negative Collateralized Portfolio Exposures: If $UseExactHWGridForMPoR$ is true, DRIP uses the portfolio value $V_i(t - \Delta t)$ $i \in I$ in each grid point that connects to the grid point at time t to calculate the collateralized portfolio value $V_{Ci}(t)$, then aggregate the expected positive exposure (EPE) and the expected negative exposure (ENE) with the probability $P_i(t - \Delta t)$ – namely

$$EPE = \sum_{i \in I} [V_{Ci}(t)]^+ P_i(t - \Delta t)$$

and

$$ENE = \sum_{i \in I} [V_{Ci}(t)]^{-} P_i(t - \Delta t)$$

respectively.

18. <u>MPoR from the Net Collateral Exposure</u>: If $UseExactHWGridForMPoR$ is false, then the expected value from all paths leading to this grid point at time t - that is, the value of the portfolio – is

$$V(t - \Delta t) = \sum_{i \in I} V_i(t) P_i(t - \Delta t)$$

This expected value can he be used to calculate the collateralized value of the portfolio in

$$V_C(t) = \begin{cases} \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} + CF(t - \Delta t, t) & V(t) > \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} \\ \Delta V(t) - H_{BANK} - MTA_{BANK} + IA_{BANK} + CF(t - \Delta t, t) & V(t) < \Delta V(t) - H_{BANK} - MTA_{BANK} - IA_{BA} \\ \Delta V(t) + CF(t - \Delta t, t) & Otherwise \end{cases}$$

Thus EPE and ENE can be calculated as $[V_C(t)]^{+}$ and $[V_C(t)]^{-}$ respectively.

19. <u>Performance Differences between the Approaches</u>: Since the expected value is calculated for each grid point at time t this approximation uses less memory, and is good enough when the number of grid points is larger at time t. The approximation in the case of calculates the positive and the negative exposures for each pair of connecting grid points, and is thus more accurate in the positive and the negative exposure calculation – however it uses more memory in the calculation.

20. <u>Margin Period Cash Flow Calculation</u>: The cash flow within the default window is calculated using the HW tree. For each grid point on each credit date P one can roll back the values of all the grid points in the next credit date which connect with the same grid point P. The

difference between the value at P and the rolled value from the next credit date is the cash flow value between these two credit dates for the grid point P.

21. Collateralized Portfolio Value Stochastic Interpolation: If the time point $t - \Delta t$ is not a grid point, the next value at time t is interpolated from the neighboring grid points, as described in the next section. The collateralized portfolio value at each grid point can be calculated as

$$V_C(t) = \begin{cases} \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} + CF(t - \Delta t, t) & V(t) > \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPT} \\ \Delta V(t) - H_{BANK} - MTA_{BANK} + IA_{BANK} + CF(t - \Delta t, t) & V(t) < \Delta V(t) - H_{BANK} - MTA_{BANK} - IA_{BA} \\ \Delta V(t) + CF(t - \Delta t, t) & Otherwise \end{cases}$$

22. Cross-Currency Swap MPoR Calculation: For the cross-currency swap portfolio model, since the model is based on Monte Carlo simulation, the portfolio value can be calculated Δt days before each simulation point.
23. Explicit vs. Interpolated Grid Values: In the same manner, if the time point $t - \Delta t$ is not a simulation date, the portfolio value at $t - \Delta t$ will be interpolated from the neighboring simulation dates, as described in the next section. The cash flow within the default window $(t - \Delta t, t]$ can be calculated for each credit date and each MC path.
24. Grid Point Collateralized Portfolio Value: The collateralized portfolio value at each grid point can be calculated using

$$V_C(t) = \begin{cases} \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPTY} + CF(t - \Delta t, t) & V(t) > \Delta V(t) + H_{CPTY} + MTA_{CPTY} - IA_{CPT} \\ \Delta V(t) - H_{BANK} - MTA_{BANK} + IA_{BANK} + CF(t - \Delta t, t) & V(t) < \Delta V(t) - H_{BANK} - MTA_{BANK} - IA_{BA} \\ \Delta V(t) + CF(t - \Delta t, t) & Otherwise \end{cases}$$

By calculating the distribution of the collateralized portfolio values, CVA/DVA can be calculated as usual.

25. Default Window Margin Call Parameters: In the CVA model for both a portfolio of single currency interest rate products and a portfolio of cross currency swaps, additional control

variables have been introduced for handling the default window and the margin call frequency in the MPoR calculation.

26. Bank/Counter Party Default Margin: They are:
    a. Counter Party Default Window => 2 weeks as default window; the default value is 14.0
    b. Bank Default Window => Default value used as an indicator only to trigger

$$BankDefaultWindow = CounterPartyDefaultWindow$$

    Default is -1.0
    c. Margin Period => Margin Call Frequency as the number of days; daily margining (i.e. -1.0) is the default.
27. Time Dependent CSA Parameters Table: The model has been enhanced to support the time-dependent CSA parameters as shown in the following table:

t_1	$CSAParameters1$
t_2	$CSAParameters2$
t_3	$CSAParameters3$

28. Piecewise Constant CSA Parameters: The CSA table for a counter party could have multiple rows; each row has a specific starting date. If there is only one row, a single set of CSA parameters is used for the entire time horizon as usual.
29. Margin Lag CSA Parameters Entry: If there are multiple rows, each row corresponds to a segment of CSA parameters with s given starting date. The collateral date $t - \Delta t$ will be used to find the CSA parameters for the exposure date t.
30. Edge Based CSA Parameters Application: For the example shown in the table, the CSA parameters #1 are used for the time period

$$t_1 \leq t - \Delta t < t_2$$

the CSA parameters #2 are used for the time period

$$t_2 \leq t - \Delta t < t_3$$

and the CSA parameters #3 are used after t_3. The time-dependent CSA parameters have been implemented in both the IRCVA as well as the cross-currency CVA models.

Exposure Interpolation

1. Margin Date Portfolio Value Availability: For calculating the collateral amount to be used for MPoR or collateralized exposure at time t, the portfolio value of the collateral group $V(t - \Delta t)$ is needed at a time prior to $t - \Delta t$. This prior portfolio $V(t - \Delta t)$ value may or may not be available in the model implementation.
2. Square Root of Time Interpolation: If this is not available, one can use the following simple interpolation from the available values of $V(t)$ and $V(s)$ based on the square root of time for achieving the constant forward volatility in the interpolation, i.e.

$$V(t - \Delta t) = V(t) - \frac{V(t) - V(s)}{\sqrt{t - s}} \Delta t$$

such that

$$t > s$$

and

$$\Delta t \geq 0$$

3. <u>Linear Time Interpolation of Variance</u>: It follows that

$$\mathbb{V}[V(t-\Delta t) - V(t)] = \frac{\mathbb{V}[V(t) - V(s)]}{t-s}\Delta t$$

   which is consistent with the assumption of constant forward volatility in the time interval $[s, t]$.
4. <u>Standard Brownian Motion Variance Interpolation</u>: Furthermore, if as a special case, is a standard Brownian motion, then then the following result is achieved:

$$\mathbb{V}[W_t - W_{t-\Delta t}] = \frac{\mathbb{V}[W_t - W_s]}{t-s}\Delta t = \Delta t$$

5. <u>Linear Time Portfolio Value Interpolation</u>: Alternatively one can use the linear interpolation as

$$V(t-\Delta t) = V(t) - \frac{V(t) - V(s)}{t-s}\Delta t$$

$$t > s$$

$$\Delta t \geq 0$$

6. <u>DRIP Brownian Bridge Interpolation Setting</u>: The interpolation type is controlled by a flag $MPoRInterpType$ in the pricing parameters. The square root of time interpolation is used in the interest rates product CVA model and cross-currency swap portfolio CVA model.
7. <u>Handling Margin Period Cash Flow</u>: If the contractual cash flow impact (due to the cash flow occurring in the time interval $(t - \Delta t, t]$) is significant, the cash flow must be handled in the

interpolation. Specifically the first cash flow $V(s)$ is removed, the interpolation is performed as described above, and finally the cash flow is added back to the interpolated value.

8. Explicit Adjustment for the Cash Flow: The final interpolated value with a cash flow $CF(s,t)$ - the cash flow within the time interval $(s,t]$ – using the square root of time interpolation for

$$t > s$$

$$\Delta t \geq 0$$

is

$$V(t-\Delta t) - V(t) = \frac{V(t) - V(s) - CF(s,t)}{\sqrt{t-s}}\Delta t + CF(s,t)\min\left(\frac{\Delta t}{t-s}, 1\right)$$

The cash flow impact is included as a part of the cash flow at risk in the chapter *CVA Cross-Asset Modeling Framework Specification*.

CVA/DVA of Muni FPA Product

1. Forward Purchase Agreement (FPA) Definition: A forward purchase/delivery agreement (FPA) obligates the provider to deliver qualifying securities to the customer at the contracted rate according to the documented schedule.
2. CTD Floating Leg Embedded Option: Typically more than one security is eligible for delivery which grants the provider the right to choose the cheapest to deliver. Muni FPA can be treated as a swap with embedded option on the floating leg.

3. Floating Leg Option Adjusted Spread: The option value for each period is converted into a floating spread so that the Muni FPA trade becomes a regular fixed vs. floating swap, which is described in detail in this section.
4. Optionality behind the CTD Feature: An elemental FPA is similar to an interest rate swap in that the cash flow in the fixed payment side is exchanged with that on the floating payment side. When the securities delivery party has options to choose the securities for delivery, the FPA contract bears options.
5. Multi-Asset Chooser Option Model: An option bearing FPA in the muni product portfolio is valued as a multi-asset chooser option with the maximum interest rate valued as follows. Let

$$R^* = \max(R_1, \cdots, R_m)$$

$$\mathbb{E}[R^*] = \mathbb{E}[\max(R_1, \cdots, R_m)] = \int_{-\infty}^{+\infty} R^* d\mathbb{P}(R^*) = \int_{0}^{+\infty} \mathbb{P}(R^* > K) dK - \int_{-\infty}^{0} \mathbb{P}(R^* \leq K) dK$$

This states that maximizing the NPV of the floating leg of an FPA contract is equivalent to maximizing the forward rate over each delivery period $[t_i, t_{i+1}]$.
6. Expression for the FPA NPV: Considering the fixed payments that may be present in an FPA contract, the NPV of an FPA contract is

$$NPV = \sum_{i=1}^{n} \mathbb{E}[R_i^*] \cdot \Delta t_i \cdot D_f(t_i) - \sum_{i=1}^{n} r \cdot \Delta T_i \cdot D_f(t_i)$$

where ΔT_i and Δt_i are the accruals of the floating side and the fixed sides respectively after taking into account their day count conventions differences.
7. FPA NPV vs. the Underlying: $D_f(t_i)$ is the discount factor from the funding curve selected according to the convention of the FPA trade. The underlying NPV is calculated under the zero-volatility assumption assuming that the maximum rate is just the maximum of the

forward rates observed on the valuation date. The time value of the FPA is just the difference of the FPA NPV and the underlying NPV.

8. Underlying vs FPA NPV Comparison: The FPA option value is converted to a static option adjusted spread. In effect the FPA spread is computed to be static before the CVA/DVA is computed for the FPA. To illustrate how large the time value of the FPA portfolio is, the FPA NPV and the underlying NPV's are calculated for each trade in a sample portfolio.
9. The FPA NPV Time Value: The total FPA NPV is $-\$142.9m$, the underlying PA NPV is $-\$186.8m$, and the time value is $\$43.9m$. For all the FPA trades, a look at the distribution of the ratios of the time values to the underlying NPV's is shown in the following table.
10. Underlying vs FPA NPV Table:

Time Value as a Percentage of the Underlying NPV	**Percent of FPA Trades**
Within 10%	78.6%
Within 20%	86.7%
Within 30%	91.2%
Within 40%	94.5%
Within 50%	95.6%
Within 60%	95.9%
Within 70%	96.4%
Within 80%	96.7%
Within 90%	96.8%
Within 100%	100.0%

11. Rational behind the FPA Approximation Scheme: It can be seen that for most trades (79%) the time values are within 10% of the underlying NPV's. Thus for most trades, the

optionality is small. This is one of the arguments for the following approximation for pricing CVA/DVA for FPA trades.

12. Multiplier/Spread Based FPA Proxy: To fir this non-standard product into the existing CVA/DVA framework, the FPA cash flow can be re-cast into an equivalent fix-float LIBOR IRS by adding the corresponding multiplier and spread, m_i and s_i, in terms of the day count conventions of the standard LIBOR index δt_i and δT_i

$$NPV = \sum_{i=1}^{n} \delta t_i \cdot \left(m_i f_{LIBOR,i} + s_i\right) \cdot D_{f,OIS}(t_{i+1}) - \sum_{i=1}^{n} \delta T_i \cdot r_i \cdot D_{f,OIS}(t_{i+1})$$

where $D_{f,OIS}(t_{i+1})$ is the discount factor from the OIS discount curve which is used in the existing CVA/DVA framework.

13. Estimating the FRA Spread/Multiplier: One chooses

$$m_i = \frac{\Delta t_i}{\delta t_i} \cdot \frac{D_f(t_i)}{D_{f,OIS}(t_{i+1})}$$

$$s_i = m_i \cdot \left(\mathbb{E}[{R_i}^*] - f_{LIBOR,i}\right)$$

for the floating leg, and

$$r_i = r \cdot \frac{\Delta T_i}{\delta T_i} \cdot \frac{D_f(t_i)}{D_{f,OIS}(t_{i+1})} \cdot \frac{Notional_i}{round(Notional_i)}$$

for the fixed leg.

14. Dynamics of the Chooser Option: For a fixed rate there is a ratio of the notionals which accounts for the convention of the delivery of the floating leg in the FPA, which is rounded to 1000, and the same notional is used for the fixed leg and the floating leg in the equivalent swap. Since the FPA NPV is the total chooser option price, including underlying NPV and

the time value, the approximation of converting multi-index instruments into a single index LIBOR instrument captures most of the dynamics.

15. <u>FPA Contract Credit Exposure Calculation</u>: As a result of this approximation, the credit exposures in an FPA contract can be priced efficiently as a single swap as described in the Section *CVA/DVA Calculation of a Single Swap* or netting with other interest rates products as described in *CVA/DVA Model for Interest Rates Products*.

CVA/DVA Model for Portfolio of Cross-Currency Swaps

1. <u>FX/Rates Market Factors</u>: This section discusses the CVA/DVA pricing for portfolios of cross-currency swaps. In this case there are additional market risk factors from the interest rate volatility for each currency, and FX volatilities.
2. <u>Monte Carlo Cross Currency Model</u>: It is more difficult to build an analytic or semi-analytic model to price the CVA/DVA as opposed to that of single currency portfolio of interest rates products. Thus, DRIP has implemented a Monte-Carlo based model for the CVA/DVA of a portfolio of cross-currency swaps. This uses a model that simulates each risk factor. The Monte Carlo model is flexible enough to combine all the underlying models from all the components.
3. <u>Components of the MC Model</u>: The portfolio can contain all kinds of swaps (fixed-fixed, fixed-floating, and floating-floating). Within the portfolio at least one swap must be cross-currency. The main components of this model are listed below.
4. <u>XCCY MC Model - Rates Volatility</u>: The interest rates in each currency is modeled using a Hull White model globally calibrated to the corresponding interest rate ATM volatility surface.
5. <u>XCCY MC Model - FX Volatility</u>: The exchange rates are calibrated using Black Scholes calibrated to ATM FX options applying the corresponding FX volatility surfaces.

6. <u>XCCY MC Model - Rates/FX</u>: The correlations among the exchange rates and the interest rates are estimated historically and assumed to be constants.
7. <u>Bank/Counter Party Credit State</u>: The hazard rate is assumed to be deterministic and directly calculated from the hazard curve.
8. <u>Bank/Counter Party Recovery Rate</u>: The recovery rate map and the CSA collateral dynamics for the margin period of risk are supported.
9. <u>Monte Carlo Path Simulation Settings</u>: The model is simulated using the Monte Carlo settings.
10. <u>Adjusted Union of Pay Dates</u>: The credit dates are generated from all payment dates, and from all streams in the portfolio. The credit dates are super-imposed step-by-step to ensure that the steps are small enough for Monte Carlo simulation.
11. <u>HW Based Rates Volatility Surfaces</u>: As discussed earlier, the Hull White model for each currency is calibrated to the given market data, in either the base scenario, or in the stressed scenario.
12. <u>Stresses/Unstressed Rates Volatility Surfaces</u>: If the market data is in the base scenario as discussed in daily pricing, the unstressed interest rate curves and the volatility surfaces are used in the Hull White model calibration; if the market data has been stressed, the stressed interest rates curves and/or stressed volatility surfaces are used in the Hull White calibration. The mean reversion speed parameter in the Hull White model is not calibrated in either the base or in the stressed scenarios.
13. <u>Stressed/Unstressed FX Volatility Dynamics</u>: The model for the exchange rates is calibrated to given FX volatility surfaces, in either the base or in the stressed scenario, respectively.
14. <u>Domestic Risk Neutral Measure Dynamics</u>: Under the domestic risk-neutral measure, the processes of interest rates and the exchange rates for $N + 1$ currencies (index 0 as domestic currency) are

$$\Delta r_0(t) = [\theta_0(t) - \kappa_0(t) r_0(t)]\Delta t + \sigma_0(t)\Delta W_0(t)$$

$$\Delta r_i(t) = \left[\theta_i(t) - \kappa_i(t) r_i(t) - \rho_{i,i+N}(t)\sigma_i(t)\eta_i(t)\right]\Delta t + \sigma_i(t)\Delta W_i(t)$$

$$\Delta[\log S_i(t)] = \left[r_0(t) - r_i(t) - \frac{1}{2}{\eta_i}^2(t)\right]\Delta t + \eta_i(t)\Delta W_{i+N}(t)$$

$$i = 1, \cdots, N$$

15. Correlations among the Brownian Processes: The Brownian motions from the rates and the FX rates $\Delta W_i(t)$

$$i = 0, \cdots, 2N$$

are correlated with each other. The correlation matrix

$$\langle \Delta W_i(t) \Delta W_j(t) \rangle = \rho_{ij} \Delta t$$

$$i, j = 1, \cdots, 2N$$

is estimated through a statistical analysis from historical data, typically by the desks and the middle office, periodically.

16. Draw of Correlated Random Variables: Simulation proceeds by the generation of correlated normal random variables. Let the correlation matrix be ρ and its Cholesky decomposition be

$$\rho = AA^T$$

By simulating uncorrelated normal variables, one can generate correlated normal variables, that is $A\epsilon$, where ϵ is a vector with each component independently drawn from the standard normal distribution.

17. Rates/FX Stochastic Evolution Increments: At each time step the variables are evolved according to the following Euler scheme (here Δt is the step size)

$$\Delta r_0(t + \Delta t) = r_0(t) + [\theta_0(t) - \kappa_0(t) r_0(t)]\Delta t + \sigma_0(t)\sqrt{\Delta t}(A\epsilon)_1$$

$$\Delta r_i(t+\Delta t) = r_i(t) + \left[\theta_i(t) - \kappa_i(t) r_i(t) - \rho_{i,i+N}(t)\sigma_i(t)\eta_i(t)\right]\Delta t + \sigma_i(t)\sqrt{\Delta t}(A\epsilon)_{i+1}$$

$$\Delta[\log S_i(t+\Delta t)] = \Delta[\log S_i(t)] + \left[r_0(t) - r_i(t) - \frac{1}{2}{\eta_i}^2(t)\right]\Delta t + \eta_i(t)\sqrt{\Delta t}(A\epsilon)_{i+N+1}$$

$$i = 1, \cdots, N$$

18. Incorporation of the Drift Adjustment: An adjustment method is used to adjust the drifts by adding an extra constant drift term for all paths at each step to make sure that the average from the MC simulation matches that from the discount curve for each currency.
19. MC Path Martingale Test Verification: This will guarantee that the underlying NPV from the MC simulation matches that with that of the underlying pricer. More detailed description of the processes for the interest rates and the exchange rates can be found in the DRIP Fixed Income Specification.
20. Path-wise Gross Adjusted Credit Exposure: The total aggregated credit exposure is then calculated as follows:

$$\begin{aligned} CVA + DVA \\ &\approx \sum_n \mathbb{E}\left[\left\{1 - R_{CPTY}\left(V^+(t_{n+1})\right)\right\} \cdot V^+(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{CPTY} < t_n + \Delta t_n) \\ &- \sum_n \mathbb{E}\left[\left\{1 - R_{BANK}\left(V^-(t_{n+1})\right)\right\} \cdot V^-(t_{n+1})\right] \cdot \mathbb{P}(t_n \le \tau_{BANK} < t_n + \Delta t_n) \end{aligned}$$

where the option values $V^+(t_{n+1}), V^-(t_{n+1})$ on t_{n+1} (end of period) are used and calculated using MC simulation. The recovery rate map is then used to calculate the recovery given an exposure.

Cross-Currency Swaps with MTM Legs for Notional Rates

1. MTM XCCY Portfolio Swaps Extension: The CVA/DVA model for the portfolio of cross-currency swaps has been extended to cover the cross-currency swaps with Mark-To-Market (MTM) legs for notional resets.
2. The MTM Notional Reset Lag: For example, a EURUSD cross-currency swaps has deterministic notionals for the EUR leg, and MTM notionals for the USD leg, in which the notional for each accrual period is reset at the beginning of each accrual period from the forward FX rate.
3. Handling Float-Float XCCY Swaps: In particular, special consideration needs to be paid for float-float XCCY swaps with notional resets. The uncertainty of the notionals for the USD leg will increase the complexity of the XCCYCVA model, and the existing frameworks based on MC simulation and equivalent cash flows cannot be applied directly.
4. Simulation of the Discount/Forward Numeraire: In the existing CVA model, for each leg, the equivalent cash flows are generated by a discount curve and a forward curve on each credit date once, and after that, only the simulated discount curve is needed to evaluate the leg NPV for each MC path, and this feature of the modeling speeds up the CVA.
5. Non-deterministic Equivalent Cash Flows: The equivalent cash flow approach will not be used for the notional reset legs, and the MC simulation will be directly used for the underlying pricing. The equivalent cash flow approach only works for the legs with deterministic notionals. At high level, there are two major components of the MC simulation model for the notional reset legs.
6. Simulating Discount and Forward Curves: The first is the simulation of the discount and the forward curve in the MC simulation framework.
7. Pricing of Notional Reset Legs: The second is to price the notional reset leg given the simulated discount and the forward curves above. DRIP Fixed Income Specification contains the mechanics of pricing the MTM legs. As long as the discount and the forward curves have been simulated, the MTM legs can be priced using the forward FX rates documented there.

8. Notional Date MTM Reset Lags: Because of the special feature of the exposures of MTM legs on notional reset dates, more credit dates have been added for valuing CVA/DVA. In addition to the accrual start and the accrual end dates for each period, 7 days after the accrual start and 7 days before the accrual end are also added.
9. Rates Curve Hull White Model: In the current MC framework, each discount curve in each currency has been modeled in terms of a Hull-White model. As usual, the state variables are simulated from the valuation date to each credit date.
10. Construction of the Discount/Forward Curve: A discount curve is constructed for each path using the simulated rates, and the Hull White model with the discount dates from all payment dates, at all notional payment dates, and all notional spot dates, to avoid interpolation. The assumption made here is that the spread cure is static, i.e., the simulated forward curve can be constructed from the simulated discount curve and a static spread curve constructed from the discount curve and the forward curve.
11. Use of Discount/Forward Curves: For optimization purposes, and to speed up the calculations, the discount and the forward curve for the notional reset leg have to be handled in the following way.
12. The Static Spread Based Forward Curve: Construct a static spread forward curve from the discount and the forward curves.
13. Simulating the Interest Rate/FX Curve: Simulate the interest rates and the FX rate to each credit date.
14. Hull White Discount Curve Construction: Construct a discount curve from the simulated rates and the discount factors from the Hull White model for each MC path.
15. Spread Based Forward Curve Construction: Construct a forward curve from the simulated discount curve and the static spread forward curve.
16. NPV from the Simulated Discount/Forward: Finally calculate the NPV from the simulated discount curve and the simulated forward curve for each path on each credit date for each notional leg.
17. Treatment of Non-MTM Legs: For all the non-MTM legs, the algorithm based on equivalent cash flows is kept the same. After each leg on each path is priced for each credit date, the

NPV is aggregated from all the legs on each credit date, and all the remaining steps for the CVA/DVA valuation is the same as in the existing model.

FX Delta and Gamma Calculation and PnL

1. Base Quote Pair FX Sensitivities: FX delta and gamma are important risk measures for portfolio hedge and PnL analysis for counter party trades with underlying of cross-currency swap portfolios. This section discusses with FX delta and gamma in two cases for base/quote currency pairs, USD as quote currency (as in GBPUSD) and USD as in base currency (like USDJPY), respectively.
2. FX Delta/Gamma Sensitivity Increment: The FX delta risk is calculated by shifting up the FX rate by 1%

$$\Delta FX = 0.1 \times FX$$

or down 1%

$$\Delta FX = -0.1 \times FX$$

of the FX rate. The average FX delta is the average of FX deltas from up and down shifts.
3. Base Currency Base Case NPV: For the case of USD as the quote currency in the base/quote currency pair, like GBP/USD, the FX rate is quoted as

$$GBPUSD = 1.5$$

4. <u>Base Currency Bump Case NPV</u>: Let NPV in USD be $NPV_{USD}(FX)$. After shifting the FX rate the NPV becomes $NPV_{USD}(FX + \Delta FX)$. The NPV difference on shifting the FX rate is $NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)$
5. <u>Base Currency FX Delta PnL</u>: Let the FX delta in GBP be D_{GBP}. The FX risk on these trades is equivalent to holding this amount in GBP. The value change in USD after the FX shift for this amount of GBP is

$$D_{GBP} \times (FX + \Delta FX - FX) = D_{GBP} \cdot \Delta FX$$

6. <u>Base Currency FX Position Delta</u>: By equating these two risk quantities, the FX Delta is GBP becomes

$$D_{GBP} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX}$$

7. <u>Base Currency Position FX Gamma</u>: FX delta can be calculated by up shift and down shift. The FX gamma is defined by

$$G_{GBP} = \frac{D_{GBP,Up} - D_{GBP,Down}}{\Delta FX}$$

8. <u>Quote Currency Delta/Gamma PnL</u>: For PnL, the daily NPV change due to FX change is

$$PnL_{USD} = NPV_{USD}\big(FX(T+1)\big) - NPV_{USD}\big(FX(T)\big)$$

By Taylor expansion

$$PnL_{USD} = NPV'_{USD}(FX)[FX(T+1) - FX(T)] + \frac{1}{2} NPV''_{USD}(FX)[FX(T+1) - FX(T)]^2$$

9. Position FX Delta and Gamma: The first and the second derivatives of $NPV_{USD}\big(FX(T)\big)$ with respect to FX can be approximated by the FX Delta and FX Gamma. According to

$$D_{GBP} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX}$$

and

$$G_{GBP} = \frac{D_{GBP,Up} - D_{GBP,Down}}{\Delta FX}$$

one has

$$NPV'_{USD}(FX) \approx D_{GBP}$$

and

$$NPV''_{USD}(FX) \approx D'_{GBP} \approx \frac{D_{GBP,Up} - D_{GBP,Down}}{\Delta FX} = G_{GBP}$$

10. Delta Gamma Based FX PnL: Finally

$$PnL_{USD} = D_{GBP}[FX(T+1) - FX(T)] + \frac{1}{2} G_{GBP}[FX(T+1) - FX(T)]^2$$

11. Daily Change in FX Delta: The daily change in FX delta

$$\Delta D_{GBP} = D_{GBP}\big(FX(T+1)\big) - D_{GBP}\big(FX(T)\big)$$

is

$$D'_{GBP} = D'_{GBP}[FX(T+1) - FX(T)] \approx G_{GBP}[FX(T+1) - FX(T)]$$

12. Base Currency Daily FX PnL: For the case of USD as the base currency in the base/quote pair, the FX rate is quoted as

$$USDJPY = 120$$

The NPV difference by shifting the FX rate is $NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)$ in USD.

13. The *Other Currency* Delta PnL: Let FX delta in JPY be D_{JPY}. The FX risk of this trade is equivalent to holding this amount of JPY. The value of change in USD after FX shift for this amount of JPY is

$$D_{JPY}\left(\frac{1}{FX + \Delta FX} - \frac{1}{FX}\right) = D_{JPY}\frac{-\Delta FX}{FX(FX + \Delta FX)} \approx D_{JPY} \times \frac{-\Delta FX}{FX^2}$$

14. The Quote Currency FX Delta: By equating these two risk quantities the FX Delta in JPY becomes

$$D_{JPY} = \frac{NPV_{JPY}(FX + \Delta FX) - NPV_{JPY}(FX)}{\Delta FX} \times (-FX^2)$$

15. The Base Currency FX Delta: Then the FX delta in the base currency USD is

$$D_{USD} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX} \times FX$$

16. The Base Currency FX Gamma: The FX gamma is then calculated by the up/down shifts of the delta

$$G_{USD} = \frac{D_{USD,Up} - D_{USD,Down}}{\Delta FX}$$

17. Base Currency Delta/Gamma PnL: For PnL the daily NPV change due to FX change is

$$PnL_{USD} = NPV_{USD}\big(FX(T+1)\big) - NPV_{USD}\big(FX(T)\big)$$

By Taylor expansion

$$PnL_{USD} = NPV'_{USD}(FX)[FX(T+1) - FX(T)] + \frac{1}{2}NPV''_{USD}(FX)[FX(T+1) - FX(T)]^2$$

18. Base Currency Position Delta/Gamma: The first and the second derivatives of $NPV_{USD}\big(FX(T)\big)$ with respect to the FX can be approximated in terms of the delta and the gamma. According to

$$D_{USD} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX} \times FX$$

and

$$G_{USD} = \frac{D_{USD,Up} - D_{USD,Down}}{\Delta FX}$$

one has

$$NPV'_{USD}\big(FX(T)\big) \approx \frac{D_{USD}}{FX}$$

and

$$NPV''_{USD}\big(FX(T)\big) \approx \frac{G_{USD}}{FX}$$

19. The Position Delta/Gamma PnL: Finally

$$PnL_{USD} = \frac{D_{USD}}{FX(T)}[FX(T+1) - FX(T)] + \frac{1}{2}\frac{G_{USD}}{FX}[FX(T+1) - FX(T)]^2$$
$$= D_{USD}\frac{FX(T+1) - FX(T)}{FX(T)} + \frac{1}{2}G_{USD}\frac{[FX(T+1) - FX(T)]^2}{FX}$$

20. The FX Delta Daily Change: The daily change of FX delta is

$$\Delta D_{USD} = D_{USD}\big(FX(T+1)\big) - D_{USD}\big(FX(T)\big)$$

is

$$\Delta D_{USD} = D'_{USD}[FX(T+1) - FX(T)] = G_{USD}[FX(T+1) - FX(T)]$$

21. GBP/USD FX Delta/Gamma: In summary the GBP/USD FX deltas and gammas are calculated by

$$D_{GBP} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX}$$

$$D_{USD} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX} \times (-FX)$$

$$G_{GBP} = \frac{D_{GBP,Up} - D_{GBP,Down}}{\Delta FX}$$

22. USD/JPY FX Delta/Gamma:

$$D_{JPY} = \frac{NPV_{JPY}(FX + \Delta FX) - NPV_{JPY}(FX)}{\Delta FX} \times (-FX^2)$$

$$D_{USD} = \frac{NPV_{USD}(FX + \Delta FX) - NPV_{USD}(FX)}{\Delta FX} \times FX$$

$$G_{USD} = \frac{D_{USD,Up} - D_{USD,Down}}{\Delta FX}$$

23. Position FX Delta and Gamma: With the FX average delta and gamma the daily PnL and the change of FX delta from the FX rate movement can be calculated from the day T to the next day $T + 1$ respectively as follows.
24. GBP/USD Delta Gamma PnL:

$$PnL_{USD} = \overline{D}_{GBP}[FX(T+1) - FX(T)] + \frac{1}{2} G_{GBP}[FX(T+1) - FX(T)]^2$$

$$\Delta D_{GBP} = G_{GBP}[FX(T+1) - FX(T)]$$

25. USD/JPY Delta Gamma PnL:

$$PnL_{USD} = \overline{D}_{USD} \frac{FX(T+1) - FX(T)}{FX(T)} + \frac{1}{2} G_{GBP} \frac{[FX(T+1) - FX(T)]^2}{FX}$$

$$\Delta D_{USD} = G_{USD}[FX(T+1) - FX(T)]$$

Negative Rate Distribution in XCCY swap CVA Model

1. Negative Rates from Monte Carlo: The XCCY swap CVA is based on Monte Carlo simulation. At each time step the interest rates in each currency is evolved to the next time

step. By counting the paths with negative rates the statistics on negative rates on each credit date can be generated.

2. Generation of Negative Rate Statistics: A distribution of negative rates for each currency has been added in the model output. The same set of credit dates as in the CVA/DVA profile is used to show the percent of negative rates on each credit date.

CVA/DVA of Credit Products

1. Factor Copula Based Static Model: Two methods are provided to evaluate CVA/DVA in the portfolio of CDS. The first method is based on a static model, where a factor copula is used to generate the correlated default times.
2. Brute Force Based Hybrid Approach: This method allows pricing risk in a hybrid approach where the forward values of the CDS portfolio can be produced externally, and the NPV's consumed by DRIP, from where a lower estimate of risk – consistent with the model and the input correlation – can be calculated.
3. Estimating the Lower/Upper Bounds: An upper estimate can also be made in the hybrid approach, although it is to be noted that a better calculation – using a recursive method not available in the hybrid approach – tends to be closer to the lower than to the upper estimate.
4. Joint Market Factor Evolution Model: The second method is a Monte Carlo based simulation that follows the joint correlated evolution of the short rate, the hazard rate intensities of the credit names reference in the underlying portfolio, and also the intensities of the bank and the counter party. A reduced dimensionality (4 factor) has also been implemented for comparison.
5. Bank/Counter Party Entity Defaults: In the simulation one can observer either the defaults of the bank or the counter party, and track the default of the credit names referenced in the portfolio, which allows computation of the payoff

$$Adjustment(t \leq \tau < t + \Delta t)$$
$$= [(1 - R_{CPTY})V^{+}] \cdot \mathrm{I}_{t \leq \tau_{CPTY} < t + \Delta t} - [(1 - R_{BANK})V^{-}] \cdot \mathrm{I}_{t \leq \tau_{BANK} < t + \Delta t}$$
$$= CVA(t \leq \tau < t + \Delta t) + DVA(t \leq \tau < t + \Delta t)$$

in each simulation path. The credit dynamics can be set to either HW or CIR++. While both tend to produce similar results at small and moderate volatilities, CIR++ is preferred.

6. HW vs. CIR++ Model Comparison: This is because in HW the short rate has a finite probability of becoming negative with typical credit market volatilities, while the CIR++ model parameters can be chosen in such a way to ensure that the short rate is positive.
7. Spread Based Dynamic Model Parameters: In general use of the dynamic model is recommended as it addresses several shortcomings of the static model. The dynamic model parameters can be estimated from historical information, as they involve spread correlation and spread volatilities.

An approximate Method to combine Credit Exposure Profiles with a given Correlation

1. Cross XVA Model Exposure Combination: This chapter formulates an approximate method to combine credit exposure profiles from two different CVA/DVA models to estimate the impact of cross-asset correlation and netting in credit valuation. For example, for a counter party with interest rate and commodity exposures, each credit exposure for each asset class can be calculated from the corresponding CVA/DVA model; the approximation method can combine credit exposure profiles from the interest rate CVA/DVA and the commodity CVA/DVA models.
2. Combination of Positive/Negative Exposures: Each model provides period-wise positive and negative credit exposures, following which additional steps are needed for pre-processing the

credit profiles before combining them. The first step is to normalize the credit dates that result from the union across the two profiles.

3. Union of Credit Dates/Exposures: Then in each profile, a linear interpolation is done of the expected positive (EPE) and the expected negative (ENE) exposures for the new dates. After this step the profiles share the same credit dates.
4. Expected Positive/Negative Vertex Exposures: On each credit date the expected positive and negative credit exposures are

$$EPE = \mathbb{E}[P^{+}] = a$$

$$ENE = \mathbb{E}[P^{-}] = b$$

where P is the portfolio value, P^{+} is the positive exposure, P^{-} is the negative exposure, and

$$a \geq 0$$

and

$$b \geq 0$$

5. Exposure Mean and Variance Fits: The assumption is that the portfolio value P is normally distributed. With positive and negative exposures the two parameters of the normal distribution are backed out. Let

$$\mathbb{P} \sim \mathcal{N}(\mu, \sigma^2)$$

with mean μ and variance σ^2 of a normal distribution.

6. Single Factor Exposure Mean/Variance: There are two equations with two unknowns.

$$\mathbb{E}[P^{+}] = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right)$$

$$\mathbb{E}[P^{+} + P^{-}] = \mu$$

where $\Phi(\cdot)$ is the cumulative probability distribution function, and $\phi(\cdot)$ is the probability density function for the standard Normal distribution. two equations to solve for μ and σ are

$$\mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right) = a$$

$$\mu = a + b$$

7. <u>Exposure Variance Estimation – Special Cases</u>: From this it follows that

$$\mu = a + b$$

To find σ one defines the function

$$f(x) = \Phi(x) + \frac{1}{x}\phi(x)$$

and the following special cases immediately result:

a. If

$$a = b = 0$$

then

$$\sigma = 0$$

b. If

$$a = 0$$

$$b \neq 0$$

then

$$\sigma = 0$$

c. If

$$a \neq 0$$

$$b = 0$$

then

$$\sigma = 0$$

d. If

$$a + b = 0$$

then

$$\sigma = \frac{a}{\phi(0)}$$

8. <u>Exposure Variance Estimation – General Cases</u>: In general one finds x such that

$$\Phi(x) + \frac{1}{x}\phi(x) = \frac{a}{a + b}$$

so that

$$\sigma = \frac{\mu}{x}$$

Note that given the shape of the function $f(x)$ in the range $(-\infty, +\infty)$ one can always find the root through $1D$ search.

9. Profile Specific Mean and Variance: After extracting two parameters for the normal distribution of each asset, the normal distribution for each asset becomes

$$\mathbb{P}_1 \sim \mathcal{N}(\mu_1, {\sigma_1}^2)$$

$$\mathbb{P}_2 \sim \mathcal{N}(\mu_2, {\sigma_2}^2)$$

One can combine the exposures as

$$\mathbb{P} = \mathbb{P}_1 + \mathbb{P}_2$$

with a correlation ρ.

10. Combining the Separate Factor Exposures: Since the combination of a normal distribution is also a normal distribution, one has

$$\mu = \mu_1 + \mu_2$$

$$\sigma^2 = {\sigma_1}^2 + {\sigma_2}^2 + 2\rho\sigma_1\sigma_2$$

Thus the combined normal distribution is

$$\mathbb{P} \sim \mathcal{N}(\mu, \sigma^2) \sim \mathcal{N}(\mu_1 + \mu_2, {\sigma_1}^2 + {\sigma_2}^2 + 2\rho\sigma_1\sigma_2)$$

Thus the positive and the negative exposures can be expressed as

$$\mathbb{E}[P^+] = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\phi\left(\frac{\mu}{\sigma}\right)$$

$$\mathbb{E}[P^-] = \mu - \mathbb{E}[P^+]$$

11. Calculation of CVA and DVA: Finally the CVA/DVA can be calculated with exposures, recovery rates, and the probabilities of default as

$$CVA = \sum_{i=1}^{n} \mathbb{E}[P^+(t_i)] \cdot [1 - R_{CP}(t_i)] \cdot S_{CP}(t_i)$$

$$DVA = \sum_{i=1}^{n} \mathbb{E}[P^-(t_i)] \cdot [1 - R_{BANK}(t_i)] \cdot S_{BANK}(t_i)$$

where t_i

$$i = 1, \cdots, n$$

is the credit date, R_i is the recovery rate, and $S_{CP}(t_i)/S_{BANK}(t_i)$ are the CP/bank probabilities of default calculated from the hazard curve.

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Cross Asset CVA Modeling, Testing, and Validation

Executive Summary of the Framework

1. Overview of the Chapter: This chapter presents the methodology design of a modular cross-asset CVA and DVA framework for credit, FX, equity, commodity, and IR OTC derivatives. Contents of this class will be referenced by the models in specific asset classes.
2. Flexible Cross Asset Framework: The focus here is on providing a methodology design of a flexible cross-asset CVA (credit valuation adjustment) framework.
3. Line of Business Model Integration: In this design, the CVA models for credit, FX, equity, commodity, and IR OTC (over-the-counter) derivatives can be first implemented in the cross-asset CVA framework in parallel on an individual asset basis or an individual line-of-business (LOB) basis, and then can be integrated to obtain the true cross-asset CVA model (minimization of the throw-away work).
4. Out-of-Scope - SFT/TBA: The exchange traded or listed derivatives, SFT's (structured finance transactions), and non-derivatives trades are not in the scope of the current methodology. TBA's (to-be-announced) will be covered in a later phase.
5. Out-of-Scope - Monolines/CDPC's: OTC derivatives facing monolines and CDPC's (credit derivatives product companies) will be handled separately (possibly using conservative approximations), as they require additional modeling methodologies.
6. Out-of-Scope - Trusts/SPVs: Currently no handling is done for OTC derivatives trades facing trusts and SPV's (special purpose vehicles) for CDO's or repackaging.
7. Centrally Generated Correlated Random Numbers: A key feature of this framework is the usage of properly correlated and centrally generated random numbers (or Gaussian variates) as inputs for generating future market simulations that are naturally correlated with other

asset classes (as opposed to generating their own random numbers that are not correlated with other asset classes).

8. Capturing Right/Wrong Way Risk: General wrong-way/right-way risks based on the correlation of the counter party (CP) credit spreads (or default probabilities) with the market are captured in the same way.
9. Using Product Data/Models: Another important feature of this framework is to leverage the desk models and the data as much as feasible, including calling the underlying trade prices (with performance and other enhancements as needed) directly to price the underlying trades on simulated market paths as of future dates for the purposes of calculating the exposures for CVA.
10. Ensuring Consistency of the Simulations: This ensures the consistency of the underlying desk models and the CVA models at time 0, and lays down the foundation for such consistency in the simulation and the exposure calculation at future time steps.
11. Treatment of the Intermediate Results: The saving and retrieving if the intermediate results is an important part of the framework to make the integration of the LOB CVA models into a true cross-asset CVA modeling framework relatively easier.
12. Flexibility across other Application Types: The framework is also designed for flexibility across other applications such as FVA (funding value adjustment), liquidity risk, and contingent funding/liquidity derivatives.
13. Extension of CVA to DVA: While this chapter focusses on CVA, it also provides the modeling requirements for DVA (Debt Valuation Adjustment). These models will also be used for the firm's CVA and DVA valuation and risk management.

Features of the LOB CVA Models

1. Accommodating Intra-Asset Class Correlation: Correlation within each asset class such credit, FX, equity, commodity, and IR (but not the cross-asset correlation), with no interest rate simulations for non-IR derivatives and no FX simulations for non-FX derivatives.
2. Multi-Factor Term Structure Solutions: Multi-factor term structure solutions are used where applicable. An exception is that one-factor simulation is used (contingent on immateriality) for credit derivatives (excluding those facing monolines and CDPC's).
3. Simulation and Pricing of Volatility Skews: Volatility skews are used in market simulation and trade pricing models when applicable.
    a. It is recommended that, in general, no stochastic volatility be deployed in production, except for the limited cases where there are major concentrated volatility risks (including variance and volatility swaps) or for the PFE (potential future exposures) calculation.
    b. Simplified, uncorrelated, non-arbitrage free stochastic volatility simulation is used for materiality analysis only – not recommended in production – and also applies to the materiality analysis of other non-simulated risk factors.
4. Handling Wrong/Right Way Risks:
    a. Limited handling of specific wrong way and right way risks, such as the right-way equity call spreads.
    b. Limited handling of general wrong way/right way risks, with simplifying approximations for large counterparties.
    c. They will be systematically handled in true cross-asset CVA models.
5. Cross Asset CVA Handling #1: For a CP netting and/or collateral group with trades across assets, before the true cross-asset CVA models become available, the CP netting and/or collateral groups are artificially broken into multiple sub-groups with each group containing only trades from a single asset class. These are then run independently for the CVA.
6. Cross-Asset CVA Handling #2: The same collateral rules are applied to each sub-group independently with proper collateral allocation. Collaboration is needed across other groups such as the Counter Party Credit Risk Analytics (CCRA) for proper collateral allocation, if needed.

7. Market Implied Default and Recovery: Market implied default probabilities (and recovery rates) and used, along with the appropriate waterfall logic.
8. Underlying Product CVA Pricer Improvement:
   a. Pricing speed optimization – for example, for vanilla swaps, XCCY swaps, and concealable swaps.
   b. Handling of trades with physical settlement, MTM reset, or accrual based convention, or possibly other future trade event.
   c. Cash flow handling.
9. Proxy Pricers for Trade Pricing: Approximated fallback trade pricing models with conservative measures will be used in very limited cases:
   a. On complex trades with limited materiality that cannot be priced efficiently and accurately for CVA.
   b. On trades with limited materiality that cannot be priced accurately due to data quality issues.
   c. Collaboration with CCRA.

Cross Asset CVA Model Features

1. Centralized Correlated Random Number Generation: Centralized correlated random numbers (or Gaussian variates) for use by the CVA model for each asset class for capturing the cross-correlation.
2. Centralized Simulation of IR/FX: Use the simulated IR and FX markets from the IR and the FX CVA models in the other CVA models for all asset classes.
3. Consistent Cross Asset Exposure Aggregation: Aggregate the counter party exposures consistently across all asset classes for computing the CVA and the DVA.

4. Generic Right/Wrong Way Risks: Systematic handling of the generic wrong-way/right way risks based on the correlation between the bank and the counter party credit spreads (or default probabilities) with the market risk factors for all asset classes.
5. Specific Right/Wrong Way Risks:
    a. Enhanced handling of specific wrong-way/right-way risks with systematic identifications of self-referencing trades.
    b. Productionization of the tactical approximations for the wrong-way/right-way risks.
6. Handling Margin Period of Risk:
    a. Improvement on handling the trade contractual cash flow at risk in the default window.
    b. Improvement on handling non-daily margin frequency.
7. Simplified Materiality Analysis Risk Run: Simplified, uncorrelated, and non-arbitrage free stochastic simulation of the non-simulated risk factors for materiality analysis only (and not for daily production runs).
8. Handling Additional Collateral and Credit: Handle additional collateral and credit terms, as needed.
9. Accuracy/Stability with Martingale Resampling: Martingale resampling for improving accuracy and numerical stability.
10. Replication Model for Trade Pricing:
    a. Use the underlying hedge portfolio to replicate or approximate complex trades that cannot be otherwise priced for CVA.
    b. This is the first waterfall of the fallback trade pricing model, if the original pricing model fails. If the replication model also fails, one falls back to the proxy pricer for trade pricing.

Overview

1. Definition of CVA and DVA: CVA (credit valuation adjustment) is the arbitrage free price (or present value) of the counter party credit risk with respect to the bank's counter party default. DVA (debt valuation adjustment) is the arbitrage free price (or present value) of the counter party credit risk with respect to bank's own default for a portfolio of OTC derivatives.
2. Counter Party/Bank Total Adjustments: CVA and DVA together are used as an adjustment to the default free price of the OTC derivatives (from the underlying LOB models) to achieve proper fair market value.
3. Challenges with the CVA/DVA Estimation: The major challenges in valuing the CVA and the DVA are the non-linear portfolio effects (including all the trades in a counter party portfolio) arising from, for example, the portfolio option-like payoff (due to the default settlement asymmetry), portfolio based netting and collateral agreements, and other credit terms, which require significant modeling effort in addition to pricing underlying trades. The LOB models are heavily leveraged for valuing CVA and DVA.
4. CVA/DVA Calculation Framework: The high level framework for valuing CVA and DVA is as follows.
5. Centralized Correlated Random Numbers Generation: The centralized correlated random numbers (or Gaussian variates) is used for:
    a. Handling correlations within each asset class and across assets
    b. Before using this functionality, each LOB CVA can optionally generate its own random numbers (or Gaussian variates).
6. Material Market Factors Generation/Simulation: The arbitrage free simulation of all the material market risk factors takes the centralized correlated random numbers (or Gaussian variates) as inputs. It also leverages the underling LOB models.
7. Approximation for Immaterial Market Factors: The market generation/simulation also employs approximations for immaterial market risk factors (including no-arbitrage-free and/or uncorrelated simulations for materiality analysis).
8. Trade Valuation No Arbitrage Principle: The arbitrage-free valuation of all material trades in each counter party portfolio on each of the above simulated market paths, and as of each simulation time step, is achieved by calling the underlying desk pricers directly – with

improvements in some cases. Proper handling needs to be done of the physical settlement, the MTM resets, the accrual conventions, or possibly other future trade events.

9. Simulation Vertex Cash Flow Computation: The pricing model also needs to compute the cash flow amount in the payment currency (discounted to the simulation time step) with payment dates for each leg (possibly separately) of each trade on each simulated market path and as of each simulation time step.
10. Trade Valuation using Replication Model:
    a. Use the underlying trade portfolio to replicate or approximate complex trades that cannot be otherwise priced for the CVA.
    b. This is also helpful in speeding up CVA and DVA computation for the major dealers.
    c. However, this needs to pass back testing.
11. Trade Valuation under Proxy Pricers:
    a. Approximate fallback trade pricing models with conservative measures to be used in very limited cases.
        i. On complex trades with limited materiality that cannot be accurately and efficiently priced for CVA
        ii. On trades with limited materiality that cannot be priced accurately for CVA due to data quality issues
    b. Conservatives measures with no netting allowed with any other trades, no DVA allowed.
    c. Does need to pass back-testing.
12. Counter Party Exposure Valuation/Aggregation: Valuation of the counter party exposures (positive and negative exposures separately) by aggregating the trade values corresponding to a given counter party taking into account netting, collateral, default window, margin period of risk (MPoR), as well as other credit terms.
13. Margin/Collateral Exposure Valuation/Aggregation: Collateral handling for the variation margin (VM) and the independent amount (IA) or the initial margin (IM).The time-0 collateral balance should be the actual collateral balance or the required collateral balance in the base case. This is important for the CCAR scenarios. Also required in the handling of the cash flow at risk as part of MPoR.

14. Cross Asset Exposure Valuation/Aggregation: Before the true cross-asset CVA model gets used, the trades of each asset class of a CP netting and/or collateral group needs to be separated into artificial netting and/or collateral groups with proper collateral allocation.
15. CVA/DVA No Arbitrage Principle:
    a. Market implied default probabilities and recoveries, if available, and the waterfall logic.
    b. Valuation of CVA and DVA as arbitrage-free expected default losses combining the counter party exposures, default probabilities, and recovery (of the bank and the counter party).
16. Accuracy and Stability using Martingale Resampling: The underlying models need to provide martingale targets, such as the PV of the underlying trade, PV of the forward starting underlying trade, PV of the call and the put options on the underlying trade.
17. CVA/DVA Development/Run Tests:
    a. Convergence testing (by varying the number of simulation paths, time steps, and the random seeds).
    b. Smoothness testing on the CVA and the DVA, and the risk metrics.
    c. Various Boundary Cases Testing.
    d. Various Martingale Testing.
    e. P&L explanation testing.
    f. Back-testing.
18. Other Functionality – CVA/DVA Scenarios:
    a. Interactive Live and Proposed Trade (and what-if) CVA and DVA calculator
    b. Scenario analysis engine – particularly for CCAR
19. Other Functionality – PnL and Risk:
    a. Risk Metrics
    b. PnL Explanation (based on the risks) and PnL attribution (based on the actual market moves)
20. Asset LOB CVA Schematic Sketch: The schematic composition of a LOB CVA framework for a particular asset class contains four dynamic modules – market generation, trade

valuation, exposure valuation and aggregation, and CVA and DVA valuation – to be run daily.

21. Cross Asset CVA Schematic Sketch: The schematic of the cross-asset CVA modeling framework – that is integration LOB CVA models with further enhancements – follows. It consists of four dynamic modules (market generation, trade valuation, exposure valuation and aggregation, and CVA and DVA valuation to be run daily) and one quasi-dynamic module (centralized random number generation to be run on demand, but not necessarily daily).
22. Persistence Retrieval - CVA Component Calculation: The saving and retrieval of the intermediate results are an important part of the framework.
23. Dependence #1 - Intra/Inter Correlations: The centralized random number generator produces the correlation for each asset class, and for each cross-asset correlation.
24. Dependence #2 - IR CVA Metrics: The dependency of the simulations and the trade valuations of the non-IR CVA models on IR needs to be modeled.
25. Dependence #3 - FX CVA Metrics: The dependency of the trade valuation, exposure valuation and aggregation, and possibly CVA and DVA valuation on the FX rate must be modeled.
26. Dependence #4 - Entity Credit Spreads: The dependency of the CVA and the DVA valuation (and possibly exposure aggregation and valuation) on the simulated bank and the counter party credit spreads (and possibly default events) for the wrong-way and the right-ay risks need to be modeled.
27. Dependence #3 - Martingale Resampling/Testing: Martingale resampling and testing needs to be performed across all modules, and at all levels.

Overview of the Mathematical Definitions – CVA

1. USD LIBOR Risk Neutral Numeraire: The notations used here are as follows. The expectation $\mathbb{E}_0^Q$ is used under the USD LIBOR risk neutral probability measure with β_t as the time-t value of a money market account – with

$$\beta_0 = 1$$

that is consistent with the discounting of the uncollateralized underlying trades.
2. Netting Group Cash Flow Dates: t_i is the default time of the CP corresponding to the i^{th} CP netting group with recovery on default of R_i. T_i is the maximum final payment date of all trades in the i^{th} netting group, or the maximum mutual mandatory break date of the i^{th} CP netting group, whichever is less.
3. Netting Group Cash Flow Values: $V_i(t)$ is the time-t cash price of all the trades in the i^{th} CP netting group (converted to USD using the then FX spot rate).
4. Netting Group Gap Cash Flows: $C_i(t)$ is the time-t value of the cash flow at risk, which is the sum of all the contractual cash flows of all the trades in the i^{th} CP netting group (converted to USD with the then FX spot rate) with payment dates t_p satisfying $\max(t - \Delta t_{i,d})$ where $\Delta t_{i,d}$ is the default window. Initially one can approximate $C_i(t)$ by the sum of all cash flows with payment date t_p satisfying

$$t_p = t$$

5. Netting Group Cash Flow Collateral: $Collateral_i(t)$ is the time-t value of the collateral available for mitigating the counter party (or the bank) default risk (or subject to the counter party or the bank default risk) which will be discussed in more detail later.
6. Netting Group CP Default Probability: $P_{i,D}(t)$ is the time- t value of the cumulative probability of default of the CP corresponding to the i^{th} CP netting group accumulating to time-t; $P_{i,D}(0, t)$ is the time-0 value of the cumulative default probability of the i^{th} CP netting group accumulating to time-t.

7. <u>The Netting Group CVA #1</u>: The CVA due to the CP default corresponding to the i^{th} CP netting group is given by

$$CVA_i = -\mathbb{E}_0^Q\left[\frac{\{V_i(t_i) + C_i(t_i) - Collateral_i(t_i)\}^+}{\beta(t_i)}(1 - R_i)\mathbb{I}_{\{0 \le t_i \le T_i\}}\right]$$

$$= -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{V_i(t_i) + C_i(t_i) - Collateral_i(t_i)\}^+}{\beta(t_i)}(1 - R_i)\mathbb{I}_{t \le t_i \le t+dt}\right]$$

or

$$CVA_i = -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t)\}^+}{\beta(t_i)}(1 - R_i)\mathbb{I}_{t \le t_i \le t+dt}\right]$$

with

$$\tilde{V}_i(t) \equiv V_i(t_i) + C_i(t_i) - Collateral_i(t_i)$$

8. <u>The Netting Group CVA #2</u>: If there is no CP default event correlation, the above can be reduced to

$$CVA_i = -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t)\}^+}{\beta(t)}(1 - R_i)\mathbb{I}_{t \le t_i \le t+dt}\right] = -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t)\}^+}{\beta(t)}(1 - R_i)dP_{i,D}(t)\right]$$

which, if there is no CP default probability (or spread) correlation, can be further reduced to

$$CVA_i = -\int_0^{T_i} \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^+}{\beta(t)}(1 - R_i)dP_{i,0,D}(t)\right]$$

which, if the CP default recovery is constant, can be even further reduced to

$$CVA_i = -\int_0^{T_i} \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^+}{\beta(t)}\right](1 - R_i)dP_{i,0,D}(t) = -\int_0^{T_i} EPE_i(t)(1 - R_i)dP_{i,0,D}(t)$$

with the discounted expected positive exposure (EPE) given by

$$EPE_i(t) = \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^+}{\beta(t)}\right]$$

the discounted expected negative exposure (ENE) given by

$$ENE_i(t) = \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^-}{\beta(t)}\right]$$

and the total discounted expected positive exposure (TEPE) given by

$$TEPE_i = -\int_0^{T_i} EPE_i(t)dt$$

and the average discounted expected positive exposure (AEPE) given by

$$AEPE_i(t) = \frac{TEPE_i}{T_i}$$

9. <u>The Netting Group PDE Formulation</u>: Further the PFE at p-percentile is defined as

$$PFE_i(p, t) \equiv \inf\left\{x | \mathbb{P}\left[\frac{\{\tilde{V}_i(t)\}^+}{\beta(t)} \le x\right] \ge p\right\}$$

and the maximum PFE (MPFE) as

$$MPFE_i(p) \equiv \max_{\{t\}} PFE_i(p, t)$$

Overview of the Mathematical Definitions – DVA

1. Netting Group Bank Default Time: The additional notations used here are as follows: t_{BANK} is the default time of the bank with recovery R_{BANK}; T_i is the maximum final payment date of all trades in the i^{th} CP netting group, or the maximum mutual mandatory break date of the i^{th} CP netting group, whichever is less.
2. Netting Group Bank Default Probability: $P_{BANK,Def}(t)$ is the time- value of the cumulative self-default probability (accumulating to time t), and $P_{BANK,Def}(0, t)$ is the time 0 value of the cumulative self-default probability (accumulating to time t).
3. The Netting Group DVA #1: The DVA due to the bank default corresponding to the i^{th} CP netting group is given by

$$DVA_i = -\mathbb{E}_0^Q\left[\frac{\{V_i(t_{BANK}) + C_i(t_{BANK}) - Collateral_i(t_{BANK})\}^-}{\beta(t_{BANK})}(1 - R_{BANK})\mathbb{I}_{0 \le t_{BANK} \le T_i}\right]$$

$$= -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{V_i(t) + C_i(t) - Collateral_i(t)\}^-}{\beta(t_i)}(1 - R_{BANK})\mathbb{I}_{t \le t_{BANK} \le t+dt}\right]$$

or

$$DVA_i = -\mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}(1 - R_{BANK})\mathbb{I}_{t \le t_{BANK} \le t+dt}\right]$$

with

$$\tilde{V}_i(t) = V_i(t) + C_i(t) - Collateral_i(t)$$

which, if there is no self-default event correlation, can be reduced to

$$\begin{aligned} DVA_i &= -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}(1-R_{BANK})\mathbb{E}_t^Q\left[\mathbb{I}_{t\le t_{BANK}\le t+dt}\right]\right] \\ &= -\mathbb{E}_0^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}(1-R_{BANK})d\mathbb{P}_{BANK,D}(0,t)\right] \end{aligned}$$

which, if there is no self-default probability (or spread) correlation, can be further reduced to

$$DVA_i = -\int_0^{T_i} \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}(1-R_{BANK})\right]d\mathbb{P}_{BANK,D}(0,t)$$

which, if the self-default recovery is constant, can be even further reduced to

$$\begin{aligned} DVA_i &= -\int_0^{T_i} \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}\right](1-R_{BANK})d\mathbb{P}_{BANK,D}(0,t) \\ &= -\int_0^{T_i} ENE_i(t)\,(1-R_{BANK})d\mathbb{P}_{BANK,D}(0,t) \end{aligned}$$

with the discounted expected negative exposure (ENE) given by

$$ENE_i(t) = \mathbb{E}_0^Q\left[\frac{\{\tilde{V}_i(t)\}^-}{\beta(t_i)}\right]$$

the discounted total expected negative exposure (TENE) given by

$$TENE_i = \int_0^{T_i} ENE_i(t) dt$$

and the discounted average expected negative exposure AENE given by

$$AENE_i = \frac{TENE_i}{T_i}$$

4. <u>The Netting Group DVA #2</u>: For convenience the discounted expected exposure EE is defined as

$$EE_i(t) = -\mathbb{E}_0^Q \left[\frac{V_i(t) + C_i(t) - Collateral_i(t)}{\beta(t)}\right]$$

and

$$EE_i(t) = EPE_i(t) + ENE_i(t)$$

5. <u>The Netting Group PFNE Expression</u>: The p-percentile potential future negative exposure (PFNE) is further defined as

$$PFNE_i(p, t) \equiv \sup\left(x \middle| \mathbb{P}\left[\left\{\frac{\tilde{V}_i(t)}{\beta(t)} \geq x\right\} \leq p\right]\right)$$

and minimum PFNE as

$$MPFNE_i(p) \equiv \underset{\{t\}}{max} \; PFNE_i(p, t)$$

Collateral

1. Counter Party/Margin Consolidated Window: The notations used here are as follows. $\Delta t_{i,t_d}$ is the total default window based on the default window (uncollateralized case) $\Delta t_{i,d}$, and the margin call time period $\Delta t_{i,m}$ that is

$$\Delta t_{i,t_d} = \Delta t_{i,m} + \Delta t_{i,d} - 1D$$

with $1D$ denoting one business day making

$$\Delta t_{i,t_d} = \Delta t_{i,d}$$

for daily margin period.

2. Start of the Total Default Window:

$$t_{i,t_d} = \max\left(t - \Delta t_{i,t_d}, 0\right)$$

is the time at the beginning of the total default window, $V_{i,t_{i,t_d}}$ is the time t_{i,t_d} values of all trades in the i^{th} CP collateral group, $IA_{R,i}\left(t_{i,t_d}\right)$ - a positive number – the time t_{i,t_d} IA (independent amount) requirements governing the counter party collateral posting, $Th_{E,i}\left(t_{i,t_d}\right)$ - the time t_{i,t_d} effective collateral threshold governing the counter party – the i^{th} CP collateral group – collateral posting, which is the sum of the counter party threshold and the minimum transfer amounts (MTA) both of which are model inputs.

3. Intuition behind Default Window Start: Intuitively, the collateral is to be posted if the exposure plus IA cushion exceeds the effective collateral threshold.

4. Bank Independent Amount/Collateral Threshold: Similarly $IA_{R,BANK}(t_{i,t_d})$ and $Th_{E,BANK}(t_{i,t_d})$ - both negative numbers – are the IA requirements and the effective collateral threshold (which is the sum of the threshold and the MTA) governing the bank's collateral posting (which are negative numbers).
5. Collateral IA/IM Vertex Balances: $Collateral_{BALANCE,i}(t_{i,t_d})$, $IA_{BALANCE,i}(t_{i,t_d})$, and $VM_{BALANCE,i}(t_{i,t_d})$ are the time t_{i,t_d} value of balance of the total collateral, IA, and VM (variation margin). These can be positive or negative numbers, with positive numbers indicating that the bank receives collateral and negative numbers indicating that the bank posts collateral.
6. First Order Collateral Effect #1: The first-order impact of the collateral is the mitigation of the counter party default risk. More specifically when the counter party posts collateral to the bank, the bank's risk to the counter party default is reduced – with the exposure reduced by the amount of the collateral the bank receives.
7. First Order Collateral Effect #2: Similarly when the bank posts collateral to the counter party, the counter party's exposure to the bank's default risk is reduced (with the absolute value of the exposure reduced by the absolute value of the collateral posted).
8. Second Order Collateral Effect #1: The second order effect of the collateral is that a part of the collateral is subject to the counter party default risk if the collateral is not segregated. More specifically, when the bank posts collateral to the counter party, and when it expects the counter party to return some of the collateral back (for example, if the receivable amount decreases), the amount of collateral to be returned to the bank is subject to the counter party default risk, if the collateral is not segregated.
9. Second Order Collateral Effect #2: Similarly when the counter party posts collateral to the bank, and the bank is expected to return some collateral to the counter party (for example if the payable amount decreases), the amount of collateral to be returned to the counter party is subject to the bank's default risk, if the collateral is not segregated.
10. Computing the Required Collateral Balance: The time t required collateral balance $Collateral_{B,i}(t)$ for the i^{th} CP netting group is given by the following collateral rules – taking into account the margin period of risk. As a starting point, the assumption here is that the collateral group and the netting group are identical.

11. Counter Party Required Collateral Balance: When the bank has net default exposures to the counter party (the i^{th} counter party collateral group), the scenario for the counter party to post collateral, that is, if:

$$V_i(t_{i,t_d}) + IA_{R,i}(t_{i,t_d}) - Th_{E,i}(t_{i,t_d}) > 0$$

then

$$Collateral_{B,i}(t_{i,t_d}) = V_i(t_{i,t_d}) + IA_{R,i}(t_{i,t_d}) - Th_{E,i}(t_{i,t_d})$$

$$IA_{BANK,i}(t_{i,t_d}) = \min[Collateral_{BANK,i}(t_{i,t_d}), IA_{R,i}(t_{i,t_d})]$$

$$VM_{BANK,i}(t_{i,t_d}) = Collateral_{BANK,i}(t_{i,t_d}) - IA_{BANK,i}(t_{i,t_d})$$

12. Required Collateral Balance for Bank: Similarly when the counter party has net default exposures to the bank (in the i^{th} CP collateral group), the scenario for the bank to post collateral, that is, if:

$$V_i(t_{i,t_d}) + IA_{R,BANK}(t_{i,t_d}) - Th_{E,BANK}(t_{i,t_d}) < 0$$

then

$$Collateral_{B,i}(t_{i,t_d}) = V_i(t_{i,t_d}) + IA_{R,BANK}(t_{i,t_d}) - Th_{E,BANK}(t_{i,t_d})$$

$$IA_{BANK,i}(t_{i,t_d}) = \max[Collateral_{BANK,i}(t_{i,t_d}), IA_{R,BANK}(t_{i,t_d})]$$

$$VM_{BANK,i}(t_{i,t_d}) = Collateral_{BANK,i}(t_{i,t_d}) - IA_{BANK,i}(t_{i,t_d})$$

13. Collateral Balance under other Scenarios: In other scenarios

$$Collateral_{BANK,i}\left(t_{i,t_d}\right) = 0$$

14. Collateral Balance without Segregation Effects: Ignoring the collateral segregation effects (which would be conservative for the CVA) results in

$$Collateral_i(t) = Collateral_{BANK,i}\left(t_{i,t_d}\right)$$

15. Dual Party IA/IM Posting: Currently DRIP assumes that if one party posts IA/IM then the other party posts no collateral. Both parties posting segregated IA/IM will be considered in future phases.
16. Importance of the Initial Collateral Balance: The time-0 collateral balance should be the actual collateral balance, or the required collateral balance in the base case. This is important for the CCAR scenarios.
17. Hierarchy of the Aggregation Groups: In general, a counter party portfolio (for example, based on a legal entity) contains zero or more netting groups and/or non-netted trades. A netting group in turn contains zero or more collateral groups, and/or trades with trade level collateral, and/or uncollateralized trades. A collateral group contains a group of trades for which a particular set of collateral rules – or CSA – is applied.
18. Enforceability of Collateral/Netting Groups: Finally the netting enforceability and the collateral enforceability also need to be considered.

Summary of Model Inputs

1. Materiality Criterion Based Market Inputs: The model inputs include the following categories based on the materiality criteria above.

2. Market Data - Underlying Trade Pricing: All market data needed for underlying desk pricing (leveraging the underlying desk data).
3. Market Data - Implied Volatility Surface: Approximated implied volatilities based on proxies or benchmarks, if the true implied volatilities are not available.
4. Market Data - Asset Class History: Historical data for all (or most of) the market risk factors in all asset classes.
5. Market Data – CDS/Spread Data: CDS spread and recovery (or the equivalent) for each counter party, the bank's CDS spread and recovery, and the implied volatilities – or approximated implied volatilities – of these credit spreads.
6. Market Data - Counter Party History: Historical data for the CDS spread (or the equivalent) for each counter party.
7. Market Data - Bank Spread History: Historical data of the bank CDS spread (wrt each counter party, if possible).
8. Trade Data - Product Contractual Details: All trade data needed for the underlying desk trade pricing – leveraging the underlying desk data.
9. Trade Data - Operational/Settlement Details: Information on physical settlement, MTM reset, or accrual based convention, or possibly other future trade events.
10. Bank/Counter Party Netting Enforceability: Netting and netting enforceability reference data settings, for each counter party and bank.
11. Bank/Counter Party CSA Settings: All CSA data, such as variation margin rules, IA/IM rules, mandatory and optional mutual breaks, rating based contingencies, collateral segregation flag, and rehypothecation flag.
12. Bank/Counter Party Collateral Enforceability: Collateral enforceability reference data settings, for each counter party and the bank.
13. Bank/Counter Party Credit Settings: Credit ratings, GICS (Global Industry Classification Standard) at all levels.
14. Collateral/Netting Group Trade Lists:
    a. Complete trade list for each netting or collateral group or for non-netted and/or uncollateralized trades
    b. Trade population for each CP netting and collateral group

15. Asset/Trade Specific Reference Data: Trade reference data for each trade.
    a. Trade type, asset class, and underlying model for each trade.
    b. Trading book, accounts, and the bank entity.
16. Assorted Set of Martingale Targets: Martingale targets such as the PV of the underlying trade, PV of the forward starting underlying trade, and the PV of the call and the put options on the underlying trade for the martingale testing and re-sampling under the based and the stressed scenarios.
17. Base/Stressed Scenarios Trade PV: Underlying trade PV and risks under the base and the stressed scenarios.
    a. To be used by the replication model using the underlying hedge portfolio.
    b. To be used by the proxy pricers.

Summary of Model Outputs

1. Final Results #1 CVA/DVA: For production runs, for each counter party,
    a. CVA and DVA
    b. CVA and DVA time profile (or CVA and DVA by time period)
2. Final Results #2 EPE/ENE:
    a. EPE and ENE exposures time profile
    b. PFE time profile
    c. Uncollateralized EPE and ENE exposures time profile
3. Final Results #3 Defaults/Collateral:
    a. Time-0 Default Probabilities time profile for the bank and the counter parties
    b. Posted and received collateral time profile
4. Intermediate Results - Factors/XVA Measures:
    a. Entire market simulations with the risk factor values on all the simulation time steps and paths

   b. The entire trade prices, cash flows, and exposures on all simulation time steps and paths
5. Intermediate Results - PnL Attributions/Scenarios: Risks, scenario analysis, PnL explanations based on the risks, and PnL attribution based on the actual market moves.

Centralized Random Number Generation

1. Intra vs. Cross Asset Correlation: This scheme is for handling correlations within each asset class, as well as correlations across asset classes.
2. Asset/Risk Factor Historical Correlation: First the correlation matric within each asset class and across assets is obtained with the historical time series data of all the risk factoes, or implied or specified correlations, if available (with proper functional transformation, depending upon the simulation model).
3. PCA within Single Risk Family: For risk factors with term structure (except for credit derivatives), all the tenors in the correlation matrix are included, and then the PCA is performed to obtain the equivalent driving factors.
4. Cholesky Based Correlated Random Draws: With the above correlation matrix, and using the Cholesky decomposition, one can generate the centralized correlated random numbers (or Gaussian variates or incremental Brownians) represented as $\mathcal{N}(0, 1)$, that is, Gaussian variates with zero mean and unit standard deviation.
5. Correlated Brownians for Asset Increments: The simulation model for each asset class then takes the correlated random numbers as inputs or driving factors for generating future market simulations that are naturally correlated within each asset class and across asset classes.
6. Credit Spread/Market Factor Correlations: The same methodology is used to correlate the spread for each counter party and the bank to the market risk factors.

7. Cross Asset Correlation Validity Universe: This cross-asset correlation is applied only to material risk factors, and in the case of multi-factor models, only to the most dominant factor of each model.

Factor Model for Correlation Handling

1. Idiosyncratic vs. Systemic Risk Factors: The correlation structure for a large number of correlated risk factors can be conveniently parameterized by a factor model with reduced dimensionality. More specifically all risk factors are driven by a much smaller number of driving factors or "market" factors and their idiosyncratic factors.
2. Sector Grouping of Risk Factors: An optional grouping of the risk factors can further reduce the number of factors needed. Such grouping can be based on, for example, credit ratings, industries, and geographical regions for credit risk factors, and industries and geographical regions for equity risk factors.
3. Factor Loadings/Correlation Mathematical Formulation: The mathematical justification and details are as follows.
4. Linear Decomposition of the Risk Grid: Let R_{ij} be the j^{th} risk factor in group i with market factor M_i and idiosyncratic factor ϵ_j, then one has

$$R_{ij} = \beta_{ij} M_i + \alpha_j \epsilon_j$$

where M_i and ϵ_j are standard normal or Gaussian random variables with mean zero and standard deviation one, or $\mathcal{N}(0, 1)$, with ϵ_j being uncorrelated with all other random variables, and β_{ij} and α_j being constant coefficients. For convenience, with

$$\alpha_j^{\ 2} = 1 - \beta_{ij}^{\ 2}$$

R_{ij} has a standard deviation of one.

5. Correlation between Risk Factor Pairs: In general the correlations between a given pair of risk factors is given by

$$\langle R_{ij} R_{kl} \rangle = \beta_{ij} \beta_{kl} \langle M_i M_k \rangle$$

6. Single Group Risk Factor Correlation: The grouping is a further simplification in that

$$\beta_{ij} = \sqrt{\rho_i}$$

$$0 \leq \rho_i \leq 1$$

and thus all the risk factors in group i have the same pairwise correlation, that is

$$\langle R_{ij} R_{ik} \rangle = \rho_i$$

7. Cross Group Risk Factor Correlation: Therefore the correlation of a given pair of risk factors is simplified and given by

$$\langle R_{ij} R_{kl} \rangle = \sqrt{\rho_i \rho_k} \langle M_i M_k \rangle$$

8. Entries of the Correlation Matrix: In this case the correlation of the risk factors can be reduced to a much smaller group correlation matrix ρ_i as the diagonal entry and $\sqrt{\rho_i \rho_k} \langle M_i M_k \rangle$ as the off-diagonal entry.
9. Simplified Construction of the Matrix: This is not a block matrix, but a simple matrix, which simplifies the system input implementation. The correlation matrix of the risk factors derived from the group correlation matrix is indeed a block matrix.

10. <u>Credit Spread Group Correlation Matrix</u>: More specifically the procedure for obtaining credit spread group correlation matrix is as follows.
11. <u>Correlation Group On-Diagonal Entries</u>: The diagonal entry of the group correlation matrix is the constant pairwise correlation of the credit names in the group.
    a. It can be calculated by the average of pairwise correlation of selected names in the group.
    b. If the group contains only one credit name then the diagonal entry would be one.
    c. The valid range for this correlation is from zero to one inclusive.
12. <u>Correlation Group Off-Diagonal Entries</u>: The off-diagonal entries of the group correlation matrix are the correlation among two different groups.
    a. They can be obtained by the average of selected credit names or indices in one group selected with selected credit names or indices in the other group.
    b. The valid range for these correlations is from negative one to positive one inclusive.
13. <u>Historical Estimation of the Entries</u>: For the above correlations estimations, if feasible, historical returns correlations with five years or more of historical data of all the CDS spreads (including bank senior unsecured CDS, CP credit spreads, and ABS CDS and CMBS CDS, is applicable, as well as credit indices) with five year tenor.

Market Generation and Monte Carlo Simulation

1. <u>Material Market Risk Factors Simulation</u>: The next few sections deal with arbitrage-free Monte Carlo simulations of all material market risk factors leveraging the underlying LOB models.
2. <u>Simulation across Time Steps/Paths</u>: The Monte-Carlo simulations are performed with the given set of non-uniform time steps and a given number of simulation paths under the local currency risk neutral probability measures consistent with the underlying models with an

option to take the centralized correlated random numbers as input or to generate the random numbers internally.

General Scope of the Simulation

1. Materiality Measures for Risk Factors: It is important to first define the materiality measures for the risk factors so that one can determine which risk factors to simulate or simulate more accurately.
2. Primary Market Risk Factor Simulation: The primary risk factors are defined as those corresponding to the outright delta risks of the underlying OTC derivatives desks for each asset class, such as the outright rate curves for the IR derivatives, FX spots for FX derivatives, stock and stock index prices for equity derivatives, outright commodity futures for commodity derivatives, and credit curves and credit index curves for credit derivatives.
3. Secondary Market Risk Factor Simulation: Further the secondary market risk factors are defined as the spread curves or basis curves to the primary market risk factors. In addition volatility and correlation and the like are defined as higher order risk factors.
4. Primary Risk Factor Materiality Measure: The materiality measure of a primary risk factor is defined using its gross delta, and more specifically, as the percentage of the gross OTC derivatives delta in its asset class relative to the total gross OTC derivatives delta from all the primary risk factors in the asset class.
5. In/Cross Asset Factors Simulation: DRIP simulates all the primary risk factors, and simulates the top 95% of the primary risk factors in each asset class with proper in-asset (historical) correlation and cross-asset (historical) correlation.
6. Simulating Secondary Market Risk Factors: Also simulated are secondary market risk factors, typically with a mean-reverting one factor model with each simulated secondary risk factor without cross-asset correlation and only with correlation to the primary risk factor.

7. Higher Order Risk Factor Simulation: DRIP does not simulate higher order risk factors in general. Therefore stochastic volatility risk model is not recommended for production runs, except in limited cases where there are major concentrations of volatility risks (including variance and volatility swaps). Simplified stochastic volatility models are recommended only for the purposes of materiality analysis and modeling.
8. Implying Future/Forward Volatility/Skew: In addition the future of forward volatility and volatility skews will need to be derived or approximated for valuing trades using the Black or the Black-Scholes model (or the like).
9. Results from the IR CVA Simulation: For non-IR assets, interest rates are not simulated, but instead the IR simulation results will be leveraged from the IR CVA. For simulating a risk factor with a term structure, two or more factors (for level, slope, and so on) are implemented, except for the credit simulation due to the immaterial CVA (excluding trades facing monolines and CDPC's).

Correlations in Monte Carlo Simulations

1. In-Asset Risk Factor Correlation: The LOB CVA model handles the historical (or implied or specified) correlation among all the simulated risk factors within an asset class by generating their own random numbers with an option to retrieve the centrally generated random number.
2. The Cross-Asset Risk Factor Correlation: The cross-asset CVA models further handle the cross-asset correlation by using the centrally generated random numbers (representing correlated incremental Brownian motions) as the simulation driving factors. The same methodology is used to correlate the spreads of each counter party and the bank to the market factors.

Credit Risk Factors Simulation

1. XVA Models for Credit Derivatives: For credit simulations, the current models for credit derivatives CVA and DVA is leveraged, and used for credit derivatives trades that are not facing monolines and CDPC's. One factor simulation is used based on immateriality.
2. Spread Simulation for Bank/CP: This (or a similar model) will be used for simulating the credit spread of each counter party and the bank. While the counter party or the bank default events may not need to be simulated, the simulation of the default events of the underlying credits is desirable.

FX Risk Factors Simulation

1. Underlying LOB FX Simulation Models: The underlying LOB FX models (such as the local volatility model or the like) are leveraged for simulating the FX rates, which are a one-factor model with volatility skews. The implied volatility will be simulated with a simplified stochastic volatility model shown below for the purposes of materiality analysis and monitoring only.

Equity Risk Factors Simulation

1. Underlying LOB Equity Simulation Models: The underlying LOB equity simulation models (such as the local volatility models or the like) are leveraged for simulating the equity prices

– these are a one-factor model with volatility skews. The dividend rate, if material, will also be simulated, but this need not be arbitrage free. The implied volatility will be simulated with a simplified stochastic volatility model shown below for the purposes of materiality analysis and monitoring only.

Commodity Risk Factors Simulation

1. Underlying LOB Commodity Simulation Models: The underlying LOB models with multiple factors and volatility skews are leveraged. The implied volatility will be simulated with a simplified stochastic volatility model shown below for the purposes of materiality analysis and monitoring only.

IR Risk Factors Simulation

1. Underlying LOB IR Simulation Models: The underlying LOB models with multiple factors and volatility skews are leveraged. In addition to simulating the LIBOR or the OIS including FX simulations from FX CVA, DRIP also has developed additional one factor models for inflation, BMA/LIBOR ratio, and possibly 1M/3M spread – some of which may not be fully arbitrage-free – with proper correlation. The implied volatility will be simulated with a simplified stochastic volatility model shown below for the purposes of materiality analysis and monitoring only.

Simplified Stochastic Volatility Model

1. Materiality Analysis using Stochastic Volatility: The simplified stochastic volatility model is for the purposes of materiality analysis only. It is a mean-reverting one-factor model that models a multiplier to be applied to the implied volatility for trade valuation, and, if feasible, for market simulation. There is one such model for each of the relevant volatilities with no correlation among the driving factors of this model and the other market risk factors.

Interpolation in Monte Carlo Simulation

1. Broken Date Risk Factor Interpolation: The simulated risk factors may be needed on dates not coinciding with the simulation time steps, in which case additional interpolations may need to be performed. While Brownian Bridge interpolation is preferred, interpolation using square-root-of-time is also acceptable.

Trade Valuation

1. Arbitrage-free Single Trade Valuation: This section deals with arbitrage-free valuations (in terms of cash price) of all trades in the payment currency (in each counter party netting group

and collateral group) on each of the above simulated market paths and as of each simulation time step by using the (improved) trade pricing models.

2. Projected Forward Cash Flow Payments: The pricing model also needs to compute the cash flow amounts in the payment currency (discounted to the simulation time step) with payment dates for each leg (possibly separately) of each trade on ach simulated market path and as of each simulation time step.
3. Improvements to the Underlying Pricers: There is a need for improving the underlying models for the CVA purposes, including pricing speed optimization (for vanilla swaps, cross currency swaps, and cancellable swaps), handling of the trades with physical settlements, MTJ resets or other accrual based conventions, or possibly other future trade events, and for cash flow handling.
4. Speed Optimization: The underlying models need to save the intermediate results for reuse in subsequent CVA trade valuations to significantly improve the speed performance.
5. Pricing Time Horizon: For CVA it is critical to handle trades with proper settlement, MTM reset, or accrual based convention by pricing all the way to the end of each cash flow payment date (rather than only to the option expiry dates or the reset dates as in the underlying portfolio).
6. Cash Flow Valuation: In addition to prices, the underlying model needs to have an option to adjust current and future cash flows for each leg and for each trade in the payment currency and the corresponding payment dates (within a given date range).
7. Martingale Targets: Time-0 price for each trade under the base and the stressed scenarios is an important martingale target and needs to be made available to CVA models at run time. Other martingale targets will be discussed in the future.
8. Analytical Valuation: This uses an underlying LOB model to price trades on each of the simulated market paths and as of each simulation time step. In addition, the future of forward volatility or volatility skews will need to be derived or approximated for valuing trades using the Black or Black-Scholes model, or the like.
9. Least Squares Monte Carlo: This optionally uses the underlying LOB Monte Carlo (or American Monte Carlo) model, if available, to price trades on each of the above simulated market paths and as of each simulation time step.

