Collateral Valuation and XVA Generation in DRIP

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Collateralized Agreements and Derivatives Valuation

Background

1. Background: While economies without risk-free rates have been considered in the past (Black (1972)), typical derivatives pricing treatments have assumed the existence of such rates as a matter of course (e.g., Duffie (2001)).
2. Holy Grail of Curve Construction: Combining multiple curves, partial collateralization involving multiple currencies, with liquidity, counter party risk, funding, and credit risk factored in into a dynamic approach is treated in a variety of papers (Pallavicini and Tarenghi (2010), Fujii, Shimada, and Takahashi (2010a), Fujii, Shimada, and Takahashi (2010b), Fujii, Shimada, and Takahashi (2010c), Fujii and Takahashi (2011a, 2011b), Castagna (2012), Henrard (2013)).
3. Treatments of CVA/DVA: Partial collateralization results in non-zero counter-party risk, and these cases are covered in Burgard and Kjaer (2011a, 2011b), Brigo, Pallavicini, Buescu, and Liu (2012), Crpey (2012a), Crpey (2012b). Considerations regarding the risk of an "average" counter-party are treated in Morini (2009).

Introduction and Motivation

1. Counter party Credit Risk Free Asset: Closest to a counterparty credit-risk free asset is an asset that is fully collateralized on a continuous basis (ISDA (2009), ISDA (2011), Sawyer (2011), Piterbarg (2012)), i.e., the collateralized asset produces cash flows that are continuous with changes in both the derivative MTM and the collateral coupon. Macey (2011) and Piterbarg (2012) illustrate how to retain the traditional risk-neutral valuation in a collateralized context.

2. Collateralized Asset Process: At the inception of a fully collateralized trade, there is no cash exchange, i.e., the upfront payment amount is returned back as collateral. Further, in exchange for the continuous pay streams above, the trade can be cancelled at any time with zero net value for either side.
3. Price of a Collateralized Asset: The price of a collateralized asset is effectively the outstanding level of the collateral account, i.e., a collateralized transaction is an asset with a zero-drift price process and with the given cumulative dividend flows (Duffie (2001)).
4. Collateral Cash Flows: $V(t)$ is the asset price paid by A to B, and B posts this amount back as collateral. A now pays the contractual collateral coupon flow $c(t)$ back to B. In time unit Δt, the cash flow that is exchanged (i.e., paid to A) is $V(t+\Delta t) - V(t) - c(t)V(t)\Delta t$, i.e.,

$$\Delta\chi(t) = \Delta V(t) - c(t)V(t)\Delta t$$

   Once this is exchanged, the transaction can terminate, and A can keep the collateral.

Two Collateralized Assets

1. Setup: Assume that each of the assets follows its corresponding real-world measures, but are exposed to the same risk factor ΔW, i.e.,

$$\Delta V_i(t) = \mu_i(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W$$

   for

$$i = 1, 2$$

2. Hedge Portfolio: Say that the corresponding collateralized account for each of these assets has the dynamics

$$\Delta\chi_i(t) = \Delta V_i(t) - c(t)V_i(t)\Delta t$$

Construct a hedge portfolio using $-\sigma_1(t)V_1(t)$ of asset 2 and $+\sigma_2(t)V_2(t)$ of asset 1. The net change in the real world collateralized portfolio of these two assets is:

$$\Delta\chi_{12}(t) = \sigma_2(t)V_2(t)[\Delta V_1(t) - c(t)V_1(t)\Delta t] - \sigma_1(t)V_1(t)[\Delta V_2(t) - c(t)V_2(t)\Delta t]$$

$$\Delta\chi_{12}(t) = V_1(t)V_2(t)[\sigma_2(t)\{\mu_1(t) - c(t)\} - \sigma_1(t)\{\mu_2(t) - c(t)\}]\Delta t$$

3. Application of the Collateral Rules: The above amount is known at time t, and maybe exchanged at $t + \Delta t$, at zero additional cost to either party. Thus, the only way both can enter into this transaction is if the net cash flow is zero (this is the collateralized version of no arbitrage). This produces

$$\frac{\mu_1(t) - c(t)}{\sigma_1(t)} = \frac{\mu_2(t) - c(t)}{\sigma_2(t)}$$

4. Differences with Traditional Risk Neutral Pricing: The main difference is: in the traditional risk-neutral pricing, the hedged portfolio grows at the "risk-free" rate. In collateralized pricing, the COLLATERALIZED + HEDGED portfolio grows at ZERO rate (i.e., does not grow at all) after incremental netting! Therefore the "risk-free" rate does not enter into this setting at all.
5. Measure Change: Create a new measure t where

$$\Delta W_Q = \Delta W + \frac{\mu_i(t) - c(t)}{\sigma_i(t)}$$

In this new measure, the individual assets grow as

$$\Delta V_i(t) = c(t)V_i(t)\Delta t + \sigma_i(t)V_i(t)\Delta W_Q$$

using which we estimate $V_i(t)$ as

$$V_i(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T c(s)ds} V_i(T) \right]$$

As may be observed, measure Q looks like the traditional risk neutral measure.

6. Different Collateral Rates: The collateral rates $c_i(t)$ can be asset-specific within changing any of our principal conclusions, and $V_i(t)$ now becomes

$$V_i(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T c_i(s)ds} V_i(T) \right]$$

Examples would be, say, a stock collateralized at its repo rate (or other funding rate), while the derivative would be collateralized at it collateral rate (e.g., Piterbarg (2010)).

7. Other Variants: Other collateralization variants include varying collateral processes, different counter-parties etc. Typically all these only end up varying the drift, thus you get

$$\frac{P_1(t,T)}{P_2(t,T)} = \frac{\mathbb{E}_t^{Q_1} \left[e^{-\int_t^T c_1(s)ds} V(T) \right]}{\mathbb{E}_t^{Q_2} \left[e^{-\int_t^T c_2(s)ds} V(T) \right]}$$

Of course, the collateralization drift can also be stochastic. This measure change from collateralization scheme #1 to collateralization scheme #2 induces a drift to the scheme #2 as

$$\mathbb{E}_t^Q \left[e^{-\int_t^T [c_2(s)-c_1(s)]ds} V(T) \right]$$

8. Many Collateralized Assets: Will quickly flip through this, as Piterbarg (2012) spells out the details. N-dimensional asset $\vec{V}$ possesses the real-world dynamics

$$\Delta\vec{V}(t) = \vec{\mu}^T(t)\vec{V}(t)\Delta t + \vec{\sigma}^T(t)\vec{V}(t)\Delta\vec{W}$$

A linearly combined weight set $\vec{w}$ of the hedge portfolio satisfies the constraint

$$\vec{w}^T \vec{\sigma} = 0$$

Using the collateral cash flow matching arguments presented above, we get

$$\vec{w}^T \left[\vec{\mu}^T \vec{V} - \vec{c}^T \vec{V} \right] = 0$$

Measure Change => As before, there exists a measure Q with the drift vector $\vec{c}$, one for each asset, such that an adjustment $\vec{\lambda}$ can be made to the real world measure making it

$$\Delta \vec{V}(t) = \vec{c}^T(t) \vec{V}(t) \Delta t + \vec{\sigma}^T(t) \vec{V}(t) \left[\Delta \vec{W} + \vec{\lambda} \Delta t \right]$$

such that $\Delta \vec{W} + \vec{\lambda} \Delta t$ can become drift-less. Once again, in this new measure, the individual assets follow

$$V_i(t) = \mathbb{E}_t^Q \left[e^{-\int_t^T c_i(s) ds} V_i(T) \right]$$

Setup of the Collateral Curve Dynamics

1. Short-Rate Collateral Curve: Piterbarg (2010) considers the risk-free curve for lending, a curve that corresponds to the safest available collateral (cash). The corresponding short rate is denoted by $r_C(t)$ – where C stands for CSA, since the assumption is that this is the agreed upon overnight rate paid among collateral dealers under the CSA.
2. HJM Parametrization of the Collateral Curve: It is convenient to parametrize term curves in terms of discount factors $P_C(t, T)$

$$0 \leq t \leq T < \infty$$

standard HJM theory applies with the following dynamics for the yield curve:

$$\frac{\Delta P_C(t, T)}{P_C(t, T)} = r_C(t) \Delta t - \sigma_C(t, T)^T \Delta W_C(t)$$

where W_C is a d-dimensional Brownina motion under the risk-neutral measure P and σ_C is a vector-valued, d-dimensional stochastic process.

3. Asset-specific Funding/Repo Rate: Piterbarg (2010) considers derivative contracts on a particular asset where the price process is denoted

$$S(t); t \geq 0$$

The short-rate on funding secured by this asset is r_R (R for repo).

4. Rates for Unsecured Funding: Finally the short-rate for unsecured funding is denoted

$$r_C(t), t \geq 0$$

As a rule, it would be expected that

$$r_C(t) \leq r_R(t) \leq r_F(t)$$

5. Funding Spread as a Default Premium: The existence of non-zero short rate spreads between short-rates of different collateral can be cast in the language of credit risk, by introducing joint defaults between the bank and the various assets used as collateral for funding.
6. Default Intensity of the Bank: In particular, the funding spread

$$s_F(t) \triangleq r_F(t) - r_C(t)$$

can be thought of as the stochastic intensity of default of the bank. The dynamics of the intensity is pursued in Gregory (2009) and Burgard and Kjaer (2009), while Piterbarg (2009) postulates the dynamics of the funding curves directly instead.

Collateralized Black-Scholes Formulation

1. Dynamics of the Derivative Value: This section examines how the regular Black-Scholes pricing methodology changes in the presence of CSA. Let $S(t)$ be an asset that follows in real-world the dynamics

$$\frac{\Delta S(t)}{S(t)} = \mu_S(t)\Delta t + \sigma_S(t)\Delta W_S(t)$$

2. Full Change in the Derivative Value: Let $V(t, S)$ be a derivative on the asset. By Ito's lemma it follows that

$$\Delta V(t) = \mathcal{L}\big(V(t)\big)\Delta t + \mathcal{X}(t)\Delta V(t)$$

where $\mathcal{L}$ is the standard pricing operator

$$\mathcal{L} = \frac{\partial}{\partial t} + \frac{1}{2}{\sigma_S}^2(t)S^2\frac{\partial^2}{\partial S^2}$$

and $\mathcal{X}(t)$ is the option's delta

$$\mathcal{X}(t) = \frac{\partial V(t)}{\partial S}$$

3. Full/Partial Collateral Cash Account: Let $C(t)$ be the collateral, i.e., the cash held in the collateral account, at time t against the derivative. For flexibility this amount may be different from $V(t)$.
4. Replicating Portfolio for the Derivative Payoff: To replicate the derivative at time t we hold $\mathcal{X}(t)$ units of the asset and $\gamma(t)$ units of cash. The value of the replication portfolio, which we denote by $\Pi(t)$ is equal to

$$V(t) = \Pi(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

where

$$\gamma(t) = C(t) + [V(t) - C(t)] - \mathcal{X}(t)S(t)$$

5. Decomposition of the Cash Account: The cash amount $\gamma(t)$ is split among a number of accounts;
   a. Amount $C(t)$ is in collateral
   b. Amount $V(t) - C(t)$ needs to be borrowed/lent from the treasury desk
   c. Amount $\mathcal{X}(t)S(t)$ is borrowed to finance the purchase of $\mathcal{X}(t)$ assets. It is secured by the assets purchased.
   d. The assets pay dividend at the rate $r_D(t)$
6. Full Growth of the Cash Account: The growth of all the cash accounts is given by

$$g(t)\Delta t = [r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)\mathcal{X}(t)S(t)]\Delta t$$

7. Applying the Self-Financing Criterion: On the other hand, from

$$V(t) = \mathcal{X}(t)S(t) + \gamma(t)$$

using the self-financing criterion one gets

$$g(t)\Delta t = \Delta V(t) - \mathcal{X}(t)\Delta S(t)$$

which becomes, by Ito's lemma

$$\Delta V(t) - \mathcal{X}(t)\Delta S(t) = \mathcal{L}\big(V(t)\big)\Delta t = \left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}\sigma_S{}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]\Delta t$$

8. Funding/Collateral Derivative Valuation PDE: Thus one obtains

$$\left[\frac{\partial V(t)}{\partial t} + \frac{1}{2}\sigma_S{}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}\right]$$
$$= r_C(t)C(t) + r_F(t)\{V(t) - C(t)\} - r_R(t)\mathcal{X}(t)S(t) + r_D(t)S(t)\frac{\partial V(t)}{\partial S}$$

which after re-arrangement results in

$$\frac{\partial V(t)}{\partial t} + [r_R(t) - r_D(t)]S(t)\frac{\partial V(t)}{\partial S} + \frac{1}{2}\sigma_S{}^2(t)S^2\frac{\partial^2 V(t)}{\partial S^2}$$
$$= r_F(t)V(t) - C(t)[r_F(t) - r_C(t)]$$

9. Solution Using Feynman-Kac Integral: The solution may be obtained by essentially following the steps that lead to the Feynman-Kac formula (Karatzas and Shreve (1997)) and is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}V(T) + \int_t^T e^{-\int_t^u r_F(v)dv}\{r_F(u) - r_C(u)\}C(u)du\right]$$

in the measure in which the asset grows at the rate $r_R(t) - r_D(t)$, that is

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

10. The Right "Risk-Free" Rate: Note that if the probability space is rich enough, it can be taken to be the same risk-neutral measure P in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T\Delta W_C(t)$$

Thus this derivation validates the view of Barden (2009) (and Hull (2006)) that the repo rate $r_R(t)$ is the right "risk-free" rate to use when valuing derivatives on $S(t)$.

Collateralization and Funding Derivative Valuation

1. Incremental Change in the Derivative Value: By re-arranging the Feynman-Kac expression above for $V(t)$ one obtains another useful expression for the valuation of the derivative:

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

It can be seen that

$$\mathbb{E}_t[\Delta V(T)] = [r_F(t)V(t) - \{r_F(t) - r_C(t)\}C(t)]\Delta t = [r_F(t)V(t) - s_F(t)C(t)]\Delta t$$

2. Derivative Value Under Full Collateralization: Thus the rate of growth in the derivative security is the funding spread $s_F(t)$ applied to the collateral. In particular, if the collateral is equal to the value $V(t)$ then

$$\mathbb{E}_t[\Delta V(T)] = r_C(t)C(t)\Delta t$$

and therefore

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T)\right]$$

and the derivative value grows at the risk free (i.e., collateral) rate. The final value is the only payment that appears in the discounted expression as the other payments net out given the assumption of full collateralization. This is consistent with the drift in

$$\frac{\Delta P_C(t,T)}{P_C(t,T)} = r_C(t)\Delta t - \sigma_C(t,T)^T\Delta W_C(t)$$

as $P_C(t,T)$ corresponds to the deposits secured by cash collateral.

3. Derived Value Under Unsecured Trading: On the other hand if the collateral is zero then

$$\mathbb{E}_t[\Delta V(T)] = r_F(t)V(t)\Delta t$$

and the rate of growth is equal to the bank's unsecured funding rate, or, using the credit risk language, adjusted for the probability of the bank default.

4. Collateral and Funding Measure Numeraires: Therefore the case

$$C = V$$

could be handled using the measure that corresponds to the risk free bond

$$P_C(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as a numeraire, and likewise, the case

$$C = 0$$

corresponds to the risky bond

$$P_F(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

as a numeraire.

5. Portfolio Effects of the Collateral Position: When two dealers are trading with each other the collateral is applied to the overall value of the portfolio between them with the positive exposures on some trades offsetting the negative exposures on the other trades (so-called netting). Hence the valuation of individual trades should take into account the collateral position of the whole portfolio.

6. Simplification of Full Collateralization/Trading: Fortunately in the simples case of the collateral being a linear function of the exact value of the portfolio - the case that includes both the no-collateral case

$$C = 0$$

as well as the full collateral case

$$C = V$$

- the value of the portfolio is just the sum of the values of the individual trades (with the collateral attributed to the trades by the same linear function). This easily follows from the pricing formula linearity of C and V in

$$V(t) = \mathbb{E}_t \left[e^{-\int_t^T r_C(u)du} V(T) - \int_t^T e^{-\int_t^u r_C(v)dv} \{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du \right]$$

Collateral PDE Formulation

1. PDE Collateralization Treatments: Bjork (2009), Piterbarg (2010), Castagna (2011), Fujii and Takahashi (2011a, 2011b), Henrard (2012), Piterbarg (2012), Ametrano and Bianchetti (2013), and Han, He, and Zhang (2013) extend the no-arbitrage to the collateralization case.
2. Review of Derivative PDE Using Replication: The derivative that is replicated using n assets and a bond via

$$V = nS + B$$

undergoes the evolution through the self-financing formulation

$$\Delta V = n\Delta S + \Delta B$$

This is matched to the derivative change

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S$$

Equating the two, setting

$$\frac{\partial V}{\partial S} = n$$

to eliminate stochasticity, and noticing that

$$\Delta B = rB\Delta t$$

we get

$$\left[\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\frac{1}{r} = B$$

Using the expression for V, this may be re-composed as the Black-Scholes PDE from

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_S{}^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} = rV$$

(Harrison and Kreps (1979), Harrison and Pliska (1981), Harrison and Pliska (1983)).

3. Derivative Replication with Collateral Account: The replication strategy now involves the assets, the bank funding account, and the collateral account.

$$V = aS + B + C$$

where a and B are the number of assets and the bank funding notional account, respectively, and C is the collateral account. Under perfect collateralization

$$C \equiv V$$

Further

$$\Delta C = r_C V \Delta t$$

$$\Delta B = r_f B \Delta t$$

and

$$\Delta S = \mu_S S \Delta t + \sigma_S S \Delta W$$

4. <u>Derivative Value Change</u>: Applying the self-financing condition

$$\Delta V = a \Delta S + \Delta B + \Delta C$$

Using the perfect collateral condition we get

$$V = aS + B + V$$

which implies

$$B = -aS$$

Thus

$$\Delta V = a \Delta S + r_f B \Delta t + r_C V \Delta t$$

results in

$$\Delta V = a\Delta S - ar_f S\Delta t + r_C V\Delta t$$

We refer to the quantity

$$\Gamma\left(r_C, r_f\right) = -ar_f S\Delta t + r_C V\Delta t$$

as the cash account.

5. <u>The Collateralization PDE</u>:

$$\Delta V = \left[\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2}\right]\Delta t + \frac{\partial V}{\partial S}\Delta S = a\Delta S - ar_f S\Delta t + r_C V\Delta t$$

Setting

$$a = \frac{\partial V}{\partial S}$$

we get

$$\frac{\partial V}{\partial t} + \frac{1}{2}{\sigma_S}^2 S^2 \frac{\partial^2 V}{\partial S^2} + r_f S\frac{\partial V}{\partial S} = r_C V$$

Re-casting using the appropriate measure terminology, we get

$$V(S,t) = \mathbb{E}_t^{Q^f}[D_C(S,T)V(S,T)]$$

where

$$D_C(t,T) = e^{-\int_t^T r_C(u)du}$$

Forward Contract Valuation

1. Repo'd Zero Strike Call Option:
    a. Asset Delivery at Future Time => Possibly the simplest derivative contract on an asset is the promise to deliver this asset at a given future time T. The contract should be seen as a zero strike call option with expiry T. In the standard theory, of course, the value of the derivative is the same as the value of the asset itself (in the absence of dividends).
    b. Forwards Value and Derivative Value => The payout of the derivative is given by

$$V(T) = S(T)$$

and the value at time t, assuming no CSA, is given by

$$V_{ZSC}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]$$

On the other hand, if

$$r_D(t) \neq 0$$

then

$$S(t) = \mathbb{E}_t\left[e^{-\int_t^T r_R(u)du} S(T)\right]$$

as follows from

$$\frac{\Delta S(t)}{S(t)} = [r_R(t) - r_D(t)]\Delta t + \sigma_S(t)\Delta W_S(t)$$

and clearly

$$V_{ZSC}(t) \neq S(t)$$

c. Repo Impact on the Value => The difference in the value between the derivative and the asset are now easily understood as the zero-strike call-option carries the credit risk of the bank, while the asset $S(t)$ does not. Or, in the language of funding, the asset $S(t)$ can be used to secure the funding – which is reflected in the corresponding repo rate applied – while $V_{ZSC}(t)$ cannot be used for such a purpose.