10. Interpolation through Grids: This leverages the underlying LOB PDE (or other) grid model; the simulated market risk models are used to interpolated through the grid prices for the CVA trade valuation. A modified Arrow-Debreu price is needed to handle the expected path dependencies.

Sparse Grid or Sub-Sampling

1. Forward Exposure using LOB Model: This section deals with speeding up the brute force pricing of a trade whereby the underlying desk model is called on each of the simulated market paths and as of each time step.
2. Sparse Grid to Space/Time: Brute force is used only to price a sparse grid of market states and time steps, which will be either pre-defined or come from a sub-sampling of the simulated market paths (that is, by selecting a subset of the simulated market paths and/or a subset of simulation time steps).
3. Broken Sate Exposure Interpolation/Regression: DRIP uses linear or square-root-of-time interpolation (where the trade depends predominantly on one simulated variable) or regression (where the trade depends on multiple dominant simulated variables) to derive or approximate the trade prices on the simulated path and the time step where the brute force price is not available. Trade cash flows and events need to be handled properly.

Replication Model with Risks or Scenario Values

1. CVA/DVA on Replicated Portfolio: Replication (mostly dynamic) is the foundation of arbitrage derivatives pricing, which can also be applied to CVA/DVA for exposure replication as an option. Intuitively the replication model uses the underlying LOB hedging portfolio to replicate the original complicated trade which cannot be easily and efficiently directly priced for CVA/DVA.
2. Resulting XVA Calculation Speed Up: Conceptually the trade valuation for CVA/DVA is approximated by valuing the simpler replication portfolio. This is also helpful in significantly speeding up the CVA/DVA computation for the major dealers.
3. Quasi Static Replication Snapshot: For CVA/DVA static replication is used, in that today's risks or scenarios are used for replication, but the replication can be updated in daily runs (on business days).
4. Tenor Bucket Based Exposure Sensitivities: It is important to point out that bucketed risks or scenarios with respect to interest rates (and other) tenors are used to capture the time profile of future exposures. For instance, today's zero delta and gamma for the bucket of 10Y to 15Y indicates zero exposure to the future time horizon of 10Y to 15Y, to the second order.
5. Weights of the Replicating Portfolio: A more general approach and formulation is scenario based by approximately matching the scenario NPV of the original trades with the replication portfolio. In other words, given the error objective function under the pre-defined scenarios as

$$E = \sum_i \left\{ NPV(i) - \sum_j w_j NPV_j(i) \right\}^2$$

one minimizes it with respect to the time independent weights or notional $\{w_j\}$ of the replication instruments where $NPV(i)$ and $NPV_j(i)$ are the NPV of the original trade and the NPV of the j^{th} replication instrument respectively, under the i^{th} scenario.
6. Portfolio Corresponding to Error Minimization: The minimization leads to

$$\frac{\partial E}{\partial w_k} = 0$$

or

$$\sum_i \left\{ NPV(i) - \sum_j w_j NPV_j(i) \right\} NPV_k(i) = 0$$

7. Replication Model for the Collateralized Cases: The replication model is particularly useful for collateralized cases where the change in the NPV is more than the NPV in itself.
8. Replication Portfolio on the Trade: One way to incorporate the replication model into the CVA/DVA model is to replace the original trade with the replication portfolio.
9. Basket Options on the Replicated Portfolios: Another way to value CVA/DVA exposures is as the prices of the basket options (or forward staring basket options for the collateralized cases) on the replication portfolio, which can be approximated by the standard quadratic approximation, in which cases the market simulations are not needed; this, in turn, significantly simplifies the system implementation. This can be further simplified by using only linear instruments.
10. Scenario Specific Replication Weights Calibration: Ideally, fir CCAR and stresses in general, the weights or the notionals $\{w_j\}$ of the replication instruments should be recalibrated for each stress scenario.
11. Replication Model with Similar Trades: This is a special case where the original trade is approximated using a similar trade (or trades) with the same NPV and very close risks. One interesting example is an European or a Bermudan option to enter into a physically settled swap that can be easily replicated by the corresponding swap plus a European or a Bermudan option to cancel the swap.
12. Testing, Monitoring, and Conservative Measures: The validity of the replication model can be confirmed with Martingale testing, bench-marking against accurate models, and back-testing after production release when enough data is accumulated. Some testing must be included in the periodic ongoing monitoring. Optionally the DVA can be zeroed out for further conservativeness.

Proxy Pricer

1. Motivation behind the Proxy Pricer: While the models described above will capture the overwhelming majority of the risks/CVA of the trades, there will be trades with relatively significant risks/CVA that cannot be priced by the models and other approximations (for example, due to significant modeling and technology challenges), in which case DRIP will fall back to the following proxy pricers/approximations.
2. Choice of the Proxy Pricer: Additional conservative measures (described later) are applied to these trades. Depending on the availability of the data and the system limitations, a particular proxy pricer will be chosen for a given trade.
3. Testing for Measure Conservativeness: Benchmark testing must be performed against more accurate implementation in selected trades, as well as back-to-back testing to demonstrate the conservativeness.
4. Suitable Proxy Pricer Methodology Candidates: Good proxy pricer candidates include:
    a. Properly calibrated RFM (Risk Factor Methodology)
    b. Basel SA-CCR (the standardized approach for measuring counter party credit risk exposures), see Bank for International Settlements (2014)
    c. ISDA SIMM (Standard Initial Margin Model), see ISDA (2017) for collateralized cases.
5. The Fall Back Proxy Pricer: Before the above (and other) proxy pricers are formulated, one further falls back to the following proxy pricer methodologies. The conservativeness needs to be demonstrated and monitored via simplified back-testing, as outlined a few sections below.

Basel Notional Conversion Factor Based Proxy Pricer

1. Basel Notional Conversion Factor Scheme: The essence of the Basel notional conversion factor based proxy pricer is to approximate the add-on uncollateralized future positive trade exposure at a given time with its notional at that time, multiplied by the conversion factor as shown in the table below specified by Basel, see tables 4 and 5 in Bank for International Settlements (2005).
2. Basel Notional Conversion Factor Table: Conversion Factor Matrix for OTC Derivatives Contracts

Remaining Maturity	One Year or Less	Over One Year to Five Years	Over Five Years
Interest Rate	0.0%	0.5%	1.5%
Foreign Exchange and Gold	1.0%	5.0%	7.5%
Credit (Investment Grade Reference Obligor)	5.0%	5.0%	5.0%
Credit (non-investment Grade Reference Obligor)	10.0%	10.0%	10.0%
Equity	6.0%	8.0%	10.0%
Precious Metals (Except Gold)	7.0%	7.0%	7.0%
Other	10.0%	12.0%	15.0%

3. Basel Proxy Positive Exposure Expression: With this proxy pricer methodology the time-t NPV of the j^{th} trade in the i^{th} counter party netting group is approximated by

$$V_{ij}(t) \approx \alpha k N_{ij}\left(t_{i,t_d}\right)\frac{3}{2}\sqrt{t} + V_{ij}(0)\beta(t)$$

where $V_{ij}(0)$ and $N_{ij}\left(t_{i,t_d}\right)$ are the time-0 dollar NPV and the time t_{i,t_d} dollar notional of the trade, k is the conversion factor from table 2, and α is set by default to one – and can be set to 1.4 for further conservativeness.

4. Forward Time Funding Numeraire Discounting:

$$t_{i,t_d} = \max\left(t - \Delta t_{i,t_d}, 0\right)$$

is the time at the beginning of the total default window. The same as before, $\beta(t)$ is the time-t value of the money-market account with

$$\beta(0) = 1$$

that is consistent with the discounting of the underlying uncollateralized trades.

5. Volatility of the Forward Exposure: By default k is set to 15% - the most conservative conversion factor – unless the conservativeness of the other conversion factors can be validated using the simplified back-testing outlined earlier.
6. Diffusion Component of the Positive Exposure: The $\sqrt{t}$ in the above formula comes from the fact that the conversion factor is used to estimate the exposure or variation of future NPV of an OTC distribution in a one-year time horizon (Federal Register (2013)), and the assumption of the normal distribution that the exposure would grow approximately proportional to $\sqrt{t}$.
7. Normalization over the Exposure Horizon: The $\frac{3}{2}$ in front of the $\sqrt{t}$ comes from the integrated exposure below:

$$\frac{3}{2}\int_{0}^{1}\sqrt{t}dt = \int_{0}^{1} dt = 1$$

8. Forward Exposure under Uncollateralized Case: For the uncollateralized case the time-t trade level discounted positive exposure EPE for j^{th} trade in the i^{th} counter party netting group is approximated by

$$EPE_{ij}(t) \approx \max\left(\frac{\alpha k N_{ij}\left(t_{i,t_d}\right)\frac{3}{2}\sqrt{t}}{\beta(t)} + V_{ij}(0), 0\right)$$

and the time-t trade level discounted negative exposure ENE is set to zero for further conservativeness, i.e.,

$$EPE_{ij}(t) \approx 0$$

based on the conservativeness measures described below.

9. Forward Exposure under Collateralized Case: For the collateralized case, also needed is the future NPV at the collateralized date for j^{th} trade in the i^{th} counter party netting group, that is $V_{ij}\left(t, t_{i,t_d}\right)$ at time t_{i,t_d}, which can be approximated as

$$V_{ij}\left(t_{i,t_d}\right) \approx \alpha k N_{ij}\left(t_{i,t_d}\right)\frac{3}{2}\left(\sqrt{t} - \sqrt{t - t_{i,t_d}}\right) + V_{ij}(0)\beta(t)$$

10. No Complexity from Netting/Aggregation: The motivation behind the above formula is to achieve the desired collateral exposure without any special handling in the aggregation and the CVA/DVA calculation models.
11. Zero Threshold, MTA, IA/IM: For instance in the case of zero collateral thresholds, zero MTA, and zero IA/IM, the desired trade-level collateral exposure can be obtained as

$$\tilde{V}_{ij}(t_{i,t_d}) \approx \alpha k N_{ij}(t_{i,t_d}) \frac{3}{2} \sqrt{t - t_{i,t_d}}$$

12. CSA Specific Thresholds and MTAs: In general the trade level collateralized exposure is computed using the existing CSA terms (such as collateral thresholds and MTAs) except that the CP IA/IM is set to zero (if they are not zero already) for further conservativeness.
13. Segregated Bank IA/IM Collateral: The initial focus is on the cases where the IA/IM that the bank posts, if any, are segregated (and thus with negligible CP risk). Future phases will deal with the proper allocation of the IA/IM.
14. Zero Negative Collateralized Exposures: In addition the negative collateralized exposures from these trades are set to zero, as described in the few sections below.
15. Forward Time NPV Calculation Parameters: When DRIP calculates the time-t NPV, there are two pricing parameters, which can be changed according to different product types. The table below contains a summary of them.
16. Forward Time NPV Calculation Parameters Table: Trade Level Pricing Parameters

Parameter Name	**Description**	**Default Value**	**Comments**
Zero Out Negative NPV	If TRUE set MTM $V_{ij}(0)$ to zero when the original MTM is negative	FALSE	
Duration Shortening	If TRUE multiply a duration shortening factor to time-t trade values	TRUE	This is set to true for swap-like trades (with periodic payments and zero clean price right before its final maturity) and for collateralized Asian-like trades (2 weeks exposure decreases

			with time and is minimal close to its final maturity since trade NPV is largely dependent on fixings)
Use Absolute Value Notional	If TRUE set the notionals to absolute values	0TRUE	This flag provides the ability to test long/short trade portfolios

17. Expression for Duration Shortening Factor: The duration shortening factor $DSF_{ij}\left(t_{i,t_d}\right)$ is defined as

$$DSF_{ij}\left(t_{i,t_d}\right) \approx \max\left(1 - \frac{t_{i,t_d}}{T_j}, 0\right)$$

where T_j is the risk end time for the j^{th} trade.

18. DSF Impact on the Exposure: With duration shortening the NPV of the j^{th} trade on the j^{th} netting group on exposure date t and collateral date t_{i,t_d} is given by

$$V_{ij}(t) \approx DSF_{ij}\left(t_{i,t_d}\right)\left[\alpha k N_{ij}\left(t_{i,t_d}\right)\frac{3}{2}\sqrt{t} + V_{ij}(0)\beta(t)\right]$$

$$V_{ij}(t) \approx DSF_{ij}\left(t_{i,t_d}\right)\left[\alpha k N_{ij}\left(t_{i,t_d}\right)\frac{3}{2}\left(\sqrt{t} - \sqrt{t - t_{i,t_d}}\right) + V_{ij}(0)\beta(t)\right]$$

NPV Based Proxy Pricer

1. <u>NPV Based Proxy Pricer – Rationale</u>: The essence of the NPV based proxy pricer is to approximate the discounted positive trade exposure with its NPV – if positive – with linear amortization to zero to final maturity for swap-like trades (with positive payments and zero clean price before its final maturity) or with constant extrapolation to its final maturity for other trades. This is mainly applicable to uncollateralized cases.
2. <u>EPE for NPV Based Proxy</u>: More specifically, for swap-like trades, the time-t discounted positive exposure for the j^{th} trade on the j^{th} netting group is approximated as

$$EPE_{ij}\left(t_{i,t_d}\right) \approx {V_{ij}}^{+}(0) \max\left(1 - \frac{t_{i,t_d}}{T_j}, 0\right)$$

and for other trades

$$EPE_{ij}(t) \approx {V_{ij}}^{+}(0)$$

3. <u>ENE for NPV Based Proxy</u>: For all trades

$$ENE_{ij}(t) = 0$$

which is to zero out the negative exposures for conservativeness.
4. <u>Conservative Measures for NPV Proxy</u>: Further conservative measures of the proxy pricer – as described in a later section – include no netting benefit to the proxy trades with other non-proxy trades regardless of the netting agreement (by placing the proxy trades in their own netting group) and reduced collateral benefit by placing the proxy trades in their own collateral group (with no IM//IA).

Testing, Monitoring, Conservative Measures, and Materiality Analysis

1. Accuracy Testing against Selected Trades: Benchmark testing and back testing will be performed against more accurate implementation on selected trades.
2. Trade Level Conservative Measure Setting: Trade level conservative measures with zero negative exposure and grossing will be used on the trades with proxy pricers. For further conservatives the entire DVA of a CP netting group containing any proxy trade can be zeroed out.
3. Segregation of the Bank IA/IM: DRIP recommends that the user monitor and ensure that the IA/IM be set to zero for further conservativeness, and that the IA/IM bank posts, if any, be segregated – and thus have negligible counter party risks.
4. Operational Monitoring of the Proxy Pricer: One of the following will need to be monitored and ensured:
    a. The total CVA of the proxy trades is less than 5% of the total CVA of the firm.
    b. The gross delta of the trades is less than 5% of the total gross delta per each asset class.
    c. The gross disastrous stress impact of these trades is less than 5% of the total gross disastrous stress impact per asset class.
    d. Optionally – in certain cases – the gross delta of these is less than 5% of the total gross per asset class for each counter party (for simplicity).
5. Monitoring Proxy Contribution to the CVA: Optionally the percentage contribution of the proxy CVA (CVA from the proxy trades) to the total CVA can be monitored against the percentage contribution of the proxy trade gross notional for each asset class (for example, IR, FX, equity, commodity, and credit).
6. Proxy Trades and CCAR Scenarios: Proxy trades need to participate in the time-0 exposure calculation and time-0 collateral requirement – following the netting and the collateral agreements – for CCAR analysis and reporting.

7. Proxy Pricer PFE Estimation Enhancement: Proxy pricer for PFE purposes will be developed as needed.

Exposure Valuation and Aggregation

1. Collateral, Margining, and Netting Groups: This section deals with the valuation of the counter party exposures (positive and negative exposures separately) by aggregating the trade values taking into account netting, collateral, default window, margin period of risk, as well as some other credit terms such as mandatory mutual break.
2. Modeling Credit Ratings Contingent Transitions: Credit ratings related contingencies, such as ratings based collateral thresholds, will be optionally modeled with contingent credit ratings transitions.
3. Segregation, Re-couponing and Early Termination: Collateral segregation, re-couponing, and early termination are not considered in this section, which would be conservative for the CVA.
4. Constant/Proportional IA/IM Handling: For IA/IM, this section handles constant IA/IM. The VaR (value-at-risk) or risk-based IA/IM, if any, are approximated by the notional proportional IA/IM.
5. Treatment using Industry Standard Approaches: Other credit terms will be handled based on industry standard approaches.

Exposure Interpolation

1. Exposure Interpolation on Broken Dates: The dates for the margin period of risk may not coincide with the simulation time steps; in this case some interpolations need to be performed. While Brownian Bridge interpolation is preferred, interpolation using square-root-of-time is also acceptable.
2. Exposure Value at Margin Date: For calculating the collateral amount to be used for MPoR (or the collateralized exposure at time t), the portfolio value of the collateral group $V(t - \Delta t)$ needs to be calculated at a prior time $t - \Delta t$. The prior value $V(t - \Delta t)$ may or may not be readily available in the model implementation.
3. Square Root of Time Interpolation: If it is not available the following simple interpolation from the available values of $V(t)$ and $V(s)$ based on the square root of time for achieving the constant forward volatility in the interpolation will be used, that is

$$V(t - \Delta t) = V(t) - \frac{V(t) - V(s)}{\sqrt{t - s}} \Delta t$$

4. Generalized Margin Date Exposure Variance: It follows that

$$\mathbb{V}[V(t - \Delta t) - V(t)] = \frac{\mathbb{V}[V(t) - V(s)]}{t - s} \Delta t$$

which is consistent with the assumption of constant forward volatility in the time interval $[s, t]$.

5. Brownian Margin Date Exposure Variance: Furthermore, if $V(t)$ is a standard Brownian motion as a special case, that is, if

$$V(t) = W(t)$$

then the following desired result is obtained.

$$\mathbb{V}[W_t - W_{t-\Delta t}] = \frac{\mathbb{V}[W_t - W_s]}{t - s} \Delta t = \Delta t$$

6. Contractual Cash Flow Handling #1: This interpolation does not handle the contractual cash flow impact – due to the cash flows occurring in the time interval $[t - \Delta t, t]$ - which typically should not be significant unless the counter party is a major dealer, or under special cases.
7. Contractual Cash Flow Handling #2: If the cash flow impact is significant, it needs to be deducted from the discounted expected cash flows in the time interval $[t - \Delta t, t]$ from $V(s)$ before performing the above interpolation. Furthermore the cash flows need to be handled explicitly as part of the cash flows at risk as shown in the earlier sections.

Conservative Measures

1. Trade Level Conservative Measures with Zero Negative Exposure and Grossing: This section develops a conservative measure with zero negative exposure and grossing and an optional remediation of the CVA and DVA model inaccuracies. The essence of the conservative measure is as follows.
2. Zero DVA and No Netting: For trade types that cannot be modeled accurately enough, the associated DVA benefit can be optionally zeroed out, and the netting benefit disallowed.
3. Enforced Conservative IA/IM Terms: More specifically, for each trade with a conservative measure, the exposure at the trade level is priced with grossing (i.e., without netting any trades) using the existing CSA terms (such as collateral thresholds and MTA's) except that the CP IA/IM is set to zero if they are not zero already, for these trades for further conservativeness.
4. Segregated Bank IA/IM Posting: Initially the focus is on the cases where the bank IA/IM posts, if any, are segregated, and thus with negligible CP risk. In the future phases the IA/IM will be properly allocated. In addition the negative exposures from these trades will be set to zero.

5. Break down of the Netting Sets: As shown in the previous sections that time t exposure corresponding to the i^{th} CP netting group is given by

$$\tilde{V}_i(t) \equiv V_i(t) + C_i(t) - Collateral_i(t)$$

which can be broken at the j^{th} trade level as

$$\tilde{V}_i(t) \equiv \sum_j \left[V_{ij}(t) + C_{ij}(t)\right] - Collateral_i(t)$$

which can be further separated into values from regularly modeled trades (with variables with ^) and the trades with the conservatives measures (CM) as

$$\tilde{V}_i(t) \equiv \hat{V}_i(t) + \hat{C}_i(t) - \widehat{Collateral}_i(t) + \sum_{j \in \{Trades\ with\ CM\}} \left[V_{ij}(t) + C_{ij}(t) - Collateral_{ij}(t)\right]$$

6. The Conservative Measure Position Exposure: The positive exposure with the conservative measure (CM) is given by

$$\tilde{V}_{i,CM}{}^{+}(t) \equiv \left[\hat{V}_i(t) + \hat{C}_i(t) - \widehat{Collateral}_i(t)\right]^{+} + \sum_{j \in \{Trades\ with\ CM\}} \left[V_{ij}(t) + C_{ij}(t) - Collateral_{ij}(t)\right]^{+}$$

7. Conservativeness of the Positive Exposure: Recognizing that, by construction

$$V_i(t) + C_i(t) = \hat{V}_i(t) + \hat{C}_i(t) + \sum_{j \in \{Trades\ with\ CM\}} \left[V_{ij}(t) + C_{ij}(t)\right]$$

and for the uncollateralized or the fully collateralized cases (with CP and bank thresholds and zero CP and bank MTA's)

$$Collateral_i(t) = \widehat{Collateral}_i(t) + \sum_{j \in \{Trades\ with\ CM\}} Collateral_{ij}(t)$$

one has

$$\begin{aligned} \tilde{V}_{i,CM}{}^{+}(t) \geq \tilde{V}_i{}^{+}(t) \\ = \Bigg\{ \hat{V}_i(t) + \hat{C}_i(t) - \widehat{Collateral}_i(t) \\ + \sum_{j \in \{Trades\ with\ CM\}} \left[V_{ij}(t) + C_{ij}(t) - Collateral_{ij}(t) \right] \Bigg\}^{+} \end{aligned}$$

which demonstrates the conservatives of the CVA – the focus of CCAR.

8. Conservativeness of the Negative Exposures: The negative exposure with the conservative measure (CM) is given by

$$\tilde{V}_{i,CM}{}^{-}(t) = \left[\hat{V}_i(t) + \hat{C}_i(t) - \widehat{Collateral}_i(t) \right]^{-}$$

9. Conservativeness of the CVA and the DVA: In general, while mathematically the conservativeness of the DVA cannot always be guaranteed (as is also true for CVA in case of partial collateralization), in practice the CVA and the CVA+DVA are conservative as this conservative measure is overly conservative; further monitoring will be provided for this.
10. Zeroing out of the DVA: For further conservativeness, the entire DVA of a CP netting group containing any trade with a trade-level conservative measure can also be optionally zeroed out.

Portfolio Level Conservative Measure for Cross-Asset or Cross-System Portfolios

1. Cross-Asset Portfolio Breakdown: For a CP netting and/or collateral group with trades across assets, before the true cross-asset CVA models become available, the CP netting group and/or collateral groups can be artificially broken down into multiple sub-groups with each group containing trades from only a single asset class, and these are then independently run for CVA and DVA.
2. Applying the Conservative Collateral Measures: The same collateral rules for each sub-group are applied independently and the CP IA/IM are set to zero if they are not zero already, for these portfolios for further conservativeness.
3. Segregation of Bank IA/IM: The focus is on cases where the bank IA/IM posts, if any, are segregated, and thus with negligible CP risk. Future releases will properly allocate the IA/IM.
4. Conservativeness of the Collateralized/Uncollateralized Cases: The above is conservative for the CVA – the focus of CCAR – for the uncollateralized or the fully collateralized cases (with zero CP and bank thresholds and zero CP and bank MTA's) similar to the analysis of the previous section.
5. Conservativeness of the Portfolio CVA/DVA: While mathematically, in general, one cannot always guarantee the conservatives of the DVA (and of the CVA in case of partial collateralization), in practice the CVA and the CVA+DVA are conservative as this conservative emasure is overly conservative; further monitoring will be needed for this.
6. Zeroing Out of Portfolio DVA: Optionally, as a further conservativeness measure, the bank marginal default probability is bounded to be no greater than that of the CP and the bank threshold/MTA is bounded to be no greater than that of the CP – in terms of absolute values – respectively.

Monitoring and Materiality Analysis

1. Zero IA/IM and Segregation: Monitoring should be done to ensure that the IA/IM are set to zero for conservativeness, and the IA/IM bank posts, if any, are segregated (and thus with negligible CP risk).
2. Monitoring and Demonstration of Conservativeness: For trade-level conservatives the monitoring needs to ensure that the total CVA with the conservative measure is less than 5% of the total CVA of the bank. The conservativeness of the CVA and the CVA+DVA under the conservative measure needs to be demonstrated in comparison with those without the conservative measures.
3. Portfolio Level Conservativeness Monitoring/Verification: For the portfolio level conservative measure, the conservativeness of the CVA and the CVA+DVA also needs to be monitored and ensured.

CVA/DVA Valuation

1. Market Parameters for CVA Calculation: This section details the valuation of the CVA and the DVA – calculated from the expected default losses from the counter party exposures, the default probabilities, and the recoveries of the counter party and the bank.
2. Market Implied Default Probabilities and Recoveries: The following waterfall is used based on the data availability for deriving the default probabilities for each of the counter parties and the bank.
    a. CDS spread, recovery swaps
    b. CDS spread, CDS recovery
    c. Equivalent CDS spread derived from bonds or loans.

    More details are covered on the section on CVA/DVA curve construction.

Overview of Wrong-Way and Right-Way Risks

1. Right and Wrong Way Risks: Wrong way and right way risks refer to the co-movements or the co-dependency of the CP credit quality or the default event with the CP exposure, and possibly with the CP default recovery or the CP recovery rate.
2. Situations Causing Wrong-Way Risk: More specifically wrong way risk (WWR) refers to one or more of the scenarios below.
3. Higher CP Default with Higher CP Exposure: Higher likelihood of C default corresponding to higher CP exposure – suggesting joint diffusion modeling of the CP default rate and the market.
4. Exposure with/without CP Default: The exposure given CP actual default being higher than that without CP actual default (suggesting jump modeling of the CP default event and the market).
5. Lower CP Recovery/Higher CP Default: Lower CP default recovery (or recovery rate) corresponding to higher likelihood of CP default or actual CP default (suggesting additional joint modeling of the state-dependent CP default recovery or recovery rate).
6. Situations Causing Right Way Risk: The right way risk (RWR), on the other hand, refers to one or more of the situations below.
7. Higher CP Default - Lower CP Exposure: Higher likelihood of CP default corresponding to lower CP exposure (suggesting the joint diffusion modeling of the CP default rate and the market).
8. Exposure With/Without CP Default: The exposure given CP actual default being lower than that without actual CP default (suggesting joint jump modeling of the CP default rate and the market).

9. Higher CP Recovery on Higher CP Default: Higher CP default recovery (or recovery rate) corresponding to higher joint likelihood of CP default (suggesting additional joint modeling of the state-dependent CP default recovery or recovery rate).

Specific Wrong Way and Right Way Risks

1. Self-Referencing OTC Derivatives Trades: Specific wrong-way and right-way risks arise from (direct or indirect) self-referencing OTC derivatives trades.
2. Example - CP Selling Self-Options: For example, a CP selling puts on its own stock (for example, as a stock buy-back strategy) is a specific wrong-way risk. A CP selling calls on its own stock (as part of a tax optimization strategy on convertible bonds, for example) is a specific right-way risk.
3. Example - CP Self Index Trades: Similarly the counter party can buy or sell credit protection on a credit index of which it is a constituent. Thus selling credit protection constitutes a specific wrong-way risk and buying credit protection constitutes a specific right-way risk.
4. Example - EM CP Self-FX: Furthermore an emerging market sovereign counter party can enter (FX or cross-currency trades) referencing their local currency. It is a specific wrong-way risk if the emerging market sovereign counter party has a long position on its local currency (versus a stronger currency) and it has a specific right-way risk if the emerging market sovereign counterparty has a short position in its local currency versus a stronger currency (for example as a hedge to its bond issuance in foreign/stronger currencies).
5. Identifying the Self-Referencing Dependency: The first step towards handling the specific wrong-way and the right-way risks is to identify the dependency of the payoff of the OTC derivatives on the counter party (or the related entities).
6. WWR/RWR for Collateralized Transactions: While the impact of the specific wrong-way and the right-way risks is normally significant for uncollateralized transactions, it is not negligible for collateralized transactions.

7. Specific WWR: Self-Referencing CSA: The specific wrong-way risk can also arise from self-referencing CSA (Credit Support Annex) whereby, for example, the collateral threshold governing the collateral posting can reduce, or the counter-party posted IA/IM can increase, if the counter party's credit rating is downgraded below a certain level, both resulting in the counter party posting more collateral – and thus less effective counterparty exposure. This is quite challenging to model.

General Wrong-Way and Right-Way Risks

1. CP State Co-dependence on Market State: General wrong-way and right-way risks arise from the market drive co-movements or the co-dependency of the counterparty credit quality or default with the counter party exposure.
2. WWR Long/Short Positive Co-movement: For instance it tends to be a general wrong-way risk if the counter party has a long position in the commodity market, or the equity market, or the credit market, or a short position in the interest rate market, and *vice versa*.
3. Market Driven Collateral Quality Change: General wrong-way risk can also arise from the deterioration of the quality of the collateral that the counter party posts resulting from market conditions. More specifically it is a general wrong-way risk if the value of the collateral declines when the counter party has a higher likelihood of default.
4. Example OTC Derivatives Collateral Trade: This is particularly important for OTC derivatives trades facing the SPV's (special purpose vehicles) for CDO's or repackaging, where the only collateral available for mitigating the counter party credit risk is the collateral held in the SPV. This is also applicable to OTC derivatives that trade as a part of the package of a loan (both backed by the property as a collateral).
5. Collateral Call Impact on the CP: Another situation that needs to be considered it that, while collateralized transactions can significantly reduce the counter party risk, calling collateral

beyond a certain amount (due to adverse market moves) may actually drive the counter party into default.

6. WWR/RWR for Collateralized Transactions: While the impact of the general wrong-way and the right-way risks is normally significant for uncollateralized transactions, it is not negligible for collateralized transactions.
7. State Dependent CP Default/Recovery: The state-dependent CP default or recovery rate refers to the CP's net asset available to cover the claims upon the actual CP default.

Extreme Wrong-Way Risks

1. Extreme WWR Credit Derivatives Trades: Extreme wrong-way risks arise from uncollateralized OTC derivatives trades – particularly on the CDO senior and the super-senior tranches – facing monolines or CDPC's (credit derivatives product companies), whereby the monolines or the CDPC's sold protection.
2. Thinly Capitalized Monolines and CDPC's: Since monolines and CDPC's are very thinly capitalized entities, the moment credit protection buyers claim default against them (due to underlying obligor defaults and/or CDO tranche attaching (which will most likely happen under extreme or severe market-stress conditions where credit/default correlation can be typically very high), they will most likely be driven to default, as their capital will be wiped out.
3. Monolines/CDPC's Wrong-Way Risks: In the above case, as such the default loss severity would be extremely high, wiping out almost the entire NPV of the credit protection trades. In some cases monolines or CDPC's may also buy credit protection, in which case it would be general right-way risk.

Methodology Survey of WWR and RWR Models

1. Approaches behind RWR/WWR Methodologies: Mercurio and Li (2015) and Li and Mercurio (2015) provide good surveys of the RWR and the WWR methodologies, which essentially model the CP default rate correlated with the market together with the market jump upon CP default, via jump-diffusion models.
2. Implementing Models of Increasing Complexity: This chapter breaks down these models into a series of models with increasing complexity and incorporates them in a phased approach, starting two sections below. Also implemented is a model for state-dependent or random CP default recovery or recovery rate.
3. Decomposition into the CVA Components: A Merton structured default model can be fit into this framework, for example, for estimating the jump size. A mixture model where the CVA is broken into an *independent* part and a *completely dependent* part is useful for intuition building.
4. Systems Impact Arising out of Model Development: The model implementation has various implications to the system, such as computing and serving multiple sets of exposures, and data-handling including identification of self-referencing trades and credit risk collateral.
5. Counter Party Default/Market Copula Correlations: For the modeling flavors that model the CP default times correlated with the market (similar to the CDO copula methodology) or the CP default time correlated with the market plus CP default rate correlated with the market, more elaborate implementation (as well as the corresponding business intuition) is required.

Identification and Monitoring of WWR and RWR CPs

1. Identification of Generic RWR/WWR: The generic WWR/RWR CPs be identified based on historical correlation of CP spreads with CP exposures, through:

a. The historical data of the CP underlying portfolio NPV and the CP 5Y spread, or better yet
b. The historical data of the independent CVA CP Spread01 and the CP (5Y) spread.

2. Monitoring Generic/Specific WWR/RWR: The materiality of the WWR or the RWR CVA can be monitored using the methods specified two sections below.
3. Specific WWR/RWR Identification #1: For specific WWR and RWR CP's, they can be identified via self-referencing trades/CPs directly from the trade data.
4. Specific RWR/WWR Identification #2: Indirectly specific RWR/WWR can be identified via stress testing by setting the market data related to the CP, such as the stock price of the CP (or a related entity), credit spread of the CP (or a related entity), or the currency of the CP domiciled country, to a distressed level. The CP's having response to such stress tests are the specific WWR and the RWR CPs.
5. Identification of the CSA WWR/RWR: WWR and RWR related to collateral values and/or the CP's net asset available for default recovery can be identified via a fundamental analysis of the CP's balance sheet.

Mathematical Modeling of Wrong-Way and Right-Way Risks

1. CP Exposure/Default Convexity Adjustment: This section starts with WWR and RWR modeling using simple approaches, such as convexity adjustment and default scenario exposure. The default scenario exposure is to estimate the exposure given CP default by analyzing the ranges of values of the relevant risk factors under the CP default and computing the CP exposures by setting the relevant risk factors to the default scenario values.
2. Simplified Approach for Materiality Analysis: For instance under the scenario of CP default, the default scenario value of its stock price can be set to zero. The purpose of these simple approaches is mainly materiality analysis.

3. Accurate RWR/WWR Modeling Approaches: More accurate and general approaches to the RWR and the WWR modeling require the modeling of the CP jump-to-default (and/or jump-to-default of the bank) in addition to the existing diffusion processes. More details will be added in line with Mercurio and Li (2015) and Li and Mercurio (2015) with possible simplifications and additions.

A Simple WWR/RWR Model Based on the Convexity Adjustment to the CP Portfolio Value

1. Convexity Adjustment Based WWR/RWR: This section formulates a simple model to capture the general WWR and the RWR based on convexity adjustment – through the change of probability measure – on the CP portfolio value (leaving the CP default probabilities unchanged) addressing the diffusion based general RWR/WWR.
2. Uncorrelated Incremental Contribution to CVA: Starting with a CVA model without considering WWR or RWR, the probability of default and the portfolio value are uncorrelated and the following expectation can be evaluated separately.

$$\mathbb{E}_Q[\mathbb{I}_{t<\tau\leq t+\Delta t}{V_\tau}^+] = \mathbb{E}_Q[\mathbb{I}_{t<\tau\leq t+\Delta t}] \cdot \mathbb{E}_Q[{V_\tau}^+] = PD \times EPE$$

where

$$PD = \mathbb{E}_Q[\mathbb{I}_{t<\tau\leq t+\Delta t}]$$

is the probability of default, and

$$EPE = \mathbb{E}_Q[{V_\tau}^+]$$

is the expected positive exposure.

3. Correlated Incremental Contribution to CVA: When the probability of default and the portfolio value are correlated, one has

$$\mathbb{E}_Q[\mathbb{I}_{t<\tau\leq t+\Delta t}V_\tau{}^+] = PD \times \mathbb{E}_Q\left[\frac{\mathbb{I}_{t<\tau\leq t+\Delta t}}{PD} \times V_\tau{}^+\right]$$

   This formula is similar to the NPV in the forward default measure if PD is interpreted as a "discount factor".

4. Normal Exposure Lognormal Default Probability: The following assumptions are made in this simple model.
   a. The default probability is related to the credit spread, which is lognormally distributed.
   b. On each credit date the portfolio value is normally distributed.
   c. The portfolio value and the credit spread are correlated.

5. Incremental CVA Calculation Parameter Estimates: The volatility of the probability of default (or the credit spread) needs to be estimated, along with the volatility of the portfolio values and their correlations. The following parameters are estimated:
   a. D => The NPV delta for 1% of the risk factor shift
   b. σ_{CS} => The counter party credit spread lognormal volatility
   c. σ_{RF} => The underlying risk factor lognormal volatility
   d. ρ => The correlation of the counter party credit spread and the underlying risk factor

6. Risk Factor vs. Exposure Linearity: Under a linear approximation of the portfolio value in terms of the risk factor, the normal volatility of the portfolio is

$$\sigma_V = 100D\sigma_{RF}$$

   The hazard rate is assumed to related to the credit spread as

$$h = \frac{CS}{1 - R}$$

where R is the recovery rate.

7. Conditional Expectation of the CVA Increment: The expectation formula involving correlated random variables is now applied. Let x_1 and x_2 be two normally distributed random variables with a correlation ρ; for any function $g(x_2)$ the following expectation results:

$$\mathbb{E}[e^{x_1} g(x_2) | (x_1, x_2) \sim \mathcal{N}(\mu_1, {\sigma_1}^2, \mu_2, {\sigma_2}^2, \rho)]$$
$$= \mathbb{E}[e^{x_1} | x_1 \sim \mathcal{N}(\mu_1, {\sigma_1}^2)] \cdot \mathbb{E}[g(x_2) | x_2 \sim \mathcal{N}(\mu_2 + \rho\sigma_1\sigma_2, {\sigma_2}^2)]$$

8. Incorporating Correlations between x_1 and x_2: Note that the expectations on the right hand side are evaluated in a one-dimensional normal distribution in x_1 and x_2 separately, but x_2 has an extra term in the mean. This will be proved at the end of this section.
9. De facto Change of Measure: Application of this formula to above results in

$$\mathbb{E}_Q[\mathbb{I}_{t<\tau\le t+\Delta t} {V_\tau}^+] = PD \times \mathbb{E}_P[{V_\tau}^+]$$

Under a new measure $\mathbb{E}_P[\cdot]$ V_τ has an extra drift term $\rho\sigma_{CS} \cdot 100D\sigma_{RF}$ but the same volatility.

10. Estimating the WWR/RWR Impact: One application of this model is to estimate the impact of WWR and RWR based on the results from a model without the feature if WWR and RWR. The following steps can be performed to estimate the impact.
11. Estimating Exposure μ and σ^2: On each credit date it is assumed that the value of the portfolio is normally distributed. The two parameters of the normal distribution $\mathcal{N}(\mu, \sigma^2)$ - the mean μ and the variance σ^2 - can be backed out from ENE and EPE.
12. Incorporating the Convexity Correction Drift: The extra drift term can be backed out of the above formula and added to the mean μ.
13. Recalculating the EPE and the ENE: The EPE and the ENE can be evaluated from the normal options pricer with an extra drift term and the same normal volatility.

14. <u>Steps for Estimating μ/σ^2</u>: In the first step, note that

$$EPE = \mathbb{E}_Q[V_\tau^+ \mid V_\tau \sim \mathcal{N}(\mu, \sigma^2)]$$

$$ENE = \mathbb{E}_Q[V_\tau^- \mid V_\tau \sim \mathcal{N}(\mu, \sigma^2)]$$

resulting in

$$EPE + ENE = \mu$$

$$EPE = \mu \cdot \mathcal{N}\left(\frac{\mu}{\sigma}\right) + \sigma \cdot \phi\left(\frac{\mu}{\sigma}\right)$$

where $\mathcal{N}(\cdot)$ is the cumulative probability distribution function and $\phi(\cdot)$ is the probability density function for the standard normal distribution.

15. <u>Explicit Estimation of μ and σ^2</u>: There are then two equations to solve for μ and σ. It follows immediately that

$$\mu = EPE + ENE$$

Specific functions are available in the DRIP normal option pricer to estimate σ.

16. <u>Convexity Correction Drift Pick-Up</u>: I the second step the extra term under the new measure $\mathbb{E}_P[\cdot]$ is calculated. V_τ is normally distributed with the extra drift term

$$V_\tau \sim \mathcal{N}(\mu + \rho\sigma_{CS} \cdot 100D\sigma_{RF}, \sigma^2)$$

17. <u>Readjusted Values for EPE/ENE</u>: Let the updated drift term be

$$\tilde{\mu} = \mu + \rho\sigma_{CS} \cdot 100D\sigma_{RF}$$

Therefore the EPE and the ENE with the consideration of WWR and RWR are

$$\widetilde{EPE} = \mathbb{E}_P[V_\tau{}^+ \mid V_\tau \sim \mathcal{N}(\tilde{\mu}, \sigma^2)] = \tilde{\mu} \cdot \mathcal{N}\left(\frac{\tilde{\mu}}{\sigma}\right) + \sigma \cdot \phi\left(\frac{\tilde{\mu}}{\sigma}\right)$$

$$\widetilde{ENE} = \tilde{\mu} - \widetilde{EPE}$$

18. <u>Re-estimation of CVA and DVA</u>: Given the updated EPE and ENE, the CVA and the DVA can be calculated as follows, respectively.

$$\widetilde{CVA} = \widetilde{EPE} \times PD_{CP} \times (1 - R_{CP}) \times D_f(\cdot)$$

$$\widetilde{DVA} = \widetilde{ENE} \times PD_{BANK} \times (1 - R_{BANK}) \times D_f(\cdot)$$

19. <u>Derivation of the Convexity Adjustment</u>: Note the formula

$$\begin{aligned}\mathbb{E}[e^{x_1} g(x_2) | (x_1, x_2) \sim \mathcal{N}(\mu_1, {\sigma_1}^2, \mu_2, {\sigma_2}^2, \rho)]& \\ = \mathbb{E}[e^{x_1} | x_1 \sim \mathcal{N}(\mu_1, {\sigma_1}^2)] \cdot \mathbb{E}[g(x_2) | x_2 \sim \mathcal{N}(\mu_2 + \rho\sigma_1\sigma_2, {\sigma_2}^2)]&\end{aligned}$$

is derived. The derivation can be done writing the probability density function in x_2 and the conditional probability density function of x_1 given x_2 as

$$f(x_1, x_2) = f(x_2, \mu_2, {\sigma_2}^2) \cdot f\left(x_1, \mu_1 + \rho\frac{\sigma_1}{\sigma_2}[x_2 - \mu_2], [1 - \rho^2]{\sigma_1}^2\right)$$

then performing the integration in x_1 first.

20. <u>Measure Change using Girsanov Theorem</u>: Alternatively it can be obtained from the Girsanov theorem using a measure change. Casting x_1 and x_2 in terms of two uncorrelated standard normal variables ϵ_1 and ϵ_2 helps perform in the integration in each variable separately:

$$x_1 = \mu_1 + \sigma_1 \left(\rho\epsilon_2 + \sqrt{1-\rho^2}\epsilon_1\right)$$

$$x_2 = \mu_2 + \sigma_2\epsilon_2$$

21. Decomposition using Dependent/Independent Wanderers: The variables are separated to perform the expectation:

$$\begin{aligned}\mathbb{E}[e^{x_1} g(x_2)] &= \mathbb{E}\left[e^{\mu_1 + \sigma_1\sqrt{1-\rho^2}\epsilon_1}\right] \cdot \mathbb{E}[e^{\rho\sigma_1\epsilon_2} g(\mu_2 + \sigma_2\epsilon_2)] \\ &= e^{\mu_1 + \frac{1}{2}{\sigma_1}^2 - \frac{1}{2}\rho^2{\sigma_1}^2} \cdot \mathbb{E}[e^{\rho\sigma_1\epsilon_2} g(\mu_2 + \sigma_2\epsilon_2)] \\ &= \mathbb{E}[e^{x_1}] \cdot \mathbb{E}\left[e^{\rho\sigma_1\epsilon_2 - \frac{1}{2}\rho^2{\sigma_1}^2} g(\mu_2 + \sigma_2\epsilon_2)\right]\end{aligned}$$

22. Application of the Girsanov Theorem: Note that the Girsanov theorem states that

$$\mathbb{E}_{\epsilon \sim \mathcal{N}(0,1)}\left[e^{\alpha\epsilon - \frac{1}{2}\alpha^2} g(\epsilon)\right] = \mathbb{E}_{\epsilon-\alpha \sim \mathcal{N}(0,1)}[g(\epsilon)]$$

which results in

$$\mathbb{E}[e^{x_1} g(x_2)] = \mathbb{E}[e^{x_1} | x_1 \sim \mathcal{N}(\mu_1, {\sigma_1}^2)] \cdot \mathbb{E}[g(x_2) | x_2 \sim \mathcal{N}(\mu_2 + \rho\sigma_1\sigma_2, {\sigma_2}^2)]$$

23. Using the Radon-Nikodyn Derivative: Thus the change of measure can be applied directly to obtain the extra drift term

$$\mathbb{E}_Q[e^{x_1} g(x_2)] = \mathbb{E}_Q[e^{x_1}] \cdot \mathbb{E}_P[g(x_2)]$$

where the Radon-Nikodyn derivative is

$$\frac{dP}{dQ} = \frac{e^{x_1}}{\mathbb{E}_Q[e^{x_1}]}$$

where x_1 and x_2 are the values of the following two processes $x_1(t)$ and $x_2(t)$ at time instant 1:

$$\Delta x_i(t) = \mu_i \Delta t + \sigma_i \Delta z_i$$

$$i = 1, 2$$

$$\Delta z_1 \cdot \Delta z_2 = \rho \Delta t$$

24. Corresponding Radon-Nikodyn Measure Change: The measure change can be realized from a change of numeraire from

$$B(t) \equiv 1$$

to

$$P(t) = \mathbb{E}\left[e^{x_1(1)} | t;\ x_1(t)\right]$$

using

$$\frac{dP}{dQ} = \frac{P(1)/P(0)}{B(1)/B(0)}$$

25. Value of the Convexity Adjustment: Note that

$$\frac{\Delta P(t)}{P(t)} = \sigma_1 \Delta z_1$$

(Brigo and Mercurio (2007)) and the extra drift term for x_2 under the numeraire $P(t)$ is

$$\Delta(\mu_2 \Delta t) = \Delta x_2 \cdot \Delta \log\left(\frac{P(t)}{B(t)}\right) = (\mu_2 \Delta t + \sigma_2 \Delta z_2) \cdot \sigma_1 \Delta z_1 = \rho \sigma_1 \sigma_2 \Delta t$$

A Simple WWR/RWR Model with Approximated CP Probability Distribution

1. Constant Exposure CP Default Correlation: The mode in the previous section is based on the constant correlation between the CP portfolio value and the CP default probability, either directly or through the delta approximation (and performing the drift adjustment on the drift of the CP portfolio value).
2. Constant Factor/CP Default Correlation: The model in this section is based on constant correlation between the CP portfolio underlying the risk factors and the CP default probability, which is more accurate because it can handle state-dependency and time-dependency between the CP portfolio value and CP default probability arising from non-linear underlying trades, underlying trades duration shortening over time, and long/short underlying trades in CP portfolios.
3. Approximate of the CP Default Distribution: The essence of the model in this section is to approximate the CP default probability distribution leaving the CP underlying values unchanged, addressing the diffusion based general WWR/RWR.
4. CVA for the i^{th} CP: Recalling from the CVA formula for the i^{th} CP

$$CVA_i = -\mathbb{E}_0{}^Q\left[\int_0^{T_i} \frac{\tilde{V}_{i,t}{}^+}{\beta_t}(1 - R_i)\mathbb{E}_0{}^Q\left[\mathbb{I}_{t<t_i<t+dt}\right]\right] = -\mathbb{E}_0{}^Q\left[\int_0^{T_i} \frac{\tilde{V}_{i,t}{}^+}{\beta_t}(1 - R_i) dP_{i,t}{}^D\right]$$

5. Distribution of the Probability Decay: It is assumed that the CP recovery rate R_i is non-stochastic, and that the CP default rate $\frac{dP_{i,t}{}^D}{dt}$ follows the following log-normal process (assuming that the interest rate is non-stochastic):

$$\frac{dP_{i,t}{}^D}{dt} = \frac{dP_{i,0,t}{}^D}{dt} e^{\frac{\sigma_i{}^{CP}}{\sqrt{\vec{w}_i{}^T \rho_M \vec{w}_i + \alpha_i{}^2}} \cdot \left(\vec{w}_i{}^T \cdot \vec{\boldsymbol{W}}_t{}^M + \alpha_i W_{it}\right) - \frac{1}{2}\left(\sigma_i{}^{CP}\right)^2 t}$$

   where $\frac{dP_{i,0,t}{}^D}{dt}$ is the forward CP default rate, $\vec{\boldsymbol{W}}_t{}^M$ is the column vector of the standard Brownian motions driving the market risk factors, which in turn drive the exposures $\frac{\tilde{V}_{i,t}}{\beta_t}$, W_{it} is a standard Brownian motion – independent of the market Brownian motion $\vec{\boldsymbol{W}}_t{}^M$ - representing the idiosyncratic factor of the CP default rate, $\sigma_i{}^{CP}$ is the constant log-normal volatility of the CP default rate, $\vec{w}_i$ is a column vector of constant weights and α_i is a constant weight, which can be determined from the correlations of the CP default rates and the market Brownian motions $\vec{\boldsymbol{W}}_t{}^M$ and their constant correlation matrix ρ_M among the Brownian motions $\vec{\boldsymbol{W}}_t{}^M$.

6. The Corresponding WWR/RWR CVA: Consequently the WWR/RWR CVA is given by

$$CVA_i = -\mathbb{E}_0{}^Q\left[\int_0^{T_i} \frac{\tilde{V}_{i,t}{}^+}{\beta_t}(1 - R_i)\frac{dP_{i,0,t}{}^D}{dt} e^{\frac{\sigma_i{}^{CP}}{\sqrt{\vec{w}_i{}^T \rho_M \vec{w}_i + \alpha_i{}^2}} \cdot \left(\vec{w}_i{}^T \cdot \vec{\boldsymbol{W}}_t{}^M + \alpha_i W_{it}\right) - \frac{1}{2}\left(\sigma_i{}^{CP}\right)^2 t} dt\right]$$

7. Extraction of the Market Brownians: To back-fit this model into existing independent CVA models the market Brownians $\vec{\boldsymbol{W}}_t{}^M$ can be backed out from the market risk factors that drive the existing independent CVA models (on a lattice or in a Monte Carlo simulation).

A Simple CVA Model with Approximated CP Collateral under Stress Scenarios

1. Collateral Value CP Default Correlation: The methodology in this section addresses the situation whereby the collateral posted by – or available from - a CP is limited to the asset in a particular entity. In this case the general WWR/RWR can occur due to the co-dependency of the CP collateral quality and value, or the CP's effective default recovery, with the CP default and/or credit spread.
2. CVA for the i^{th} CP: This is particularly applicable to CP's with no liquid CDS available for hedging. The earlier formula for the CVA for i^{th} CP is recalled:

$$CVA_i = -\mathbb{E}_0{}^Q\left[\int_0^{T_i} \frac{\{\tilde{V}_i(t) + C_i(t) - Col_i(t)\}^+}{\beta_t}(1 - R_i)\mathbb{E}_0{}^Q\left[\mathbb{I}_{t<t_i<t+dt}\right]\right]$$

3. Estimating Collateral Value under Stress: The essence of this methodology on WWR is to use the CP posted collateral value $Col_i(t)^+$ estimated under some stress scenarios.
4. The Requested Total Initial Collateral: For the case where the swap/trade facing the CP is part of a package of a *pari passu* loan (for example, for converting the loan from a floating coupon to a fixed coupon), one can start with a minimum total required collateral $Col_{i,TOTAL}(t)$ covering both the swap and the *pari passu* loan, before the CP defaults, given by

$$Col_{i,TOTAL}(t) = \frac{N_i(0) + EPE_i(0)}{LTV_i}$$

where $N_i(0)$ is the notional of the loan at the initiation, $EPE_i(0)$ is the EPE of the *pari passu* swap at the initiation, and LTV_i is the minimum required loan-to-value ratio. $N_i(t)$ is the notional of the loan – which is the same as that of the swap, and $LTV_i(t)$ – typically a constant – is the maximum required LTV (loan-to-value) ratio.

5. Collateral Devaluation upon CP Default: Upon CP default the minimum total collateral $Col_{i,TOTAL,DEFAULT}(t)$ devalues to

$$Col_{i,TOTAL,DEFAULT}(t) = (1 + J_i) \cdot Col_{i,TOTAL}(t)$$

where J_i is the constant percentage of the total collateral value upon CP defaults, which can be estimated under some stress scenarios such as the market condition under the 2008 crisis or the CCAR stress scenarios, and optionally the worst case is to be chosen on the conservative side.

6. Collateral Devaluation Effective Recovery Rate: Thus the effective recovery rate $\tilde{R}_i(t)$ for both the swap and the *pari passu* loan is given – when

$$V_i(t) + C(t) > 0$$

by

$$\tilde{R}_i(t) = \frac{Col_{i,TOTAL,DEFAULT}(t)}{N_i(t) + V_i(t) + C(t)} = \frac{N_i(0) + EPE_i(0)}{LTV_i} \frac{1 + J_i}{N_i(t) + V_i(t) + C(t)}$$

7. The WWR CVA Risk Formulation: Therefore the WWR CVA risk formula can be re-written as

$$CVA_i = -\mathbb{E}_0{}^Q \left[\int_0^{T_i} \frac{\{\tilde{V}_i(t) + C_i(t)\}^+}{\beta_t} \{1 - R_i(t)\} \mathbb{E}_0{}^Q \left[\mathbb{I}_{t < t_i < t + dt} \right] \right]$$

which is similar to the uncollateralized case, but with a state-dependent and a time-dependent recovery rate.

8. <u>Effective Recovery Rate Approximation #1</u>: The effective recovery rate can be approximated by the following conservative formula when

$$V_i(t) + C(t) > 0$$

by

$$\tilde{R}_i(t) \approx \frac{N_i(0)}{LTV_i} \frac{1 + J_i}{N_i(t) + V_i(t) + C(t)}$$

9. <u>Effective Recovery Rate Approximation #2</u>: For a back-of-the-envelope calculation one can use

$$\tilde{R}_i(t) \approx \frac{1 + J_i}{LTV_i}$$

10. <u>The Original CP Recovery Rate</u>: Since the CP default recovery is explicitly handled, the original CP recovery rate $R_i(t)$ in the expressions for the CVA is set to zero for the CVA exposure calculation in the above equation – the recovery rate in the CP credit curve should not be changed.
11. <u>No Change to the DVA</u>: The expression for the DVA should not change, but needs to properly handle the segregation status of the collateral posted by the CP.
12. <u>Collateralized Exposures under Stress Scenarios</u>: Optionally the exposures under the stressed scenarios can also be used.
13. <u>A Simple WWR Model with Approximated CP Collateral under Stress Scenarios with Distribution</u>: The methodology in this section addresses the same situation as the previous one, except that the worst case will not be used some stress scenarios. Rather the CP collateral value distribution estimated under various stress scenarios will be used.

A Simple Specific WWR and RWR Model with Approximated Exposures upon CP Default

1. General/Specific WWR and RWR: The methodology in this section addresses the specific and general WWR and RWR whereby the impact to the market due to the CP default can be reasonably estimated. In this case the exposures used in the CVA and the DVA calculations can be estimated by setting the market variables to the estimated values upon CP default.
2. Override of the CVA/DVA Recoveries: For simplicity of the system implementation, the CVA/DVA recoveries above can be optionally overridden to handle the above – although the recovery rates for the counter parties and the bank should not be changed.
3. Example CP Tax Optimization Strategy: For instance in the tax optimization strategy for the convertible bonds, the CP buys (or enters a debit) a call spread on its own stock from the bank – which is typically treated as two separate calls without netting.
4. Tax Optimization Strategy Recovery Overrides: Technically it can be argued that the CVA and the DVA should be virtually zero in this case, as if the CP were in default the stock price would go to zero, and if the bank were to default the market would drop significantly, and so would the CP's stock. The override of the CVA and the DVA recoveries would be used to handle this – although the conservativeness would need to be demonstrated with monitoring.

Default and Market Risk Metrics

1. DTZ and DTR Metrics Definitions: Default-to-zero (DTZ) and default-to-recovery (DTR) are important risk metrics for risk management. DTZ represents a loss if the CP defaults immediately with 0% recovery and DTR represents a loss if the CP defaults immediately with CP's marked recovery rate (R_{CP}).

2. Exposures for DTZ and DTR: They are defined as

$$DTZ = -\max(PortfolioValue - CollateralValue, 0) - NPV$$

$$DTR = -\max(PortfolioValue - CollateralValue, 0) - (1 - R_{CP})NPV$$

   where $PortfolioValue$ is the underlying time-0 CP portfolio MTM from the CVA model, $CollateralValue$ is the time-0 collateral that the bank receives, if positive, and the time-0 collateral that the bank posts, if negative, and

$$NPV = CVA + DVA$$

   They are generated from the CVA model directly.
3. DTZ/DTR Calculation Assumption #1: One major assumption is that the time-0 collateral is directly calculated from the CVA model, not the actual collateral, and as such the CVA and the DVA do not work under stress scenarios.
4. DTZ/DTR Calculation Assumption #2: Further, DRIP does not handle the potential segregation of the collateral (nor IA/IM), which does not have any impact for the DTZ/DTR for the case where the bank receives collateral from the counter party, and is on the conservative side when the bank posts collateral to the counter party.
5. Market Risk Metrics: The technology system – not DRIP – needs to produce risk numbers via bumping and revaluation, such as:
   a. Theta
   b. Portfolio Delta
   c. Market Delta
   d. Market Gamma
   e. Cross Gamma
   f. Vega
   g. Cega (Correlation Sensitivity)

Martingale Resampling and Testing

1. Martingale Resampling and Numerical Adjustment: This is for improving accuracy and numerical stability by numerically enforcing the necessary martingale conditions. More details will be added in line with Tang and Li (2007).
2. CVA Model Self-Consistency Testing: This section formulates various model testing cases for model self-consistency testing and testing against known target values, mostly based on the martingale arbitrage pricing theories, which is why they are also called martingale tests.
3. Subset Based Automated Test Model: A subset of these tests will also be used for automated model monitoring.
4. Supplementing the Model Validation Procedures: It is important to note that not all these tests apply to all trades or trade types, depending on the availability of the martingale targets. These are meant to supplement – not to replace – the testing required by the model validation procedure.
5. Formulating the Issue Materiality Criterion: In addition materiality criterion is formulated (for example, based on exposures and CVA/DVA), and if any issue is deemed immaterial, no issue resolution is needed. But the issue needs to be included in the monitoring tests on an on-going basis to ensure that it remains immaterial.
6. Notional/Exposure Based Trade Selection: Selected trades are normally tested, typically based on top notional and/or top AEPE/AENE.
7. Accuracy vs. Precision Issue Identification: Optionally the errors are also tested against the two times Monte Carlo error to determine whether the issue is an accuracy issue or a precision issue, or it has random or systematic errors.

At the Underlying Trade Level

1. Trade Level Martingale Relationships Testing: At the underlying trade level, martingale testing is used to test the trade level martingale relationships outlined below (if the martingale targets are available), which does not apply to all trades.
2. Market and Self Consistency Tests: These tests are used to test the accuracy of the CVA pricing models and/or the accuracy of the combination of the CVA simulation and the pricing models against the LOB pricing models (for market consistency) and against the CVA pricing models for self-consistency.
3. Approximation Models for Materiality Analysis: Approximated models for materiality analysis only (such as the stochastic volatility models for materiality analysis only) are not in scope for these tests.
4. CVA vs. LOB Model PV: The time 0 PV of every trade from the CVA model is tested against that from the underlying LOB model (with proper handling of the initial trade cash flows).
5. PV Error Metrics Representation Choice: It is useful to represent such PV differences in terms of ratios to risk metrics such as IR01, CS01, vega, and so on, to convert the PV differences into running basis points (or volatility), which are more intuitive.
6. Martingale Target – Future Uncollateralized Exposure: The martingale target of the discounted future uncollateralized expected exposure profile (including both the positive and the negative exposures) equals the PV of the corresponding forward starting swaps (with proper handling of the trade cash flows).
7. Martingale Target - Uncollateralized EPE/ENE: The martingale target of the discounted future uncollateralized expected positive or negative exposure profile of a swap equals the PV of the corresponding options to enter or cancel the remaining swap (with proper handling of the trade cash flows).
8. Martingale Target - Uncollateralized European Swaption: The martingale target of the discounted future uncollateralized expected exposure profile of a European swaption is a constant (after the premium is paid) and equals its PV. The same applies to all trade types with one net cash flow as the payoff.

9. Martingale Target - Collateralized Swap Exposure: The martingale target of the discounted future fully collateralized expected positive or negative exposure profile of a swap approximately equals the PV of the corresponding forward starting (two week) options to enter (or cancel) the remaining swap (with proper handling of the trade cash flows).
10. Handling the Trade Cash Flows: It is important to note that the CVA pricing models may be configured to include the trade cash flows to be paid at the pricing date, while the underlying pricing models do not. This requires proper handling of the trade cash flows.
11. Asset Class Trade Category Specification: For each asset class (for example, IR, FX, equity, commodity, and credit), the following trade categories are defined.
12. Vanilla Non-Volatility Trade Category: For linear or approximated linear trades with insignificant volatility sensitivity, such as swaps, forwards, non-tranched CDS (including structured CDS, CDS index, and ABS index), LIBOR in-arrears, and CMS swaps.
13. Volatility Based Trade Category Specification: For options and trades with embedded options, such as European, Bermudan, American, and exotic options, and cancellable swaps.
14. Stochastic Volatility and Correlation Categories: Stochastic volatility category for trades specifically requiring stochastic volatility modeling, and correlation category for CDO (collateralized debt obligations) and tranches.
15. Test Specific Dollar Error Specification: For each of the tests the dollar error e' is defined based on the specifics of the test.
16. Error using Dominant Risk Metrics: It is useful to express such dollar error e' in terms of ratios to dominant risk metrics such as IR01, bank or counter party CS01, vega, and so on, to convert the PV differences to running basis points (or a fraction of delta or volatility point), which are more intuitive.
17. Generalized Risk Based Relative Error: Further DRIP defines the risk based error (with respect to the risk metric R) and the relative error in general (with respect to the numeraire R) as

$$e = \left|\frac{e'}{R}\right|$$

18. Compendium of Common Error Thresholds: The thresholds for these errors are defined as follows:

$$e_{IR01} = 3\ bp$$

$$e_{IR01\_AGG} = 1.5\ bp$$

$$e_{CS01} = 3\ bp$$

$$e_{CS01\_AGG} = 1.5\ bp$$

$$e_{VEGA} = 1.5\ Black\ Volatility\ Points$$

$$e_{VEGA\_SV} = 3\ Black\ Volatility\ Points$$

$$e_{NOTIONAL} = 2\%$$

$$e_{EXP} = \min\left(\max\left(5\%, \frac{1\%}{\sigma_B}\right), 10\%\right)$$

where σ_B is the relevant Black ATM volatility driving the exposures.

$$e'_{MAX} = \$10\ MM$$

$$e'_{MIN} = \$100\ K$$

$$e'_{RECOVERY} = 1.5\ Black\ Volatility\ Points$$

19. Trade Level Error Threshold Breach: Unless otherwise indicated, if the error of any test at the individual trade level exceeds the relevant threshold, then the model for that trade is deemed to have an issue, subject to the following materiality criterion.
20. Extreme OTM Error Threshold Breach: In some cases, the trades being tested are quiet off-the-money, which may result in failed percentage errors.
21. OTM Error Breach - ATM Testing: In this case the corresponding ATM trades (or those with less off-the-money trades) are required to pass the tests and show smoothness of exposures as a function of moneyness.
22. Thresholds as Worst-Case Errors: It is to be noted that these error thresholds are more like the worst-case errors, and the overall CVA errors are much smaller, partially due to the averaging or the diversification benefit of a counter party portfolio. DRIP model developers will continue to improve the model accuracy to the extent practical.

Known Limitations of the CVA Models

1. IR CVA Model Expected Accuracy: The rationale of setting such thresholds is mainly based on the expected accuracy or the known limitations of the IR CVA model in the cross-asset framework, which will account for the vast majority of the CVA among all asset classes; such thresholds are used for other CVA models as approximations when applicable.
2. Trade Level IR CVA Accuracy: More specifically such thresholds are roughly estimated by the accuracy of the trade-level pricing in the IR CVA model.
3. Non-vanilla IR Trades Accuracy: For trade-level pricing accuracy, the error for the PV of vanilla non-vol IR trades is mainly from various interpolations (for example, in the discount curves). The error is roughly estimated to be a few basis points. Therefore DRIP sets it to

$$e_{IR01} = 3\ bp$$

4. Volatility Category IR Trades Accuracy: On the other hand the error for the PV of volatility category IR trades is mainly from the volatility calibration or the specification error, which is roughly estimated to be about one Black volatility point. Therefore DRIP sets

$$e_{VEGA} = 1.5\ Black\ Volatility\ Points$$

5. Driving Brownian Motion Factor Exposure: For the exposure error e_{EXP}, DRIP estimates it with one Black volatility point error for an ATM option with 10% to 20% Black volatility, which is about 5% to 10% ($\frac{1}{20}$ to $\frac{1}{10}$).
6. Portfolio Diversification Counter Party Error Impact: The counter party level error is typically smaller due to portfolio diversification.
7. Inconsistency between Simulation and Valuation: Another general limitation of the CVA model is that the market generation/simulation and the trade valuation in the CVA model are not 100% consistent (in terms of dynamic processes, for example). An important purpose of the martingale test is to quantify such inconsistency and its impact on the CVA.

Materiality Criteria

1. Trade Level Model Immateriality Criterion: The issue of the model for that trade is deemed immaterial and no issue resolution is needed, regardless of other testing results, if

$$e' \leq e'_{MIN}$$

2. Trade Level CVA Credit Error: As an example, for a trade with 5 year duration with a counter party with 200 bp credit spread, the trade level CVA error would be about

$$e'_{MIN} \times 200\ \frac{bp}{Y} \times 5Y \approx \$100\ K$$

which is immaterial.

3. Trade Level CVA Recovery Error: In an extreme case where the counter party is extremely close to default (with 40% recovery) the trade level error would be about

$$e'_{MIN} \times (1 - 40\%) \approx \$60\ K$$

which is also not material.

4. Ongoing Monitoring of Immateriality: The counter party level CVA is typically smaller due to portfolio diversification. But the issue needs to be included in the monitoring tests on an on-going basis to ensure that it remains immaterial.
5. Trade Level Model Materiality Criterion: On the other hand the issue of the model for that trade is deemed material and the model developer needs to follow the issue resolution process regardless of other testing results if

$$e' \geq e'_{MAX}$$

6. Trade Level Dollar Materiality Threshold: In this case for a trade with five year duration and a counter party with 200 bp credit spread, the trade level CVA would be

$$e'_{MAX} \times 200\ \frac{bp}{Y} \times 5Y \approx \$1\ MM$$

which is material. Again the counter party level CVA error is typically smaller due to portfolio diversification.

CVA-PV-Underlying-PV Test

1. Desk CVA Time-0 PV Reconciliation: For selected trades in scope for CVA of every trade type, DRIP tests trade-by-trade that time-0 PV or the time-0 uncollateralized exposure from the CVA model equals the PV from the LOB model with proper handling of the initial or time-0 trade cash flows as listed below, as the CVA PV includes the time-0 cash flow and the LOB does not.
2. Limited Test of CVA Accuracy: The purpose of this is to test one aspect of the accuracy of the CVA pricing models against the LOB pricing models.
3. Absolute and Relative Risk Errors: For the comparison of the CVA PV PV_{CVA} and the LOB PV PV_{LOB} the dollar error is defined as

$$e' = |PV_{CVA} - C - PV_{LOB}|$$

   and the risk based error as

$$e = \left|\frac{e'}{R}\right|$$

   where C is the time-0 cash flow of the trade, if PV_{CVA} is configured to include the cash flow. Otherwise C is set to 0.
4. IR Swaps Risk Metrics Numeraire: For vanilla non-vol IR swaps, IR01 is used as the risk metrics R, and it is required that the error

$$e \leq e_{IR01}$$

5. CDS Trade Risk Metrics Numeraire: For non-tranched CDS trades, CS01 is used as the risk metrics R, and it is required that the error

$$e \leq e_{CS01}$$

6. Credit CS01 Numeraire Choice Caveat: However CS01 test may become unreliable when the reference credit is distressed, as in that case the CS01 becomes very small.
7. Recovery Rate Sensitivity Metrics Numeraire: DRIP then uses the recovery sensitivity – by bumping the recovery rate by one percentage point – as the risk metrics R and requires that the error

$$e \leq e_{RECOVERY}$$

8. Volatility Sensitivity Risk Metrics Numeraire: For volatility category trades across all assets, DRIP uses vega as the risk metrics R and requires that the error

$$e \leq e_{VEGA}$$

9. ATM Vega as the Volatility Numeraire: For this trade one has the option of using the ATM vega of the corresponding ATM trade (or the maximum absolute value of the vega of the corresponding trade) before it is declared a failure.
10. Optional Notional Based Materiality Criterion: For trades not covered by the above risk-based errors, or trades failing the risk-metrics based tests due to the risk metrics being small under special conditions, DRIP requires that

$$e \leq e_{NOTIONAL}$$

with

$$R = NOTIONAL$$

the trade notional. Optionally a stronger test may be used by setting

$$R = PV_{LOB}$$

The materiality criteria list above apply.

One Cash Flow Test

1. European Swaption Uncollateralized Expected Exposure: For selected trades in the scope for CVA, DRIP tests that the discounted future uncollateralized expected exposure (EE) profile of a European swaption is a constant – after the premium is paid – and equals its PV.
2. Single Cash Flow Payoff: The same applies to all European options and all trade types with one net cash flow as the payoff.
3. Purpose - CVA Models Accuracy Testing: The purpose of this is to test the accuracy of the CVA simulation models – in the case of simple test trades – or the accuracy of the combination of the CVA simulation models and the pricing models – in the case of complex trades – against the LOB pricing models.
4. Replacement of the Martingale Tests: With simple test trades whose payoff is a linear function of the simulated risk factor, this test can replace the martingale tests on the simulation.
5. CVA Volatility Skew Modeling Testing: With European options this tests the accuracy and the consistency of the volatility skew modeling in the CVA simulation models and the pricing models.
6. Expected Exposure under Different Filtrations: Recall that in this case

$$EE(t) = \mathbb{E}_0{}^Q\left[\frac{V(t)}{\beta(t)}\right]$$

and

$$\frac{V(t)}{\beta(t)} = \mathbb{E}_t{}^Q\left[\frac{V(T)}{\beta(T)}\right]$$

where $EE(t)$ is the uncollateralized exposure at time-t, $V(t)$ and $V(T)$ represent the price of the trade at time-t and time-T respectively, with T being the final payment of the trade.

7. Applying the Iterated Expectations Theorem: With the iterated expectations theorem one has

$$EE(t) = \mathbb{E}_0{}^Q\left[\frac{V(t)}{\beta(t)}\right] = \mathbb{E}_0{}^Q\left[\mathbb{E}_t{}^Q\left[\frac{V(T)}{\beta(T)}\right]\right] = \mathbb{E}_0{}^Q\left[\frac{V(T)}{\beta(T)}\right] = V_0$$

8. Absolute and Relative Error Thresholds: The dollar error is defined as

$$e' = \frac{1}{T}\int_0^T |EE(t) - V_0|dt$$

and the relative error as

$$e = \frac{e'}{\max(AEPE, -AENE)}$$

It is required that the error

$$e \leq e_{EXP}$$

9. Applying Vega Based Error Analysis: In the case of test failure, vega-based error analysis may be optionally performed, as later discussed in a later section. The materiality criteria above apply.

Swap-Forward-Swap Test

1. Forward Swap Exposure PV Reconciliation: For selected trades in scope for CVA DRIP tests that the discounted future uncollateralized expected exposure (EE) profile of a swap equals the PV of the corresponding forward starting swap – with proper handling of the trade cash flows as shown below.
2. Purpose - CVA Models Accuracy Testing: The purpose of this is to test the accuracy of the CVA simulation models in the case of simple test trades, or the accuracy of the combination of the CVA simulation models and the pricing models in the case of complex trades against the LOB pricing models.
3. Linear Sum of Payoffs: The same applies to all types of trades whose payoff is a linear combination of future payoffs, such as CDS, commodity swaps, and caps/floors.
4. Expected Filtration of Linear Sums: For such a trade

$$EE(t) = \mathbb{E}_0{}^Q\left[\frac{V(t)}{\beta(t)}\right]$$

and

$$\frac{V(t)}{\beta(t)} = \mathbb{E}_t{}^Q\left[\sum_{i=i(t)}^{N}\frac{V(t_i)}{\beta(t_i)}\right]$$

where $EE(t)$ is the uncollateralized expected exposure (EE) at time-t, $V(t_i)$ and $V(T)$ represent the price of the trade at time t_i and time T, with T being the final payment date of the trade.

5. Application of the Iterated Expectations: Applying the iterated expectations theorem the above becomes

$$EE(t) = \mathbb{E}_0{}^Q\left[\frac{V(t)}{\beta(t)}\right] = \mathbb{E}_0{}^Q\left[\mathbb{E}_t{}^Q\left[\sum_{i=i(t)}^{N}\frac{V(t_i)}{\beta(t_i)}\right]\right] = \mathbb{E}_0{}^Q\left[\sum_{i=i(t)}^{N}\frac{V(t_i)}{\beta(t_i)}\right] = V_0$$

where there is no trade cash flow at time-t. Otherwise the above can be used as an approximation for testing purposes.

6. Impact of the Knock-Out: For CDS and other knock-out trades, this assumes that if the underlying credit defaults or the knock-out event happens on or before the forward starting date, the forward starting CDS or trade would terminate with zero value.
7. Relative and Absolute Error Thresholds: The dollar error is defined as

$$e' = \frac{1}{T}\int_0^T |EE(t) - V_0| dt$$

and the relative error is defined as

$$e = \frac{e'}{\max(AEPE, -AENE)}$$

8. Derivation of the Martingale Target: For some trade types such as CDO, the martingale target $V_0(t)$ may not be readily available, in which case it may be derived from

$$V_0(t) = V_0(t_{MAT} = T) - V_0(t_{MAT} = t)$$

where $V_0(T)$ is the PV of the original underlying trade with original maturity T and $V_0(t_{MAT} = t)$ the PV of the original underlying trade with modified maturity t and excluding principal payments, and with all other payments to be the same.
9. Black Volatility Driving Process Error: It is required that

$$e \leq e_{EXP}$$

10. Martingale Targets for American Options: Similarly for some trade types including American and Bermudan options, the martingale target $V_0(t)$ can be approximated as

$$V_0(t) \approx V_0(t_{MAT} = T) \cdot \min\left(\frac{V_0(t_{MAT} = T) - V_0(t_{MAT} = t)}{V_{0E}(t_{MAT} = T) - V_{0E}(t_{MAT} = t)}, 1.\right)$$

where $V_0(t_{MAT} = T)$ and $V_0(t_{MAT} = t)$ have the same meaning as in

$$V_0(t) = V_0(t_{MAT} = T) - V_0(t_{MAT} = t)$$

and $V_{0E}(t_{MAT} = T)$ is the PV of the European counterpart of the original trade, i.e., the only difference is in the possibility of early exercise.

11. Intuition behind the Expression #1: Likewise, $V_{0E}(t_{MAT} = T)$ is the PV of the counterpart $V_0(t_{MAT} = T)$. The term $\min\left(\frac{V_0(t_{MAT}=T)-V_0(t_{MAT}=t)}{V_{0E}(t_{MAT}=T)-V_{0E}(t_{MAT}=t)}, 1.\right)$ can be approximately interpreted as the probability of the original trade not being exercised at time t.
12. Intuition behind the Expression #2: The intuition behind

$$V_0(t) \approx V_0(t_{MAT} = T) \cdot \min\left(\frac{V_0(t_{MAT} = T) - V_0(t_{MAT} = t)}{V_{0E}(t_{MAT} = T) - V_{0E}(t_{MAT} = t)}, 1.\right)$$

is as follows. If the early exercise compared to t for the trade is favorable, which is equivalent to small exposure at t, then $V_0(t_{MAT} = t)$ is close to $V_0(t_{MAT} = T)$ so the right hand side is also small.

13. Intuition behind the Expression #3: On the other hand, if there is no incentive for early exercise compared to t, then the exposure at time t is very large. $V_0(t_{MAT} = t)$ tends to be small relatively compared to $V_0(t_{MAT} = T)$, so $V_0(t)$ is large. DRIP requires that the error

$$e \leq 2 \cdot e_{EXP}$$

14. Vega Based Error Analysis Validation: In case of test failures, vega based error analysis is optionally performed, as discussed in the corresponding section. The materiality criterion list above applies.

Swap Swaption Test

1. Swap EPE/ENE Swaption PV: For selected trades in scope for CVA, DRIP tests that the discounted future uncollateralized expected positive or negative exposures EPE or ENE profile of a swap equals the PV of the corresponding options to enter or cancel the remaining swap.
2. Test of Underlying Swaption Model: The same applies to all trade types for which the corresponding options to enter and to cancel the underlying trades are available from the LOB models.
3. Accuracy of the Volatility Skew Models: The purpose of this is to test the accuracy of the volatility skew modeling in the CVA simulation models, and the volatility skew consistency of the CVA simulation and the pricing models.
4. Trade Level EPE and ENE: For such a trade

$$EPE(t) = \mathbb{E}_0{}^Q\left[\frac{V^+(t)}{\beta(t)}\right] = V_0{}^{OPT-ENTER}(t)$$

and

$$ENE(t) = \mathbb{E}_0{}^Q\left[\frac{V^-(t)}{\beta(t)}\right] = -V_0{}^{OPT-CANCEL}(t)$$

where $V_0{}^{OPT-ENTER}(t)$ and $V_0{}^{OPT-CANCEL}(t)$ are the PV's of the option to enter and to cancel the underlying trades respectively, with expiry at time t when there is no other trade cash flow at time t. Otherwise the above can be used as an approximation for testing purposes.

5. Absolute and Relative Error Thresholds: The dollar errors are defined as

$$e_{EPE}{}' = \frac{1}{T}\int_0^T \left|EPE(t) - V_0{}^{OPT-ENTER}(t)\right| dt$$

$$e_{ENE}{}' = \frac{1}{T}\int_0^T \left|ENE(t) - V_0{}^{OPT-CANCEL}(t)\right| dt$$

and the relative errors as

$$e_{EPE} = \left|\frac{e_{EPE}{}'}{\max(AEPE, -AENE)}\right|$$

$$e_{ENE} = \left|\frac{e_{ENE}{}'}{\max(AEPE, -AENE)}\right|$$

6. EPE and ENE Exposure Thresholds: DRIP requires that the error

$$e_{EPE} \leq e_{EXP}$$

and

$$e_{ENE} \leq e_{EXP}$$

7. Vega Based Error Analysis Validation: In case of test failure, the vega based error analysis is optionally performed. The materiality criteria list above applies.

Swap Forward Swaption Test

1. Swap Exposure Forward Swaption Test: For selected trades in the scope for CVA, DRIP tests that the discounted future fully collateralized expected positive or negative exposure EPE or ENE profile of a swap approximately equals the PV of the corresponding forward starting options.
2. Settlement Window as Margin Period: These options are to enter or cancel the remaining swap with the time to expiry equal to the length of the total default window Δt_{td} - or 14 calendar days by default – when there is no contractual cash flow in the default window (and the collateral threshold and the minimum transfer amount, the MTA, are all zero).
3. Margin Period of Risk Test: The purpose of this test is to test the accuracy of the MPoR (margin period of risk).
4. MPoR Based EPE and ENE: More specifically for such a trade

$$EPE(t) = \mathbb{E}_0{}^Q\left[\frac{\left(V_t - V_{t-\Delta t_{td}} + C_{t-\Delta t_{td},t}\right)^+}{\beta_t}\right] = V_{0,t,\Delta t_{td}}{}^{FSOPT-ENTER}$$

and

$$ENE(t) = \mathbb{E}_0{}^Q\left[\frac{\left(V_t - V_{t-\Delta t_{td}} + C_{t-\Delta t_{td},t}\right)^-}{\beta_t}\right] = -V_{0,t,\Delta t_{td}}{}^{FSOPT-CANCEL}$$

when

$$C_{t-\Delta t_{td},t} = 0$$

where $C_{t-\Delta t_{td},t}$ is the cash flow of the total default window conditioned on time t, ${V_{0,t,\Delta t_{td}}}^{FSOPT-ENTER}$ and ${V_{0,t,\Delta t_{td}}}^{FSOPT-CANCEL}$ are the PV's of the forward starting options to enter and to cancel the underlying trades respectively, with starting time at $t - \Delta t_{td}$ and time to expiry of Δt_{td}, again conditioned on time t.

5. Corresponding Absolute and Relative Errors: The dollar errors are defined as

$$e_{EPE}' = \frac{1}{T}\int_0^T \left|EPE(t) - {V_0}^{FSOPT-ENTER}(t)\right| dt$$

$$e_{ENE}' = \frac{1}{T}\int_0^T \left|ENE(t) - {V_0}^{FSOPT-CANCEL}(t)\right| dt$$

and the relative errors as

$$e_{EPE} = \left|\frac{e_{EPE}'}{\max(AEPE, -AENE)}\right|$$

$$e_{ENE} = \left|\frac{e_{ENE}'}{\max(AEPE, -AENE)}\right|$$

6. Approximations to the Option Prices: In most cases ${V_{0,t,\Delta t_{td}}}^{FSOPT-ENTER}$ and ${V_{0,t,\Delta t_{td}}}^{FSOPT-CANCEL}$ are not readily available, but DRIP uses simple approximations using a Black model.
7. Delta Based MPoR Valuation Gap: For a linear trade that depends upon on dominant risk factor R_t one has the following delta risk approximation ignoring the cash flow effects:

$$\frac{V_t - V_{t-\Delta t_{td}}}{\beta_t} \approx D_{t-\Delta t_{td}} \Delta R_t \approx D_{t-\Delta t_{td}} \sigma_{t,t_{td}} R_t \Delta W_t$$

where $D_{t-\Delta t_{td}}$ is the delta risk corresponding to the time $t - \Delta t_{td}$ of the forward starting trade, R_t is the time-t forward value, $\sigma_{t,t_{td}}$ is the time-0 forward Black volatility, all as of time-0 and assumed non-stochastic, and ΔW_t is a standard Brownian motion. The drift terms are ignored due to the relatively short time period Δt_{td}.