2. No-CSA Forwards Valuation:
    a. Forwards Contract without CSA => This section considers a forward contract on $S(t)$ where at a time t the bank agrees to deliver the asset at time T against a cash payment at time T.
    b. No-CSA Forward Contract Definition => A no-CSA forward contract could be seen as a derivative with payout $S(T) - F_{NoCSA}(t, T)$ at a time T where $F_{NoCSA}(t, T)$ is the forward price at a time t for a delivery at T. As the forward contract is cost free, we have by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

that

$$0 = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - F_{NoCSA}(t, T)\}\right]$$

so we get

$$F_{NoCSA}(t, T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

c. Valuation of the No-CSA Forward => From the above expression for $F_{NoCSA}(t,T)$ define

$$P_F(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]$$

Note that this is essentially a credit-risky bond issued by the bank. Thus the expression for $F_{NoCSA}(t,T)$ can be re-written as

$$F_{NoCSA}(t,T) = \widetilde{\mathbb{E}}_t^T[S(T)]$$

where the measure $\tilde{P}_T$ is defined by the numeraire $P_F(t,T)$ as

$$e^{-\int_0^t r_F(u)du} P_F(t,T) = \mathbb{E}_t\left[e^{-\int_0^T r_F(u)du}\right]$$

is a P-martingale. Thereby $F_{NoCSA}(t,T)$ is a $\tilde{P}$-martingale.

d. No-CSA Forward Probability Measure => Note that the value of the asset under no-CSA at time t is given by

$$\mathbb{E}_t[\Delta V(T)] = r_F(t)V(t)\Delta t$$

to be

$$V(t) = \mathbb{E}_t\left[e^{-\int_0^T r_F(u)du} V(T)\right] = P_F(t,T)\widetilde{\mathbb{E}}_t^T[V(T)]$$

so it could be calculated simply by taking the expected value of the payout in the risky T-forward measure.

3. <u>Forwards Contract with CSA</u>:

a. Full CSA Forward Contract Definition => Now let us consider a forward contract covered by a CSA where we assume that the collateral posted C is always equal to the value of the contract.

b. Valuation of the CSA-Based Forward => Let the forward price $F_{CSA}(t,T)$ be fixed at t; then the value from

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T) - \int_t^T e^{-\int_t^u r_C(v)dv}\{r_F(u) - r_C(u)\}\{V(u) - C(u)\}du\right]$$

is given by

$$0 = V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}V(T)\right] = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right]$$

so we get

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

Comparing this with

$$F_{NoCSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\right]}$$

we see that

$$F_{NoCSA}(t,T) \neq F_{CSA}(t,T)$$

By the arguments similar to the no-CSA case we obtain

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

where P^T is the standard T-forward measure – that is a measure defined by

$$P_C(t,T) \triangleq \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]$$

as its numeraire.

c. CSA Based Forward Probability Measure => Note that the value of an asset under CSA at a time t with a payout $V(t)$ is given by

$$V(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} V(T)\right] = P_C(t,T) = \mathbb{E}_t^T[V(T)]$$

so it could be simply calculated by taking the expected value of the payout in the risk-free T-forward measure.

4. <u>Calculating CSA Convexity Adjustment</u>:
   a. The Funding Basis Spread Numeraire => This section focusses on the difference between the CSA and the non-CSA forward prices. It can be seen that

$$F_{NoCSA}(t,T) = \widetilde{\mathbb{E}}_t^T[S(T)] = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_F(u)du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} e^{-\int_t^T \{r_F(u)-r_C(u)\}du} S(T)\right]}{P_F(t,T)}$$

$$= \frac{P_C(t,T)}{P_F(t,T)} \mathbb{E}_t^T\left[e^{-\int_t^T s_F(u)du} S(T)\right] = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)} S(T)\right]$$

where

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

is a P^T-martingale, as

$$M(t,T) = \mathbb{E}_t^T \left[e^{-\int_t^T s_F(u) du} \right]$$

b. CSA vs. no-CSA Convexity => It can be noted trivially that

$$\mathbb{E}_t^T \left[\frac{M(T,T)}{M(t,T)} \right] = 1$$

so

$$F_{NoCSA}(t,T) - F_{CSA}(t,T) = \mathbb{E}_t^T \left[\left\{ \frac{M(T,T)}{M(t,T)} - \mathbb{E}_t^T \left[\frac{M(T,T)}{M(t,T)} \right] \right\} \{S(t,T) - F_{CSA}(t,T)\} \right]$$
$$= \frac{1}{M(t,T)} Covariance_t^T [M(T,T), F_{CSA}(T,T)]$$

c. Funding Spread Dynamical Model => To obtain the actual value of the adjustment the joint dynamics of $s_F(u)$ and $S(u), u \geq t$ needs to be postulated. A simple model presented later shows the results of these corrections.

5. Futures vs. CSA Forward Contracts:
    a. Futures vs. CSA Forward – Similarity => At first sight the forward contract under CSA looks like a futures contract on the asset. With the futures contract, the daily price difference gets credited/debited to the margin account. In the same way, as the forward prices move, a CSA forward contract also specifies that money exchanges hands.
    b. Futures vs. CSA Forward – Differences => There is, however, an important difference. Consider the value of the forward contract at

$$t' > t$$

a contract that was entered at time t, so

$$V(t) = 0$$

Then

$$\begin{aligned} V(t') &= \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\{S(T) - F_{CSA}(t,T)\}\right] \\ &= \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du} S(T)\right] - \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\right] F_{CSA}(t,T) \end{aligned}$$

From

$$F_{CSA}(t,T) = \frac{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du} S(T)\right]}{\mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\right]}$$

one gets

$$V(t') - V(t) = \mathbb{E}_{t'}\left[e^{-\int_{t'}^{T} r_C(u)du}\right]\{F_{CSA}(t',T) - F_{CSA}(t,T)\}$$

so the difference between the contract values on t and t' that exchanges hands on t' is equal to the discounted T difference in the forward prices. For a futures contract the difference will not be discounted.

c. Futures vs. CSA Forward Convexity => Therefore the types of convexity seen in the futures contract are different from those seen in the CSA vs. non-CSA forward contracts, a conclusion different from those reached by Johannes and Sundaresan (2007).

European Style Options

1. CSA vs. non-CSA Pricing:
    a. Basic European Option Pricing Setup => Consider a European style option on $S(T)$ with a strike K. Depending on the presence of absence of CSA we get two prices:

$$V_{CSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_C(u)du}\{S(T) - K\}^+\right]$$

    and

$$V_{NoCSA}(t) = \mathbb{E}_t\left[e^{-\int_t^T r_F(u)du}\{S(T) - K\}^+\right]$$

    where for the CSA case we assume that the collateral posted C is always equal to the option value V_{CSA}.
    b. CSA vs. non-CSA Numeraires => By the same measure change arguments as in the previous sections we get

$$V_{CSA}(t) = P_C(t,T)\mathbb{E}_t^T[\{S(T) - K\}^+]$$

    and

$$V_{NoCSA}(t) = P_F(t,T)\widetilde{\mathbb{E}}_t^T[\{S(T) - K\}^+]$$

    c. CSA vs. non-CSA Raw Moments => The difference between the measures $\tilde{P}_t^T$ and P_t^T not only manifests itself in the mean of $S(T)$ – as already established – but also reveals itself in the characteristics of the distribution of $S(\cdot)$ such as its variance and higher moments.
2. Distribution Impact of Convexity Adjustment:
    a. No-CSA European Option Price => To see ow a change of measure affects the distribution of $S(\cdot)$, using

$$F_{NoCSA}(t,T) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right]$$

one has

$$V_{NoCSA}(t) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\{S(T)-K\}^+\right]$$

where $M(t,T)$ is defined as

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)}e^{-\int_t^T s_F(u)du}$$

b. Conditional on $S(T)$ Option Price => From this, by conditioning on $S(T)$ one obtains

$$V_{NoCSA}(t) = P_F(t,T)\mathbb{E}_t^T\left[\alpha\big(t,T,S(T)\big)\{S(T)-K\}^+\right]$$

where the deterministic function $\alpha(t,T,x)$ is given by

$$\alpha(t,T,x) = \mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}|S(T)=x\right]$$

c. Linearization of the Conditional Funding Basis => Using the approach of Antonov and Arneguy (2009), Piterbarg (2010) approximates the function $\alpha(t,T,x)$ by a function that is linear in x;

$$\alpha(t,T,x) = \alpha_0(t,T) + \alpha_1(t,T)x$$

and obtains α_0 and α_1 by minimizing the squared differences while using the fact that

$$F_{CSA}(t,T) = \mathbb{E}_t^T[S(T)]$$

and

$$\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}\right] = 1$$

as

$$\alpha_1(t,T) = \frac{\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right] - F_{CSA}(t,T)}{Variance_t^T[S(T)]}$$

and

$$\alpha_0(t,T) = 1 - \alpha_1(t,T)F_{CSA}(t,T)$$

d. Slop of the Conditional Funding Basis Distribution => Recognizing the term $\mathbb{E}_t^T\left[\frac{M(T,T)}{M(t,T)}S(t,T)\right] - F_{CSA}(t,T)$ as the convexity adjustment of the forward between the n-CSA and the CSA versions of $F(t,T)$ one can write

$$\alpha_1(t,T) = \frac{F_{NoCSA}(t,T) - F_{CSA}(t,T)}{Variance_t^T[S(T)]}$$

e. Collateral vs. Funding Measure Relation => Differentiating

$$V_{NoCSA}(t) = P_F(t,T)\mathbb{E}_t^T\left[\alpha\big(t,T,S(T)\big)\{S(T) - K\}^+\right]$$

twice with respect to K one obtains the probability density functions (PDFs) of $S(T)$ under the two measures as

$$\tilde{P}_t^T(S(T) \in [K, K + \Delta K]) = [\alpha_0(t,T) + \alpha_1(t,T)K]P_t^T(S(T) \in [K, K + \Delta K])$$

So the PDF of $S(T)$ under the no-CSA measure is obtained by the density of $S(T)$ under the CSA measure by multiplying it with a linear function. It is not hard to see that the main impact of such a transformation is on the slope of the volatility smile of $S(\cdot)$.

3. Stochastic Funding Model: This section considers a simple stochastic funding model to estimate the impact of collateral rules on forwards and options. Consider an asset that follows a log-normal process

$$\frac{\Delta S(t)}{S(t)} \cong \mathcal{O}(\Delta t) + \sigma_S \Delta W_S(t)$$

and a funding spread that follows a simple one-factor Gaussian model of interest rates

$$\Delta s_F(t) = -\aleph_F[\theta - s_F(t)]\Delta t + \sigma_F \Delta W_F(t)$$

with

$$\langle \Delta W_F(t) \Delta W_S(t) \rangle = \rho \Delta t$$

where ρ is the correlation between the asset price and the funding spread.

a. Evolution of F_{CSA} under P => It is also assumed for simplicity that $r_C(t)$ and $r_R(t)$ are deterministic, and

$$r_D(t) = 0$$

Then

$$F_{CSA}(t, T) = \mathbb{E}_t^T[S(T)]$$

and

$$\frac{\Delta F_{CSA}(t,T)}{F_{CSA}(t,T)} = \sigma_S \Delta W_S(t)$$

with $W_S(t)$ being a Brownian motion under the risk-neutral measure P.

b. Evolution of the Credit-Risky Numeraire => On the other hand

$$\frac{\Delta P_F(t)}{P_F(t)} = \mathcal{O}(\Delta t) - \sigma_F b(T-t)\Delta W_F(t)$$

where

$$b(T-t) = \frac{1 - e^{-\aleph_F(T-t)}}{\aleph_F}$$

c. Evolution of the Funding Spread => As $M(t,T)$ is a martingale under P (since $r_C(t)$ is deterministic, the measures P and P_T coincide) one has from

$$M(t,T) \triangleq \frac{P_F(t,T)}{P_C(t,T)} e^{-\int_t^T s_F(u)du}$$

that

$$\frac{\Delta M(t,T)}{M(t,T)} = -\sigma_F b(T-t)\Delta W_F(t)$$

d. Evolution of the Convexity Adjustment => Both $M(t,T)$ and $F_{CSA}(t,T)$ are martingales under P so it follows that

$$\frac{\Delta[M(t,T)F_{CSA}(t,T)]}{M(t,T)F_{CSA}(t,T)} = \mathcal{O}(\Delta t) + \rho\sigma_S\sigma_F b(T-t)\Delta W_F(t)$$

Using the fact that

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = \mathbb{E}\left[\frac{M(T,T)}{M(0,T)}\{F_{CSA}(T,T) - F_{CSA}(0,T)\}\right]$$

one gets

$$F_{NoCSA}(0,T) = F_{CSA}(0,T)e^{-\int_0^T \rho\sigma_S\sigma_F b(T-t)dt} = F_{CSA}(0,T)e^{-\rho\sigma_S\sigma_F\frac{T-b(t)}{\aleph_F}}$$

and in the case

$$\aleph_F = 0$$

$$F_{NoCSA}(0,T) - F_{CSA}(0,T) = F_{CSA}(0,T)\left[e^{-\frac{1}{2}\rho\sigma_S\sigma_F T^2} - 1\right]$$

e. Tenor Dependence of Convexity Correction => Note that the adjustment grows roughly as T^2. A similar formula was obtained by Barden (2009) using a model in which the funding spread was functionally linked to the value of the asset.

Cross Currency Model

1. <u>LCH.ClearNet Collateral Rules</u>: Single currency trades (currently mostly swaps) are collateralized in their own currencies, but multi-currency trades (e.g., cross currency swaps) are typically collateralized in USD.
2. <u>Building Blocks</u>: The building blocks typically are a) Domestic-Currency Collateralized Domestic Zero Coupon Bonds, b) Foreign-Currency Collateralized Foreign Zero Coupon Bonds, c) Collateralized FX Contracts. In practice, the former (the collateralized zeros) may not trade, whereas collateralized FX contracts typically do.

3. Foreign Bonds Collateralized in Domestic Currency: Consider a foreign zero-coupon bond collateralized with domestic collateral. The price of this zero coupon bond in foreign currency is $P_{f,d}(t)$. If $X(t)$ is the forex rate (i.e., the number of domestic units per foreign unit), the collateral account cash flow growth is

$$\Delta\chi_i(t) = \Delta\left[P_{f,d}(t)X(t)\right] - c_d(t)\left[P_{f,d}(t)X(t)\right]\Delta t$$

where $c_d(t)$ is the domestic collateral rate.

4. Collateralization of the FX: If $r_{d,f}(t)$ is the rate agreed on a domestic loan collateralized by foreign collateral, then the FX collateral account cash flow growth is

$$\Delta\chi_X(t) = \Delta X(t) - X(t)\Delta t$$

The contention by Piterbarg (2012) is that there is no relation between the collateralization rates $r_{d,f}(t)$, $c_d(t)$, and $c_f(t)$.

5. Collateralization Using Domestic Collateral:

$$\begin{bmatrix} \dfrac{\Delta X(t)}{X(t)} \\ \dfrac{\Delta P_{d,d}(t)}{P_{d,d}(t)} \\ \dfrac{\Delta P_{f,d}(t)}{P_{f,d}(t)} \end{bmatrix} = \begin{bmatrix} r_{f,d}(t) \\ c_d(t) \\ c_f(t) \end{bmatrix} \Delta t + \sigma_d \Delta W_d$$

Thus, under the domestic collateralization risk-neutral measure Q_d we have the following:

$$X(t) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T r_{d,f}(s)ds} X(T)\right]$$

$$P_{d,d}(t,T) = \mathbb{E}_t^{Q_d}\left[e^{-\int_t^T c_d(s)ds}\right]$$

$$P_{f,d}(t,T) = \frac{1}{X(t)} \mathbb{E}_t^{Q_d} \left[e^{-\int_t^T c_d(s) ds} X(T) \right]$$

6. <u>Collateralization Using Foreign Collateral</u>:

$$\begin{bmatrix} \frac{\Delta\left[\frac{1}{X(t)}\right]}{\left[\frac{1}{X(t)}\right]} \\ \frac{\Delta P_{f,f}(t)}{P_{f,f}(t)} \\ \frac{\Delta P_{d,f}(t)}{P_{d,f}(t)} \end{bmatrix} = \begin{bmatrix} -r_{d,f}(t) \\ c_f(t) \\ c_d(t) \end{bmatrix} \Delta t + \sigma_f \Delta W_f$$

Thus, under the foreign collateralization risk-neutral measure Q_f we have the following:

$$\frac{1}{X(t)} = \mathbb{E}_t^{Q_f} \left[e^{\int_t^T r_{d,f}(s) ds} \frac{1}{X(T)} \right]$$

$$P_{f,f}(t,T) = \mathbb{E}_t^{Q_f} \left[e^{-\int_t^T c_f(s) ds} \right]$$

$$P_{d,f}(t,T) = X(t) \mathbb{E}_t^{Q_d} \left[e^{-\int_t^T c_d(s) ds} \frac{1}{X(T)} \right]$$

7. <u>$P_{d,f}(t,T)$ and $P_{f,d}(t,T)$ Numeraires</u>: These are effectively the cross-currency, oppositely collateralized numeraires, i.e., one unit of domestic/foreign currency collateralized using the corresponding foreign/domestic collateral. Thus these numeraires, as such, can form the basis for cross-currency discount curves employed in cross-currency swaps. Further, while these building blocks are primarily only discounting oriented – securities with forward/floater leg may also require a quanto adjustment to be applied.
8. <u>Cross Currency Model Parameters</u>: All the model parameters and the process dynamical parameters in the set of equations above can be independently observed.

9. "Implied" Cross Currency Risk Free Rate: The measure change from Q_d to Q_f under the Q_d measure is captured by the Q_d martingale

$$M(t) = \frac{\partial Q_f}{\partial Q_d} = e^{-\int_t^T r_{d,f}(s)ds} \frac{X(t)}{X(0)}$$

Thus, the corresponding growth rate $r_{d,f}(t)$ also helps clarify the references to the "cross-currency risk-free Rates" (e.g., Fujii and Takahashi (2011a, 2011b)) – viz., they are instantaneous FX collateralization rate using the foreign collateral.

10. Forward Forex Contract Collateralized with Domestic Collateral: This contract pays

$$X(T) - K$$

in the domestic currency, and is collateralized using the domestic collateral. Thus

$$\mathbb{E}_t^{Q_{f,d}}[X(T) - K] = X(t)P_{f,d}(t,T) - KP_{d,d}(t,T)$$

Therefore, the par strike K for this contract is

$$K = \frac{X(t)P_{f,d}(t,T)}{P_{d,d}(t,T)}$$

11. Forward Forex Contract Collateralized with Foreign Collateral: This contract pays $1 - \frac{K}{X(T)}$ in the foreign currency, and is collateralized using the foreign collateral. Thus

$$\mathbb{E}_t^{Q_f}\left[1 - \frac{K}{X(T)}\right] = P_{f,f}(t,T) - K\frac{P_{d,f}(t,T)}{X(T)}$$

Therefore, the par strike K for this contract is

$$K = \frac{X(T)P_{f,f}(t,T)}{P_{d,f}(t,T)}$$

12. Same Currency Collateralization:

$$V_{d,d}(t,T) = \mathbb{E}_t^{Q_{d,d}{}^T}[X(T) - K] = P_{d,d}(t,T)[X(T) - K]$$

and

$$V_{f,f}(t,T) = \mathbb{E}_t^{Q_{f,f}{}^T}\left[1 - \frac{K}{X(T)}\right] = \left[1 - \frac{K}{X(T)}\right]P_{f,f}(t,T)$$

No rocket science, really, with simple forwards. Question, however, is that whether $X(T)$ would ever be domestically collateralized, and that whether $\frac{1}{X(T)}$ would ever be collateralized in foreign currency. Same currency collateralization is uncommon presumably for these reasons.

13. Market Quotes for Collateralized Forex Forwards: Strictly speaking, all Forex Forwards should always be collateralized using either foreign or domestic collateral. Thus, the Forward Prices should be different depending on the collateralization currency. However, this DOES NOT appear to be the market practice, as the quotes are independent of collateral.

Collateral Choice Model

1. Setup: Here, an American style path-dependent collateral is chosen at every incremental step by opting for the collateral among the choices available that maximizes the incremental collateral cash flow.
2. Motivation: Collateralization at the domestic collateral accrual rate is c_d. On switching over to the foreign collateral, the rate becomes $c_f + r_{d,f}$. Thus at each time step we want to maximize the incremental collateral cash flow

$$\max(c_d, c_f + r_{d,f}) = c_d + \max(c_f + r_{d,f} - c_d, 0)$$

We begin by setting

$$q_{d,f} = c_f + r_{d,f} - c_d$$

3. Dynamics of $Q_{d,f}$: Consider the dynamics of

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

This entity has a drift

$$q_{d,f} = c_f + r_{d,f} - c_d$$

First of all, the dynamics of $c_d(t)$, $r_{d,f}(t)$, and $c_f(t)$ may be worked out using one of several typically accepted practices – e.g., the HJM-type dynamics, or an even more simplified Hull-White type dynamics.