8. Margin Period Risk Estimate: Thus

$$V_{0,t,\Delta t_{td}}{}^{FSOPT-ENTER} = \mathbb{E}_0{}^Q\left[\frac{\left(V_t - V_{t-\Delta t_{td}}\right)^+}{\beta_t}\right] = \mathbb{E}_0{}^Q\left[\left(D_{t-\Delta t_{td}}\sigma_{t,t_{td}}R_t\Delta W_t\right)^+\right]$$

$$= \frac{\left|D_{t-\Delta t_{td}}\sigma_{t,t_{td}}R_t\sqrt{\Delta t_{td}}\right|}{\sqrt{2\pi}}$$

$$V_{0,t,\Delta t_{td}}{}^{FSOPT-CANCEL} = -\mathbb{E}_0{}^Q\left[\frac{\left(V_t - V_{t-\Delta t_{td}}\right)^-}{\beta_t}\right] = -\mathbb{E}_0{}^Q\left[\left(D_{t-\Delta t_{td}}\sigma_{t,t_{td}}R_t\Delta W_t\right)^-\right]$$

$$= \frac{\left|D_{t-\Delta t_{td}}\sigma_{t,t_{td}}R_t\sqrt{\Delta t_{td}}\right|}{\sqrt{2\pi}}$$

9. Expanded EPE and ENE Thresholds: Due to various testing approximations involved, DRIP allows more tolerance and requires that the errors be within

$$e_{EPE} \le 2 \cdot e_{EXP}$$

and

$$e_{ENE} \le 2 \cdot e_{EXP}$$

10. Validation using Vega Based Analysis: In case of test failure, vega based error analysis is optionally performed, as is discussed in a later section.
11. Linear Trade Forward Exposure Test: The following swap forward-swap tests are also performed. For a linear trade the positive and the negative exposures should be

approximately symmetric when there is no contractual cash flow in the default window, even when the collateral threshold and the MTA are not zero.

12. Forward Swap Exposure Test Errors: This can also be used as a simpler test when the time-0 forward Black volatility is not readily available. The dollar errors, ignoring the cash flow effects, are defined as

$$e_{EE}{}' = \frac{1}{T}\int_0^T |EPE(t) + ENE(t)|dt$$

and the relative errors as

$$e_{EE} = \left|\frac{e_{EE}{}'}{\max(AEPE, -AENE)}\right|$$

13. Expected Exposure Error Threshold Extension: Due to various testing approximations involved, more error tolerance is allowed, and the error is required to be

$$e_{EE} \leq 2 \cdot e_{EXP}$$

14. Non-linear Trade Exposure Test: For a non-linear trade depending on one dominant factor, such as an option, smoothness analysis is first performed by changing the strike or moneyness. Next the moneyness or strike is set to be way in the money so that the trade becomes approximately linear. The above procedure is then repeated.

Cash Flow Martingale Target

1. Individual Cash Flows – Special Considerations: The cash flow martingale target is generally the PV of the individual cash flow. Certain cash flows, such as default payments of credit derivatives, require special attention as they tend to be more significant in terms of the order of magnitude.
2. CDS Expected Default Cash Flow: For example for a CDS with notional N and recovery R the martingale target for the expected default payment cash flow – assuming zero settlement delay – to the protection buyer is given by

$$C_{t-\Delta t_{td},t} = N(1-R)\left(P_{0,t}^{D} - P_{0,t-\Delta t_{td}}^{D}\right)$$

   where $P_{0,t}^{D}$ is the time-0 cumulative default probability up to time-t of the reference credit of the CDS. The cash flow martingale target is independent of the collateral thresholds and the MTA.

Vega Based Error Analysis

1. ATM Options Vega Risk Metrics: In case of the martingale test failure, optionally the vega based error analysis is performed. The time weighted vega of the corresponding ATM options is chosen as the risk metrics R.
2. Time Weighted Vega Definition/Formulation: The time weighted vega $Vega_{ATM}$ is defined as

$$Vega_{ATM} = \frac{1}{T}\int_{0}^{T} Vega_{ATM}(t)dt$$

where T is the final payment date of the trade and $Vega_{ATM}(t)$ is the ATM vega of the martingale target corresponding to the exposure (EPE/ENE) at time t conditional on time 0.

3. Rationale behind Time Weighted Vega: ATM vega is used because it is the maximum absolute of the value of the vega of the corresponding trade with different moneyness.
4. Vega Based EPE/ENE Errors: The vega based errors are

$$e_{EPE,VEGA} = \left|\frac{e_{EPE}'}{Vega_{ATM}}\right|$$

$$e_{ENE,VEGA} = \left|\frac{e_{ENE}'}{Vega_{ATM}}\right|$$

5. Vega Based EPE/ENE Thresholds: DRIP requires that the errors

$$e_{EPE,VEGA} \leq e_{VEGA}$$

and

$$e_{ENE,VEGA} \leq e_{VEGA}$$

where the threshold e_{VEGA} is defined from

$$e_{VEGA} = 1.5\ Black\ Volatility\ Points$$

6. Vega Based Error Analysis - Illustration: This section uses the commodity trade valuation as an example to illustrate the details of $Vega_{ATM}(t)$ for the vega-based error analysis for the case of one cash flow test, swap-forward-swap test, and swap-swaption test.
7. ATM Commodity Swap Option Vega: For swaps $Vega_{ATM}(t)$ is the vega for the option to enter the ATM swap with expiry at time t.

8. European Swap Options ATM Vega: For one cash flow options such as European swaptions and options

$$Vega_{ATM}(t) = Vega_{ATM}(0)$$

with discounting. Therefore

$$Vega_{ATM} = Vega_{ATM}(0)$$

9. Estimating the Initial ATM Vega: In case $Vega_{ATM}(0)$ cannot be directly retrieved from the underlying trade, it can be obtained from an option with the strike set as the time-0 forward price.
10. Options with Multiple Cash Flows: For options with multiple cash flows, such as the commodity Asian options with monthly payments, $Vega_{ATM}(t)$ is the ATM vega for the remaining options at time t. To simplify the analysis, $Vega_{ATM}(0)$ may be used as an approximation for $Vega_{ATM}$.

At the State Variable Level

1. State Variable Level Martingale Testing: At the state variable level, martingale testing is used to optionally test the following martingale relationships under LIBOR discounting. The purpose of these tests is to gain the transparency and intuition on the CVA simulation results.
2. Failure of CVA/Pricing Models: These tests are optional and considered only of the combination of the CVA simulation and the pricing models in the next section for simple/linear trades fail. The success criterion for these tests is that the error is less than two times the Monte Carlo error.

3. Criterion for Materiality Issues Treatment: In rare cases the simulations not satisfying this requirement are considered to have material issues only if they are deemed to have issues relevant issues by other tests in the later sections.
4. Forward LIBOR Rate Martingale Test: Forward LIBOR is a martingale under the corresponding forward measure.
5. Forward Swap Rate Martingale Test: Forward swap rate is a martingale under the corresponding annuity measure.
6. Forward FX Rate Martingale Test: Forward FX rate – the price of a unit of foreign currency in terms of the domestic currency – is a martingale under the corresponding domestic forward measure.
7. Forward CDS Rate Martingale Test: Forward CDS rate is a martingale under the corresponding survival annuity measure.
8. Forward Default Probability Martingale Test: Default probability of a given credit name is a martingale under the risk neutral measure of the same currency when the interest rate is deterministic.
9. Risk Neutral Measure Future Prices: Future prices are martingales under the risk neutral measure of the same currency.
10. Discounted Stock Price Martingale Test: Properly discounted stock prices (plus all the dividends paid in the relevant time interval) are a martingale under the risk neutral measure of the same currency.
11. Input/Realized Volatility Profile Reconciliations: The volatility profile is reported as a function of the simulation time of selected risk factors and market observable variables realized in the CVA simulation. These are compared with the observed or input ATM volatility if applicable. The success criterion for these tests is that the error is less than 1.5 Black volatility points.
12. Input/Realized Correlation Profile Reconciliations: The correlation and the covariance profile are reported as a function of the simulation time of selected risk factors and market observed variables realized in a CVA simulation; these are compared with the input correlation or covariance, if applicable. The success criterion for this test is that the relative error is less than 5%.

Portfolio Based Testing

1. Dollar Error of the Portfolio: For selected or all counter party portfolios DRIP performs the CVA-PV-underlying-PV test for each counter party portfolio and defines the dollar error as

$$e' = |PV_{CVA} - C_P - PV_{UND}|$$

   where C_P is the time-0 cash flow of the counter party portfolio, if PV_{CVA} is configured to include this cash flow. Otherwise C_P is set to 0.
2. Relative Error of the Portfolio: The relative error is further defined a

$$e = \frac{e'}{\max(MPFE_{97.5\%}, -MPFNE_{2.5\%})}$$

   where $MPFE_{97.5\%}$ and $MPFNE_{2.5\%}$ are the maximum PFE at 97.5 percentile and the minimum potential future negative exposure at 2.5 percentile of the CP portfolio respectively.
3. DRIP Portfolio Dollar Error Threshold: DRIP requires that the error

$$e \leq e_{EXP}$$

   The above list of materiality criteria applies.
4. Portfolio Error across Trade Categories: Optionally, for all the trades in a given trade category, for the comparison of the CVA PV $PV_{CVA,i}$ and the underlying PV $PV_{UND,i}$ for each trade i the dollar error is defined as

$$e' = \sum_i \left| PV_{CVA,i} - C_i - PV_{UND,i} \right|$$

and the risk based error as

$$e = \frac{e'}{\sum_i |R_i|}$$

where the summation is over all trades in a given trade category, C_i is the time-0 cash flow of the trade if $PV_{CVA,i}$ is configured to include this cash flow. Otherwise C_i is set to 0.

5. IR01 Based Portfolio Error Threshold: For vanilla non-vol IR swaps IR01 is used as the risk metrics and the error is required to be

$$e \le e_{IR01\_AGG}$$

6. CS01 Based Portfolio Error Threshold: For non-tranched CDS trades DRIP uses CS01 as the risk metrics and requires that the error

$$e \le e_{CS01}$$

This test may become unreliable when the reference credit is distressed, as in this case CS01 becomes very small. The recovery sensitivity will then be used as the risk metrics R (by bumping the recovery rate by one percentage point) and it is required that

$$e \le e_{RECOVERY}$$

7. Vega Based Portfolio Error Threshold: For volatility category trades of all assets, vega is used as the risk metrics, and it is required that

$$e \le e_{VEGA}$$

For this test one has the option of using the vega of the corresponding ATM trades (or the maximum absolute value of the vega of each of the corresponding trades) before it is declared a failure.

8. Notional Based Portfolio Error Threshold: For trades not covered by the above risk-based errors, or the trades failing the risk metrics tests due to the risk metrics being small under special conditions, it is required that

$$e \leq e_{NOT}$$

with

$$R_i = N_i$$

the dollar notional trade i. Optionally a stringer test may be used by setting

$$R_i = PV_{UND,i}$$

9. Portfolio Level Trade Martingale Test: Optionally DRIP applies all the trade level martingale methodologies to elected counter party portfolios to perform portfolio level martingale testing. In these cases the martingale targets are derived from the portfolio forward PV's and the corresponding basket option prices valued with quadratic approximations.
10. Approximations Based Portfolio Martingale Targets: For the cases with martingale targets involving approximations such as quadratic approximations, the error thresholds are doubled. The above list of materiality criteria applies.

Materiality Analysis of Unmodeled or Failed Trades

1. Inability to Model Certain Trades: In limited cases – due to the limitations in the system or in the models – certain trades cannot be modeled even with the proxy pricers formulated above, which simply fail in production runs.
2. Unmodeled Trades Extended Materiality Analysis: In these cases a simple exposure-based or CVA-based materiality analysis can be formulated along with their estimates for such trades in terms of their potential impact on their overall exposures or CVA per asset class, or in terms of their potential impact on exposures or CVA per counter party (per netting or collateral group to be more precise).
3. Incorporation into Remediation Frameworks: This materiality analysis will be used in conjunction with the model control procedures, such as certain trigger event procedures, to determine remediation actions and plans.
4. Materiality Metric used in DRIP: More specifically the materiality is measured by the ratio of the materiality metric such as USD equivalent NPV, notional, and delta between unmodeled or failed trades and trades that can be modeled.
5. Modeled vs Unmodeled Trade Portfolio: Considered in this section are two portfolios per asset class (for example, credit CVA):
    a. Trades that can be modeled
    b. Trades that cannot be modeled or simply fail in production runs.
6. Maturity and Duration Weighted Delta: For the i^{th} trade in each portfolio, N_i denotes its notional, V_i its NPV, D_i the delta risk, D_{i,T_M} the time to maturity (T_M) weighted delta risk, and $D_{i,DUR}$ for the duration (Dur) weighted delta risk – if the duration can be defined – such that

$$D_{i,T_M} = D_i \times T_M$$

$$D_{i,DUR} = D_i \times DUR$$

7. Portfolio Gross/Net Notional/NPV: The following quantities are defined for each portfolio:

$$N_P = \sum_i |N_i|$$

$$V_P = \sum_i V_i$$

and

$$V_{P,ABS} = \sum_i |V_i|$$

where i is the summation over all the trades in the portfolio. So N_P is the total notional of the portfolio, V_P is the net NPV of the portfolio, and $V_{P,ABS}$ is the gross NPV of the portfolio.

8. Portfolio Gross and Net Delta: The net and gross delta risk of each portfolio D_P and $D_{P,ABS}$ are defined as

$$D_P = \sum_i D_i$$

and

$$D_{P,ABS} = \sum_i |D_i|$$

9. Correlated/Uncorrelated Aggregated Portfolio Duration: The portfolio time to maturity weighted delta risk D_{P,T_M} and the duration weighted delta risk (if the duration can be defined for all the underlying trades) $D_{P,DUR}$ are calculated by

$$D_{P,T_M}{}^{0-CORR} = \sqrt{\sum_i D_{i,T_M}{}^2}$$

$$D_{P,DUR}{}^{0-CORR} = \sqrt{\sum_i D_{i,DUR}{}^2}$$

if the pair-wise correlation between the risk factors of the underlying trades is assumed to be zero, or

$$D_{P,T_M}{}^{1-CORR} = \sum_i \left|D_{i,T_M}\right|$$

$$D_{P,DUR}{}^{1-CORR} = \sum_i \left|D_{i,DUR}\right|$$

if the pair-wise correlation is assumed to be one.

10. <u>Unmodeled Error Portfolio Materiality Metric</u>: The error corresponding to the j^{th} materiality metric per asset class is defined as the ratio of the j^{th} materiality metric between the unmodeled portfolio and the modeled portfolio

$$e_j{}^{UN-MODEL} = \frac{MM_{p,j}{}^{UN-MODEL}}{MM_{p,j}{}^{MODEL}}$$

with $MM_{p,j}$ in

$$\left\{N_P, V_P, V_{P,ABS}, D_P, D_{P,ABS}, D_{P,T_M}{}^{0-CORR}, D_{P,T_M}{}^{1-CORR}, D_{P,DUR}{}^{0-CORR}, D_{P,DUR}{}^{1-CORR}\right\}$$

11. <u>Effective Unmodeled Error Materiality Metric</u>: The error corresponding to the entire asset class is defined as

$$e^{UN-MODEL} = \max_{j} e_j^{UN-MODEL}$$

with running over all materiality metrics in

$$\left\{N_P, V_P, V_{P,ABS}, D_P, D_{P,ABS}, D_{P,T_M}{}^{0-CORR}, D_{P,T_M}{}^{1-CORR}, D_{P,DUR}{}^{0-CORR}, D_{P,DUR}{}^{1-CORR}\right\}$$

12. Portfolio Unmodeled Materiality Metric Threshold: The unmodeled or failed trades are considered immaterial if

$$e^{UN-MODEL} \geq 5\%$$

13. FX/Equity/Commodity Delta Risk: For FX, equity, and commodity trades, the delta risk is the USD equivalent delta risk of the derivatives trade with respect to each of the relevant underlying risk factors.
14. Delta Risk for Credit Trades: For credit trades DRIP uses USD equivalent CS01 as the delta risk.
15. Single/Cross Currency Rates Trades: For single currency rates trades DRIP uses the USD equivalent IR01 or MM01 as the delta risk, and for cross currency (XCCY) IR trades, DRIP uses USD equivalent FX delta as the delta risk.
16. Delta Risk for the Basket Trades: For basket trades, DRIP uses the parallel delta risk computed by bumping all the relevant risk factors simultaneously.
17. Criteria Based Unmodeled Trades Aggregation: Similar methodology can be optionally applied to analyze the materiality of exposures or CVA per counter party (per netting or collateral group to be more precise) due to unmodeled or failed trades.

Materiality Analysis Based on P&L

1. Causes of Unstable Trade Valuations: In some cases numerical inaccuracies such as Monte Carlo noises may cause unstable valuations, risks/Greeks, and/or poor P&L explanations.
2. Verifying Origins of Numerical Inaccuracies: The first step is to confirm that such instabilities are dominated by numerical inaccuracies, and not by other fundamental model deficiencies, by demonstrating that:
    a. Either the instability or the error is within two standard deviations of the Monte Carlo error
    b. Or a significantly improved stability with refined numerical parameters, such as more Monte Carlo paths.
3. Materiality Impact on Net CVA: After this the materiality of the net CVA (CVA + DVA) P&L and the unexplained P&L impact for the entire trading books needs to be evaluated.
4. Breach of the Materiality Threshold: If these P&L's or the unexplained P&L's are less than the threshold outlined in typical contingent group credit policies and controls, they are considered to be immaterial.
5. Periodic Monitoring to Verify Immateriality: No further actions need to be taken except for periodic monitoring to ensure that they remain immaterial going forward. The main motivation for not using more Monte Carlo simulation paths is computational performance and efficiency.

CVA/DVA Testing

1. CP/Bank/Market Zero Correlation: For the special case of zero correlation among the counter party and the bank credit spreads and the underlying market (that is without wrong-way or right-way risks) for selected counter parties or trades the model CVA and DVA are tested using

$$CVA_i = -\int_0^{T_i} \mathbb{E}_0{}^Q \left[\frac{\tilde{V}_{i,t}{}^+}{\beta_t}\right] (1 - R_i) dP_{i,0,t}{}^D = -\int_0^{T_i} EPE_{i,t}(1 - R_i) dP_{i,0,t}{}^D$$

and

$$EPE_{i,t} = \mathbb{E}_0{}^Q \left[\frac{\tilde{V}_{i,t}{}^-}{\beta_t}\right]$$

based on the spot date's bank and counter party hazard curves.

2. Selected Trades Tested for Correlation: For the special case of non-zero correlation trades (that is with some wrong-way or right-way risks), for selected counter parties or trades, the model CVA and the DVA smoothness with respect to the correlation is tested. Also tested are the model CVA and DVA against the convexity adjustment and CCDA (contingent CDS), if available.
3. Smoothness Test for MPoR XVA: In addition to all other smoothness tests, DRIP further performs smoothness tests for collateralized/MPoR CVA and DVA with respect to the collateral thresholds, IA, MTA, and default window size $\Delta t_{i,td}$ as defined before.
4. CVA/DVA Error Impact Thresholds: For various approximations, DRIP requires that the error impact on CVA be

$$e \leq e_{CP,CS01}$$

and separately in DVA it be

$$e \leq e_{BANK,CS01}$$

unless the conservative measure is used.

5. Application of the Materiality Criterion: The above list of materiality criteria applies.

Stress/CCAR Testing

1. General Stress Testing – Smoothness Runs: For general stress testing further smoothness testing can be performed as a function of various bump sizes for each of the important market data and model parameters for the following: CVA PV (for selected trades at the trade level), CVA and DVA (for selected counter parties at the counter party level).
2. CCAR Testing - CVA PV Runs: For CCAR testing one can perform the above CVA-PV-Underlying-PV test under each of the CCAR scenarios for selected underlying trades and selected counter party portfolios in scope for CVA, and define the dollar error between the CVA PV and the underlying PV under the CCAR scenario as

$$e' = \left|PV_{CVA,CCAR} - C - PV_{UND,CCAR}\right|$$

   where C is the time-0 cash flow of the trade or of the counter party portfolio under the CCAR scenario if $PV_{CVA,CCAR}$ is configured to include this cash flow. Otherwise C is set to zero.
3. CCAR Testing - Relative Error Threshold: One can further define

$$e = \frac{|e'|}{NOT}$$

4. CCAR Testing - Dollar Notional Threshold: DRIP requires that

$$e \leq e_{NOT}$$

   where NOT is the trade dollar notional (for trade-level testing) or the total gross trade dollar notional (for the counter party level testing). The above list of materiality applies.

5. General Stress CVA PV Testing: For the general stress testing, in addition to the smoothness testing, one can perform the CVA-PV-underlying-PV test under various pre-defined scenarios (typically with stress on a single risk factor at a time) and define the dollar error between the CVA PV ($PV_{CVA,STRESS}$) and the underlying PV ($PV_{UND,STRESS}$) under the stress scenario as

$$e' = \left|PV_{CVA,STRESS} - C - PV_{UND,STRESS}\right|$$

   where C is the time-0 cash flow of the trade under the CCAR scenario if $PV_{CVA,STRESS}$ is configured to include the cash flow. Otherwise C is set to zero.

6. General Testing - Relative Error Threshold: One can further define

$$e = \frac{|e'|}{NOT}$$

7. General Testing - Dollar Notional Threshold: DRIP requires that

$$e \leq e_{NOT}$$

   where NOT is the trade dollar notional (for trade-level testing) or the total gross trade dollar notional (for the counter party level testing). The above list of materiality applies.

8. Thresholds for Full Test Suite: Optionally all of the above test are performed under the stress and the CCAR scenarios, but with error thresholds doubled under CCAR-like scenarios. The above list of materiality criteria applies.

Back Testing

1. Definition of Model Back Testing: Model back testing refers to comparing the model's output against the realized values, which is also referred to as the historical time series.
2. Statistical Approach to Back Testing: Model back-testing uses a statistical method to reach a conclusion as to whether the model should be rejected under a given confidence level. The statistical method can be based on either a quantile or a distribution.
3. Quantile-Based Model Back Testing: The quantile based back testing method involves essentially counting the number of exceptions over a test time window. As such the quantile based back testing method id simple to implement and easy to analyze but does not provide distributional detail.
4. Applying Quantile-Based Back Testing: The quantile based back testing has been the standard methodology for VaR back testing. This methodology can be directly applied to PFE back testing due to the mathematical similarity between VaR and PFE. For EPE ever part of the exposure distribution is important.
5. Distribution Based Model Back Testing: A distribution based method for exposure back testing, while more powerful, is also more complicated, both in theory development and system implementation.
6. Back Testing the Exposure Models: For CVA the exposure models may be subject to back testing to ensure that the exposure model used to calculate the CVA is sound. Initially DRIP will use the quantile based method for exposure back testing.
7. Static Hypothetical Portfolios - Back Testing: Back testing should be conducted on both real counter party portfolios as well as hypothetical portfolios. Static hypothetical portfolios avoid the issue of portfolio change that actual portfolios are likely to encounter, as well as enable tailored back testing to identify whether a particular type of instrument is modeled correctly.
8. Static Hypothetical Portfolios - Extended Testing: In addition the use of static hypothetical portfolios is an effective way to meaningfully test the predictive abilities of exposure models over long time horizons.
9. Back Testing on Real Portfolios: Back testing on real counter party portfolios evaluates the model performance on actual counter party exposures taking into account portfolio changes over time.

10. Testing Single Trades and Portfolios: Note that hypothetical portfolio back testing includes back testing on single trades, and portfolio back testing includes both collateralized and uncollateralized portfolios.
11. Historical Basis behind Back Testing: Back testing in general (except for some special cases such as simplified back testing) will be performed after the models have been released into production, when enough data has been accumulated.

Quantile Based Exposure Testing

1. Selecting the Test Time Window: Quantile based back testing involves the following tests. First a test window $\{t_1, \cdots, t_N\}$ containing N time nodes is selected.
2. Risk Horizon and Confidence Level: A risk horizon h and a confidence level CL are selected.
3. Identification of the Observation Window: Given the test window and the horizon, the observation window is $\{T_1, \cdots, T_N\}$ with

$$T_k = t_k + h$$

$$k = 1, \cdots, N$$

containing observations of the portfolio value

$$V_k = V(T_k)$$

the realized portfolio value, as well as the collateral value if applicable.
4. PFE at the Observation Node: The PFE value at the observation time T_k - PFE_k – is the CL percentile of the model distribution of the portfolio value T_k conditional on t_k. PFE is floored at 0.

5. Running Counter of the Exceptions: If

$$PFE_k < V_k$$

it is counted as an exception.

6. Threshold Breach of the Exception: The final three steps above are repeated for

$$k = 1, \cdots, N$$

to obtain the total number of exceptions M. If

$$M < Th_{CL}$$

the model is deemed to have passed at CL; otherwise the test is deemed to have failed. Th_{CL} is the critical number corresponding to CL. The number of exceptions must not exceed Th_{CL} in order for the test to be deemed a pass.

7. Testing across Multiple Quantile Thresholds: For the purpose of EPE back testing, the above quantile based back testing can be repeated for a number of CL as back testing on the quantile model function of the portfolio value distribution provides a more detailed picture of the closeness between the model distribution and the realizations.

Simplified Back Testing

1. Variance Tally across Simulation Runs: The essence of simplified back testing is to compare the model produced future NPV variance, expected positive exposure, or tail distribution against historical NPV variance for a given trade, or a given portfolio of trades.
2. Model vs. Realized NPV Distribution: if the model produced future NPV variance is greater than (or comparable to) the historical NPV variance, then it is a favorable indication that the model does not under-estimate the historical volatility.
3. Example - Basel Based Proxy Pricer: The following section describes how to perform simplified back-testing of the Basel notional conversion factor based proxy pricer.
4. Collateralized EPE using Black Model: First the EPE of a collateralized trade is calculated using the Black normal model. For a linear trade dependent upon one dominant risk factor R_t, the delta risk approximation is provided from

$$\frac{V_t - V_{t-\Delta t_{td}}}{\beta_t} \approx D_{t-\Delta t_{td}} \Delta R_t \approx D_{t-\Delta t_{td}} \sigma_{t,t_{td}} R_t \Delta W_t$$

5. Expression for the Black EPE: Thus in calculation one has

$$EPE_t = \mathbb{E}_0^Q \left[\frac{\left(V_t - V_{t-\Delta t_{td}}\right)^+}{\beta_t} \right] \approx \mathbb{E}_0^Q \left[\max\left(N_{t-\Delta t_{td}} \sigma_{t,t_{td}} \Delta W_t, 0\right) \right] \approx \mathbb{E}_0^Q [\max(N_0 \sigma \Delta W_t, 0)]$$

$$= \frac{N_0 \sigma \sqrt{t}}{\sqrt{2\pi}}$$

   where N_0 is the dollar notional at time-0, σ is the historical volatility of the trade value normalized by the notional, and ΔW_t is the standard Brownian motion in Δt_{td}.
6. Accommodating the Duration Shortening Effects: It needs to be mentioned here that the duration shortening effect needs to be considered for swap-like trades (with periodic payments and zero clean price right before its final maturity) and for Asian-like trades (volatility decreases with time and is minimal close to the trade maturity since trade NPV largely depends on fixings).

7. Historical Volatility with Duration Shortening: Considering duration shortening the historical volatility for 1Y duration is calculated by

$$\sigma = (T - t_0)\frac{Std\left\{\dfrac{\Delta U_{t_i}}{T - t_i}\right\}}{\sqrt{\Delta t_i}}$$

where T is the maturity and t_0 is time-0.

8. Trade NPV/Notional Time Series: For a trade with two historical time series $\{V_{t_i}\}$ and $\{N_{t_i}\}$ (V is for NPV and N is for notional) one defines

$$\Delta U_{t_i} = \frac{V_{t_i}}{N_{t_i}} - \frac{V_{t_{i-1}}}{N_{t_{i-1}}}$$

to calculate the standard deviation of the time series $\left\{\frac{\Delta U_{t_i}}{T - t_i}\right\}$.

9. Time Scaling of the Volatility Estimate: It is then annualized by multiplying by $\frac{1}{\sqrt{\Delta t_i}}$ where

$$\Delta t_i = t_i - t_{i-1}$$

In calculation the time (T, t_i) is presented in terms of years.

10. Non-duration Shortened Volatility Estimate: For other trades σ is calculated using

$$\sigma = \frac{Std\{\Delta U_{t_i}\}}{\sqrt{\Delta t_i}}$$

11. EPE of an Uncollateralized Trade: Similarly the EPE of an uncollateralized trade can be calculated (ignoring the cash flow effects) using the Black Normal model by

$$EPE_t = \mathbb{E}_0^Q\left[\frac{{V_t}^+}{\beta_t}\right] \approx \mathbb{E}_0^Q[\max(V_0 + V_{DRIFT} + D_t\Delta R_t, 0)]$$
$$\approx N_0\mathbb{E}_0^Q\left[\max\left(\frac{V_0}{N_0} + \mu\Delta t + \sigma\Delta W_t, 0\right)\right]$$

where D_t is the delta risk of the corresponding time t of the forward starting trade, and ΔW_t is the standard Brownian motion in $[0, t]$.

12. Regression Estimate of the Drift: The drift μ is obtained by performing a linear regression analysis of $\left[\frac{V_{t_i}}{N_{t_i}} - \frac{V_{t_{i-1}}}{N_{t_{i-1}}}\right]$ vs. $\{t_i - t_0\}$. It is the slope of the regression function.
13. EPE for the Proxy Pricer: The EPE is next calculated using the proxy pricer.
14. Collateralized Trade Proxy Pricer EPE: For collateralized trades one uses

$$EPE_t = \mathbb{E}_0^Q\left[\frac{\left(V_t - V_{t-\Delta t_{td}}\right)^+}{\beta_t}\right] \approx \mathbb{E}_0^Q\left[\max\left(\frac{\alpha k N_{t_{td}}\frac{3}{2}\sqrt{t - t_{td}}}{\beta_t}, 0\right)\right]$$
$$\approx \mathbb{E}_0^Q\left[\max\left(\frac{\alpha k N_0\frac{3}{2}\sqrt{t - t_{td}}}{\beta_t}, 0\right)\right]$$

where α by default is set to 1.0 and k is set to 15%.

15. Uncollateralized Trade Proxy Pricer EPE: For uncollateralized trades one uses

$$EPE_t = \mathbb{E}_0^Q\left[\frac{{V_t}^+}{\beta_t}\right] \approx \mathbb{E}_0^Q\left[\max\left(\frac{\alpha k N_{t_{td}}\frac{3}{2}\sqrt{t}}{\beta_t} + V_0, 0\right)\right] \approx \mathbb{E}_0^Q\left[\max\left(\frac{\alpha k N_0\frac{3}{2}\sqrt{t}}{\beta_t} + V_0, 0\right)\right]$$

16. Proxy Pricer vs Black EPE: If the EPE calculated by the proxy pricer is larger than the EPE calculated by the Black Normal model, the simplified back testing is deemed to have passed. Otherwise further analysis is needed before one can conclude that the proxy pricer model under-estimates historical volatility.

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Modeling Counterparty Credit Exposure in the Presence of Margin Agreements

Abstract

1. Margin Based Credit Exposure Reduction: Margin agreements as a means of reducing counterparty credit exposure.
2. Calculating MPoR Collateralized Exposures:
    a. Collateralized Exposure and Margin Period of Risk
    b. Semi-analytical Method for calculating collateralized EE
3. Analysis of Basel Exposure Methods: Analysis of Basel *Shortcut* method for Effective EPE.

Margin Agreements as a Means of Reducing Counterparty Credit Exposure

1. Definition of Counterparty Credit Risk: *Counterparty Credit Risk* is the risk that a counterparty in an *OTC* derivative transaction will default prior to the expiration of the contract and will be unable to make all contractual payments. *Exchange-traded* derivatives bear no counterparty risk.
2. Counterparty vs. Lending Risk Difference: The primary feature that distinguishes counterpart risk from lending risk is the uncertainty of exposure at any future time. For a loan, the exposure at any future date is the outstanding balance, which is certain – not taking into

account pre-payments. For a derivative, the exposure at any future date is the replacement cost, which is determined by the market value at that date, and is, therefore, uncertain.

3. Bilateral Nature of the Counterparty Risk: Since derivative portfolio value can be both positive and negative, counterparty risk is *bilateral*.
4. Spot/Forward Contract Market Value: Market value for counter i with counterparty is known only at the current date

$$t = 0$$

For any future date t this value $V_i(t)$ is uncertain and should be assumed random.

5. Replacement Cost at Counterparty Default: If a counterparty defaults at a time τ prior to the contract maturity, the economic loss is equal to the replacement loss of the contract. If

$$V_i(\tau) > 0$$

the dealer does not receive anything from the defaulted counterparty, but has to pay $V_i(\tau)$ to another counterparty to replace the contract. If

$$V_i(\tau) < 0$$

the dealer receives $V_i(\tau)$ from another counterparty, but has to forward this amount to the defaulted counterparty.

6. Forward Exposure at Contract Level: Combining these two scenarios the contract-level exposure $E_i(t)$ at time t is specified according to

$$E_i(t) = \max(V_i(t), 0)$$

7. Definition of Counterparty Level Exposure: *Counterparty level exposure* at a future time t can be defined as the loss experienced by the dealer if the counterparty defaults at time t under the assumption of no recovery.

8. Unmitigated Counterpart Level Positive Exposure: It the counterparty risk is not mitigated in any way, the *counterparty level* exposure equals the sum of *contract-level* exposure:

$$E(t) = \sum_{i} E_i(t) = \max(V_i(t), 0)$$

9. Counterparty Level Exposure under Netting: If there are *netting agreements*, derivatives with positive values at the time of the default offset the ones with negative values within each netting set NS_k so that the *counterparty-level exposure* is

$$E(t) = \sum_{k} E_{NS_k}(t) = \sum_{k} \max\left(\sum_{i \in NS_k} V_i(t)\,, 0\right)$$

   Each non-nettable trade represents a netting set.

10. Purpose of a Margin Agreement: Margin agreements allow for further reductions to counterparty level exposure.
11. Bilateral Posting under Margin Agreements: Margin agreement is a legally binding contract between two counterparties that requires one or both counterparties to post collateral under certain conditions. A margin threshold is defined for one (unilateral agreement) or both (bilateral agreement) counterparties. If the difference between the net portfolio value and the already posted collateral exceeds the threshold, the counterparty must provide sufficient collateral to cover this excess (subject to the minimum transfer amount).
12. Threshold Value in Margin Agreements: The threshold value depends primarily on the margin value of the counterparty.
13. Netting Set Level Margin Agreement: Assuming that every margin agreement requires a netting agreement, the exposure to the counterparty is

$$E_C(t) = \sum_{k} \max\left(\sum_{i \in NS_k} V_i(t) - C_k(t), 0\right)$$

where $C_k(t)$ is the market value of the collateral for the netting set NS_k at time t. If the netting set is not covered by a margin agreement, then

$$C_k(t) = 0$$

14. Netting Set Collateralized Portfolio Value: To simplify the notations, consider a single netting set

$$E_C(t) = \max(V_C(t), 0)$$

where $V_C(t)$ is the collateralized portfolio value at time t given by

$$V_C(t) = V(t) - C(t) = \sum_i V_i(t) - C(t)$$

Collateralized Exposure and Margin Period of Risk

1. Collateral as Excess Portfolio Value: Collateral covers the excess portfolio value $V(t)$ over the threshold H:

$$V_C(t) = \max(V(t) - H, 0) = -\min(H - V(t), 0)$$

2. Expression for the Collateralized Portfolio Value: Therefore the collateralized portfolio value is

$$V_C(t) = V(t) - C(t) = \min(V(t), H)$$

3. Floor/Ceiling of the Collateralized Exposure:

$$E_C(t) = \max(V_C(t), 0) = \begin{cases} 0 & V(t) < 0 \\ V(t) & 0 < V(t) < H \\ H & V(t) > H \end{cases}$$

   is limited from above and by zero from below.
4. Margin Period of Risk (MPoR): Even with a daily margin call frequency, there is a significant period delay known as the *margin period of risk (MPoR)* between a margin call that the counterparty does not respond to and the start of the default procedures. Margin calls can be disputed, and it may take several days to realize that the counterparty is defaulting rather than disputing the call. Further there is a grace period after the counterparty issues a notice of default. During this grace period the dealer and/or the counterparty may still post collateral.
5. Counterparty Variation Margin Posting Delay: Thus the collateral available at time t is determined by the portfolio value at time $t - \Delta t$.
6. Delay Dependence - Call Frequency and Product Liquidity: While Δt is not known with certainty, it is usually assumed to be a fixed number. The assume value of Δt depends on the margin call frequency and the trade liquidity.
7. MPoR Start - Collateral/Portfolio Values: Suppose that at time $t - \Delta t$ the unilateral collateral value is $C(t - \Delta t)$ and the portfolio value is $V(t - \Delta t)$.
8. Posted Counterparty Collateral at t: Then the amount $\Delta C(t)$ that should be posted at time t is

$$\Delta C(t) = \max\big(V(t - \Delta t) - C(t - \Delta t) - H, -C(t - \Delta t)\big)$$

   negative $\Delta C(t)$ means that the collateral will be returned to the counterparty.
9. Unilateral Counterparty Collateral at t: The unilateral counterparty collateral $C(t)$ available at time t is

$$C(t) = C(t - \Delta t) + \Delta C(t) = \max(V(t - \Delta t) - H, 0)$$

10. The Total Collateralized Portfolio Value: The collateralized portfolio value is

$$V_C(t) = V(t) - C(t) = \min\big(V(t), H + \Delta V(t)\big)$$

where

$$\Delta V(t) = V(t) - V(t - \Delta t)$$

11. Monte Carlo Primary Simulation Points: Suppose one has a set of *primary* simulation points $\{t_k\}$ for modeling non-collateralized exposure.
12. Monte-Carlo Look-back Points: For each

$$t_k > \Delta t$$

define a lookback point at time $t_k - \Delta t$
13. Monte Carlo Primary Plus Lookback: The task is to simulate the non-collateralized portfolio value along the path that includes both the *primary* and the *lookback* simulation times.
14. Collateral PLUS Collateralized/Uncollateralized Portfolio Values: Given $V(t_{k-1})$ and $C(t_{k-1})$ one calculates:
    a. Uncollateralized Portfolio Value $V(t_k - \Delta t)$ at the next lookback time $t_k - \Delta t$
    b. Uncollateralized Portfolio Value $V(t_k)$ at the next primary time t_k
    c. Collateral at t_k:

$$C(t_k) = \max(V(t_k - \Delta t) - H, 0)$$

    d. Collateralized Portfolio Value at t_k:

$$C(t_k) = V(t_k) - C(t_k)$$

e. Collateralized Exposure at t_k:

$$E_C(t_k) = \max(V_C(t_k), 0)$$

15. Simulating the Collateralized Portfolio Value: Collateralized threshold can go above the threshold due to MPR and MTA.

Semi-Analytical Method for Collateralized EE

1. Portfolio Value at Primary Points: Assume that the simulation is only run for the primary time points t and the portfolio distribution has been obtained in the form of M quantities $V_j(t)$, where j (from 1 to M) designates different scenarios.
2. Evaluating the Unconditional Portfolio Distribution: From the set $\{V_j(t)\}$ once can estimate the unconditional expectation $\mu(t)$ and standard deviation $\sigma(t)$ of the portfolio value, as well as any other distributional parameter.
3. Collateralized EE at Lookback Points: Can the collateralized EE profile be estimated without simulating the portfolio value at the lookback time points $\{V_j(t - \Delta t)\}$?
4. Collateralized EE Conditional on Path: Collateralized EE can be represented as

$$EE_C(t) = \mathbb{E}[EE_{C,j}(t)]$$

where $EE_{C,j}(t)$ is the collateralized EE conditional on $V_{C,j}(t)$:

$$EE_{C,j}(t) = \mathbb{E}\left[\max\left(V_{C,j}(t), 0\right) | V_j(t)\right]$$

5. The Conditional Collateralized Portfolio Value: The collateralized portfolio value $V_{C,j}(t)$ is

$$V_{C,j}(t) = \min\left(V_j(t), H + V_j(t) - V_j(t - \Delta t)\right)$$

6. Goal - Computing the Collateralized EE Analytically: If $EE_{C,j}(t)$ can be computed analytically, the *unconditionally collateralized EE* can be obtained as a simple average of $EE_{C,j}(t)$ across all scenarios j
7. Assumption of Normal Portfolio Value: Assume that the portfolio value $V(t)$ at time t is normally distributed with mean $\mu(t)$ and standard deviation $\sigma(t)$.
8. Brownian Bridge for Secondary Nodes: One can construct a *Brownian Bridge* from $V(0)$ to $V_j(t)$.
9. $V_j(t - \Delta t)$ Mean and Standard Deviation: Conditional on $V_j(t)$, $V_j(t - \Delta t)$ has a *normal distribution* with *expectation*

$$\alpha_j(t) = \frac{\Delta t}{t} V(0) + \frac{t - \Delta t}{t} V_j(t)$$

and *standard deviation*

$$\beta_j(t) = \sigma(t)\sqrt{\frac{\Delta t(t - \Delta t)}{t^2}}$$

10. Closed Form Conditional Collateralized EE: *Conditional Collateralized EE* can be obtained in a closed form.
11. Piece-wise Constant Local Volatility: It is assumed that, conditional on $V_j(t)$, the distribution of $V_j(t - \Delta t)$ is normal, but $\sigma(t)$ will be replaced by the local quantity $\sigma_{LOC}(t)$.
12. Portfolio Value Monotonically Increasing with Z: The portfolio value $V(t)$ at time t is described using

$$V(t) = \vartheta(t, Z)$$

where $\vartheta(t, Z)$ is a monotonically increasing function of the standard normal random variable Z.

13. <u>The *Equivalent Normal* Portfolio Process</u>: A *normal equivalent* portfolio process is defined as

$$W(t) = \omega(t, Z) = \mu(t) + \sigma(t)Z$$

14. <u>Density Scaling to determine $\sigma_{LOC}(t)$</u>: To obtain $\sigma_{LOC}(t)$, $\sigma(t)$ will be scaled by the probability densities of $W(t)$ and $V(t)$.
15. <u>Standard Deviation Scaled Probability Density</u>: The probability density of the quantity X is denoted via $f_X(\cdot)$ and the standard deviation is scaled according to

$$\sigma_{LOC}(t) = \frac{f_{W(t)}\big(\omega(t, Z)\big)}{f_{V(t)}\big(\vartheta(t, Z)\big)} \sigma(t)$$

16. <u>Changing Variables from W/V to Z</u>: Changing the variables from $V(t)$ and $W(t)$ to Z, one gets

$$f_{V(t)}\big(\vartheta(t, Z)\big) = \frac{\phi(Z)}{\partial\vartheta(t, Z)/\partial Z}$$

$$f_{W(t)}\big(\omega(t, Z)\big) = \frac{\phi(Z)}{\sigma(t)}$$

17. <u>Substitution to the Definition of $\sigma_{LOC}(t)$</u>: Substituting to the definition of $\sigma_{LOC}(t)$ above gives

$$\sigma_{LOC}(t) = \frac{\partial\vartheta(t, Z)}{\partial Z}$$

18. Estimating CDF - The Base Methodology: The values of Z_j corresponding to $V_j(t)$ can be obtained from

$$Z_j = \Phi^{-1}\left(F_{V(t)}\left(V_j(t)\right)\right)$$

19. Estimating the CDF - Sorting the Realizations: One sorts the array $V_j(t)$ in increasing order so that

$$V_{[j(k)]}(t) = V_{k,SORTED}(t)$$

where $j(k)$ is the sorting index.

20. Estimating CDF - Piecewise Constant Jump: From the sorted array, one can build a piecewise constant CDF that jumps by $\frac{1}{M}$ as $V(t)$ crosses any of the simulated values.

$$F_{V(t)}\left(V_j(t)\right) \approx \frac{1}{2}\frac{k-1}{M} + \frac{1}{2}\frac{k}{M} = \frac{2k-1}{2M} \rightarrow \frac{k-0.5}{M}$$

where 0.5 is the de-facto bias reducer.

21. Estimation of the Weiner Wanderer: Now one can obtain Z_j corresponding to $V_j(t)$ as

$$Z_{[j(k)]} = \Phi^{-1}\left(\frac{2k-1}{2M}\right)$$

22. Estimating the Local Standard Deviation: Local standard deviation $\sigma_{LOC,j}(t)$ can be estimated as

$$\sigma_{LOC,[j(k)]}(t) \equiv \sigma_{LOC}\left(t, Z_{[j(k)]}\right) \approx \frac{V_{[j(k+\Delta k)]}(t) - V_{[j(k-\Delta k)]}(t)}{Z_{[j(k+\Delta k)]} - Z_{[j(k-\Delta k)]}}$$

23. <u>Choice of the Different Amount Δk</u>: The offset Δk should not be too small (too much noise) or too large (loss of *locality*). This range works apparently well (Pykhtin (2009)):

$$20 \leq \Delta k \leq 0.05M$$

24. <u>The Brownian Bridge Mean and σ</u>: Similar to the above it is assumed that, conditional on $V_j(t)$, $V_j(t - \Delta t)$ has a *normal distribution* with *expectation*

$$\alpha_j(t) = \frac{\Delta t}{t} V(0) + \frac{t - \Delta t}{t} V_j(t)$$

and *standard deviation*

$$\beta_j(t) = \sigma(t) \sqrt{\frac{\Delta t(t - \Delta t)}{t^2}}$$

25. <u>The Collateralized Exposure Mean and σ</u>: The *collateralized exposure* depends on $\Delta V_j(t)$, which is also normal conditional on $V_j(t)$ with the same standard deviation $\beta_j(t)$ and expectation $\alpha_{C,j}(t)$ given by

$$\alpha_{C,j}(t) = V_j(t) - \alpha_j(t) = \frac{\Delta t}{t}\left[V_j(t) - V(0)\right]$$

26. <u>Collateralized EE Conditional on j</u>: Collateralized EE conditional on scenario j at time t is

$$EE_{C,j}(t) = \mathbb{E}\left[\max\left(\min\left(V_j(t), H + \Delta V_j(t)\right), 0\right) | V_j(t)\right]$$

27. <u>Collateralized EE on Negative Exposure</u>: $EE_{C,j}(t)$ equals *zero* whenever

$$V_j(t) > 0$$

so that

$$EE_{C,j}(t)=\mathbb{I}_{V_j(t)>0}\mathbb{E}\left[\min\left(V_j(t),H+\Delta V_j(t)\right)|V_j(t)\right]$$

28. Integral Form for Collateralized EE: Since $\Delta V_j(t)$ has a normal distribution, one can write

$$EE_{C,j}(t)=\mathbb{I}_{V_j(t)>0}\int_{-\infty}^{+\infty}\min\left(V_j(t),H+\alpha_{C,j}(t)+\beta_j(t)Z\right)\phi(Z)dZ$$

$$=\mathbb{I}_{V_j(t_k)>0}\left\{\int_{-d_2}^{-d_1}\left[H+\alpha_{C,j}(t)+\beta_j(t)Z\right]\phi(Z)dZ+V_j(t)\int_{-d_1}^{+\infty}\phi(Z)dZ\right\}$$

29. Conditional Collateralized EE Closed Form: Evaluating the integrals, one obtains

$$EE_{C,j}(t)=\mathbb{I}_{V_j(t_k)>0}\left\{\left[H+\alpha_{C,j}(t)\right][\Phi(d_2)-\Phi(d_1)]+\beta_j(t)[\phi(d_2)-\phi(d_1)]+V_j(t)\Phi(d_1)\right\}$$

where

$$d_1=\frac{H+\alpha_{C,j}(t)-V_j(t)}{\beta_j(t)}$$

$$d_2=\frac{H+\alpha_{C,j}(t)}{\beta_j(t)}$$

Analysis of Basel "Shortcut" Method for Collateralized Effective EPE

1. Basel 2 Exposure Capital Requirements: Basel 2 minimal capital requirements for the counterparty risk are determined by wholesale exposure rules with exposure at default obtained from expected exposure profile as follows.
2. Exposure at Default - Basel Variants:
    a. Expected Exposure (EE) – Expected Exposure Profile (EE)
    b. Expected Positive Exposure (EPE) –

$$EPE = \int_0^{1\ Year} EE(t)dt$$

    c. Effective EE -

$$Effective\ EE(t_k) = \max\big(EE(t_k), Effective\ EE(t_{k-1})\big)$$

    d. Effective EPE –

$$Effective\ EPE = \int_0^{1\ Year} Effective\ EE(t)dt$$

    e. Exposure at Default (EAD) –

$$EAD = \alpha \times Effective\ EPE$$

3. Incorporating the Margin Agreement: For collateralized counterparties, the netting set level Effective EPE must incorporate the effect of margin agreement.
4. Effective EPE using Internal Model of Collateral: Collateralized Effective EPE can be calculated using an *internal model of collateral*.

5. Basel 2 Simple and Conservative Shortcut: Alternatively dealers can use a *simple and conservative approximation* to the effective EPE, and sets the effective EPE for a margined counterparty equal to the lesser of:
   a. The *Threshold*, if positive, under the margin agreement *plus* an *add-on* that reflects the potential increase in exposure over the margin period of risk. The *add-on* is computed as the *expected increase in the netting set's exposure* beginning from the current exposure of zero over the margin period of risk.
   b. *Effective EPE without a margin agreement*.
6. Derivation of the "Shortcut" Method: The Basel "Shortcut" method can be obtained as follows:

$$
\begin{aligned}
EE_C(t) = \mathbb{E}\left[\max\left(\min\left(V(t), H + \Delta V(t)\right), 0\right)\right] &= \mathbb{E}[\min(E(t), H + \max(\Delta V(t), -H))] \\
&\le \mathbb{E}[\min(E(t), H + \max(\Delta V(t), 0))] \le \min(EE(t), H + \mathbb{E}[\max(\Delta V(t), 0)]) \\
&\approx \min(EE(t), H + \mathbb{E}[\max(\Delta V(\Delta t), 0)]) \equiv EE_{C,BSM}(t)
\end{aligned}
$$

7. Enhancing the Exposure Conservativeness: Time averaging adds more conservativeness:

$$
\frac{1}{T}\int_0^T EE_{C,BSM}(t)dt \le \min(EPE, H + \mathbb{E}[\max(\Delta V(\Delta t), 0)])
$$

Conclusion

1. Margin Agreements for Risk Mitigation: Margin agreements are important risk mitigation tools that need to be modeled accurately.
2. Complete MC Doubles Simulation Time: Full Monte Carlo is the most flexible approach, but requires simulating trade values at secondary time points, thus doubling the simulation time.

3. Semi-Analytical Approach Avoids That: Pykhtin (2009) has presented an accurate semi-analytical approach of calculating the EE that avoids doubling of the simulation time.
4. Basel 2 Shortcuts are too Conservative: Basel 2 "Shortcut" method for Effective EPE has sound theoretical grounds, but is too conservative.

References

- Pykhtin, M. (2009): Modeling Counterparty Credit Exposure in the Presence of Margin Agreements

Re-thinking the Margin Period Risk

Abstract

1. Enhanced CSA Collateral Exposure Model: Andersen, Pykhtin (2017) describe a new framework for collateral exposure modeling under an ISDA Master Agreement with Credit Support Annex. The proposed model captures the legal and the operational aspects of default in considerably greater detail than models currently used by most practitioners, while remaining fully tractable and computationally feasible.
2. Legal Rights Exercise/Deferral Choices: Specifically, it considers the remedies and the suspension rights available within these legal agreements; the firm's policies of availing itself of these rights; and the typical time it takes to exercise them in practice.
3. Significantly Higher Credit Exposure Revealed: The inclusion of these effects is shown to produce a significantly higher credit exposure for representative portfolios compared to the currently used models. The increase is especially pronounced when dynamic initial margin is also present.

Introduction

1. Margin Period of Risk Overview: In modeling the exposure of collateralized positions, it is well recognized that credit default cannot be treated as a one-time event. Rather the entire

sequence of events following up to the default and beyond need to be considered, from the last successful margin call in advance of the eventual default to the time when the amount of loss becomes known – in the industry parlance, *crystallized*. These events unfold over a period of time called the *margin period of risk* (MPoR).

2. Range of Model Applicability: To properly identify the exposure during the MPoR, a detailed understanding of the contractual obligations is essential. In their paper, Andersen, Pykhtin, and Sokol (2017) focus on collateralized exposures under bilateral trading relationships governed by the *ISDA Master Agreement* (IMA) and its *Credit Support Annex*. The IMA, by far, is the most common legal contract for bilateral over-the-counter (OTC) derivatives trading, although other agreements are sometimes used (such as national forms of agreements used in some jurisdictions for domestic trading). The analysis by Andersen, Pykhtin, and Sokol (2017) is expected to apply to a broad class of contracts, although the model assumptions should be re-examined to confirm that the key legal provisions remain substantially the same as IMA.
3. Refinement for Legal/Operational Impact: It should be noted that the modeling of default exposure and close-out risk arising from a non-zero MPoR has received a fair amount of attention in the past (see, e.g., Gibson (2005), Pykhtin (2009, 2010), and Brigo, Capponi, Pallavicini, and Papatheodorou (2011)), although most past analysis has been conducted under very strong simplifying assumptions about the trade and the margin flows during the MPoR. One exception is Bocker and Schroder (2011), which contains elements of a more descriptive framework, including recognition of the role played by the cash flows close to the default event. Andersen, Pykhtin, and Sokol (2017) use a more detailed framework for legal and operational behavior to refine the classical models for collateralized exposure modeling.
4. Variation Margin Operational Timelines: This chapter is organized as follows. The fundamentals of variation margin posting are first outlined, and the classical collateralized exposure model is then presented. The full timelines of events likely to transpire during a credit default are then discussed from both legal and operational perspectives. This sets the stage for the proposal of a condensed representation of the timeline suitable for analytical and numerical work. The resulting setup results in a more significantly nuanced and flexible definition of the collateralized trading exposure. As fixing the actual model parameters (i.e.,

calibrating the MPoR model) requires taking a stance on operational procedures and corporate behavior, the next section discusses how such parametrizations may be done in practice, for various levels of overall model prudence and counterparty types.

5. Numerical Computation of Collateralized Exposure: Subsequently, the model is fleshed out in more detail, especially as it pertains to numerical implementations and quantitative comparisons with the classical model. As a starting point, exposure models are formulated in mathematical terms, and the key differences to the classical models are highlighted by means of brute-force Monte-Carlo simulations. Computational techniques permitting efficient model implementation are introduced subsequently, along with several test results. Applications to portfolios with risk-based initial margins are briefly discussed, and conclusions are finally drawn.

The Fundamentals of Variation Margin: Basic Definitions

1. Types of Margin: Initial/Variation: In bilateral OTC derivatives trading, it is common for parties to require posting of collateral to mitigate excessive exposures. Although the initial margin is discussed briefly in a later section, this section focuses primarily on the variation margin (VM) as a form of collateral that is regularly re-adjusted based on the changing value of the bilateral portfolio. The VM is calculated and settled in time according to a set of CSA rules discussed a few sections down.
2. Dealer and Client VM Timelines: For concreteness, throughout this chapter, the exposure of dealer D to a client C with whom D engages in bilateral OTC trading under the IMA/CSA legal framework is considered. C is referred to as the *defaulting party*, and D is *the dealer* or the *non-defaulting* party. All present value and exposure amounts throughout this chapter will be calculated from the viewpoint of D. Let the default-free market value to D of the securities portfolio at time t be $V(t)$ and let $A_D(t)$ and $A_C(t)$ be the collateral support amounts stipulated by the CSA to be posted to D and C respectively. In the absence of initial

margin it is virtually always the case that only one of A_D or A_C is positive, i.e., only one party will be required to post margin at a given point in time.

3. Net Collateral and its Posting: Assuming that collateral is netted (rather than posted by both parties in full and held in segregated accounts or by a third party), the total collateral amount in D's possession may be calculated as of time t as

$$c(t) = A_C(t) - A_D(t)$$

Assuming also that the collateral may be treated as *pari passu* with the derivatives portfolio itself for the purposes of bankruptcy claim, it is common to denote the positive part of the difference $V(t) - c(t)$ as the *exposure* $E(t)$:

$$E(t) = [V(t) - c(t)]^+$$

$$c(t) = A_C(t) - A_D(t)$$

where the notation

$$x^+ = \max(x, 0)$$

is used. Normally both the collateral and the portfolio would be treated together as a senior unsecured claim of D against the bankruptcy estate of C. There are several time lags and practical complications that render the above exposure and collateral expressions an imprecise measure, and they shall be substantially refined later on. In particular it is emphasized that the collateral computed at time t is generally not transferred to D until several days after t.

4. VM Designated to Track Portfolio Value: The type of VM encountered in the CSA is typically designed to broadly track the value of the portfolio between the parties, thereby ensuring that $E(t)$ in

$$E(t) = [V(t) - c(t)]^+$$

$$c(t) = A_C(t) - A_D(t)$$

does not grow to be excessive. However, to avoid unnecessary operational expenses, it is common to introduce language in the CSA to relax margin transfer requirements if the amounts are sufficiently small.

5. Reducing the Collateral Posting Events: To that end the typical CSA language for collateral calculations will stipulate:
    a. Collateral posting requirements by each party, h_D and h_C, representing the minimum amount of exposure before D or C, respectively, is required to post collateral
    b. A minimum transfer amount (MTA) establishing a minimum valid amount of a margin call
    c. Rounding, which rounds collateral movements to some reasonable unit (say $1,000).
6. Incorporating Thresholds into Collateral Expressions: Formally the effects of thresholds on the stipulated collateral may be written as

$$A_D(t) = [-V(t) - h_D]^+$$

$$A_C(t) = [V(t) - h_C]^+$$

with the net stipulated credit support amount assigned to D being

$$c(t) = A_C(t) - A_D(t)$$

as before. The actual availability of this amount is then subject to the path dependent effects on collateral by MTA and rounding, of which the former has significant effect only for zero or very small thresholds, and the latter is usually negligible. Both have been omitted in the equation above.

7. Unilateral and Asymmetric Collateral Requirements: Most CSAs are bilateral in nature, but unilateral CSAs do exist in which only one of the two parties is required to post collateral. A CSA may be formally bilateral, but highly asymmetric, requiring both parties to post collateral but with vastly different thresholds, e.g.

$$h_D = \$20\ mm$$

vs.

$$h_C = \$2\ mm$$

Typically, even for asymmetric CSAs, the MTAs and the rounding are the same for both parties.

Margin Calls and Cash Flows

1. Margining Frequency of the Collateral Process: From an exposure perspective, the frequency with which the amount of collateral is adjusted – the *re-margining frequency* – is a critical component of the CSA. Following the financial crisis, most new IMA/CSAs, especially between major financial institutions, have been using daily re-margining frequency in order to reduce the amount by which the exposure can change relative to the collateral between the margin calls. However many small financial institutions or buy-side clients may not be able to cope with the operational burden of frequent margin calls and will often negotiate longer re-margining frequencies, e.g., weekly, monthly, or even longer.
2. Events Constituting the Margining Process: The amount of collateral held by the parties is adjusted to their stipulated values A_D and A_C via the mechanism of a margin call. Many models for exposure treat the margin call as an instantaneous event, taking place on the re-

margining date and completed instantaneously. In practice the margin call is a chain of events that takes several days to complete. With daily re-margining, several such chains run concurrently in an *interlaced* manner; even as one margin call is yet to be settled, another one already may be initiated. The time lag of this settlement process, long with the inherent lag of the re-margining schedule, means that the changes in the VM are always running behind the changes in the portfolio value. This, in turn, implies that the idealized expressions such as

$$E(t) = [V(t) - c(t)]^{+}$$

$$c(t) = A_C(t) - A_D(t)$$

are inaccurate. The detailed events involved in an initiation and the eventual settlement of a margin call will be discussed in a later section.

3. Underlying Trade Cash Flow: With both default processes and margin settlements being non-instantaneous events, it becomes relevant to track what payment flows take place – or not – during the periods close to a default. Two types of payments are needed here. The first type – called using the term *trade flows*, covers the contractual cash flows, physical settlements, and other forms of asset transfers related to the trade themselves. These terms are spelt out in trade documents and *term sheets* for each trade. The term *trade flows* rather than *cash flows* is used to emphasize that term sheets may involve flows other than cash – such as transfers of other non-cash assets, e.g. commodities, physical settlements resulting from the creation of new trades from old ones, e.g. exercise of a physically settled swaption into a swap. A missed trade flow is a serious event under the IMA, and a failure to pay can rapidly result in a default and trade termination unless cured properly. Any missed trade flow is, of course, part of the non-defaulting party's claim.
4. CSA Specified Margin Cash Flow: The second type of flows is that that arises from the exchange of collateral between the parties – *margin flows*. The legal treatment of the margin flows is determined by the IMA/CSA, rather than by the trade documentation between the parties. For purposes of this treatment, the most important aspect of the IMA/CSA is the relatively mild treatment it affords to a party that misses a margin flow. Indeed, partially

missing a margin payment is a common occurrence, as disputes about margin amounts happen regularly, and sometimes persist for years.

5. Delays causing the Default Termination: During a collateral dispute, the CSA protocol calls for the payment of the undisputed components of the collateral, but there is of course the possibility that there will be no undisputed component at all, if one party's counter-proposals are sufficiently frivolous. Should suspicious about *gaming* arise, the CSA does contain a methodology to stop disputes through market quotations, but the resulting leakage of the position information is often a good deterrent to its use. As such, there is potential for abuse by firms experiencing financial difficulties, and a good possibility that such abuse can go on for some time before the dealer takes further efforts to end it. This, in turn, may result in a fairly long period of time between the last fully settled margin call and the eventual termination of a portfolio due to default.

Revised Exposure Definition

1. Stipulated vs. Realized Collateral Amount: In light of the discussion above, this section makes a first effort at improving

$$E(t) = [V(t) - c(t)]^+$$

$$c(t) = A_C(t) - A_D(t)$$

For this consider a default of C at time τ following an early termination of the trade portfolio at time

$$t \geq \tau$$

At time t, let $K(t)$ be the collateral B can *actually* rely on for the portfolio termination; this amount will very likely differ from the CSA stipulated amount $C(t)$ – and from $C(\tau)$ for that matter – due to the margin transfer time lags and some degree of non-performance by C.

2. Exposure Enhanced by Trade Flow: In addition, it is possible that some trade flows are missed; denote their value at time t, including accrued interest, as $UTF(t)$. The exposure generated by a default at time

$$\tau \leq t$$

may be re-defined as

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

Notice that the expression anchors the exposure at the termination date rather than at the default date τ - this will be treated at a later section. For later use, the time-0 expectation of the future time-t exposure is defined as

$$EE(t) = \mathbb{E}_0[E(t)]$$

where $\mathbb{E}$ is the expectation operator in a relevant probability measure.

3. Impact of the Margin Timelines: Determining how $K(t)$ can differ from $c(t)$, and how large can realistically $UTF(t)$ be, will require a more detailed understanding of the settlement and the margining processes, a topic that will be treated in detail in a later section. The next goes about determining how classical approaches go about modeling $K(t)$ and $UTF(t)$ in

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

Classical Model for Collateralized Exposure – Assumptions about Margin Flows

1. The Naïve Collateralized Exposure Model: A naïve, and now outdated, model for collateralized exposure follows the definition

$$E(t) = [V(t) - c(t)]^{+}$$

$$c(t) = A_C(t) - A_D(t)$$

literally, and assumes that the collateral available is exactly equal to its prescribed value at time t. That is, in the language of

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

it is assumed that

$$K(t) = c(t)$$

In addition, the parties are assumed to pay off all of the trade flows as described

$$UTF(t) = 0$$

and it is assumed that the termination date in

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

is the default date τ, i.e., there is no lag between the default date and the termination date. In this model, the assumption of loss crystallized at a time t is the function of the portfolio value at a single point $V(t)$ and does not depend on the earlier history $V(\cdot)$ In the limit of *perfect CSA* where

$$c(t) = V(t)$$

the collateralized exposure in such a model is exactly zero.

2. Lag Induced Classical Exposure Model: Assuming

$$K(t) = c(t)$$

is an idealization that ignores the non-instantaneous nature of collateral settlement protocols and does not capture the fact that firms under stress may stop fully honoring margin calls, resulting in a divergence between the portfolio value and the collateral value at some time lag δ before the termination of the portfolio. In what is denoted here as the *Classical Model* (see, for example, Pykhtin (2010)), this particular lag effect is captured by modifying

$$E(t) = [V(t) - c(t)]^{+}$$

$$c(t) = A_C(t) - A_D(t)$$

to

$$E(t) = [V(t) - K(t)]^{+}$$

$$K(t) = c(t - \delta)$$

So, for instance, for a CSA with thresholds h_D and h_C and from

$$A_D(t) = [-V(t) - h_D]^{+}$$

$$A_C(t) = [V(t) - h_C]^+$$

one gets

$$K(t) = [V(t-\delta) - h_C(t-\delta)]^+ - [-V(t-\delta) - h_D(t-\delta)]^+$$

3. Drawbacks of the Classical Exposure Model: Having a mechanism for capturing the divergence between the collateral and the portfolio value is an important improvement over the older method described above, and the classical model has gained widespread acceptance for both the CVA (Credit Valuation Adjustment) and the regulatory calculations. Nevertheless, it hinges on a number of assumptions that are unrealistic. For instance,

$$E(t) = [V(t) - K(t)]^+$$

$$K(t) = c(t-\delta)$$

   assumes that both D and C will simultaneously stop paying margin at time $t-\delta$ freezing the margin level over the entire MPoR. In reality, if the party due to post collateral at $t-\delta$ happens to be the non-defaulting party D, it will often continue doing so for some time even in the presence of the news about the possible impending default of C. And should C miss a few margin payments (maybe under the guise of a dispute), D would often continue to post collateral while it evaluates its options. This creates an asymmetry between posting and receiving collateral that the classical model fails to recognize.
4. Impact of Lag on Exposure: In

$$E(t) = [V(t) - K(t)]^+$$

$$K(t) = c(t-\delta)$$

the lag parameter δ is clearly critical; the larger δ is, the more $V(t)$ may pull from the frozen margin value at time $t - \delta$ and bigger the expected exposure will become. In practice, the determination of δ will often be done in a simplistic manner, e.g. by using a fixed lag (10 BD is common), or, more realistically, by adding a universal time delay to the re-margining frequency of the CSA in question. This practice is echoed in regulatory guidelines, e.g., in Basel 3 accord where MPoR is set to the re-margining frequency minus 1 BD plus an MPoR floor that defaults to 10 BD. The MPoR floor must be increased in certain cases, e.g., for large netting sets, illiquid trades, illiquid collateral, and recent collateral disputes – however, the increase is specified as a multiplier relative to the same default. With a high proportion of individually negotiated and amended features in real life IMA/CSAs, using a *one size fits all* assumption may, however, lead to significant inaccuracies.

Assumptions about Trade Flows

1. <u>The Classical+ Collateral Exposure Model</u>: Because large trade flows after the start of MPoR may no longer be followed by collateral adjustment, they have the potential to either extinguish or exacerbate the exposure. For this reason, the model assumptions with respect to the date when either party suspends the trade flows are likely to have a significant impact on the counterparty credit loss. In one common interpretation of the classical model, it is simply assumed that both D and C will continue to pay all the trade flows during the entire MPoR. As a consequence, the unpaid trade flow term $UTF(t)$ in

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

will be zero, consistent with

$$E(t) = [V(t) - K(t)]^{+}$$

$$K(t) = c(t - \delta)$$

For ease of reference this version of classical model is denoted as Classical+.

2. The Classical- Collateral Exposure Model: In another, less common, version of the Classical Model, the assumption is that both C and D will stop paying trade flows at the moment the MPoR commences, i.e., at time $t - \delta$. In this case, the unpaid trade flows are set equal to

$$UTF(t) = TF_{NET}(t;\ (t - \delta, t])$$

where $TF_{NET}(t;\ (t', t"])$ is the time t value of all net trade flows scheduled to be paid in the interval $(t', t"]$. Note that the time is measured in discrete units of business days, such that the notation $(u, s]$ is equivalent to $[u + 1BD, s]$. Further, if t is after the margin flow date, the trade flow value accrues from the payment date to t at a contractually specified rate. This version of the classical model is denoted Classical-; it is associated with an exposure definition of

$$E(t) = [V(t) + TF_{NET}(t;\ (t - \delta, t]) - c(t - \delta)]^+$$

3. Inadequacies of the Classical Exposure Models: In practice neither the Classical+ exposure equation

$$E(t) = [V(t) - K(t)]^+$$

$$K(t) = c(t - \delta)$$

nor the Classical- exposure equation

$$E(t) = [V(t) + TF_{NET}(t;\ (t - \delta, t]) - c(t - \delta)]^+$$

are accurate representations of reality. Trade flows are likely to be paid at least by D at the beginning of the MPoR, and are likely not to be paid by C at least at its end. For instance, due to the CSA protocol for collateral calculations (next section), there is typically a $3\ BD$ lag between the start of an MPoR – the market observation date for the last full margin payment – and the date when D definitely observes that C has missed paying a margin flow; during this period D would always make all trade payments unless C commits any additional contract violations. Even after D has determined that C has missed a margin payment, D's nominal rights to suspend payments following a breach would, as mentioned earlier, not always be exercised aggressively. Legal reviews, operational delays, and grace periods can further delay the time when D stops paying trade flows to C.

4. Accrual of Missed Trade Flows: Another trade flow effect arises during the last $2 - 3$ days of the MPoR (just prior to termination) where C has already defaulted and neither party is likely making trade payments. Here, the IMA stipulates that the missed trade flows within this period accrue forward at a contractually specified rate and become part of the bankruptcy claim. This gives rise to a termination period in addition to the $UTF(t)$ term, in turn leading to an adjustment of the exposure.

Full Timeline of IMA/CSA Events

1. Details of the IMA/CSA Processes: Loosely speaking, the IMA concerns itself with the events of default, termination, and close out, and the CSA governs the collateral exchanges, including the concrete rules for collateral amount calculations and posting frequencies. While this chapter so far has touched on the workings of the IMA/CSA in the previous sections, model construction hereafter will require more detailed knowledge of certain provisions regarding the normal exchange of collateral, the legal options available in the case of missed payments, and the common dealer policies with respect to availing itself of these options. A

detailed exposition of the IMA and the CSA legal complexities can be found in multiple sources, including at http://www.isda.org; here only a brief summary to the extent necessary to develop the model is provided. The focus is on the development of a plausible timeline of events taking place around the default, and the subsequent portfolio termination.

Events Prior to Default

1. Calculated vs. Actual Collateral Amounts: It is assumed that the dealer D is the Calculation Agent for the computation of the collateral amounts. As before, A_C and A_D denote the *prescribed* collateral amounts for C and D; as discussed they may differ from the *actually* available collateral amounts M_C and M_D if one of the parties fails to make the margin flow or changes the prescribed amount.
2. CSA Specified Margin Process Timelines: The following list describes the complete sequence of events taking place at times $T_0, T_1, \cdots$ The next section simplifies and condenses these into a tractable model.
3. Collateral/Portfolio Valuation Date T_0: The timeline begins at T_0, the as-of-date at which the value of the portfolio and its collateral are measured, for usage in the T_1 evaluation of the formulas for the Credit Support Amount – plainly, the amount of collateral. Typically, T_0 is the close of business on the business day before T_1.
4. Honored Collateral Invocation Date T_1: For the purposes of this treatment, T_1 is used to refer to the last *undisputed and respected* Valuation Date prior to default. At time T_1, besides officially determining $A_c(T_0)$ and $A_C(T_0)$, the dealer D calculates the incremental payment amounts to itself and to C as

$$m_D = A_D(T_0) - M_D(T_0)$$

and

$$m_C = A_C(T_0) - M_C(T_0)$$

respectively. Taking into account any minimum transfer amounts, the transfer amounts m_D and m_C should normally be communicated by D to C prior to a Notification Time (e.g., 1 PM local time).

5. Collateral Transfer Initiation Date T_2: After receiving the notice of the calculated collateral amount, C must initiate the transfers of the sufficient amount of eligible collateral on the payment date T_2. Assuming that D managed to have the collateral amount notification sent to C prior to the Notification Time, T_2 defaults to 1 BD after T_1. If D is late in its notification, T_2 would be 2 BD after T_1. It is assumed here that the required amounts – recalling that they were calculated at T_1 using market data at time T_0 – are all settled without incident at T_2. However, T_2 will be the *last* time that margin flows settle normally before the default takes place.
6. Non honored Collateral Calculation Date T_3: Let T_3 denote the next scheduled valuation date after T_1. If α is the average scheduled time between collateral calculations, one approximately has – ignoring business calendar effects –

$$T_3 \approx T_1 + \alpha$$

   At T_3 – hopefully before the Notification Time - D will be able to send payment notice to C, but C has fallen into financial distress and will not be able – or willing – to pay further margin flows. Should C simply fail to pay collateral at the next payment date, a *Credit Support Default* could be triggered shortly thereafter (non-payment of collateral is associated with a 2 BD grace period). To prevent this from happening, it is, as discussed earlier, likely that C would attempt to stall by disputing the result of the T_3 collateral calculation by D.
7. Potential Event of Default τ: Exactly how long the margin dispute is allowed to proceed is a largely a behavioral question that requires some knowledge of D's credit policies and its willingness to risk legal disputes with C. Additionally one needs to consider to what extend C

is able to conceal its position of financial stress by using dispute tactics, or, say, blaming operational issues on its inability to pay collateral. Ultimately, however, D will conclude that C is in default of its margin flows (a Credit Support Default), or C will commit a serious contract violation such as failing to make a trade-related payment. At that point D will conclude that a *Potential Event of Default* (PFD) has occurred. The time of this event is identified as the true *default time* τ.

8. Client PED Communication Date T_4: Once the PED has taken place, D needs to formally communicate it to C, in writing. Taking into account mail/courier delays, legal reviews, and other operational lags, it is likely that the communication time, denoted T_4, takes place at a slight delay to the PED.
9. Event of Default Date T_5: After the receipt of the PED notice, C will be granted a brief period of time to cure the PED. The length of this *cure period* is specified in the IMA and depends on both the type of the PED and the specific IMA. For instance, of the PED in question is Failure to Pay, the default cure period is 3 BD in the 1992 IMA and 1 BD in the 2002 IMA – this may very well be overridden in the actual documents. At the end of the cure period – here denoted T_5 – and *Event of Default* (ED) formally crystallizes. It is emphasized here at the T_5 (the *official* default time) is *not* associated with the true default time τ; instead τ is equated to the time of the actual default (the PED) that, after contractual formalities, will lead to the default of C.
10. ED Communication (T_6) and ETD Designation Dates (T_7): After the ED has taken place, D will inform C of the ED at time

$$T_6 \geq T_5$$

and may, at time

$$T_7 \geq T_6$$

elect to designate an *Early Termination Date* (ETD).

11. Early Termination Date T_8: The ETD is denoted T_8; per the IMA it is required that $T_8 \in [T_7, T_7 + 20D]$ The ETD constitutes the as-of-date for the termination of C's portfolio and collateral positions. Many dealers will aim for speedy resolution in order to minimize market risk, and will therefore aim to set the ETD as early as possible. There are, however, cases where this may not be optimal, as described in the Section below.
12. Post Client ETD Establishment Events: Once the portfolio claim has been established as of the ETD, the value of any collateral and unpaid trade flows held by C is added to the amount owed to D. Paragraph 8 of the CSA then allows D to liquidate any securities collateral in its possession and to apply the proceeds against the amount it is owed. Should the collateral be insufficient to cover what is owed to D, the residual amount will be submitted as a claim in C's insolvency. The claim is usually challenged by the insolvency representative, and where parties cannot agree, may be referred to court. It can sometimes take a long time before the claim is resolved by the bankruptcy courts and the realized recovery becomes known. The interest on the recovery amount for this time is added to the awarded amount. Note that this chapter focusses exclusively on modeling the magnitude of exposure and bankruptcy claim, and does not challenge the established way of modeling the amount and the timing of the eventual recovery using a loss-given-default (LGD) fraction.

Some Behavioral and Legal Aspects

1. Margin Exposure Modeling Parametrization Components: With the timeline just having been established, it remains for it to be tied with a proper model for exposure. In order to do so, as already mentioned, the timeline needs to be combined with coherent assumptions about the dealer and the client behavior in each sub-period. The assumptions should be determined not only by the rights available under the IMA/CSA, but also by the degree of operational efficiencies in serving notices and getting legal opinions, and also by the level of prudence

injected into the assumptions about the dealer ability and willingness to strictly uphold contractual terms within each client group as it pertains to margin flows and disputes.

2. Issues with Exercising Suspension Rights: From a legal rights perspective, the most important observation is that once notice of a PED has been served (time T_4) the so called *suspension rights* of IMA (Section 2(a)(iii)) and the CSA (Paragraph 4(a)) will allow D to suspend all trade- and collateral- related payments to C until the PED has been cured. The extent to which the suspension rights are actually exercised, however, is quiet situational. A particular danger is that D exercises its suspension rights due to a Potential Event of Default (PED), but that subsequently the PED is ruled to be not valid. Should this happen, the dealer can inadvertently commit a breach of contract which, especially in the presence of cross-default provisions, can have serious consequences for the dealer.
3. Choice of Designating an ETD: Another, somewhat counter-intuitive, reason for D not to enforce its suspension rights is tied to IMA Section 2(a)(iii)which can sometimes make it favorable for D to *never* designate an ETD. Indeed, if D owes C money, it would seem a reasonable course of action for D to simply:
    a. Never designate and ETD, and
    b. Suspend all the payments in the portfolio until the default gets *cured* – which most likely will never happen.

    This tactic basically allows D to walk away from its obligations on the portfolio when C defaults, effectively making D a windfall gain.
4. Jurisdiction Legality of the ETD Delays: The strategy of delaying the ETD is perpetuity has been tested by UK courts and found legal – although contract language has been proposed by ISDA to prevent the issue. In the US, however, local *safe haven* laws have been ruled to prevent ETD's for more than about one year. Still a one-year delay may prove tempting if D has a big negative exposure to C and is unwilling to immediately fund the large cash flow needed to settle. As most large dealers are presumably unlikely to play legal games with the ETD, this topic shall not be considered further here, but note that there is room to make more aggressive model assumptions around the ETD's than is done here.

Simplified Timeline of IMA/CSA Events

1. Motivation for the Timeline Simplification: It should be evident from the preceding section that the full timeline of IMA/CSA reviewed earlier is in many ways different, and more complex, than what is assumed in the Classical- and the Classical+ versions of the classical model. However, it is equally evident that the timeline is too complex to be modeled in every detail. This section offers a simplification of the timeline designed to extract the events most important for exposure modeling. The resulting model offers several important improvements over the classical model, while remaining practical and computationally feasible.

Identification of Key Time Periods

1. Classical MPOR Start/End Dates: To recap, first the classical model only considers two dates in the timeline of default; the start and the end of the MPoR. The start of the MPoR, denoted by $t - \delta$, is defined as the last observation date for which the margin was settled in full (a few days after the observation date). The end of the MPoR, denoted by t, is the observation date on which D's claim is established. Note that t coincides with the IMA's *Early Termination Date* (ETD) discussed earlier.
2. Classical Model Lag Length Error: In the classical model there is no clear distinction between the observation and the payment dates, making it difficult to cleanly capture the trade flow effects. For instance, in the classical version of the model, $t - \delta$ denotes both the last observation date as well as the dates on which all trade flows cease. In reality, the last margin observation date is unlikely to be contentious and trigger stoppage of trade flows, as

the margin payment to which the observation corresponds to will only be missed by C several business days later. Specifically, if the market data is observed on day 0, the valuation is performed in day 1, then only on day 2 (or 3 if the notification was late) is the initiation of the actual payment expected to take place. The length of this lag is of the same order of magnitude as typical assumptions for the length of the MPoR, and can be a source of considerable model error if not handled properly.

3. Delineating Observations and Payment Dates: In the simplified timeline proposed here, care is kept to take care of the distinction between the observation and the payment dates, and also to consider the possibility that D may take the action of stopping a particular type of flow at a different time than C does. Accordingly, the model includes two potentially different observation dates for which D and C later settle their margin flows in full for the last time; and two potentially different dates when they pay their trade flows respectively for the last time. The end of the MPoR is defined as in the same way as in the classical model, to coincide with the ETD. The table below summarizes the notation for the five dates in the simplified timeline.
4. Notation for the Dates in the Simplified Timeline:

Event	**Date Type**	**Notation**
Observation Date for the Last Margin Flow from C	Observation	$t_C = t - \delta_C$
Observation Date for the Last Margin Flow from D	Observation	$t_D = t - \delta_D$
Observation Date for the Last Trade Flow Payment from C	Settlement	$t_C{}' = t - \delta_C{}'$
Observation Date for the Last Trade Flow Payment from D	Settlement	$t_D{}' = t - \delta_D{}'$
ETD	Observation	t

5. Current Scheme MPoR Start Date: The start of the MPoR in the current model is $t - \delta$, which in the notation of table above may be defined symmetrically as

$$\delta = \max(\delta_C, \delta_D)$$

C is always expected to stop posting margin no later than the non-defaulting party D, and therefore on would very likely have

$$\delta_C \geq \delta_D$$

and

$$\delta = \delta_C$$

6. <u>Exposure Model Timeline Lag Choices</u>: The second column in the table above specifies which of the dates is the observation date, and which is the settlement or the payment date. According to the notation established in the table, δ_C and δ_D are the lengths of time preceding the ETD during which changes in the portfolio values no longer result in collateral payments by C and D, respectively. Similarly, $\delta_C{}'$ and $\delta_D{}'$ are the lengths of time preceding the ETD during which the respective parties do not pay trade flows. In, say, a Classical 10 day MPoR model

$$\delta_C = \delta_D = 10\ BD$$

with

$$\delta_C' = \delta_D' = 0$$

for Classical+ and

$$\delta_C' = \delta_D' = 10\ BD$$

for Classical-.

Establishing the Sequence of Events

1. Order of the MPoR Events: *A priori*, the four events in the Table between the start event and the end event of the MPoR can occur in any order. However, this section will now explain why the table very likely shows the proper sequence of events.
2. Time Lag between Margin/Trade Flows: As discussed earlier, missing trade flows are recognized as a more serious breach of contractual obligations than missing margin flows, especially since the latter may take the form of a margin valuation dispute. Therefore, it is reasonable to assume that neither party will stop paying the trade flows before stopping the payment of margin flows. Accounting for the margin settlement lag between the observation date and the payment date, this yields

$$\delta_C' \leq \delta_C - Margin\ Settlement\ Lag$$

$$\delta_D' \leq \delta_D - Margin\ Settlement\ Lag$$

3. Lag between Dealer/Client Events: It is also reasonable to assume that either of the two types of flows is first missed by the defaulting party C, and then only by the non-defaulting party D. This leads to the following additional constraints on the sequence of events within the timeline:

$$\delta_C \geq \delta_D$$

$$\delta_C' \geq \delta_D'$$

4. Client Settlement vs. Dealer Observation: Except in rare and unique situations such as outright operational failures, D would not continue to pay margin flows once C commits a more serious violation by missing a trade flow, resulting in

$$\delta_C' \leq \delta_D - Margin\ Settlement\ Lag$$

Combining these inequalities results in the chronological order of events shown in the table above.

Evaluation of the Client Survival Probability

1. Client Survival at MPoR Start: As was the case for the classical model, the setup anchors the exposure date t at the termination date ETD, at the very end of the MPoR. The ETD is the same for both parties, and constitutes a convenient reference point for aligning the actions of one party against those of the other. It needs to be emphasized that the ETD for which the exposure is evaluated does not coincide with the date at which the survival probability is evaluated, e.g. for the computation of the CVA. In the simplified timeline, the counterparty survival probability should be evaluated for $t - \delta_C{}'$, the last date when C stops paying trade flows – effectively assuming that the default is due to failure-to-pay. Hence, if $EE(t)$ is the expected exposure anchored at the ETD t, then the incremental contribution to the unilateral CVA from time t is, under suitable assumptions, $EE(t) \cdot \Delta\mathbb{P}_{t-\delta_C}[\cdot]$ where $\mathbb{P}$ is the survival probability under the model's measure – later sections contain concrete examples.
2. Client Survival at MPoR End: Evaluating the default probability at the anchor date t rather than at $t - \delta_C{}'$ will introduce the slight error in computing the survival probability. While this error is relatively small and is often ignored by practitioners, it takes virtually no effort, and has no impact on model efficiency, to evaluate the survival probability at the right date.

Timeline Calibration

1. Client Customization of IMA/CSA Specifications: As mentioned earlier, the specific IMA/CSA terms for a given counterparty should always be ideally examined in detail, so that any non-standard provisions may be analyzed by their impact on the timeline. For those cases where such bespoke timeline construction is not practical (typically for operational reasons), two standard (*reference*) timelines are proposed here. This will allow the demonstration of the thought process behind the timeline calibration, and will provide some useful base cases for later numerical tests.
2. Parametrizing the Client and the Dealer Timelines: While factors such as portfolio size and dispute history with the counterparty should, of course, be considered in establishing the MPoR, an equally important consideration in calibrating the model is the nature of expected response by B to a missed margin or trade flow by C. Even under plain vanilla IMA/CSA terms, experience shows that the reactions to contract breaches are subject to both human and institutional idiosyncracies, rendering the MPoR quiet variable.
3. Aggressive vs. Conservative Timeline Parametrization: Recognizing that "one size does not fit all", two different calibrations shall therefore be considered; one *aggressive* which assumes the best case scenario for rapidly recognizing the impending default, and taking swift action; and one *conservative*, which takes into consideration not only a likely delay in recognizing that the counterparty default is imminent, but also the possibility that the bank may not aggressively enforce its legal rights afforded under IMA and CSA in order to avoid damaging its reputation. In both scenarios daily re-margining is assumed – if a CSA calls for less frequent margin calls than this, the MPoR must be lengthened accordingly.

Aggressive Calibration

1. Applicability of the Aggressive Calibration: The aggressive calibration applies to trading relationship between two counterparties that both have string operational competence, and where there is little reputational risk associated with swift and aggressive enforcement of the non-defaulting party's legal rights against the defaulting party.
2. Inter-dealer Monitoring and Call-outs: A good example would be trading between two large dealers, both willing to aggressively defend against a possible credit loss. The credit officers here are assumed to be diligent in the monitoring of their counterparties, and generally be able to see a default developing, rather than be caught by surprise.
3. Full Application of Operational Sophistication: Under aggressive calibration, the event of C missing or disputing a margin call by any non-trivial amount will, given C's sophistication, immediately alert D that an impending default is likely. D will not be misled by claims of valuation disputes or other excuses, and will send a Notice of Credit Support Default under the IMA/CSA the next business day after the breach of the margin agreement. At the same time, to protect itself further, D will stop both the margin and the trade flows. The counterparty is assumed to simultaneously stop paying margin and trade flows as well, so that no further payments of any kind are exchanged by the parties.
4. Elimination of Settlement/Herstatt Risk: The simultaneous action by both parties in the Aggressive scenario to stop paying the trade flows at the earliest possible moment results in the elimination of all *settlement* risk – the possibility that the dealer may continue paying on its trade flow obligations while not receiving promised payments in return. In the context of cross-currency trades, this type of settlement risk is frequently referred to as the *Herstatt* risk, after the bank that caused large counterparty losses in this manner (https://en.wikipedia.org/wiki/Settlement_risk). Such risk shall be captured in the Conservative Calibration case below, and shall be discussed in more detail in a later section.
5. Timeline of the corresponding MPoR: Despite D's immediate and aggressive response, the MPoR will still be fairly long due to the way the IMA/CSA operates in practice. In particular,

notice that the first period in the simplified timeline is between the last observation date for which the margin was fully settled, and the first date for which C misses a margin flow.

6. Breakdown of the CSA Steps: As it takes at least 2 business days to settle a margin payment, plus 1 business day between the last margin that was successfully settled and the first margin payment that was not, a minimum of 3 business days will accrue from the start of an MPoR and a margin-related PED. Further, once the margin flow is missed, D must send at least 2 notices and permit a grace period – usually 2 business days – to cure the violation before an event of default (ED) has officially taken place and an ETD has been designated.
7. Comparison with Classical MPoR Timeline: Since an ETD cannot be designated prior to the event of default, it is unlikely that an MPoR can ever be less than 7 business days. It is remarkable that even under the most aggressive set of assumptions, the MPoR is still only 3 business days shorter than the classical 2-week MPoR.
8. Detailed Breakdown of the Aggressive Timeline: The detailed taxonomy of the aggressive timeline is listed in the table down below, and essentially splits the MPoR into two sections; a margin delay period of 3 business days, and a default resolution period of 4 business days. During the latter period, C and D cease paying on the first day, leaving a period of 3 business days where neither party makes any payments. Notice that it is assumed that the ETD is declared to coincide with the ED, i.e., the dealer will terminate as quickly as legally possible.

Conservative Calibration

1. Non-aggressive Enforcement of CSA Rights: The conservative calibration is intended to cover the situation where the dealer's enforcement of its rights under the IMA/CSA is deliberate and cautious, rather than swift. There may be several reasons for such a situation, sometime acing in tandem.
2. Applicable Clients - Less Sophisticated Participants: First, a dealer, if overly trigger-happy, can gain a market-wide reputation as being rigid and litigious, potentially causing clients to

seek other trading partners. In fact, should aggressive legal maneuvers be applied to counterparties that may be considered "unsophisticated", there is even a potential for the dealer to be perceived as predatory by the larger public.

3. <u>"Leakage" of Dealer Positions</u>: Second, there are situations where exercising the legal rights would cause an unattractive leakage of information into the broader market. As indicated earlier, this may happen for instance if the formal collateral dispute methodology of Paragraph 5 of the IDSA CSA is activated; the market poll inherent in the methodology would inevitably reveal the positions held with the counterparty to competing dealers.
4. <u>Ramification of Aggressive Legal Exercise</u>: Third, sometimes an aggressive interpretation of the legal rights can backfire in the form of lawsuits and counter-measures by the counterparty. For example, even when the dealer may have the rights to withhold payments (e.g., under Section 2(a)(iii)), it would often elect to not exercise this right immediately out of concern that a counter-ED would be raised against it or that withholding payments would exacerbate the liquidity situation of the counterparty potentially exposing the dealer to liabilities and lawsuits.
5. <u>Damage from "Improper PED" Rulings</u>: As mentioned, a particular danger is that the dealer exercises its suspension rights due to a Potential Event of Default (PED), but that subsequently the PED is ruled to not be valid. Should this happen, the dealer can inadvertently commit a breach of contract.
6. <u>Limitations with the Dealers' Operational Capacity</u>: Of course, even if a dealer may potentially be willing to aggressively exercise its rights, it may not have the operational capacity to do so quickly. For example, the dealer may not be able to perform the required legal review on a short notice, or may not always have the efficiency to get the notices mailed out at the earliest possible date. On top of this there is always potential for technology related and human errors and oversights.
7. <u>Timeline Incurred by Conservative Calibration</u>: While it is harder to get concrete data to estimate a reasonable timeline for the Conservative case (this case being dependent not only on the IMA/CSA details, but also on the specifics of the dealer's reputational considerations), under a perfectly reasonable set of assumptions the MPoR ends up being more than twice as long as for the Aggressive case above. Under this calibration choice, the Conservative

scenario assumes that the totality of the margin dispute negotiations, operational delays, human errors, legal reviews etc., adds up to 8 business days, yielding an MPoR of a total of 15 business days.

8. Typical Conservative CSA Event Timeline: One plausible scenario with daily re-margining could be:
    a. $t - 15 \implies D$ observes the portfolio value as needed for the margin transfer amount #1 as of $t - 15$.
    b. $t - 14 \implies D$ sends margin call #1 to C; D observes a margin transfer amount #2.
    c. $t - 13 \implies D$ sends margin call #2 to C; C honors margin call #1; D observes a margin transfer amount #3.
    d. $t - 12 \implies C$ fails to honor margin call #2 and initiates dispute; D tries to resolve the dispute while still paying and calculating the margin.
    e. $t - 7 \implies C$ fails to make a trade payment.
    f. $t - 6 \implies D$ stops paying margin and sends a PED notice.
    g. $t - 5 \implies C$ receives PED; D keeps making trade payments.
    h. $t - 3 \implies$ The PED is not cured.
    i. $t - 2 \implies D$ stops making trade payments and sends an ED notice to C, designating t as the ETD.
    j. $t \implies$ ETD.
9. Current/Interlacing Outstanding Margin Process: Notice that a number of different margin processes are simultaneously active (denoted #1, #2, and #3), reflecting the interlacing nature of the daily margin calls. Also, unlike the earlier Aggressive Calibration, the above scenario explicitly involves settlement risk, as a time period exists only where D pays trade flows (from $t - 7$ to $t - 3$, both dates inclusive).
10. Dealer/Client Payment/Settlement Lags: To translate the scenario above into the notation of the earlier sections, first notice that

$$\delta_C = 15$$

since the observation date of the last margin call (#1) honored by C is $t-15$. Second, as D makes its last possible margin call at $t-7$ based on an observation at time $t-9$

$$\delta_D = 9$$

Third, as C fails to make a trade payment at $t-7$, C's last payment date is $t-8$, and therefore

$$\delta_C' = 8$$

And finally since D stops its trade payments at $t-2$

$$\delta_D' = 3$$

Summary and Comparison of Timelines

1. Classical+/Classical-/Aggressive/Conservative Parametrizations: Using the notation above, the Aggressive and the Conservative scenarios are presented in the table below. For reference, the Classical+ and the Classical- versions of the classical model are presented in the table as well. Note that the 10 BD assumption of the classical MPoR lies between the two calibration choices proposed, and is closer to the Aggressive scenario.
2. MPoR Periods for CSA's with Daily Re-margining:

Parameter	Conservative	Aggressive	Classical+	Classical-
δ_C	15 BD	7 BD	10 BD	10 BD

δ_D	9 BD	6 BD	10 BD	10 BD
$\delta_C{}'$	8 BD	4 BD	0 BD	10 BD
$\delta_D{}'$	3 BD	4 BD	0 BD	10 BD

3. Caveats over Aggressive/Conservative Parameters: The Aggressive and the Conservative parameter choices represent two opposite types of dealer-client relationships, and may also be used as two limit scenarios for materiality and model risk analysis. Of course, the best approach would always be to set the model parameters based on prudent analysis of the firm's historical default resolution timelines, to the extent that it is practically feasible. The model could also conceivably treat the various time lags as random variables to be simulated as part of the exposure computations; yet it is debatable whether increasing the number of model parameters this way is warranted in practice.

Unpaid Margin Flows and Margin Flow Gap

1. Margin and Trade Flow Gaps: To formulate the model in more precise mathematical terms, this section returns to

$$E(t) = [V(t) + UTF(t) - K(t)]^{+}$$

and considers how to draw the analysis of the previous sections to reasonably specify both the collateral amount $K(t)$ as well as the value $UTF(t)$ of net unpaid cash flows.
2. Client Last Margin Posting Date: As with the classical model, it is assumed that the MPoR starts at time

$$t_C = t - \delta_C$$

the portfolio observation date associated with the last regular collateral posting by C. Recall that the classical model further assumes that D will stop posting collateral simultaneously with C so that

$$K(t) = c(t_C)$$

where $c(t_C)$ denotes the CSA prescribed collateral support amount calculated from the market data observed at time t_C. This is to be compared with

$$E(t) = [V(t) - K(t)]^+$$

$$K(t) = c(t - \delta)$$

3. Definition of the Margin Flow Gap: In contrast to

$$K(t) = c(t_C)$$

this model assumes that D will continue posting and returning collateral to C for all contractual margin observations dates t_i whenever required by the CSA, after

$$t_C = t - \delta_C$$

and up to and including the observation date

$$t_D = t - \delta_D$$

The presence of an observation period of non-zero length for which D is posting and returning collateral but C is not is referred to as *margin flow gap*.

4. Choosing the Collateral Computation Date: Here, it is always expected that

$$t_D \geq t_C$$

which therefore in effect assumes the possibility of a time interval $(t_D, t_C]$ where only D honors its margin requirements. In this interval, D can match its contractually stipulated amounts $c(t_i)$ only when they involve transfers from D to C. This asymmetry results in D holding at time t the *smallest* collateral computed in the observation interval $[t_C, t_D]$ i.e.

$$K(t) = \min_{t_i \in [t_C, t_D]} c(t_i)$$

5. Implications of the Collateral Date Choice: The *worst case* form of the above provides a less optimistic view on available collateral than on classical modeling, resulting in larger exposure whenever there are multiple collateral observation dates in $[t_C, t_D]$ – it is assumed that one of the observation dates t_i always coincides with the start of the MPoR t_C. All other things being equal, the difference in exposure relative to the classical model will increase with $\delta_C - \delta_D$. If $\delta_C - \delta_D$ is kept constant, the difference will increase with more frequent re-margining. Note that

$$K(t) = \min_{t_i \in [t_C, t_D]} c(t_i)$$

matches

$$K(t) = c(t_C)$$

when

$$\delta_C = \delta_D$$

Unpaid Trade Flows and Trade Flow Gap

1. Origin of the Trade Flow Gap: According to the assumptions in the earlier sections, the last date when C is still paying the trade flows is

$$t_C' = t - \delta_C'$$

and the last date when D is still trade flows is

$$t_D' = t - \delta_D' \geq t_C'$$

The period when D is still paying trade flows while C does not is referred to as the *trade flow gap*.

2. Projecting the Unpaid Trade Flow Value: The value of the net trade flows unpaid by the termination date t can be expressed using the notation established so far as

$$\begin{aligned} UTF(t) &= TF_{C \to D}(t;\ (t_C', t]) + TF_{D \to C}(t;\ (t_D', t]) \\ &= TF_{C \to D}(t;\ (t_C', t_D']) + TF_{NET}(t;\ (t_D', t]) \end{aligned}$$

where an arrow indicates the direction of the trade flows and $C \to D$ ($D \to C$) trade flows have positive (negative) sign.

3. Same Date Dealer/Client Flows: In calculating the above equation, care needs to be taken on how the trade flows are aggregated and accrued to the termination date t. Cash flows of opposite direction scheduled to be paid in the same currency on the same date (for instance, the two legs of an ordinary single-currency interest-rate swap) in the period $(t_D', t]$ are aggregated (netted) at the cash flow date, therefore only the aggregated amount – their

difference – enters in to the above equation. The aggregated amount of the missed cash flows should be accrued to time t at the interest rate of the currency in question, and then converted to D's domestic currency.

4. <u>Different Currency Dealer/Client Flows</u>: Cash flows in opposite direction scheduled to be paid in *different* currencies on the same date (for instance, the two legs of a cross-currency interest rate swap) are *not* netted at the cash flow date. The missed cash flow amounts in each currency should be accrued to time t at the relevant interest rates, and then converted to D's domestic currency.
5. <u>Physically Settled Dealer/Client Flows</u>: The value of each asset flow (for instance, a swap that would result from exercising a physically settled swaption) should be obtained through pricing at time t of the undelivered asset in D's domestic currency. Generally, asset flows are not aggregated.
6. <u>Collateral available at Termination Date</u>: To analyze the impact of these assumptions on the expression for $UTF(t)$ above, consider for simplicity a zero-threshold margin agreement with no MTA/rounding. Then, from

$$K(t) = \underset{t_i \in [t_C, t_D]}{min} c(t_i)$$

the collateral available to D at the termination date can be written as

$$K(t) = V(t_{COL})$$

$$t_{COL} = \underset{t_i \in [t_C, t_D]}{min} V(t_i)$$

7. <u>Corresponding CSA Implied Client Exposure</u>: Substituting

$$UTF(t) = TF_{C \to D}(t;\ (t_C{}', t_D{}']) + TF_{NET}(t;\ (t_D{}', t])$$

and

$$K(t) = V(t_{COL})$$

$$t_{COL} = \min_{t_i \in [t_C, t_D]} V(t_i)$$

into

$$E(t) = [V(t) + UTF(t) - K(t)]^+$$

yields, for the simple CSA considered

$$E(t) = [V(t) - V(t_{COL}) + TF_{C\to D}(t;\ (t_C{}', t_D{}']) + TF_{NET}(t;\ (t_D{}', t])]^+$$

8. <u>CSA Implied Client Exposure Components</u>: $E(t)$ above implies that trading flows from D to C can occur within the MPoR. Thus trade flows have the potential to generate large spikes in the exposure profiles – especially in the presence of a trade flow gap where only D pays trade flows. To see this, the exposure components of $E(t)$ can be further drilled down as follows. First, ignoring the minor discounting effects inside the MPoR, the portfolio value at time t_{COL} can be represented as a sum of the portfolio's forward value V_F to time t and the value of all the trade flows taking place <u>after t_{COL}</u> and <u>up to and including t</u>:

$$V(t_{COL}) = V_F(t_{COL}; t) + TF_{NET}(t_{COL};\ (t_{COL}, t])$$

9. <u>Trade Flow Post Collateral Date</u>: This may be further re-written as

$$\begin{aligned} &TF_{NET}(t_{COL};\ (t_{COL}, t]) \\ &\qquad = TF_{NET}(t_{COL};\ (t_{COL}, t_C{}']) + TF_{C\to D}(t_{COL};\ (t_C{}', t_D{}']) \\ &\qquad + TF_{D\to C}(t_{COL};\ (t_C{}', t_D{}']) + TF_{NET}(t_{COL};\ (t_D{}', t]) \end{aligned}$$

which, together with

$$V(t_{COL}) = V_F(t_{COL}; t) + TF_{NET}(t_{COL};\ (t_{COL}, t])$$

allows re-stating

$$E(t) = [V(t) - V(t_{COL}) + TF_{C \to D}(t;\ (t_C', t_D']) + TF_{NET}(t;\ (t_D', t])]^+$$

in the following form:

$$\begin{aligned} E(t) = \{ & V(t) - V_F(t_{COL}; t) + TF_{C \to D}(t;\ (t_C', t_D']) - TF_{C \to D}(t_{COL};\ (t_C', t_D']) \\ & + TF_{NET}(t;\ (t_D', t]) - TF_{NET}(t_{COL};\ (t_D', t]) - TF_{NET}(t_{COL};\ (t_{COL}, t_C']) \\ & - TF_{D \to C}(t_{COL};\ (t_C', t_D'])\}^+ \end{aligned}$$

10. <u>The Market Driven Portfolio Change $t_{COL} \to t$</u>: The terms in

$$E(t) = [V(t) - V(t_{COL}) + TF_{C \to D}(t;\ (t_C', t_D']) + TF_{NET}(t;\ (t_D', t])]^+$$

have been re-arranged into five separate components, corresponding to the five different contributions to the exposures. The first is

$$V(t) - V_F(t_{COL}; t)$$

- the change of the portfolio forward value to time t driven by the change in the market factors between t_{COL} and t. This term is driven by the volatility of the market factors between t_{COL} and t; it produces no spikes in the expected exposure profile.

11. <u>The Market Driven Trade Flow $t_C' \to t_D'$</u>:

$$TF_{C \to D}(t;\ (t_C', t_D']) - TF_{C \to D}(t_{COL};\ (t_C', t_D'])$$

represents the change of the value of the trade flows scheduled to be paid – but actually unpaid – by C in the interval $(t_C{}', t_D{}']$ resulting from the change of the market factors between t_{COL} and t. This term is driven by the volatility of the market factors between t_{COL} and t; it produces no spikes in the expected exposure profiles.

12. The Market Driven Trade Flow $t_D{}' \rightarrow t$:

$$TF_{NET}(t;\ (t_D{}', t]) - TF_{NET}(t_{COL};\ (t_D{}', t])$$

    is the change of the value of the net trade flows between C and D scheduled to be paid – but actually unpaid – in the interval $(t_D{}', t]$ resulting from the change in the market factors between t_{COL} and t. This term likewise produces no spikes in the expected exposure profile.

13. Net Trade Flow between $t_{COL} \rightarrow t_C{}'$:

$$-TF_{NET}(t_{COL};\ (t_{COL}, t_C{}'])$$

    is the negative value of the net trade flow between C and D scheduled to be paid – and actually paid – in the interval $(t_{COL}, t_C{}']$. Paths where D is the net payer – so that TF is negative – contribute to the upward spikes in the EE profile.

14. Dealer Trade Flow across $t_C{}' \rightarrow t_D{}'$: The negative value of the trade flows scheduled to be paid – and actually paid – by C to D in the interval $(t_C{}', t_D{}']$. Whenever such trade flows are present D is always the payer, leading to upward spikes in the EE profile. Furthermore, in some cases, the spikes arising from this term can be of extreme magnitude, e.g., the scheduled notional exchange in a cross-currency swap where D pays the full notional, but receives nothing.

Numerical Examples

1. Illustrative Exposures and CVA Magnitudes: To gain intuition for the model, this section presents exposure profiles and CVA metrics for several trade and portfolio examples, using both the Aggressive and the Conservative calibrations. The focus is on ordinary and cross-currency swaps, as these instruments are the primary sources of exposures in most dealers. For all the numerical examples, the stochastic yield curves are driven by one-factor Hull-White model; for cross-currency swap examples, the FX rate is assumed to follow a Black Scholes model.
2. Single Swap Classical/Conservative Exposures: Andersen, Pykhtin, and Sokol (2017) first examine how the model differs from the classical exposure approach. They use Monte Carlo simulation on a USD 10 MM 1-year par-valued vanilla interest rate swap to compare exposures of the Conservative calibration with those computed against the Classical+ and the Classical- models – see the Table above. To make comparisons more meaningful, they override the default setting of 10 business days for the Classical model, and instead set it equal to 15 business days – the length of the MPoR for the Conservative calibration.
3. Single Swap - Classical Exposure Estimations: As they note in their figures, the Classical- calibration is, of course, the least conservative setting, as it ignores both the effect of trade flows and that of the margin asymmetry. The Classical+ calibration tracks the Classical- calibration at most times, but contains noticeable spikes around the last 3 cash flow dates. No spikes occur in their figure on the first quarterly cash flow date, as they assume that the floating rate is fixed at the fixed rate, making the net cash flow zero in all scenarios.
4. Single Swap Conservative Model - MPoR End: The conservative calibration results also contain spikes around the cash flow dates, although these differ from the Classical+ calibration in several ways. First, the Conservative calibration always recognizes that there will be a part towards the end of the MPoR (after time $t_D{}'$) where C and D will both have stopped paying margin and coupons; as a result, the spikes of the Conservative calibration start later (here: 3 business days) than those of the Classical+ calibration.
5. Single Swap Conservative Model - Trade Flows: Second, the initial part of the spike – in the period from $t_C{}'$ to $t_D{}'$ - is substantially higher for the Conservative calibration due to the assumption of only D paying cash flows in this sub-period. The remainder of the spike is comparable in height to the Classical+ spike.

6. Single Swap - Classical vs. Conservative: *Between* spikes the Conservative calibration produces higher exposures than both the Classical- and the Classical+ methods – by around 40%. This is, of course, a consequence of the *worst case* margin asymmetry mechanism in

$$K(t) = \min_{t_i \in [t_C, t_D]} c(t_i)$$

the effect of which will grow with the diffusion volatility of the rate process. Of course the last coupon period again has no exposure between spikes, since the volatility of swap prices vanishes after the last coupon rate fixing.

7. Single Swap: Aggressive Calibration Exposures: Comparison the Aggressive calibration to the Classical+ and the Classical- calibrations are qualitatively similar. A detailed comparison is therefore skipped here, but Andersen, Pykhtin, and Sokol note that the pick-up in exposure from margin asymmetry falls to about 15%, rather than the 40% observed for the Conservative calibration – a result of the fact that, for Aggressive calibration, the *worst case* margin result is established over much fewer days. Andersen, Pykhtin, and Sokol (2017) demonstrate the comparison of the exposure profiles for the Aggressive and the Conservative calibrations in their graphs; as expected, the Conservative calibration leads to both bigger and wider exposure spikes, as well as to higher exposure levels between spikes.
8. Single Swap - Maturity/Coupon Effect: While instructive, the 1-year vanilla swap example is quiet benign exposure-wise; not only is the instrument very short-dated, it also allows for netting of coupons on trade-flow dates, thereby reducing the effects of trade flow spikes. Andersen, Pykhtin, and Sokol (2017) relax both effects by increasing the maturity of the swap, and by making the fixed and the floating legs pay on different schedules – and illustrate the exposure results in a separate figure.
9. Single Swap - Coupon Payment Mismatch: The upward exposure spikes occur twice per year, whenever the dealer must make a semi-annual fixed payment. On the dates when the counterparty makes a quarterly floating payment that is not accompanied by a fixed payment by the dealer, a narrow *downward* spike emerges, due to the delay in transferring the coupon back to the counterparty through the margin mechanism.

10. Single Swap - Impact of Maturity: The exposure between spikes is also much larger, a consequence of the higher volatility of the 10-year swap compared to the 1-year swap. Of course, as the swap nears its maturity, its duration and volatility die out, so the non-spike exposure profile predictably gets pulled to zero at the 10-year date. Also, as predicted, the Aggressive calibration produces much lower exposures than the Conservative calibration, by nearly a factor of 2.
11. Single Cross-Currency Swap - Herstatt Risk: A more extreme form of trade flow spikes will occur for cross-currency swaps, where neither the coupon nor the final payment can be netted. The notional payment, in particular, can induce a very significant payment exposure spike (the Herstatt Risk), whenever the exposure model allows for a trade flow gap. To recall, the Conservative calibration has a trade flow gap, but the Aggressive calibration does not.
12. Single Cross-Currency Swap - Coupon Mismatch: As confirmed by Andersen, Pykhtin, and Sokol (2017), the exposure for a conservative calibration has a very large spike that is not present in the Aggressive calibration. Like Conservative calibration, Aggressive calibration will, of course, still produce spikes at the cash flow dates, due to margin effects.
13. Single Cross-Currency Swap - Principal Mismatch: As a consequence, the principal exchange is likely fa away from break-even, resulting in a large exposure spike at maturity. Although smaller than for the Conservative calibration, the spike at maturity is also present for the Aggressive calibration; while both C and D pay the principal exchange, C does not make the margin transfer for the balance of the principal payments.

Portfolio Results

1. Single Swap Portfolio: Setup Overview: For individual trades, the presence of localized spikes in the exposure profiles may ultimately have a relatively modest impact on the credit risk metrics, such as the CVA – after all, the likelihood of the counterparty default in a

narrow time interval around quarterly or semi-annual cash flow event is typically low. For a *portfolio* of swaps, however, the spikes will add up and affect the net exposure profile nearly everywhere.

2. Single Swap Portfolio - Draw Algorithm: To illustrate this, Andersen, Pykhtin, and Sokol (2017) picked 50 interest rate swaps with quarterly floating rate payments and semi-annual fixed rate payments of 2%. The terms of the swap were randomized as follows:
    a. Notionals of the swap are sampled uniformly on the interval from 0 to USD 1 MM.
    b. Duration of the fixed leg payments – payer or receiver – is random.
    c. Start date of each swap is subject to a random offset to avoid complete MPoR overlaps.
    d. Swap maturities are scaled uniformly on the interval from 1 to 10 years.
3. Single Swap Portfolio: Aggressive vs Conservative: They also illustrate the resulting exposure profile in a separate figure. Both the Conservative profile, and to a lesser extent, the Aggressive profile include frequent spikes around the trade-flow times above the *baseline* exposure level. As seen in the next section, these spikes make a significant contribution to the CVA metrics. As before, the exposure under the Conservative calibration is twice as large as that under the Aggressive calibration.
4. XCCY Swaps Portfolios Generation Algorithm: To repeat the portfolio results with a cross-currency swap, Andersen, Pykhtin, and Sokol (2017) constructed a 50 deal portfolio by randomization, using the following rules.
    a. EUR notionals are sampled uniformly in the interval from 0 to USD 10 MM
    b. USD notionals are 1.5 times the EUR notionals
    c. EUR leg has a fixed semi-annual coupon of 3%, and the USD leg floating quarterly coupon
    d. Direction of the fixed leg payments (payer or receiver) is random
    e. Start date of each swap is subject to a random offset to avoid complete MPoR overlaps
    f. Swap maturities are sampled uniformly in the interval from 1 to 10 years
5. XCCY Swap Portfolio: Conservative vs. Aggressive: As shown in their figure for an expected exposure for a 10Y cross-currency swap, Andersen, Pykhtin, and Sokol (2017)

generated the swaps within the portfolio such that the principal exchanged and the fixed coupon are not at-the-money, to mimic a typical situation corresponding to a portfolio of seasoned trades. As demonstrated in another figure for the expected exposure of the cross-currency swap portfolio, the exposure for the conservative calibration is, as expected, dominated by a series of Herstatt risk spikes, one per swap in the portfolio.

CVA Results

1. CVA Computation from Expected Exposure: As mentioned earlier, a common use of the expected exposure results is the computation of the CVA. Under suitable assumptions, D's unilateral CVA may be computed from the expected exposure (EE) profile as

$$CVA = (1 - R) \int_0^\infty P(u + \delta_C{'}) EE(u + \delta_C{'}) dX(u)$$

   where R is the recovery rate, $P(t)$ is the time-0 discount factor to time t, and $X(t)$ is the time-0 survival probability of C to time t. As discussed before, the exposure profile here is offset by $\delta_C{'}$ to properly align it with the default events.
2. CVA Metrics Dealer/Client Settings: The CVA metric serves as a convenient condensation of the exposure profiles of the previous two sections into single numbers, and Andersen, Pykhtin, and Sokol (2017) tabulate the CVA numbers for the corresponding instruments/portfolios. The CVA integral was discretized using a daily grid, assumed at

$$R = 40\%$$

   and the forward default intensity is left constant at 2.5% such that

$$X(t) = e^{-0.025t}$$

For reference, the table also includes the results of the Classical method, with the MPoR length equal to both that of the Aggressive calibration (7 BD) and the Conservative Calibration (15 BD).

3. CVA Comparison - Classical/Conservative/Aggressive: Their results confirm what was seen earlier. For instance, the CVA for the Aggressive calibration is 50% to 70% smaller than that for the Conservative calibration. In Addition, the CVA of the Conservative calibration is between 50% and 100% larger than that of the Classical+ calibration – at similar MPoR – which in turn is larger than the CVA for the Classical- calibration by around 5% to 25%. Not surprisingly the CVA results for the XCCY portfolio are particularly high in the Conservative calibration due to the Herstatt risk.

Improvement of the Computation Times

1. Computation Speed-Up using Coarse Grids: In exposure calculations for realistic portfolios, horizons can be very long, often exceeding 30 years. For such lengthy horizons, brute-force Monte-Carlo exposures on a daily, or even weekly, time grid will often be prohibitively slow. It is therefore common to use daily simulation steps only for the earliest parts of the exposure profile (e.g., the first month), and then gradually increase the step-length over time to monthly or quarterly, in order to keep the total number of simulation dates manageable. Unfortunately, such a coarsening of the time-grid will inevitably fail to capture both the *worst case* margin effect and the trade spikes that are key to the exposure model.
2. Coarse Grid Lookback Analysis: The next two sections look at ways to capture exposure without having to resort to brute-force daily simulation. A common speed-up technique for the Classical model – the Coarse Grid Look-back Model – is first reviewed, and its

shortcomings and pitfalls are highlighted. An improved practical technique based on Brownian Bridge is the proposed.

The Coarse Grid Lookback Method and its Shortcomings

1. Layout of the Coarse Grid: Assume that the portfolio is not computed daily, but instead on a coarse grid $\{s_j\}$ where j runs from 1 to J. This section uses s rather than t to distinguish the model grid from the daily margin calculation grid.

   Points on the Coarse Grid: In the classical model, the collateral depends only on the portfolio value at the start and at the end of the MPoR, i.e., $s_j - \delta$ and s_j, as is seen from

$$E(t) = [V(t) - K(t)]^+$$

$$K(t) = c(t - \delta)$$

$$K(t) = [V(t - \delta) - h_C(t - \delta)]^+ - [-V(t - \delta) - h_D(t - \delta)]^+$$

   where MPoR is usually around

$$\delta = 10\ BD$$

for CSA's with daily margining. To achieve acceptable computational performance, the time step of the coarse model grid $s_j - s_{j-1}$ must be significantly greater than the length of the MPoR. This, however, would preclude one from establishing a portfolio value at $s_j - \delta$.

2. Introducing a Lookback Node: The coarse grid lookback method deals with this issue by simply adding a second *lookback* time point $s_j - \delta$ to all *primary* measurement times s_j, in effect replacing each node on the coarse model grid by a pair of closely spaced nodes. For each simulated portfolio path, the portfolio value at the lookback point is then used to determine the collateral available at the corresponding primary time point.
3. Slowdown due to the Lookback: The Coarse Grid Lookback Scheme causes, at worst, a factor of $\times$ 2 slowdown relative to valuing the portfolio once per node of the Coarse model grid. If even a $\times$ 2 performance loss is not acceptable, a Brownian Bridge constructed between the primary coarse grid nodes can be used to interpolate the value of the portfolio at each lookback point, see, for example, Pykhtin (2009). Notice that the use of the Brownian Bridge for this purpose should not be confused with its use in the next section.
4. Shortcoming of the Model: The Coarse Grid Lookback method is a common way of addressing the mismatch between the long time-step of the coarse model grid and the much shorter MPoR. Similar to the commonly used models of uncollateralized exposure, the method produces accurate – with respect to the underlying assumptions of the Classical model – exposure numbers at the coarse grid time points, but provides no information on the exposure between the grid points.
5. Collateralized vs. Uncollateralized Grid Exposures: For uncollateralized positions, the exposure profiles are reasonably smooth, so one can safely interpolate between the grid points for calculating integral quantities, such as the CVA. In collateralized case, however, one cannot rely on such interpolations because the true exposures, as has been seen above, is likely to have spikes and jumps between the grid points. The Coarse Grid Lookback method has no means to determining the position or the magnitude of the irregularities between the grid points, and thus, is not suitable for CVA or capital calculations.
6. Classical+ Model - Coarse Grid Impact: To briefly expand on this, consider the Classical+ version of the classical model. Here it is assumed that all trade flows are paid within the

MPoR, where, as shown before, trade flows often result in exposure spikes. Exposure profiles computed from daily time steps would consequently show spikes from all trade flows until the maturity of the portfolio.

7. Trade Flows outside the Simulated MPoR's: In contrast, in a typical implementation with sparsely spaced MPoR's, only trade flows that happen to be within a sparsely simulated MPoR's may result in spikes; the exposure profile would then miss all other flows.
8. Simulation Calendar Impact on Exposure: Furthermore, as the location of the simulation point will likely change with the advancement in the calendar time, trade flows would move in and out of the simulated MPoR's, and the exposure profile one report on any given day may very well differ significantly from those that were reported the day before. This in turn causes CVA or risk capital to exhibit significant, and entirely spurious, oscillations.
9. Classical- Model - Coarse Grid Impact: While the Classical- exposure model does not exhibit outright spikes, its exposure profiles still exhibit jumps around significant trade flows. The classical coarse grained implementation would not be able to resolve the position of these jumps, instead only showing the conservative jumps between two exposure measurement points often separated by many months. This creates another source of instability, present in both the Classical- and the Classical+ versions of the classical model.
10. Illustration using the Forward CVA: To illustrate the effects described above, Andersen, Pykhtin, and Sokol (2017) define the concept of time t forward CVA, denoted CVA_t, obtained by
    a. Changing the lower integration limit in

$$CVA = (1 - R)\int_{0}^{\infty} P(u + \delta_C{}')EE(u + \delta_C{}')dX(u)$$

    from 0 to t, and

    b. Dividing the result by $P(t)X(t)$.

Using the same portfolio of 50 EUR-USD cross-currency swaps, they show the t-dependence of CVA_t on a daily grid to portfolio maturity.

11. Spurious Oscillations from Moving Windows: As CVA is an integral of exposures, spikes in exposure profile profiles should result in jumps rather than oscillations in CVA_t. However, when one of the Coarse Grid Lookback method's sparsely located *MPoR window* moves past a large trade flow, the contribution to the CVA temporarily increases only to drop back when the window moves past the large trade flow. As illustrated by Andersen, Pykhtin, and Sokol (2017), such oscillations are spurious and their presence is highly unattractive when CVA is computed and reported as part of daily P&L.

Brownian Bridge Method

1. Brute Force Portfolio Value Simulation: Overcoming the deficiencies outlined in the previous section is, unfortunately, prohibitively expensive for large portfolios, mostly due to the expense of repricing the entire portfolio at each simulation path and each observation date.
2. Daily Simulation of Risk Factors: On the other hand, merely simulating the risk factors at a daily resolution is generally feasible, as the number of the simulated risk factors is typically relatively small (i.e., several hundred) and the equations driving the risk factor dynamics are usually simple.
3. Generation of Daily Trade Flows: Furthermore, having produced risk factors on a daily grid, one can normally also produce all realized trade flows along each path because trade flows, unlike trade prices, are usually simple functions of the realized risk factors.
4. Risk Factors under Daily Resolution: Based on these observations, Andersen, Pykhtin, and Sokol (2017) propose the following algorithm for generating paths of portfolio values and trade flows on a daily time grid. First, simulate paths of market risk factors with daily resolution.
5. Trade Flow under Daily Resolution: For each path m, use the simulated market risk factors to calculate trade flows on the path with daily resolution.

6. Coarse Grid Path Portfolio Valuation: For each path m and each coarse portfolio valuation time point s_j $(j = 1, \cdots, J)$ use the simulated risk factors to calculate portfolio value on the path $V_m(s_j)$
7. Trade Flow Adjusted Forward Value: For each path m and each time point s_j use the trade flows realized on the path between times s_{j-1} and s_j to calculate the *forward* to s_j portfolio value $V_m'(s_{j-1};\ s_j)$:

$$V_m'(s_{j-1};\ s_j) = V_m(s_{j-1}) - TF_{m,NET}(s_j;\ (s_{j-1}, s_j])$$

   Note that $V_m'(s_{j-1};\ s_j)$ is not a true forward value because the realized trade flows are subtracted from the s_{j-1} portfolio value rather than the true forward value being calculated at time s_{j-1}.
8. Portfolio Value Local Variance Estimation: For each path m and each portfolio measurement time point s_j compute the local variance ${\sigma_m}^2(t_{j-1})$ for the portfolio value *diffusion* $V_m(s_j) - V_m'(s_{j-1};\ s_j)$ via a kernel regression estimator – e.g., the Nadaraya-Watson Gaussian kernel estimator (Nadaraya (1964), Watson (1964)) conditional on the realized value of $V_m'(s_{j-1};\ s_j)$. The selection of bandwidth for the kernels is covered in, e.g., Jones, Marron, and Sheather (1996). In their numerical results, Andersen, Pykhtin, and Sokol (2017) use the *Silverman's Rule of Thumb* (Silverman (1986)).The term *diffusion* is used to indicate that the portfolio value change has been defined to avoid any discontinuities resulting from trade flows.
9. Brownian Bridge Local Interpolation Scheme: For each path m and each exposure measurement time point s_j, simulate an independent, daily sampled, Brownian Bridge process (see, for instance, Glasserman (2004)) that starts from the value $V_m'(s_{j-1};\ s_j)$ at time s_{j-1} and ends at the value $V_m(s_j)$ at time s_j. The volatility of the underlying Brownian motion should be set to $\sigma_m(s_{j-1})$.
10. Brownian Bridge Portfolio Value Approximation: For each path m and each exposure measurement time point s_j, the portfolio values for each time u of the daily grid in the

interval (s_{j-1}, s_j) are approximated from the simulated Brownian bridge $BB_m(u)$ by adding the trade flows realized along the path m between the times u and s_j:

$$V_{m,APPROX}(u) = BB_m(u) + TF_{m,NET}(s_j;\ (u, s_j])$$

11. Rational behind Brownian Bridge Methodology: In a nutshell, the algorithm above uses a Brownian bridge process to interpolate portfolio values from a coarse grid in a manner that ensures that intermediate trade flow events are handled accurately. The algorithm produces paths of portfolio values and trade flows in a daily time grid, wherefore exposure can be calculated as described earlier with daily resolution and overlapping MPoR's. Furthermore, daily sampling allows for further refinements of the proposed model by consistently incorporating thresholds, minimum transfer amount, and rounding.
12. Brownian Bridge Portfolio Weiner Increment: A key assumption made by the Brownian Bridge algorithm is that the portfolio value process within the interpolation interval is a combination of an approximately normal *diffusion* overlaid by trade flows. For Weiner process models without risk factor jumps, this approximation is accurate in the limit of infinitesimal interpolation interval, and is often a satisfactory approximation for monthly or even quarterly interpolation steps.
13. Brownian Bridge Approximation Error #1: Nevertheless, the presence of trade flows that depend on the values of the risk factors between the end points introduces two types of errors. Suppose that there is a trade flow at an end point that depends on the risk factor value at the date when it is paid. The independence of the Brownian Bridge process from the risk factor processes that drive that trade flow would result in an error in the expected exposure profile around the trade flow date. This error is largest for trade flows in the middle of the interpolation interval and disappears for trade flows near the ends of the interval.
14. Brownian Bridge Approximation Error #2: Suppose that there is a trade flow that occurs at the end point of an interpolation interval, but whose values depend entirely on the realization of the risk factor within the interpolation interval. A typical example would be a vanilla interest rate swap where the floating leg payment being paid at the end of the interpolation interval depends on the interest rate on a date within the interval. Even in the absence of a

trade flow within the interpolation interval, the volatility of the swap value drops at the floating rate fixing date as some of the uncertainty is resolved. Thus the *true* swap value process has two volatility values; a higher value before the rate fixing date and a lower value after the rate fixing date. In contrast the approximation algorithm assumes a single value of volatility obtained via kernel regression between the end points. Similar to the de-correlation error discussed above, the error resulting from this volatility mismatch is largest for fixing dates in the middle of the interpolation interval and disappears for fixing dates near the end points.

15. <u>Trade Flow at Mid-Interval</u>: To illustrate the two errors above, Andersen, Pykhtin, and Sokol (2017) compute the expected exposure profile for a one year interest rate swap when a monthly grid for full valuation is situated so that the payments/fixing dates sit roughly in the middle of the interpolation interval, thus maximizing the error of the Brownian Bridge algorithm.
16. <u>Unbiased Nature of the Error</u>: While there are, as expected, some error around the trade flow dates, they are acceptable in magnitude and overall unbiased, in the sense that the over-estimation of the exposure is about as frequent as the under-estimation of the exposure. For, say, CVA purposes, the Brownian Bridge results would therefore be quite accurate.
17. <u>Trade Flows at Interval End</u>: Andersen, Pykhtin, and Sokol (2017) also compute the expected exposure profiles when the monthly valuation points are aligned with the rate fixing/payment dates. In this case, Brownian Bridge approximation is nearly exact.
18. <u>Choice of Valuation Grid Location</u>: Of course, in practice such alignment is only possible for a single trade or a small netting set, and not for large portfolios where trade flows will occur daily. Yet, even for large netting sets the calculation accuracy will improve if the interpolation pillars are aligned with the largest trade flows (e.g., principal exchange dates for the largest notional amounts). In practice, errors can be typically expected to be somewhere between the two extremes discussed above.
19. <u>Performance Gains from Brownian Bridge</u>: While the exact speed up provided by the Brownian Bridge method depends on the implementation, for most portfolios the overhead of building the Brownian Bridge at a daily resolution is negligible compared to computing the exposure on the model’s coarse grid.

20. Comparison with Coarse Grid Lookback: In this case, the computational effort of the daily Brownian Bridge method is about half the computational effort of the Coarse Grid Lookback method, as the former does not require adding a *lookback* point to each of the primary coarse grids. Thus the Brownian Bridge is both faster and significantly more accurate than the Standard Coarse Grid Lookback method.

Initial Margin

1. Role of IM: Extra Protection: The posting of initial margin (IM), in addition to the regular variation margin collateral (VM), provides dealers with a mechanism to gain additional default protection. The practice of posting IM has been around for many years, typically with IM being computed on trade inception on a trade level basis.
2. Modeling Static Initial Margin Exposure: This type of IM is entirely deterministic and normally either stays fixed over the lifetime of a trade or amortizes down according to a pre-specified schedule. As a consequence, modeling the impact on the exposure is trivial; for the exposure points of interest all trade level IM amounts are summed across the netting set and the total – which is the same for all paths – is subtracted on the portfolio value from each path.
3. Dynamically Refreshed Initial Margin (DIM): A more interesting type of IM is dynamically refreshed to cover portfolio-level close-out risk at some high percentile, often 99%. This type of margin is routinely applied by Clearinghouses (CCPs) and by margin lenders, and will also soon be required by regulators for inter-dealer OTC transactions.
4. BCBS IOSCO Initial Margin Rules: In particular, in 2015 BCBS and IOSCO issued a final framework on margin requirements (BCBS and IOSCO (2015)) under which two covered entities that are counterparties in non-centrally cleared derivatives are required to:
    a. Exchange VM under a zero threshold margin agreement, and
    b. Post IM to each other without netting the amounts.

Covered entities include all financial firms and systematically important non-financial firms. Central banks and sovereigns are not covered entities.

5. Third Party Management of IM: IM must be held in a default remote way, e.g., by a custodian, so that IM posted by the counter-party should be immediately available to it should the other counter-party default.
6. Internal Model/Standardized Schedule IM: Under the BCBS and IOSCO rules, regulatory VM can be calculated by an internal model or by lookup in a standardized schedule.
7. Internal Models Based IM Calculation: If an internal model is used, the calculation must be made at the netting set level as the value-at-risk at the 99% confidence level. The horizon used in this calculation equals 10 business days for daily exchange of VM or 9 business days plus a re-margining period for less frequent exchange of VM.
8. Denial of Cross-Asset Netting: Diversification across distinct asset classes is not recognized, and the IM internal model must be calibrated to a period of stress of each of the asset classes.
9. Handling Adjustments to the IM: The required levels of IM are changed as the cash flows are paid, new trades are booked, or markets move. To accommodate this, dealers would call for more IM or return the excess IM.
10. Complexities Associated with the IM Estimation: For trades done with CCPs or under the new BCBS-IOSCO rules, one must find a way to estimate the future IM requirements for each simulated path. No matter how simple the IM VaR model is, it will likely be difficult to perform such calculations in practice if one wants to incorporate all the restrictions and twists of the IM rules; stress calibration, limited diversification allowance, and, for CCP's, add-ons for credit downgrades and concentration risk.
11. Estimating Simplified Version of IM: However, it is possible to utilize the model in this chapter to calculate the counter-party exposures if one ignores these complications. Note that ignoring such complications is conservative, as it will always lead to a *lower* of IM, and therefore, to a *higher* level of exposure.
12. t_C as IM Delivery Date: To calculate the exposure at time t the assumption here is that the last observation date for which C would deliver VM to D is

$$t_C = t - \delta_C$$

It is reasonable to assume that this date is also the last date at which C would deliver IM to a custodian.

13. Simplified IM Mechanics Timeline: To simplify modeling, it is assumed that the custodian would not return any amount to C for observation dates after $t - \delta_C$. Thus, to calculate exposure at time t, IM on a path has to be estimated from the dynamics of the exposure model as of time $t - \delta_C$
14. t_C IM Estimate using Gaussian Portfolio Evaluation: Assuming, as is common in practice, that the portfolio values are locally Gaussian, it suffices to know the local volatility for the portfolio value for the period $[t - \delta_C, t]$ estimated at $t - \delta_C$. Denoting the IM horizon by δ_{IM} and the local volatility of the portfolio value at time u on path m via $\sigma_m(u)$, the IM available to D at the ETD date t on path m is given by

$$IM_m(t - \delta) = \sigma_m(t - \delta)\sqrt{\delta_{IM}}\Phi^{-1}(q)$$

where q is a confidence level – often 99% - and $\Phi^{-1}(\cdot)$ is the inverse of the standard normal cumulative distribution function.

15. Kernel Regression Based Local Volatility: Estimating the local volatility can be done via kernel regression, as in the previous section. If the portfolio value is simulated at both $t - \delta_C$ and t, the kernel regression for could be run on the $P\&L$ $V(t) - V(t - \delta_C) + TF_{NET}(t;\ (t - \delta_C, t])$ conditional on the realization of the portfolio value on path m at the beginning of the $MPoR$ $V_m(t - \delta_C)$. If one does not calculate the portfolio value at the beginning of the $MPoR$ but uses the fast approximation outlined earlier instead, $\sigma_m(t - \delta_C)$ can be set equal to the local volatility estimated for the time interval that encloses the given $MPoR$ $[t - \delta_C, t]$.
16. Brownian Bridge IM Plus VM: Thus, the Brownian Bridge framework can now produce not only the collateralized exposure under VM alone, but also a reasonable estimate of the collateralized exposure under a combination of VM and IM.
17. IM Timing and Transfer Mechanics: In calculating the IM, an important consideration is the timing and the mechanics of the adjustment to the IM when C misses a margin flow or a trade flow.

18. Assumption - IM Return to the Client: For instance, when a large trade reaches maturity, the portfolio VaR may be reduced, in which case, some of the IM posted by C must be refunded. The issue of whether this refund can be delayed due to an ongoing margin dispute is not yet fully resolved. To simplify the calculations, it is assumed that no part of the IM is returned to C during the $MPoR$.
19. 10Y OTC Swap VM + IM EE: To show some numerical results, Andersen, Pykhtin, and Sokol (2017) consider the individual trades and portfolios of the earlier section. They use the case of a 10Y vanilla swap for which they calculate the impact of IM on exposure.
20. Time Horizon IM Mechanism Impact: As is evident from their calculations, the IM mechanism strongly reduces exposures away from trade flows, but near the trade flow dates the protection gets progressively weaker and disappears almost completely for the last couple of trade flows. The reason for this uneven benefit of IM on this trade is that the 10 day VaR of this trade bears no direct relationship to the size of the trade flows that determines the exposure spikes in the model.
21. Inter/Intra-Spike IM Exposures: The variance of the $P\&L$ reduces as the swap approaches maturity so that the amount of IM on a given path is also reduced. However, the size of the trade flows is not reduced, but can actually grow with simulation time as larger and larger realizations of the floating rates are possible. Thus, when the swap approaches maturity the amount of IM is greatly reduced relative to the trade flows, so exposure spikes grow larger, while the *diffusion* component of the exposure becomes smaller.
22. Cross-Currency Swap VM + IM EE: Andersen, Pykhtin, and Sokol (2917) compute the impact of IM on the vanilla swap and the cross currency swap portfolios described earlier. As can be seen there the IM strongly suppresses the diffusion component of the portfolio value changes, but proves inadequate in reducing the spikes of exposure for both single currency, and especially, cross-currency portfolios.

Conclusion

1. Fully Collateralized Counterparty Exposure: Industry standard models for collateralized credit risk are well-known to produce non-negligible counterparty credit risk exposure, even under full collateralization of the variation margin. This exposure essentially arises due to the inevitable operational an legal delays (margin period of risk, or $MPoR$) that are *baked into* the workings of ISDA contracts that govern OTC trading.
2. Classical Implementations of the $MPoR$: In the most common industry implementation, the length of the $MPoR$, and precisely what transpires inside it, is, however, often treated in a highly stylized fashion. Often the $MPoR$ is set equal to 10 business days for little reason other than tradition, and often counter-parties are assumed to have oddly synchronized behavior inside the $MPoR$.
3. The Classical+ and Classical- Implementations: For instance, one common approach – denoted Classical- - assumes that the $MPoR$ and the trade flows by both counter-parties terminate at the beginning of the $MPoR$, but the trade flows terminate simultaneously at the end of the $MPoR$. Surprisingly, the Classical+ and the Classical- approaches continue to co-exists in the market, and neither has become the sole market practice.
4. Reasons for the Popularity of the Classical Models: One reason for this state of affairs is that the two models correspond to different choices for the trade-off between implementation complexity and the model stability; Classical+ is easier to implement but is prone to spurious spikes in the daily CVA P&L – as demonstrated earlier – whereas Classical- is more difficult to implement, but is free from such spikes.
5. Objectives of *Well* Designed Models: Ultimately, of course, a model should be selected not on the basis of the implementation ease or on the properties of a specific numerical technique, but on the basis of how well the model captures the legal and the behavioral aspects of the events around a counter-party default. The term *well* means different things in different applications of the exposure model. For regulatory capital purposes, prudence and conservatism may, for instance, be as important as outright precision.
6. Inadequacies of the Classical Approach: To this end, even a cursory analysis suggests that the perfect synchronicity of the Classical $\pm$ models cannot be supported in reality. For instance, due to the way the CSA works in practice, the non-defaulting party will need at

least 3 days after a portfolio valuation date to determine for sure that the corresponding margin payment by its counterparty will not be honored.

7. Detailed Analysis of $MPoR$ Timeline: This chapter carefully dissects the $MPoR$ into a full timeline around the default event, starting with the missed margin call and culminating at the post-default valuation date at which the termination value of the portfolio is established.
8. Model Parameters of the Timeline: For modeling purposes, the timeline of the model has been condensed into 4 model parameters, each specified as the number of days prior to the termination for the events below – in contrast the classical model has only one parameter – the full length of the $MPoR$.
9. Dealer/Client Trade/Margin Dates:
    a. The last market data measurement for which the margin flow is received (δ_C) and paid (δ_B) as prescribed.
    b. The last date when the defaulting party ($\delta_C{}'$) and the dealer ($\delta_D{}'$) make the trade payments as prescribed.
10. Legal Operational Basis behind the Parameters: As shown, each of these parameters has a legal and/or operational interpretation, enabling calibration from the CSA and from the operational setup of the dealer. Note that the proposed model parameterization includes the Classical+ and the Classical- models as the limit cases.
11. Aggressive CSA Timeline for the $MPoR$: For indicative purposes, two particular models are described – Aggressive and Conservative. For former assumes that the non-defaulting dealer always operates at an optimal operational level, and will enforce the legal provisions of the ISDA legal contracts as strictly as possible.
12. Conservative CSA for $MPoR$ Timeline: The latter will allow for some slack in the operations of the dealer, to allow for manual checks of calculations, legal reviews, *gaming* behavior of the counterparty, and so forth.
13. Aggressive/Conservative Timeline Exposure Comparison: The Conservative model setting obviously produces higher exposures than the Aggressive setting, for the following reasons.
    a. The Conservative setting has a longer overall length of $MPoR$
    b. The Conservative setting has a margin flow period where the dealer pays, but does not receive, margin flows

c. The Conservative setting, unlike the Aggressive setting, contains a trade flow gap period where the dealer pays, but does not receive, trade flows

14. Comparison of Margin Flow Exposures: In their numerical tests, Andersen, Pykhtin, and Sokol (2017) found that the first two factors of the Conservative setting to have approximately twice the exposure of both the Aggressive and the Classical $\pm$ settings away from the dates of large trade flows.
15. Comparison of Trade Flow Exposures: The last factor, i.e., the presence of a large trade flow gap may cause exposure spikes of extremely large magnitudes under the Conservative calibration. Despite the fairly short duration of these spikes, they may easily add up to very significant CVA contributions, especially for cross-currency trades with principal exchange – the Herstatt risk.
16. Past Realizations of Trade Flow Default: Credit losses due to trade flow gaps materialized in practice during the financial crisis – especially due to the Lehmann Brothers' default – so their incorporation into the model is both prudent and realistic.
17. Impracticality of Daily Simulation Schemes: Detailed tracking of the margin and the trade flow payments requires the stochastic modeling of the trade portfolio on a daily grid. As brute-force simulations at such a resolution are often impractically slow, it is important that numerical techniques be devised to speed up the calculations.
18. Kernel Regression on *Stripped* Cash Flows: While the focus of this chapter was mainly on establishing the fundamental principles for margin exposure, it also proposed an acceleration method based on kernel regression and applied Brownian Bridge to portfolio values *stripped* of cash flows.
19. Impact on Different Product Types: For ordinary and cross-currency swaps this chapter demonstrates that this method is both accurate and much faster than either brute-force simulation or standard acceleration techniques of the desired model. Further improvements in the acceleration techniques, and expansion of applicability into more exotic products, is an area of future research.
20. Initial Margin – Classical- Settings Impact: Under suitable assumptions, kernel regression may also be used to embed risk-based initial margin into exposure simulations. As

demonstrated in the final section of the chapter, initial margin at 99% exposure greatly succeeds in reducing bilateral exposure for the Classical- calibration.

21. Initial Margin Trade Flow Impact: For all other calibration choices, and especially for the Conservative setting, the reduction in counterparty exposure afforded by initial margin fails around the time of large trade flows, when a sudden change of exposure following an initial trade flow exceeds the initial margin level.
22. Initial Margin Maturity Decay Impact: Note that the already inadequate level of IM protection deteriorates around the maturity of the portfolio, where the local volatility of the trade flow value decreases, but the trade flows themselves do not. Overall accurate modeling of the events within the $MPoR$ becomes critically important for portfolios covered by dynamic IM.

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A Sound Modeling and Back Testing Framework for Forecasting Initial Margin

Abstract

1. Mandatory Margins for OTC Transactions: The introduction of mandatory margining for bilateral OTC transactions is significantly affecting the derivatives market, particularly in light of the additional funding costs that financial institutions could face.
2. Initial Margin Forecast Models Backtest: This chapter details the approach by Anfuso, Aziz, Loukopoulos, and Giltinan (2017) for a consistent framework, applicable equally to cleared and non-cleared portfolios, to develop and backtest forecasting models for initial margin.

Introduction

1. BCBS-IOSCO Mandatory Margining Guidelines: Since the publication of the new Basel Committee on Banking Supervision and the International Organizations of Securities Commissions (BCBS-IOSCO) guidance for mandatory margining for non-cleared OTC derivatives (Basel Committee on Banking Supervision (2015)) there has been a growing interest in the industry regarding the development of dynamic initial margin models (DIM) – see, for example, Green Kenyon (2015), Andersen, Pykhtin, and Sokol (2017b). By *DIM*

model this chapter refers to any model that can be used to forecast portfolio initial margin requirements.

2. Protection Afforded by BCBS-IOSCO: The business case for such a development is at least two fold. First, the BCBS-IOSCO IMR (B-IMR) rules are expected to protect against potential future exposure at a high-level of confidence (99%) and will substantially affect funding costs, XVA, and capital.
3. IM and VM Based Margining: Second, B-IMR has set a clear incentive for clearing; extensive margining in the form of variation margin (VM) and initial margin (IM) is the main element of the central counter-party (CCP) risk management as well.
4. IMR Impact on Bilateral + Cleared: Therefore, for both bilateral and cleared derivatives, current and future IMR significantly affects the probability and the risk profile of a given trade.
5. B-IMR Case Study - Performance Evaluation: This chapter considers B-IMR as a case study, and shows how to include a suitably parsimonious DIM model on the exposure calculation. It also proposes an end-to-end framework and also defines a methodology to backtest model performance.
6. Organization of this Chapter: This chapter is organized as follows. First, the DIM model for forecasting future IMR is presented. Then methodologies for two distinct levels of back-testing analysis are presented. Finally, conclusions are drawn.

How to Construct a DIM Model

1. Applications of the DIM Model: A DIM model can be used for various purposes. In the computation of the counter-party credit risk (CCR), capital exposure, or credit valuation adjustment (CVA), the DIM model should forecast, in a path-by-path basis, the amount of posted and received IM at any revaluation point.

2. Path Specific IMR Estimation: For this specific application, the key ability of the model is to associate a realistic IMR to any simulated market scenario based on a mapping that makes use of a set of characteristics of the path.
3. RFE Dependence on the DIM: The DIM model is *a priori* agnostic to the underlying risk factor evolution (RFE) models to generate the exposure paths (as shall be seen, dependencies may arise, if for example, the DIM is computed on the same paths that are generated for the exposure).
4. Cross-Probability Measure IMR Distribution: It is a different story if the goal is to predict the IMR distribution (IMRD) at future horizons, either in the real-world P or the market-implied Q measures.
5. IMRD Dependence on the RFE: In this context, the key feature of the model is to associate the right probability weight with a given IMR scenario; hence the forecast IMRD also becomes a measure of the accuracy if the IMRD models (which ultimately determine the likelihood of different market scenarios).
6. P vs. Q Measure IMRD: The distinction between the two cases will become clear later on, in the discussion of how to assess model performance.
7. ISDA SIMM BCBS IOSCO IM: The remainder of this chapter considers the BCBS-IOSCO IM as a case study. For the B-IMR, the current industry proposal is the International Swaps and Derivatives Association Standard Initial Margin Model (SIMM) – a static aggregation methodology to compute the IMR based on first-order delta-vega trade sensitivities (International Swaps and Derivatives Association (2016)).
8. Challenges with SIMM Monte Carlo: The exact replication of SIMM in a capital exposure or an XVA Monte Carlo framework requires in-simulation portfolio sensitivities to a large set of underlying risk factors, which is very challenging in most production implementations.
9. Andersen-Pykhtin-Sokol IM Proposal: Since the exposure simulation provides the portfolio mark-to-market (MTM) on the default (time t) and closeout (time $t + MPoR$, where $MPoR$ is the *margin period of risk*) grids, Andersen, Pykhtin, and Sokol (2017b) have proposed using this information to infer path-wise the size of any percentile of the local $\Delta MTM(t, t + MPoR, Path_i)$ distribution, based on a regression that uses the simulated portfolio $MTM(t)$ as a regression variable.

10. Andersen-Pykhtin-Sokol Proposal Assumptions: The

$$\Delta MTM(t, t + MPoR) = MTM(t + MPoR) - MTM(t)$$

distributed is constructed assuming that no cash flow takes place between the default and the closeout. For a critical review of this assumption, see Andersen, Pykhtin, and Sokol (2017a).

11. Enhancing the Andersen-Pykhtin-Sokol Model: This model can be further improved by adding more descriptive variables to the regression, e.g., values at the default time of the selected risk factors of the portfolio.
12. Optimization: Re-using Exposure Paths: For the DIM model, the following features are desirable. First the DIM should consume the same number of paths as the exposure simulation, to minimize the computational burden.
13. DIM Optimization – B-IMR SIMM Reconciliation: Second, the output of the DIM model should reconcile with the known IMR value for

$$t = 0$$

i.e.

$$IM(Path_i, 0) = IMR_{SIMM}(0)$$

for all i.

14. Key Aspects of IOSCO/SIMM: Before proceeding, this section notes some of the key aspects of the BCBS-IOSCO margining guidelines, and, consequently, of the ISDA SIMM Model (International Swaps and Derivatives Association (2016)).
15. Andersen-Pykhtin-Sokol Proposal Assumptions: First, the $MPoR$ for the IM calculation of a daily margined counter-party is 10 BD. This may differ from the capital exposure calculation, in which, for example

$$MPoR = 20\ BD$$

if the number of trades in the portfolio exceeds 5,000.

16. No Netting across the Asset Classes: Second, the B-IMR in the Basel Committee on Banking Supervision (2015) prescribes calculating the IM by segregating trades from different asset classes. Tis feature is reflected in the SIMM model design.
17. SIMM Methodology Market Volatility Independence: Finally, the SIMM methodology consumes trade sensitivities as its only inputs and has a static calibration that is not sensitive to market volatility.
18. Regression on the ΔMTM Distribution: For the IM calculation, the starting point is similar to that of Andersen, Pykhtin, and Sokol (2017a), i.e.
    a. A regression methodology based on path's $MTM(t)$ is used to compute the moments of the local $\Delta MTM(t, t + MPoR, Path_i)$ distribution, and
    b. $\Delta MTM(t, t + MPoR, Path_i)$ is assumed to be a given probability distribution that can be fully characterized by its first two moments – the drift and the volatility. Additionally, since the drift is immaterial over the $MPoR$ horizon, it is not computed and set to 0.
19. Quadratic Regressor for Local Volatility: There are multiple regression schemes that can be used to determine the local volatility $\sigma(i, t)$ The present analysis follows the standard American Monte Carlo literature (Longstaff and Schwartz (2001)) and uses a least squares method (LSM) with a polynomial basis:

$$\sigma^2(i, t) = \mathbb{E}[\Delta MTM^2(i, t) \mid MTM(i, t)] = \sum_{k=0}^{n} a_{\sigma k} MTM^k(i, t)$$

$$IM_{R/P,U}(i, t) = \Phi^{-1}\big(0.99/0.01, \mu = 0, \sigma = \sigma(i, t)\big)$$

where R/P indicates received and poste, respectively. In this implementation, the n in

$$\sigma^2(i,t) = \mathbb{E}[\Delta MTM^2(i,t) \mid MTM(i,t)] = \sum_{k=0}^{n} a_{\sigma k} MTM^k(i,t)$$

is set equal to 2, i.e., a polynomial regression of order 2 is used.

20. Calculating the Unnormalized IM Value: The unnormalized posted and received $IM_{R/P,U}(i,t)$ and calculated analytically in

$$\sigma^2(i,t) = \mathbb{E}[\Delta MTM^2(i,t) \mid MTM(i,t)] = \sum_{k=0}^{n} a_{\sigma k} MTM^k(i,t)$$

$$IM_{R/P,U}(i,t) = \Phi^{-1}\big(0.99/0.01, \mu = 0, \sigma = \sigma(i,t)\big)$$

by applying the inverse of the cumulative distribution $\Phi^{-1}(x,\mu,\sigma)$ to the appropriate quantiles; $\Phi(x,\mu,\sigma)$ being the probability distribution that models the local $\Delta MTM(t, t + MPoR, Path_i)$.

21. Note on the Distributional Assumptions: The precise choice of Φ does not play a crucial role, since the difference in the quantiles among the distribution can be compensated in calibration be applying the appropriate scaling factors (see the $\alpha_{R/P}(t)$ functions below). For simplicity, in the below Φ is assumed to be normal.

22. Comparative Performance of the LSM: It is observed that the LSM method performs well compared to the more sophisticated kernel methods such as Nadaraya-Watson, which is used in Andersen, Pykhtin, and Sokol (2017a), and it has the advantage of being parameter free and cheaper from a computational stand-point.

23. Applying $t = 0, MPoR$ and SIMM Reconcilers: The next step accounts for the

$$t = 0$$

reconciliation as well as the mismatch between SIMM and the exposure model calibrations – see the corresponding items above.

24. De-normalizing using IM Scaling Parameters: These issues can be tackled by scaling $IM_{R/P,U}(i,t)$ with suitable normalization functions $\alpha_{R/P}(t)$:

$$IM_{R/P}(i,t) = \alpha_{R/P}(t) \times IM_{R/P,U}(i,t)$$

$$\alpha_{R/P}(t) = [1 - h_{R/P}(t)] \times \sqrt{\frac{10\ BD}{MPoR}} \times \left[\alpha_{R/P,\infty} + \left(\alpha_{R/P,0} - \alpha_{R/P,\infty}\right)e^{-\beta_{R/P}(t)t}\right]$$

$$\alpha_{R/P,0} = \sqrt{\frac{MPoR}{10\ BD}} \times \frac{IMR_{R/P,SIMM}(t=0)}{q\left(0.99/0.01\,, \Delta MTM(0, MPoR)\right)}$$

25. Differential Calibration for Posted/Received IM: In

$$\alpha_{R/P}(t) = [1 - h_{R/P}(t)] \times \sqrt{\frac{10\ BD}{MPoR}} \times \left[\alpha_{R/P,\infty} + \left(\alpha_{R/P,0} - \alpha_{R/P,\infty}\right)e^{-\beta_{R/P}(t)t}\right]$$

$$\beta_{R/P}(t) > 0$$

and

$$h_{R/P}(t) < 1$$

with

$$h_{R/P}(t=0) = 0$$

are four functions to be calibrated – two for received and two for posted IM’s. As will become clearer later in this chapter, the model calibration generally differs for received and posted DIM models.

26. Scaling IM using RFE MPoR: In

$$IM_{R/P}(i,t) = \alpha_{R/P}(t) \times IM_{R/P,U}(i,t)$$

$$\alpha_{R/P}(t) = \left[1 - h_{R/P}(t)\right] \times \sqrt{\frac{10\ BD}{MPoR}} \times \left[\alpha_{R/P,\infty} + \left(\alpha_{R/P,0} - \alpha_{R/P,\infty}\right)e^{-\beta_{R/P}(t)t}\right]$$

$MPoR$ indicates the $MPoR$ relevant for the Basel III exposure. The ratio of $MPoR$ to 10 BD accounts for the VM vs. IM margin period, and it is taken as a square root because the underlying models are typically Brownian, at least for short horizons.

27. Components of the $\alpha_{R/P}(t)$ Term: In

$$\alpha_{R/P,0} = \sqrt{\frac{MPoR}{10\ BD}} \times \frac{IMR_{R/P,SIMM}(t=0)}{q\left(0.99/0.01\,, \Delta MTM(0, MPoR)\right)}$$

$IMR_{R/P,SIMM}(t=0)$ are the $IMR_{R/P}$ computed at

$$t = 0$$

using SIMM; $\Delta MTM(0, MPoR)$ is the distribution of the MTM variations over the first $MPoR$; and $q(x,y)$ is a function that gives quantile x for the distribution y.

28. $t = 0$ chosen to match SIMM: The values of the normalization functions $\alpha_{R/P}(t)$ at

$$t = 0$$

are chosen in order to reconcile $IM_{R/P}(i,t)$ with the starting SIMM IMR.

29. Mean-reverting Nature of the Volatility: The functional form of $\alpha_{R/P}(t)$ at

$$t > 0$$

is dictated by what is empirically observed, as is illustrated by Anfuso, Aziz, Loukopoulos, and Giltinan (2017); accurate RFE models, in both P and Q measures, have either a volatility term structure or an underlying stochastic volatility process that accounts for the mean-reverting behavior to the normal market conditions generally observed from extremely low or high volatility.

30. Reconciliation with Static SIMM Methodology: Since the SIMM calibration is static (independence of market volatility for SIMM), the

$$t = 0$$

reconciliation factor is not independent of the market volatility, and thus not necessarily adequate for the long-term mean level.

31. Volatility Reducing Mean-reversion Speed: Hence, $\alpha_{R/P}(t)$ is an interpolant between the

$$t = 0$$

scaling driven by $\alpha_{R/P,0}$ and the long-erm scaling driven by $\alpha_{R/P,\infty}$, where the functions $\beta_{R/P}(t)$ are the mean-reverting speeds.

32. Estimating from the Long-End: The values of $\alpha_{R/P,\infty}$ can be inferred by a historical analysis of a group of portfolios, or it can be *ad hoc* calibrated, e.g., by computing a different $\Delta MTM(0, MPoR)$ distribution in

$$\alpha_{R/P,0} = \sqrt{\frac{MPoR}{10\ BD}} \times \frac{IMR_{R/P,SIMM}(t = 0)}{q\big(0.99/0.01\,, \Delta MTM(0, MPoR)\big)}$$

using the long-end of the risk-factor implied volatility curves and solving the equivalent scaling equations for $\alpha_{R/P,\infty}$.

33. Interpreting the Haircut $h_{R/P}(t)$ Term: As will be seen below, the interpretation of $h_{R/P}(t)$ can vary depending on the intended application of the model.
34. $h_{R/P}(t)$ for Capital/Risk Models: For capital and risk models, $h_{R/P}(t)$ are two capital and risk functions that can be used to reduce the number of back-testing exceptions (see below) and ensure that the DIM model is conservatively calibrated.
35. $h_{R/P}(t)$ for the XVA Models: For XVA pricing, $h_{R/P}(t)$ can be fine-tuned – together with $\beta_{R/P}(t)$ - to maximize the accuracy of the forecast based on historical performance.
36. Lack of Asset Class Netting: Note that owing to the *No netting across Asset Classes* clause, the $IM_{R/P,x}(i,t)$ can be computed on a stand-alone basis for every asset class x defined by SIMM (IR/FX, equity, qualified and non-qualified credit, commodity) without any additional exposure runs. The total $IM_{R/P}(i,t)$ is then given by the sum of the $IM_{R/P,x}(i,t)$ values.
37. Historical vs. Computed IM Calibrations: A comparison between the forecasts of the DIM model defined in

$$\sigma^2(i,t) = \mathbb{E}[\Delta MTM^2(i,t) \mid MTM(i,t)] = \sum_{k=0}^{n} a_{\sigma k} MTM^k(i,t)$$

$$IM_{R/P,U}(i,t) = \Phi^{-1}\big(0.99/0.01, \mu = 0, \sigma = \sigma(i,t)\big)$$

$$IM_{R/P}(i,t) = \alpha_{R/P}(t) \times IM_{R/P,U}(i,t)$$

$$\alpha_{R/P}(t) = \big[1 - h_{R/P}(t)\big] \times \sqrt{\frac{10\ BD}{MPoR}} \times \left[\alpha_{R/P,\infty} + \big(\alpha_{R/P,0} - \alpha_{R/P,\infty}\big)e^{-\beta_{R/P}(t)t}\right]$$

$$\alpha_{R/P,0} = \sqrt{\frac{MPoR}{10\ BD}} \times \frac{IMR_{R/P,SIMM}(t=0)}{q\big(0.99/0.01\,, \Delta MTM(0, MPoR)\big)}$$

and the historical IMR realizations computed with the SIMM methodology is shown in Anfuso, Aziz, Loukopoulos, and Giltinan (2017) where alternative scaling approaches are considered.

38. Criteria Utilized in the Comparison: A comparison is performed at different forecasting horizons using 7 years of historical data, monthly sampling, and averaging among a wide representation of single-trade portfolios for the posted and the received IM cases.
39. $\mathcal{L}_1$ Error Metric Choice: For a given portfolio/horizon, the chosen error metric is given by $\mathbb{E}_{t_k}\left[\frac{|F_{R/P}(t_k+h)-G_{R/P}(t_k+h)|}{G_{R/P}(t_k+h)}\right]$ where $\mathbb{E}_{t_k}[\cdot]$ indicates an average across historical sampling dates – the definitions of $F_{R/P}$ and $G_{R/P}$ are contained below. Here and throughout this chapter, t_k is used in place of t whenever the same quantity is computed at multiple sampling dates.
40. Comparison of the Tested Universe: The tested universe is made up of 102 single-trade portfolios. The products considered, always at-the-money and of different maturities, include cross-currency swaps, IR swaps, FX options, and FX forwards – approximately 75% of the population is made up of

$$\Delta = 1$$

trades.
41. Calibrated Estimates of the Parameters: As is made evident by Anfuso, Aziz, Loukopoulos, and Giltinan (2017), the proposed term structure of $\alpha_{R/P}(t)$ improves the accuracy of the forecast by a significant amount – they also provide the actual calibration used for their analysis.
42. Conservative Calibration of the Haircut Function: Below contains further discussions on the range of values that the haircut functions $h_{R/P}(t)$ are expected to take for a conservative calibration of DIM to be used for regulatory exposure.
43. Comparison with CCP IMR: Finally, as an outlook, Anfuso, Aziz, Loukopoulos, and Giltinan (2017) show the error metrics for the case of CCP IMR where the Dim forecasts are

compared against the Portfolio Approaches to Interest Rate Scenarios (Pairs: LCH.ClearNet) and historical value-at-risk (HVaR; Chicago Mercantile Exchange) realizations.

44. <u>Prototype Replications of CCP Methodologies</u>: The realizations are based on prototype replications of the market risk components of the CCP IM methodologies.
45. <u>Universe Used for the CCP Tests</u>: The forecasting capability of the model is tested separately for Pairs and HVaR IMR as well as for 22 single-trade portfolios (IRS trades of different maturities and currencies). The error at any given horizon is obtained by averaging among 22×2 cases.
46. <u>Accuracy of the Proposed Scaling</u>: Without fine tuning the calibration any further, the time-dependent scaling $\alpha_{R/P}(t)$ drives a major improvement in the accuracy of the forecasts with respect to the alternative approaches.

How to Backtest a DIM Model

1. <u>Assessing Model for Different Applications</u>: The discussion so far has focused on a DIM model for B-IMR without being too specific about how to assess the model performance for different applications, such as CVA and margin valuation adjustment (MVA) pricing, liquidity coverage ratio/net stable funding ratio (LCR/NSFR) monitoring (Basel Committee on Capital Supervision (2013)), and capital exposure.
2. <u>Estimating the IMR Distribution Accurately</u>: As mentioned above, depending upon which application one considers, it may or may not be important to have an accurate assessment of the distribution of *the simulated IM requirements* value (IMRD).
3. <u>Backtesting to measure DIM Performance</u>: This chapter introduces two distinct levels of backtesting that can measure the DIM model performance in two topical cases:
    a. DIM applications that do not depend directly on the IMRD (such as capital exposure and the CVA), and

b. DIM applications that directly depend on the IMRD (such as MVA calculation and LCR/NSFR monitoring).

The methodologies are presented below, with a focus on the P-measure applications.

Backtesting DIM Mapping Functions (for Capital Exposure and CVA)

1. Review of the Monte-Carlo Framework: In a Monte-Carlo simulation framework, the exposure is computed by determining the MTM values of a given portfolio on a large number of forward looking risk-factor scenarios.
2. Adequacy of Forecasts across Scenarios: To ensure that a DIM model is sound, one should verify that the IM forecasts associated with the future simulation scenarios are adequate for a sensible variety of forecasting horizons as well as initial and terminal market conditions.
3. Setting up a Suitable Backtesting Framework: A suitable historical backtesting framework so as to statistically assess the performance of the model by comparing the DIM forecast with the realistic exact IMR, e.g., in the case of B-IMR calculated according to the SIMM methodology – for a representative sample of historical dates as well as market conditions and portfolios.
4. Generic IMR of a Portfolio: Let us first define generic IMR of a portfolio p as

$$IMR = g_{R/P}\left(t = t_\alpha, \Pi = \Pi\big(p(t_\alpha)\big), \vec{M}_g = \vec{M}_g(t_\alpha)\right)$$

The terms are as follows.

5. Posted/Received IMR Computation Algorithm: The functions g_R and g_P represent the exact algorithm used to compute the IMR for the posted and the received IM's, respectively (e.g., such as SIMM for B-IMR, or in the case of the CCP's, IM methodologies such as Standard Portfolio Analysis of Risk (SPAN), Pairs, or HVaR).

6. Date of the IMR Valuation:

$$t = t_\alpha$$

is the time at which the IMR portfolio p is determined.

7. Portfolio Trade Population at t_α: $\Pi\big(p(t_\alpha)\big)$ is the trade population of portfolio p at time t_α.
8. Market State Information at t_α: $\vec{M}_g(t_\alpha)$ is a generic state variable that characterizes all of the $T \leq t_\alpha$ market information required for the computation of the IMR.
9. DIM Forecast of the Portfolio: Similarly, the DIM forecast for the future IMR of a portfolio p can be defined as

$$DIM = f_{R/P}\left(t_0 = t_k, t = t_k + h, \vec{r}, \qquad \Pi = \Pi\big(p(t_k)\big), \vec{M}_{DIM} = \vec{M}_{DIM}(t_k)\right)$$

The terms are as follows.

10. Posted/Received DIM Computation Algorithm: The functions f_R and f_P represent the DIM forecast for the posted and the received IM's, respectively.
11. Date of the DIM Forecast:

$$t_0 = t_k$$

is the date time at which the DIM forecast is computed.

12. Horizon of the DIM Forecast:

$$t = t_k + h$$

is the time for which the IMR is forecast – over a forecasting horizon

$$h = t - t_0$$

13. Predictor Set of Market Variables: $\vec{r}$ - the *predictor* – is a set of market variables whose forecasted values on a given scenario are consumed by the DIM models as input to infer the IMR.
14. $\vec{r}$ as Simulated Portfolio MTM: The exact choice of $\vec{r}$ depends on the DIM model. For the one considered previously, $\vec{r}$ is simply given by the simulated MTM of the portfolio.
15. Market State Information at t_k: $\vec{M}_{DIM}(t_k)$ is the generic state variable characterizing all the

$$T \leq t_k$$

market information required for the computation of the DIM forecast.
16. Portfolio Trade Population at t_k: $\Pi(\cdot)$ is defined as before.
17. Caveats around f_R and f_P: Despite being computed using the stochastic RFE models, f_R and f_P are not probability distributions, as they do not carry any information regarding the probability weight of a given received/posted IM value. $f_{R/P}$ are instead mapping functions between the set $\vec{r}$ chosen as predictor and the forecast value for IM.
18. Confidence Level Based DIM Calibration: In terms of $g_{R/P}$ and $f_{R/P}$ one can define exception counting tests. The underlying assumption is that the DIM model is calibrated at a given confidence level (CL); therefore it can be tested as a $VaR(CL)$ model.
19. Model Conservatism Linked to CL: This comes naturally in the context of real-world *P* applications, such as capital exposure or liquidity monitoring, where a notion of model conservatism, and hence of exception, is applicable, since the model will be conservative whenever it understates (overstates) posted (received) IM.
20. The Portfolio Backtesting Algorithm Steps: For a portfolio p, a single forecasting day t_k, and a forecasting horizon h, one can proceed as follows.
21. t_k Estimate of the Forecast Functions: The forecast functions $f_{R/P}$ computed at time t_k are $f_{R/P}\left(t_0 = t_k, t = t_k + h, \vec{r},\ \Pi = \Pi\left(p(t_k)\right), \vec{M}_{DIM} = \vec{M}_{DIM}(t_k)\right)$ Note that $f_{R/P}$ depends exclusively on the predictor $\vec{r}$ –

$$\vec{r} = MTM$$

for the case considered above.

22. Impact of the Horizon on Predictor/Portfolio: The realized value of the predictor

$$\vec{r} = \vec{R}$$

is determined. For the model considered above, $\vec{R}$ is given by the portfolio value $p(t_k + h)$ where the trade population $\Pi\big(p(t_k + h)\big)$ at $t_k + h$ differs from t_k only because of portfolio aging. Aside from aging, no other portfolio adjustments are made.