- Using

$$Q_{d,f} = \frac{P_{d,f}}{P_{f,f}}$$

it is fairly straightforward to show that

$$\frac{\Delta Q_{d,f}}{Q_{d,f}} = q_{d,f}(t)\Delta t + \sigma_q(t)\Delta W_q$$

where

$$\sigma_q(t)\Delta W_q = \sigma_f(t)\Delta W_f + \sigma_X(t)\Delta W_X - \sigma_d(t)\Delta W_d$$

4. Piterbarg (2012) Expression for $q_{d,f}(t)$: Piterbarg (2012) employs a combination of HJM machinery as listed above and additional techniques outlined in Andersen and Piterbarg (2010) to obtain $q_{d,f}(t)$.
5. Collateral Choice - Deterministic $q_{d,f}(t)$: If $q_{d,f}(t)$ is deterministic, there will be no optionality involved; however, depending upon the sign of $q_{d,f}(t)$, there will be a collateral switch at each time increment. Piterbarg (2012) demonstrates this in his framework by turning the volatility explicitly down to zero.
6. Deterministic and Incremental Curve Decay Collateral: If the collateral discounting path choice can be proxied using a "curve roll up" phenomenon, the collateral choice discount factor becomes

$$P_{d,CC}(t_0, t_n) = \prod_{i=1}^{n} \min\left(\left\{P_{d,j}(t_{i-1}, t_i)\right\}_{j=1}^{r}\right)$$

where

$$j = 1, \cdots, r$$

are the r possible collateral choices, $j = 0$ is the domestic collateral curve, $P_{d,j}(t_{i-1}, t_i)$ is the discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the foreign collateral j, and $P_{d,CC}(t_{i-1}, t_i)$ is the collateral choice discount factor between t_{i-1} and t_i for one unit of domestic currency collateralized using the most appropriate incremental collateral. Note that this discount curve is artificial and deterministic.

- Advantages of using deterministic collateral choices => All the advantages stem from the computational simplicity. They are:
    - More than one collateral currency may be used, thus optimizing over the multiple collateral choices (USD, GBP, EUR, JPY, etc.)
    - Empirical Curve Representations using splining techniques may be usable

7. Valuing the Collateral Choice Option: The value we seek is of the form

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}V(T)\right]$$

where $V(T)$ is the terminal payoff at the time instant T. It may be a fixed amount (i.e., the fixed swap rate) or a variable amount (the floating swap coupon).

- Closed Form => Typically

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

has to be computed using Monte-Carlo or a PDE, therefore we seek an alternative fast analytic approximation. By Jensen's inequality, Piterbarg (2012) noticed that

$$\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right] \geq e^{-\int_t^T \max(q_{d,f}(s),0)ds}$$

This approximation may be used to compute the fixed leg value for the swap above. For the floater leg, the term $V(T)$ may be pushed outside to a separated expectation to get

$$P_{d,d}(0,T)\mathbb{E}_t^{Q_d^T}[V(T)]\mathbb{E}_t^{Q_d^T}\left[e^{-\int_t^T \max(q_{d,f}(s),0)ds}\right]$$

Piterbarg (2012) performs a full set of comparison to demonstrate that these approximations behave favorably with the Monte-Carlo under several situations.

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Centralized Random Number Generator

Introduction

1. Centralized Random Number Generation Methodology: This chapter describes the methodology and the implementation of the Centralized Random Generator, and its usage in DRIP's XVA framework.
2. Market Risk Factor Random Numbers: The Central Random Generator is the central price for establishing a cross-asset framework in DRIP's XVA framework. In the market generation step of the XVA framework, each market risk factor is simulated using the centralized random numbers.

Centralized Random Number Generator

1. Centralized Random Number Generation Module: The centralized random number generation module is responsible for producing the incremental shocks of major market risk factors between consecutive CVA simulation dates, taking into account the correlations between these risk factors.
2. Factor/Time/Path Random Increments: Each factor includes a sufficient number of time steps, paths, and components if needed (principal components or multiple factors). The module outputs standard normal random numbers to represent the shocks (increments) at each time step for each path and each component.

3. Cross Sectional Factor Correlations Incorporation: For each individual risk factor, random factors are mutually independent across time steps, paths, and components. For distinct market risk factors, the random numbers incorporate cross-sectional correlations between them.

Data Structures

1. The Random Sequence Component ID: For any risk factor, each random number produced is labeled by a component ID (tenor ID), a time step index, and a path index. These components are typically strings, such as "PC1" (the first principal component), or "3M" (the 3M tenor).
2. The Random Sequence Date Index: A time step index is an ordinal number to denote the order of the CVA dates, starting from 0. The CVA dates are also attached to the random number matrix.
3. The Random Sequence Path Index: The path index is also an ordinal number used to label the paths in the simulation, starting from 0.

Factor Model for Correlation Handling

1. Two-Tier Risk Factor Model: To model the correlation structure of the risk factors, a two-tier factor-risk model is used. The risk factors are classified into two groups – the primary (driving) market risk factors and the secondary risk factors – which are determined by their importance in terms of their market risk impact as well as the bank's exposure.
2. Modeling Primary/Secondary Risk Factors: The correlation between the primary market risk factors are modeled using a full correlation matrix. The secondary market risk factors are

modeled as dependent factors on the primary market risk factor so that their correlations are derived from the latter.

Variance Reduction

1. Generating the Antithetic Variable Sequence: The random number generator is capable of sampling the antithetic variable sequence. It is a common practice to use antithetic variables in Monte Carlo as a variance reduction technique. In the output of random numbers with antithetic variables, the $(2p+1)^{th}$ path is the negative of the $(2p)^{th}$ path

$$p = 0, 1, 2, \cdots$$

2. Variance Reduction with Quadratic Resampling: In some instances, it is desired that the standard deviation of the generated random numbers to be exactly equal to one, to reduce the Monte Carlo errors. This can be achieved by quadratic resampling.
3. Quadratic Resampling - The Two Steps: For a sequence of random numbers $r_0, r_1, \cdots, r_{T-1}$ the quadratic resampling follows the two step procedure outlined below.
4. Sub-dividing the sequence into Blocks: First, the random numbers sequence is divide into blocks of size N where N is typically large, but no greater than the length of the random number sequence T. If T is not a multiple of N, then the last block will be incomplete, i.e., it contains less than N numbers, and it will be excluded from the resampling.
5. Mean-Centering across the Block: Without loss of generality, it is assumed that the sequence is generated with antithetic sampling, i.e.,

$$r_{2p+1} = -r_{2p}$$

and N is even, so that for each complete block, the average is zero. Otherwise a mean-centering adjustment can be made to make it zero.

6. Variance Scaling inside the Block: Then inside each complete block $r_{iN}, r_{iN+1}, \cdots, r_{(i+1)N-1}$

$$0 \le i \le \left[\frac{T}{N}\right] - 1$$

let

$$s_i = \sqrt{\frac{1}{N}\sum_{j=0}^{N-1} {r_{iN+j}}^2}$$

denote the standard deviation.

7. Unbiased Estimate of the Standard Deviation: Another definition of the standard deviation is

$$s_i = \sqrt{\frac{1}{N-1}\sum_{j=0}^{N-1} {r_{iN+j}}^2}$$

These two definitions yield little difference when N is large.

8. The Quadratically Resampled Random Sequence: The quadratically resampled random numbers then are

$$\tilde{r}_{iN+j} = \frac{r_{iN+j}}{s_i}$$

$$j = 0, \cdots, N-1$$

It is straightforward to verify that taking a sequence of length $\{\tilde{r}_0, \cdots, \tilde{r}_{kN-1}\}$, it standard deviation is 1, where k is a positive integer.

Implementation – The Scope of the Risk Factors

1. The Primary Market Risk Factors: The risk factors considered in the random number generator are aligned with the asset classes, including the bench-mark interest rates (forward curves), foreign exchange rates, equity indexes and stock prices, commodity futures prices, CDS indexes, inflation rates, etc. and they are consistent with the risk drivers in typical productions of risk reports in each asset class.
2. Pre-generation of Random Numbers: To hasten the CVA generation, given the large number of risk factors, the random numbers for the risk factors are pre-generated before the CVA market generation step is initiated. I particular, random numbers can be pre-generated and stored in a system infrastructure and can be accessed upon request.
3. On-Demand Random Number Generation: For those risk factors that do not have these pre-generated random numbers, random numbers can be generated at the run time when the CVA market generation is initiated.
4. Rates Primary Market Risk Factor: Forward rate curves for the major currencies are considered the risk factors, which includes the ones listed below.
5. 3M LIBOR Forward Rate States: Forward rate curves for 3M LIBOR for USD, EUR, GBP, JPY, and CHF are the primary risk factors.
6. xM LIBOR Forward Rate States: Forward rate curves for LIBOR at other tenors.
7. Non-G5 DM IBOR Forwards: Forward rate curves for other major currencies that are not listed in LIBOR (such as AUD, CAD, etc.)
8. FX Primary Market Risk Factor: The spot FX rate for G10 currencies EUR, JPY, GBP, CHF, AUD, NZD, CAD, SEK, and NOK. For example, in Calypso, the foreign exchange rates all use USD as the base currency. In addition, other currencies that are available in Calypso are included, including CNY, DKK, HKD, etc.
9. Equity Primary Market Risk Factor: The risk factors in the equity space is primarily S&P 500, since over 80% of the gross equity deltas for typical equity derivatives books is in .SPX.

10. Equity Secondary Market Risk Factor: Other equity indices and individual stocks, such as AAPL, BRK.B, MET etc. are assigned as secondary (dependent) risk factors.
11. Commodity Primary Market Risk Factor: The risk factors are commodities futures curves such as WTI, BRT, and CGOLD. Each commodities futures type will have at least one futures curve being assigned as the primary risk factor – the rest can be assigned as secondary risk factors. For important basis curves (e.g., natural gas at major locations), they will also be assigned as the primary risk factor.
12. Credit Primary Market Risk Factor: Credit default swap indexes, such as CDX.NA.IG, CDX.NA.HY, iTRAXX Europe, etc. are used as primary risk factors.
13. Inflation Primary Market Risk Factor: CPI-U (consumer price index for all urban consumers) is included as the primary market factor for inflation risk. CPI-U is the reference index for TIPS and other inflation swaps.