23. Forecast Received/Posted IMR Estimate: The forecast values for the received and the posted IM's are computed as

$$F_{R/P}(t_k + h) = f_{R/P}\left(t_0 = t_k, t = t_k + h, \vec{r}, \qquad \Pi = \Pi\big(p(t_k)\big), \vec{M}_{DIM} = \vec{M}_{DIM}(t_k)\right)$$

24. Forecast of the Received/Posted IM Estimate: The realized values for the received and the posted IM's are computed as

$$G_{R/P}(t_k + h) = g_{R/P}\left(t = t_k + h, \Pi = \Pi\big(p(t_k + h)\big), \vec{M}_g = \vec{M}_g(t_k + h)\right)$$

25. Exception Case: F/G Mismatch Conservatism: The forecast and the realized values are then compared. The received and the posted DIM models are considered independently, and a backtesting exception occurs whenever $F_R(F_P)$ is larger (smaller) than $G_R(G_P)$. As discussed above, this definition of exception follows from the applicability of a notion of model conservatism.
26. Detecting the Backtesting Exception History: Applying the above steps to multiple sampling points t_k one can detect back-testing exceptions for the considered history.
27. Dimensionality Reduction for the Comparison: The key step is the estimate of the posted/received IMR forecast, where the dimensionality of the forecast is reduced – from a

function to a value – making use of the realized value of the predictor, and, hence, allowing for a comparison with the realized IMR.

28. Determining the Test p-value using TVS: The determination of the test p-value requires additional knowledge of the Test Value Statistics (TVS), which can be derived numerically if the forecasting horizons are overlapping (Anfuso, Karyampas, and Nawroth (2017)).
29. Caveats behind Blind TVS Usage: In the latter situation, it can happen that a single change from one volatility regime to another may trigger multiple correlated exceptions; hence the TVS should adjust the beck-testing assessments for the presence of false positives.
30. Accuracy of the $\alpha_{R/P}(t)$ Scaling: The single trade portfolios seen earlier have been tested by Anfuso, Aziz, Loukopoulos, and Giltinan (2017) using the SIMM DIM models with the three choices of scaling discussed earlier. The results confirm the greater accuracy of the term structure scaling of $\alpha_{R/P}(t)$.
31. Accuracy in the Presence of Haircut: In fact, for the same level of the haircut function

$$h_{R/P}(t > 0) = \pm 0.25$$

positive/negative for posted/received – a much lower number of exceptions is detected.

32. Realistic Values for the Haircut: Anfuso, Aziz, Loukopoulos, and Giltinan (2017) also observe that, in this regard, for realistic diversified portfolios and calibration targets of

$$CL = 95\%$$

the functions $h_{R/P}(t)$ take values typically in the range of $10 - 40\%$.

33. Assumptions Underlying the Haircut Assumption: The range of values for $h_{R/P}(t)$ has been calibrated using

$$\beta_{R/P}(t) = 1$$

and

$$\alpha_{R/P,\infty}(t) = 1$$

Both assumptions are broadly consistent with historical data.

34. IOSCO results in Over-collateralization: Note also that the goal of the BCBS-IOSCO regulations is to ensure that the netting sets are largely over-collateralized as a consequence of:
    a. The high confidence level at which the IM is computed, and
    b. The separate requirements for IM and VM.
35. Impact of Over-collateralization: Hence, the exposure generating scenarios are tail events, and the effect on capital exposure of a conservative haircut applied to the received IM is rather limited in absolute terms.
36. Over-collateralization Impact on Exposure: This issue is demonstrated by Anfuso, Aziz, Loukopoulos, and Giltinan (2017) where the expected exposure (EE) at a given horizon t is shown as a function of $h_R(t)$ – the haircut to be applied to the received IM collateral – for different distributional assumptions on $\Delta MTM(t, t + MPoR)$.
37. Distribution Dependence on Haircut Functions: In particular, they compute the expected exposure for

$$h_R(t) = 0$$

and

$$h_R(t) = 1$$

indicating full IM collateral benefit or no benefit at all – and take the unscaled IM as the 99^{th} percentile of the corresponding distribution. For different classes of the ΔMTM distribution, the exposure reduction is practically unaffected up to haircuts of $\approx 50\%$.

Backtesting the IMRD for MVA and LCR/NSFR

1. MC Based DIM IMR Distributions: The same Monte Carlo framework can be used in combination with a DIM model to forecast the IMD at any future horizon – implicit here are the models in which the DIM is not always constant across the scenarios. The applications of the IMRD are multiple.
2. Some Applications using the IMRD: The following are two examples that apply equally to the cases of B-IMR and CCP IMR:
    a. Future IM funding costs in the P measure, i.e., the MVA
    b. Future IM funding costs in the Q measure, e.g., in relation to LCR and NSFR regulations (Basel Committee on Banking Supervisions (2013))
3. Numerically Forecasting the IMR Distributions: The focus here is on the forecasts on the P-measure – tackling the case of the Q-measure may require a suitable generalization of Jackson (2013). The main difference with the backtesting approach discussed above is that the new model forecasts are the numerical distributions of the simulated IMR values.
4. Scenario-specific IM Forecasting: These can be obtained for a given horizon by associating every simulated scenario with its corresponding IMR forecast, computed according to the given DIM model.
5. Posted/Received IMR Density CDF: Using the notation introduced previously, the numerical representations of the received/posted IMRD cumulative density functions (CDF's) of a portfolio p for a forecasting day t_k and a horizon h are given by

$$CDF_{R/P}(x, t_k, h) = \frac{\#\{v \in \mathbb{V} \mid v \leq x\}}{N_{\mathbb{V}}} \; \forall \, \vec{r}_\omega \in \Omega$$

$$\mathbb{V} = \left\{f_{R/P}\left(t_0 = t_k, t = t_k + h, \vec{r}, \qquad \Pi = \Pi\big(p(t_k)\big), \vec{M}_{DIM} = \vec{M}_{DIM}(t_k)\right)\right\}$$

6. Terms of the CDF Expression: In

$$CDF_{R/P}(x, t_k, h) = \frac{\#\{v \in \mathbb{V} \mid v \leq x\}}{N_{\mathbb{V}}}$$

$N_{\mathbb{V}}$ is the total number of scenarios. In

$$\mathbb{V} = \left\{f_{R/P}\left(t_0 = t_k, t = t_k + h, \vec{r}, \qquad \Pi = \Pi\big(p(t_k)\big), \vec{M}_{DIM} = \vec{M}_{DIM}(t_k)\right) \forall\, \vec{r}_\omega \in \Omega\right\}$$

$f_{R/P}$ are the functions computed using the DIM model, $\vec{r}_\omega$ are the scenarios for the predictor – the portfolio MTM values in the case originally discussed, and Ω is the ensemble of $\vec{r}_\omega$ spanned by the Monte Carlo simulation.

7. Suitability of IMRD for Backtesting: The IMRD in this form is directly suited for historical backtesting using the Probability Integral Transformation (PIT) framework (Diebold, Gunther, and Tay (1998)).
8. Forecasting Horizon PIT Time Series: Referring to the formalism described in one can derive the PIT time series $\tau_{R/P}$ for a portfolio p for a given forecasting horizon h and backtesting history $\mathcal{H}_{BT}$ as:

$$\tau_{R/P} = CDF\left(g_{R/P}\left(t = t_k + h, \Pi = \Pi\big(p(t_k + h)\big), \vec{M}_g = \vec{M}_g(t_k + h)\right), t_k, h\right) \forall\, t_k \in \mathcal{H}_{BT}$$

9. Samples from the Actual IMR Algorithm: In the expression for $\tau_{R/P}$ above, $g_{R/P}$ is the exact IMR algorithm for the IMR methodology that is to be forecast – defined as

$$IMR = g_{R/P}\left(t = t_\alpha, \Pi = \Pi\big(p(t_\alpha)\big), \vec{M}_g = \vec{M}_g(t_\alpha)\right)$$

   and t_α are the sampling points in $\mathcal{H}_{BT}$.
10. Probability of t_k-realized IMR: Every element of the PIT time series $\tau_{R/P}$ corresponds to the probability of the realized IMR at time $t_k + h$ according to the DIM forecast built at t_k.
11. Backtesting of the Portfolio Models - Variations: As discussed extensively in Anfuso, Karyampas, and Nawroth (2017) one can backtest $\tau_{R/P}$ using uniformity tests. In particular,

analogous to what was shown in Anfuso, Karyampas, and Nawroth (2017) for portfolio backtesting in the context of capital exposure models, one can use test metrics that do not penalize conservative modeling – i.e., models overstating/understating posted/received IM. In all cases the appropriate TVS can be derived using numerical Monte Carlo simulations.

12. Factors affecting the Backtesting: In this setup the performance of a DIM is not done in isolation. The backtesting results will be mostly affected by the following.
13. Impact of $\vec{r}$ on Backtesting: As discussed earlier, $\vec{r}$ is the predictor used to associate an IMR with a given scenario/valuation time point. If $\vec{r}$ is a poor indicator for the IMR, the DIM forecast will consequently be poor.
14. Mapping of $\vec{r}$ to IMR: If the mapping model is not accurate, then the IMR associated with a given scenario will be inaccurate. For example, the models defined in

$$\sigma^2(i,t) = \mathbb{E}[\Delta MTM^2(i,t) \mid MTM(i,t)] = \sum_{k=0}^{n} a_{\sigma k} MTM^k(i,t)$$

$$IM_{R/P,U}(i,t) = \Phi^{-1}\big(0.99/0.01, \mu = 0, \sigma = \sigma(i,t)\big)$$

$$IM_{R/P}(i,t) = \alpha_{R/P}(t) \times IM_{R/P,U}(i,t)$$

$$\alpha_{R/P}(t) = \big[1 - h_{R/P}(t)\big] \times \sqrt{\frac{10\ BD}{MPoR}} \times \big[\alpha_{R/P,\infty} + \big(\alpha_{R/P,0} - \alpha_{R/P,\infty}\big)e^{-\beta_{R/P}(t)t}\big]$$

$$\alpha_{R/P,0} = \sqrt{\frac{MPoR}{10\ BD}} \times \frac{IMR_{R/P,SIMM}(t=0)}{q\big(0.99/0.01\,, \Delta MTM(0, MPoR)\big)}$$

include scaling functions to calibrate the calculated DIM to the observed

$$t = 0$$

IMR. The performance of the model is therefore dependent on the robustness of this calibration at future points in time.

15. RFE Models used for $\vec{r}$: The models ultimately determine the probability of a given IMR scenario. It may so happen that the mapping functions $f_{R/P}$ are accurate but the probabilities for the underlying scenarios for $\vec{r}$ are misstated, and, hence, cause backtesting failures.
16. Differential Impact of Backtesting Criterion: Note that
    a. The choice of $\vec{r}$, and
    b. The mapping

$$\vec{r} \rightarrow IMR$$

are also relevant to the backtesting methodology discussed earlier in this chapter. RFE models used for $\vec{r}$, however, are particular to this backtesting variance, since it concerns the probability weights of the IMRD.

Conclusion

1. Framework to Develop/Backtest DIM: This chapter has presented a complete framework to backtest and develop DIM models. The focus has been on B-IMR and SIMM, and the chapter has shown how to obtain forward-looking IM's from the simulated exposure paths using simple aggregation methods.
2. Applicability of the Proposed Model: The proposed model is suitable for both XVA pricing and capital exposure calculations; the haircut functions in

$$\alpha_{R/P}(t) = \left[1 - h_{R/P}(t)\right] \times \sqrt{\frac{10\ BD}{MPoR}} \times \left[\alpha_{R/P,\infty} + \left(\alpha_{R/P,0} - \alpha_{R/P,\infty}\right)e^{-\beta_{R/P}(t)t}\right]$$

can be used to either improve the accuracy (pricing) or to ensure the conservatism of the forecast (capital).

3. CCR Capital using DIM Models: If a financial institution were to compute CCR exposure using internal model methods (IMM), the employment of a DIM could reduce the CCR capital significantly, even after the application of a conservative haircut.
4. Over-collateralization inherent in Basel SA-CCR: This should be compared with the regulatory alternative SA-CCR, where the benefits from over-collateralization are largely curbed (Anfuso and Karyampas (2015)).
5. Backtesting Methodology to Estimate Performance: As part of the proposed framework, this chapter introduced a backtesting methodology that is able to measure model performance for different applications of DIM.
6. Agnosticity of DIM to the Underlying IMR: The DIM model and the backtesting methodology presented are agnostic to the underlying IMR algorithm, and they can be applied in other contexts such as CCP IM methodologies.

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Exposure Aggregation and XVA Calculation in Cross-Asset Model

Introduction

1. Exposure Aggregation/XVA Calculation Methodology: This section presents a general methodology for exposure aggregation and XVA calculation in the cross-asset CVA model framework.
2. Cross Asset Counter Party Model: The cross-asset counter-party model prices X valuation adjustment (XVA) including credit valuation adjustment (CVA) and debt valuation adjustment (DVA) for multi-asset portfolios.
3. Multi Stage Monte Carlo Simulations: The model is based on the Monte-Carlo simulation in five stages.
    a. Centralized Random Number Generation
    b. Market Data Simulation
    c. Trade Valuation
    d. Counter Party Level Aggregation
    e. XVA Calculation
4. Centralized Random Number Market Scenarios: The centralized random number generator generates all the random numbers for all the risk factors. The future market scenarios are generated for each model for each product category.
5. Calculation of the Future Exposures: The trade values for the future time points are computed for the simulated market scenarios for the specific model in each category. The future exposure of the portfolio is aggregated from the trade level, and the probabilities of default are calculated from the hazard curves.

6. Comprehensive Estimation of CVA/DVA: The CVA and the DVA can then be calculated from the expected future exposures, the probabilities of default, and the recovery rates. This model systematically handles wrong-way and right-way risks, cross-asset correlations, and cross-asset integration.
7. Chapter Motivation, Scope, and Purpose: The general description of the cross-asset model can be found in a previous chapter. This chapter will present the methodology for future exposure aggregation and XVA calculation, specifically.

Netting and Aggregation

1. Netting Group vs. Collateral Group – Definition: In general, a counter party could contain multiple netting groups, and a netting group could contain multiple collateral groups. The collateral group contains clauses such as threshold, eligible collateral, and minimum transfer amount.
2. Definition of the Uncollateralized Groups: The trades not covered by any collateralized group are called uncollateralized, or equivalently, belong to a special collateral group whose collateral amount is always zero.
3. Contractually Specified Allowed Netting Types: Most agreements allow netting such that receivables and payables offset each other in the calculation of the exposure. This feature is called contractual netting. Other agreements disallow netting, and are called contractual non-netting. For contractual non-netting, each trade is treated as a netting group with a single trade. There is usually no collateral for contractual non-netting.
4. Treatment of Non-enforceable Contractual Netting: In contractual netting, under some circumstances (for example, if the counter-party is domiciled in emerging markets), the netting is deemed non-enforceable, and no netting is applied in the calculation of the positive exposure. However, in the case of negative exposure, netting is always applied.

5. Contractual Non-Netting Negative Exposures: For contractual non-netting counter-parties, for the purpose of being conservative, currently netting is applied in the calculation of negative exposures. This will be re-visited in the future.
6. Cross Collateral Group Collateral Amount Calculation: To calculate the exposure at the counter-party level, first the collateral amount for each collateral group is calculated, based on the values of all the trades within the collateral group and the CSA.
7. Cross Netting Group Exposure Aggregation: Then the exposure on the netting group level is calculated depending on the contractual netting and the netting enforceability. No netting is allowed across netting groups. In other words, each netting group should be treated similar to an individual counter-party. The results will be used to price the XVA in the next stage.
8. Value across Collateral/Netting Groups: Let V_{ij} be the j^{th} trade in the i^{th} collateral group, and

$$T_{i,CPTY} > 0$$

and

$$T_{i,BANK} < 0$$

be the effective collateral thresholds for the counter-party and the bank, respectively. Then to total portfolio value at time t in the netting group is

$$V(t) = \sum_{i.j} V_{ij}(t)$$

9. Margin Period Collateral Amount Estimation: The collateral amount from the i^{th} collateral group used for calculating the collateralized portfolio exposure at time t is calculated at time $t - \delta$ where δ is the default window with a default value of 14 calendar days.

$$C_i(t) = \sum_j [V_{ij}(t-\delta) - T_{i,CPTY}]^+ + \sum_j [V_{ij}(t-\delta) - T_{i,BANK}]^-$$

10. Explicitly Specified "Current Collateral Balance": When the collateral date is before or on the valuation date, i.e.,

$$t - \delta \le T_{VAL}$$

and optional input – the "Current Collateral Balance" – can be used (this is the collateral balance at time 0). If this input is present, it will be used directly instead of calculating the time 0 collateral as shown above.

11. Cash Flow Adjusted Collateralized Portfolio: Then the collateralized portfolio amount including the cash-flow in the netting group at time t is

$$V_C(t) = V(t) + CF(t) - C(t) = \sum_{i.j} [V_{ij}(t) + CF_{ij}(t)] - \sum_i C_i(t)$$

$$= \sum_i \left\{ \sum_j [V_{ij}(t) + CF_{ij}(t)] - C_i(t) \right\}$$

Here the cash flow $CF(t)$ is the cash flow in the time window $(t-\delta, t]$, inclusive of the payment at t and exclusive of the payment at $t-\delta$

12. Contractual Netting Positive/Negative Exposures: For contractual netting, when netting is enforceable, the collateralized positive exposure and the negative exposure for the netting group is

$$ColPositiveExp(t) = {V_C}^+(t) = \left(\sum_i \left\{ \sum_j [V_{ij}(t) + CF_{ij}(t)] - C_i(t) \right\} \right)^+$$

$$ColNegativeExp(t) = V_C{}^{-}(t) = \left(\sum_i \left\{\sum_j \left[V_{ij}(t) + CF_{ij}(t)\right] - C_i(t)\right\}\right)^{-}$$

13. Contractual Unenforceable Netting Positive Exposure: When the netting is not enforceable, the collateralized positive exposure is calculated as

$$\begin{aligned} &ColPositiveExp(t) \\ &= \sum_i \left(\left\{\sum_j \left[V_{ij}(t) + CF_{ij}(t)\right]^{+} - C_i{}^{+}(t)\right\}^{+} \right. \\ &\left. + \left\{\sum_j \left[V_{ij}(t) + CF_{ij}(t)\right]^{-} - C_i{}^{-}(t)\right\}^{+}\right) \end{aligned}$$

14. Contractual Unenforceable Netting Negative Exposure: The collateralized negative exposure is still calculated as before, i.e.,

$$ColNegativeExp(t) = \left(\sum_i \left\{\sum_j \left[V_{ij}(t) + CF_{ij}(t)\right] - C_i(t)\right\}\right)^{-}$$

15. Contractual Non Netting Uncollateralized Exposures: For contractual non-netting, there is no collateral, and the positive and the negative exposures become

$$UncolPositiveExp(t) = \sum_{i,j} \left[V_{ij}(t) + CF_{ij}(t)\right]^{+}$$

$$UncolNegativeExp(t) = \sum_{i,j} \left[V_{ij}(t) + CF_{ij}(t)\right]^{-}$$

16. Collateralized Aggregation across Netting Groups: Finally, if a counter-party has multiple netting groups, then the collateralized positive exposure and the negative exposure for the counter-party are simply a summation across all the netting groups.

$$ColPositiveExp_{CPTY}(t) = \sum_{ng} ColPositiveExp_{ng}(t)$$

$$ColNegativeExp_{CPTY}(t) = \sum_{ng} ColNegativeExp_{ng}(t)$$

17. Uncollateralized Aggregation across Netting Groups: For uncollateralized exposures, the above formulas hold true, except that the collateral amounts are set to zero;

$$C_i(t) = C_i(t) = C_i(t) = 0$$

After the netting set aggregation, six aggregated measures at each time and in each path are sown in the table below.

18. Exposure Related Time/Path Measures:

#	**Measure**	**Description**
1	Positive Exposure	Positive Exposures of a Portfolio
2	Collateralized Positive Exposure	Positive Exposures of a Collateralized Portfolio
3	Negative Exposure	Negative Exposures of a Portfolio
4	Collateralized Negative Exposure	Negative Exposures of a Collateralized Portfolio
5	Collateral Amount	Collateral Amounts
6	Cash Flow	Cash Flow Amounts

19. Aggregator Inputs - Counter Party Information: Counter Party Reference Data including netting groups, collateral groups, threshold, minimum transfer amount, independent amount, current collateral balance, contractual netting flag, netting enforceability flag, and trade ID in each collateral group.
20. Aggregator Inputs - Default Window Settings:
    a. Counter Party Default Window => The counter-party's default window. 2 weeks as a default value (default is 14.0)
    b. Bank Default Window => The bank's default window. The default window is used as an indicator only to trigger the counter party window == the bank default window.
21. Aggregator Inputs MPoR Estimation Settings:
    a. Margin Period => Margin Call Frequency; Daily Margin as Default Value; Default is 1.0
    b. MPoRInterpType => Portfolio Value Interpolation Type when calculating MPoR. Valid values currently are: LINEAR, SQRT_T, and BBRIDGE. BBRIDGE is set as default.

Aggregation for Trades Priced in Proxy Models

1. User-Specified Netting/Collateral Configuration: For trades priced from proxy models, one can use different aggregation methods. This example starts with 12 trades in one netting group and across two collateral groups as shown in the table below.

Trade ID	Collateral Group	Aggregation Type	Netting Group (User Specified)
1	1	A	1

2	1	A	1
3	1	B	1
4	1	B	1
5	1	Non-Proxy	1
6	1	Non-Proxy	1
7	2	A	1
8	2	A	1
9	2	B	1
10	2	B	1
11	2	Non-Proxy	1
12	2	Non-Proxy	1

2. Method #1 – Netting/Collateral Groups: This section considers three proposals to handle trades with proxy models. The first method is to separate each trade with a proxy aggregation type into a final netting group as shown below.

Trade ID	**Collateral Group**	**Aggregation Type**	**Netting Group (User Specified)**	**Netting Group (Final)**
1	1	A	1	2
2	1	A	1	3
3	1	B	1	4
4	1	B	1	5
5	1	Non-Proxy	1	1

6	1	Non-Proxy	1	1
7	2	A	1	6
8	2	A	1	7
9	2	B	1	8
10	2	B	1	9
11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

3. <u>Method #2 – Netting/Collateral Groups</u>: The second method is to group trades with different aggregation types in each collateral group into different netting groups as shown below.

Trade ID	**Collateral Group**	**Aggregation Type**	**Netting Group (User Specified)**	**Netting Group (Final)**
1	1	A	1	2
2	1	A	1	2
3	1	B	1	3
4	1	B	1	3
5	1	Non-Proxy	1	1
6	1	Non-Proxy	1	1
7	2	A	1	4
8	2	A	1	4
9	2	B	1	5
10	2	B	1	5

11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

4. <u>Method #3 – Netting/Collateral Groups</u>: The third method is to group trades with different aggregation types into different netting groups as shown below.

Trade ID	Collateral Group	Aggregation Type	Netting Group (User Specified)	Netting Group (Final)
1	1	A	1	2
2	1	A	1	2
3	1	B	1	3
4	1	B	1	3
5	1	Non-Proxy	1	1
6	1	Non-Proxy	1	1
7	2	A	1	2
8	2	A	1	2
9	2	B	1	3
10	2	B	1	3
11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

Three Points Brownian Bridge Interpolation

1. Intermediate Broken Date Exposure Calculation: When calculating the margin period of risk, on each exposure date, one needs to find a trade value δ days before the exposure date, where δ is the default window with a default value of 14 calendar days.
2. Brownian Bridge Based Exposure Interpolation: The trade value on that date may not be available from simulation. In this case on needs an interpolation method to interpolate the trade values from the neighboring simulation dates. This chapter uses the three point Brownian Bridge method – it is described as follows.
3. Brownian Bridge - The Grid Points: Given the values on three dates $\{(t_1, V_1), (t_2, V_2), (t_3, V_3)\}$ the intention is to interpolate the value on date t such that

$$t_2 < t < t_3$$

The trade value is assumed to follow the Brownian motion

$$\Delta V_t = \sigma \Delta W_t$$

4. Normally Distributed Brownian Bridge Factor: According to the Brownian Bridge interpolation in the interval (t_1, t_3)

$$V_t = \frac{t_3 - t}{t_3 - t_1} V_1 + \frac{t - t_1}{t_3 - t_1} V_3 + \sqrt{\frac{(t_3 - t)(t - t_1)}{t_3 - t_1}} v$$

where v is a Brownian Bridge factor from the normal distribution $\mathcal{N}(0, \sigma^2)$

5. v Chosen to satisfy V_2: One chooses v so that the interpolation at t_2 matches the value at t_2. Namely, the value at t_2 is

$$V_2 = \frac{t_3 - t_2}{t_3 - t_1} V_1 + \frac{t_2 - t_1}{t_3 - t_1} V_3 + \sqrt{\frac{(t_3 - t_2)(t_2 - t_1)}{t_3 - t_1}} v$$

6. Estimation of the Brownian Bridge Factor: Therefore ν is determined as

$$\nu = \sqrt{\frac{t_3 - t_1}{(t_3 - t_2)(t_2 - t_1)}}\left[V_2 - \frac{t_3 - t_2}{t_3 - t_1}V_1 - \frac{t_2 - t_1}{t_3 - t_1}V_3\right]$$

XVA Calculation

1. Aggregation over Counter-party Trades: Given the result of the aggregation, one has the counter-party level trade values at each time point and in each path. The next step is to proceed to calculate the XVA's.
2. Counter-Party/Bank Default Dates: Let $t_0, \cdots, t_N$ be the credit dates, where the first date t_0 is the valuation date. τ_{CPTY} and τ_{BANK} are the default dates for the counter-party and the bank, respectively.
3. Collateralized Expected Positive/Negative Exposures: Let $ColEPE(t_n)$ and $ColENE(t_n)$ be the collateralized expected positive exposure and the collateralized expected negative exposure respectively on the credit date t_n; let $D_f(t_n)$ be the discount factor on the credit date t_n.
4. CVA from Collateralized Positive Exposure: The CVA is then calculated as

$$\begin{aligned} CVA = -\sum_{n=0}^{N-1} &\left[w \cdot ColEPE(t_n) \cdot D_f(t_n) + (1-w) \cdot ColEPE(t_{n-1}) \cdot D_f(t_{n-1})\right] \\ &\times \left\{1 - R_{CPTY}\left(w \cdot ColEPE(t_n) + (1-w) \cdot ColEPE(t_{n-1}) \cdot \frac{D_f(t_{n-1})}{D_f(t_n)}\right)\right\} \\ &\times Prob(t_n \le \tau_{CPTY} < t_{n+1}) \end{aligned}$$

5. DVA from Collateralized Positive Exposure: The DVA is calculated as

$$DVA = -\sum_{n=0}^{N-1}\left[w \cdot ColENE(t_n) \cdot D_f(t_n) + (1-w) \cdot ColENE(t_{n-1}) \cdot D_f(t_{n-1})\right]$$
$$\times \left\{1 - R_{BANK}\left(w \cdot ColENE(t_n) + (1-w) \cdot ColENE(t_{n-1}) \cdot \frac{D_f(t_{n-1})}{D_f(t_n)}\right)\right\}$$
$$\times Prob(t_n \leq \tau_{BANK} < t_{n+1})$$

6. Bank/Counter-Party Recovery Maps: $R_{BANK}(\cdot)$ and $R_{CPTY}(\cdot)$ are the recovery mapping functions for the counter-party and the bank, respectively.
7. Exposure Weights over Period Vertexes: w is the weight of the period end exposure – default value being 0.5, which indicates that the CVA/DVA is calculates using the average exposure between the beginning and the end of each period.
8. Bank/Counter-Party Default Distribution: The distribution of defaults can be inferred from a hazard curve easily. The "discount factors" calculated from the hazard curves are just survival probabilities. Thus the probability of default between t_n and t_{n+1} is given by

$$Prob(t_n \leq \tau < t_{n+1}) = DF_{Hazard}(t_n) - DF_{Hazard}(t_{n+1})$$

9. Portfolio Trajectory Period Vertex Measures: In addition to the CVA/DVA calculations, several other summary measures may be computed, as shown in the table below.

#	**Measure**	**Description**
1	CVA + DVA	NPV
2	CVA	CVA
3	DVA	DVA
4	Portfolio Value (MC)	Portfolio Value from Monte Carlo
5	CVA + DVA MC Error	NPV MC Error

6	CVA MC Error	CVA MC Error
7	DVA MC Error	DVA MC Error
8	MaxPosPFE	Maximum Positive PFE
9	AvgPosPFE	Average Positive PFE
10	MaxNegPFE	Maximum Negative PFE
11	AvgNegPFE	Average Negative PFE
12	AEPE	Average Expected Positive Exposure
13	AEPE MC Error	AEPE MC Error
14	AENE	Average Expected Negative Exposure
15	AENE MC Error	AENE MC Error
16	Col AEPE	Collateralized AEPE
17	Col AEPE MC Error	Collateralized AEPE MC Error
18	Col AENE	Collateralized AENE
19	Col AENE MC Error	Collateralized AENE MC Error
20	DTZ	$-\max(PortfolioValue - Collateral, 0) - NPV$
21	DTR	$-\max(PortfolioValue - Collateral, 0) - (1 - RR) \times NPV$

10. <u>Portfolio Trajectory Period Edge Measures</u>: For each time period the CVA/DVA are computed and the following measures are stored.
    a. Vertex Start
    b. Vertex End
    c. Vertex Unconditional Counter Party Default Probabilities

d. Vertex Counter Party Default Probabilities Conditional on Bank Survival
e. Edge CVA
f. Vertex Positive Exposures
g. Vertex Discount Factors
h. Vertex Counter Party Recovery Rate
i. Vertex Unconditional Bank Default Probabilities
j. Vertex Bank Default Probabilities Conditional on Counter Party Survival
k. Edge DVA
l. Vertex Negative Exposures
m. Vertex Bank Recovery Rate
n. Vertex Positive PFE
o. Vertex Negative PFE
p. Vertex Collateralized EPE
q. Vertex Collateralized ENE
r. Vertex Collateral
s. Vertex Positive Collateral
t. Vertex Negative Collateral
u. Vertex Cash Flows
v. Vertex Positive Cash Flow
w. Vertex Negative Cash Flow
x. Vertex EPE MC Error
y. Vertex EPE Percent MC Error
z. Vertex ENE MC Error
aa. Vertex ENE Percent MC Error
bb. Vertex Collateralized EPE MC Error
cc. Vertex Collateralized EPE Percent MC Error
dd. Vertex Collateralized ENE MC Error
ee. Vertex Collateralized ENE Percent MC Error
ff. Vertex Start EPE
gg. Vertex Start ENE

hh. Vertex Start Collateralized EPE

ii. Vertex Start Collateralized ENE

11. <u>XVA Calculation Input Parameters</u>:

    a. Credit Data => Credit Information for both parties; hazard curves and recovery rates/recovery maps

    b. Discount Curves or simulated discount factor paths

    c. Threshold to indicate PFE – default is set to 0.975

    d. An indicator (TRUE/FALSE) on whether to use conditional default probability or unconditional – default is unconditional.

Prudent Adjustments

Abstract

1. Inadequacy of Complete Markets Approach: The ongoing controversy about whether or not FVA and KVA should be an adjustment to fair valuation originates from the attempts to shoehorn the metrics quantifying market incompleteness into the traditional valuation paradigm based on complete markets.
2. Prudent Valuation in Incomplete Markets: After reviewing the concept of fair valuation, Albanese and Syrkin (2016) introduce the concept of prudent valuation in incomplete markets and discuss what went lost in translation in the FVA/KVA debate.

Fair Valuations

1. Traits of Complete Markets: In complete markets all conceivable trades can be executed; all payoffs can be perfectly replicated and there is no need for risk capital; REPO markets exist for all contingent claims and there are no funding costs for carrying derivatives.
2. Complete Markets - Trades/Participants/Mechanisms: In complete markets, fair value of trades equals its cost of replication. Trades can be priced in isolation at levels that do not depend on the portfolio holdings. All economic agents are equivalent; there is no difference between a broker-dealer, a corporate client, a lender, or a dealer share-holder. All market participants value assets identically. The value of a trade can be readily assessed as its exit

price, i.e., the amount that one can sell it for to any other market participant. Prices are also entirely insensitive to regulatory requirements regarding capital and collateralization.

3. Rationale behind the Use of MMT: When markets are complete, Modigliani and Miller (1958) argue that all trades clear at levels both parties consider as fair, i.e., without wealth transfers (day-one gains). According to this argument, if an investment decision were to increase the wealth of a dealer as a whole but reduce the wealth of the share-holders, manage should still opt to take it, as share-holders in this case can retire the entire debt of the firm to compensate for the wealth transfer. According to this reasoning, dealers should optimally have no leverage at all.
4. Computation of the Fair Value: Mathematically, fair valuations are uniquely computed as discounted expectations of future cash flows, simulated consistently with all market information.

Wealth Transfers and Prudent Valuation

1. Wealth Transfer in Incomplete Markets: Although real markets are very different from complete markets, the mathematical formulation of fair value as a discounted expectation is still useful as a global-anchor, market-wide measure of value which could not possibly give rise to any arbitrage opportunity, even if a market-competing infrastructure came into existence. However, there is no real reason to expect that, in general, trades actually clear at fair value levels, i.e., without a wealth transfer between the parties.
2. Quantifying the Wealth Transfer Components: XVA metrics (e.g., FVA, KVA etc.) quantify such wealth transfers, i.e., they measure the degree by which the fair value requires modification. Trades by broker-dealers typically transfer wealth between creditors, counter-parties, and share-holders. Managers reflect wealth transfers in prudent valuations as it is their mandate to preserve the risk capital that funding costs could deplete. Prudent calculations also need to enable a sustainable dividend distribution policy. The difference

between prudent and fair valuations is given by the XVA metrics. A concrete implementation of the XVAs – out of the potential candidates – reflects the specific choice for completing the incomplete markets.

3. XVA Metrics Guiding the Operating Decisions: Hull and White (2016) agree that FVA is a wealth transfer from the share-holders to creditors, as proposed in Albanese, Andersen , and Iabichino (2015) Albanese, Caenazzo, and Crepey (2016), and Andersen, Duffie, and Song (2017). However, they argue that the wealth transfer should at least in principle be captured in the fair valuation of the dealer debt, and trades should clear at the fair valuation levels. In Albanese, Andersen , and Iabichino (2015) Albanese, Caenazzo, and Crepey (2016), and Andersen, Duffie, and Song (2017), it is stressed instead that virtually all decisions taken by dealer managers are predicated on prudent valuations and are sensitive to XVA metrics.
4. Rationale behind the Hull and White (2016) Hypothesis: The argument revolves around the expectation that there is a constant and statistically predictable deal flow. Under this assumption, wealth transfers from the bond-holders to share-holders would optimally occur at the point of the debt issuance. Creditors would grant a discount equivalent to all future incremental FVA due to future trades. Banks would then recognize this discount as a gain and provision it as FVA reserve capital at the time of debt issuance.
5. Trade Flow and Capital Depletion: Since trade flow uncertainty in itself is a risk factor with potentially adverse effect on capital, Albanese and Syrkin (2016) believe that it is prudent to model portfolios on a run-off basis, i.e., treat each new trade as if it were the last one ever to be made. If a new trade triggers wealth transfers from share-holders to creditors, then the dealer must pass on to the end client the wealth transfer amount, which is the incremental FVA, in order to not deplete the capital.
6. Usage of the KVA Metric: The run-off assumption is also relevant in case of the KVA metric. Under Solvency II, KVA is called the risk-margin, and is a form of the loss-absorbing capital that is sourced from clients, retained at the inception and distributed gradually with time. Dividends are modeled as being proportional to the Economic Capital requirements by a proportionality factor called the hurdle rate. The KVA is then computed on a run-off basis contingent on the dealer never defaulting.
7. Principles behind Prudent Trade Valuation: In summary, the prudent valuation of a trade:

a. Does not entail a DVA benefit
b. Contains FVA adjustments to offset funding costs
c. Contains a KVA adjustment that eventually flows into the dividend stream

8. Bilateral Nature of Prudent Valuation: Since funding strategies involve portfolio-wide re-hypothecation and risk capita is calculated for the entire book, the FVA and the KVA are meaningful only if calculated for the entire portfolio, not individual trades. Strictly speaking, one can only talk about incremental prudent valuation of a trade when it is added to a portfolio. In particular, while fair-valuations are marker wide numbers, prudent valuations are bilateral and entity-specific. Polling for transaction prices yields the average for prudent valuations, but has no bearing on fair valuations.

Loss in Translation

1. Share-holders Interests vs. Manager's Incentives: Each and every decision that a broker-dealer takes uses the prudent valuation metrics – fair valuations are insufficient. By introducing FVA and KVA the industry seeks to embed the cost of funding and the cost of capital into the valuation process in order to align the interests of the share-holders with the incentives of the managers.
2. Treating the Funding/Capital Charges as Adjustments: The banking industry struggles to achieve this objective with the traditional accounting framework, which is designed around capital markets. Since the only lever available to dealers is fair valuation, the FVA and the KVA are interpreted as adjustments, which flies in the face of finance theory.
3. Inadequacy of the Fair Value Metric: Albanese and Syrkin (2016) concur with Hull and White (2016) in that what the market academics refer to as fair value for the derivative portfolios should not entail FVA and KVA adjustments. However, they also believe that fair valuations (in the academic sense of the word) are too aggressive for reporting purposes, and prudent valuations should be used.

4. Redefining a *Prudential* Fair Value: Albanese and Syrkin (2016) see no particular contradiction in embedding the FVA in a redefinition of the fair value. Pathological situations occur only when the FVA is computed by symmetrically discounting a spread over OIS, since portfolios dominated by liabilities give rise to fictitious gains as portfolio FVA is negative. But if the FVA is valued asymmetrically as in Albanese, Andersen, and Iabichino (2015), there is no contradiction in embedding the FVA value into a redefinition of the fair valuation.
5. Intricacies Embedded in the KVA Hurdle Rate: However, if the same procedure above is followed for the FVAs, one would arrive at undesirable consequences. For instance a change of target hurdle rate by managers would automatically trigger a write-off. It would be more appropriate if the dealers monitored the level of retained earnings on a mark-to-market basis, to determine an *implied* hurdle rate for which the KVA would equal the retained earnings. On this basis, dealers may report to investors both the *implied* and the target hurdle rates, immunizing the earnings and the capital ratios from having any dependency in the hurdle rate.
6. The KVA *Local Equilibrium* Adiabat: The introduction of the KVA has a cascading effect on the market dynamics. Two price metrics co-exist: market-wide fair valuations and local bilateral prudential valuations. The latter embed XVA adjustments and aim at preserving the capital while seeking to generate a sustainable ROE. As trades are executed, the economy only achieves a local optimum restricted to the dealer and the clients involved in the trade. Achieving a market-wide optimum requires a cascade of optimizing whereby further gains are achieved by re-allocating risk across all other financial institutions. A sequence of *XVA Compression* steps is then required to achieve a global optimum.
7. Relaxation onto the Global Adiabat: Given the interconnectedness of the financial system, the occurrence of each trade creates, generally speaking, a perturbation with market-wide impact triggering a cascade of bilateral and multi-lateral optimization steps for credit and collateral strategies. All complex systems showcase similar features whereby local optima are achieved rather quickly, while global relaxation to the equilibrium is a gradual and more slower process.

8. XVA Metrics ad Local Decision Drivers: In conclusion, prudent valuations and their embedded metrics are the local drivers that drive the decision making process of economic agents as they constantly strive to achieve optimality in a local, bilateral/entity-specific sense. As markets evolve towards a global optimum and market-competing infrastructure evolves too, in the limiting case of vanishing market efficiencies, valuations will converge to their theoretical market-wide fair values. However, current markets are a long way away from the theoretical market efficiency.

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CVA And Funding Adjustments PDE

Counterparty Risk and Funding Costs

1. PDE Derivation of Adjustments – Approach: Burgard and Kjaer (2012a) derive the partial differential equation (PDE) representation for the value of financial derivatives with bilateral counterparty risks and funding costs. The model is very general in that the funding rate for lending and borrowing and the MTM value at default can be exogenously specified.
2. Buy-Back of Own Bonds: The buying back of a party's own bonds is a key part of the delta hedging strategy; they discuss how the cash account of the replication strategy provides sufficient funds for this.
3. Full Value as Default Payout: First they consider the case where the mark-to-market at default is given by the derivative, which includes the counterparty risk. They find that the resulting pricing PDE becomes non-linear, except in those special cases where the non-linear terms vanish and the Feynman-Kac representation of the total value can be obtained. In these cases the total value of the derivative can be decompose into a default-free part and a bilateral credit valuation and funding adjustment part.
4. Fair Derivative Value at Payout: Next they assume that the MTM value at default is given by the fair economic value of the derivative. This time the resulting PDE is linear and the corresponding Feynman-Kac representation is used to decompose the total value of the derivative into a default-free value plus bilateral credit valuation and funding cost adjustments.

Motivation, Literature Scan, and Approach

1. Counterparty Credit Risk Definition: Counterparty credit risk implicitly embedded in derivative contracts has become increasingly relevant in recent market conditions. This risk represents the possibility that the counterparty defaults while owing money under the terms of a derivative contract, or more precisely, if the mark-market value of the derivative is positive to the seller at the time of the default of a counterparty.
2. OTC Counterparty Risk Mitigation: While for exchange traded products the counterparty risk is mitigated by the presence of the exchange as an intermediary, this is not the case for OTC products. For these a number of different techniques are being used to mitigate the counterparty risk, most commonly by means of netting agreements and collateral mechanisms. The details of these agreements are, for example, published by the International Swaps and Derivatives (ISDA) 2002 Master Agreement.
3. Bilateral Counterparty Credit Risk: However the counterparty faces a similar risk of the seller defaulting when the mark-to-market value is positive to the counterparty. Taking into account the credit risk of both the parties is commonly referred to as considering the bilateral counterparty risk. When doing so the value of the derivative to its seller is influenced by its own credit quality.
4. Counterparty Credit Risk Coverage: Papers, books, and book chapters that develop techniques for the valuation of the derivatives and derivative portfolio under counterparty risk include, but are not limited to, Jarrow and Turnbull (1995), Jarrow and Yu (2001), Brigo and Mercurio (2007), Li and Tang (2007), Pykhtin and Zhu (2007), Alavian, Ding, Whitehead, and Laiddicina (2008), Gregory (2009), and Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009).
5. Counterparty Risk With Funding Costs: There are other areas where the credit of the seller is relevant, in particular in terms of the MTM accounting of its own debt, as well as the effect it has on its funding costs. Piterbarg (2010) discusses the funding costs on derivative valuations when collateral has to be posted. Thus Burgard and Kjaer (2012a) combine the effects of the seller's credit on its own funding costs with that on the bilateral counterparty risk into a unified framework.
6. Black-Scholes PDE Formulation Extension: Further they use the hedging argument to derive extensions to the Black-Scholes PDE in the presence of bilateral counterparty risk in the

presence of bilateral jump-to-default model and include funding considerations in the financing of the hedge positions. In addition they consider two scenarios for the determination of the derivative MTM at default – namely that recovery is on the total risky value or that it is on the counterparty riskless value. The latter corresponds to the most common approach taken in the literature.

7. <u>The ISDA 2002 Master Agreement</u>: The total value of the derivative will then depend upon which of the 2 MTM values is used at default. For contracts following the ISDA 2002 Master Agreement the value of the derivatives upon the default of one of the counterparties is determined by a dealer poll. There is no reference to the counterparties, and one could reasonably expect the derivative value to be the counterparty riskless value, i.e., the second case considered.
8. <u>Risky Derivative Value at Payout</u>: In the case where the default-risky derivative value is used as the mark-to-market, Burgard and Kjaer (2012a) derive a pricing PDA that is in general non-linear and demonstrate that the unknown risky price can be found by solving a non-linear integral equation. Under certain conditions on the payoff the non-linear terms vanish and the Feynman-Kac representation of the resulting linear PDE is examined.
9. <u>Fair Derivative Value at Payout</u>: In the case where the counterparty derivative price is used as the mark-to-market, the resulting pricing PDE is linear. As in the first case the Feynman-Kac representation can be used to decompose the risky derivative value into a counterparty risk-free part, a funding part, and a bilateral credit valuation adjustment (CVA) part.
10. <u>Granular Hedging Accounts and Strategies</u>: By using a fine-grained hedging strategy to derive their results Burgard and Kjaer (2012a) ensure that the hedging costs of all considered risk factors are included in their derivative price such that the decomposition of the risky price is a generalization of the result commonly found in literature. Further they get explicit expressions for hedges, which is important for risk management.
11. <u>Own Credit Hedging Risk Caveats</u>: There have been discussions about how a seller can hedge out its own credit risk – Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009) contain a summary. The strategy described by Burgard and Kjaer (2012a) includes the repurchase by the seller of its own bonds to hedge out its own credit risk. On the face of it this may seem like a futile approach since of the bond purchase were funded by issuing more

debt (i.e., more bonds), the seller would have in effect achieved nothing in terms of hedging its own credit risk.

12. Differentiated Buy-Back Funding Strategy: However the replication strategy presented shows how the funding for the purchase of the seller's own bond is achieved through the cash account of the hedging strategy. The hedging strategy (including the premium of the derivative) generates the cash needed to repurchase the sellers' own bond.
13. Multiple Assets and Netting Extensions: Although all the results presented in Burgard and Kjaer (2012a) are for one derivative on one underlying asset following the specified dynamics, extension to the situation of a netted derivatives portfolio on several underlying following generalized diffusion dynamics is straightforward.

Notation, Symbology, and Key PDEs

1. **The Nature of the Derivative**: Consider a derivative contract $\hat{V}$ between a seller B and a counterparty C that may both default. The asset S is not affected by the default of either B or C and is assumed to follow a Markov process with a generator $\mathcal{A}_t$. Similarly we let V denote the same derivative between parties that cannot default.
2. **Payout on a Default**: At the default of either the counterparty or the seller the value of the derivative to the seller $\hat{V}$ is determine by using an MTM rule M which may equal $\hat{V}$ or V. Throughout the convention used is that the positive derivative values correspond to seller assets and counterparty liabilities.
3. **Notations, Parameter Definitions, and Caveats**:
    a. r => Risk-free rate
    b. r_B => Yield on a Recovery-less Bond of Seller B
    c. r_C => Yield on a Recovery-less Bond of Counterparty C
    d. λ_B =>

$$\lambda_B = r_B - r$$

e. λ_C =>

$$\lambda_C = r_C - r$$

f. r_F => Seller funding rate for the borrowed cash on the seller's derivative replication cash account.

$$r_F = r$$

if the derivative can be used as a collateral.

$$r_F = r + (1 - R_B)\lambda_B$$

if the derivative cannot be used as a collateral.

g. s_F =>

$$s_F = r_F - r$$

h. R_B => Recovery on the derivative MTM value in case the seller B defaults

i. R_C => Recovery on the derivative MTM value in case the counterparty C defaults

4. PDE for $\hat{V}$ when $M = \hat{V}$: When the MTM at default is given by

$$M = \hat{V}$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

5. **PDE for $\hat{V}$ when $M = V$**: When the MTM at default is given by

$$M = V$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)\hat{V}^- - (\lambda_B + R_C \lambda_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

6. **Credit Funding Adjustment when $M = V$**: Let

$$M = V$$

and

$$r_F = r + s_F$$

Then

$$\hat{V} = V + U$$

and the credit funding adjustment U is given by

$$\begin{aligned} U(t, S) = &-(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^-\big(u, S(u)\big)\right] du \\ &- (1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^-\big(u, S(u)\big)\right] du \\ &- \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^+\big(u, S(u)\big)\right] du \end{aligned}$$

where

$$D_k(t,u) = e^{-\int_t^u k(v)dv}$$

is the discount factor between times t and u using the rate k. If

$$s_F = 0$$

then U is identical to the regular bilateral CVA derived in many of the papers.

7. Justification of the Buy-Back Strategy: Another important result of Burgard and Kjaer (2012a) is the justification on which the seller's own-credit risk can be taken into account. In the hedging strategy considered the risk is hedged out by the seller buying back its own bonds. It is shown that the cash needed for doing so is generated through the replication strategy.

Model Setup and the Derivation of the Bilateral Risky PDE

1. Underlying Traded Assets and Securities: Consider an economy with the following four traded assets:
    a. P_R => Default-risk free zero-coupon bond
    b. P_B => Default-risky, zero-recovery, zero-coupon bond of party B
    c. P_C => Default-risky, zero-recovery, zero-coupon bond of counterparty C
    d. S => The Spot Asset with no Default Risk
2. Zero-Recovery Bonds Building Blocks: Both the risky bonds P_B and P_C pay \$1 at some future date T if the issuing counterparty has not defaulted, and 0 otherwise. These simplistic bonds are useful for modeling and can be used as building blocks for more complex bonds, including the ones with zero recovery.

3. Dynamics of the Underlying Assets: The processes for he assets P_B, P_C, P_R, and S under their corresponding probability measures are specified by

$$\frac{\Delta P_B}{P_B} = r_B(t)\Delta t - \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C(t)\Delta t - \Delta J_C$$

$$\frac{\Delta P_R}{P_R} = r(t)\Delta t$$

$$\frac{\Delta S}{S} = \mu(t)\Delta t - \sigma(t)\Delta W(t)$$

where $W(t)$ is a Weiner process,

$$r(t) > 0$$

$$r_B(t) > 0$$

$$r_C(t) > 0$$

and

$$\sigma(t) > 0$$

are deterministic functions of t and J_B and J_C are two point processes that jump from 0 to 1 on the default of B and C respectively.

4. Asset Independence of the Seller/Counterparty: The assumptions above indicate that the hedging can be done by P_B and P_C alone, and later this assumption is relaxed. Further the

spot asset price S is assumed to be unaffected by a default of either B or C. Finally B is taken to be the *seller* and C the *counterparty* respectively.

5. Terminal Derivative Payout at Maturity: It is assumed that the parties B and C enter a derivative on the spot asset S that pays the seller B the amount

$$H(S) \in \mathbb{R}$$

at maturity T. Thus in this convention the payout scenario

$$H(S) \geq 0$$

means that the seller receives cash or asset from the counterparty.

6. Credit Risky Derivative Present Value: The value of the derivative to the seller at time t is denoted $\hat{V}(t, S, J_B, J_C)$ and depends on the spot asset S of the underlying and the default states J_B and J_C respectively of the seller B and the counterparty C. Analogously $V(t, S)$ denotes the value to the seller of the same derivative as if it were a transaction between two default-free counterparties.
7. Default Derivative Close-out Claim: When party B or C default in general the mark-to-market on the derivative determines the close-out or the claim on the position. However the precise nature of this depends on the contractual details and the mechanism by which the mark-market is determined.
8. 2002 ISDA Master Agreement Specification: The 2002 ISDA Master Agreement specifies that the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the point of view of the surviving party) just prior to default, whereas the full mark-to-market has to be paid to the defaulting party if the mark-to-market value is negative (from the view of the surviving party). The master agreement specifies a dealer poll mechanism to establish the mark-to-market to the seller at default $M(t, S)$ without referring to the names of the counterparties involved in the derivative transaction.
9. Handling Imprecise ISDA Close-outs: From the above one would expect $M(t, S)$ to be close to $V(t, S)$ even though it is unclear if the dealers in the poll may or may not include their

funding costs in the derivative prices. In case the ISDA Master Agreement is not followed there may be other mechanisms involved. Therefore Burgard and Kjaer (2012a) derive the PDE for the general case $M(t, S)$ and consider the two special cases

$$M(t, S) = \hat{V}(t, S, 0, 0)$$

and

$$M(t, S) = V(t, S)$$

10. Default Close-out Boundary Conditions: Let

$$R_B \in [0, 1]$$

and

$$R_C \in [0, 1]$$

denote the recovery rates on the derivatives positions for parties B and C respectively – for now they are taken to be deterministic. The above discussions result in the following boundary conditions:

$$\hat{V}(t, S, 1, 0) = M^{+}(t, S) + R_B M^{-}(t, S)$$

when the seller defaults first and

$$\hat{V}(t, S, 0, 1) = R_C M^{+}(t, S) + M^{-}(t, S)$$

when the counterparty defaults first. Li and Tang (2009), Gregory (2009), and the vast majority of the papers on the valuation of the counterparty risk use

$$M(t, S) = V(t, S)$$

11. The Black Scholes Replicating Portfolio: As in the usual Black-Scholes framework the derivative s hedged with a self-financing portfolio that covers all the risk factors of the model. In this case the portfolio Π the seller sets up consists of $\alpha_S(t)$ units of S, $\alpha_B(t)$ units of P_B, $\alpha_C(t)$ units of P_C, and $\beta(t)$ units of cash such that the portfolio value at time t replicates the value of the derivative to the seller, i.e.,

$$\hat{V}(t) + \Pi(t) = 0$$

Thus

$$-\hat{V}(t) = \Pi(t) = \alpha_S(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t)$$

12. Counterparty Credit Hedge Ratio: Before proceeding note that when

$$\hat{V} \geq 0$$

the seller will incur a loss at the counterparty default. To hedge this loss P_C needs to be shorted, so we expect that

$$\alpha_C \leq 0$$

Assuming that the seller can borrow the bond P_C at a rate close to the risk-free rate r through a repurchase agreement, the spread λ_C between the rate r_C on the bond and the cost of financing the hedge position in C can be approximated to

$$\lambda_C = r_C - r$$

Since P_C is a bond with zero recovery this spread corresponds to the default intensity of C.

13. Own Credit Risk Hedge Ratio: On the other hand, if

$$\hat{V} < 0$$

the seller will gain at own default, which can be hedged by buying back P_B bonds, so one expects that

$$\alpha_C \geq 0$$

in this case. For this to work, enough cash must be generated, and any remaining cash after the purchase of P_B bonds is invested in such a way that it does not generate additional credit risk to the seller, i.e., any remaining positive cash generates a yield at the risk free rate of r.

14. Growth in the Cash Account: Imposing that the portfolio $\Pi(t)$ is self-financing implies that

$$-\Delta\hat{V}(t) = \alpha_S(t)\Delta S(t) + \alpha_B(t)\Delta P_B(t) + \alpha_C(t)\Delta P_C(t) + \Delta\bar{\beta}(t)$$

where the growth in the cash $\Delta\bar{\beta}(t)$ may be decomposed into

$$\Delta\bar{\beta}(t) = \Delta\bar{\beta}_S(t) + \Delta\bar{\beta}_F(t) + \Delta\bar{\beta}_C(t)$$

i.e., it is composed of three parts – the asset cash growth, the growth from the counterparty cash flow, and the growth from a unified (bond + cash) account.

15. Instantaneous Growth vs. Portfolio Rebalancing: The growth above is the growth in the cash account before re-balancing of the portfolio. The self-financing condition ensures that after Δt the rebalancing can happen at zero overall cost. This distinction is clarified in detail by Brigo, Buescu, Pallavicini, and Liu (2012).

16. Growth From the Asset Position $\Delta\bar{\beta}_S(t)$: The asset position provides a dividend income of $\alpha_S(t)\gamma_S(t)S(t)$ at a financing cost of $-\alpha_S(t)q_S(t)S(t)$ so

$$\Delta\bar{\beta}_S(t) = \alpha_S(t)[\gamma_S(t) - q_S(t)]S(t)\Delta t$$

The value of $q_S(t)$ may depend on the risk-free rate $r(t)$ and the repo rate of $S(t)$.

17. Growth From the Counterparty Position $\Delta\bar{\beta}_C(t)$: Using the arguments above, the seller will short the counterparty bonds using a repurchase agreement and incur financing costs of

$$\Delta\bar{\beta}_C(t) = -\alpha_C(t)r(t)P_C(t)\Delta t$$

Note the use of $r(t)$ instead of $r_C(t)$.

18. Positive/Negative Cash Account Asymmetry on $\Delta\bar{\beta}_F(t)$: From the above analysis any surplus cash held by the user after the own-bonds have been purchased must earn the risk-free rate $r(t)$ in order to not introduce any further credit risk to the seller. If borrowing money the seller needs to pay the rate $r_F(t)$. Thus the own bonds/cash account uses a funding/hedging scheme that is not symmetric.
19. Unsecured Funding vs. Derivative as Collateral: For this funding rate, the following two cases are distinct; where the derivative itself can be used as a collateral for the required funding, on no haircut $r_F(t)$ is set to $r(t)$. If however the derivative cannot be used as a collateral, the funding rate is set to the yield of the unsecured seller bond with recovery R_B, i.e.,

$$r_F(t) = r(t) + (1 - R_B)\lambda_B$$

20. Own Bond Cash Position Growth: In practice the second instance above is the more realistic case. Keeping $r_F(t)$ general for now,

$$\Delta\bar{\beta}_F(t) = \left[r(t)\left(-\hat{V} - \alpha_B P_B\right)^+ + r_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\right]\Delta t$$
$$= r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\Delta t$$

where the funding spread

$$s_F = r_F - r$$

i.e.,

$$s_F = 0$$

if the derivative cannot be used as a collateral, and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

21. Cumulative Cross Cash Growth Account: From the above analysis it follows that the total change in the cash account (here t is dropped from the notation where applicable to improve clarity) is given by

$$\Delta\bar{\beta} = (\gamma_S - q_S)\alpha_S S \Delta t + \left[r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-}\right]\Delta t - r\alpha_C P_C \Delta t$$

22. Change of the Replication Portfolio Value: Using the $\Delta\bar{\beta}$ computed above in

$$-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B \Delta P_B + \alpha_C \Delta P_C + \Delta\bar{\beta}$$

one gets

$$\begin{aligned}
-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B P_B (r_B \Delta t - \Delta J_B) + \alpha_C P_C (r_C \Delta t - \Delta J_C) + \Delta\bar{\beta} \\
+ \left[(\gamma_S - q_S)\alpha_S S + r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} - r\alpha_C P_C\right]\Delta t \\
= \left[-r\hat{V} + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} + (\gamma_S - q_S)\alpha_S S + (r_B - r)\alpha_B P_B \right. \\
\left. + (r_C - r)\alpha_C P_C\right]\Delta t - \alpha_B P_B \Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S
\end{aligned}$$

23. Ito's Lemma for Jump Diffusion: On the other hand, by Ito's lemma for jump diffusion and the assumption that a simultaneous jump of both the seller and the counterparty is a zero-probability event, the derivative value moves by

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial S^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

can be computed from

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

24. Asset and Bond Hedge Ratios: Equating the $\Delta\hat{V}$ from the cash growth and the Ito's lemmas, the following choice of α_S, α_B, and α_C eliminates all risks in the portfolio:

$$\alpha_S = \frac{\partial\hat{V}}{\partial S}$$

$$\alpha_B = \frac{\mathcal{D}\hat{V}_B}{P_B} = -\frac{\hat{V} - (M^+ + R_B M^-)}{P_B}$$

$$\alpha_C = \frac{\mathcal{D}\hat{V}_C}{P_C} = -\frac{\hat{V} - (M^- + R_C M^+)}{P_C}$$

25. Cash Account Evolution Expression: Hence the cash account evolution

$$\Delta\bar{\beta}_F(t) = r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^- \Delta t$$

can be rewritten as

$$\Delta\bar{\beta}_F(t) = [-r(t) R_B M^- - r_F(t) M^+]\Delta t$$

so the amount of cash deposited by the seller at the risk-free rate equals $-rR_B M^-$ and the amount borrowed at the funding rate r_F equals $-M^+$

26. Derivative as Collateral Black Scholes: Introducing the parabolic differential operator $\mathcal{A}_t$ as

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial S^2} + (\gamma_S - q_S) S \frac{\partial\hat{V}}{\partial S}$$

it follows that $\hat{V}$ is the solution to the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B \mathcal{D}\hat{V}_B - \lambda_C \mathcal{D}\hat{V}_C$$

$$\hat{V}(T, S) = H(S)$$

where

$$\lambda_B = r_B - r$$

and

$$\lambda_C = r_C - r$$

27. Incorporation of the Boundary Condition: Inserting

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with the boundary conditions

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B \mathcal{D}\hat{V}_B - \lambda_C \mathcal{D}\hat{V}_C$$

finally gives

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

$$\hat{V}(T, S) = H(S)$$

where the relation

$$\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ = (R_B M^- + M^+)^+ = M^+$$

has been used.

28. Fair Derivative Black Scholes Value: In contrast the risk-free value V satisfies the regular Black-Scholes PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = 0$$

$$V(T, S) = H(S)$$

Thus if one interprets λ_B and λ_C as effective default rates then the difference between $\hat{V}$ and V may be interpreted as follows.

29. Self/Counterparty Default Impact: The term $(\lambda_B + \lambda_C)\hat{V}$ is the additional growth rate the seller B requires on the risky asset $\hat{V}$ to compensate for the risk that default of either the seller the seller or the counterparty will terminate the derivative contract.
30. Receivables Funding Impact Hedge Strategy: The term $s_F M^+$ is the additional funding cost for negative values of the cash account for the hedging strategy.
31. Own Default Close-out: The term $-\lambda_B(R_B M^- + M^+)$ is the adjustment in the growth rate that the seller can accept because of the cash flows occurring at own default.
32. Counterparty Default Close-out: The term $-\lambda_C(R_C M^+ + M^-)$ is the adjustment in the growth rate that the seller can accept because of the cash flow occurring at the counterparty default.
33. Modeling of the Extinguisher Trade: The terms $(\lambda_B + \lambda_C)\hat{V}$, $-\lambda_B(R_B M^- + M^+)$, and $-\lambda_C(R_C M^+ + M^-)$ are related to counterparty risk whereas the term $s_F M^+$ represents the funding cost. From this interpretation it follows that the PDE for a so-called *extinguisher trade*, whereby it is agreed that no party gets anything at default, is obtained by removing the terms $-\lambda_B(R_B M^- + M^+)$ and $-\lambda_C(R_C M^+ + M^-)$ from

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

Using $\hat{V}(T, S)$ As Mark-to-Market at Default

1. Conceptually Simple Payout Condition: This section considers the case where the payments in default are based on $\hat{V}(T, S)$ so that

$$M(T, S) = \hat{V}(T, S)$$

in the boundary condition

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

Conceptually this is the simpler case since if the defaulting party is in the money with respect to a derivative contract, then there is no additional impact on the profit and the loss at the point of default.

2. Simplification of the PDE for $\hat{V}$: Similarly if the surviving party is in the money with respect to the derivative contract, then its loss is simply $(1 - R)\hat{V}$. In this case

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

reduces to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ - (1 - R_B)\lambda_B \hat{V}^- - (1 - R_C)\lambda_C \hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

where

$$s_F = 0$$

if the derivative can be posted as collateral and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

3. Own Credit And Counterparty Hedges: Moreover the hedge ratios α_B and α_C are given by

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

and

$$\alpha_C = -\frac{(1 - R_C)\hat{V}^+}{P_C}$$

so that

$$\alpha_B \geq 0$$

and

$$\alpha_C \geq 0$$

and the replication strategy generates enough cash $-\hat{V}^-$ for the seller to purchase back its own bonds.

4. Interpretation of the Own Credit Hedge: The cash available to the seller is $-\hat{V}^-$, of which the fraction $1 - R_B$ is invested in buying back the recovery-less bond B and the fraction R_B is invested risk-free. This is equivalent to investing a total amount of $-\hat{V}^-$ into purchasing a seller bond $\bar{B}$ with recovery R_B.
5. Credit Valuation Adjustment (CVA) Formulation: In the counterparty risk literature it is customary to write

$$\hat{V} = V + U$$

where U is called the credit valuation adjustment or the CVA. Inserting this representation into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

$$\frac{\partial V}{\partial t} + \mathcal{A}_tV - rV = 0$$

$$V(T, S) = H(S)$$

yields

$$\frac{\partial U}{\partial t} + \mathcal{A}_tU - rU = s_F(V + U)^+ + (1 - R_B)\lambda_B(V + U)^- + (1 - R_C)\lambda_C(V + U)^+$$

$$U(T, S) = 0$$

where V is known and acts as the source term.

6. Funding + CVA Feynman-Kac Integral: Furthermore one may apply the Feynman-Kac integral to the PDE for U, which with the additional assumption of deterministic interest rates produces the following non-linear integral equation:

$$U(t, S) = -(1 - R_B)\int_t^T \lambda_B(u) D_r(t, u)\mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^-\right]du$$
$$- (1 - R_C)\int_t^T \lambda_C(u) D_r(t, u)\mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^+\right]du$$
$$- \int_t^T s_F(u) D_r(t, u)\mathbb{E}_t\left[\left\{V\big(u, S(u)\big) + U\big(u, S(u)\big)\right\}^+\right]du$$

It follows that one can compute U first by computing V and then solving either the non-linear PDE above or the integral equation.

7. Receivable/Payable + Funding Numeraire: Before proceeding with the study of the two cases

$$s_F = 0$$

and

$$s_F = (1 - R_B)\lambda_B$$

it is worthwhile to examine a few instances where $\hat{V}$ corresponds to 1 UNIT seller receivable/payable, where those bonds may be with and without recovery.

8. The Seller sells 1 Unit Payable to the Counterparty:
    a. Zero-recovery Payable Numeraire => The first case considered is one unit sold by the seller to the counterparty C. In this situation

$$\hat{V} = \hat{V}^{-} = -1$$

and

$$R_B = 0$$

Since only deterministic rates and spreads are considered, there is no risk with respect to the underlying market factors, and the term $\mathcal{A}_t \hat{V}$ vanishes, so

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^{+} + (1 - R_B)\lambda_B \hat{V}^{-} + (1 - R_C)\lambda_C \hat{V}^{+}$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} = (r + \lambda_B)\hat{V} = r_B \hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T r_B(s) ds}$$

as expected for

$$\hat{V}(T, S) = -1$$

b. Non-zero Recovery Payable Numeraire => If on the other hand the recovery

$$R_B \neq 0$$

then

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

becomes

$$\frac{\partial \hat{V}}{\partial t} = \{r + (1 - R_B)\lambda_B\}\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T \{r(s) + (1 - R_B)\lambda_B(s)\} ds}$$

As expected the rate $r + (1 - R_B)\lambda_B$ payable on a bond with recovery is equal to the unsecured funding rate r_F that the seller has to pay on negative cash balances when the derivative cannot be posted as a collateral.

9. <u>The Seller Buys 1 Unit Receivable From C</u>:

a. Zero-Recovery Receivable Funding Numeraire => If $\hat{V}$ describes the purchase of 1 Unit receivable by the seller from the counterparty (i.e., the seller lends to the counterparty without recovery) then

$$\hat{V} = \hat{V}^{-} = +1$$

and

$$R_C = 0$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^{+} + (1 - R_B)\lambda_B \hat{V}^{-} + (1 - R_C)\lambda_C \hat{V}^{+}$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} = (r_F + \lambda_C)\hat{V} = [r_F + (r_C - r)]\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

b. Zero-Recovery Receivable Collateralized Numeraire => In this case, if the seller can use the derivative (i.e., the loan asset) as collateral for the funding of its short position within its replication strategy, then (neglecting haircuts)

$$r_F = r$$

the risk-free rate. Then the net result in this case is then

$$\frac{\partial \hat{V}}{\partial t} = r_C \hat{V}$$

so

$$\hat{V}(t) = -e^{-\int_t^T r_C(s)ds}$$

as expected for

$$\hat{V}_C(t) = P_C(t)$$

c. Non-zero Recovery Numeraire => If on the other hand

$$R_C \neq 0$$

then

$$\hat{V}(t) = -e^{-\int_t^T \{r(s)+(1-R_C)\lambda_C(s)\}ds}$$

as expected.

10. <u>The Case $r_F = r$</u>:
    a. PDE With Derivatives as Collateral => If the derivative can be posted as collateral the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

which is a non-linear PDE that needs to be solved numerically unless

$$\hat{V} \geq 0$$

or

$$\hat{V} \leq 0$$

b. Feynman-Kac Integral for $\hat{V}(t, S) \leq 0$ => Assuming that

$$\hat{V} \leq 0$$

i.e., the seller sold an option to the counterparty so

$$H(S) \leq 0$$

and that all rates are deterministic the Feynman-Kac representation of $\hat{V}$ is given by

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t, T) H\big(S(T)\big)\right]$$

where

$$D_k(t, T) = -e^{-\int_t^T k(s)ds}$$

is the discount factor over $[t, T]$ given that rate k.

c. Own-Credit Adjustment for Payables => Alternatively if for

$$\hat{V} \leq 0$$

we insert the ansatz

$$\hat{V} = V + U_0$$

where the zero subscript in U_0 indicates that the CVA U_0 is computed at

$$s_F = 0$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T,S) = H(S)$$

and apply the Feynman-Kac theorem, and finally use that

$$V(t,S) = D_r(t,u)\mathbb{E}_t\big[V\big(u,S(u)\big)\big]$$

one then gets

$$U_0(t,S) = -V(t,S)\left[\int_t^T (1-R_B)\lambda_B(u)D_{(1-R_B)\lambda_B(u)}(t,u)du\right]$$

d. Counterparty Adjustment for Receivables => When

$$\hat{V} \geq 0$$

i.e., the "seller" bought an option, symmetry yields that

$$U_0(t,S) = -V(t,S)\left[\int_t^T (1-R_C)\lambda_C(u)D_{(1-R_C)\lambda_C(u)}(t,u)du\right]$$

Thus one concludes that if

$$\hat{V} \leq 0$$

then U_0 depends only on the credit of the seller, whereas if

$$\hat{V} \geq 0$$

it depends only on the counterparty credit.

11. The Case $r_F = r + (1 - R_B)\lambda_B$:

a. The Derivative Cannot Serve as Collateral => If the derivative cannot be posted as collateral then

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\}\hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

which again is a non-linear PDE.

b. Feynman-Kac Integral for $\hat{V}(t, S) \leq 0$ => For

$$\hat{V} \leq 0$$

one writes

$$\hat{V} = V + U$$

and it is easy to see that

$$U = U_0$$

given in

$$U_0(t,S) = -V(t,S)\left[\int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B(u)}(t,u)du\right]$$

so $\hat{V}$ is given by

$$\hat{V}(t,S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H\big(S(T)\big)\right]$$

c. Feynman-Kac Integral for $\hat{V}(t,S) \geq 0$ => If

$$\hat{V}(t,S) \geq 0$$

then

$$\hat{V}(t,S) = \mathbb{E}_t\left[D_{r+k}(t,T)H\big(S(T)\big)\right]$$

where

$$k = (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$$

d. Own/Counterparty Credit Adjustment => Analogous to the case

$$r_F = r$$

one may make the ansatz

$$\hat{V} = V + U$$

and show that

$$U(t,S) = -V(t,S)\int_t^T k(u)D_k(t,u)du$$

Comparing this $U(t,S)$ with

$$U_0(t,S) = -V(t,S)\int_t^T (1-R_C)\lambda_C(u)D_{(1-R_C)\lambda_C}(t,u)du$$

shows that when the "seller" buys an option from the counterparty it encounters an additional funding spread

$$s_F = (1-R_B)\lambda_B$$

Using $V(T,S)$ As Mark-to-Market at Default

1. <u>$M(t,S) = V(t,S)$ Case for the PDE for $\hat{V}$</u>: This section considers the scenario where the payments in the case of default are based on V and hence use

$$M(t,S) = V(t,S)$$

in the boundary conditions

$$\hat{V}(t,S,1,0) = M^+(t,S) + R_B M^-(t,S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

The PDE for $\hat{V}$

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

then becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)V^- - (R_C \lambda_C + \lambda_B)V^+ + s_F V^+$$

$$\hat{V}(T, S) = H(S)$$

2. Own-Credit and Counterparty Hedges: The above PDE is linear and has a source term on the right hand side. On writing

$$\hat{V} = V + U$$

the hedge ratios become

$$\alpha_B = \frac{U + (1 - R_B)V^-}{P_B}$$

and

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C}$$

Comparing α_B with

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

shows that in the current case default triggers a windfall cash flow of U that needs to be taken into account in the hedging strategy.

3. Valuation Adjustment Feynman Kac Integrals: Writing

$$\hat{V} = V + U$$

also gives the following PDE for U:

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+$$

$$U(T, S) = 0$$

Thus application of the Feynman-Kac theorem yields

$$U(t, S) = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^-\left(u, S(u)\right)\right]du$$

$$- (1 - R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^+\left(u, S(u)\right)\right]du$$

$$+ \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t\left[V^+\left(u, S(u)\right)\right]du$$

The CVA U can be calculated using $V(t,S)$ as a known source term when solving the PDE in the integral or in the differential form.

4. Derivatives as Cash Account Collateral: In the case where we can use the derivative as a collateral for the funding of our cash account, i.e.,

$$s_F = 0$$

the last term in the expression for $U(t,S)$ above vanishes and the equation reduces to the regular bilateral CVA derived in many papers and books cited earlier, for example Gregory (2009).

5. Zero Funding Cost Credit Valuation Adjustment: The bilateral benefits in this case does not come from any own default, but from being able to use the cash generated from the hedging strategy and buy back own bonds, thus generating an excess return of $(1 - R_B)\lambda_B$. This CVA that corresponds to the case

$$M = V$$

and

$$s_F = 0$$

is denoted U_0.

6. Unsecured Funding Rate Valuation Adjustment: In practice, however, the derivative cannot normally be used as a collateral and

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\big[V^-\big(u,S(u)\big)\big]du$$
$$-(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\big[V^+\big(u,S(u)\big)\big]du$$
$$+\int_t^T s_F(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\big[V^+\big(u,S(u)\big)\big]du$$

gives a consistent adjustment of the derivative prices for bilateral counterparty risk and funding costs. In the specific case where the funding spread corresponds to that of the unsecured B bond (with recovery R_B), i.e.,

$$s_F = (1-R_B)\lambda_B$$

the first and the third terms may be merged and U may be re-written as

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\big[V\big(u,S(u)\big)\big]du$$
$$-(1-R_C)\int_t^T \lambda_C(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\big[V^+\big(u,S(u)\big)\big]du$$

7. Payable and Receivable Funding Impact: The first and the last term of the previous equation now not only contain the bilateral asset described above, but also the funding liability arising from the fact that the higher rate

$$r_F = r + (1-R_B)\lambda_B$$

is paid when borrowing for the hedging strategy's cash account.

Funding and Default Payoff Examples

1. Funding and Default Payoff Scenarios: In this section the total derivative value $\hat{V}$ for a call option bought by the "seller" is computed in the following 4 cases:
   a. Case I =>

$$M = \hat{V}$$

   and

$$s_F = 0$$

   b. Case II =>

$$M = \hat{V}$$

   and

$$s_F = (1 - R_B)\lambda_B$$

   c. Case III =>

$$M = V$$

   and

$$s_F = 0$$

   d. Case IV =>

$$M = V$$

and

$$s_F = (1 - R_B)\lambda_B$$

2. <u>One-Sided "Bought" Call CVA</u>: A bought call is a one-sided trade that satisfies

$$V \geq 0$$

and

$$\hat{V} \geq 0$$

and furthermore if the rates are constant the CVAs U/U_0 for the four cases above simplifies to

a. Case I =>

$$U_0 = -V(t,S)\left[1 - e^{-(1-R_C)\lambda_C(T-t)}\right]$$

b. Case II =>

$$U = -V(t,S)\left[1 - e^{-\{(1-R_B)\lambda_B+(1-R_C)\lambda_C\}(T-t)}\right]$$

c. Case III =>

$$U_0 = -V(t,S)\frac{(1 - R_C)\lambda_C}{\lambda_B + \lambda_C}\left[1 - e^{-(\lambda_B+\lambda_C)(T-t)}\right]$$

d. Case IV =>

$$U = -V(t,S)\frac{(1-R_B)\lambda_B + (1-R_C)\lambda_C}{\lambda_B + \lambda_C}\left[1 - e^{-(\lambda_B+\lambda_C)(T-t)}\right]$$

3. CVA and Funding Impact Analysis: Thus all the four CVA's are linear in $V(t,S)$. All are negative since the seller faces counterparty risks and funding costs when

$$s_F = 0$$

but does not have any bilateral asset because of the one-sidedness of the option payoff. Further the effect of the funding cost is significantly larger than choosing

$$M = \hat{V}$$

or

$$M = V$$

for a bought option. For a sold option the impact of the funding cost does not have any effect.

Counterparty Funding and PDE Extensions

1. Liquid Markets CVA/Funding Impact: The valuation adjustments presented here are particularly relevant when pricing interest rate swaps and vanilla options since these markets are very liquid and having an analytical model that does not take fully all costs into account may consume all profits from a deal depending upon the funding and the credit spreads.
2. Derivatives with more General Payouts: Though the examinations above were conducted on a simple one-asset, one-derivative Black Scholes framework, the results can be immediately

extended to derivatives with more general payments than $H(S(T))$. These could be Asian options or interest rate swaps.

3. <u>Netted Portfolios with Multiple Trades</u>: In this case the vales V and $\hat{V}$ represent the net derivatives portfolio value rather than the value of a single derivative.
4. <u>Generalized Multi-asset Diffusion Dynamics for Multiple Underlyings</u>: The only restriction is that the asset price SDE's satisfy technical conditions such that the option pricing PDE (now multi-dimensional) admits a unique solution given by the Feynman-Kac representation. Note that if the number of assets exceeds two or three it is computationally more efficient to compute the CVA using Monte-Carlo simulation combined with numerical integration rather than solving the high-dimensional PDE.
5. <u>Stochastic Interest Rates</u>: This is essential for interest rate derivatives, and the effect would be that the discounting in the CVA expression would happen inside the expectation operator.
6. <u>Stochastic Hazard Rates</u>: One way of introducing default time dependence and right/wrong way risk would be to make λ_B and λ_C stochastic and correlate them with each other and with other market factors. This would, again, simply imply that we do not move the discount factors outside of the expectation operator in

$$
\begin{aligned}
U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_r(t,u) \mathbb{E}_t\left[\{V(u,S(u)) + U(u,S(u))\}^-\right] du \\
- (1-R_C)\int_t^T \lambda_C(u) D_r(t,u) \mathbb{E}_t\left[\{V(u,S(u)) + U(u,S(u))\}^+\right] du \\
- \int_t^T s_F(u) D_r(t,u) \mathbb{E}_t\left[\{V(u,S(u)) + U(u,S(u))\}^+\right] du
\end{aligned}
$$

or in

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\big(u,S(u)\big)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$
$$+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

Also the generator $\mathcal{A}_t$ would incorporate terms corresponding to the new stochastic state variables.

7. Explicitly Specified Default Time Dependence: Another way of specifying default time dependence is by specifying simultaneous defaults. This could be done by setting J_0, J_1, and J_2 be independent point processes and then setting

$$J_B = J_0 + J_1$$

and

$$J_C = J_0 + J_2$$

This approach is the well-known Marshall-Olkin copula and would require some basket default instrument for perfect replication. The hazard rates λ_0, λ_1, and λ_2 of J_0, J_1, and J_2 could be made stochastic in which case the right and the wrong way risk can be modeled as well.

Balance Sheet and Funding Cost Management

1. Funding Position Dependent Cost Adjustment: In both Piterbarg (2010) and Burgard and Kjaer (2012a) the size of the funding cost adjustment is dependent on the specific way the

funding is achieved and thus gives rise to prices that are dependent on the funding position of the issuer. The counterparty would clear the price with the best funding position.

2. Funding Cost as Seller Windfall: Burgard and Kjaer (2012b) show that the funding cost term is related to the windfall to the issuer's bondholders upon default of the issuer. This leads them to examine the impact of the derivative asset and the funding positions on the balance sheet from within the confines of a simple balance sheet model.
3. Funding Strategy Balance Sheet Impact: They demonstrate that this impact on the balance sheet and the overall funding position of the issuer reduces the effective marginal funding spread for new positions to zero.
4. Strategies for Balance Sheet Impact Neutralization: Burgard and Kjaer (2012b) discuss two strategies for how the balance sheet impact can directly be neutralized, mitigating the need for a funding cost adjustment to a derivatives price. If such strategies can be put into practice, they lead back to a state where the symmetric prices between the issuer and the counterparty can be achieved.

Unified Framework for Bilateral Counterparty Risk and Funding Adjustments

1. CVA, DVA, and FVA Expressions: Burgard and Kjaer (2012b) list the explicit integral representations for the CVA, the DVA, and the FCA.

$$U(t, S) = CVA + DVA + FCA$$

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^+\big(u, S(u)\big)\right] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V^-\big(u, S(u)\big)\right] du$$

$$FVA = -\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V^+\left(u, S(u)\right)\right] du$$

2. <u>Analysis of the XVA Expressions</u>: It is clear from the above expression that while both the CVA and the FVA are related to the credit position of the issuer, they do not double-count the issuer's credit, but capture the exposures of the mark-to-market value of the derivatives of the opposite sign and as such are opposite sides of the same coin. The DVA term itself can be seen as a funding benefit as it arises from the issuer using a positive cash account to buy back its own bonds, earning the spread on it while at the same time hedging out own credit risk on the derivatives position.
3. <u>Own Credit Hedge Using Bonds</u>: It should be noted that the hedging strategy leading to the set of expressions above involves the issuer re-purchasing its own bonds. It does not involve any dealings in the CDS. The term λ_B is the spread of a zero-recovery bond over the risk-free rate. It is not the hazard rate derived from the CDS market. Thus if there is a basis between the bonds and the CDS for the issuer B, it is the bond market that counts in determining the λ_B used in the DVA and the FVA terms.
4. <u>Contribution of U to the Hedges</u>: The contributions of U to the hedge ratios

$$\alpha_B = \frac{U + (1 - R_B)V^-}{P_B}$$

and

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C}$$

on the other hand comes from the fact that, upon default the close-out amount of V differs from the risky value of the derivative just prior to the default by the amount U because the credit and the funding adjustments for the trade disappear on default of the counterparty or the issuer. Thus a credit risk on the full amount of U needs to be hedged out and this is achieved by means of taking positions in zero-recovery bonds B and C corresponding to the full value of U.

5. Symmetry between the CVA and the DVA: Without the FVA term the risky value $\hat{V}$ would be symmetric in B and C, i.e., the counterparty and the issuer, following the same methodology, would agree on the same price. If however the funding costs for the replication strategy are included, and the funding spread is non-zero, then the two parties hedging out their risks and pricing in their funding costs would not agree on the price. A counterparty that wants to buy this derivative would buy it from the seller with the smallest FVA value, i.e., the lowest funding cost.
6. Balance Sheet Funding Costs Mitigation: Burgard and Kjaer (2012b) also clarify the origins of the FVA term within the ambit of their framework in more detail, and outline different ways of how the funding costs can be mitigated. Doing so successfully can reduce the FVA cost to zero and produce prices that are independent of the funding costs of the issuer, and therefore symmetric.

Simple Model for the Impact of Derivative Asset on Balance Sheet and Funding

1. Balance Sheet and Funding Model: As discussed earlier, derivatives and their funding positions contribute to the issuer asset and liability positions on the issuer default. Therefore they themselves should impact the funding costs of then issuer. Burgard and Kjaer (2012b) quantify this feedback effect using a simple balance sheet and funding model.
2. Pre-trade Asset and Liability: Assuming that, as in a reduced form credit model, the default of the issuer is driven by an instantaneous default process with default intensity λ. Prior to entering into the derivative contract, let A_0 be the expected assets on default of the issuer and L_0 be the liabilities, so that the expected recovery on default is

$$R_0 = \frac{A_0}{L_0}$$

3. Pre-trade Instantaneous Incremental Funding Cost: Within this simple setup, the funding spread s_F of the issuer over the risk-free rate that compensates for the expected loss upon its default is

$$s_F = (1 - R_0)\lambda$$

Thus the instantaneous funding cost f_0 over the time Δt for a total liability L_0 is

$$f_0 \Delta t = [r + (1 - R_0)\lambda] L_0 \Delta t$$

4. Addition of a Derivative Transaction: Let the seller now add a derivative with a positive value d as an asset, resulting in total assets of

$$A_1 = A_0 + d$$

The positive value d corresponds to $-V^-$ in the previous analysis. The corresponding negative cash is funded by adding a corresponding liability giving a total new liability of

$$L_1 = L_0 + d$$

Thus the new expected recovery is now

$$R_1 = \frac{A_1}{L_1} = \frac{A_0 + d}{L_0 + d}$$

5. Post-Trade Incremental Funding Cost: Assuming that the seller has hedged the market and the counterparty risk, the addition of the derivative asset does not change the default intensity of the seller. Thus the instantaneous funding cost after adding the derivative is

$$
\begin{aligned}
f_1\Delta t = [r + (1 - R_1)\lambda]L_1\Delta t &= r(L_0 + d)\Delta t + (L_1 - A_1)\lambda\Delta t \\
&= rL_0\Delta t + rd\Delta t + (L_0 - A_0)\lambda\Delta t = rd\Delta t + rL_0\Delta t + (1 - R_0)\lambda L_0\Delta t \\
&= rd\Delta t + f_0\Delta t
\end{aligned}
$$

6. Effective Incremental Post-Trade Funding Cost: Thus the effective funding cost for the additional liability d is $rd\Delta t$. While the new liability d draws a new funding spread $(1 - R_1)\lambda$ the change on the recovery and its effect on the funding of the total liabilities results in an effective funding rate for d that is the risk-free rate. Thus within this balance sheet model the spread s_F is zero.
7. Balance Sheet Funding Cost Mitigation: While this balance sheet model is somewhat simplistic, it shows that the proper accounting for the effects of the derivative assets on the balance sheet can mitigate the funding costs and bring the FVA terms down to zero. With a vanishing FVA term, the equation

$$U = CVA + DVA$$

yields an adjustment U and a risky value $\hat{V}$ that are symmetric between the issuer B and the counterparty C.
8. Operational Challenges with the above Approach: Practically, however, the challenge is an operational one in that the benefit of the balance sheet impact is difficult to pin down at the moment of trading and hedging the derivative contract and therefore difficult to allocate as a benefit to the derivatives trading desk.

Balance Sheet Management to Mitigate Funding Costs

1. Issuer Windfall and Liability Matchup: Another way to shield the balance sheet from the impact of both the derivative asset and the funding liability is to actively manage the balance

sheet in such a way that the windfall from the derivative asset and the funding position upon default of the issuer is balanced out by a corresponding liability.

2. Multiple Classes of Issuer Bonds: The above can be achieved if the issuer can freely trade two of its own bonds P_1 and P_2 with different recovery rates R_1 and R_2, i.e., different seniority.
3. Dynamics of the Bonds and the Assets: The setup is changed from before in that there are now 4 hedging instruments - P_1, P_2, P_C, and S. All the positive and negative cash in the cash account is invested/raised by buying back and/or issuing P_1/P_2 bonds. The assets follow the dynamics

$$\frac{\Delta P_1}{P_1} = r_1 \Delta t - (1 - R_1) \Delta J_B$$

$$\frac{\Delta P_2}{P_2} = r_2 \Delta t - (1 - R_2) \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C \Delta t - \Delta J_C$$

$$\frac{\Delta S}{S} = \mu \Delta t - \sigma \Delta W$$

where

$$R_1 \in [0, 1)$$

and

$$R_2 \in [0, 1)$$

and

$$R_1 < R_2$$

Neither of the recoveries R_1 or R_2 need equal the derivative recovery rate R_B.

4. Payoff Replication Hedge Portfolio: As before the replicating hedge portfolio Π is setup and given by

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \alpha_S S + \beta_S + \beta_C$$

where

$$\beta_S = -\alpha_S S$$

is the funding account for the asset position and

$$\beta_C = -\alpha_C P_C$$

is the funding position for the P_C bonds.

5. Application of the Self-Financing Criterion: The fact that Π is meant to be a replicating self-financing hedge portfolio implies that

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 = -\hat{V}$$

$$\Delta\Pi = -\Delta\hat{V}$$

so repeating the delta-hedging arguments of Burgard and Kjaer (2012a) and defining

$$s_1 = r_1 - r$$

and

$$s_2 = r_2 - r$$

yields

$$\alpha_S = -\frac{\partial \hat{V}}{\partial S}$$

$$\alpha_1(1 - R_1)P_1 + \alpha_2(1 - R_2)P_2 = \mathcal{D}\hat{V}_B$$

$$\alpha_C P_C = \mathcal{D}\hat{V}_C$$

$$\alpha_1 s_1 P_1 + \alpha_2 s_2 P_2 + \alpha_C \lambda_C P_C = -\frac{\partial \hat{V}}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial t^2} + r\hat{V} - (q_S - \gamma_S)S\frac{\partial \hat{V}}{\partial S}$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with $\hat{V}(t, S, 1, 0)$ and $\hat{V}(t, S, 0, 1)$ given by

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = M^-(t, S) + R_C M^+(t, S)$$

with

$$M = V$$

6. Hedge Ratios and Consolidated PDEs: From the above equations one can determine α_1 and α_2 to be

$$\alpha_1 = -\frac{R_2\hat{V} - V^+ - R_B V^-}{(R_2 - R_1)P_1}$$

$$\alpha_2 = -\frac{-R_1\hat{V} + V^+ + R_B V^-}{(R_2 - R_1)P_2}$$

which implies the following pricing PDE:

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_1\frac{R_2\hat{V} - V^+ - R_B V^-}{R_2 - R_1} + s_2\frac{-R_1\hat{V} + V^+ + R_B V^-}{R_2 - R_1} - \lambda_C\left(V^- + R_C V^+ - \hat{V}\right)$$

7. Zero Bond Funding Spread Basis: If furthermore one assumes zero differential between the bond basis and the funding spread, i.e.,

$$s_1 = (1 - R_1)\lambda_B$$

and

$$s_2 = (1 - R_2)\lambda_B$$

then the previous equation simplifies to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)V^- - (\lambda_B + R_C \lambda_C)V^+$$

8. Issue Senior and Repurchase Junior: Comparing this PDE with

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)V^- - (\lambda_B + R_C \lambda_C)V^+ + s_F V^+$$

   implies financing of the negative cash account at vanishing spread

$$s_F = 0$$

   In practice the strategy involves issuing the senior P_2 bonds and using some of the proceeds to re-purchase the junior and hence higher-yields P_1 bonds.
9. Excess Return Balancing the Funding Costs: The excess return generated by this strategy exactly offsets the funding costs so the net financing rate becomes r. At the same time the combined positions of the P_1 and the P_2 bonds ensures there is no windfall to the bondholders in the case of default of the issuer while V is positive (and the cash account negative).
10. Junior/Senior Positions at Default: These are shown in the table below which summarizes the total bond positions at the issuers own default and which partially offsets the value of the derivative defined in

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

   and

$$\hat{V}(t, S, 0, 1) = M^-(t, S) + R_C M^+(t, S)$$

P_1 Position Value	$\frac{R_1 R_2 \hat{V} - R_1 V^+ - R_1 R_B V^-}{R_2 - R_1}$
P_2 Position Value	$-\frac{-R_2 R_1 \hat{V} + R_2 V^+ + R_2 R_B V^-}{R_2 - R_1}$
Total Position Value	$-(V^+ + R_B V^-)$

11. Default Windfall Balancing out Funding Costs: Thus if the issuer is able to offset the impact of the derivative and its funding on the balance with combination of going long senior bonds and shot junior bonds, then the windfall is effectively monetized while the issuer is alive, and by doing so the funding cost term is reduced to zero.

Funding Strategies and Costs Impact

1. Abstract: The economic values of derivatives depends on their funding costs, because they can result in windfalls or shortfalls to bondholders on their firm's default. But this depends on not just who is funding them, but how – so the resulting adjustments depend on the funding strategy deployed. It is another layer of complexity to derivatives pricing, as argued by Burgard and Kjaer (2013).
2. Derivative ITM vs OTM Funding: Assuming that the issuer can hedge its own default when the derivative position is in the money and so provides funding is more realistic than assuming that the issuer can freely dynamically trade spread positions on its own bonds. When the derivative is out of the money and the issuer requires funding, a post-default windfall to the issuer's estate is generated. In that case a funding cost adjustment (FCA) is added in to compensate. There have been a flurry of papers prosing alternative approaches from different authors, including Morini and Prampolini (2012), Brigo, Pallavicini, and Perini (2012), Crepey (2013a, 2013b), and perhaps most famously Hull and White (2012a) that use risk-neutral valuation principles to examine the question.
3. Classical Price with Bilateral CVA: Such an approach, discounting all expected cash flows at the risk free rate results in the classical price with the bilateral CVA. But this disregards the

preferences of the different stake holders regarding the value of the pre- and post- own-default cash flows. This is justified if all the risks – including own default – are hedgeable so that net post-default cash flows are zero. But, as mentioned, they are not.

4. Impact on the Shareholders: Shareholders are primarily interested in the pre-default cash flows of derivatives and their hedges, but post-default cash flows matter for bondholders, as they contribute to the recovery realized. Shareholders only care about the latter through the balance sheet effects that are in practice hard to realize and account for.
5. Selective Disregarding of Default Cash flows: Some authors have considered cases where the post-default cash flows on the funding leg are disregarded, but not the ones of the derivative. But it is not clear why some post-default cash flows should be disregarded but not others, and without specifying the funding strategies, the resulting recursive relations cannot be solved.
6. Crepey's Generalization Approach: Crepey's generalization of the original Markovian framework to non-Markovian processes using backward stochastic equations is elegant, but difficult to solve explicitly.
7. Selective Hedging of Default Cash Flows: Burgard and Kjaer (2013) look at funding strategies in terms of holding or issuing own bonds. The strategies hedge out some but not all cash flows at own default. The economic value of a derivative to the shareholders is then given by assuming that they disregard any remaining post-default cash flows and pre-default balance sheet effects.
8. Implementation of the Custom Funding Strategies: The funding cost adjustment is then given by the discounted expected value of the post-default cash flows. The strategies previously considered are then special cases of those considered by Burgard and Kjaer (2013), thus generalizing the previous work. Dealers can consider their own strategies and decide which adjustments represent the economic funding costs they expect to change while in business.
9. CSA Variants and Set-offs: Burgard and Kjaer (2013) also consider boundary conditions covering practical cases such as one-way or two-way credit support annexes (CSAs) governing collateral agreements and the so called set-offs. Set-offs are particularly interesting as they mitigate the need for funding cost adjustments. Their explicit calculations and numerical results show that different funding strategies can yield quiet different funding adjustments and asymmetries in the absence of set-of provisions.

Generalized Semi-Replication and Pricing PDE

1. Imperfect Own Credit Hedge: Consider a derivative contract, possibly collateralized, between an issuer B and a counterparty C with an economic value $\hat{V}$ that incorporates the risk of the counterparty and the issuer and any net funding costs the issuer may encounter prior to own default. This section describes a general semi-replication strategy that the issuer can deploy to perfectly hedge out any market factors and counterparty default, but which may provide a perfect hedge in the event of the issuer's "own" default.
2. Portfolio of Tradeable Instruments: The tradeable instruments used in this strategy are a counterparty zero-coupon zero-recovery bond P_C, two issuer "own" bonds P_1 and P_2 of different seniorities, i.e., different recoveries R_1 and R_2 respectively, and a market instrument S that can be used to hedge out the market factor for the derivative contract (e.g., stock).
3. Underlying Assets and Market Factors Dynamics: The setup can be easily extended to many market factors. The following standard dynamics for these instruments are assumed

$$i = 1,2$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C {P_C}^{-} \Delta t - {P_C}^{-} \Delta J_C$$

$$\Delta P_i = r_i {P_i}^{-} \Delta t - (1 - R_i) {P_i}^{-} \Delta J_B$$

where J_B and J_C are the default indicators for B and C respectively, and

$$P_{i/C}^{-} = P_{i/C}(t^{-})$$

are the pre-default bond prices.

4. Risk-Neutral Bond-Funding Spread: Without loss of generality, P_1 is the junior bond, i.e.,

$$R_1 < R_2$$

and

$$r_1 > r_2$$

In case of zero basis between bonds of different seniority, it is trivial to show that

$$r_i - r = (1 - R_B)\lambda_B$$

where r is the risk-free rate, λ_B corresponds to the spread of a potentially hypothetical zero-recovery zero-coupon bond of the issuer.

5. Generalized Boundary Conditions at Default: Let $\hat{V}(t, S, J_B, J_C)$ be the total economic value of the derivative to the issuer. Using Kjaer (2011) the general boundary conditions at default of the issuer or the counterparty are given by

$$\hat{V}(t, S, 1, 0) = g_B(M_B, X)$$

if B defaults first, and

$$\hat{V}(t, S, 0, 1) = g_C(M_C, X)$$

if C defaults first with general close-out amounts M_B and M_C, and collateral X.

6. Collateral Extensions to Boundary Conditions: If

$$M_B = M_C = V$$

these boundary conditions are called “regular”, where V is the classic Black-Scholes price of the derivative, i.e., without counterparty and own-default risks and no funding costs. For example the regular bilateral boundary conditions with collateral are defined as

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

7. Extensions to Alternate Close-outs: Burgard and Kjaer (2012a) consider alternate close-out cases for

$$M_B = M_C$$

and Brigo and Morini (2011) extend this to the cases where

$$M_B \neq M_C$$

and include the cost of funding in the close-out amounts. Separately Burgard and Kjaer (2012c) apply the present framework to such funding aware close-outs.

8. Regular Bilateral Uncollateralized Close-outs: An example for the close-out functions are the regular bilateral close-outs with collateral, which are described by

$$g_B = V^+ + R_B V^-$$

and

$$g_C = R_C V^+ + V^-$$

Later other examples such as one-way CSA and set-offs are examined.

Semi-Replication

1. The Semi-replication Hedge Portfolio: For the semi-replication the hedge portfolio Π is set up as

$$\Pi(t) = \alpha_S(t)S(t) + \alpha_1(t)P_1(t) + \alpha_2(t)P_2(t) + \alpha_C(t)P_C(t) + \beta_S(t) + \beta_C(t) - X(t)$$

with $\alpha_S(t)$ units of $S(t)$, $\alpha_{1/2}(t)$, and $\alpha_C(t)$ units of own and counterparty bonds respectively, cash accounts $\beta_S(t)$ and $\beta_C(t)$, and a collateral account $X(t)$.

2. The Asset Financing Cash Accounts: The cash accounts β_S and β_C are used to finance the S and the P_C positions, i.e.,

$$\alpha_C P_C + \beta_C = 0$$

and

$$\alpha_S S + \beta_S = 0$$

and assumed to pay net rates of $(q_S - \gamma_S)$ and q_C respectively, where γ_S may be the dividend income. The hedge positions may be collateralized or repo'ed, so q_S and q_C maybe the collateral or the repos rates respectively.

3. The Replication Portfolio Funding Constraint: The derivative collateral balances X are assumed to be fully re-hypothecable and pay the amount r_X and

$$X > 0$$

corresponds to the counterparty having posted the amount X with the issuer. The strategy shall be designed such that

$$\hat{V} + \Pi = 0$$

except possibly at the issuer default. The issuer bond positions $\alpha_1 P_1$ and $\alpha_2 P_2$ are used to finance/invest any remaining cash that is not funded via the collateral, which yields the following *funding constraint*

$$\hat{V} - \Pi + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

4. Evolution of the Derivative Hedge Portfolio: The evolution of the hedge portfolio Π defined in

$$\Pi = \alpha_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta_S + \beta_C - X$$

is given by

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

where ΔS, ΔP_1, ΔP_2, and ΔP_C are given earlier and $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ are changes to the cash and the collateral accounts, excluding rebalancing.

5. Funding Costs for Cash Accounts: As in Burgard and Kjaer (2012a) the hedge account β_S is collateralized with the financing rate q_S and income via dividend γ_S.
6. Growth in the Cash Accounts: Likewise the counterparty bond position is assumed to be setup via a repo transaction costing a repo rate q_C. The derivative collateral account is assumed to cost a collateral rate r_X. Excluding rebalancing these yield the following increments in the accounts:

$$\Delta\bar{\beta}_S = \alpha_S S(\gamma_S - q_S)\Delta t$$

$$\Delta\bar{\beta}_C = -\alpha_C q_C P_C \Delta t$$

$$\Delta\bar{X} = -r_X X \Delta t$$

7. Incremental Change in the Derivative Portfolio: With the pre- and the post-default values of the issuer bond position given by

$$P = \alpha_1 P_1 + \alpha_2 P_2$$

and

$$P_D = \bar{R}_1 \alpha_1 P_1 + \bar{R}_2 \alpha_2 P_2$$

respectively, inserting

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C P_C{}^{-} \Delta t - P_C{}^{-} \Delta J_C$$

and

$$\Delta P_i = r_i P_i{}^{-} \Delta t - (1 - R_i) P_i{}^{-} \Delta J_B$$

and the expressions for $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ into

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

results in

$$\Delta\bar{\Pi} = [r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C\alpha_C P_C + \alpha_S S(\gamma_S - q_S) - r_X X]\Delta t + (P_D - P)\Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S$$

where

$$\lambda_C \equiv r_C - q_C$$

is the spread of the zero-coupon bond price P_C yield over its repo rate, i.e., the financing rate of the counterparty default position.

8. Ito's Lemma for the Derivative Contract: The evolution of the derivative $\Delta\hat{V}$ on the other hand is given by Ito's lemma for jump diffusions as

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

with

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) = g_B - \hat{V}$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) = g_C - \hat{V}$$

9. The Consolidated Derivative Portfolio Increment: Combining the evolution of the derivative in

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

with the hedge portfolio

$$\Delta\bar{\Pi} = [r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\alpha_CP_C + \alpha_SS(\gamma_S - q_S) - r_XX]\Delta t + (P_D - P)\Delta J_B - \alpha_CP_C\Delta J_C + \alpha_S\Delta S$$

gives

$$\Delta\hat{V} + \Delta\bar{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \frac{1}{2}\sigma^2S^2\frac{\partial^2\hat{V}}{\partial t^2} + r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\alpha_CP_C + \alpha_SS\frac{\partial\hat{V}}{\partial S}(\gamma_S - q_S) - r_XX\right]\Delta t + (g_B + P_D - X)\Delta J_B + \left(\mathcal{D}\hat{V}_C - \alpha_CP_C\right)\Delta J_C + \left(\frac{\partial\hat{V}}{\partial S} + \alpha_S\right)\Delta S$$

where the term in front of ΔJ_B follows from the fact that

$$\hat{V} - P + X = 0$$

from the funding constraint

$$\hat{V} - X + \alpha_1P_1 + \alpha_2P_2 = 0$$

10. Asset and Counterparty Hedges: From the derivatives portfolio increment we can eliminate the stock price and the counterparty risks by choosing

$$\mathcal{D}\hat{V}_C = \alpha_CP_C$$

and

$$\alpha_S = -\frac{\partial\hat{V}}{\partial S}$$

which yields

$$\Delta\hat{V} + \Delta\bar{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1P_1 + r_2\alpha_2P_2 + \lambda_C\mathcal{D}\hat{V}_C - r_XX\right]\Delta t + (g_B + P_D - X)\Delta J_B$$

where

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2S^2\frac{\partial^2\hat{V}}{\partial t^2} + \alpha_SS\frac{\partial\hat{V}}{\partial S}(\gamma_S - q_S)$$

11. <u>Incorporating the Hedge Ratios</u>: On using the zero bond basis relation

$$r_i - r = (1 - R_B)\lambda_B$$

$$\hat{V} - X + \alpha_1P_1 + \alpha_2P_2 = 0$$

and

$$\mathcal{D}\hat{V}_C = \alpha_C - \hat{V}$$

the above becomes

$$\Delta\hat{V} + \Delta\bar{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_XX + \lambda_Bg_B + \lambda_Cg_C - \epsilon_h\lambda_B\right]\Delta t + \epsilon_h\Delta J_B$$

where

$$s_X = r_X - r$$

and

$$\epsilon_h = g_B + P_D - X$$

12. Own-Credit Default Hedging Error: From the jump term above it follows that upon issuer default there is a hedge error of size ϵ_h. While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h \lambda_B$ per unit time.
13. Hedge Error Windfall and Shortfall: It can thus be seen that the combination of the derivative $\hat{V}$ and the hedge portfolio Π is risk free as long as the issuer is alive. At issuer default the jump term $\epsilon_h \Delta J_B$ gives rise to a hedge error of size ϵ_h. The hedge error can be a windfall or a shortfall and its size depends on the post-default value of the own bond portfolio, and thus the funding strategy employed.
14. Windfall/Shortfall Accrual Gain/Bleed: While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h \lambda_B$ per unit time. Tis can be seen as the running spread to pay for the potential windfall/shortfall upon issuer default.
15. Collateral/Hedge Error Derivative PDE: Since the issuer wants the strategy to evolve in a self-financing fashion while he is alive the total drift term above should become zero. This produces the following PDE for the risky economic value $\hat{V}$ for the derivative:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \epsilon_h \lambda_B$$

$$\hat{V}(T, S) = H(S)$$

where $H(S)$ is the payout of the derivative at maturity.

16. PDE for the Valuation Adjustment: To estimate the correction

$$U = \hat{V} - V$$

to the risk-free Black Scholes price V the Black-Scholes PDE for V is used to get the PDE for U to be

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C) U = s_X X - \lambda_B (g_B - V) - \lambda_C (g_C - V) + \epsilon_h \lambda_B$$

$$U(T, S) = H(S)$$

17. Decomposition of U into Components: Applying the Feynman-Kac theorem to this PDE gives

$$U = CVA + DVA + FVA + COLVA$$

with

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V(u) - g_C\big(V(u), X(u)\big)\right] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t\left[V(u) - g_B\big(V(u), X(u)\big)\right] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[X(u)] du$$

$$D_k(t, u) = e^{-\int_t^u k(v) dv}$$

is the discount factor between t and u for a rate k. The measure of the expectations in these equations is such that S drifts at the rate $q_S - \gamma_S$. The sum of DVA and FCA is sometimes referred to as FVA.

18. Symmetry of CVA, DVA, and $COLVA$: Here the sum of the CVA, the DVA, and the $COLVA$ is symmetric in that is identical – with sign flipped – when computed by the issuer and the counterparty, respectively.
19. Lack of Symmetry in FCA: The $FCA - \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$ on the other hand is the discounted survival probability weighted expected value of the hedge error ϵ_h implied by the semi-replication strategy chosen. Because the hedge error on own default is different for the issuer and for the counterparty, the FCA is not symmetric. This is a generalization of the result presented in Burgard and Kjaer (2012b) for regular bilateral close-outs and a particular choice of bonds that states that the FCA is the cost of generating a windfall to the issuer bondholders in the case of a default. If the issuer wants to break-even while being alive, the cost has to be included in the derivative price charged to the counterparty.
20. Asset Addition Funding Rate Impact: The analysis above assumes that the funding rates r_1 and r_2 remain unaffected by the addition of the derivative asset and the funding positions. Burgard and Kjaer (2012b) have noted that the presence of potential windfall has a positive balance sheet effect; it improves the recovery rate to the bondholders and therefore the funding spread of the issuer should go down. Hull and White (2012a) have used a similar argument.
21. Balance Sheet Funding Cost Mitigation: For a simple balance sheet model with floating funding costs Burgard and Kjaer (2012b) have demonstrated that this effect can result in an effective marginal funding rate that corresponds to the risk-free rate.
22. Practical Challenges with the Mitigation: However as discussed there, in practice the balance sheet effect on the funding costs is rather indirect, fraught with accounting issues, and in general only feed through over time. Therefore the current treatment assumes that the issuer disregards this effect.

Examples of Different Bond Portfolios

1. Strategies Generating Different Hedge Errors: Using the general framework developed above Burgard and Kjaer (2013) provide three different examples of semi-replication strategies that generate different hedge errors ϵ_h and therefore different valuation adjustments.
2. Zero FCA and Windfall Only: The first strategy, if employed, allows for perfect replication and generates zero FCA. The second one is equivalent to the setup used in Burgard and Kjaer (2012a) for the bilateral close-outs and ensures that there is never a shortfall at issuer default and only a windfall (potentially).
3. Windfall or Shortfall Generation Strategy: The third strategy assumes hedging with a single issuer bond. It generates both potential windfall and potential loses post-default and is an extension of the model derived in Piterbarg (2010).
4. Strategy Economic Value and Adjustments: The different strategies generate different economic values – and therefore different adjustments – to the issuer while he is alive. They demonstrate the assumptions implicitly being made when using different adjustment formulas in practice.
5. Bilateral and Funding Curve Close-outs: Throughout it is assumed that the close-out value is V, i.e., that

$$M_B = M_C = V$$

   Funding aware close-outs are discussed separately in Burgard and Kjaer (2012c).
6. Generalized Bond Portfolio Hedge Ratio:

$$\alpha_1 = \frac{\epsilon_h + R_2\hat{V} - g_B + (1 - R_2)X}{(R_1 - R_2)P_1}$$

$$\alpha_2 = \frac{\epsilon_h + R_1\hat{V} - g_B + (1 - R_1)X}{(R_2 - R_1)P_2}$$

Perfect Replication – The FCA Vanishes

7. Hedging Windfalls and Shortfalls: The first case considered is the one where the issuer is able to perfectly hedge out the windfall/shortfall at own default. This corresponds to the case discussed in the Section *Balance-Sheet Management to Mitigate Funding Costs* in Burgard and Kjaer (2012b) and also covers the risk-neutral approach outlined in Hull and White (2012a).
8. PDE Corresponding to the Perfect Hedge: Perfect hedge is equivalent to the hedge error ϵ_h being zero, i.e.,

$$g_B + P_D - X = 0$$

The valuation PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

Correspondingly the FCA given by

$$FCA = \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

vanishes.

9. Corresponding Own Portfolio Hedge Ratios: The hedge ratios α_1 and α_2 that achieve this perfect replication are determined by the no-windfall condition

$$g_B + P_D - X = 0$$

and the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

These two conditions provide two equations that can be solved to find

$$\alpha_1 = \frac{R_2 \hat{V} - g_B + (1 - R_2) X}{(R_1 - R_2) P_1}$$

$$\alpha_2 = \frac{R_1 \hat{V} - g_B + (1 - R_1) X}{(R_2 - R_1) P_2}$$

10. Valuation Adjustment Feynman-Kac Integrals: As an example, for the regular bilateral conditions

$$g_B = (V - X)^+ + R_B (V - X)^- + X$$

and

$$g_C = R_C (V - X)^+ + (V - X)^- + X$$

these adjustments specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r + \lambda_B + \lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

with

$$FCA = 0$$

and

$$COLVA = 0$$

if

$$r_X = r$$

11. Case of Classical Bilateral CVA: If the derivatives are not collateralized, i.e.,

$$X = 0$$

these adjustments correspond to the classical bilateral CVA. Thus the classical bilateral CVA can be achieved when perfect replication is possible. As mentioned, in practice such a dynamic balance sheet management via actively traded spread options between junior and senior bonds is in general not a viable option.

Semi-Replication with No Shortfall at Own-Default

1. Collateralized without Shortfall at Default: Burgard and Kjaer (2013) demonstrate a bond portfolio and hedging strategy that constitute an equivalent to the one presented in Burgard

and Kjaer (2012b) extended to more general conditions, including the possibility of collateral.

2. Dynamic Trading of Two Bonds: It still involves dynamic trading of two bonds, but is more conservative than the dynamic trading strategy of the previous section as it does not aim at monetizing the potential windfall upon own default by entering into an offsetting position between the two bonds.
3. Bilateral CVA Plus a DVA: While this generates potential windfalls at own default, it does not generate shortfalls, and has the additional advantage that for regular bilateral close-outs without collateral it results in the usual bilateral CVA adjustment plus a funding cost adjustment, so presents a simple extension to the existing framework, where the derivatives dealer does not think he can monetize the windfall.
4. P_1/P_2 Buy-Sell Strategy: The strategy involves a zero recovery bond P_1 with

$$R_1 = 0$$

and a recovery bond

$$R_2 = R_B$$

The issuer runs the following bond positions:

    a. Invest of fund the difference between $\hat{V}$ and V by buying or issuing P_1 bonds.
    b. Hold the number of P_2 bonds given the following funding constraint:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

5. Corresponding Own-Portfolio Hedge Ratios: The strategy is this defined by the following values of α_1 and α_2:

$$\alpha_1 P_1 = -(\hat{V} - V) = U$$

$$\alpha_2 P_2 = -\alpha_1 P_1 - \hat{V} + X = -(V - X)$$

6. <u>Collateral Adjusted Junior Bond Hedge</u>: The strategy is thus symmetric between positive and negative funding, and the risk-free value V not covered by the collateral X is funded/invested via own unsecured bonds with recovery R_B. Only the adjustment U, which falls away on own-default, is funded/invested via a zero-recovery bond. As such this strategy looks more palatable from a regulatory and accounting perspective than the perfect replication strategy which attempts to actively extract the funding spread from the balance sheet by issuing own senior bonds to buy back own junior bonds.
7. <u>Regular Bilateral Close-out Case</u>: This bond portfolio is equivalent to the one described in Burgard and Kjaer (2012a, 2012b) for the bilateral close-outs considered there. This section analyzes the setup in more detail for the case of regular bilateral close-out given in

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

For this case the hedge error ϵ_h specializes to

$$\epsilon_h = (1 - R_B)(V - X)^+$$

which is always a windfall – possibly zero – to the bondholders of the issuer.
8. <u>Zero-Recovery Own Bond Off-setter</u>: Therefore this strategy is characterized by the ability of the issuer to perfectly hedge out the difference U between the risky value of the derivative before own default and the close-out amount after own default by means of trading own bonds at zero-recovery.
9. <u>R_B Recovery Own Bond Balance</u>: The remainder, i.e., the difference between the close-out and the collateral is invested/funded using own bonds with recovery R_B.This part generates

the windfall ϵ_h to the issuer's bondholders when $V - X$ is in the money and the issuer defaults.

10. <u>CVA, DVA, FCA, and $COLVA$</u>: The CVA, the DVA, the FCA, and the $COLVA$ specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

11. <u>Consolidation of the DVA and the FCA</u>: It is possible to combine the DVA (a funding benefit) and FCA (a funding cost) into a funding value FVA as

$$FVA = DVA + FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[V(u) - X(u)] du$$

Note that Hull and White (2012a, 2012b) refer to this FCA as FVA. Burgard and Kjaer (2013) use the notation that is consistent with their previous papers and with Gregory (2012).

12. <u>Uncollateralized Classical Bilateral CVA/DVA</u>: For uncollateralized derivatives

$$X = 0$$

and the $COLVA$ term vanishes. The CVA and the DVA then correspond to the classical bilateral CVA. The FCA term provides the funding cost adjustment on top.

13. Gold-Plated Two-Way CSA: For gold-plated two-way CSAs where

$$X = V$$

the CVA, the DVA, and the FVA terms are all zero. If the collateral rate is the risk-free rate, then the $COLVA$ – which represents the spread earned by the issuer – also vanishes. In this case the risky price $\hat{V}$ of the derivative becomes the risk-free price V as well. The intuition is that the collateral cash is exactly what is needed to fund the hedge and eliminate all counterparty risk and as a consequence

$$\alpha_1 = \alpha_2 = 0$$

This corresponds to the result for fully collateralized trades of Piterbarg (2010).

14. One-Way CSA Issuer Posting: Another special case worth considering is that of a one-way CSA whereby the issuer only posts collateral when the risk-free value of the trade is out-of-the-money, i.e.,

$$X = V^-$$

One-way CSAs are common when the issuer trades with sovereign counterparties or with sovereign-like public entities.

15. One-Way CSA Valuation Adjustments: In this case the adjustments specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)^+] du$$

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)^+] du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)^-] du$$

Unsurprisingly the introduction of one-way CSA makes the DVA vanish while leaving the CVA and the FCA terms unchanged.

16. Implication of One-Way CSA: Unlike the uncollateralized case any cash available when the derivative is out-of-the-money must be handed over as collateral and thus cannot be used to generate a funding benefit. And unlike two-way CSA there is no influx of collateral cash that can be used to fund the hedge when the derivative is in-the-money. The issuer is thus faced with the uncollateralized and the 2-way cases, and need to charge a higher price to the counterparty to compensate for that in order to break even.

Set-offs

1. Definition of the Set-off Mechanism: It is also instructive to study the case of the so-called set-offs. A set-off is a legal agreement that allows the surviving party to settle the outstanding derivative claims of the defaulting party by means of supplying the bonds of the defaulting party at nominal value rather than cash.
2. Boundary Conditions for Set-offs: Since post-default these bonds trade at their recoveries, this type of settlement is valuable to the surviving party. Explicitly for regular bilateral set-offs without collateral, the boundary conditions are given by

$$g_B = R_B V$$

and

$$g_C = R_C V$$

Inserting this into

$$\alpha_1 P_1 = -U$$

and

$$\alpha_2 P_2 = -(V - X)$$

implies that the hedge error ϵ_h disappears.

3. Set-offs CVA and DVA: The CVA and the DVA adjustments in this case are given by

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[V(u)] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[V(u)] du$$

4. Issuer and Counterparty Price Symmetry: Significantly with

$$\epsilon_h = 0$$

the FCA term vanishes resulting in symmetric prices between the issuer and the counterparty. Thus adoption of close-outs would mitigate the need for economic funding for adjustments.

Semi-Replication with a Single Bond

1. Strategy P_1/P_2 Hedge Ratios: This section considers a very simple strategy where the issuer uses a single own bond with recovery R_F, i.e.,

$$\alpha_1 P_1 = 0$$

and

$$\alpha_2 P_2 = -\left(\hat{V} - X\right) = -(V + U - X)$$

where the second line follows from the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

For aesthetic reasons the remaining bond P_F and its yield are relabeled

$$r_F = r + s_F$$

The hedge ratios imply that the issuer raises all necessary net cash by issuing P_F-bonds and invests any surplus net cash by repurchasing the same bonds.

2. Insufficient Hedge Degrees of Freedom: With a single bond, once the funding constraint is fulfilled, there are no degrees of freedom left for the issuer to hedge out his own default. This is in contrast to the previous setup of strategy I where the issuer is able to hedge out its own default risk when the trade is out-of-the-money at least.
3. Hedge Error Produced by the Strategy: For the own bond portfolios with the hedge ratios above the hedge error ϵ_h amount to

$$\epsilon_h = g_B + P_D - X = g_B - R_F\hat{V} - (1 - R_F)X$$

where the default value of the own bond portfolio given by

$$P_D = -R_F\left(\hat{V} - X\right)$$

has been used.

4. <u>Recursive Setup Formulation for the FCA</u>: Using this hedge error in the general adjustment expression

$$FCA = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[\epsilon_h(u)]du$$

yields a recursive relationship. This is because

$$\hat{V} = V + U$$

appearing on the RHS of the expression for ϵ_h above includes the contribution from U and thus the FCA itself.

5. <u>Elimination of Recursion with the Original PDE</u>: The best way to deal with this situation is by going back to the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

and insert the hedge error above to obtain

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r_F + r_C)\hat{V} = -(r_F - r_C)X - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

6. Uncollateralized Discounting-with- Funding Approach: The boundary condition g_B does not enter – this is because in this strategy there is no attempt to hedge own default. It is worth noting that for uncollateralized trades, i.e., trades with

$$X = 0$$

and zero counterparty risk –

$$\lambda_C = 0$$

– the PDE specializes to a simple funding-with-discounting approach as in Piterbarg (2010).
7. Special Case – Discounting-with-Funding: Thus discounting-with-funding is a special case of this strategy and assumes that the issuer deals with any funding requirement or surplus by using a single funding instrument and is happy to generate a windfall or shortfall upon own default.
8. PDE for Gross Valuation Adjustment: Similarly inserting the hedging error

$$\epsilon_h = g_B + P_D - X = g_B - R_F\hat{V} - (1 - R_F)X$$

into the PDE

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_B(g_B - V) - \lambda_C(g_C - V) + \lambda_B \epsilon_h$$

$$U(T, S) = 0$$

gives the adjustment U for this strategy as

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_C)U = -\lambda_C(g_C - V) + s_F(V - X) + s_X X$$

$$U(T, S) = 0$$

9. <u>Collateralized Regular Bilateral Close-outs</u>: This step carries out an analysis of the regular bilateral close-outs with collateral as given in

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

For these

$$g_C - V = -(1 - R_C)(V - X)^+$$

Applying the Feynman-Kac theorem we obtain

$$U = CVA_F + DVA_F + FCA_F + COLVA_F$$

with

$$CVA_F = -(1 - R_C)\int_t^T \lambda_C(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^+]du$$

$$DVA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^-]du$$

$$FCA_F = -\int_t^T s_F(u) D_{r_F+\lambda_C}(t,u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$COLVA_F = -\int_t^T s_X(u) D_{r_F+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

10. <u>DVA_F and FCA_F as FVA_F</u>: Combining DVA_F and FCA_F into FVA_F results in

$$FVA_F = DVA_F + FCA_F = -\int_t^T s_F(u) D_{r_F+\lambda_C}(t,u) \mathbb{E}_t[V(u) - X(u)] du$$

11. <u>Comparison with the No Shortfall Case</u>: These adjustments are very similar to the ones in the strategy described in *semi-replication with no shortfall at own default* except that the discounting used is $D_{r_F+\lambda_C}(t,u)$ rather than $D_{r_F+\lambda_B+\lambda_C}(t,u)$. There is no reference to λ_B, only to the funding rate r_F of the P_F bond used in the own-bond portfolio of the semi-replication strategy.
12. <u>Inapplicability as Generalized Valuation Adjustment</u>: It should be noted that the adjustments CVA_F, DVA_F, FCA_F, and $COLVA_F$ are not direct specializations of the general adjustments defined in

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_C\big(V(u), X(u)\big)\right] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_B\big(V(u), X(u)\big)\right] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

In particular FCA_F does not correspond anymore to the discounted expectation of the hedge error ϵ_h upon default.

13. <u>FCA As Collateralized Adjustment Difference</u>: Obviously the expectation above for FCA_F still equals the difference between $CVA_F + FVA_F + COLVA_F$ and the classical bilateral CVA with collateral. The FCA of

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

can thus be calculated as

$$FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$$

The adjustments of the special cases of uncollateralized, gold-plated 2-way CSA, and 1-way CSA can then be derived easily equivalently to those in the strategy *semi-replication with no shortfall at own default*.

14. <u>Simplicity and Relevance of the Strategy</u>: The strategy specified above is thus very simple to understand and implement. It is also of practical relevance not least because dealers who simply discount by the funding rate assume this strategy implicitly (for zero counterparty risk), including potential windfalls and shortfalls to their estate upon own default.

Burgard and Kjaer (2013) Case Study

1. Strategy Specific Valuation Asymmetry Estimation: Burgard and Kjaer (2013) provide an illustrative case study for the generalized bilateral CVA – the sum of CVA from

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_C\big(V(u), X(u)\big)\big] du$$

DVA from

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_B\big(V(u), X(u)\big)\big] du$$

and the generalized FCA from

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

and the valuation asymmetries for strategies I and II computed from the perspective of the issuer and the counterparty (i.e., with counterparty's FCA) respectively together with the issuer's hedge error $\epsilon_h(t_0)$ in case of immediate default of the issuer for three sample trades. For strategy II the FCA is computed as per

$$FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$$

2. ITM, ATM, and OTM Swaps: The sample trades are 10Y swaps with \$100m notional where the issuer pays fixed and receives 6M LIBOR floating. Burgard and Kjaer (2013) consider different close-out provisions for 3 fixed rates – 3.093% (OTM), 2.693% (ATM), and 2.293% (ITM).
3. Dealer Specific Funding Value Adjustments: The adjustments computed from the perspective of the counterparty show that if both sides include their funding costs their economic values

may be far apart and the two parties may not agree on the deal. If the counterparty does not include the funding costs in general it will deal with the issuer with the lowest funding costs.

4. Differences between Strategies I and II: As shown in Burgard and Kjaer (2013) the differences between the adjustments of the strategies I and II are in general not particularly big (they increase with funding rate), but as expected strategy II has potentially significant shortfalls upon issuer default, whereas strategy I only generates windfalls.
5. Impact of using Setoffs: As discussed earlier, when using set-offs the impact of the funding cost is mitigated. When following the strategy I the FCA vanishes completely and symmetric prices are obtained. Even when implementing strategy II the FCA prices are pretty small and the prices are close to being symmetric. Setoff close-outs are an attractive way of mitigating the need for funding cost adjustments.
6. Asymmetric Valuation and Hedge Error: Burgard and Kjaer (2013) show how asymmetric valuation and hedging error can vary across funding strategies I and II outlined above, and how they interact with CSA's and set-offs. Each in turn prices an out-of-the money (OTM), at-the-money (ATM), and in-the-money (ITM) 10Y \$100m swap with the OTM and the ITM swaps 40 bp either side of the ATM level.
7. Impact of the Issuer Bond Spread: Issuer bond spreads are considered at 100 bp and 500 bp respectively, while the counterparty spread is set constant at 300 bp. Total adjustments are calculated for the issuer and the counterparty as well as the bilateral CVA and the hedging error. Uncollateralized one-way CSAs in the counterparty's favor, as well as cases including a set-off are also considered.
8. Adjustment Impact between the Sides: The adjustments differ between the two counterparties creating an asymmetry in the derivative's valuation. The degree of this and the resulting hedging error depends on the funding and the collateralization strategy. In the 100 bp case the uncollateralized swap has a valuation asymmetry of 50 bp of the notional with the magnitude decreasing from OTM through ATM to ITM.
9. One-Way CSA Adjustment Impact: The introduction of the one-way CSA increases the size of the issuer's adjustment but reduces the asymmetry to 20-30 bp with the amount of reduction skewed from the opposite direction from ITM to OTM. This is because under the

one-way CSA the issuer has to post more collateral for an OTC swap, thus reducing the funding benefit.

10. XVA Metrics for Strategy I: Strategy I has zero hedge error for OTM and ITM, but produces roughly 2% notional hedging error in the ITM case regardless of the existence of the CSA. In the presence of the set-off the FCA is eliminated, and so the total adjustment is equal to the bilateral CVA.
11. XVA Metrics for Strategy II: Strategy II is more complex, but the valuation asymmetry is dramatically reduced – by roughly a factor of 10 in the case study. The hedging error is also reduced with the biggest reduction coming for the ITM case. When the issuer is ITM and defaults, the setoff implies that the counterparty can pay back the full present value of the trade using the issuer bond notional rather than cash, which reduces the post-default bond-holder windfall.
12. Directional Dependence on the Issuer Spread: The interesting point is that the quantitative findings of the study are independent of the issuer's bond spread – only the magnitudes change with a greater proportional reduction in the valuation asymmetry and the hedging error of the strategy II in the presence of a set-off, for instance – as can be seen from the results for the 500 bp case.

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Accounting for OTC Derivatives: Funding Adjustments and Re-hypothecation Option

Status of Current FCA/FBA Accounting

1. Motivation: Banks hold and routinely exercise the option of freely re-hypothecating variation margin across counter-parties and trades.
2. FCA/FBA Accounting Standards: However, the emerging FCA/FBA cost accounting metrics for funding costs are mostly formulated in terms of netting set specific metrics that fail to properly account for re-hypothecation benefits to common Equity Tier 1 Capital (CET1).
3. Double Counting in FCA/FBA: Additionally, the FCA/FBA standard introduces a double counting issue between the funding benefits and the DVA which leads ultimately to the violation of the fundamental accounting tenet of asset-liability symmetry.
4. FVA/FDA Accounting Objectives: Albanese and Andersen (2014) propose an alternative accounting framework meant to rectify some of the problems in existing standards. This new accounting method, which they call FVA/FDA, explicitly incorporates the re-hypothecation option into its definition of funding costs, and maintains consistency with the Modigliani-Miller Theorem, with fair-value and asset-liability symmetry principles, and with Basel III rules for DVA and equity capital.
5. Pricing at the CET1 Indifference Level: They argue that derivative pricing necessitates an incremental assessment of capital structure impact on new trades and propose that the entry prices should be struck at the indifference level for CET1.
6. Departure From FCA/FBA Accounting: Unlike the FCA/FBA method, the FVA/FDA accounting does not result in outright net-income write-offs due to funding costs.

Comparison Between FCA/FBA and FDA/FVA

1. Where are they similar: FCA/FBA accounting and FVA/FDA accounting lead to very similar and quantitatively close conclusions in the particular case of a portfolio consisting of a single netting set and a single trade.
2. Portfolio Sets and Portfolio Sizing: However material differences arise in the case of large portfolios. Albanese and Andersen (2014) discuss a case study with a representative portfolio whereby the CET1 adjustments for funding are 3 times as large as the ones required in FVA/FDA accounting.
3. Impact on Prices and VAs: After the portfolio effects are accounted for, incremental entry prices for individual trades differ materially between the FCA/FBA and the FVA/FDA methods, with the FCA/FBA accounting often displaying sizeable and risky pricing biases between derivative payables and receivables.
4. Abbreviations and Expansions:

Abbreviation	Expansion
A	Asset Account
BCBS	Basel Committee on Banking Supervision
CA	Contra-Asset
CCP	Central Counter-party (i.e., a Clearing House)
CDS	Credit Default Swap
CET1	Common Equity Tier 1 Capital
CL	Contra-Liability
CFD	Central Funding Desk
CSA	ISDA Credit Support Annex Agreement
CVA	Credit Valuation Adjustment, same as FTDCVA

CVA_{CL}	Contra-Liability entry for Credit Valuation Adjustment
DVA	Debt Valuation Adjustment
DVA2	Funding Debt Adjustment (same as FDA)
EE	Expected Exposure
ENE	Expected Negative Exposure
EPE	Expected Positive Exposure
FBA	Funding Benefit Adjustment
FCA	Funding Cost Adjustment
FDA	Funding Liability Adjustment (same as DVA2)
FTDCVA	First-to-default CVA, same as CVA
FTP	Funding Transfer Pricing
FVA	Funding Valuation Adjustment
PFV	Portfolio Fair Valuation
KVA	Capital Valuation Adjustment
L	Liability Account
OIS Rate	Overnight Index Swap Rate
OTC	Over-the-Counter
RE	Retained Earnings
REPO	Re-purchase Agreement
RHO	Re-hypothecation Option Benefit
SFVA	Symmetric Funding Value Adjustment

UCVA	Unilateral CVA
VM	Variation Margin
XVA	"X" Valuation Adjustment (short-hand for all valuation adjustments, such as CVA, DVA, FVA, etc.)

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Funding and Re-Hypothecation Adjustment - Motivation

OTC vs. Repo Markets

1. Repo Market Infrastructure: One key indicator of trade liquidity is the existence of an efficient REPO market infrastructure in support of market-making activities. In a liquid markets, such as those trading government bonds, security acquisitions may be financed by reverse REPO trades.
2. Funding in OTC Derivatives Market: The OTC derivatives market, on the other hand, is not directly supported by a REPO infrastructure, and uncollateralized derivatives must instead be financed by other means.
3. OTC Hedges Variation Margin Funding: As derivatives receivables are an inefficient form of collateral, the rates banks face when funding variation margin (VM) for their hedges for uncollateralized derivatives are typically close to those for unsecured funding.

Modus Operandi of Funding Desks

1. Funding the Derivatives Hedging Cost: Without an efficient REPO market infrastructure, managers cannot prevent or hedge the wealth transfer with unsecured derivatives trading. Instead they must seek to recoup the loss to shareholders by passing on the cost to the clients.
2. The Funding Cost Transfer Chain: At a more granular level, this cost transfer commonly goes through a chain that starts when a bank treasury issues unsecured debt to raise funds needed for uncollateralized derivatives trading activities.
3. The CFD/CVA Desks: In the standard bank setup, the running spread cost of the bank issuance is subsequently passed by the treasury to a central funding desk (CFD) which

consolidates the management of the funding costs on behalf of the bank's trading functions. In many cases the CFD is merged with the CVA desk in a unified trading operation.

4. FTP Policy and Client Pricing: The CFD, in turn, transfers costs of funding to business line desks on an upfront basis by implementing a funding transfer pricing policy. Finally business line desks charge the costs to clients by embedding them into deal structures.

MMT And Asset-Liability Symmetry

1. Modigliani-Miller Theorem and Indifference Pricing: According to the famous Modigliani-Miller Theorem (MMT – Modigliani and Miller (1958)) the indifference price of new trade should not depend the rate on which the bank funds the VM on the hedges.
2. Wealth Transfer Across the Capital Structure: However, inefficient funding strategies may still give rise to wealth transfer across the capital structure of the bank, directed from the shareholders to the senior creditors (Albanese and Iabichino (2013)).
3. Funding Charges in Accounting Statements: While estimates for funding costs have been used and incorporated into funding costs informally for decades, it is only post-crisis that the banks have attempted to recognize these costs in official accounting statements by adding funding related valuation adjustments (FVA) to the existing CVA and DVA credit risk adjustments.
4. FVAs and Asset Write-downs: Despite unresolved controversies in the literature, in the last quarter of 2013 funding costs were reflected in the accounts at various banks including JP Morgan, Deutsche Bank, Nomura, and others. Not only did these give rise to very material adjustments to CET1, they also led to, due to asymmetries in the fair-value adjustments, asset write-downs that at least in one case exceeded $1 billion (JP Morgan Press Release (2013)).

Rigorous Framework For Funding Costs

1. Impact of Basel Committee Recommendations: The question of how and whether to include funding costs in accounts is complicated by the recent decision of the national regulators to accept the recommendation by the Basel Committee (Basel Committee On Banking Supervision (2012)) and mandate the exclusion of the DVA and other own-credit benefits from CET1.
2. Impact of DVA Capital Exclusion: The decision is relevant in the context of funding strategies as popular accounting methods for funding costs are inter-twined with the definition (or sometimes re-definition) of the DVA. In a realistic case-study example, Albanese and Andersen (2014) show that the impact of the DVA capital exclusion is to triple the CET1 deductions for funding whenever the FCA/FBA accounting method is used.
3. Motivation for the Rigorous Funding Valuation: Further, Albanese and Andersen (2014) show that such adverse impacts are to a large extent due to logical faults in the FCA/FBA accounting, and are not justified in a more rigorous framework for funding costs.

Funding Set VM RHO Computation

1. Rationale Behind the VM RHO Computation: The calculation of the CET1 deductions and the FTP amounts within lines of businesses are essentially always model based and depend strongly on a variety of assumptions and approximations.
2. VM Re-hypothecation Across Netting Sets: Of particular relevance here is how the re-hypothecation option (RHO) for the VM is treated. As banks are allowed to re-hypothecate across netting sets in the same business line portfolio, the RHO is valuable and routinely exercised through the shifting of the cash collateral hedges from receivable hedges to payable hedges.
3. Funding Set Portfolio RHO Valuation: As a consequence, rigorously computed funding cost adjustment is an aggregate portfolio level amount that cannot be linearly decomposed across netting sets. Specifically re-hypothecation dictates that funding costs be calculated through a portfolio level simulation with scenarios that are shared at the *finding set* level, whereby a

funding set is defined as a portfolio of unsecured or partly collateralized trades and their corresponding hedges among which the variation margin can be re-hypothecated.

Shortcomings of Traditional CVA Systems

1. Funding Cost Valuation Implementation Challenges: Modeling challenges here get intertwined with technology implementation difficulties. Funding cost calculations that require aggregation of trade and collateral values across netting are rarely a good fit for traditional CVA systems optimized for individual netting sets.
2. CVA Systems Retrofit for RHO: In particular modeling of re-hypothecation using common distributed computing setups is often awkward compared to an in-memory architecture where all counter-party credits are simulated dynamically and all scenarios are shared.
3. Approximations to Funding Cost Calculations: To bypass technical implementation difficulties and to re-use grid-based CVA systems, many market participants have implemented an approximate accounting method for funding costs based on metrics that are additive over netting sets.
4. FCA/FBA Netting Set Additivity: The popular FCA/FBA accounting method was designed with linear aggregation in mind. However the netting set additivity inherent in the FCA/FBA accounting typically overlaps between the funding benefits (as captured by the FBA) and the DVA on the derivative payables.
5. Adjusting for DVA Double Counting: The resulting double-counting is handled normally by an outright replacement of DVA with FBA (Castagna (2011), Caccia (2013)). This replacement inevitably intertwines the DVA on payables (a CET1 deduction) with the RHO (which should instead *add* to CET1).

Addressing the Shortcomings of FCA/FBA Accounting

1. Rigorous Modeling of the RHO: To address the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) examine an alternate accounting method – denoted FVA/FDA – on which the RHO is modeled rigorously and care is taken to make entries of financial statements as meaningful as possible.
2. "Going Concern" Viewpoint of Accounting: Reconciling financial statements is often a delicate task as the accrual principles and the "going concern" viewpoint of financial accounting inevitably clashes with the notion of fair market value, DVA, and balance-sheet wealth transfers.
3. Accounting Theory Consistency of the FVA/FDA: Unlike the FCA/FBA Accounting method, the FVA/FDA accounting method is simultaneously consistent with the MMT, the risk-neutral pricing, and the general accounting principles such as asset-liability symmetry. Further in FVA/FDA accounting there is no double-counting of the DVA, and the funding cost adjustment to CET1 has the correct directional dependence with respect to the bank's own credit spread.
4. Impact on Income, CET1, and Price: Consistently with the MMT, the FVA/FDA funding value adjustments do not impact income, but do affect both CET1 and the entry price levels. This should provide sufficient incentives for the trading personnel to manage their funding costs properly, something that cannot be said for classical accounting principles that ignore funding costs entirely.

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Albanese and Andersen (2014) Results Summary

Valuation Adjustment Estimation Framework Setup

1. Book of Uncollateralized OTC Trades: Albanese and Andersen (2014) consider an OTC book containing trades with multiple unsecured counterparties, alongside back-to-back hedges with dealers or clearing houses. The unsecured counterparties do not post VM in full while hedges are fully collateralized.
2. Rate on the VM Collateral: In case the unsecured book is a net receivable, the hedge book is net payable, and the bank needs to procure VM to hedge counterparties. In this situation the bank receives a rate of OIS on the collateral posted as VM.
3. Definition of Funding Value Adjustment: The funding value adjustment (FVA) is defined as the PV of the carry cost of funding VM, net of the OIS receipts on the posted collateral.

OTC Books Funding Set Decomposition

1. Definition of the Derivative Funding Sets: OTC books are decomposed into *funding sets*, defined as trade sets for which VM for hedges can be re-hypothecated across all trades. Funding sets may span a large number (possibly thousands) of netting sets.
2. FVA Additivity over Funding Sets: The FVA is additive over funding sets, but not over netting sets. Therefore the valuation is difficult to carry out with standard CVA systems, and the industry is currently focused on using simpler alternatives which are linear over netting sets.
3. Funding Costs over Netting Sets: Funding costs would be additive over netting sets if either one of the following two mutually exclusive hypothesis hold:

a. HY1 => Re-hypothecation is possible only between hedges to trades in individual netting sets;
b. HY2 => The collateral received from hedges to each payable netting set can be fully re-hypothecated as VM for hedges against other receivable netting sets.

4. Complete Cross netting Set Re-hypothecation: This leads to a symmetric FVA (SFVA) which recognizes a re-hypothecation benefit to all VM received, whereas HY1 leads to the FCA metric that aggregates funding costs linearly over all netting sets. The metric

$$FBA = FCA - SFVA$$

measures the difference between the funding costs under assumptions HY1 and HY2.
5. Shortcomings of FCA and SFVA: Note that in the above expression the FCA generally overstates the funding costs because it neglects the RHO for hedges to unsecured trades in different netting sets. The SFVA on the other hand has errors of opposite sign since it overvalues the RHO.

Inconsistent Booking Under the FCA/FBA

1. Overlap of FBA and DVA: In the interpretation of the FBA, it is often noted that the FBA overlaps with the DVA on payables – in their case study, Albanese and Andersen (2014) note that the FBA is about 20% larger than the standard DVA on payables.
2. DVA CL Replacement with the FBA: As a result advocates of FCA/FBA accounting remove the regular DVA contra-liability (CL) entry on the financial accounting statement and effectively replace it with the FBA number. In fact to better comply with the accounting laws, the CL entry may be broken into two pieces – the DVA, plus a new "funding" term equal to $FBA - DVA$. The net CL entry, however, is still FBA.
3. New Accounting Rules on DVA: DVA on payables is always recognized as a gain on the income statement and until 2012 could theoretically be considered a contribution to CET1.

Under FCA/FBA accounting, this would ultimately lead to a net CET1 reduction relating to funding equal to the SFVA.

4. Payables DVA Contribution to CET1: In 2012, the DVA on payables was de-recognized as contributing to CET1 (Basel Committee On Banking Supervision (2012)). As the FBA is basically re-classified as DVA, this effectively prevents the FBA from contributing to CET1 and sets the overall CET1 deduction for funding equal to the FCA.
5. DVA Cross Netting Set Re-hypothecation Benefit: As a consequence the re-hypothecation benefits across netting sets are ignored altogether in CET1. As discussed in Albanese and Andersen (2014), this has material impact on both accounting, management, and trading decisions.
6. FCA/FBA Accounting Compromise Solution: The compromise solution in FCA/FBA accounting includes the following:
    a. Enter FCA and unilateral CVA (UCVA) as CET1 deductions.
    b. Eliminate DVA on payables from accounts, replace it with FBA, and enter this amount as a contra-liability (CL) adjustment not contributing to CET1.
    c. Transfer UCVA and SFVA to clients.
    d. As we explain later in the paper, the end result is that the FTP's are struck at the indifference levels to the income.
7. Breakage of the Asset Liability Symmetry: Besides the issues that have already been mentioned, it is clear from the booking rules above that the FCA/FBA accounting implies a loss of asset-liability symmetry, since the DVA on payables is eliminated on favor of the FBA even though CVA is supplemented with (rather eliminated in favor of) FCA. The lack of symmetry is problematic from an accounting standpoint and contradicts FASB 159 adopted in 2007.
8. SFVA as a CET1 Deduction: A possible way around this asymmetry involves deducting SFVA (rather than FCA) from the equity capital. In this case the DVA double counting issue manifests itself by the fact that the SFVA inherits from the DVA a wrong-way sensitivity with respect to the own-credit of the bank, i.e., it may decrease (causing the CET1 to increase) whenever the bank credit deteriorates.

9. FTP Policies in FCA/FBA Accounting: As mentioned above, the funding related FTP policies in FCA/FBA accounting normally pass through the SFVA amount, i.e., include FBA benefits. Prior to 2012 rules, this could be argued to be reasonable from a share-holder perspective (as proxied by CET1, at least). Yet, since FBA has been currently demoted to the status of contra-liability that is not recognized in equity capital considerations, the FCA/FBA FTP policy induces deal-flow volatility to CET1.
10. Interpretation of the FCA/FBA FTP Policy: One way to interpret the effect above is that the FCA/FBA FTP policies are based on indifference pricing to the overall firm (including senior creditors), rather than just share-holders as is normally desired.
11. Inaccuracies of the FCA/FBA Accounting: In addition to the above-mentioned undesirable side-effects, numerical experiments show that the net FTP amounts are often too large in absolute value, despite the inclusion of the FBA benefits. This effect is due to large inaccuracies in modeling VM re-hypothecation and leads to incorrect firm-level hedge ratios for market risk.

Improvements Offered by the FVA/FDA Accounting

1. Proposals to Overcome the FCA/FBA Drawbacks: To overcome the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) propose an accounting methodology that:
    a. Reflects and justifies Basel III regulatory requirements regarding counterparty credit risk;
    b. Is consistent with generally accepted accounting principles;
    c. Is consistent with the tenets of classical finance theory, such as the MMT and risk neutral valuation.

   Within this framework they then consistently value cash flow streams for VM funding and re-hypothecation strategies.
2. CET1 as a Shareholder Proxy: Their proposal fundamentally uses CET1 as a proxy for shareholder value and defines FVA as a discounted expectation of future funding costs

occurring whenever there is an overall deficit of the VM at the book level. Future scenarios where there is a net excess of OTC collateral do *not* contribute to the FVA.

3. Corporate Finance Interpretation of FVA/FDA: A corporate finance interpretation of this FVA metric equates it to the present value of the wealth transfer from the bank shareholders to the bank senior creditors as a result of the bank entering into OTC trades with unsecured funding.
4. Validity of Asset Liability Symmetry: Importantly the FVA definition is such the funding adjustments are entirely divorced from the DVA on payables, wherefore the asset-liability symmetry still holds and the own-credit metric of the FVA has the correct sign.
5. Embedding of RHO in FVA: Since the RHO is embedded in the valuation of the FVA, the FVA amount is much smaller than the FVA amount in the case of portfolios of realistic size – about one third as large in the case study portfolio of Albanese and Andersen (2014).
6. Consistency of the FVA/FDA with MMT: In FVA accounting, the MMT is satisfied, as the FVA is accompanied by an offsetting CL adjustment which does not overlap with the DVA on payables. This contra-liability is named DVA2 by Hull and White (2014) and is referred to as FDA by Albanese and Andersen (2014). In the case of FCA/FBA accounting the term DVA2 is not meaningful because the approximations involved break the MMT and compromise a rigorous capital structure interpretation.
7. CVA Fair Valuation in FVA/FDA: Within the FVA/FDA framework, the fair valuation of CVA is most naturally a bilateral one (sometimes known as first-to-default CVA, or FTDCVA). As this fair value contains a DVA-like element of self-default benefit, guidelines in the Basel Committee On Banking Supervision (2012) suggest that FTDCVA cannot be directly deducted from CET1.
8. UCVA, FTDCVA, and CVA-CL Metrics: From the FTDCVA number a unilateral CVA (UCVA) number can be split out and it may be recorded as a contra-asset (CA) adjustment that is subtracted from CET1. The remaining "self-CVA" term is listed as a contra-liability (CL) and is to be excluded from CET1.
9. Elements of FVA/FDA Accounting - Summary: In summary the FVA/FDA accounting with rigorous RHO modeling includes the following elements:

a. The UCVA and the FVA are both entered as CA adjustments and CET1 deductions recognizing the full benefit of the RHO to CET1;
b. The DVA and the FDA are both entered as CL adjustments, as is the part of the FTDCVA that involves benefits from the bank defaults. None of the CL adjustments are to be counted for the CET1 purposes.
c. The FTP is designed to immunize the CET1 from deal-flow volatility and to transfer the incremental costs of FVA and UCVA capital deductions to clients.

10. Income, CET1, and MMT Impact: Note that a) and b) preserve the standard CVA-DVA accounting at the net income level, as the FVA and the FDA adjustments cancel against each other. CET1, however, is affected by the funding costs.
11. Conservative Computation of the FTP: The FTP rule in the FVA/FDA accounting aims at preserving the CET1 capital, a principal that is fundamentally more conservative than the one followed in FCA/FBA accounting, where one only insists that new trades not have a negative impact on the income.
12. CET1 Impact from Deal Flow Volatility: In FVA/FDA accounting, deal flow still engenders volatility of the contra-liabilities (such as the DVA) as well as the fair value of the bank itself. However, due to our alignment of CET1 and the shareholder value, mitigating this volatility is irrelevant from the viewpoint of the shareholder.
13. FCA/FBA vs. FVA/FDA FTP Comparison: Notwithstanding the stronger FTP requirement, the fact that the RHO is properly modeled means that the FTP amounts obtained in the FVA/FDA accounting are generally quiet reasonable and often materially smaller than those in the FCA/FBA methodology.
14. FCA/FBA vs. FVA/FDA Derivatives Valuation: Relative to FVA/FDA, FCA/FBA accounting is observed to systematically undervalue the derivatives payables and over-value derivatives receivables, thus potentially giving rise to biased sub-optimal positioning of the OTC book.
15. Non-linear RHO Funding Set Contribution: It should be emphasized that under the FVA/FDA accounting, the notion of an individual unsecured trade price loses its meaning because all trades within the same funding set contribute non linearly to the RHO of the

funding set. To a lesser extent, individual unsecured trade price loses its meaning in FCA/FBA accounting as well, since here the smallest possible additive unit is a netting set.

16. Valuation Across Entire Funding Sets: It is therefore possible to conclude that in order to correctly account for funding adjustments, one needs to value derivatives in the context of *entire* funding sets.
17. Funding Set Level Scenario Simulation: In FVA/FDA accounting, in order to account for collateral thresholds and to model re-hypothecation benefits correctly, whenever w new possible trade is priced, one needs to evolve dynamically the full portfolio valuation along with all the CDS curves for all counterparties and compute book level incremental statistics.

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CET1 Capital Deductions in Basel III and Capital Structure Considerations

CET1 Deductions

1. DVA as Full CET1 Deduction: The BCBS recommended in 2012 that the DVA be fully deducted from CET1.
2. BCBS Statement on DVA Impact: The relevant wording from the Basel Committee on Banking Supervision (2012) is:

   Therefore, after considering all the views, the Basel Committee is of the view that all DVA's for derivatives should be fully deducted in the calculation of CET1. The deduction of the DVA's is to occur at each reporting date and requires deducting the spread premium over the risk-free rate for derivative liabilities. In effect, this would require banks to value their derivatives for CET1 purposes as if they (but not their counterparties) were risk free, and to deduct unrealized gains both at the inception of the derivative trade and afterwards, when the creditworthiness of the bank deteriorates.
3. Credit Quality Impact On CET1: The BCBS rule ensures that a bank cannot claim increases in CET1 solely due to the deterioration in its own credit quality, consistently with the spirit of Basel III accord (Basel Committee On Banking Supervision (2011)).
4. Impact on CVA and FVA: While Basel Committee On Banking Supervision (2012) nominally only deals with the derivatives liability and the DVA, it is generally understood that the disallowance of the CET1 increases from deteriorating bank credit is a universal principal that extends to CVA and FVA as well.
5. BCBS Language Relevant to CVA: For instance, for CVA some relevant language is (from Federal Register (2014)):

   CVA equals the credit valuation adjustment that the bank has recognized in its balance sheet valuation of any OTC derivatives contract in its netting set. For the purposes of this *paragraph, CVA does not include any adjustments to CET1 attributable to changes in its own credit risk since the inception of the transaction with the counterparty.*

6. Unilateral CVA vs. Bilateral FTDCVA: From a modeling standpoint, adherence to this particular language can be achieved by deducting from the capital a unilateral UCVA metric as opposed to a smaller bilateral FTDCVA.

"Going Concern" or Defaultable Banks?

1. Regulatory Intent Behind the Deductions: The intent of the regulator regarding both the DVA and the CVA capital deductions is to exclude from CET1 the present value of the cash flows that benefit the bank after the default.
2. Regulatory Notion of "Going Concern": Care must be taken to not extend the regulatory exclusion to a fair value setting. For instance, the regulatory notion of valuation from the viewpoint of "going concern", in which the bank is assumed to be unable to default, is clearly at odds with both the reality and with the objective of consistent market pricing.
3. Unintended Consequence of "Going Concern": If taken literally and applied out the intended context, this no-default assumption has the unwanted and undesirable side effect of increasing the prices on the bank-issued debt and significantly lowering the bank's funding costs.
4. Funding Spread as a Liquidity Spread: We note in particular that if one were to consistently assume that the bank cannot default, then the spread separating the bank funding rates from the OIS rates would need to be interpreted as a liquidity spread. This is one of the possible financial interpretation behind the FCA/FBA accounting rules, along with an alternate approach based on modeling debt buy-back strategies.
5. Trouble with the Liquidity Spread Approach: The liquidity spread assumption is hardly defensible; typical funding spreads are in the range of 50-400 bp while typical liquidity spreads are below 5 bp.

6. Funding Spread without Credit Risk: It would be very difficult to construct a financial interpretation to funding spreads which does not involve the credit risk of the bank. The debt buy-back argument is more subtle and is discussed later.
7. No-default View on Funding Considerations: Besides making fair-value considerations awkward, the no bank default view is not reasonable for funding considerations either. Specifically the FVA is a fair valuation of a cash flow stream resulting from a funding strategy that the bank clearly cannot implement past its own default; once the bank is in a state of default, its funding spread ins infinite, and the bank is unable to borrow funds on an unsecured basis or to conduct most other trading activities. As such any correct measure for funding costs must inescapably reference default by the bank (and its counterparties, for that matter).
8. Wealth Transfer across the Capital Structure: From yet another angle, FVA admits the financial interpretation as an internal wealth transfer across the capital structure of the bank resulting from the implementation of a funding strategy and is not the price of an asset sold to the counterparty on which the bank can default. However, wealth transfers can stop once the default occurs and the equity holders are wiped out.

Categorization of Cash-flow Streams

1. Accounting Merger/Unification Viewpoint Framework: Since a straight "going concern" assumption is inadequate for evolving a comprehensive accounting framework, Albanese and Andersen (2014) put together an alternate framework which reproduces and justifies the regulator mandated CET1 deductions for CVA and DVA, but which is also meaningful from the viewpoints of funding costs, classical Finance Theory, and the generally accepted accounting principles.
2. Fundamental Cash flow Steam Types: For this purpose, Albanese and Andersen (2014) propose that the cash-flow streams be fundamentally classified into the following 5 types:
    a. CF1 => Contractually promised cash flow streams excluding all bank and counterparty credit risk events;

b. CF2 => Trade-related cash flows resulting from counter party defaults, but excluding bank default events;
c. CF3 => Trade related cash flows resulting from the bank default;
d. CF4 => Cash flows streams derived from dynamic trading strategies (such as funding strategies) implemented by the banks and taking place prior to the bank default;
e. CF5 => Cash flow steams deriving from dynamic trading strategies (such as funding strategies) implemented by the bank and taking place at or after the default of the bank.

3. Derivative Contractual Agreement Cash Flows: Any derivatives contract can always be split into separate contractual agreements generating cash flows to types CF1, CF2, and CF3.
4. Dynamic Trading Strategy Cash Flows: Similarly cash flows arising from any dynamic trading strategy can be modeled as a split between types CF4 and CF5.
5. Cash Flows Contributing to CET1: We assume that the splits have been carried out such that each unit of account referring to either a counter party contract or a trading strategy is matched to the relevant cash flow stream. We then designate the units of account whose underlying cash flow streams are of the types CF1, CF2, and CF4 as contributing to CET1, while units of account whose cash flow streams are of type CF3 and CF5 do not contribute to CET1.
6. Counterparty Contract vs. Trading Strategy: A key difference between a counterparty contract and a trading strategy is that the former is settled with a counterparty at the time of the bank default while the latter simply terminates at that point in time. The reason why the 2 cases are treated differently is that a contractually promised cash flow reflects an obligation by the bank that extends to the last maturity, independently of whether a bank defaults or remains a going concern until then.
7. Trading Flows at Bank Default: Cash flows deriving from trading strategies cannot be implemented past the time of the default of the bank, at which time the bank goes into receivership and is unable to carry out normal trading activities. Nevertheless the implementation of the trading strategies prior to default has consequences on a post-default basis which have an impact on the default claim held by senior creditors. These are cash flows of type CF4.

8. Cash Flow Streams:
    a. Collateralized Transactions => Collateralized transactions involve the cash flow streams of type CF1 which are immune from counterparty credit risk.
    b. UCVA => The UCVA refers to a cash flow stream of type CF2 and is, effectively, the price of a CVA protection contract promised by the bank, excluding the effects of bank default.
    c. DVA => The DVA refers to cash flow streams of the type CF3 as it represents the benefit the senior creditors of the bank obtain from the default of the bank on derivative liabilities.
    d. CVA CL => The CVA contra-liability is the DVA component of the CVA; like the ordinary DVA on liabilities it is a cash flow of type CF3.
    e. FVA => The FVA can be interpreted as the price of the strategy of borrowing VM collateral up to the time the defaults (after which it becomes impossible to borrow any further). This is a cash flow stream of type CF4.
    f. FDA => The FDA is the post-default benefit to the senior creditors deriving from owning a title to the portfolio of derivative receivables whose hedges were funded on an unsecured basis. The FDA corresponds to a cash flow steam of type CF5.
9. Corporate Finance Interpretation of CET1: The cash flow rules above are designed to allow for a Corporate Finance Interpretation of CET1 as a proxy for the value of the bank assets to the shareholders – or at least the contributions to CET1 from derivatives trading activities.
10. Cash Flows after Bank Default: In particular, since the shareholders are indifferent to the cash flows occurring at or after the bank default, such cash flow streams should not contribute to CET1 – excluding here the feedback effects of the type discussed in Burgard and Kjaer (2011). Also note that by ensuring that the cash flow types of CF3 have no impact on CET1, we achieve the stated regulatory objective of deducting both FVA and UCVA from CET1.
11. RHO Exercise Impact on CET1: By the same token, funding costs are assessed at the market level, and the benefits resulting from the exercise of the RHO at times prior to the bank default contributes positively to CET1.

12. Self-Default Benefit - Unilateral CVA: One particular rationale for the regulatory mandate to deduct full UCVA from CET1 was to avoid the scenario where an increase in the bank's own credit spread could lead to a higher CET1.
13. Bank Default Continuation Funding Spread: One may ask whether a similar principle applies to FVA, necessitating the definition of a unilateral FVA. However such a remedy would be difficult to justify within a consistent modeling framework, since we would have to introduce the notion of continuation funding spread for the bank at and after its own default.
14. Peer Proxy Continuation Funding Spread: While such a notion could potentially be based on average or minimum peer spreads (post bank default), satisfying the Basel III principles does not depend on it as the FVA defined here normally increases as a function of the bank's own credit spread.
15. Credit Spread Impact on FVA: The FVA is impacted by rising bank credit spreads in 2 different directions; rising spreads tend to decrease FVA because funding costs are cut short by a bank default; yet rising spreads also tend to increase the FVA as the spread paid on unsecured lending increases. Albanese and Andersen (2014) show that in normal circumstances the latter effect dominates.
16. Own-Credit Sensitivity in FCA/FBA: Albanese and Andersen (2014) also show that this *not* the case for SFVA, which is the metric for funding costs that are transferred to the clients in the form of FTP in FCA/FBA accounting. Since the SFVA can be shown to have own-credit sensitivities of the wrong sign when applied to portfolios containing mostly payables, its use as a CET1 deduction would be problematic from a regulatory standpoint. For this reason alone, the CET1 deduction in FCA/FBA accounting must be the full FCA.

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Accounting Principles, Units of Accounts, and Valuation Adjustment Metrics

Accounting Rules

1. Accounting Principles for Derivative Portfolios: We list here the key accounting principles for derivatives portfolios. KPMG (2011) and PWC (2011) contain a discussion on them.
2. Units of Account:
    a. Trading Securities (AP0) => Derivatives contracts are categorized as trading securities and must be listed on the balance sheet on their fair market value. Changes to the fair market value are registered as income or loss on the income statement.
    b. Units of Account and Linearity (AP1) => Financial portfolios are decomposed into elementary *units of account* satisfying a linearity property, in the sense that the fair valuation of the portfolio is the sum of the fair valuations of the units of account.
    c. Fair Value as an Exit Price (AP2) => The *fair value* of a unit of account is defined in IFRS13 as the price that would be received if one were to sell the unit of account in an orderly trade between the market participants at the measurement date. This is called the *exit price* of the unit of account. In the determination of the exit prices neither transaction costs (such as broker commissions) nor entity-specific production costs are considered as part of the fair valuation.
    d. Model Based Valuation (AP3) => When a unit of account is not tradeable in the active market, its fair value may be determined using a model based valuation technique. In this case an income based approach is allowed, where the cash flows of the trades present valued. This includes cash flows associated with the default of one of the counterparties.

e. Symmetry Principle (AP4) => When (as is often the case) there is no active market for the liability transfer, IASB accounting standards require that the fair value of the liability be determined from the perspective of the asset investor.
f. Non-Performance Risk (AP5) => According to FASB 157, the benefit to senior creditors on the non-performance of the liabilities should be captured. Non-performance risk includes the effects of credit risk, as well as any other factors the influence the likelihood of fulfilling contractual obligations.

3. Application to Derivative Security Accounting: To consider the application of these principles for derivatives securities accounting, Albanese and Andersen (2014) consider the situation where a bank B transacts in partially collateralized OTC derivatives with a set of n credit-risky counterparties.
4. Collateral Posting and CSA Agreements: Each counterparty holds a portfolio of derivatives under an over-arching CSA agreement involving a netting clause, and possibly, but not necessarily, collateral posting obligations for the variation margin (VM).
5. ISDA 2009 Closeout Amount Protocols: In the absence of full collateralization, closeout protocols in place to govern settlements whenever a bank or a counterparty defaults need to be explicitly considered; standard protocols are the ISDA Market Quotation Close-out Convention of 1992 (most common) and the ISDA Close-out Amount Protocol of 2009. Note that closeout protocols are relevant for the fair valuation because of the accounting principles AP3 and AP5 above.
6. Decomposition of the Bank OTC Portfolio: Before discussing any valuation metrics, it needs to be clarified how the bank OTC portfolio is linearly decomposed into units of account.
7. Decomposition of the Unsecured Derivative Contract: The above question is non-trivial as an unsecured derivative contract in isolation generally cannot be considered a proper unit of account= because netting clauses cause valuations to depend on the netting set to which the trade belongs.
8. Funding Sets vs. Netting Sets: The RHO for VM causes valuation to depend on funding sets defined as the largest set of trades among which re-hypothecation is possible. Notice that funding sets can cut across netting sets as it is entirely possible that the VM is re-hypothecated separately across distinct business lines contributing to the same netting sets.

9. Fully Collateralized Counterparty Derivative Trades: Fully collateralized trades done with traders or with CCP's generally have negligible credit risk, and shall be considered here to be default free; fir valuation of such trades is additive at the trade level. The valuation of the unsecured derivative assumed to be fully collateralized in cash is referred to as its *default-free valuation*.
10. Bank Counterparty Credit Valuation Adjustments: The impact of the counterparty and the bank credit risk on valuations is then captured by other units of accounts called *adjustments*. Adjustments make reference to sub-portfolios as opposed to individual trades and can be interpreted as the valuation of derivatives referring to sub-portfolios as underlying.
11. The Asset Account (A): Albanese and Andersen (2014) construct a reference to stream-lined accounting framework of OTC derivatives based on six balance sheet accounts. The Asset Account (A) refers to the receivable units of account referring to cash-flow streams of the type CF1.
12. The Liabilities Account (L): This balance sheet account includes payable units of account referring to cash flow streams of type CF1.
13. The Contra Asset Account (CA): This balance sheet account includes payables adjustments with underlying cash-flow streams of types CF2 and CF4; these are deducted from CET1.
14. The Contra Liabilities Account (CL): This balance sheet account includes receivable adjustments referring to the cash flow streams of the type CF3 and CF5; these do not contribute to CET1.
15. The Retained Earnings Account (RE): This balance sheet account includes provisions that are set aside to meet future obligations such as funding costs and credit default losses. These entries do contribute to CET1.
16. The Equity Account (PFV): This balance sheet account is defined in such a way that the basic accounting equation

$$A + RE - CA = PFV + L - CL$$

is satisfied. The Equity Account has the meaning of Portfolio Fair Valuation and the variation of the PFV over an accounting period is called the *Income*.

17. The CET1 Capital Measure Components: The CET1 is a capital measure that requires the exclusion of the value of units of accounts referencing cash-flows of the type CF3 and CF5 taking place at or after the time of default of the bank B. As those types of cash flows are captured in the CL account, we exclude this account from the common equity and write

$$CET1 = PFV - CL = A - L - CA + RE$$

18. CET1 as Bank Shareholder Value: Within their framework, Albanese and Andersen (2014) interpret CET1 as the value of the bank to the shareholders, while PFV is the combined value to the shareholders and the senior creditors. Here the term "senior creditors" is used to refer to a class of creditors which is different from collateral lenders, and which either have priority or are at the same level of seniority as the collateral lenders.
19. Computation of the CET1 Deduction: Upon entering into a derivative transaction, the CET1 is subjected to an incremental deduction denoted by ΔCA and is augmented through earnings by the FTP amount received from the clients over and above the default-free valuation. Attributing the FTP to the Retained Earnings (RE) account the CET1 variation is given by

$$\Delta CET1 = -\Delta CA + \Delta RE$$

20. Computation of the Income Increment: A nee trade also affects the CL adjustments by an incremental amount ΔCL, an amount tied to the benefits of the bank default and therefore only having an impact on the senior creditors' wealth. The ΔCL term is excluded from CET1, but affects the income and the fair valuation of the bank.

$$\Delta PFV = \Delta Income = \Delta CET1 + \Delta CL = -\Delta CA + \Delta CL + \Delta RE$$

Contra-Asset and Contra-Liability Accounting for Credit Risk

1. CVA for the i^{th} Netting Set: To examine the CA and the CL accounts more closely, consider first the fair valuation of the credit risk for the i^{th}netting set, denoted by CVA_i. The quantity can be interpreted as the value of the default protection contract implicitly sold by the bank to the counterparty i, with the notional set to the netting set value at the time of the counterparty default.
2. Decomposition of the CVA Components: As shown in Albanese and Andersen (2014), the precise valuation methodology to be used depends on the closeout rules. The total CVA is computed by summing the CVA_i across n netting sets and is commonly split into 2 components:

$$CVA = FTDCVA = UCVA - CVA_{CL}$$

3. The Unilateral CVA Component – UCVA: The UCVA component is booked as a CA adjustment and is a unilateral CVA metric independent of the closeout rules, i.e., it is the present value of all the counterparty credit risk losses resulting from the default of the counterparty computed under the assumption that the bank does not default.
4. The CL CVA Component CVA_{CL}: The CVA_{CL} is loosely speaking the DVA component of the CVA, i.e., it is the benefit the bank senior creditors receive at the time of the bank default by, in effect, no longer accepting to sustain future counterparty credit losses. It is booked as a CL adjustment. The magnitude of the CVA_{CL} is closely linked to the closeout specifications; Albanese and Andersen (2014) give the relevant equation for CVA_{CL} using the ISDA 1992 closeout rules.
5. Non-Performance Risk Accounting Principle: According to the accounting principle AP5 on non-performance risk, cash flows taking place at or after the default of the bank should be present valued and accounted for.
6. Symmetry Principle Applied towards DVA: By virtue of principle AP4, the value of the default protection sold implicitly by the unsecured counterparty to the bank should be valued as the CVA assessed by the counterparty against the bank. This amount is the DVA for payables, the reporting of which was mandated by FASB 159 in 2007.

7. DVA Impact on CET1 Numbers: DVA enters accounts as a CL adjustment, and as is references cash flows ensuing a bank default, it is excluded from CET1. A split such as the one seen for CVA is not meaningful for DVA, as this quantity only involves post-default cash flows (with no direct relevance to equity holders and to the capital).
8. "Going Concern" Impact on the CVA: It should be noted here in passing that not all banks account for the CVA_{CL} term, as they effectively equate the CVA with the UCVA term. While this "going concern" definition is convenient in a number of ways (e.g., regulatory and accounting definitions of CVA are better aligned), it is hard to argue that it is correct or consistent with DVA accounting.