Implementation – The Correlation Matrix

1. Correlations between Primary Market Factors: The correlation coefficients between the primary market risk factors are obtained from multiple sources as shown in the table below, e.g., the correlations between rates and FX spot are obtained from Calypso.
2. Source of DRIP Correlation Coefficients:

	Rates	**FX**	**Commodity**	**Equity**	**Credit**	**Inflation**
Rates	Calypso	Calypso	Historical	Historical	Historical	Historical
FX	Calypso	Calypso	Historical	Historical	Historical	Historical
Commodity	Historical	Historical	Perimeter Historical	Historical	Historical	Historical

Equity	Historical	Historical	Historical	Historical	Historical	Historical
Credit	Historical	Historical	Historical	Historical	Historical	Historical
Inflation	Historical	Historical	Historical	Historical	Historical	Historical

3. Correlations across Major Commodities Futures: Correlations between the prompt contracts of the major commodities futures are obtained from Perimeter (market implied correlation from European basket options on commodity pairs), while all other correlation coefficients are estimated from historical data.
4. Log-Normal Historical Returns Correlation: Pearson correlation with multiple years of historical data based on lognormal returns are computed. Correlation matrix can typically be updated on a monthly basis (or as needed).
5. Validity of Pooled Correlation Matrices: As some of the correlation coefficients are obtained from Calypso/Perimeter, and others are calculated from historical data, when they are pooled, the validity of the correlation matrices (e.g., positive definiteness, as well as numerical singularity) needs to be checked.
6. Conversion to Nearest Correlation Matrix: To handle numerical singularity, Jackel's approach is used; it finds a similar matrix that is positive definite (replacing non-positive eigenvalues with a small positive value).
7. Centralized Random Number Generation Customization:

Parameter	**Description**
OutputRequest	Outputs desired
Name	CRNG Name
ScenarioName	Scenario Name
TimeSteps	Array of Time Steps
NumPaths	Number of Paths

useQuadResample	Whether Quadratic resampling is used
isAntithetic	Whether Antithetic Paths are Simulated
PrimaryFactors	Primary Market Factors
Correlation	Correlation between the Primary Market Factors
SecondaryFactors	
rfRequestedFactors	Risk Factors to Generate Random Numbers for
Storage	Storage Type for the Output
Seed	Seed for the Random Number Generator
URI	URI for the Storage
BlockSize	Block Size used in Quadratic Resampling
BumpCorrFactors	
BumpCorrAmount	
BumpBetaFactors	
BumpBetaAmount	Bump Size for the Beta Coefficients
MarketDate	Optional, Run Date for the Random Numbers

Testing

1. Characterizing Monte Carlo Sequence Errors: By design the cross-correlations between the time steps are zero, and the cross-correlations between the risk factors are input correlations.

Consider the Monte-Carlo error; the actual correlations will only be close to the theoretical values.

2. Risk Factors Cross Correlation Errors: The results for an initial test are contained in the table below. For cross-correlation between the different risk factors, the simulation error is about 1%. For cross-correlation between the steps the simulation error is about 3%.
3. Risk Factor Serial Correlation Errors: The first column is the input correlation coefficients between these risk factors and the USD 3M LIBOR. The second column contains the correlation coefficients of the generated random numbers. Column 3 contains the differences. Column 4 contains the average step-wise correlation for the first 5 steps. The calculation is based on 1,000 paths and 5 time steps.

Risk Factor	**Input Correlation with USD_LIBOR_3M**	**Simulated Correlation**	**Difference**	**Average Cross-Step Correlation**
EUR_LIBOR_3M	16.1%	15.8%	-0.3%	-1.8%
JPY_LIBOR_3M	24.5%	26.1%	1.6%	-3.3%
GBP_LIBOR_3M	25.9%	25.7%	-0.2%	-0.8%
CHF_LIBOR_3M	35.2%	36.2%	1.0%	-0.05%
EUR FX	16.6%	15.5%	-1.0%	-0.07%
JPY FX	0.3%	-0.3%	-0.6%	1.5%
GBP FX	11.4%	12.4%	1.0%	0.06%
CHF FX	3.8%	2.8%	-1.0%	-1.6%

4. Historical vs. Model Correlation Comparison: For equities, only one primary risk factor (SPX) is used in the factor model. The table above gives the results for comparing the factor

model correlations and the true historical correlations for AAPL and .STOXX50E. As can be seen the differences are not significant.

	Lognormal 1Y	**Lognormal 5Y**	**Normal 1Y**	**Normal 5Y**
Historical Correlation	30.29%	27.48%	30.95%	26.38%
Factor Model Correlation	31.37%	33.76%	31.45%	32.34%
Difference (Factor Minus Historical)	1.09%	6.28%	0.51%	5.96%

Random Number Generators

1. Pseudo-Random Number Sequence Generators: The random numbers used in computer programs are pseudo-random, which means that they are generated in a predictable fashion using algorithms. These algorithms can generally create long runs of numbers with good properties, but eventually the sequence repeats.
2. Seed induced Pseudo-Random Sequence: The series of values generated by such algorithms is generally determined by a fixed number called a seed. If one gives the program the same seed twice, it produces the same sequence of random numbers. As these numbers are by no means “completely random”, they are sometimes referred to as pseudo-random numbers.
3. RNG as a Finite State Machine: A random number generator (RNG) can be described as a state machine; each time you ask for a random number, the state changes, so that the random number is different the next time you ask. In fact, since the RNG runs on a computer, which is finite, it must be a finite state machine.

4. Components of the RNG State: Formally, and RNG is defined by three components.
   a. An initial state s_0
   b. A transition function S on the states such that

$$s_{k+1} = S(s_k)$$

   for all

$$k \geq 0$$

   c. An output function V on the states such that for all

$$k \geq 0$$

   $V(s_k)$ is the random number at invocation k of the RNG.
5. Algorithm #1 - Linear Congruential Operator: One of the most common RNG's is the linear congruential generator, which uses recurrence based on modular integer arithmetic

$$s_{k+1} = (a s_k + b) \; mod \; M$$

   With appropriately selected a and b the sequence can achieve the maximum period M.
6. Algorithm #1 - L'Ecuyer's Recursive Generator: By combining multiple recursive sequences, a generator can have a large state space with good randomness properties, such as the L'Ecuyer's multiple recursive generator MRG32k3a

$$y_{1,n} = \left(a_{12} y_{1,n-2} + a_{13} y_{1,n-3}\right) mod \; m_1$$

$$y_{2,n} = \left(a_{21} y_{2,n-1} + a_{23} y_{2,n-3}\right) mod \; m_2$$

$$s_n = y_{1,n} + y_{2,n}$$

for all

$$n \geq 3$$

where

$$a_{12} = 1403580$$

$$a_{13} = -810728$$

$$m_1 = 2^{32} - 209$$

$$a_{21} = 527612$$

$$a_{23} = -1370589$$

$$m_2 = 2^{32} - 22853$$

This generator has a period length of approximately 2^{191} ($\approx 10^{57}$)

7. Algorithm #2 - Shift Register Generator: Another important class of RNG is the shift register generator which takes the form

$$x_{n+k} = \sum_{i=0}^{k-1} a_i x_{n+i} (mod\ 2)$$

where the x_n's and the a_i's are either 0 or 1.

8. Algorithm #2 - The Mersenne Twister: The maximum period of $2^k - 1$ can be achieved using as few as two non-zero values of a_i. This leads to a very fast RNG. The famous Mersenne Twister (MT19937) belongs to this class.

Multi-Stream RNG's

1. Eliminating the Seed-Path Overlap: In Monte Carlo multiple independent random number sequences are usually required. A naïve way of getting multiple independent random number seeds is by setting different seeds. It is simple to implement, but there is no guarantee that truly independent sequences are obtained, as multiple sequences may overlap.
2. Multi-Stream Seed Path Overlap: An ideal solution is to use multi-stream random number generator. This is achieved by splitting the random number sequence into multiple, uncorrelated streams.
3. Seed Path Overlap Elimination Schemes: The most commonly used methods are the skip-ahead method and the leap-frog method. Suppose that the threads are numbered from 0 to $N-1$ and the elements in the original sequence are numbered from 0 to $T-1$.
4. Simple Skip Ahead Overlap Elimination: Start thread t

$$0 \le t \le N-1$$

   at spot $T \times \frac{t}{N}$ in the sequence, and let each thread step through its length $\left\lceil \frac{T}{N} \right\rceil$ sub-sequence.
5. Leapfrog Path Overlap Elimination: Start thread t at spot t in the sub-sequence, and let each thread skip N elements at a time in the sequence.
6. Hybrid Seed Path Overlap Elimination: A skip-ahead is performed at the thread block level, and within one block each thread does a leapfrog generation.
7. Parallelizability of the Multi-Stream Generators: The key feature of the multi-stream RNG's is that each stream can be generated independently of the others. This makes them CUDA friendly and ready to be used in parallel algorithms.

Exposure Aggregation and XVA Calculation in Cross-Asset Model

Introduction

1. Exposure Aggregation/XVA Calculation Methodology: This section presents a general methodology for exposure aggregation and XVA calculation in the cross-asset CVA model framework.
2. Cross Asset Counter Party Model: The cross-asset counter-party model prices X valuation adjustment (XVA) including credit valuation adjustment (CVA) and debt valuation adjustment (DVA) for multi-asset portfolios.
3. Multi Stage Monte Carlo Simulations: The model is based on the Monte-Carlo simulation in five stages.
    a. Centralized Random Number Generation
    b. Market Data Simulation
    c. Trade Valuation
    d. Counter Party Level Aggregation
    e. XVA Calculation
4. Centralized Random Number Market Scenarios: The centralized random number generator generates all the random numbers for all the risk factors. The future market scenarios are generated for each model for each product category.
5. Calculation of the Future Exposures: The trade values for the future time points are computed for the simulated market scenarios for the specific model in each category. The future exposure of the portfolio is aggregated from the trade level, and the probabilities of default are calculated from the hazard curves.
6. Comprehensive Estimation of CVA/DVA: The CVA and the DVA can then be calculated from the expected future exposures, the probabilities of default, and the recovery rates. This

model systematically handles wrong-way and right-way risks, cross-asset correlations, and cross-asset integration.

7. Chapter Motivation, Scope, and Purpose: The general description of the cross-asset model can be found in a previous chapter. This chapter will present the methodology for future exposure aggregation and XVA calculation, specifically.

Netting and Aggregation

1. Netting Group vs. Collateral Group – Definition: In general, a counter party could contain multiple netting groups, and a netting group could contain multiple collateral groups. The collateral group contains clauses such as threshold, eligible collateral, and minimum transfer amount.
2. Definition of the Uncollateralized Groups: The trades not covered by any collateralized group are called uncollateralized, or equivalently, belong to a special collateral group whose collateral amount is always zero.
3. Contractually Specified Allowed Netting Types: Most agreements allow netting such that receivables and payables offset each other in the calculation of the exposure. This feature is called contractual netting. Other agreements disallow netting, and are called contractual non-netting. For contractual non-netting, each trade is treated as a netting group with a single trade. There is usually no collateral for contractual non-netting.
4. Treatment of Non-enforceable Contractual Netting: In contractual netting, under some circumstances (for example, if the counter-party is domiciled in emerging markets), the netting is deemed non-enforceable, and no netting is applied in the calculation of the positive exposure. However, in the case of negative exposure, netting is always applied.
5. Contractual Non-Netting Negative Exposures: For contractual non-netting counter-parties, for the purpose of being conservative, currently netting is applied in the calculation of negative exposures. This will be re-visited in the future.