Contra-Asset and Contra-Liability Accounting for Funding

1. CA and CL Credit Risk: Accounting for the credit risk through the UCVA (contra-asset entry) and the $DVA + CVA_{CL}$ term (the contra-liability term) as seen before is a fairly well-established practice even if there are minor differences in the way banks sometimes define CVA and DVA.
2. Supplementing the CVA and the DVA Metrics: As discussed before, recently the CVA and the DVA risk metrics have been supplemented by the quantities meant to account for funding cash flows to which the banks are subject to through their postings of the VM on the hedges.
3. Nature of the Posted VM: VM is normally paid in cash (or to a lesser extent) in highly liquid short term government debt and maybe re-hypothecated across trades within the same funding set. The bank posts VM on collateralized derivative liabilities and receives VM on collateralized derivative assets.
4. Funding Adjustment Unit of Account: For funding costs, we work with a unit of account at the funding set level and define FVA as the discounted value of the book level funding costs arising whenever the funding set has a net collateral deficit. Future states of the world whereby the funding set is a net receiver of the VM are modeled as *not* contributing to funding assets.

5. Handling the Different VM States: On the one hand the excess cash deposited as VM can earn a riskless rate of OIS. But on the other hand, derivative counterparties pledging VM are also entitled to interest rate payments at a matching OIS rate. In total, the states of the world where the bank enjoys accumulation of excess VM collateral are modeled as having zero funding costs.
6. Alternate VM Cash Management Strategies: As discussed in Albanese and Andersen (2014), the zero benefit assumption around excess collateral maybe considered a conservative assumption, and some researchers have considered strategies where the bank buys back long-term debt with excess VM (Burgard and Kjaer (2011a, 2011b, 2011c, 2013)).
7. VM Strategies Capital Structure Impact: However these strategies are not very difficult to implement in practice because the VM is very volatile, they also have zero impact on income if one insists on MMT consistency. As a consequence, if the deleveraging transactions occur at fair valuation, wealth is not transferred between share-holders and the senior creditors, i.e., the benefit is not truly there.
8. Origin of the Funding Cost Impact: Neglecting basis spreads, funding costs for the VM procurement are non-zero because the collateral lenders receive only partial recovery upon bank default. If we denote the recovery rate to collateral lenders with ${R_B}^C$, then the case where there is a perfectly functioning REPO market for derivatives would have

$$ {R_B}^C = 1 $$

9. FVA Definition and CET1 Impact: In reality

$$ {R_B}^C < 1 $$

   i.e., recovery is only partial because of market inefficiencies and the spreads for collateralized borrowing are very close the spreads for unsecured borrowing. The FVA is defined as the discounted expectation of the funding costs up until the time of bank default. Hence the FVA is booked as a CA adjustment and a CET1 deduction.

10. Origin of the Funding Benefit Impact: When short term debt is issued to fund the VM collateral, the lenders providing the funds are exposed to the bank default risk. The flip-side of the risk is the DVA-like benefit held by the bank. To account for it, we introduce an FDA entry as the present value of the depreciation of the collateral debt at the time of the bank default, due to incomplete recovery.
11. FDA Definitions and the CET1 Impact: That is the FDA is the present value of $1 - {R_B}^C$ times the notional borrowed for VM funding purposes, received at the time of the bank default. Since the FDA makes reference to cash flows happening at or after the bank default, the FDA is booked as a CL adjustment and excluded from the CET1.
12. Shareholder Debt Holder Transfers: For the MMT to hold, the FVA must equal the FDA. In this case, the funding strategies, and in particular the value of ${R_B}^C$ do not affect the fair valuation of the bank. However if ${R_B}^C$ is strictly less than 100% then the deal flow induces a wealth transfer from the shareholders to the senior creditors.
13. FVA/FDA Cash Flow Transfer View: In accounting terms if we decrease the value of ${R_B}^C$ from 1 down to 0, a portion of CET1 is gradually demoted to the status of contra-liability, reflecting the wealth transfer from the shareholders to the senior creditors. The FVA may therefore be considered the wealth transfer lost from the shareholders, while the FDA is the amount earned by the senior creditors.

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Accounting Cash Flows

Accounting Cash Flow Setup Framework

1. XVA Metrics Valuation Formulas Setup: Having so far been limited to a largely qualitative accounting discussion, the next step is to follow the treatment set out in Albanese and Andersen (2014) and proceed to provide concrete valuation formulas for the XVA metrics. For this the precise funding and credit related flows taking place in OTC derivatives trading need to be considered, both before and after the default of the counterparties and the bank itself.
2. Modeling Details in the XVA Computation: Since elaborate cash flow details can obscure the main concepts, this section omits certain less essential minutiae, such as flows associated with ratings dependent CSA thresholds and credit-risk sensitive closeout conditions. This is later followed by a more complete cash flow representation, much of which is implemented in the case study carried out by Albanese and Andersen (2014).
3. Transmitting XVA Charges to Clients: Whenever the bank enters into an unsecured trade, the XVA adjustments are accompanied by charges to clients which can be structured in various ways. In their treatment Albanese and Andersen (2014) assume that the XVA related FTP payments are simply upfront and due at the inception of the trade in question.
4. Nature and Type of FTP: As described earlier, business line trading desks, in fact, typically pay FTP charges to the CVA/CFD desks on an upfront basis. However, assuming upfront structure for the client charges is a stylized approximation, since often costs are paid in the instalments embedded in the unsecured derivative structure itself.
5. Default Free Counterparty Portfolio Value: Working within the modeling framework briefly described before, let us introduce the notation

$$V_i^U(t) \ \forall \ i = 1, \cdots, n$$

for the default valuation at the time t of the portfolio held with the counterparty

$$i = 1, \cdots, n$$

computed by neglecting all funding and credit risks including that of the bank itself.

6. Fully Collateralized Netting Set Value: That is, ${V_i}^U(t)$ represents the value of the netting set i in a world where all trades are collateralized in full. ${V_i}^U(t)$ is set to 0 in case the i^{th} counterparty is in a state of default at time t.
7. Counterparty Value Netting Set Additivity: Assuming no closeout risk and no initial margin, the fair valuation of fully collateralized books is additive over individual trades and therefore also additive over netting sets, i.e., we can meaningfully define a total default-free portfolio value of

$$V^U(t) = \sum_i {V_i}^U(t)$$

8. Hedging of the Unsecured Trade: For simplicity's sake for now we assume that the unsecured trades are hedged on a precise back-to-back basis with the hedge trades having a valuation ${V_i}^H(t)$ equal to exactly $-{V_i}^U(t)$. Indeed, one way to interpret the role of the CFD and the CVA desks is that, after the appropriate compensation, they allow the lines of business desks to hedge risk as if there were not funding or credit risk.
9. Value of the Hedge Trades: The common value of the default-free unsecured trades and hedges is denoted as follows:

$$V_i(t) = -{V_i}^H(t) = {V_i}^U(t)$$

It is also assumed for now that neither the bank nor the counterparty post any VM to each other; later these hypotheses are relaxed and extensions including partial VM postings with CSA collateral thresholds are considered.

10. VM Re-hypothecation Across Hedge Trades: Banks typically have a separate funding set for each business line and jurisdiction. Within each such dedicated book, banks always exercise the RHO for the VM across hedges (albeit not necessarily optimally – collateral systems of many banks tend to be rudimentary).
11. Net VM Across Hedge Trades: The net VM posted at any given time is given by the sum

$$C_{VM}(t) = \sum_{i} C_{VM,i}(t)$$

where

$$C_{VM,i}(t) = V_i(t)\mathbb{I}_{t<\tau_B}\mathbb{I}_{t<\tau_i}$$

(this is a simplification, as there is typically a material VM from the CVA hedges – this complication is discussed down later).
12. Impact of the Bank/Counterparty Default: Notice the presence of the indicator functions $\mathbb{I}_{t<\tau_i}$ and $\mathbb{I}_{t<\tau_B}$ above, reflecting the fact that the default of either the bank or the counterparty results in an immediate settlement of collateralized hedges. The sign convention is that if $C_{VM}(t)$ is positive (negative) then the bank is a net poster (receiver) of the collateral on the hedges.
13. Assumption of the REPO Rate Value: For simplicity's sake it is assumed that the difference between the OIS rate and the REPO rate for general collateral is quantitatively immaterial and denote both rates with $r_{OIS}(t)$.
14. Recovery Rates Across Debt Classes: Another assumption is that the bank has at least two classes of debt. One is unsecured senior debt with recovery rate R_B used for regulatory costs, initial margin, and administrative costs. The other is debt used to finance VM collateral imperfectly secured by derivative receivables. This second class of debt is modeled as having a recovery rate ${R_B}^C$, a funding rate equal to $r_B(t)$ and a spread over OIS equal to

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

15. Estimation of the Funding Spread: If there existed an efficient REPO market for unsecured OTC derivatives, collateral lenders would be guaranteed a full recovery on the VM, i.e., we would have

$$R_B{}^C = 1$$

and the funding spread would be 0. In general

$$R_B{}^C \leq 1$$

and the risk neutral valuation of the overnight funding spread is given by

$$s_B(t) = \left(1 - R_B{}^C\right)\lambda_B(t)$$

where $\lambda_B(t)$ is the probability of the rate of default at the time t (Lando (1998)). In practice an additional liquidity spread may apply, but this basis is ignored for the present discussion.

    a. FVA Spread vs. FDA Spread => As pointed out by Morini and Prampolini (2011), FVA/FDA indifference assumes that the CDS-bond basis is zero. This is not the case in practice, since FVA is calculated from bond yield spread whereas FDA should, in theory, be calculated from the CDS spread.

16. Hybrid Debt Class Type Assumptions: It needs to be stressed that the assumption that the debt is divided into 2 classes is only formal and does not restrict the generality of the argument. In the general case one can still assume that the traded debt securities are hybrids between the theoretical bonds of the two types considered.

Cash Flows Related to VM Funding

1. Ignore CVA-related Funding Costs: We start by considering the funding flows on the interval $[t, t + \Delta t]$. To simplify the exposition the funding costs for any VM arising from the default hedges that the CVA desk may have entered into are ignored.
2. Borrowing Cost for the VM Funding: Suppose first that the net VM collateral $C_{VM}(t)$ posted by the bank is *positive*, whereby the bank needs to borrow to fund its overall VM position. In this scenario the bank treasury is assumed to issue short-term unsecured debt into the market to raise the necessary funds.
3. Funding Rate on the Collateral: More specifically, if in the time interval $[t, t + \Delta t]$ the bank has to fund a net collateral shortage

$$C_{VM}(t) > 0$$

the treasury issues $C_{VM}(t)$ worth of short-term debt for this purpose, either unsecured or backed by derivative receivables. Then interest charge on the unsecured debt in the time interval $[t, t + \Delta t]$ at the CFD funding rate $r_B(t)$ is $C_{VM}(t)r_B(t)\Delta t$.
4. CFD Role in the Bank Setup: The required VM collateral is then routed through the inter-dealer positions where an interest-rate amount of $C_{VM}(t)r_{OIS}(t)\Delta t$ is received back. As illustrated pictorially in Albanese and Andersen (2014), when

$$C_{VM}(t) > 0$$

the CFD experiences a net negative cash flow in the amount of $-C_{VM}(t)s_B(t)\Delta t$. FVA is defined as the present value of this negative carry over the lifetime of the funding set or until the time of bank default, whichever comes first.
5. Interest Rate on VM Receivables: In case the total collateral requirement for the VM is *negative*, i.e.

$$C_{VM}(t) < 0$$

the treasury OTC funding program is not called upon. In this case we assume that the bank treasury would invest the excess collateral in short-term securities, yielding on average (nearly) OIS levels. As the bank is liable to paying OIS on hedge counterparties on the net collateral received, the bank is therefore assumed to receive no benefits and face no costs due to VM posting obligations when

$$C_{VM}(t) < 0$$

6. Handling the Net Excess Collateral: In principle one can imagine that the net excess collateral would be passed by the CFD to the treasury, which would in turn use the funds to retire outstanding long-term deb. If this were the case on would imagine that the CFD receives from the treasury a benefit based on the interest-savings on the long-term debt, resulting in positive carry.
7. Rapid Fluctuation of the VM Levels: The above assumption is an aggressive one as VM varies greatly in short time scales and generally constitutes an unstable base from which to retire debt. In practice it is far easier to let debt mature than to retire long-dated bonds on the secondary market.
8. Art of Treasury Liquidity Management: Normally collateral is over-provisioned allowing for buffers as the treasury needs to manage liquidity prudently. There are additional costs to collateral over-provisioning and risk management of the liquidity buffers, all of which are ignored here.
9. Funding Rates and Strategies - Assumption: Besides the above “neutral” assumptions about investment returns, the definition of FDA used here makes additional assumptions about treasury funding strategies and funding rates. The specified ones are listed below.
    a. Worst-case Funding Rate Scenario => The worst scenario regarding funding rates is assumed, i.e., that derivative receivables cannot be passed as collateral and that VM borrowing is entirely unsecured.
    b. Best Case Over-provisioning Scenario => The best-case scenario regrading over-provisioning is also assumed, that is the bank procures on an overnight basis only the cash collateral that is strictly needed for VM borrowing, not more.

10. <u>FVA and Funding Strategy Link</u>: Both of the above assumptions are reasonable for accounting purposes, although they admittedly represent idealizations of the actual funding behavior (which no doubt varies from bank to bank). In this view, it is important to note that FVA calculations are non-unique and tied to the *actual* funding strategy to which a firm commits, a fundamental difference from the ideas underlying fair value pricing of derivatives.
11. <u>Derivative Contract Fair Value Theory</u>: According to classical finance theory, if there exists a replication strategy, then the cost of replication/hedging equals the fair value. Hence if one replication strategy is theoretically possible, then the cost of implementing it ought to equal the fair value *whether or not* the strategy is actually implemented. If more than one replication strategy can potentially be implemented, absence of arbitrage indicates that the cost of implementing each one equals the fair value.
12. <u>MMT on Funding Cost Impact</u>: For funding cost valuation, the MMT tells us that the fair value of funding strategies for trades entered at fair value is zero, as FVA does not represent the fair value of an asset, but instead the wealth transfer amount from the shareholders to the senior creditors.

Cash Flows at Counterparty Default

1. <u>Counterparty Triggered Default Flows</u>: If τ_i is the default time of the counterparty i, let $D_i(\tau_i)$ be the default cash flow received from the bank from the counterparty i as part of the default triggered closeout of the i^{th} portfolio. $D_i(\tau_i)$ is governed by the ISDA agreement between the bank and the i^{th} counterparty.
2. <u>ISDA Standard Closeout Assumption</u>: For simplicity's sake it is assumed that the closeout procedure follows the ISDA Market Quotation Protocol of 1992, and extensions to the other protocols are covered later.
3. <u>Applying the ISDA 1992 Protocol</u>: In the prevailing interpretation of the ISDA 1992 closeout protocol, if either the counterparty or the bank itself default prior to the maturity of the trade

portfolio, then the portfolio is settled at default free levels, i.e., XVA adjustments are excluded from the calculation of the settlement amount.

4. Counterparty Default Cash Flow: In other words, the default cash flows at time τ_i received the bank from counterparty i is

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i(\tau_i) \geq 0} R_i V_i(\tau_i) + \mathbb{I}_{V_i(\tau_i) < 0} V_i(\tau_i)\right]$$

where τ_B is the default time of the bank and

$$R_i \in [0, 1]$$

is the recovery rate received from the counterparty i. Observe the presence of the default indicator $\mathbb{I}_{\tau_i < \tau_B}$ in this expression, a reflection of the fact that if the bank defaults prior to the counterparty i, then no cash flow takes place at time τ_i since the unsecured derivative portfolio held with counterparty i is assumed to have been unwound at τ_B.

5. Sign of the Counterparty Default Cash Flow: $D_i(\tau_i)$ may be positive or negative, and can be considered an inherent part of the portfolio flows of counterparty i. Notice that if

$$V_i(\tau_i) \geq 0$$

and

$$\tau_B > \tau_i$$

the above equation represents a loss to bank B. The present value of this loss is the CVA_i for portfolio i.

6. Hedging the Counterparty Risk: As indicated earlier, the cash flow formula above assumes that the counterparty credit risk above is left unhedged. In reality the CVA desk seeks to layoff the counterparty credit risk through the purchase of credit hedges such as single names and index CDS.

7. CVA Hedge Collateral VM Pool: This setup does not alter our conclusions about how to account for CVA, but it does mean that CVA hedges are typically entered with dealers/exchanges on a fully collateralized basis, which in itself produces VM postings that can be introduced into the overall VM "pool" generated by the client trades.

Cash Flows at Bank Default

1. Collateral Impact on Bank Default: At the bank default time τ_B, any posted collateral amount stays with the holder, and all the inter-dealer positions are torn up, with no economic impact to any dealer counterparty. Similarly the default of a dealer used as a counterparty for hedging purposes has no impact as all credit risk is covered by collateral.
2. Explicit Cash Flows: The positions with n counterparties are typically settled at ISDA terms, leading to a loss for those counterparties that have a previous exposure to the bank. Under the standard ISDA 1992 terms, the effective default cash flow at time τ_B received by the bank from counterparty i may be written as

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{V_i(\tau_B) \geq 0} V_i(\tau_B) + \mathbb{I}_{V_i(\tau_B) < 0} R_B V_i(\tau_B)\right]$$

where

$$R_B \in [0, 1]$$

is the recovery rate for the bank B.
3. Credit Losses to Counterparty: Notice that the loss to counterparty is therefore $-(1 - R_B)V_i(\tau_B)^-$. Senior bank creditors, who in aggregate have a claim on derivative receivables of value $V_i(\tau_B)^-$ for the i^{th} counterparty, would similarly recover a fraction $R_B V_i(\tau_B)^-$ of their claim.
4. DVA Gain to the Bank: Again for the sake of simplicity it is assumed that the recovery rate on the senior debt and on the derivatives portfolio are identical. Notice that if

$$V_i(\tau_B) < 0$$

then the equation for $D_i(\tau_B)$ represents a gain for the bank relative to a fully collateralized payout of $V_i(\tau_B)$. The present value of this gain is the DVA_i associated with counterparty i.

5. Non additivity of the VM Collateral: Finally note that if is the recovery rate for the VM collateral lenders, upon defaulting the bank does not return (i.e., effectively receives) the following amount in VM cash:

$$D_i(\tau_B) = \left(1 - {R_B}^C\right)\left[\sum_i V_i(\tau_B)^-\right]$$

Since the total amount of VM collateral borrowed is reduced by exercising the RHO, the amounts lent and recovered from the collateral lenders cannot be represented as a linear sum over netting sets.

References

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Credit and Funding Valuation Adjustments

Introduction

1. Cash Flow Impact On Accounting: Having outlined all relevant cash flows above, the next step is to work out how these cash flows are valued, and how they affect the capital structure of the OTC book. In particular, this section looks at in detail the relevant XVA metrics needed for the accounting treatment outlined earlier.

CVA and DVA

1. Bilateral FTD CVA Formulation: The cash flows triggered by the counterparty defaults were under ISDA 1992 were discussed earlier. The loss in value due to counterparty i is measured by the present value of the protection against the value loss inherent in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i(\tau_i) \geq 0} R_i V_i(\tau_i) + \mathbb{I}_{V_i(\tau_i) < 0} V_i(\tau_i)\right]$$

2. Expression for CVA_i or $FTDCVA_i$:

$$CVA_i = FTDCVA_i = \mathbb{E}\left[e^{-\int_0^{\tau_i} r_{OIS}(u)du}\mathbb{I}_{\tau_i<\tau_B}(1-R_i)V_i(\tau_i)^+\right]$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t r_{OIS}(u)du}\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]\mathbb{Q}(\tau_i \in [t, t+dt])$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]dt$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)+\lambda_B(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

3. <u>Applying a Stochastic Default Intensity</u>: Here $\mathbb{E}[\cdot]$ denotes the expectation in the risk neutral measure $\mathbb{Q}$ and $\lambda_i(t)$ is the (possibly stochastic) default intensity for counterparty i. The concept of default intensity was introduced by Lando (1998) in the context of reduced for models based on Cox processes, but has become meaningful (and quite useful) also for many structural processes.
4. <u>Unilateral CVA Contra-Asset Adjustment</u>: As discussed in the previous section, the CVA entry to be booked as a contra-asset is the unilateral UCVA metric given by the expression

$$UCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

without the indicator function $\mathbb{I}_{t<\tau_B}$.

5. <u>CVA Contra-Liability Adjustment Expression</u>: This entity is accompanied by a CL adjustment defined as follows:

$$CVA_{CL,i} = UCVA_i - FTDCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

6. <u>DVA Contra-Liability Adjustment Formulation</u>: The $CVA_{CL,i}$ can be interpreted as the "DVA of the CVA", i.e., the benefit the senior creditors have on the default of the bank on the

default protection contract implicitly sold to the counterparties. The more substantial benefit associated with the option of defaulting on the underlying derivatives is instead captured by a DVA term defined as

$$DVA_i = \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \mathbb{I}_{t<\tau_i} \lambda_B(t)(1-R_B)V_i(t)^-\right] dt$$

7. Valuation Across Netting Sets: CVA and DVA numbers maybe added across counterparties giving rise to

$$FTDCVA = CVA = \sum_i CVA_i$$

$$UCVA = \sum_i UCVA_i$$

$$CVA_{CL} = \sum_i CVA_{CL,i}$$

$$DVA = \sum_i DVA_i$$

8. Accounting of CA/CL Adjustments: The booking of these quantities as CA and CL adjustments was covered earlier and is tabulated below.

FVA and FDA

1. Funding Payments Cash Flow Stream: As explained earlier, to find fair valuation expressions for the FVAs, one needs to value a continuous cash flow stream for funding costs at the rate given by the spread in

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

It was argued earlier that the contribution to this cost in the time period $[t, t + \Delta t]$ is $C_{VM}(t)^+ s_B(t) \Delta t$.

2. Funding Cost Valuation Expression: The discounted present value of the future funding costs is defined as follows:

$$FVA = \mathbb{E}\left[\int_0^\infty e^{-\int_0^t r_{OIS}(u) du} C_{VM}(t)^+ s_B(t) dt\right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u) + \lambda_B(u)\} du} \left\{\sum_i V_i(t) \mathbb{I}_{t<\tau_i}\right\}^+ s_B(t) dt\right]$$

where the second equality follows form the definition of the net collateral $C_{VM}(t)$ in

$$C_{VM}(t) = \sum_i V_i(t) \mathbb{I}_{t<\tau_i} \mathbb{I}_{t<\tau_B}$$

As explained earlier we include the cash flows only up to the time of the default of the bank.

3. Accommodating the Timing Element of Default: The are a plethora of competing definitions of FVA, some of which deliberately avoid the complication of a default timing element. Carver (2013) contains details on the "dark art" of FVA definitions.
4. Funding Spreads for VM Receivables: Due to the lack of infrastructure to guarantee water-tight collateralization mechanics with unsecured derivatives receivables as underlying, funding spreads for VM collateral are observed to be near unsecured levels even in the rare cases where derivatives receivables are nominally mentioned as collateral.

5. PV of the Funding Benefits: The present value of the gain that senior creditors of the bank gain due to the inability of the collateral lenders to recover in full on default is referred to as FDA here. To compute this quantity, observe that at the time of a bank default, senior creditors will gain a benefit equal to the fraction $1 - {R_B}^C$ of the pool of derivative receivables, while a fraction ${R_B}^C$ goes to the collateral lenders.
6. FDA Valuation Adjustment Formulation: From the above we have that

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t r_{OIS}(u)du}\left(1 - {R_B}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$= \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - {R_B}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

Thanks to the risk neutral valuation relation

$$s_B(t) = \left(1 - {R_B}^C\right)\lambda_B(t)$$

for the bank credit spreads, we find

$$FVA = FDA$$

consistent with the discussion in the earlier section.
7. MMT Consistency of FVA/FDA Accounting: The FVA and the FDA contribute with opposite signs to the bank fair valuation and cancel each other on the bank balance sheet, in agreement with the MMT. In other words, the fair valuation of the bank assets is indifferent to the funding strategy employed, and therefore to the FVA and the FDA. This relation was first noticed in Hull and White (2012), where the FVA is called DVA2 and the equation

$$FVA = FDA$$

was stated as a direct consequence of the MMT.

FCA and FBA

1. Motivation Behind the FCA/FBA Accounting: FCA/FBA accounting is an approximation to the funding value adjustment motivated by a desire to base the methodology upon netting set specific metrics that are computable by means of traditional CVA systems. This methodology is in a sense an extension to large books of the work in Piterbarg (2010) and Burgard and Kjaer (2011) which focused on the case of individual trades treated in isolation and not in a portfolio context.
2. FCA/FBA Methodology - SFVA Computation: To better understand the logic behind FCA/FBA accounting, we note that if hypothetically the bank was never a net receiver of VM collateral and there were no collateral thresholds, then the funding cost of a cash flow $X(T)$ would be represented as

$$SFVA_X = V_X(0) - {V_X}^*(0)$$

where

$$V_X(t) = \mathbb{E}_t\left[e^{-\int_t^T r_{OIS}(u)du} X(T)\right]$$

and

$${V_X}^*(t) = \mathbb{E}_t\left[e^{-\int_t^T r_B(u)du} X(T)\right]$$

where $\mathbb{E}_t$ represents the time t expectation in the risk-neutral measure. One interpretation of this formula is that the CFD desk has access to borrowing and lending lines at a funding rate of $r_B(t)$ and is not subject to any credit risk.

3. Application of the Feynman-Kac Theorem: By the Feynman-Kac Theorem we have that

$$\mathbb{E}_0\left[e^{-\int_0^T r_B(u)du} X(T)\right]$$

$$= \mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du} X(T) - \int_0^T e^{-\int_0^t r_{OIS}(u)du}\{r_B(t) - r_{OIS}(t)\} V_X^*(t)dt\right]$$

and also by symmetry

$$\mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du} X(T)\right] = \mathbb{E}_0\left[e^{-\int_0^T r_B(u)du} X(T) - \int_0^T e^{-\int_0^t r_B(u)du}\{r_{OIS}(t) - r_B(t)\} V_X(t)dt\right]$$

4. Simplification of the SFVA Calculation: It follows that

$$SFVA_X = V_X(0) - V_X^*(0)$$

may be rewritten as

$$SFVA_X = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du} s_B(t) V_X^*(t)dt\right] = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_B(u)du} s_B(t) V_X(t)dt\right]$$

In practice, the dependence on $V_X^*(t)$ in the exposure integral is inconvenient, so the second equality is probably the most useful in applications.

5. Applying SFVA to the Netting Set: Extending the result above to the more general portfolio case with collateralized hedges, we write the SFVA for counterparty i as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)dt\right]$$

Let us remark that assuming there are no collateral thresholds, the $SFVA_i$ decomposes further into a sum over the SFVA of trades contained in the i^{th} netting set. Collateral thresholds break this property and make the SFVA a netting set specific amount.

6. Decomposition into FCA/FBA Metrics: The netting set specific SFVA can be decomposed into cost and benefit components as follows.

$$SFVA_i = FCA_i - FBA_i$$

where

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

and

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

We also introduce the aggregate amounts

$$SFVA = \sum_i SFVA_i$$

$$FCA = \sum_i FCA_i$$

$$FBA = \sum_i FBA_i$$

CA and CL Adjustments

1. Lack of Consistency in FCA/FBA: While, as described earlier, FVA/FDA accounting cleanly splits the CA and CL adjustments, the lack of consistency in the FCA/FBA accounting requires some effort to strike a reasonable compromise between conflicting assumptions. For this purpose, notice that the FBA in

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

is quiet similar to the DVA in

$$DVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \mathbb{I}_{t<\tau_i} \lambda_B(t)(1-R_B)V_i(t)^-\right] dt$$

yet it differs from it in two ways.

2. Effective Discount Rate in FCA/FBA: First the FBA expression does not contain the indicator function $\mathbb{I}_{t<\tau_i}$. Second, the FBA losses are discounted at the rate

$$r_B(t) = r_{OIS}(t) + \lambda_B(t)(1-R_B)$$

rather than at (effectively) the rate $r_{OIS}(t) + \lambda_B(t)$. Only in the unlikely case of

$$R_B = 0$$

does the discounting rule in the two expressions match. In general we have that

$$FBA > DVA$$

3. FBA as a CL Adjustment: The FBA partially accounts for the RHO, but always contains a large overlap with the standard DVA on payables. Since the two components of FBA cannot be disentangled from each other, the full FBA entry is normally configured as a CL adjustment, similar to the way DVA is treated. On the other hand the FCA is clearly a CA deduction from CET1 that adds to the usual CVA deduction.
4. Comparison of FCA/FDA and FCA/FBA:

	CA Adjustment	**CL Adjustment**
FVA/FDA	$UCVA + FVA$	$CVA_{CL} + DVA + FDA$
FCA/FBA	$UCVA + FCA$	$CVA_{CL} + FBA$

5. FCA/FBA vs. FDA/FVA CA Adjustments: To comment on the relative magnitudes of the CA and CL adjustments between the two methods, notice that since the FCA/FBA approximation effectively amounts to recognizing the re-hypothecation benefits only within the individual netting sets, one expects that the FCA is much larger than the FVA, i.e., the FCA/FBA accounting produces much larger CA deductions than the FVA/FDA accounting.
6. FVA/FDA vs FCA/FBA CL Adjustments: As for the CL adjustments, note that the CL adjustment under the FCA/FBA accounting effectively drops the DVA term in order to avoid double-counting. The case study conducted by Albanese and Andersen (2014) confirms these conclusions. They also quantify the RHO amount, the degree of symmetry breaking in the FCA/FBA accounting, and the difference between FBA and DVA.

Own Credit Sensitivities

1. Expected Positive Exposure Valuation Formulation: As discussed earlier CET1 deduction should generally not decrease when the bank spreads increase. To investigate the sensitivity of the FVA to the bank spread, it is necessary to estimate the discounted Expected Positive Exposure (EPE) up to time t as

$$EPE(t) = \mathbb{E}\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where T (the upper integration limit) is the longest maturity on any trade in the funding set in question.

2. <u>FVA Funding Spread Impact Theorem</u>: Assume that $s_B(t)$ is independent of $r_{OIS}(t)$ and $\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+$ and let $s_B(t)$ be subject to a perturbation of the type

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

where ϵ is a scalar and

$$h(t) \geq 0$$

is a bounded deterministic function. Also define the Gateaux derivative

$$\mathcal{D}_h FVA = \left.\frac{\partial FVA}{\partial \epsilon}\right|_{\epsilon=0}$$

Then

$$\mathcal{D}_h FVA \geq 0$$

if and only if

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1-{R_B}^C}s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

3. <u>FVA Funding Spread Impact - Proof</u>: Defining

$$q_B = \frac{1}{1 - {R_B}^C}$$

and re-casting FVA as a function of ϵ in

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

$$FVA = \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s) + q_B[s_B(s) + \epsilon h(s)]\} ds} \{s_B(t) + \epsilon h(t)\} \left\{\sum_i V_i(t) \mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where we have used

$$s_B(t) = \left(1 - {R_B}^C\right)\lambda_B(t)$$

Straightforward calculus shows that

$$\begin{aligned}
\mathcal{D}_h FVA = \left.\frac{\partial FVA}{\partial \epsilon}\right|_{\epsilon=0} \\
&= \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s) + \lambda_B(s)\} ds} \left\{h(t) \right.\right. \\
&\left.\left. - q_B s_B(t) \int_0^t h(s)\, ds\right\} \left\{\sum_i V_i(t) \mathbb{I}_{t<\tau_i}\right\}^+ dt\right] \\
&= \int_0^T EPE(t) \mathbb{E}\left[e^{-\int_0^t \lambda_B(s) ds} \left\{h(t) - q_B s_B(t) \int_0^t h(s)\, ds\right\}\right] dt
\end{aligned}$$

where the second equality follows from the independence assumption. The criterion

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - q_B s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

then follows.

4. Gateaux Function Lower Bound Corollary: In the setting of the previous theorem, if

$$h(t) \geq \frac{1}{1 - {R_B}^C}\frac{\mathbb{E}\left[s_B(t)e^{-\int_0^t \lambda_B(s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\right]}\int_0^t h(s)\,ds \geq 0\ \forall\ t \in [0, T]$$

then

$$\mathcal{D}_h FVA \geq 0$$

5. Gateaux Function Lower Bound Proof: This corollary follows directly from

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C}s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

and from the fact that

$$EPE(t) \geq 0$$

Further it turns out that it is a sufficient but conservative condition that ensures

$$\mathcal{D}_h FVA \geq 0$$

6. Right-way Regulatory Sensitivity Criterion: Suppose for instance that h and s_B are positive constants, i.e., we have a flat credit curve that is being shifted up in a parallel fashion. "Right-way" regulatory sensitivity is guaranteed by

$$h(t) \geq \frac{1}{1 - {R_B}^C} \frac{\mathbb{E}\left[s_B(t) e^{-\int_0^t \lambda_B(s) ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s) ds}\right]} \int_0^t h(s)\, ds \geq 0 \ \forall\ t \in [0, T]$$

if

$$h \geq \frac{s_B h T}{1 - {R_B}^C}$$

or

$$s_B \leq \frac{1 - {R_B}^C}{T}$$

7. Right-way Sensitivity Typical Behavior: Some typical values for the constants in $\frac{1 - {R_B}^C}{T}$ are

$${R_B}^C = 40\%$$

and

$$T = 10$$

which yields

$$s_B \leq 6\%$$

a condition that is rarely violated (common values for s_B are $1 - 2\%$).

8. Alternate Right-way Regulatory Sensitivity: Alternatively, if we assume

$$EPE(t) \approx EPE$$

is approximately constant in

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C} s_B(t) \int_0^t h(s)\, ds\right\}\right] dt \geq 0$$

then it can be written (again for constant h and s_B)

$$\mathcal{D}_h FVA \approx EPE \cdot \int_0^T e^{-\lambda_B t}\left(h - \frac{1}{1 - {R_B}^C} s_B t\right) dt \approx EPE \cdot hT\left[1 - \frac{1}{2} q_B s_B T\right]$$

which leads to a bound that is twice as loose as before

$$s_B \leq \frac{2\left(1 - {R_B}^C\right)}{T}$$

9. Normal Situation FVA Regulatory Sensitivity: In most situations the $EPE(t)$ terms in

$$\mathcal{D}_h FVA = \int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - q_B s_B(t) \int_0^t h(s)\, ds\right\}\right] dt$$

typically peak long before T, which would widen the bound even further. All in all it is safe to say that in normal conditions FVA increases when s_B is decreased.

10. SFVA Funding Spread Perturbation Impact: If we repeat the arguments behind the FVA funding spread impact theorem for the SFVA defined in

$$SFVA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t)V_i(t)dt\right]$$

the result is easily seen to be

$$\mathcal{D}_h SFVA = \mathbb{E}\left[\int_0^{T} e^{-\int_0^t \{r_{OIS}(u)+s_B(u)\}du}\left\{h(t) - s_B(t)\int_0^{t} h(s)\,ds\right\}\left\{\sum_{i=1}^{n} V_i(t)\right\}dt\right]$$

11. Normal Conditions SFVA Regulatory Sensitivity: In case $h(t)$ satisfies the conditions above that ensure the positivity of $\mathcal{D}_h FVA$ and in situations where the funding sets are prevailingly net payables, the SFVA has wrong sign sensitivities and is unsuitable as a CET1 deduction. This issue is expanded later on.

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Triggers and Close-out Adjustments

Introduction

1. Motivation: This section provides a greater detail and a realistic analysis of the cash flows and the close out protocols.
2. Default-Free Fair Valuation: Let

$$V_i^U(t)\ \forall\ i = 1, \cdots, n$$

be the default free valuation at time t of the portfolio held with counterparty i, i.e., the valuation computed by neglecting the upfront XVA payments, and also neglecting credit risks and funding costs. The default free value of the unsecured book is additive over trades, in particular

$$V^U(t) = \sum_i V_i^U(t)$$

3. FTP Transfer via Super-Replication: Hedges are typically struck at par and engender cash flows across time that super-replicate the cash flows of the unsecured trades. Strict super-replication is required in order to ensure a positive return on equity.
4. Value Adjustment on Super-Replication: In the case of swaps adjustments are usually embedded as a fixed spread in addition to either the fixed leg or the floating leg or both. The cash flows are such that typically the default free valuation of the unsecured portfolio is positive at inception.
5. CFD Routing of the Hedge Flows: The hedge typically consists of one or a portfolio of collateralized swaps entered in the inter-dealer market, initially struck at par and whose cash

flows super replicate those of the unsecured trade. The excess cash flows are then routed to the CVA desk and the CFD desk at the time when they are received.

6. Reducing of the VM Posting Impact: In the case of swaptions and FX options premia are typically paid at maturity and struck at the level for which the present value of the option are zero. This structure lessens the amount of VM to be exchanged. Unsecured trade often pay an upfront premium which is added to the book cash account. By transforming a portion of the upfront payment into a payment at the option maturity on achieves super-replication in this case as well. Excess cash flows are given by the XVA adjustments.
7. Hedge Portfolios across Netting Sets: In general hedges are not specific to individual netting sets. Even if hedges are entered initially on a deal-specific basis, the compression cycles of the collateralized swap portfolio reduce the number of hedge trades and obfuscates the attribution of the individual netting sets.
8. Super Replicated Hedge Portfolio Value: Nevertheless by means of a hedge attribution analysis, it is possible to arrive at the concept of the value of the hedge book for a single portfolio i – we call it ${V_i}^H(t)$. Super-replication is achieved when

$${V_i}^H(t) \leq {V_i}^U(t)$$

The present value of the difference ${V_i}^U(t) - {V_i}^H(t)$ is the FTP.

9. Value of the Hedge Book: The value of the hedge book is denoted as

$${V_i}^H(t) = \sum_i {V_i}^H(t)$$

10. Gap Risk on Collateralized Hedges: As the inter-dealer market and the exchanges require the posting of the collateral in full, the hedge trades are affected by the bank/counterparty credit risk only because of the gap risk exposure. For simplicity's sake the gap risk on collateralized hedges is neglected here since the analysis and the notations become very heavy. However a professional system implementation should account for it as the effect can be quite material.

Collateral Triggers and Close-outs

1. CSA Based Collateral Trigger Levels: Typically CSA collateral agreements include time-dependent collateral trigger levels $\Gamma_i(t)$ that dependent on the counterparty rating level and is such that, if the exposure of the counterparty surpasses $\Gamma_i(t)$ then the counterparty is obliged to post collateral above that threshold. The bank has a similar trigger level $\Gamma_{B,i}(t)$ for each counterparty i. Initial margin on unsecured trade is not a common practice as they would be ineffective unless thresholds are struck at virtually zero level.
2. Collateral Trigger Impact on VM: The equation for the variation margin accounts can be refined for ratings dependent collateral thresholds and also for the fact that variation margins are received on the CVA hedges. The more general expression is

$$C_{VM,i}(t) = \mathbb{I}_{\tau_B > t}\mathbb{I}_{\tau_i > t}\left[-V_i{}^H(t) + \left\{V_i{}^U(t) - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t) - \Gamma_{B,i}(t)\right\}^- - CVA_{i,CA}(t)\right]$$
$$\approx \mathbb{I}_{\tau_B > t}\mathbb{I}_{\tau_i > t}\left[-V_i{}^H(t) + \left\{V_i{}^U(t) - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t) - \Gamma_{B,i}(t)\right\}^-\right]$$

   where again the approximation is implemented in the Albanese and Andersen (2014) case study.
3. Generalized Hedge VM Funding Rate: The equation

$$s_B(t) = r_B(t) - r_{OIS}(t)$$

   for the spread $s_B(t)$ can also be extended. In general $s_B(t)$ should defined as the spread between the funding rate of the bank on the short-term debt instruments used for the purpose of financing variation margin and the rate received on the VM posted. In general the interest rate received on the VM posted may differ from OIS only by a small spread.
4. Trigger Based FVA/FDA Impact: The portfolio FVA is given by (note the VM related alteration)

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

The FDA is sensitive to the value of bank receivables at the time of the bank impact and is given by

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - R_B{}^C\right)\lambda_B(t)\left\{\sum_i \max\left(\min\left(V_i{}^U(t), \Gamma_i(t)\right), -\Gamma_{B,i}(t)\right)\right\}^+ dt\right]$$

5. Composite Trigger-Based FVA Formulation: On the above basis, the expression for asymmetric FVA from the bank perspective in

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

needs to be amended to

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i \left[-V_i{}^H(t)\mathbb{I}_{\tau_i>t} + \left\{V_i{}^U(t)\mathbb{I}_{\tau_i>t} - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t)\mathbb{I}_{\tau_i>t} - \Gamma_{B,i}(t)\right\}^-\right]\right\}^+ s_B(t)dt\right]$$

This equation accounts for the VM to be posted to or received from unsecured derivative counterparties depending on the thresholds.

6. Asset-Liability Super Replication Mismatch: Notice that FVA and FBA are not precisely equal but they are very close. Because of the inequality

$$V_i^H(t) \leq V_i^U(t)$$

it turns out

$$FVA \leq FDA$$

This discrepancy is due to the fact that the FDA payments are embedded in the deal structure, and at the time of the bank default, the senior creditors still hold a claim to future FTP payments for funding without the obligation (or even the capability) to continue hedging.

7. Super Replication Impact on MMT: All in all, since banks enact slightly super-replicating strategies (as opposed to precise replication), in order to have a positive return on equity, the FDA tends to be slightly larger than the FVA.
8. Wealth Transfer From Derivative Counterparties: The above can be viewed as a wealth transfer from the derivatives counterparties to the senior creditors. The MMT is still not invalid as the game is still zero sum, as long as one includes in the analysis not only shareholders and senior creditors but also the derivatives counterparties.
9. Trigger Based FCA/FBA Impact: The expression for $C_{VM,i}(t)$ above can be used to extend

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

as

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^+ dt\right]$$

while

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

becomes

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^- dt\right]$$

Incorporating ISDA 1992 Close-outs

1. Impact of the ISDA Closeouts: The representation of the cash flows at the time of counterparty defaults under the ISDA 1992 closeout protocol in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i^U(\tau_i) \geq 0} R_i V_i^U(\tau_i) + \mathbb{I}_{V_i^U(\tau_i) < 0} V_i^U(\tau_i)\right]$$

is modified as follows to include collateral thresholds:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i^U(\tau_i) \geq 0} R_i \min\left(V_i^U(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{V_i^U(\tau_i) < 0} V_i^U(\tau_i)\right]$$

Similarly

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{V_i^U(\tau_B) \geq 0} V_i^U(\tau_B) + \mathbb{I}_{V_i^U(\tau_B) < 0} R_B V_i^U(\tau_B)\right]$$

becomes

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{V_i^U(\tau_B) \geq 0} V_i^U(\tau_B) + \mathbb{I}_{V_i^U(\tau_B) < 0} R_B \max\left(V_i^U(\tau_B), -\Gamma_B(\tau_B)\right)\right]$$

2. Exit Price of the Trade: Under the 2009 ISDA Close-out rules, cash flows differ in that ${V_i}^U(\tau_i)$ is replaced by the exit price of the trade, including the residual DVA_i at the time of the default. On the other hand, in case the bank defaults first, then the i^{th} counterparty has a right to recover its own DVA, which is the CVA. This implies that when valuing CVA from the bank viewpoint, these default losses happen after the default of the bank itself.
3. Closeout Payments after the Default:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{{V_i}^{(+)}(\tau_i)\geq 0} R_i \min\left({V_i}^{(+)}(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{{V_i}^{(+)}(\tau_i)<0} {V_i}^{(+)}(\tau_i)\right]$$

$$D_i(\tau_B) = \mathbb{I}_{\tau_i>\tau_B}\left[\mathbb{I}_{{V_i}^{(-)}(\tau_B)\geq 0} {V_i}^{(-)}(\tau_B) + \mathbb{I}_{{V_i}^{(-)}(\tau_B)<0} R_B {V_i}^{(-)}(\tau_B)\right]$$

where

$${V_i}^{(+)}(\tau_i) = {V_i}^U(\tau_i) + DVA_i(\tau_i) \approx {V_i}^U(\tau_i)$$

and

$${V_i}^{(-)}(\tau_B) = {V_i}^U(\tau_B) - CVA_i(\tau_B) \approx {V_i}^U(\tau_B)$$

4. SFVA Estimation Under Collateral Trigger: The definition of SFVA in

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t) dt\right]$$

is extended as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) \max\left(\min\left({V_i}^U(\tau_i), \Gamma_i(\tau_i)\right), -\Gamma_B(\tau_B)\right) dt\right]$$

5. Trigger Based UCVA and DVA: The valuation formula for the UCVA under ISDA 2009 including CSA thresholds is given by

$$UCVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+, \Gamma_i(t)\right)dt\right]$$

The DVA is given by the following extension

$$DVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\lambda_B(t)(1-R_B)\left\{-\min\left(\left\{V_i{}^U(t)\right\}^-, \Gamma_B(t)\right)\right\}dt\right]$$

6. MTM ISDA Closeout Impact: Under ISDA 2009, at the time of the bank default, the bank needs to mark-to-market the derivatives by also including the CVA discount. Symmetrically, whenever the counterparty defaults the bank is also entitled to recover a DVA benefit.
7. Modified Bilateral FTD CVA Adjustment: The DVA benefit entitlement, however, typically gives rise to a negligible correction that is currently ignored in order to avoid the need for nested simulations for the calculation of the CVA itself. Hence the fair valuation formula for the CVA is

$$\begin{aligned} FTDCVA_i = \int_0^\infty \mathbb{E}\Big[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1 \\ - R_i)\min\left(\left\{V_i{}^U(t) + DVA_i(t)\right\}^+, \Gamma_i(t)\right)\Big]dt \\ \approx \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+, \Gamma_i(t)\right)\right]dt \end{aligned}$$

8. Modified CVA CL and CA: In this case we also have an alteration to the contra-liability given by the present value of the default losses occurring after the bank default, i.e.,

$$
\begin{aligned}
CVA_{CL,i} = -\int_0^{\infty} \mathbb{E}\Big[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1 \\
- R_i)\min\Big(\{V_i{}^U(t)+DVA_i(t)\}^+,\Gamma_i(t)\Big)\Big]\,dt \\
\approx -\int_0^{\infty} \mathbb{E}\Big[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1-R_i)\min\Big(\{V_i{}^U(t)\}^+,\Gamma_i(t)\Big)\Big]\,dt
\end{aligned}
$$

The sum

$$
FTDCVA_i = UCVA_i + CVA_{CL,i}
$$

is the first-to-default unilateral CVA and represents the fair valuation of the counterparty credit.

VM Rehypothecability Across Funding Sets

1. Restrictions on VM Rehypothecation: In their case study, Albanese and Andersen (2014) model rehypothecation by assuming that the VM received can be posted back to meet posting requirements by the hedges in the same funding set. This is a simplifying assumption that may have to be refined in practical implementations, For instance, there may be restrictions in the CSA preventing banks from being able to rehypothecate the VM.
2. Impact of the Rehypothecation Ban: Rehypothecation bans are sometimes triggered by a degradation in the bank credit quality. Whenever this happens, the cost of funding is effectively increased and the bank has a greater interest in "flattening out" its book.
3. Funding Costs Under Stress Conditions: A careful simulation of the funding costs should account for the resulting increase in funding costs that occur whenever such stress conditions manifest themselves.

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Entry Prices, Exit Prices, and Trade FTP

Trade and Portfolio FTP Estimation

1. The Breakeven Trade FTP: We recall that the FTP policy defines the costs or benefits that must at a minimum be passed to the clients in excess of the default-free trade values (FTP can be negative if the trade reduces the credit risk of the funding costs for the bank). In other words, the FTP defines what constitutes a "break-even" transaction and thereby determines the bank's *entry price*, i.e., the price that the bank would bid to acquire a trade or possibly a collection of trades.
2. Marginal FTP Impact on Portfolios: As mentioned earlier XVA-aware accounting systems do not aggregate portfolios linearly over trades, so FTP policies cannot operate only at the trade level but must take into consideration the effect that a new trade has on the risk profiles of existing trade aggregations (at the netting set or at the funding set levels). For this reason it is reasonable to set out FTP policies in terms of *marginal* increments to portfolio level metrics.
3. FCA/FBA vs FVA/FDA Magnitude Differences: As the choice of the accounting methods have material implications on the choice of various accounting quantities, the FTP policies are necessarily different in the FCA/FBA and the FVA/FDA frameworks. This is discussed in the next two sections. The topic of *exit prices* and fair value accounting are subsequently covered.
4. Transaction Cost Incorporation in Pricing: By convention entry prices include transaction costs and exit prices exclude them. For the purposes of this discussion the assumption throughout here is that the transaction costs are zero.

FTP For FCA/FBA Accounting

1. Netting Set Granularity For FTP: In the case of FCA/FBA accounting, the relevant units of account refer either to default-free trades or (for CVA, DVA, FCA, and FBA) to the netting sets. There are no funding set level units of account in the FCA/FBA method.
2. Credit and Funding Cost Components: In the FCA/FBA accounting, the standard FTP policy is to transfer to the clients at a minimum the incremental amount

$$FTP = \Delta UCVA + \Delta SFVA$$

$\Delta UCVA$ and $\Delta SFVA$ cover the counterparty credit risk and the funding charges respectively, and are paid by the business line trading desk to the CVA and CFD desks.
3. FTP Impact on Equity Capital: From

$$\Delta CET1 = -\Delta CA + \Delta RE$$

we see that the incremental impact of a new trade on the equity capital is

$$\Delta CET1 = -\Delta CA + FTP = -\Delta UCVA - \Delta FCA + \Delta UCVA + \Delta SFVA = -\Delta FBA$$

while the change in liabilities is

$$\Delta CL = \Delta FBA$$

By design the FTP policy in FCA/FBA accounting ensures that the impact on income from adding a new trade is zero, i.e.,

$$\Delta PFV = \Delta CET1 + \Delta CL = 0$$

4. FTP Alignment with the Shareholders' Wealth: By tailoring the FTP in such a way that new trades have no impact on income, the deals flows induces volatility in CET1, as is evidenced from

$$\Delta CET1 = -\Delta FBA$$

As CET1 is a proxy for shareholders wealth, it may be argued that

$$FTP = \Delta UCVA + \Delta SFVA$$

does not fully align executive incentives with shareholders' interests. An FTP policy designed for CET1 indifference would, however, not be viable for FCA/FBA accounting due to the sheer size of the full FCA, as is demonstrated in the case study by Albanese and Andersen (2014).

FTP For FVA/FDA Accounting

1. FTP as a CET1 Enhancer: In FVA/FDA accounting, FVA and UCVA are CET1 deductions and DVA, FVA, and CVA_{CL} are CL adjustments. The upfront charge for a given trade that a given business line desk needs to pay the CFD desk in order to ensure that CET1 stays constant is

$$FTP = \Delta UCVA + \Delta FVA$$

With this choice

$$\Delta CET1 = 0$$

2. Netting and Funding Set Granularity: The calculation of FTP involves assessing the marginal impact on two separate units of account:
    a. The netting set associated with the counterparty to the trade in question.

b. The overall funding set under which the VM may be re-hypothecated. This calculation in particular is not always trivial – this is treated later.

3. FTP Shareholder Centric View: By focusing on CET1

$$FTP = \Delta UCVA + \Delta FVA$$

effectively expresses a share-holder centric view in the computation of the FTP, where the proxy for "true" deal in the FTP is only the shareholder part of the overall value that a trade brings to the firm.

4. FTP Impact on Net Income: For trades that involve bond holder benefit post bank default, any such benefits are ignored when passing costs to the client. When it comes to net income the FTP policy therefore has an impact given by

$$\Delta PFV = \Delta CL = \Delta DVA + \Delta FDA + \Delta CVA_{CL}$$

Again this is a marginal calculation at the level of both the netting set (DVA, CVA_{CL}) and the funding set (FVA, FDA).

5. Alignment Across Debt and Equity: It needs to be emphasized that while the FTP policy in FVA/FDA accounting is based on principles more conservative than those for FCA/FBA, the inclusion of RHO into FVA causes the funding cost component of the CA account to be materially smaller in the absolute value than FCA/FBA accounting (by about a factor of two in the case study examples of Albanese and Andersen (2014)). As a consequence CET1 immunization is practical in the case of FVA/FDA method and results in an FTP principle that aligns the interests of the bank managers and the shareholders.

Exit Prices and Fair Valuation

1. Deciding on the Unit of Account: Exit prices are important from an accounting point of view as they provide a model-independent method to access fair valuation. As a general approach, the first step is to decide what is to be considered a unit of account.
2. Auctioning for an Account Unit: If a unit of account can be transferred in a market with sufficient liquidity and without altering any of the expected cash flows an auction process may be run and a best bidder chosen.
3. Model Pricing for the Account Unit: Failing the execution of the auction process, a model-based valuation method is used with a model calibrated consistently to all available market pricing information for the relevant risk factors.
4. Consistency with the Accounting Principles: According to accounting principles AP0 and AP1 above, the exit prices discovered through the steps outlined in the general approach above are what should be recorded as fair value for derivative assets and liabilities on the balance sheet.
5. Granularity of a Single Trade: Unless trades are fully collateralized, the notion that a single trade can ever be considered a unit of account is, as discussed before, an incorrect one. So while it may be tempting to ask for the market price of a single trade, the question becomes truly meaningful only when asked about entire netting set portfolios – and sometimes even that may be too granular.
6. Granularity of a Full Portfolio: Further when discussing portfolio prices, not only must the netting set portfolios be held fixed, but so must also the credit quality of the bank and its counterparty – otherwise cash flows induced by the default are not the same. For instance, trying to assign a portfolio from a low-rated bank to a high-rated bank automatically changes the cash flow characteristics of a bilateral portfolio and violates the apples-to-apples provision in the auctioning setup above.
7. Compatibility with the Bidding Bank: Even if the bidding bank has exactly the right credit spread, the bidding bank very likely has trades with the counterparty in question. The bid is therefore not for the original portfolio in question, but for a combined portfolio of old and new trades.

8. Need to invoke Model Pricing: Due to the above effects it is exceedingly rare that an auction of the type discussed is ever practical; instead the model based pricing approach is virtually always called upon.

FVA/FDA Accounting

1. Exit Price Trade Level Granularity: As we have seen, exit price at the level of a netting set operates at too narrow a unit account, as funding costs due to re-hypothecation must still be modeled at the funding set level. In the FVA/FDA method this certainly is required for the FTP and entry price calculations as just seen above, but is not required for fair value calculations.
2. CA and CL Adjustment Cancellation: The reason for this, of course, is that cash flows associated with the funding operations are modeled as having zero net value for the firm as a whole, consistent with the MMT. This manifests itself in the cancellation of the CA and the CL adjustments for funding, such that funding costs never make it to the level of net income and net fair asset/liability values on the balance sheet.
3. Price Impact on Net Income: DVA and CVA, however, generally do not cancel out, and their difference shows up in net fair values. Specifically for the FVA/FDA accounting, the Portfolio Fair Value (PFV) of the derivatives is, according to the table seen earlier,

$$PFV = A - L - CA + CL = A - L - FTDCVA + DVA$$

   an expression that contains in $A - L$ the present values of promised cash flows, and in $-FTDCVA + DVA$, the present values of the lost cash-flows due to default.
4. Trade Level Cash Flow Focus: The above flows are focused on cash flows generated by the trades in isolation and does not reference how the trades are funded. As a consequence, as mentioned above, $A - L - FTDCVA + DVA$ is additive over netting sets. However, this is not the case for FTP expressions in FVA/FDA accounting which are aligned CET1 (a non-additive metric over netting sets).

FCA/FBA Accounting

1. Price Impact on Net Income: In FCA/FBA accounting funding costs are allowed to hit net income as the PFV is given by

$$PFV = A - L - UCVA - FCA + FBA = A - L - UCVA - SFVA$$

   This measure is a mixture of partial and incomplete cash flow valuations with a measure of funding costs thrown in. As in the expression for PFV in FVA/FDA accounting, the expression is additive over netting sets.
2. Caveat - Fair Valuation of CVA: UCVA is generally not the fair valuation of the CVA, as it ignores self-survival. Similarly the expression for PFV in FCA/FBA accounting does not contain the DVA and the approximation

$$FBA \approx DVA$$

   is a poor one.
3. Violation of the MMT Principles: As mentioned earlier it is clear that

$$PFV = A - L - UCVA - SFVA$$

   above violates both the MMT and the principle of asset-liability symmetry. One should also ask the question whether the presence of SFVA in

$$PFV = A - L - UCVA - SFVA$$

   amounts to an entity-specific cost adjustment decoupled from trade cash flows, something that violated the principles of exit pricing and normally not allowed in fair value accounting.

4. Popular Justification for the FCA/FBA Accounting: Proponents of the FCA/FBA accounting often attempt to argue away the entity specific nature of the PFV expression by suggesting that their bank's credit spreads are "representative", wherefore the SFVA represents a market average that can be used for exit pricing purposes. This is a questionable line of reasoning for several reasons.
5. Choice of Appropriate Funding Spread: First, even if the bank's funding spread is close to the market average at some point, this may cease to be the case in the future, especially if the bank itself approaches default. And second it is unclear if the market average spread is a metric that can be used for exit pricing and PFV computations – perhaps the "best" spread (i.e., high spreads for receivables and low spreads for payables) could be argued to be more appropriate.
6. Industry Standard Proxy for Funding: A related line of thought suggest that the SFVA term in

$$PFV = A - L - UCVA - SFVA$$

   is not to be computed at the bank's own spread, but at a separately marked industry spread. This removes the entity specific nature of the FCA/FBA PFV, but also decouples FBA entirely from DVA, whereby FCA/FBA accounting would violate one of the accounting principles.
7. Impact of Exogenously Marked Spread: One also wonders where the industry spread curve is supposed to come from, especially since such curves are very hard to detect at the level of individual trades (since FTP entry prices operate on netting or funding set metrics only). The idea of an exogenously marked spread is treated in detail in the next section.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Liquidity Spreads, Asset Liability Symmetry, and Alternative Allocations for Excess Collateral

Motivation

1. Assumptions Underlying the FCA/FBA Scheme: This section reviews the FCA/FBA accounting framework by examining a variety of assumptions that have been put forward to justify at least some of the elements of the FCA/FBA accounting ideas. These assumptions are quiet strong, and not necessarily realistic.
2. Working Capital Management for Derivatives: Derivatives are normally funded on a short-term basis, as the funding needs associated with derivatives trades typically exhibits considerable variation through time. The inherent variability in turn would make it unlikely that the bank's treasury department would commit to systematically using excess collateral from derivative trading to retire general term debt from the bank's liabilities.
3. FCA/FBA Working Capital Assumptions: Yet, as seen earlier, this assumption is essentially one that is required to make sense of some of the FCA/FBA accounting ideas.
4. Returns on the Excess Collateral: Earlier it was suggested that one use a more conservative and reasonable assumption that the excess collateral is simply invested in short-term risk-free investments, earning a rolling rate of $r_{OIS}(t)$. This way no shareholder gain is generated out of variable excess collateral.

Working Capital Management and Operations

1. Shareholder/Creditor Income Share: The sum of the values of the bank to the shareholders and the senior creditors add up in income statements and define the PFV of the bank. The value of the debt from collateral lenders is excluded from this calculation.

2. Handling of Excess Working Capital: Whenever the bank finds itself in the situation where it has excess cash the bank managers have an option to deleverage by buying back the senior debt. According to MMT neither the decision to leverage up with collateral lenders or de-leveraging by debt buy-back have a net impact on income. However the two decision differ in terms of wealth transfer between the shareholders and the senior creditors.
3. Fair Value Bond Buy-back: The interest paid to collateral lenders triggers a wealth transfer between the shareholders and the senior creditors of a bank and this is quantified by the FVA and the FDA. Whenever bonds are bought back/sold to from the senior creditors at their fair value, the wealth of both the shareholders and the senior creditors is not affected by the decision to de-leverage.

Equity Gain and Debt Gain

1. Equity/Debt Gain Overview and Definition: An alternative to the FVA/FDA assumption would be to assume that the short-term collateral excesses can be invested in strategies that lead to time 0 shareholder and debt-holder increases of EG ("Equity Gain") and DG ("Debt Gain") respectively.
2. MMT Consistency Across the Gains: It would generally be a stretch to build into accounting statements any "sure thing" on firm-wide profitably of investment strategies, it seems that one should at least require

$$EG + DG = 0$$

   This is of course the condition required to satisfy MMT.
3. CA and CL Impact of Gains: EG and DG are specific to whatever investment strategy that the treasury commits to, but should one somehow be able to project their values, the FVA/FDA accounting method could be adapted as follows:

$$CA\ Entries := FVA + UCVA - EG$$

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

4. Funding Definitions Impact on Accounting: This is a good time once again to point out that the funding cost definitions are not immutable, but depend strongly on the assumptions made on how the finds are raised and invested. Baking any such assumptions into the accounting numbers obliges the bank to actually follow strategies on which it based its accounting numbers.
5. Maintenance of the Asset-Liability Symmetry: As

$$EG + DG = 0$$

implies

$$EG = -DG$$

it is clear that introducing EG and DG into the FVA/FDA accounting would preserve the asset-liability symmetry and would not have any effects on the Net Income. However if $EG \neq 0$ new terms would arise at the balance sheet accounting level and would potentially affect CET1.
6. Positive EG and Debt Covenants: As bank managers should not engage in trading strategies where

$$EG < 0$$

the introduction of the EG and the DG terms would realistically only involve the cases where

$$EG > 0$$

and therefore

$$DG < 0$$

i.e., it is possible to only consider collateral investment strategies that prevent wealth transfers from shareholders to senior creditors. Such strategies are possible in principle but are non-trivial to setup since bond covenant put serious restrictions on any activities that enrich the shareholders solely at the expense of senior creditors.

7. Counterparty Credit Risk Transfer: Transferring counterparty credit risk to third party investors may prevent wealth transfers from shareholders to senior creditors. However unless such strategies are properly quantified and executed, theoretical equity gains should not be reflected in accounts just because they are possible. As a consequence the FVA/FDA assumption of

$$EG = DG = 0$$

is a more reasonable and a rigorous one.

8. Shareholder Equity Gain Formulation: Pursuing the extensions above a bit further, supposing that the investment benefits are assumed to accrue at the rate of $s_G(t)$ where $s_G(t)$ is interpreted as the spread over $r_{OIS}(t)$ returns. In this case we would write, long the same lines as

$$FVA(t) = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

9. Equity Gain as Return Rate: For the special case where

$$s_G = s_B$$

we can observe that

$$FVA - EG(s_B) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t)\left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\} dt\right]$$

where now the max operator has disappeared from the expectation. Approximating the recovery rate of the bank as zero, i.e., setting

$$R_B = 0$$

we find that

$$FVA - EG(s_B) \approx FCA - FBA = SFVA$$

10. <u>Debt Retirement using Equity Gain</u>: Of course setting

$$s_G = s_B$$

in

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t)\left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

basically amounts to debt retirement – a strategy that was questioned, so this cannot be endorsed. Nevertheless

$$FVA - EG(s_B) \approx SFVA$$

does help understand better the underpinning of the FCA, the FBA, and the SFVA terms.

11. Gain Accounting vs FCA/FBA Accounting: It needs to be emphasized, however, that setting

$$s_G = s_B$$

and following the CA and the CL entries above does not reproduce the FCA/FBA method, not even approximately. For instance, note that the FCA/FBA method sets the CA account to $CVA + FCA$ whereas

$$FVA - EG(s_B) \approx SFVA$$

results in a CA account of $CVA + SFVA$

$$CA\ Entries := FVA + UCVA - EG$$

and

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

constitutes an accounting method with asset-liability symmetry, whereas FCA/FBA method does not.

Liquidity Based Analysis and Treatment

1. Asset-Liability Symmetry without MMT: Let us note that it is possible to preserve asset-liability symmetry without satisfying the MMT. Suppose that the entire market decides that a liquidity spread of s_L - unrelated to the compensation of the default risk – applies to all the discounting operations on unsecured derivatives. Incorporation of this spread would only

mean a universal redefinition of the default-free security value V_i – something that would result in asset and earnings re-statements across firms, but would not break the asset-liability symmetry.

2. Liquidity Value Adjustment Metric Formulation: More concretely we define a Liquidity Value Adjustment (LVA) as

$$LVA(s_L) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i\,(t)\right\}dt\right]$$

$$= \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^+\right\}dt\right]$$

$$+ \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^-\right\}dt\right] \triangleq LVA_A(s_L) - LVA_L(s_L)$$

3. CA/CL Components of LVA: One obvious way of accounting for liquidity spreads would be to let $LVA_A(s_L)$ and $LVA_L(s_L)$ to be entered as contra-asset and contra-liability, respectively. It is clear here that $LVA_L(s_L)$ - unlike all other terms in

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

– is not associated with self-default or wealth transfers, and therefore should *not* be excluded from CET1. Note also that introducing the LVA would result in an earnings impact equal to the LVA.

4. LVA in FCA/FBA Accounting: A radical idea is to assume that the funding spread s_L is unrelated to default and in actuality is just a friction-type liquidity spread. In that case one may interpret the SFVA as an LVA by noting the identity

$$SFVA = LVA(s_L)$$

as well as

$$FCA = LVA_A(s_L)$$

and

$$FBA = LVA_L(s_L)$$

5. Symmetric Asset-Liability Accounting Rule: In this interpretation we would use the following accounting rule:

$$CA\ Entries := UCVA + LVA$$

$$CL\ Entries := CVA_{CL} + DVA$$

   where effectively only $CVA_{CL} + DVA$ (but not FBA) would need exclusion from the regulatory capital.
6. DVA Double Counting and Resolution: Besides stretching the imagination by assuming that the liquidity spreads are not as large as credit spreads, this rule effectively double counts the DVA. Note that this does not preserve the MMT (and results in a funding impact on the earnings), but does preserve the asset-liability symmetry. If as in FCA/FBA accounting, DVA is removed from the CL entries to avoid double counting, the accounting symmetry is broken.

Problems with the Gain Accounting

1. SFVA Wrong-Way Sensitivity Impact: The approach above is problematic not only from an accounting angle, but also due to the regulatory viewpoint due to the wrong-way sensitivity of the SFVA on the bank's own credit.

2. Misplaced Incentives from FCA/FBA Accounting: In case there were a generalized acceptance of the SFVA being interpreted as an LVA and qualified for a CET1 deduction, the FTP's computed using the SFVA would incentivize traders to buy payables and sell receivables, since they overstate both the funding costs for the latter and the funding benefits for the former.
3. Worsening Bank Credit CET1 Impact: Hence portfolios of banks with material credit spreads would drift towards being net payables. One can imagine that the SFVA for the worst funders would become negative and develop a wrong-sign sensitivity with respect to own credit. A vicious cycle may ensue since the worsening of the credit of the bank would ipso facto increase equity capital.
4. Sudden Accounting Regime Change Impact: If a blow-up occurs and the accounting standards need to be changed for all the market participants to enforce an FCA or an FVA deduction, the system-wide impact on the regulatory capital would be pervasive.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Albanese and Andersen (2014) Case Study

Case Study Setting and Purpose

1. Errors Associated with the FCA/FBA Accounting: While market participants are generally aware of the fact that the FCA/FBA accounting is approximate, little is known about the magnitude of the errors involved. This is particularly true of the RHO due to the computational challenges involved, the proper accounting of which involves simulation of the entire funding set with multiple netting sets.
2. FCA/FBA vs. FDA/FVA Accounting Comparison: In their section on case study findings, Albanese and Andersen (2014) attempt to shed some light on the materiality of the errors by using a realistic test-bed to compare the FCA/FBA and the FVA/FDA accounting results. They obtained results using $Global\ Valuation\ Esther^{TM}$, an in memory risk analytics system designed for the simulation of massive OTC portfolios.
3. Global Valuation Esther and Athena: Esther uses an mathematical framework based on operator algebras as discussed in Albanese, Bellaj, Gimonet, and Pietronero (2011) and references therein. Model calibrations were taken from $Global\ Valuation\ Athena^{TM}$ data service and refer to 7 July 2014.
4. The Fixed Income OTC Portfolio: As their test case, they use a realistic portfolio of fixed income derivatives of about 100,000 trades, 1,600 counterparties, and a variety of collateral agreements, some involving thresholds. The portfolio contains trades in 8 different currencies, including swaps, cross-currency swaps, swaptions, FX forwards, and options. The simulation entails 100 time steps at each of which they find scenarios for the default-free valuation of all the netting sets.
5. Types of Netting Sets Considered: Most netting sets in their tests are at least partially unsecured, with the collateral thresholds being either positive finite or infinite. Some large netting sets are of the unilateral "government" type, i.e., the threshold for the bank is zero

and the threshold for the counterparty is infinite. Caveat – if all netting sets were of the government type, then the FCA would equal the FVA, as the government type netting sets do not contribute to the RHO.

6. Fully Collateralized Netting Set Construction: The portfolio contains also a few fully collateralized netting sets with both thresholds at zero, but they do not contribute to funding and contribute to CVA only mildly through the closeout gap risk – which Albanese and Andersen (2014) do capture in their tests, even though they do not discuss in their paper.
7. Custom Scenario Bank CDS Curves: Since funding metrics depend crucially on the bank funding rates, Albanese and Andersen (2014) carry out the calculation for two different funding curves. One corresponds to the 5Y CDS spread of 106 bp and the other to a 5Y CDS spread of 274 bp. Funding costs are simulated dynamically and consistently with the funding curves.
8. Full Market Risk Factors Simulation: CDS curves for all counterparties are simulated dynamically, as needed for ratings dependent collateral policies (of which the test portfolios had a few) and the analysis of credit-correlation effects related to loss-distributions and stress testing. Albanese and Andersen (2014) also simulate all relevant market risk factors and derivative security prices. The various XVA metrics are evaluated by computing the XVA expressions after incorporating thresholds and closeouts.

Scenario Estimation of the XVA Metrics

1. Book Level Incremental and Cumulative Errors: Albanese and Andersen (2014) found that in their simulation trials the funding set calculations were well-behaved. As expected, they found that the incremental FVA is noisier than the book level FVA. In order to the book-level FVA errors to below the 0.5% mark and the incremental metrics within the 2% mark, it was necessary to run about 100,000 scenarios.
2. Netting Set Granularity XVA Caching: Albanese and Andersen (2014) retained the scenario information in memory aggregated at the netting set granularity. By having the scenarios cached in memory the calculation of the incremental XVA metrics was quite efficient.

Incremental metrics of interest include the FVA, the symmetric FVA, as well as the UCVA, the FTDCVA, and the DVA.

3. Funding Curve Scenario XVA Metrics: Albanese and Andersen (2014) demonstrate the XVA portfolio metrics for the two funding curve scenarios. Accounting metrics are computed using the accounting rules listed earlier. Accounting entries are computed separately for the following cases:
    a. Only counterparty credit risk is accounted for
    b. Funding is accounted for using FCA/FBA accounting
    c. Funding is accounted for using FVA/FDA accounting
4. CET1 and MMT Accounting Impact: Albanese and Andersen (2014) demonstrate that there are material difference between the FVA/FDA accounting and the FCA/FBA accounting. In particular the CET1 charges for funding are about triple in the FCA/FBA accounting as opposed to the FVA/FDA accounting. Since the FVA/FDA accounting is consistent with the MMT, the write-off one faces by adding funding entries on top of credit adjustments in NILL.
5. FVA/FDA vs. FCA/FBA CL Impact: Contra Liabilities in FVA/FDA accounting are slightly larger than in FCA/FBA accounting. This happens because although the FDA is substantially smaller than FBA, the DVA is preserved as is required by the asset-liability symmetry, and there is no overlap.
6. EE, EPE, and ENE Estimates: For FVA/FDA calculations the key statistics in measuring RHO are the Expected Exposure (EE), the Expected Positive Exposure (EPE), and the Expected Negative Exposure (ENE) of the portfolio as a whole. Using 100,000 simulation trials, Albanese and Andersen (2014) report these metrics for a range of time horizons. The book under consideration starts off as a net receivable and remains as such on average.
7. Cross Netting RHO Impact: The impact of modeling rehypothecation between hedges belonging to different netting sets is sizeable and receives contributions also from the states of the world where the book is a net payable, because even in such situations a fraction of the netting sets are receivables and the corresponding hedges receive collateral.
8. Receivable vs Payable Book Swing: Furthermore the book valuation swings to a net payable status as one can see from the fairly substantial size of the ENE. A large ENE (in this case) is

an indication that the FVA is a materially non-linear metric which cannot be represented as a simple sum over the netting sets. In other words the incremental FVA for a netting set or a trade can be computed accurately only by knowing and accounting for the positions in the entire book.

Product and Scenario Threshold Type Scenarios

1. Impact of the Swap Types Traded: In their analysis, Albanese and Andersen (2014) add their portfolio three swaps – one initially at par, one a payable, and the third initially a receivable. They then compute the XVA metrics under the following 3 scenarios.
2. Thresholded Scenario XVA Impact:
    a. The swaps are added to a netting set with a finite collateral thresholds that already contains other trades
    b. The swaps are added to a netting set with a threshold at infinity
    c. The swaps are added to a netting set which is initially empty, thus neglecting the benefits of netting
3. Netting and Collateral Threshold Impact: The FTPs differ substantially between all the cases above, and are sensitive to both collateral thresholds and netting. Bothe collateral thresholds and netting materially decrease the FTP.
4. FCA/FBA vs. FVA/FDA FTP Estimates: There are also material differences between the FTP's obtained under the FCA/FBA accounting and those using the rules of FVA/FDA accounting. In the latter case the FTPs are smaller by a factor of $1.5 - 2.0$ in absolute value. This is also remarkable from the viewpoint that the FVA/FDA method does not recognize a DVA benefit to clients. This de-recognition is however more than compensated by a correct modeling of the RHO.
5. Accounting Scheme Objective of the FTP: The incremental FTPs in the two methods achieve a different objective. In the case of the FCA/FBA method the incremental fair value of the OTC portfolio ΔPFV is NILL while the Net Equity Capital is systematically depleted,

especially in the case of payables. In FVA/FDA accounting instead the Equity Capital is stable while the Bank fair value appreciates systematically.

XVA Metric Errors and Incrementals

1. Error Rates from Simulation Runs: Albanese and Andersen (2014) display the standard errors on the simulation runs – their simulation entails 100,000 scenarios and shows that the FTP for the FVA/FDA method is around 1.2-1.7%, an error of which type would be acceptable in most circumstances. Using 10,000 scenarios would imply relative errors as large as 10% and of the order of 1 bp per annum, which we would consider as being unacceptable large.
2. Tail Loss Distribution Contribution Scenarios: Further 100,000 scenarios allow carrying out of the reverse stress testing analysis by identifying extreme scenarios which either contribute to the tail of the loss distribution or invalidate a collateral procurement strategy.
3. FVA and SFVA Trade Incrementals: Albanese and Andersen (2014) demonstrate the size of the incrementals after adding one trade of the forward FVA and SFVA as follows:

$$FVA(t) = \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t) \left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$$

and

$$SFVA(t) = \mathbb{E}\left[e^{-\int_0^t r_B(s)ds} s_B(t) \left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$$

4. Incremental SFVA vs. Incremental FVA: They observed that incremental SFVA is systematically above the incremental FVA in the case of receivables and systematically below the FVA in the case of payables. We conclude that the approximations which are intrinsic to the FCA/FBA accounting over-estimates funding costs for receivables and also over-estimates funding benefits for payables.

Estimation of the FCA/FBA – FVA/FDA Mismatch

1. Payables Derivatives Bias FCA/FBA: The bias in favor of payables in FCA/FBA accounting is a worrisome feature of this method as it induces banks to skew their exposure in favor of the payables. If the traders are incentivized with an FTP computed by this method, they are inclined to sell out of the money options for premium upfront, effectively raising collateral and subjugating themselves to the treasury, thus incurring inefficiencies.
2. Origins of the Payables Bias: The above happens because the FCA/FBA method recognizes a benefit to the excess variation margin at a rate equal to the funding spread of the bank while the FVA/FDA method does not recognize any such benefit).
3. Threshold Impact of FCA/FBA Accounting: The SFVA is a transactional amount only in cases where there are no thresholds or the thresholds are very remote and attained only with a negligible probability. For instance the two SFVAs for the case where the trade is added to an empty netting set is full but the thresholds are neglected are very close. This happens although the FCA and the FBA are very different.
4. FCA/FBA Volatility on Bank PFV: Within FCA/FBA accounting a new trade does not cause volatility in the fair valuation of the bank as a whole given by the PFV. In this case, however, each trade causes a transfer of wealth from CET1 to CL, i.e., from shareholders to senior creditors.
5. Shareholder to Senior Creditor Transfer: On average this transfer is a net loss to CET1. It MMT held this would happen for a zero FTP. The fact that it happens for a non-zero FTP is yet again a signal of the internal inconsistency of the FCA/FBA accounting.
6. Income and CET1 FVA/FDA Impact: In the case of FVA/FDA accounting instead we propose to structure the FTP in seeking to keep CET1 constant. We could also have computed the FTP in such a way to keep the income constant and this would have given rise to zero FTP's.
7. FTP in FCA/FBA vs. FVA/FDA: The FTP in FVA/FDA accounting are lower than those in FCA/FBA accounting for a net receivable book. This signals again the internal inconsistency

of the FCA/FBA accounting as the FTP policy should ensure on average that the CET1 is not depleted while it instead appears as though it is.

8. Single Swap on Single Counterparty: It is also useful to consider the case of a single swap transaction with a single counterparty. Albanese and Andersen (2014) demonstrate the CET1 changes which measure the economic value of the transaction are very close between the FCA/FBA and FDA/FVA accounting schemes. In this case the FVA/FDA methodology can be regarded as an extension of the FCA/FBA method as long as the latter is restricted to portfolios consisting of a single netting set.
9. Two Counterparties with Opposite Swaps: The second interesting particular case in Albanese and Andersen (2014) is the one described whereby there is a portfolio with two counterparties, each with a single swap position. The swaps have identical terms except that one is a payable while the other is a receivable.
10. Collateral Offset from the Swaps: What happens in this case is that the collateral received on the hedge to one swap is nearly always the exact amount the bank is required to post on the hedge to the other swap. Discrepancies arise only in scenarios where one counterparty defaults while the other is still ongoing, in which case one of the hedge positions is closed upon default.
11. Magnitude of the Asymmetric FVA: Assuming that the CDS spread curves of the counterparties are identical, and that they reach 200 bp in 5Y, the resulting asymmetric FVA is small.
12. FCA/FBA vs. FVA/FDA CET1 Deductions: Instead the FCA for each of the two counterparties is large. As a consequence the FCA/FBA accounting method gives rise to capital deductions that are 9 times larger than the deductions resulting from the FCA/FBA method.

References

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Conclusions with Funding Adjustments with RHO

Traditional Challenges with Derivative Accounting

1. Indifference Pricing of OTC Derivatives: Although funding costs may feel all too real for the derivatives trader that sees his unsecured positions bleed negative carry, well-established finance principles nevertheless insist that fair values of assets are independent of how they are funded.
2. "Going Concern" Accounting Principles Reconciliation: Reconciling the funding carry vs. the Corporate Finance Theory within the confines of the financial accounting statements is not an easy exercise, especially since traditional "going concern" accounting principles were not designed for credit-risky securities.
3. CET1 Equity Capital Regulatory Principles: Complicated matters further are the newly established regulatory principles for the CET1 equity capital that require particular care in the accounting of DVA and DVA-like adjustments.

Problems with FCA/FBA Accounting

1. Not Accounting for the RHO Handling: While some banks have put forward – and into action – the FCA/FBA method for funding cost accounting, Albanese and Andersen (2014) demonstrate that this method is not satisfactory. First FCA/FBA does not properly reflect the re-hypothecation options embedded in variation margin financing and as a result over-estimated funding related deductions from the equity capital.
2. Violation of the Asset-Liability Symmetry: Second the FCA/FBA violates the asset-liability symmetry principles of generally accepted accounting standards while breaking the Modigliani-Miller Theorem dear to financial economists.

3. Wrong-Way Bank Credit Sensitivity: Another popular variation of the FCA/FBA accounting – which involves deducting SFVA from equity capital rather than FCA – is not viable from a regulatory standpoint as the deduction has a wrong-way sensitivity with respect to the bank credit spread for portfolios which are net payables.

FVA/FDA as FCA/FBA Enhancement

1. CET1, Income, and Fair Value: Albanese and Andersen (2014) proposal for funding cost accounting aims to establish some coherence and to clear up a number of holes in the FCA/FBA accounting. In their tests this new method differs significantly from the FCA/FBA method on key accounting numbers (such as CET1, Net Income, and Fair Asset Value), yet the FVA/FDA accounting should resonate with most relevant parties.
2. MMT and Risk-Neutral Pricing: Firstly financial economists and asset pricing experts will appreciate that the accounting rules will satisfy the Modigliani-Miller Theorem and lean heavily on classic risk-neutral pricing principles.
3. Exclusion of Self-Credit Benefits: Secondly regulators will also appreciate that all self-credit benefits are collected cleanly in a Contra-Liability account that can be easily excluded from common equity for capital purposes.
4. Adherence to Accepted Accounting Principles: Thirdly accountants will appreciate the adherence to the accounting principles, and in particular to the asset-liability symmetry principle.

Trading Staff Point of View

1. Explicit Trade-Level Valuation Adjustment: For trading personnel the picture gets more complicated. On the one hand the FVA/FDA adjustment does not trigger the funding related adjustments to income and asset valuations that many prefer to see.

2. Equity Capital Based Incentive Schemes: On the other hand Albanese and Andersen (2014) describe why traders and managers drafting incentive schemes should care about the changes in CET1 than about the Net income.
3. CET1 Based FTP Estimation Schemes: In particular they highlight the link between the shareholder value and CET1 and demonstrate how a rational FTP scheme can be designed around the principle of book-level CET1 indifference pricing.

Challenges with the XVA Metric Estimation

1. Simulation Across Multiple Netting Sets: On the topic of FTP calculation there is no doubt that FVA/FDA method requires a fairly sophisticated calculation engine to support the necessary incremental FVA calculation. Being a book-level quantity the FVA (and therefore the FTP) computation involves simulating through entire books across time, involving a large number of netting sets.
2. Enhancements to Existing CVA Systems: This, in turn, requires modifications to standard CVA calculation engines that normally can only aggregate trades at the level of individual netting sets.
3. Challenges from a Computational Finance Perspective: Given the complexities involved in computing FTP's against the backdrop of an entire book position, there are challenging computational finance questions to be addressed.

Shortfalls of the FVA/FDA Scheme

1. Derivatives Focus of the FVA/FDA Accounting: While FVA/FDA leans on a number of ideas from corporate finance, Albanese and Andersen (2014) make it clear that the method is pragmatic and derivatives focused.
2. Targeted Scoping of the FVA/FDA Accounting: They do not attempt a rigorous, full-blown analysis of the balance sheet that takes into consideration many other assets and operation of

a typical bank. Neither do they consider the effects of taxes, bond covenants, dividend policies, and subtle feedback effects from investment decisions on firm-wide recovery rates (which they assume to be constant) and default probabilities.