6. Cross Collateral Group Collateral Amount Calculation: To calculate the exposure at the counter-party level, first the collateral amount for each collateral group is calculated, based on the values of all the trades within the collateral group and the CSA.
7. Cross Netting Group Exposure Aggregation: Then the exposure on the netting group level is calculated depending on the contractual netting and the netting enforceability. No netting is allowed across netting groups. In other words, each netting group should be treated similar to an individual counter-party. The results will be used to price the XVA in the next stage.
8. Value across Collateral/Netting Groups: Let V_{ij} be the j^{th} trade in the i^{th} collateral group, and

$$T_{i,CPTY} > 0$$

and

$$T_{i,BANK} < 0$$

be the effective collateral thresholds for the counter-party and the bank, respectively. Then to total portfolio value at time t in the netting group is

$$V(t) = \sum_{i.j} V_{ij}(t)$$

9. Margin Period Collateral Amount Estimation: The collateral amount from the i^{th} collateral group used for calculating the collateralized portfolio exposure at time t is calculated at time $t - \delta$ where δ is the default window with a default value of 14 calendar days.

$$C_i(t) = \sum_j \left[V_{ij}(t-\delta) - T_{i,CPTY}\right]^+ + \sum_j \left[V_{ij}(t-\delta) - T_{i,BANK}\right]^-$$

10. Explicitly Specified "Current Collateral Balance": When the collateral date is before or on the valuation date, i.e.,

$$t - \delta \leq T_{VAL}$$

and optional input – the "Current Collateral Balance" – can be used (this is the collateral balance at time 0). If this input is present, it will be used directly instead of calculating the time 0 collateral as shown above.

11. Cash Flow Adjusted Collateralized Portfolio: Then the collateralized portfolio amount including the cash-flow in the netting group at time t is

$$V_C(t) = V(t) + CF(t) - C(t) = \sum_{i.j} \left[V_{ij}(t) + CF_{ij}(t) \right] - \sum_i C_i(t)$$
$$= \sum_i \left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right] - C_i(t) \right\}$$

Here the cash flow $CF(t)$ is the cash flow in the time window $(t - \delta, t]$, inclusive of the payment at t and exclusive of the payment at $t - \delta$

12. Contractual Netting Positive/Negative Exposures: For contractual netting, when netting is enforceable, the collateralized positive exposure and the negative exposure for the netting group is

$$ColPositiveExp(t) = V_C{}^+(t) = \left(\sum_i \left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right] - C_i(t) \right\} \right)^+$$

$$ColNegativeExp(t) = V_C{}^-(t) = \left(\sum_i \left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right] - C_i(t) \right\} \right)^-$$

13. Contractual Unenforceable Netting Positive Exposure: When the netting is not enforceable, the collateralized positive exposure is calculated as

$$\begin{aligned} &ColPositiveExp(t) \\ &\quad = \sum_i \left(\left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right]^+ - C_i{}^+(t) \right\}^+ \right. \\ &\quad \left. + \left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right]^- - C_i{}^-(t) \right\}^+ \right) \end{aligned}$$

14. <u>Contractual Unenforceable Netting Negative Exposure</u>: The collateralized negative exposure is still calculated as before, i.e.,

$$ColNegativeExp(t) = \left(\sum_i \left\{ \sum_j \left[V_{ij}(t) + CF_{ij}(t) \right] - C_i(t) \right\} \right)^-$$

15. <u>Contractual Non Netting Uncollateralized Exposures</u>: For contractual non-netting, there is no collateral, and the positive and the negative exposures become

$$UncolPositiveExp(t) = \sum_{i,j} \left[V_{ij}(t) + CF_{ij}(t) \right]^+$$

$$UncolNegativeExp(t) = \sum_{i,j} \left[V_{ij}(t) + CF_{ij}(t) \right]^-$$

16. <u>Collateralized Aggregation across Netting Groups</u>: Finally, if a counter-party has multiple netting groups, then the collateralized positive exposure and the negative exposure for the counter-party are simply a summation across all the netting groups.

$$ColPositiveExp_{CPTY}(t) = \sum_{ng} ColPositiveExp_{ng}(t)$$

$$ColNegativeExp_{CPTY}(t) = \sum_{ng} ColNegativeExp_{ng}(t)$$

17. Uncollateralized Aggregation across Netting Groups: For uncollateralized exposures, the above formulas hold true, except that the collateral amounts are set to zero;

$$C_i(t) = C_i(t) = C_i(t) = 0$$

After the netting set aggregation, six aggregated measures at each time and in each path are sown in the table below.

18. Exposure Related Time/Path Measures:

#	Measure	Description
1	Positive Exposure	Positive Exposures of a Portfolio
2	Collateralized Positive Exposure	Positive Exposures of a Collateralized Portfolio
3	Negative Exposure	Negative Exposures of a Portfolio
4	Collateralized Negative Exposure	Negative Exposures of a Collateralized Portfolio
5	Collateral Amount	Collateral Amounts
6	Cash Flow	Cash Flow Amounts

19. Aggregator Inputs - Counter Party Information: Counter Party Reference Data including netting groups, collateral groups, threshold, minimum transfer amount, independent amount, current collateral balance, contractual netting flag, netting enforceability flag, and trade ID in each collateral group.
20. Aggregator Inputs - Default Window Settings:

a. Counter Party Default Window => The counter-party's default window. 2 weeks as a default value (default is 14.0)
b. Bank Default Window => The bank's default window. The default window is used as an indicator only to trigger the counter party window == the bank default window.

21. Aggregator Inputs MPoR Estimation Settings:
    a. Margin Period => Margin Call Frequency; Daily Margin as Default Value; Default is 1.0
    b. MPoRInterpType => Portfolio Value Interpolation Type when calculating MPoR. Valid values currently are: LINEAR, SQRT_T, and BBRIDGE. BBRIDGE is set as default.

Aggregation for Trades Priced in Proxy Models

1. User-Specified Netting/Collateral Configuration: For trades priced from proxy models, one can use different aggregation methods. This example starts with 12 trades in one netting group and across two collateral groups as shown in the table below.

Trade ID	**Collateral Group**	**Aggregation Type**	**Netting Group (User Specified)**
1	1	A	1
2	1	A	1
3	1	B	1
4	1	B	1
5	1	Non-Proxy	1
6	1	Non-Proxy	1

7	2	A	1
8	2	A	1
9	2	B	1
10	2	B	1
11	2	Non-Proxy	1
12	2	Non-Proxy	1

2. Method #1 – Netting/Collateral Groups: This section considers three proposals to handle trades with proxy models. The first method is to separate each trade with a proxy aggregation type into a final netting group as shown below.

Trade ID	**Collateral Group**	**Aggregation Type**	**Netting Group (User Specified)**	**Netting Group (Final)**
1	1	A	1	2
2	1	A	1	3
3	1	B	1	4
4	1	B	1	5
5	1	Non-Proxy	1	1
6	1	Non-Proxy	1	1
7	2	A	1	6
8	2	A	1	7
9	2	B	1	8
10	2	B	1	9

11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

3. Method #2 – Netting/Collateral Groups: The second method is to group trades with different aggregation types in each collateral group into different netting groups as shown below.

Trade ID	**Collateral Group**	**Aggregation Type**	**Netting Group (User Specified)**	**Netting Group (Final)**
1	1	A	1	2
2	1	A	1	2
3	1	B	1	3
4	1	B	1	3
5	1	Non-Proxy	1	1
6	1	Non-Proxy	1	1
7	2	A	1	4
8	2	A	1	4
9	2	B	1	5
10	2	B	1	5
11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

4. Method #3 – Netting/Collateral Groups: The third method is to group trades with different aggregation types into different netting groups as shown below.

Trade ID	Collateral Group	Aggregation Type	Netting Group (User Specified)	Netting Group (Final)
1	1	A	1	2
2	1	A	1	2
3	1	B	1	3
4	1	B	1	3
5	1	Non-Proxy	1	1
6	1	Non-Proxy	1	1
7	2	A	1	2
8	2	A	1	2
9	2	B	1	3
10	2	B	1	3
11	2	Non-Proxy	1	1
12	2	Non-Proxy	1	1

Three Points Brownian Bridge Interpolation

1. Intermediate Broken Date Exposure Calculation: When calculating the margin period of risk, on each exposure date, one needs to find a trade value δ days before the exposure date, where δ is the default window with a default value of 14 calendar days.
2. Brownian Bridge Based Exposure Interpolation: The trade value on that date may not be available from simulation. In this case on needs an interpolation method to interpolate the

trade values from the neighboring simulation dates. This chapter uses the three point Brownian Bridge method – it is described as follows.

3. Brownian Bridge - The Grid Points: Given the values on three dates $\{(t_1, V_1), (t_2, V_2), (t_3, V_3)\}$ the intention is to interpolate the value on date t such that

$$t_2 < t < t_3$$

The trade value is assumed to follow the Brownian motion

$$\Delta V_t = \sigma \Delta W_t$$

4. Normally Distributed Brownian Bridge Factor: According to the Brownian Bridge interpolation in the interval (t_1, t_3)

$$V_t = \frac{t_3 - t}{t_3 - t_1} V_1 + \frac{t - t_1}{t_3 - t_1} V_3 + \sqrt{\frac{(t_3 - t)(t - t_1)}{t_3 - t_1}} v$$

where v is a Brownian Bridge factor from the normal distribution $\mathcal{N}(0, \sigma^2)$

5. v Chosen to satisfy V_2: One chooses v so that the interpolation at t_2 matches the value at t_2. Namely, the value at t_2 is

$$V_2 = \frac{t_3 - t_2}{t_3 - t_1} V_1 + \frac{t_2 - t_1}{t_3 - t_1} V_3 + \sqrt{\frac{(t_3 - t_2)(t_2 - t_1)}{t_3 - t_1}} v$$

6. Estimation of the Brownian Bridge Factor: Therefore v is determined as

$$v = \sqrt{\frac{t_3 - t_1}{(t_3 - t_2)(t_2 - t_1)}} \left[V_2 - \frac{t_3 - t_2}{t_3 - t_1} V_1 - \frac{t_2 - t_1}{t_3 - t_1} V_3 \right]$$

XVA Calculation

1. Aggregation over Counter-party Trades: Given the result of the aggregation, one has the counter-party level trade values at each time point and in each path. The next step is to proceed to calculate the XVA's.
2. Counter-Party/Bank Default Dates: Let $t_0, \cdots, t_N$ be the credit dates, where the first date t_0 is the valuation date. τ_{CPTY} and τ_{BANK} are the default dates for the counter-party and the bank, respectively.
3. Collateralized Expected Positive/Negative Exposures: Let $ColEPE(t_n)$ and $ColENE(t_n)$ be the collateralized expected positive exposure and the collateralized expected negative exposure respectively on the credit date t_n; let $D_f(t_n)$ be the discount factor on the credit date t_n.
4. CVA from Collateralized Positive Exposure: The CVA is then calculated as

$$
\begin{aligned}
CVA = -\sum_{n=0}^{N-1} & \left[w \cdot ColEPE(t_n) \cdot D_f(t_n) + (1-w) \cdot ColEPE(t_{n-1}) \cdot D_f(t_{n-1})\right] \\
& \times \left\{1 - R_{CPTY}\left(w \cdot ColEPE(t_n) + (1-w) \cdot ColEPE(t_{n-1}) \cdot \frac{D_f(t_{n-1})}{D_f(t_n)}\right)\right\} \\
& \times Prob(t_n \leq \tau_{CPTY} < t_{n+1})
\end{aligned}
$$