3. Inclusive, Wide Ranging Accounting Treatment: It remains an interesting question for future whether a more rigorous and large (an necessarily complex) analysis can provide any insights that can be turned into concrete accounting rules that improve upon what they propose.

Alternate Specialized Value Adjustment Metrics

1. Capital Charge Value Adjustment: While FVA is the most prominent newcomer to the XVA alphabet soup, there are other adjustments just waiting around the block. For instance, it has been suggested that the cost of capital charges should be reflected in the deal pricing through a "capital value adjustment" or KVA (Green, Kenyon, and Dennis (2014)).
2. Trade Scenarios Requiring Initial Margin: Similarly one may consider Margin Valuation Adjustment (MVA) due to the funding cost of the initial margin (IM) for the netting sets that require IM posting. Initial Margin is required when trading with the CCPs, and due to regulatory requirement also becomes far more prevalent in the future for non-cleared products (Basel Committee on Banking Supervision (2013)).
3. MVA and KVA Estimation Complexities: The accounting for – and the associated impact of – MVA, KVA, and other metrics that come up, are topics for future research, as is their practical computation. Albanese and Andersen (2014) note that both the capital and the initial margins are complex quantities that are more involved to calculate dynamically on the path than just portfolio values (as needed for the FVA). Regression-based methods or nested simulations are likely needed here.

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The FVA Puzzle: Accounting, Risk Management, and Collateral Trading

Abstract

1. Bridge between Regulators, Traders, and Economists: In Albanese and Andersen (2014), the FVA/FDA accounting framework for funding costs was introduced, with the aim of providing an accounting method that reasonably balances the conflicting concerns of accountants, regulators, traders, and financial economists.
2. FVA/FDA vs FCA/FBA Accounting Schemes: Albanese, Andersen, and Iabichino (2014) provide a concise comparison of FVA/FDA accounting against the FCA/FBA method endorsed by several large banks. The FTP policies and the risk management implications are discussed, and the notion of funding arbitrage is quantified.

Introduction

1. Metrics for Capturing Funding Costs: The notion of charging for funding costs became painfully relevant for banks as their borrowing spreads jumped up during the financial crisis. Drawing inspiration from the works of Piterbarg (2010) and Burgard and Kjaer (2011, 2013), several banks have instituted accounting changes aimed at capturing the funding costs for uncollateralized derivatives transaction.

2. Challenges with the FCA/FBA Accounting: The prevalent FCA/FBA accounting method is simple but controversial – see Cameron (2013) – and has raised concerns about the breakage of asset-liability symmetry, double counting of DVA, and embedding entity-specific costs in exit prices. Additionally it appears that the FCA/FBA method is interpreted differently from one bank to the next.
3. Reconciliation across Regulation, Valuation, and Accounting: Albanese and Andersen (2014) proposed the FVA/FDA accounting method as a compromise between the financial asset valuation principles, the "going concern" accounting principles, and regulatory capital requirements.
4. Remedy of the Dominant Accounting Method: The method explicitly models the re-hypothecation option of variation margin on hedges, and remedies many of the theoretical inconsistencies of the FCA/FBA method.
5. Albanese and Andersen (2014) Core Ideas: Albanese, Andersen, and Iabichino (2014) review the core ideas behind Albanese and Andersen (2014) and contrast it to FCA/FBA numerically and conceptually.
6. Albanese and Andersen (2014) Main Implications: They then consider some risk management implications of funding cost accounting, and propose using CET1 simulation as a tool for hedging, collateral optimization, and reverse stress testing.
7. Albanese and Andersen (2014) Principal Strategies: They also discuss strategies to exploit funding arbitrage by means of CVA reducing trades.

CVA/DVA Accounting

1. Funding Adjustments under Classical Accounting: Before discussing the funding adjustments in detail, this section reviews how the accounting rules work for the classical case where the bank's OTC portfolio value is adjusted for credit risk.

2. OTC Portfolio Accounting Ledger Rules: First the accounting ledger rules for OTC portfolios typically assign trades with positive valuations (i.e., receivables) to an asset account, and trades with negative values (i.e., payables) to a liabilities account.
3. Accounting Ledger Rules Portfolio Value: In the absence of credit risk, the Portfolio Fair Valuation PFV to the bank holding the position is given by the default-free value of the assets A minus the default-free value of the liabilities L:

$$PFV = A - L$$

4. Counter-party Credit Risk Twists: Counter-party credit risk adds a few complexities and necessitates the introduction of *contra* accounts as well as change in the *unit of account* from individual trades to counter-party specific netting sets.
5. The Contra Asset Account Concept: Downward adjustments to the asset values from the counter-party credit risk are *Credit Valuation Adjustment* (CVA) entries in the *contra-asset* CA account. The CA value aggregates CVA across all counter-parties and is subtracted from the default-free asset values.
6. The Contra Liability Account Concept: In addition to the contra-asset account, there is also a *contra-liability* CL account which includes *Debt Valuation Adjustment* for each counterparty.
7. Total DVA of the Bank: The bank's total DVA equals the total CVA recorded by all counterparties *against* the bank, and ensures that the accounting system is symmetric and does not create wealth out of zero-rum bilateral trading. DVA entries are benefits and represent the present value of the bank's option to default on its liabilities.
8. Derivatives Balance Sheet Fair Value: To summarize the fair value associated with the derivatives portion of the balance sheet may be written as

$$PFV = A - L - CA + CL$$

9. Cash Account and Equity Component: If one includes a cash account $Cash$, the simplified balance sheet may be computed by writing the total assets as $Cash + A - CA$, with the *accounting* Equity defined as

$$Equity = Cash + PFV = Cash + A - CA - L + CL$$

10. DVA Impact on Equity Holders: While DVA is a rational and well-defined component of the bank-wide PFV, it should arguably not contribute to regulatory capital as benefits associated with a bank default are neither loss absorbing, nor do they contribute to the wealth of the bank equity holders (who are wiped out by a bank default).
11. Common Equity Tier I Capital - Definition: DVA entries in the CL are therefore excluded by the regulators – BCBS (2012) and Federal Register (2014) – from *Common Equity Tier I Capital* $CET1$. That is

$$CET1 = Equity - CL = Cash + A - CA - L$$

12. DVA De-recognition in Quotation Practices: The interpretation of DVA as not benefitting equity holders will often manifest itself in quotation practices, where it is common for traders to internally de-recognize all or part of the DVA benefits in the prices they quote to the counterparties, in effect charging DVA through to the client on top of PFV.
13. $CET1$ Reflection from Retained Earnings: If trades get done at the quoted levels, the bank will consequently day 1 trading gains that ultimately hit retained earnings, and in turn, contribute to $CET1$.

The FBA/FCA Method

1. Post Crisis Variation Margin Spike: In the aftermath of the financial crisis, funding costs for variation margin collateral that banks post on hedges against uncollateralized derivatives have escalated from tens to hundreds of basis points.

2. Motivation behind the FVA Metric: This has elevated collateral trading strategies to a role of prominence and has motivated banks to seek metrics to capture a *Funding Value Adjustment* in addition to CVA and DVA.
3. FCA/FBA Method Origin : Liquidity Gap: The commonly used FCA/FBA method of funding cost accounting can be presented in several different ways. One simple approach takes as its starting point that the bond cannot default, and in effect, that the funding spreads paid on collateral lending are due to friction and lack of liquidity.
4. Implication of the *Going Concern* Assumption: This assumption is undoubtedly a string one, as liquidity spreads are typically in the order of a handful of basis points, while bank funding spreads can run into hundreds of basis points. Nevertheless, if one accepts the premise, the derivatives can then be priced at the cost of their replication while ignoring that possibility of the bank default.
5. *Symmetric Funding Value Adjustment* Metric: Details can be found in Albanese and Andersen (2014), but ultimately the relevant funding metric for a fully uncollateralized trading involves evaluating a *symmetric funding adjustment* metric $SFVA$ given by

$$SFVA = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\right\}dt\right]$$

where $\mathbb{E}$ denotes the risk neutral expectation; $V_i(t)$ is the default-free value of the i^{th} unsecured netting set; $r_B(t)$ is the bank's funding rate; and

$$s_B(t) = r_B(t) - r_{OIS}(t)$$

is the bank's spread over the OIS rate.
6. Incorporation of Default Unwind Times: It should be noted that there have been considerable debate and disagreement about whether default events or expected unwind times should be merged into the $SFVA$ above.
7. Appropriate Proxies for Funding Spreads: It has also been advocated that the funding spread s_B should be replaced with an industry average spread or with the most favorable spread of

any bank on the market (*best funder*); the latter is presumably meant to be consistent with the auction-style *exit-price* principles favored in accounting definitions of fair value.

8. Proprietary Composite Blended Spreads: Some banks appear to be using proprietary blends of their own spreads and spreads observed for other banks.
9. The *Funding Benefit Adjustment* (*FBA*) Metric: The expression for $SFVA$ above may be split into contributions from assets and liabilities. Starting with the latter, one defines the *Funding Benefit Adjustment* (FBA) as follows:

$$FBA = -\mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i^-(t)\right\} dt\right]$$

where the negative superscript is defined as

$$x^- = \min(x, 0)$$

10. Elimination of DVA from PFV: The FBA is closely related to the unilateral DVA and gives rise to a double counting paradox that the FCA/FBA method elects to resolve by *removing* DVA from PFV altogether.
11. DVA in CA/CL Accounts: However, as DVA is required by the current accounting rules (e.g., IFRS13), FCA/FBA will normally institute this by, say, leaving the DVA in the CL account, and then adding the DVA to the CA account. Note that for PFV the DVA entries then cancel.
12. *Funding Cost Adjustment* (*FCA*) Metric: The asset component of *SFVA*

$$FCA = SFVA + FBA = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i^+(t)\right\} dt\right]$$

is known as the *Funding Cost Adjustment* and is kept as a metric for funding costs. The FCA adds to the contra-asset account and is therefore subtracted from $CET1$.

FVA/FDA Accounting

1. Notion of a Funding Set: In FVA/FDA accounting, the unit of accounting for funding cost metrics is expanded from the netting sets in the CVA/DVA accounting according to the larger notion of a *funding set*, a collection of unsecured trades across which cash received for derivatives funding can be re-hypothecated.
2. Definition of the Funding Set: Equivalently the funding set is a collection of trades for which the variation margin posted on uncollateralized hedges with dealers may be re-hypothecated. For a large bank, funding sets could, say, be defined at the legal entity level and may contain hundreds or even thousands of counterparties and netting sets.
3. The Funding Value Adjustment Metric: Assuming for simplicity all unsecured trades in the derivatives portfolio of a bank constitute one single funding set, the present value of the funding costs is now given by the following *FVA* (*Funding Value Adjustment*) metric for uncollateralized trading:

$$FCA = \mathbb{E}_0\left[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

4. Explicit Bank/Counter Party Defaults: Here the positive superscript denotes the positive part, and τ_i and τ_B are the default times of the default times of the i^{th} counterparty and the bank, respectively.
5. Situation resulting in Funding Costs: In the FBA expression above, the funding costs arise only when the portfolio valuation $\sum_i V_i(t)\mathbb{I}_{t<\tau_i}$ is positive and the bank is therefore a net poster of the variation margin in its hedges over the funding set. States of the world where the bank is a net receiver of variation margin are assigned zero benefit, as would be the case if the excess collateral were invested over the short-term at the risk-free rate.

6. Rationale behind the Re-investment Assumption: This investment assumption is prudent, and many investment strategies conjectured in the literature, such as retirement of long-term debt, are often not practical due to the volatility and the short-term nature of any excess funds generated by derivatives trading.
7. Alternate Re-investment/Cash Assumption/Strategies: Albanese and Andersen (2014) detail the extensions and a fuller discussion on the possible benefits of cash positions.
8. The Funding Debt Adjustment Metric: From an accounting standpoint the *FVA* computed in

$$FVA = \mathbb{E}_0 \left[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t) \left\{ \sum_i V_i(t) \mathbb{I}_{t<\tau_i} \right\}^+ dt \right]$$

is configured as a *contra-asset* entry. The *FVA* also has a contra-liability twin, denoted *Funding Debt Adjustment* (*FDA*).
9. Origin of the *FDA* Term: The *FDA* term fundamentally arises from conservation principles; if funding is a cost to the bank shareholders, other agents must receive a benefit of equal size.
10. Alternate Names for the *FVA*: Hull and White (2012) first introduced the *FDA* concept under the moniker *DVA2* but this treatment prefers a more descriptive term.
11. OTC Derivatives as Non-Repoable Transactions: Consequently we can think of the *FVA* costs as originating from the lack of repo market on unsecured OTC derivatives, forcing banks to borrow variation margin on hedges on an unsecured basis at a substantial spread to OIS.
12. Parity between *FVA* and *FDA*: The flip-side of unsecured borrowing is that the senior creditors are entitled to recover from a pool of unsecured derivatives receivables after a bank default. Under mild assumptions the value of this recovery option may be shown (Albanese and Andersen (2014)) to be equal to the *FVA*, i.e.,

$$FDA = FVA$$

13. Asymmetry between DVA and FDA: It should be noted that the DVA originates from similar zero-sum considerations as the FDA, but there is an important difference; while CVA

represents the wealth transfer from the bank to the counterparties due to the acceptance of the counterparty credit risk, FVA is an *internal* transfer from the bank shareholders to the bank creditors.

14. Excluding FDA Contributions to $CET1$: Hence FDA is a contra-liability term that contributes to the wealth of the bank senior creditors but should be excluded from $CET1$.
15. Consequence of FVA/FDA Mismatch: With FDA and FVA entries being recorded in equal and opposite CA and CL accounts, funding ultimately has no impact on PFV since FVA and FDA cancel. This preserves the symmetry of CVA/DVA accounting, eliminates bank-specific costs from fair valuations, and allows much of the standard finance theory to escape unscathed at the level of PFV.
16. Impact at the $CET1$ Level: At the $CET1$ level, however, we depart from the CVA/DVA accounting by requiring that FVA be deducted from $CET1$ to reflect the fact that the bank shareholders are penalized by the funding cost and do not share the FDA recovery benefit. Albanese and Andersen (2014) demonstrate that the typical own-spread sensitivities of the FVA term defined above are such that the FVA qualifies as a valid deduction from the regulatory $CET1$.
17. Bank/Counter Party Default Impact: Finally note that the definition of FVA in

$$FVA = \mathbb{E}_0\left[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

recognizes that the funding needs for a netting set vanish with either the default of the bank or the counterparty.

18. Bilateral Estimation of FCA/FBA Accounting: While the CVA and the DVA metrics are always calculated on a unilateral basis under the FCA/FBA accounting, FCA/FBA accounting is therefore best done under bilateral definitions of both CVA and DVA.
19. First To Default CVA ($FTDCVA$): Accordingly FVA/FDA accounting excludes CVA contributions after bank's own default and replaces FCA/FBA accounting's unilateral CVA ($UCVA$) definition with a bilateral CVA measure known as *first-to-default* CVA ($FTDCVA$).

20. Mathematical Definitions of $UCVA$ and $FTDCVA$: For the well-known mathematical definitions of $UCVA$ and $FTDCVA$ we refer to Albanese and Andersen (2014) which also discusses the effect of ISDA close-out protocols on CVA.

21. Regulatory Guidelines Induced Accounting Modifications: As $FTDCVA$ decreases in the bank's credit spread, a slight modification of the accounting rules above is needed in order to not run afoul of regulatory guidelines.

22. Decomposition of the $FTDCVA$ Metric: In particular, rather than registering the entire $FTDCVA$ as a CA entry, the decomposition

$$FTDCVA = UCVA - CVA_{CL}$$

is applied, and the $UCVA$ is recorded as a CA entry and the *DVA of CVA* term CVA_{CL} as a CL entry. This ensures that the self-default benefits are relegated to the CL account.

23. Decomposition of the $FTDDVA$ Metric: Similar break-outs can be entertained for the DVA, which in the FVA/FDA accounting must be computed bilaterally as the $FTDDVA$. This treatment, however, simply places the entire $FTDDVA$ in the CL account.

24. FCA/FBA and FVA/FDA Accounting Measures: A summary of the accounting entries under both the FCA/FBA and the FVA/FDA frameworks is in the table below. For both methods, PFV and $CET1$ are computed as per

$$PFV = A - L - CA + CL$$

and

$$CET1 = Equity - CL = Cash + A - CA - L$$

Note that the DVA entries in the table for the FCA/FBA and the FVA/FDA methods are unilateral ($UDVA$) and bilateral ($FTDDVA$) respectively.

25. FCA/FBA FVA/FDA CA and CL:

	CA Adjustment	CL Adjustment
FVA/FDA	$UCVA + FVA$	$CVA_{CL} + FTDDVA + FDA$
FCA/FBA	$UCVA + FCA - FBA + UDVA$	$UDVA$

Funds Transfer Pricing

1. Incremental Trade Cost FTP Policy: Managing FVA costs requires, as a minimum, a well-designed funds transfer pricing (FTP) policy for proper recognition of costs and benefits of new trades.
2. Implementation of FTP through CFD: For the discussion of the implementation of the FTP and the role of a *central funding desk* see Albanese and Andersen (2014).

FCA/FBA Accounting

1. Contributions from UCVA and SFVA: In the standard FCA/FBA accounting method, the prevailing policy is to charge the clients at a minimum the amount

$$FTP = \Delta UCVA + \Delta SFVA$$

over the basic default free trade value. Δ is used here to denote the marginal portfolio impact, so $\Delta UCVA$ and $\Delta SFVA$ cover the credit risk and the funding charge impacts respectively, as a new trade is added to the overall portfolio.

2. FTP Impact on PFV: Using the FTP above, the inflow to the cash account at the time of the trade is

$$\Delta Cash = -\Delta A + \Delta L + FTP = -\Delta A + \Delta L + \Delta UCVA + \Delta SFVA = -\Delta PFV$$

where the last equality follows from

$$PFV = A - CA - L + CL$$

3. FTP Impact on the Equity: By

$$Equity = Cash + PFV = Cash + A - CA - L + CL$$

it is clear that

$$\Delta Equity = 0$$

so the FCA/FBA FTP policy ensures that the impact on the income and the retained earnings on adding a new trade is zero.

4. FTP Impact on $CET1$: On the other hand, from

$$CET1 = Equity - CL = Cash + A - CA - L$$

the incremental impact of a new trade on $CET1$ is

$$\Delta CET1 = \Delta Equity - \Delta CL = \Delta UDVA$$

5. Management vs. Shareholders Incentive Mismatch: By tailoring the FTP in such a way that new trades have no impact on income, deal flow induces volatility on $CET1$ as is evident from the expression for $\Delta CET1$ above. Since $CET1$ is a proxy for shareholders wealth, it may be argued that

$$FTP = \Delta UCVA + \Delta SFVA$$

   does not fully align executive incentives with shareholders interests.
6. $CET1$ Indifference based on FTP Policy: An FTP policy designed for $CET1$ indifference would, however, not likely be viable with the FCA/FBA accounting due to the sheer size of the full FCA.

FVA/FDA Accounting

1. Recognition of the Wealth Transfer: As the bank enters unsecured OTC transactions, FVA/FDA accounting recognizes a transfer of wealth in the amount of

$$FVA = FBA$$

   from the bank shareholders to bank senior creditors.
2. $CET1$ Change Induced by Trade: Assuming that the bank managers act on behalf of the shareholders, entry pries need to be set such that this wealth transfer is countered by a corresponding increase in the equity account. Equivalently, charges to the client must be set such that the proxy for the shareholder wealth, $CET1$, stays constant.
3. The corresponding Trade FTP Policy: From

$$CET1 = Equity - CL = Cash + A - CA - L$$

the client charge must therefore be ΔCL above the FTP. Expressed as a surcharge on the default-free trade value, the FTP amount required to ensure that

$$\Delta CET1 = 0$$

is

$$FTP = \Delta UCVA + \Delta FVA$$

4. No DVA Benefit Pass Through: Notice that the above expression requires that the benefit of DVA terms (including the DVA part of $FTD\ CVA$) are not passed on to clients, as discussed earlier. In addition, FVA/FDA accounting adds a FVA funding term to the charge.
5. FTP Policy Impact on Income: The FTP policy of FVA/FDA accounting has a net impact on income given by

$$\Delta Equity = \Delta CL = \Delta FTDDVA + \Delta FDA + \Delta CVA_{CL}$$

6. FTP Policy Impact on Shareholders: By focusing on $CET1$

$$FTP = \Delta UCVA + \Delta FVA$$

expresses a shareholder centric view in the computation of FTP, in effect requiring that the trading strategies be self-financing only to bank shareholders and not to the bank as a whole.
7. FTP Policy Impact on the Bond Holders: For trades that involve bond holder benefits post bank default, and such benefits are ignored in trading and when quoting to the client. Again, the shareholder view is expressed here only in deal charges, and not in the recording of the bank-wide PFV.

Notes on Exit Pricing and Asset-Liability Symmetry

1. Definition of Fair Market Value: The notion of fair market value in the FVA/FDA framework accounts for the risk-neutral values of *all* flows associated with a unit of account, irrespective of the identity of the stake-holder that benefits from the flow.
2. Exit Price as a Fair Value Price: This notion preserves the asset-liability symmetry and can be considered as an exit price in a competitive auction where entities of all types, including unlevered real money funds with negligible funding costs, are allowed to participate.
3. Contrast with the Entry Price: In contrast *entry* prices in the FVA/FDA methods deviate from the fair value metric by effectively neglecting all cash flows that do not benefit the shareholders. The resulting price contains entity specific costs and should, of course, not be confused with an exit price.
4. Most Favorable Price as Entry Price: Yet, in a sufficiently illiquid market dominated by banks with pricing power, one can imagine non-competitive equilibrium where the exit price can be equated to the most favorable entry price across the banks. Usage of the most favorable bank entry price provides a conceivable alternative to the definition of the market price, but is not an unproblematic one.
5. Problem with Most Favorable Entry Price #1: First the definition is not consistent with the asset-liability symmetry principles that underpin the IFRS13 requirement of including DVA in market prices.
6. Problem with Most Favorable Entry Price #2: Second, the clean determination of the most favorable entry price for a non-trivial collection of trades and providers is subject to complex portfolio effects and is effectively impossible to determine with any accuracy. Although some pricing service data does exist, it is very limited and does not consider portfolio effects.

7. Impact on $CET1$ Write Down: It should be noted that irrespective one's precise definition of the fair value, the $CET1$ write-down in the FVA/FDA metric – which in many ways is the key metric – is unchanged.

Extensions

1. CP Collateralization and XVA Hedge VM: For ease of exposition the FVA expression

$$FVA = \mathbb{E}_0\left[\int_0^{\tau_B} e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

ignores the variation margin collected on CVA hedges and assumes that the counterparty positions are fully collateralized. A more accurate expression that captures the collateral effects will basically replace V_i in the FVA above with $V_i - C_i - UCVA_i$ where C_i is the amount of collateral posted to the bank by counter party i.
2. Multiple Spreads in One Funding Set: More details can be found in Albanese and Andersen (2014) which also entertains the notion of multiple funding rates co-existing in a funding set. Such situations can conceivably occur when one attempts to align funding spreads with the observed bond-CDS basis for the various counterparties. They may also occur whenever the various legal entities of the bank have different costs of funding, while variation margin can freely flow between them.

Balance Sheet Simulations and Reverse-Stress Testing

1. Management of $CET1$ Capital Volatility: $CET1$ capital volatility with credit and market risk needs to be managed by hedging and capital buffer provisioning.
2. Standard and Custom CCAR Scenarios: Sensitivity analysis is a viable technique here, but is of limited use for longer time horizons and for stress scenarios, such as those applied in Comprehensive Credit Analysis and Review (CCAR) assessments. Besides the standardized CCAR scenarios, regulators advise banks to identify stress conditions specific to their portfolios.
3. Simulation Based Reverse Stress Testing: Simulation-based *reverse stress testing* is useful is this context and can pinpoint the precise circumstances that are associated with capital depletion.
4. Long Term Scenario Analysis Definition: To implement a long-term scenario analysis, Albanese, Andersen, and Iabichino (2015) evolve a case-study for two years on bi-monthly interval, registering $UCVA$ variation, FVA variation, default losses, and realized funding costs.
5. Primary/Secondary Scenarios Setup: The simulation is based on 20,000 primary scenarios for which XVA metrics are computed dynamically on the path by generating 1,000 secondary scenarios at each simulation time point.
6. Secondary Scenario Evolution Time Horizon: Secondary scenarios were stepped through 100 time points over a 40 year time horizon with simulation of all market and all credit risk factors, accounting for the wrong-way risk.
7. Gross Simulation Execution Time Elapsed: In all 240 million scenarios are involved in this calculation. Using the mathematical framework described in Albanese, Bellaj, Gimonet and Pietronero (2011), Albanese, Andersen, and Iabichino (2015) are able to achieve overnight execution times using two servers.
8. Steady Book without Bank Defaults: The calculations assume that no new trades are added over time and consider only scenarios where the bank does not default.
9. Tracking $CET1$ Variation in Time: The net metric of interest was computed at each time t as the time-0 present value of

$$CET1(t) - CET1(0) = [UCVA(0) - UCVA(t)] + [FVA(0) - FVA(t)] - CDL(t) - FC(t)$$

where $CDL(t)$ is the cumulative default loss on the time interval $[0, t]$ present value to time 0, and $F(t)$ is defined as follows.

$$F(t) = \int_0^t e^{-\int_0^u r_{OIS}(s)ds} s_B(u) \left\{ \sum_i V_i(u) \mathbb{I}_{u<\tau_i} \right\}^+ du$$

10. Valuation Adjustment Sign Convention Adopted: Here A and L are assumed to be hedged perfectly on a back-to-back basis while the CVA and the FVA are unhedged. The sign convention is such that counterparty defaults and increases in FVA or CVA are all registered as negative numbers – corresponding to reductions in $CET1$.
11. Skewed Nature of $CET1$ Distributions: Albanese, Andersen, and Iabichino (2015) plot the fixed time distribution of $CET1$ losses. Due to the inevitability of the default losses, the distribution becomes increasingly skewed in the direction of losses as time horizon is extended. While Albanese, Andersen, and Iabichino (2015) conduct the simulation under the risk-neutral measure, simulations under the historical probability measure - $P - measure$ – are, of course, possible.
12. Stress Proxy - USD OIS Rate: For the reverse-stress testing purposes, tail events can be extracted from the data used to generate the previous runs. Focus may occur on any market variable in examining these scenarios, but Albanese, Andersen, and Iabichino (2015) investigate the USD OIS overnight rate in these scenarios that produce the 200 worst 2-year losses in the sample of 20,000 runs.
13. Primary (Time/Rate) Stress Proxy: Each of the 200 stress paths have been assigned the point in time on the path on which the biggest net loss was registered. Albanese, Andersen, and Iabichino (2015) noticed that the stress scenarios are characterized by interest rates which are substantially lower than their unconditional expectations.
14. Distribution of Stress Time Nodes: They also notice that quiet of few stress scenarios produce their largest losses over relatively shorter time horizons, a consequence of the fact that the portfolio has less FVA and CVA variation as trades expire over time.

Strategies for Exploiting Funding Arbitrage

1. Bank Portfolio View of MMT: As seen above unsecured OTC trades have capital structure implications as they trigger wealth transfer from share-holders to bond holders. The classical Modigliani-Miller theory ignores this aspect since the discussion is typically framed under the Efficient Market Hypothesis (EMH) under which share-holders would immunize such losses by acquiring bank debt.
2. Equity Only Capital Structure Scheme: In this view, trading decision should be pursued as long as they have a non-negative value to the bank as a whole, i.e., if the incremental PV is non-negative. Of course, if share-holders were to implement this strategy systematically, banks would develop a pure equity capital structure.
3. Capital Structure Funding Arbitrage: In practice, this is not reasonable, and FTP charges, as discussed, are needed to ensure that share-holders are immunized against losses. This, in turn, opens up the possibility of funding arbitrages.
4. Bond/Share Holders' Cross Transfer: In designing funding strategies, it should be kept in mind that bond covenants constrain outright transfer of wealth transfer from bond-holders to share-holders. A viable strategy goal should therefore be to simple prevent, or at least minimize, wealth-transfers from share-holders to bond-holders.
5. Serving as a Two-Sided Trade Intermediary: This can be accomplished in numerous ways, starting from straightforward trade intermediation strategies. For instance if the funding FTP calculated by a bank is too large to prevent a particular client trade from being viable, the bank may act as an intermediary, and for a fee, take the client to an investor with a smaller funding cost.
6. Intermediation Strategies and Margin Funding: The intermediation strategy is a special case of a general principal that investors with low funding costs (or funding axes for specific trades) may be inserted into transactions to provide funding at levels cheaper than those for

the arranging bank. Loosely, one can think of all such strategies as being examples of *margin lending*.

7. Margin Lending as Bridge Financing: A particularly simple example of margin lending is where a third party investor makes available an amount K_0 in cash to the bank for a fixed period of time (e.g., two years) up to date T.
8. Segregated Multi-Purpose Margin Lending: The amount K_0, on which interest must obviously be paid, is meant to be used as a capital buffer, by absorbing default losses arising from the OTC derivatives book and providing funds for the variation margin. Any funds not used for OTC variation margin purposes are kept in a segregated account and not re-hypothecated for other purposes.
9. OTC Book First-Loss Structure: Then, if the OTC book suffers a default loss L prior to expiry, the amount L is deducted from the capital buffer.
10. Post time-t Buffer as VM: Any capital left in the buffer at time t during the life of the trade may be used for variation margin postings to OTC hedge counterparts.
11. Segregation Event Contingent Scenario Tranche: In case the bank defaults during the lifetime of the trade, any segregated amount of the capital buffer not posted as variation margin will be repaid.
12. Custom Priority of VM Receivable: The amount of the capital buffer used as a variation margin will be repaid, with priority over all other creditors, by funds obtained by settlement of unsecured derivatives receivables.
13. Losses From Bank/CPTY Defaults: If, as a consequence of the bank default, some counter parties default and fail to settle their obligations, first losses are apportioned to the capital buffer.
14. Maintenance of the Bond Covenants: The structure above does not break *pari passu* bond covenants since the structure is a form of collateralized lending.
15. Reduction in the FVA FTP: If the finds are provided by a margin lender at a sufficiently low interest rate cost, the strategy may produce a funding arbitrage and a lower FTP. In their study, Albanese, Andersen, and Iabichino (2015) show that the FVA reduction accompanying the primary UCVA reduction can be very substantial for large funding spreads.

16. Custom Capital Structure Gearing Dimensions: For instance, an attractive gearing of better than 90% on the ratio of FVA to UCVA is achievable at a funding spread of 274 bp. Since the investment structure above naturally may be tranched, a similar ratio can arguably be applied across the entire capital structure. Additional gearing can also be provided by regulatory capital reduction.
17. Managing Targeted Book Level Risks: The strategy outlined above may be extended in many ways, and can even be arranged to eliminate CVA entirely (Albanese, Brigo, and Oertel (2013)). All of these are used by Albanese, Andersen, and Iabichino (2015) to point out that funding costs can be managed and monetized, and FVA/FDA accounting establishes clear metrics for structuring desks to work with.

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Derivatives Funding, Netting, and Accounting

Introduction, Motivation, Scope, and Synopsis

1. Portfolio Counterparty and Funding: This chapter expands the previous replication results for economic value of derivatives including funding costs and counter party risk into several counter parties and netting sets. It also identifies the specific funding and replication strategy that corresponds to a popular accounting and pricing proposal (Burgard and Kjaer (2015)).
2. Asymmetry of the Accounting Proposal: The resulting strategy is asymmetric in that the negative net cash of the derivatives portfolio is funded at the funding rate, whereas the positive net cash is invested at the risk free rate. This asymmetry makes the funding cost adjustments non-additive across the netting sets.
3. Symmetric Funding Transfer Mechanism Strategies: In contrast, funding transfer mechanisms that recycle positive cash for other funding purposes enable symmetric funding strategies, resulting in additive adjustments across funding sets. These can generate a higher economic value over the life of the trade than the one accounted for under the proposed methodology.

Model Setup and Asset Dynamics

1. Albanese and Andersen (2014) Accounting Approach: In their paper, Albanese and Andersen (2014) discuss how to account for the OTC derivatives with funding costs and counter party risks in a way that is consistent with general accounting principles. This is an important contribution to the debate around funding value adjustments.
2. Contra Asset and Contra Liability: Albanese and Andersen (2014) propose contra-asset and contra-liability terms related to counter party and funding costs that account for a total

enterprise value to the shareholders and the bondholders that is consistent with the Modigliani-Miller Theorem.

3. CET1 Enhancement Policy for the FTP: They also propose a funds transfer policy for the derivatives that keeps the Core Equity Tier 1 (CET1) constant when adding new derivatives to the book, resulting in a funding cost adjustment being applied to the price.
4. Funding Strategy Impact on CET1: Burgard and Kjaer (2013) have shown that different funding strategies correspond to different economic value to the shareholders. So what funding strategy would generate an economic value to the shareholder over the life of the trading book that corresponds to the price that keeps the CET1 constant under the accounting methodology proposed by Albanese and Andersen (2014)? And what happens if a different funding strategy is followed?
5. Funding Strategy Portfolio Effects Impact: To address these questions, this chapter uses the replication framework of Burgard and Kjaer (2011a, 2011b, 2011c, 2013). In order to study the portfolio effects, the framework is extended to two counterparties and allows for several netting sets for each. The portfolio is assumed to be within a single funding set, i.e., the whole portfolio is funded with a single funding strategy.
6. Single Currency Default Free Funding Source: For simplicity, it is assumed that the trade between the issuer and any of the counter parties are uncollateralized, and a single currency economy with a default provider of funding and hedge assets is considered.
7. Risk Free Bond in Portfolio: A third risk-free bond is added to the funding portfolio to allow for the case considered in Albanese and Andersen (2014) where excess cash is invested in risk free assets rather than being used to repurchase own bonds, or being recycled for other funding purposes of the issuer.
8. Neglecting the Feedback Effects: Like Burgard and Kjaer (2013) and Albanese and Andersen (2014), the balance sheet feedback effects are neglected. Such effects were discussed in Burgard and Kjaer (2011b, 2011c) but depend on the funding cost of the existing debt to be directly significant to the shareholders. For simplicity deterministic rates, credit, and funding spread are assumed, although the results can easily be extended in that regard.
9. Notations used in the Formulation:

i	Counter Party Index such that $i = 1, 2$
J_B, J_{Ci}	Independent Poisson Processes during Defaults of the Issuer B and the Counter Party i
S	The Underlying Asset
P_1	Risk Free Zero Coupon Bond with Rate r
P_B	Issuer Risky ZCB with rate r_B and recovery R_B
P_{Ci}	Counter Party i ZCB with rate r_{Ci} and zero recovery
β_S	Asset Cash Account with the Net Rate $\gamma_S - q_S$
β_{Ci}	Bond P_{Ci} Hedge Cash Account

10. <u>Asset/Bank/Counter Party Dynamics</u>: Next the following simple dynamics are assumed.

$$\Delta P_1 = r P_1 \Delta t$$

$$\Delta P_B = r_B P_B^{-} \Delta t - (1 - R_B) P_B^{-} \Delta J_B$$

$$\Delta P_0 = r_0 P_0^{-} \Delta t - P_0^{-} \Delta J_0$$

$$\Delta P_{Ci} = r_{Ci} P_{Ci}^{-} \Delta t - P_{Ci}^{-} \Delta J_{Ci}$$

$$\Delta S = r S \Delta t + \sigma S \Delta W$$

Balance Sheet Dynamics under Semi-Replication

1. <u>Two Counter Parties and Multiple Netting Sets</u>: To derive the replication results the derivation methodology of Burgard and Kjaer (2013) is closely followed, extending it to two counter parties and several netting sets. The issuer has entered into uncollateralized derivatives with both counter parties and for simplicity it is assumed that all trades have the

same maturity T. Let $H_{ij}(S(T))$ denote the payoff at the expiry of trades belonging to counter party i and netting set j.

2. Value of Portfolio to Bank: Different trades with a counter party may belong to different netting units. Let $\hat{V}$ be the total economic value of the portfolio including funding costs and counter party risks before the default of any of the counter parties.
3. Definition of the Economic Value: Economic value is defined to be the expected (and discounted) value that will be realized over the life of the trade by the issuer's shareholders.
4. Value on Counter Party Default: Analogously $\hat{V}_i$ is the total value of the trades with counter party i when this name is considered in isolation. This value becomes relevant once the other counter party has defaulted.
5. Value of the Unadjusted Reference Portfolio: Finally the total reference value V on top of which the valuation adjustments are computed is the value of the book with the same payoff but done on a fully collateralized basis with the collateral rate r.
6. Aggregation across Counter Party/Netting Sets: If V_{ij} is the reference value of the trades with counter party i and netting unit j then

$$V = \sum_i V_i = \sum_{i,j} V_{ij}$$

Analogously

$$V_i = \sum_j V_{ij}$$

is the reference value of the trades with the counter party i.

7. Portfolio Book Value Post Default: With this notation in place the book value immediately post a first default of issuer B or one of the counter parties C_1 or C_2 is defined by

$$g_B = \sum_{i,j} \left({V_{ij}}^+ + R_B {V_{ij}}^-\right) \equiv g_{B1} + g_{B2}$$

$$g_{C1} = \sum_j \left(R_{C1}{V_{1j}}^{+} + {V_{1j}}^{-}\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

$$g_{C2} = \sum_j \left(R_{C2}{V_{2j}}^{+} + {V_{2j}}^{-}\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

respectively, where for simplicity, regular bilateral close-outs are assumed.

8. Book Value Post Counter Party Default: What these boundary conditions say is that when the counter party k is the first to default, the issuer gets the closeout value $\bar{g}_{Ck}$ based on the close-out of the trades with *that* counter party, *plus* it continues to hold the total value (including counter party risks and funding costs) of the trades with the surviving counter party.
9. Counter Party/Bank Default Boundary Conditions: If i is the surviving counter party, if it then subsequently defaults, the boundary condition is given by $\bar{g}_{Ci}$, whereas if the issuer defaults it is given by g_{Bi}.
10. No Cross Asset Default Impact: Note that it is implicitly assumed here that the default of one counter party does not impact the value of the portfolio with the other counter party. This assumption could be relaxed by considering close-outs g_{Ck} that are more complicated functions of $\hat{V}_i$.
11. The Derivative Master Replication Portfolio: Assuming all parties are alive, a portfolio Π can be setup as

$$\Pi = \delta S + \alpha_1 P_1 + \alpha_B P_B + \alpha_{C1} P_{C1} + \alpha_{C2} P_{C2} + \beta_S + \beta_{C1} + \beta_{C2}$$

12. No Hedging at Bank Default: This portfolio includes the funding instruments and aims to replicate the derivative value $\hat{V}$ in all scenarios except possibly issuer default.
13. Financing/Counter Party Portfolio Positions: The cash accounts β_S and β_{Ci} are used to finance the S and the P_{Ci} positions, i.e.

$$\alpha_{Ci}P_{Ci} + \beta_{Ci} = 0$$

and

$$\delta S + \beta_S = 0$$

and are assumed to pay net rates of $q_S - \gamma_S$ and q_{Ci} respectively, where γ_S may be a dividend income.

14. Definition of the Funding Constraint: The hedge positions maybe collateralized or repo'ed, so q_S or q_{Ci} may be the collateral or the repo rates, respectively. Any other cash is financed/invested via the bond positions $\alpha_1 P_1$, $\alpha_B P_B$, or $\alpha_0 P_0$, which implies that the *funding constraint*

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

must hold at all times until the first default. Later on different funding strategies are specified, but for now this is kept general.

15. Eliminating Market/Counter Party Risks: Next considered is the combination of the derivative book and the hedge portfolio, where $(\delta, \alpha_{C1}, \alpha_{C2})$ have been chosen such that all the market and the counter party risks have been eliminated.
16. Evolution of the Derivative Portfolio: Going through algebra similar to that as in Burgard and Kjaer (2013), it can be shown that subject to the funding constraint

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

the combination of the derivatives and the replicating portfolio (which includes the funding instruments) evolves as

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1P_1 + r_B\alpha_BP_B + r_0\alpha_0P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

where

$$\mathcal{A}_t = \frac{1}{2}\sigma^2S^2\frac{\partial^2}{\partial S^2} + (q_S - \gamma_S)S\frac{\partial}{\partial S}$$

$$P_D = \alpha_1P_1 + r_B\alpha_BP_B$$

is the post-issuer default value of the issuer bond portfolio, and

$$\lambda_{Ci} = r_{Ci} - q_{Ci}$$

is the spread of the yield of zero coupon bonds P_{Ci} over its repo rate, i.e., the surviving rate of the counter party default hedge position.

17. Absence of Cash/Funding Basis: Under the assumption of zero basis between bonds

$$r_B = r + (1 - R_B)\lambda_B$$

where

$$\lambda_B \equiv r_0 - r$$

Inserting this into

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1P_1 + r_B\alpha_BP_B + r_0\alpha_0P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

yields

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t$$
$$+ (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

18. The Bank Default Hedge Error: It is thus evident that all risks except own default have been hedged, and as in Burgard and Kjaer (2013), the hedge error upon issuer default ϵ_h can be defined as

$$\epsilon_h = g_B + P_D = \sum_{i,j}\left({V_{ij}}^+ + R_B{V_{ij}}^-\right) + \alpha_1 P_1 + r_B\alpha_B P_B$$

subject to the funding constraint

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

19. Application of the Self Financing Criterion: While the user is still alive, the Δt terms in

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t$$
$$+ (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

should add to zero for the strategy to be self-financing. This gives a PDE for the economic value in this state.

20. Validity of the Funding Constraint: Without issuer default the total value of the portfolio $\hat{V} + \Pi$ remains zero, i.e., yielding perfect replication, while if the issuer defaults first, it jumps to the hedge error ϵ_h.
21. Counter Party Default Close Out: According to

$$g_{C1} = \sum_j \left(R_{C1} {V_{1j}}^+ + {V_{1j}}^-\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

and

$$g_{C2} = \sum_j \left(R_{C2} {V_{2j}}^+ + {V_{2j}}^-\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

the closeout amounts g_{C1} and g_{C2} depend upon the value of the portfolio with the surviving party $\hat{V}_2$ and $\hat{V}_1$ respectively. Thus the value of the portfolio with the surviving party $\hat{V}_i$ needs to be determined first.

22. Surviving Counter Party Replication Portfolio: Describing the funding strategy in this state by $(\alpha_{1i}, \alpha_{Bi}, \alpha_{0i})$ the balance sheet dynamics becomes

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$$

with hedge error

$$\epsilon_{hi} = g_{Bi} + \alpha_{1i}P_1 + \alpha_{Bi}R_B P_B = \sum_j \left({V_{ij}}^+ + R_B {V_{ij}}^-\right) + \alpha_{1i}P_1 + \alpha_{Bi}R_B P_B$$

subject to the funding constraint

$$\widehat{\hat{V}_i} + \alpha_{1i}P_1 + \alpha_{Bi}P_B + \alpha_{0i}P_0 = 0$$

Economic Values

1. Lack of Own-Default Benefit: The next step is to solve for the economic values $\hat{V}_i$ and then $\hat{V}$. As per

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

when the counterparty k has defaulted and only counterparty i is alive the shareholders earn or pay the amount $-\lambda_B\epsilon_{hi}$ per unit of time during the life of the issuer while not receiving the benefit ϵ_{hi} upon own default.

2. Economic Value of the Surviving Party: The drift $-\lambda_B\epsilon_{hi}$ but not the windfall $\epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$ should therefore be included in the *economic value* $\hat{V}_i$ to the shareholders, which satisfies the PDE

$$\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B\epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

3. Expression for Solution to $\hat{V}_i$: By Burgard and Kjaer (2013) the solution is given by

$$\hat{V}_i = \sum_j\left(V_{ij} + BLCVA_{ij}\right) + FCA_i$$

with

$$BLCVA_{ij} = -(1 - R_C)\int_t^T \lambda_{Ci}(u)D_{r+\lambda_B+\lambda_{Ci}}(t, u)\mathbb{E}_t\left[{V_{ij}}^{+}(u)\right]du$$
$$- (1 - R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_{Ci}}(t, u)\mathbb{E}_t\left[{V_{ij}}^{-}(u)\right]du \equiv FTDCVA_{ij} + FBA_{ij}$$

$$FCA_i = -\int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u)\mathbb{E}_t[\epsilon_{hi}(u)]du$$

4. Backwards Iteration from Counter Party Default: The default of the counter party k is a zero PnL event as the counter party credit risk is hedged, so we can continue backwards across the default event to the state where all the parties are alive in a self-financing fashion.
5. Funding Costs Incurred to Shareholders: Analogous to before

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

tell us that in this state the shareholders earn or pay the amount $-\lambda_B\epsilon_h$ per unit time during the life of the issuer while not receiving the benefit of the hedge error upon own default, and the drift $-\lambda_B\epsilon_h$ should also be included in the economic value $\hat{V}$ to the shareholders.
6. PDE for the Derivative Value: The value is then given as the solution to the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1}g_{C1} - \lambda_{C2}g_{C2} + \lambda_B\epsilon_h$$

$$\hat{V}(T,S) = H(S)$$

where it is to be recalled that the closeout values g_{C1} and g_{C2} as given by

$$g_{C1} = \sum_j \left(R_{C1}V_{1j}^{+} + V_{1j}^{-}\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

$$g_{C2} = \sum_j \left(R_{C2}V_{2j}^{+} + V_{2j}^{-}\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

depend on $\hat{V}_2$ and $\hat{V}_1$ respectively.

7. Solving the Coupled System of PDE's: Consequently the PDE's

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

and

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

form a coupled system of PDE's where the to the former feeds into the latter.

8. Funding Strategy Specific Value Adjustments: Based on this observation one creates the Ansatze

$$\hat{V} = V + U$$

and

$$\hat{V}_i = V_i + U_i$$

and after some details (next section) one obtains

$$\hat{V}_\alpha = V + BLCVA + FCA_\alpha$$

with

$$BLCVA = \sum_{i,j} BLCVA_{i,j}$$

and

$$\begin{aligned}
FCA_\alpha = & -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du \\
& - \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C_1}}(t,u)\left[1 - D_{\lambda_{C_2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du \\
& - \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C_2}}(t,u)\left[1 - D_{\lambda_{C_1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du \\
& - - \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C_2}}(t,u)\right]\left[1 - D_{\lambda_{C_2}}(t,u)\right] \mathbb{E}_t[0] du
\end{aligned}$$

where the dependence of the economic value $\hat{V}_\alpha$, the funding cost adjustment FCA_α, and the hedges $\epsilon_{h1,\alpha}$, $\epsilon_{h2,\alpha}$, and $\epsilon_{h,\alpha}$ on the specific funding strategy

$$\alpha = (\alpha_1, \alpha_B, \alpha_0)$$

has been made explicit. Even when there is no strategy superscript for the benefit of clarity, the dependence on the funding strategy is implicit.

9. Correspondence to the Default Scenarios: The four terms of the correspond to the following four default scenarios:
    a. No counter party has defaulted prior to the issuer default
    b. Only counter party 1 has defaulted prior to the issuer default
    c. Only counter party 2 has defaulted prior to the issuer default
    d. Both counter parties have defaulted prior to the issuer default
10. Multiple Counter Parties Valuation Adjustment: As discussed earlier as well as in Burgard and Kjaer (2013), the FCA is the expected value of the windfall or shortfall at the issuer default. The difference in the multi-name setup is that the difference is extended over the four scenarios.
11. Applicability of the Modigliani Miller Theorem: In each of the scenarios, the second additive terms of

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

are zero sum games in the expectations between the shareholders and the creditors, and hence do not affect the total value of the firm, in line with the Modigliani-Miller Theorem.
12. Terms affecting the Bond Holders: The first terms $\epsilon_h\Delta J_B$ and $\epsilon_h\Delta J_B$ affect the bond holders only, and correspond to the FDA accounting term in Albanese and Andersen (2014) – alias the DVA2 accounting term in Hull and White (2012).
13. Terms affecting the Shareholders: The second terms, the compensating drift $-\epsilon_h\lambda_B\Delta t$ and $-\epsilon_{hi}\lambda_B\Delta t$, affect the shareholders while there are live trades, and give rise to the FCA.

14. FCA Dependence on Funding Strategy: Since ϵ_h and ϵ_{hi} depend on the specific funding strategy, the FCA given by

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\epsilon_{h,\alpha}(u)] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)[1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[\epsilon_{h1,\alpha}(u)] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)[1 - D_{\lambda_{C1}}(t,u)] \mathbb{E}_t[\epsilon_{h2,\alpha}(u)] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)[1 - D_{\lambda_{C2}}(t,u)][1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[0] du$$

depends on it as well. In particular it may depend on $\hat{V}$ in a non-linear way and thus may not be explicitly computable.

15. Fee Transfer Pricing (FTP) Calculation: From the arguments above and in the last paragraph in particular it follows that the new trades should be charged with the incremental economic value $\Delta\hat{V}$ such that the shareholders do not loose money over the life of the book.
16. FCA Additivity over Netting Sets: The choice of the funding strategy also determines whether the FCA is additive over the counter parties and the netting sets or not, as will be determined in the section on funding strategies.

Derivation of the Coupled Solutions

1. Decomposition of $\hat{V}$ and Re-arrangement: One wishes to solve the coupled PDE system

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci}) \hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

and

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

Inserting the Ansatze

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U_{12}$$

into

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

yields

$$\begin{aligned}\frac{\partial \hat{V}_1}{\partial t} + \mathcal{A}_t \hat{V}_1 - (r + \lambda_B + \lambda_{C1}) \hat{V}_1 - \lambda_{C1} \hat{V}_2 + \frac{\partial \hat{V}_2}{\partial t} + \mathcal{A}_t \hat{V}_2 - (r + \lambda_B + \lambda_{C2}) \hat{V}_2 - \lambda_{C2} \hat{V}_1 \\ + \frac{\partial U_{12}}{\partial t} + \mathcal{A}_t U_{12} - (r + \lambda_B + \lambda_{C1} + \lambda_{C2}) U_{12} \\ = -\lambda_B g_{B1} - \lambda_{C1} \bar{g}_{C1} - \lambda_{C1} - \lambda_B g_{B1} - \lambda_{C1} \bar{g}_{C1} - \lambda_{C1} \hat{V}_2 - \lambda_B g_{B2} - \lambda_{C2} \bar{g}_{C2} \\ - \lambda_{C2} \hat{V}_1 + \lambda_B \epsilon_h\end{aligned}$$

2. <u>Substituting the Expressions for $\hat{V}_1 / \hat{V}_2$</u>: Identifying the terms of the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

allows the elimination of many of the terms resulting in

$$\frac{\partial U_{12}}{\partial t} + \mathcal{A}_t U_{12} - (r + \lambda_B + \lambda_{C1} + \lambda_{C2}) U_{12} = \lambda_B (\epsilon_h - \epsilon_{h1} - \epsilon_{h2})$$

$$U_{12}(T, S) = 0$$

with the solution

$$U_{12} = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{C1}+\lambda_{C2}}(t, u) \mathbb{E}_t[\epsilon_h(u) - \epsilon_{h1}(u) - \epsilon_{h2}(u)] du$$

3. Replacing the Expression for FCA: With

$$\hat{V}_i = \sum_j \left(V_{ij} + BLCVA_{ij}\right) + FCA_i$$

one can write

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U_{12} = V + U_1 + U_2 + U_{12} = V + \sum_j BLCVA_{ij} + FCA_1 + FCA_2 + U_{12}$$

and can thus define

$$FCA \equiv FCA_1 + FCA_2 + U_{12}$$

to end up with the equations

$$\hat{V}_\alpha = V + BLCVA + FCA_\alpha$$

and

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\epsilon_{h,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) [1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[\epsilon_{h1,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) [1 - D_{\lambda_{C1}}(t,u)] \mathbb{E}_t[\epsilon_{h2,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) [1 - D_{\lambda_{C2}}(t,u)] [1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[0] du$$

Fair Values

1. Correspondence with Albanese/Andersen Taxonomy: The different parts of the cash flows in

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B(g_B - \hat{V}) + \lambda_{C1}(g_{C1} - \hat{V}) + \lambda_{C2}(g_{C2} - \hat{V})\right]\Delta t$$
$$+ (g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B(g_{Bi} - \hat{V}_i) + \lambda_{Ci}(\bar{g}_{Ci} - \hat{V}_i)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

can be identified with the cash flow types CF1 to CF5 introduced in Albanese and Andersen (2014).

2. Counter Party Default Dynamic Hedging: As all the counter party risks are hedged the CF2 cash flows (counter party default) have been transferred into CF4 (dynamic hedging).
3. Windfall/Shortfall Cash Flows: As discussed, the components $(g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$ are martingales with short term jump size ϵ_h or ϵ_{hi} which are CF5 terms that represent windfalls or shortfalls to the creditor (aka the funding provider) and can be accounted for through a contra-liability term FDA, and the drift compensators $-\lambda_B\epsilon_h$ and $-\lambda_B\epsilon_{hi}$ per unit time are of CF4 types and generate the FCA (this is called FVA in Albanese and Andersen (2014)) through the life of the trade.
4. Equivalence of the FCA and the FDA: Since $(g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$ have zero expectations it follows that

$$FCA = FDA$$

The total terms of $(g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$ therefore do not contribute to the value of shareholders and bondholders.

5. Portfolio Fair Value to Bank: Referring to the combined value as the *fair value* $\hat{V}_{FV}$ it then follows that

$$\hat{V}_{FV} = \sum_{i,j}(V_{ij} + BLCVA_{ij}) \equiv \sum_{i,j}\hat{V}_{FV,ij}$$

where the bilateral CVA is given in

$$BLCVA_{ij} = -(1-R_C)\int_t^T \lambda_{Ci}(u)D_{r+\lambda_B+\lambda_{Ci}}(t,u)\mathbb{E}_t\left[V_{ij}{}^{+}(u)\right]du$$
$$-(1-R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_{Ci}}(t,u)\mathbb{E}_t\left[V_{ij}{}^{-}(u)\right]du \equiv FTDCVA_{ij} + FBA_{ij}$$

$\hat{V}_{FV}$ corresponds to the fair value that keeps PFV constant in Albanese and Andersen (2014).

Funding Strategies

1. Exploring Alternate Funding Strategies: This section considers four funding strategies. The first three strategies are described in Burgard and Kjaer (2013) and extended to the multi-name setup, so that the additivity of the adjustments may be studied. The last strategy is the one that replicates the adjustments proposed in Albanese and Andersen (2014).
2. Counter Party Default Portfolio Change: For the strategies presented here $(\alpha_1, \alpha_B, \alpha_0)$ represents the holdings prior to the first default, and $(\alpha_{1i}, \alpha_{Bi}, \alpha_{0i})$ represents that holdings when counter party k has defaulted, but counter party i and the issuer are still alive.
3. Perfect Replication Strategy Portfolio Composition: There are many strategies that imply perfect replication, i.e.,

$$\epsilon_{hi} = 0$$

and

$$\epsilon_h = 0$$

in addition to the funding constraints

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

and

$$\hat{\hat{V}}_i + \alpha_{1i}P_1 + \alpha_{Bi}P_B + \alpha_{0i}P_0 = 0$$

for example

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -V - \frac{1 - R_B}{R_B} \sum_{i,j} {V_{i,j}}^+$$

$$\alpha_0 P_0 = -U + \frac{1 - R_B}{R_B} \sum_{i,j} {V_{i,j}}^+$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -V_i - \frac{1 - R_B}{R_B} \sum_{j} {V_{i,j}}^+$$

$$\alpha_{0i} P_0 = -U_i + \frac{1 - R_B}{R_B} \sum_{i,j} {V_{i,j}}^+$$

which implies that the economic value in this case is equal to $\hat{V}_{FV}$

4. Violation of the Bond Covenants: However, as discussed in Burgard and Kjaer (2013) these funding strategies involve issuing senior bonds to repurchase risker ones in a dynamic fashion, and would, in general, violate bond covenants.
5. Strategy I in Burgard and Kjaer (2013): In a multi-name setup, the strategy I of Burgard and Kjaer (2013) corresponds to the following: example

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -V$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -V_i$$

$$\alpha_{0i} P_0 = -U_i$$

6. No Adjustment on Bank Default: This strategy is deigned to wipe out the adjustments U and U_i upon default of the issuer. As a result the hedge errors $\epsilon_{hi,I}$ and $\epsilon_{h,I}$ for this strategy are given by

$$\epsilon_{hi,I} = g_{Bi} - R_B V_i = \sum_j (1 - R_B) {V_{i,j}}^+$$

$$\epsilon_{h,I} = g_B - R_B V = \sum_{i,j} (1 - R_B) {V_{i,j}}^+ = \epsilon_{h1,I} + \epsilon_{h2,I}$$

which are always positive and thus a windfall to the bondholders.

7. Value of the Derivative to the Bank: It follows from

$$\hat{V}_\alpha = V + BLCVA + FCA_\alpha$$

and

$$
\begin{aligned}
FCA_{\alpha} = &- \int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du \\
&- \int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du \\
&- \int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) \left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du \\
&- \int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du
\end{aligned}
$$

that the economic value $\hat{V}_I$ and FCA for this strategy is given by

$$\hat{V}_I = V + BLCVA + FCA_I$$

$$FCA_I = \sum_{i,j} FCA_{ij,I}$$

with

$$FCA_{ij,I} = -\int_{t}^{T} s_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^{+}(u)\right] du$$

where

$$s_B = (1 - R_B)\lambda_B$$

8. Additivity across the Netting Sets: Hence for this funding strategy $\hat{V}$ and FCA are additive across funding sets.

9. Using Senior Unsecured Bank Bond: Strategy II of Burgard and Kjaer (2013) makes use of only senior unsecured bonds of a single recovery, and does so regardless of the sign.
10. Portfolio Composition of the Strategy: In a multi-name setup this is described by

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -\hat{V}$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -\hat{V}_i$$

$$\alpha_{0i} P_0 = 0$$

11. Hedge Errors of the Strategy: In this case, the hedge errors are given by

$$\epsilon_{hi,II} = g_{Bi} - R_B \hat{V}_i = \sum_j (1 - R_B) {V_{i,j}}^+ - R_B U_i$$

$$\epsilon_{h,II} = g_B - R_B \hat{V} = \sum_{i,j} (1 - R_B) {V_{i,j}}^+ - R_B U$$

and the FCA

$$FCA_\alpha = -\int_t^T \lambda_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C1}+\lambda_{C2}}(t,u)\mathbb{E}_t\big[\epsilon_{h,\alpha}(u)\big]du$$
$$-\int_t^T \lambda_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C1}}(t,u)\big[1-D_{\lambda_{C2}}(t,u)\big]\mathbb{E}_t\big[\epsilon_{h1,\alpha}(u)\big]du$$
$$-\int_t^T \lambda_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C2}}(t,u)\big[1-D_{\lambda_{C1}}(t,u)\big]\mathbb{E}_t\big[\epsilon_{h2,\alpha}(u)\big]du$$
$$-\int_t^T \lambda_B(u)D_{r+\lambda_B}(t,u)\big[1-D_{\lambda_{C2}}(t,u)\big]\big[1-D_{\lambda_{C2}}(t,u)\big]\mathbb{E}_t[0]du$$

becomes recursive due to its dependence on U_i and U.

12. Formulation without using the Adjustment: However, analogous to Burgard and Kjaer (2013) it is possible to rewrite

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r+\lambda_B+\lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B\epsilon_{hi}$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r+\lambda_B+\lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1}g_{C1} - \lambda_{C2}g_{C2} + \lambda_B\epsilon_h$$

as

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r_B\hat{V}_i = \lambda_{Ci}\big(\hat{V}_i - \bar{g}_{Ci}\big)$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r_B\hat{V} = \lambda_{C1}\big(\hat{V} - \bar{g}_{C1} - \hat{V}_2\big) + \lambda_{C2}\big(\hat{V} - \bar{g}_{C2} - \hat{V}_1\big)$$

with the boundary conditions

$$\hat{V}_i(T, S) = H_i(S)$$

and

$$\hat{V}(T, S) = H(S)$$

13. Solution for CVA and FVA: Solving this equation via Feynman-Kac gives

$$\hat{V}_{II} = \hat{V}_{1,II} + \hat{V}_{2,II} = V + CVA_F + FVA_F$$

where the modified adjustments

$$CVA_F = \sum_{i,j} CVA_{ij,F}$$

and

$$FVA_F = \sum_{i,j} FVA_{ij,F}$$

are additive across counter parties and netting sets with

$$CVA_{ij,F} = -(1 - R_{Ci}) \int_t^T \lambda_{Ci}(u) D_{\lambda_B + \lambda_{Ci}}(t, u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

$$FCA_{ij,F} = -\int_t^T s_B(u) D_{\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[V_{ij}{}^{+}(u)\right] du$$

14. Unconventional CVA and FVA Terms: Please note that these are not the traditional CVA and FVA terms, in particular the $CVA_{ij,F}$ is not the expected values of the losses due to the counter party default, and the $FCA_{ij,F}$ embedded in the $CVA_{ij,F}$ is not the expected value of the hedge errors upon issuer default anymore.
15. Derivative Value and the FCA_{II}: Thus these terms would not be the ones to be used in contra-asset and the contra-liability accounting entries. However since the total sum of the adjustments U are the same, it is still possible to extract the FCA by means of

$$\hat{V}_{II} = V + BLCVA + FCA_{II} = V + CVA_F + FVA_F$$

and thus

$$FCA_{II} = \sum_{i,j} \left(CVA_{ij,F} + FVA_{ij,F} - BLCVA_{ij}\right) = \sum_{i,j} FCA_{ij,II}$$

16. FCA_{II} Additivity across the Netting Sets: FCA_{II} is therefore additive across netting sets. Strategy II as defined in

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -\hat{V}$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi}P_B = -\hat{V}_i$$

$$\alpha_{0i}P_0 = 0$$

implies that the replication of the adjustments themselves are funded at the rate r_B.

17. Correspondence with the Piterbarg Methodology: Finally note that if there is no counterparty risk, then this model is equivalent to the one developed in Piterbarg (2010) where everything boils down to discounting with the r_B curve.
18. Albanese and Andersen (2014) Strategy III: In Albanese and Andersen (2014) it is assumed that any excess variation margin of a funding set has to be invested at the risk free rate r whereas any shortfall must be funded at the rate r_B albeit it is let somewhat unclear as to how the adjustments themselves are funded.
19. Differential between Investment and Funding: The simplest such strategy would be to invest the total amount $-\hat{V}$ into risk free bonds, if positive, and to fund via the unsecured bond P_B if negative.
20. The Corresponding Portfolio Allocation Strategy: This approach translates into the strategy

$$\alpha_1 P_1 = -\hat{V}^-$$

$$\alpha_B P_B = -\hat{V}^+$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i}P_1 = -\hat{V}_i^{\,-}$$

$$\alpha_{Bi}P_B = -\hat{V}_i^{\,+}$$

$$\alpha_{0i}P_0 = 0$$

21. The PDE Driving the Economic Value: It would follow that the economic value satisfies the non-linear PDE system

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i = r_B \hat{V}_i - s_B \hat{V}_i^{\,-} + \lambda_{Ci}\left(\hat{V}_i - \bar{g}_{Ci}\right)$$

$$\hat{V}_i(T, S) = H_i(S)$$

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} = r_B \hat{V} - s_B \hat{V}^- + \lambda_{C1}\left(\hat{V} - \bar{g}_{C1} - \hat{V}_2\right) + \lambda_{C2}\left(\hat{V} - \bar{g}_{C2} - \hat{V}_1\right)$$

$$\hat{V}(T, S) = H(S)$$

22. Use of Monte Carlo Methods: This non-linear PDE could be solved using Monte-Carlo methods, see e.g., Labordere (2012), but the performance of such methods is unclear for real world large portfolios such as the one studied in Albanese and Andersen (2014).
23. Nonlinear PDE Adjustment Impact: Note that due to non-linearity there may be no natural way of expressing $\hat{V}$ as a sum of V and some valuation adjustment U other than defining

$$U \equiv \hat{V} - V$$

In any case this strategy will not lead to the adjustments described in Albanese and Andersen (2014).
24. A Variation Implied Funding Strategy: The non-linearity is caused by the funding strategy being non-linear function of $\hat{V}$. The following small variation of this strategy – called strategy III henceforth – yields a system of linear PDE's:

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -V_i{}^-$$

$$\alpha_{Bi} P_B = -V_i{}^+$$

$$\alpha_{0i} P_0 = -U_i$$

where the usage of the zero-recovery bonds P_0 for the funding of the adjustments U and U_i is motivated by the fact that for closeouts based on the risk free values of the derivatives the adjustments disappear on default.

25. Hedge Errors of the Strategy: Thus funding the adjustment U via a zero recovery bond ensures that the funding disappears as well when the issuer defaults. The hedge errors in this case are given by

$$\epsilon_{hi,III} = g_{Bi} - V_i{}^- - R_B V_i{}^+ = -\sum_j (1 - R_B) V_{i,j}{}^- - (1 - R_B) V_i{}^+$$

$$\epsilon_{h,III} = g_{Bi} - V^- - R_B V^+ = -\sum_{i,j} (1 - R_B) V_{i,j}{}^- - (1 - R_B) V^+$$

and inserting these into the general formula

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\epsilon_{h,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) [1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[\epsilon_{h1,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) [1 - D_{\lambda_{C1}}(t,u)] \mathbb{E}_t[\epsilon_{h2,\alpha}(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) [1 - D_{\lambda_{C2}}(t,u)][1 - D_{\lambda_{C2}}(t,u)] \mathbb{E}_t[0] du$$

yields

$$FCA_{III} = FCA_{AA} - \sum_{i,j} FBA_{ij}$$

where FBA_{ij} is defined from

$$BLCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t[{V_{ij}}^+(u)] du$$
$$- (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t[{V_{ij}}^-(u)] du \equiv FTDCVA_{ij} + FBA_{ij}$$

and

$$FCA_{AA} = -\int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\{V_1(u) + V_2(u)\}^+] du$$

$$- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C_1}}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[{V_1}^+(u)] du$$

$$- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) \left[1 - D_{\lambda_{C_1}}(t,u)\right] \mathbb{E}_t[{V_2}^+(u)] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

where the zero bond basis relation

$$s_B = (1 - R_B)\lambda_B$$

has been used.

26. <u>Independence of the Default Poisson's</u>: FCA_{AA} corresponds to the funding cost adjustment used in Albanese and Andersen (2014) – where it is referred to as FVA – provided that the default times τ_i are given by the first jump times of independent (of each other and the market) Poisson processes with intensities λ_{Ci}.
27. <u>Non Additivity of the FCA_{AA}</u>: From

$$FCA_{AA} = -\int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\{V_1(u) + V_2(u)\}^+] du$$
$$- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[{V_1}^+(u)] du$$
$$- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) \left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t[{V_2}^+(u)] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

it can be seen that FCA_{AA} is non-additive and its calculation requires the simulation of the entire funding set in a path consistent way which would be a challenging task in terms of memory requirements.

28. Relation between FCA_{III} and FCA_{AA}: It is worth noting that the negative FBA component in the FCA_{III} of

$$FCA_{III} = FCA_{AA} - \sum_{i,j} FBA_{ij}$$

cancels the FBA component of the bilateral CVA such that the economic value may be written as

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

29. Expectation of the Hedge Error: Similar to theFCA_F in strategy II it should be noted thatFCA_{AA} is not the expected value of the hedge error at issuer default – this is given by FCA_{III}.
30. FTP Policy of Albanese and Andersen (2014): The adjustment obtained by charging the incremental $\hat{V}$ is very similar to the FTP pricing proposed in Albanese and Andersen (2014),

with the only difference being that they propose charging the increment of the slightly different economic value

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

where the unilateral CVA is defined as

$$UCVA = \sum_{i,j} UCVA_{ij}$$

with

$$UCVA_{ij} = -(1 - R_{Ci}) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^{+}(u)\right] du$$

and has replaced FTDCVA in

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

31. Incorporating Accounting and Regulatory Principles: Albanese and Andersen (2014) motivate this choice by means of accounting and regulatory principles that stipulate that CVA should be unilateral and not depend on the issuer credit quality.
32. Derivative Value to the Book: In the next section it is shown that a small, albeit more complicated modification of the funding strategy

$$\alpha_1 P_1 = -V^{-}$$

$$\alpha_B P_B = -V^{+}$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i}P_1 = -V_i^-$$

$$\alpha_{Bi}P_B = -V_i^+$$

$$\alpha_{0i}P_0 = -U_i$$

- the strategy IV – yields the economic value

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

rather than

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

33. Comparison of the FCA Magnitudes: Intuitively, this strategy is more conservative than the other strategies, in that FCA_{IV} is most negative. The section below proves that the inequality

$$FCA_{IV} < FCA_{III} \leq FCA_I$$

indeed holds.

Derivation of $\hat{V}_{IV}$

1. Funding Strategy that generates $\hat{V}_{IV}$: This section specifies the funding strategy that generates the economic value $\hat{V}_{IV}$ given in

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

2. Methodology for Calculating the Adjustment: So far in this chapter, the methodology has been to specify a funding strategy, derive the resulting economic value $\hat{V}$, and finally derive the valuation adjustment(s) by making the Ansatze

$$\hat{V} = V + U$$

3. Avoidance of Logical Formulation Fallacy: The strategy developed in this section contains the $UCVA$, so it is critical that the arguments are developed carefully to avoid the logical fallacy of *circulus in probando*.
4. Definition of the Unilateral CVA - UCVA: Let i, j be the counter party and the netting unit index, respectively. The unilateral CVA $UCVA_{ij}$ is defined as

$$UCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_{Ci}}(t, u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

which is the solution to

$$\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci}) UCVA_{ij} = (1 - R_{Ci}) \lambda_{Ci} {V_{ij}}^+$$

$$UCVA_{ij}(T, S) = 0$$

5. Counter Party/Netting Set Additivity: Furthermore the following definitions are set to hold:

$$UCVA_i = \sum_j UCVA_{ij}$$

and

$$UCVA = \sum_i UCVA_i$$

6. Motivation for the UCVA Definition: These definitions are motivated by making the issuer default-free by setting

$$\lambda_B = 0$$

in the PDE's

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_{ij}(T, S) = H_i(S)$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

in which case one obtains

$$\hat{V}_i = \sum_j \left(V_{ij} + UCVA_{ij}\right) = V_i + UCVA_i$$

and

$$\hat{V} = \sum_{i,j} \left(V_{ij} + UCVA_{ij}\right) = V + UCVA$$

7. Determination of the Funding Strategy: Before specifying the funding strategy one writes

$$\hat{V}_i = V_i + UCVA_i + U_i$$

and

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U = V + UCVA + U_1 + U_2 + U$$

where $UCVA_i$ and $UCVA$ are defined above and the remaining valuation adjustments U_i and U are unspecified for now.

8. The Funding Strategy Portfolio Components: The funding strategy is then given by

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^- - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^+$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

9. Funding of CVA/Other Adjustments: This strategy is very similar to the strategy

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -V_i^{-}$$

$$\alpha_{Bi} P_B = -V_i^{+}$$

$$\alpha_{0i} P_0 = -U_i$$

apart from that the unilateral CVA is funded at the risk-free rate and the remaining adjustments (denoted U and U_i) at the zero recovery bond rate r_0.

10. <u>The “One Counter Party” Case</u>: From

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1 P_1 + r_B\alpha_B P_B + r_0\alpha_0 P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

it follows that the PDE

$$\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B\epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

before invoking the zero basis conditions

$$r_B = r + (1 - R_B)\lambda_B$$

and

$$r_0 = r + \lambda_B$$

takes the form

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci} \hat{V}_i + r\alpha_{i1} P_1 + r_B \alpha_{iB} P_i + r_0 \alpha_{i0} P_0 = -\lambda_{Ci} \bar{g}_{Ci}$$

$$\hat{V}_i(T, S) = H_i(S)$$

11. <u>Using the Adjustment Breakdown Ansatz</u>: On inserting the funding strategy

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^- - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^+$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

and the ansatz

$$\hat{V}_i = V_i + UCVA_i + U_i$$

into

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci}\hat{V}_i + r\alpha_{i1}P_1 + r_B\alpha_{iB}P_i + r_0\alpha_{i0}P_0 = -\lambda_{Ci}\bar{g}_{Ci}$$

$$\hat{V}_i(T,S) = H_i(S)$$

one obtains

$$\begin{aligned}
\frac{\partial V_i}{\partial t} + \mathcal{A}_t V_i - rV_i + rV_i + \frac{\partial UCVA_i}{\partial t} + \mathcal{A}_t UCVA_i + \frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - r{V_i}^- - rUCVA_i \\
- [r + (1 - R_B)\lambda_B]{V_i}^+ - (r + \lambda_B)U_i + \lambda_{Ci}\bar{g}_{Ci} - \lambda_{Ci}V_i - \lambda_{Ci}UCVA_i - \lambda_{Ci}U_i \\
= \frac{\partial V_i}{\partial t} + \mathcal{A}_t V_i - rV_i \\
+ \sum_j \left[\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci})UCVA_{ij} + (1 - R_{Ci})\lambda_{Ci}{V_{ij}}^+\right] \\
+ \frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - (r + \lambda_B + \lambda_{Ci})U_i - s_B{V_i}^+
\end{aligned}$$

with

$$s_B = r_B - r$$

12. <u>Stochastic Integral Expression for U_i</u>: Recognizing that V_i and $UCVA_{ij}$ satisfy the Black-Scholes PDE and the PDE

$$\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci})UCVA_{ij} = (1 - R_{Ci})\lambda_{Ci}{V_{ij}}^+$$

$$UCVA_{ij}(T,S) = 0$$

respectively, one obtains

$$\frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - (r + \lambda_B + \lambda_{Ci}) U_i = s_B {V_i}^+$$

$$U_i(T, S) = 0$$

so it follows that

$$U_i = -\int_t^T s_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t, u) \mathbb{E}_t[{V_i}^+(u)] du$$

13. <u>U_i Interpretation as the FCA_i</u>: If one writes

$$FCA_i \equiv U_i$$

then it was shown earlier that

$$\hat{V}_i = V_i + UCVA_i + FCA_i$$

for the case of a single counter party.

14. <u>One Counter Party Formulation Recast</u>: Before proceeding to the two counter party case

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci} \hat{V}_i + r \alpha_{i1} P_1 + r_B \alpha_{iB} P_i + r_0 \alpha_{i0} P_0 = -\lambda_{Ci} \bar{g}_{Ci}$$

$$\hat{V}_i(T, S) = H_i(S)$$

is rewritten as

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci} \bar{g}_{Ci} - s_B {V_i}^+ = 0$$

which will be helpful for the two counter party case below.

15. The Two Counter Parties Case: From

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1P_1 + r_B\alpha_BP_B + r_0\alpha_0P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

it follows that the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_Bg_B - \lambda_{C1}g_{C1} - \lambda_{C2}g_{C2} + \lambda_B\epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

before invoking the zero basis conditions takes the form

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (\lambda_{C1} + \lambda_{C2})\hat{V} + r\alpha_1P_1 + r_B\alpha_BP_i + r_0\alpha_0P_0 = -\lambda_{C1}g_{C1} - \lambda_{C2}g_{C2}$$

$$\hat{V}(T, S) = H(S)$$

16. Applying the Adjustment Decomposition Ansatz: Analogous to the single counter party case, inserting the ansatz

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U$$

and the funding strategy

$$\alpha_1P_1 = -V - UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^{\ -} - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^{\ +}$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

one obtains

$$\sum_i \left[\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci}\bar{g}_{Ci} - s_B V_i^{\ +}\right] + \frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_{C1} + \lambda_{C2})U - s_B\left(V^+ - V_1^{\ +} - V_2^{\ +}\right) = 0$$

17. <u>Stochastic Integral Expression for U_i</u>: By

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci}\bar{g}_{Ci} - s_B V_i^{\ +} = 0$$

the summation above is zero so U satisfies

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_{C1} + \lambda_{C2})U - s_B\left(V^+ - V_1^{\ +} - V_2^{\ +}\right) = 0$$

$$U(T, S) = 0$$

and is thus given by

$$U = -\int_t^T s_B(u) D_{r+\lambda_B+\lambda_{C1}+\lambda_{C2}}(t,u)\mathbb{E}_t[{V_1}^+(u) + {V_2}^+(u) - V^+(u)]du$$

18. <u>Recovering the $\hat{V}_{IV}/FCA_{AA}$ Relation</u>: Wrapping up, it has been shown that

$$\hat{V}_{IV} = V_1 + V_2 + UCVA_1 + UCVA_2 + FCA_1 + FCA_2 + U$$

and as the arguments in the Albanese and Andersen strategy section (strategy III) yield

$$FCA_1 + FCA_2 + U = FCA_{AA}$$

one in fact has the equality

$$\hat{V}_{IV} = V + UCVA + FCA_{AA}$$

as claimed.

Proof of the Statement $FCA_{IV} \le FCA_{III} \le FCA_I$

1. <u>FCA_{IV} more conservative than FCA_I</u>: In this section it is shown that the FCA from strategy IV is more conservative than the FCA from strategy I.
2. <u>Comparison between FCA_{IV} and FCA_{III}</u>: From

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

and

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

it follows that

$$FCA_{IV} = FCA_{III} + UCVA - FTDCVA \leq FCA_{III}$$

3. Comparison between FCA_I and FCA_{III}: This is because

$$UCVA \leq FTDCVA$$

due to the risk-free vs. risky discounting and the positiveness of the expected exposures. So it remains to prove that FCA_{III} is more conservative than FCA_I.

4. Hedge Errors I and III: From Burgard/Kjaer's strategy I and Albanese and Andersen (2014) strategy III, the hedge errors $\epsilon_{hi,I}$, $\epsilon_{hi,III}$, $\epsilon_{h,I}$, and $\epsilon_{h,III}$ for the two strategies are given by

$$\epsilon_{hi,I} = \sum_j (1 - R_B){V_{ij}}^+$$

$$\epsilon_{hi,III} = -\sum_j (1 - R_B){V_{ij}}^- + (1 - R_B){V_i}^+$$

$$\epsilon_{h,I} = \sum_{i,j} (1 - R_B){V_{ij}}^+$$

$$\epsilon_{h,III} = -\sum_{i,j} (1 - R_B){V_{ij}}^- + (1 - R_B)V^+$$

5. Comparison across the Hedge Errors: On dropping $(1 - R_B)$ term for clarity

$$\epsilon_{h,III} = V^{+} - \sum_{i,j} V_{ij}{}^{-} = \max\left(\sum_{i,j} V_{ij}, 0\right) - \sum_{i,j} V_{ij}{}^{-}$$

$$= \max\left(\sum_{i,j} V_{ij} - \sum_{i,j} V_{ij}{}^{-}, -\sum_{i,j} V_{ij}{}^{-}\right) = \max\left(\sum_{i,j} V_{ij}{}^{+}, -\sum_{i,j} V_{ij}{}^{-}\right)$$

$$\geq \sum_{i,j} V_{ij}{}^{+} = \epsilon_{h,I}$$

6. Hedge Error $III > I$: From this proof it follows that

$$\epsilon_{h,III} \geq \epsilon_{h,I}$$

so the bondholder windfalls under strategy III are always greater than or equal to that of strategy I.

7. FCA_{III} more conservative than FCA_I: By

$$FCA_{\alpha} = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$

$$-\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$

$$-\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$

$$-\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

this translates into

$$FCA_{III} \leq FCA_I$$

so strategy III has the highest funding costs as expected.

Discussion

1. Albanese/Andersen Consistent Funding Strategy: The aim of this chapter was to find a funding strategy that generates an economic value to the shareholders that corresponds to the FTP price that keeps CET1 constant in the accounting methodology of Albanese and Andersen (2014).
2. Specification of the Funding Strategy: This strategy is to be found in the strategy III described by

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -V_i{}^-$$

$$\alpha_{Bi} P_B = -V_i{}^+$$

$$\alpha_{0i} P_0 = -U_i$$

or more precisely its refinement in

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^{\,-} - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^{\,+}$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

3. Characteristics of the Third Strategy: Being able to specify such a funding strategy means that the proposed accounting methodology is consistent with no double counting or missing terms. The main qualitative difference between this strategy and the others discussed in the section on funding strategies is that in this strategy the positive and the negative cash positions of the funding set are not treated symmetrically.
4. No Rehypothecation of Funding Flows: IN particular, apart from the adjustments themselves, a positive cash position (i.e., when V is negative) is not re-hypothecated across the funding sets, but re-invested at the risk free rate even if the issuer as a whole is funding negative.
5. Rehypothecation across Multiple Funding Sets: By contrast, in strategies I and II net positive cash in a funding set still earns funding rates and this is achieved by re-purchasing the issuer's own bonds, or equivalently, by recycling it for the issuer's other funding needs through an FTP or an economic funding process, typically involving a central treasury department.
6. Operational Challenges with Strategy III: Under strategy III the issuer becomes uncompetitive in selling derivatives that are asset like to the counter party out of a cash rich funding set as it cannot compensate the counter party for the credit risk exposure that it takes with respect to the issuer – this is because with strategy III the issuer cannot monetize DVA as a funding benefit in this funding set.
7. Potential Arbitrage with III/IV: It should also be noted that with the CET1 neutral pricing policy proposed by Albanese and Andersen (2014) an issuer with separate funding sets within the same netting set becomes arbitrageable, i.e., when one funding set is significantly

cash positive and the other is significantly cash negative, then a default-free counter party buying a derivative asset from a cash negative funding set gets a price that is lowered by the funding benefit and then could sell it back to the cash positive funding set where it is not charged the corresponding funding cost.

8. Elimination of the Bank Credit Risk: The counter party does not pick up the credit risk to the issuer in this case since the two derivatives are back-to-back and close-outs net in the same netting set.
9. Funding Sets Spanning across Netting Sets: Thus, the funding sets must span at least the netting nets, and, with internal transfer processes in place, could span the whole of the issuer business. In this case strategy III converges to strategy I as long as the issuer as a whole is funding negative, i.e., which is typically the case for a bank.
10. Albanese/Andersen Methodology on I: What happens if the accounting is based on smaller funding units, and the accounting methodology of Albanese and Andersen (2014) is applied but one of the funding strategies I or II is followed? In this case, if the issuer is able to achieve the CET1 neutral price, and then follows the funding strategy I (as an example), then the issuer achieves an economic value to the shareholders that is higher than what was originally accounted for.
11. The Three Different Economic Values: In this case, three different economic values can be found – the accounting fair value V_{FV} which is the value to the shareholder and the bond holder combined, the accounting value that makes the derivative CET1 neutral in the accounting methodology of Albanese and Andersen (2014), and the economic value $\hat{V}$ that corresponds to the funding strategy employed.
12. The Negative CET1 Impact Scenario: Note that if the issuer charges the economic value $\hat{V}$ then it will see a negative CET1 impact under this accounting methodology, but recover it over the life of the trade.
13. Funding Set Positive Cash Recycling: Ideally the accounting methodology could be modified such that the benefit of recycling positive cash balances of a given funding set to other businesses is accounted for so that the CET1 risk-neutral price is aligned with the economic value that can be achieved by recycling the funding through internal FTP processes.

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