5. DVA from Collateralized Positive Exposure: The DVA is calculated as

$$
\begin{aligned}
DVA = -\sum_{n=0}^{N-1} & \left[w \cdot ColENE(t_n) \cdot D_f(t_n) + (1-w) \cdot ColENE(t_{n-1}) \cdot D_f(t_{n-1})\right] \\
& \times \left\{1 - R_{BANK}\left(w \cdot ColENE(t_n) + (1-w) \cdot ColENE(t_{n-1}) \cdot \frac{D_f(t_{n-1})}{D_f(t_n)}\right)\right\} \\
& \times Prob(t_n \leq \tau_{BANK} < t_{n+1})
\end{aligned}
$$

6. Bank/Counter-Party Recovery Maps: $R_{BANK}(\cdot)$ and $R_{CPTY}(\cdot)$ are the recovery mapping functions for the counter-party and the bank, respectively.
7. Exposure Weights over Period Vertexes: w is the weight of the period end exposure – default value being 0.5, which indicates that the CVA/DVA is calculates using the average exposure between the beginning and the end of each period.
8. Bank/Counter-Party Default Distribution: The distribution of defaults can be inferred from a hazard curve easily. The "discount factors" calculated from the hazard curves are just survival probabilities. Thus the probability of default between t_n and t_{n+1} is given by

$$Prob(t_n \leq \tau < t_{n+1}) = DF_{Hazard}(t_n) - DF_{Hazard}(t_{n+1})$$

9. Portfolio Trajectory Period Vertex Measures: In addition to the CVA/DVA calculations, several other summary measures may be computed, as shown in the table below.

#	**Measure**	**Description**
1	CVA + DVA	NPV
2	CVA	CVA
3	DVA	DVA
4	Portfolio Value (MC)	Portfolio Value from Monte Carlo
5	CVA + DVA MC Error	NPV MC Error
6	CVA MC Error	CVA MC Error
7	DVA MC Error	DVA MC Error
8	MaxPosPFE	Maximum Positive PFE
9	AvgPosPFE	Average Positive PFE
10	MaxNegPFE	Maximum Negative PFE
11	AvgNegPFE	Average Negative PFE

12	AEPE	Average Expected Positive Exposure
13	AEPE MC Error	AEPE MC Error
14	AENE	Average Expected Negative Exposure
15	AENE MC Error	AENE MC Error
16	Col AEPE	Collateralized AEPE
17	Col AEPE MC Error	Collateralized AEPE MC Error
18	Col AENE	Collateralized AENE
19	Col AENE MC Error	Collateralized AENE MC Error
20	DTZ	$-\max(PortfolioValue - Collateral, 0) - NPV$
21	DTR	$-\max(PortfolioValue - Collateral, 0) - (1 - RR) \times NPV$

10. <u>Portfolio Trajectory Period Edge Measures</u>: For each time period the CVA/DVA are computed and the following measures are stored.
    a. Vertex Start
    b. Vertex End
    c. Vertex Unconditional Counter Party Default Probabilities
    d. Vertex Counter Party Default Probabilities Conditional on Bank Survival
    e. Edge CVA
    f. Vertex Positive Exposures
    g. Vertex Discount Factors
    h. Vertex Counter Party Recovery Rate
    i. Vertex Unconditional Bank Default Probabilities
    j. Vertex Bank Default Probabilities Conditional on Counter Party Survival
    k. Edge DVA
    l. Vertex Negative Exposures

m. Vertex Bank Recovery Rate
n. Vertex Positive PFE
o. Vertex Negative PFE
p. Vertex Collateralized EPE
q. Vertex Collateralized ENE
r. Vertex Collateral
s. Vertex Positive Collateral
t. Vertex Negative Collateral
u. Vertex Cash Flows
v. Vertex Positive Cash Flow
w. Vertex Negative Cash Flow
x. Vertex EPE MC Error
y. Vertex EPE Percent MC Error
z. Vertex ENE MC Error
aa. Vertex ENE Percent MC Error
bb. Vertex Collateralized EPE MC Error
cc. Vertex Collateralized EPE Percent MC Error
dd. Vertex Collateralized ENE MC Error
ee. Vertex Collateralized ENE Percent MC Error
ff. Vertex Start EPE
gg. Vertex Start ENE
hh. Vertex Start Collateralized EPE
ii. Vertex Start Collateralized ENE

11. XVA Calculation Input Parameters:
    a. Credit Data => Credit Information for both parties; hazard curves and recovery rates/recovery maps
    b. Discount Curves or simulated discount factor paths
    c. Threshold to indicate PFE – default is set to 0.975
    d. An indicator (TRUE/FALSE) on whether to use conditional default probability or unconditional – default is unconditional.

CVA And Funding Adjustments PDE

Counterparty Risk and Funding Costs

1. PDE Derivation of Adjustments – Approach: Burgard and Kjaer (2012a) derive the partial differential equation (PDE) representation for the value of financial derivatives with bilateral counterparty risks and funding costs. The model is very general in that the funding rate for lending and borrowing and the MTM value at default can be exogenously specified.
2. Buy-Back of Own Bonds: The buying back of a party's own bonds is a key part of the delta hedging strategy; they discuss how the cash account of the replication strategy provides sufficient funds for this.
3. Full Value as Default Payout: First they consider the case where the mark-to-market at default is given by the derivative, which includes the counterparty risk. They find that the resulting pricing PDE becomes non-linear, except in those special cases where the non-linear terms vanish and the Feynman-Kac representation of the total value can be obtained. In these cases the total value of the derivative can be decompose into a default-free part and a bilateral credit valuation and funding adjustment part.
4. Fair Derivative Value at Payout: Next they assume that the MTM value at default is given by the fair economic value of the derivative. This time the resulting PDE is linear and the corresponding Feynman-Kac representation is used to decompose the total value of the derivative into a default-free value plus bilateral credit valuation and funding cost adjustments.

Motivation, Literature Scan, and Approach

1. Counterparty Credit Risk Definition: Counterparty credit risk implicitly embedded in derivative contracts has become increasingly relevant in recent market conditions. This risk represents the possibility that the counterparty defaults while owing money under the terms of a derivative contract, or more precisely, if the mark-market value of the derivative is positive to the seller at the time of the default of a counterparty.
2. OTC Counterparty Risk Mitigation: While for exchange traded products the counterparty risk is mitigated by the presence of the exchange as an intermediary, this is not the case for OTC products. For these a number of different techniques are being used to mitigate the counterparty risk, most commonly by means of netting agreements and collateral mechanisms. The details of these agreements are, for example, published by the International Swaps and Derivatives (ISDA) 2002 Master Agreement.
3. Bilateral Counterparty Credit Risk: However the counterparty faces a similar risk of the seller defaulting when the mark-to-market value is positive to the counterparty. Taking into account the credit risk of both the parties is commonly referred to as considering the bilateral counterparty risk. When doing so the value of the derivative to its seller is influenced by its own credit quality.
4. Counterparty Credit Risk Coverage: Papers, books, and book chapters that develop techniques for the valuation of the derivatives and derivative portfolio under counterparty risk include, but are not limited to, Jarrow and Turnbull (1995), Jarrow and Yu (2001), Brigo and Mercurio (2007), Li and Tang (2007), Pykhtin and Zhu (2007), Alavian, Ding, Whitehead, and Laiddicina (2008), Gregory (2009), and Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009).
5. Counterparty Risk With Funding Costs: There are other areas where the credit of the seller is relevant, in particular in terms of the MTM accounting of its own debt, as well as the effect it has on its funding costs. Piterbarg (2010) discusses the funding costs on derivative valuations when collateral has to be posted. Thus Burgard and Kjaer (2012a) combine the effects of the seller's credit on its own funding costs with that on the bilateral counterparty risk into a unified framework.
6. Black-Scholes PDE Formulation Extension: Further they use the hedging argument to derive extensions to the Black-Scholes PDE in the presence of bilateral counterparty risk in the presence of bilateral jump-to-default model and include funding considerations in the

financing of the hedge positions. In addition they consider two scenarios for the determination of the derivative MTM at default – namely that recovery is on the total risky value or that it is on the counterparty riskless value. The latter corresponds to the most common approach taken in the literature.

7. The ISDA 2002 Master Agreement: The total value of the derivative will then depend upon which of the 2 MTM values is used at default. For contracts following the ISDA 2002 Master Agreement the value of the derivatives upon the default of one of the counterparties is determined by a dealer poll. There is no reference to the counterparties, and one could reasonably expect the derivative value to be the counterparty riskless value, i.e., the second case considered.
8. Risky Derivative Value at Payout: In the case where the default-risky derivative value is used as the mark-to-market, Burgard and Kjaer (2012a) derive a pricing PDA that is in general non-linear and demonstrate that the unknown risky price can be found by solving a non-linear integral equation. Under certain conditions on the payoff the non-linear terms vanish and the Feynman-Kac representation of the resulting linear PDE is examined.
9. Fair Derivative Value at Payout: In the case where the counterparty derivative price is used as the mark-to-market, the resulting pricing PDE is linear. As in the first case the Feynman-Kac representation can be used to decompose the risky derivative value into a counterparty risk-free part, a funding part, and a bilateral credit valuation adjustment (CVA) part.
10. Granular Hedging Accounts and Strategies: By using a fine-grained hedging strategy to derive their results Burgard and Kjaer (2012a) ensure that the hedging costs of all considered risk factors are included in their derivative price such that the decomposition of the risky price is a generalization of the result commonly found in literature. Further they get explicit expressions for hedges, which is important for risk management.
11. Own Credit Hedging Risk Caveats: There have been discussions about how a seller can hedge out its own credit risk – Cesari, Aquilina, Charpillon, Filipovic, Lee, and Manda (2009) contain a summary. The strategy described by Burgard and Kjaer (2012a) includes the repurchase by the seller of its own bonds to hedge out its own credit risk. On the face of it this may seem like a futile approach since of the bond purchase were funded by issuing more debt (i.e., more bonds), the seller would have in effect achieved nothing in terms of hedging its own credit risk.

12. Differentiated Buy-Back Funding Strategy: However the replication strategy presented shows how the funding for the purchase of the seller's own bond is achieved through the cash account of the hedging strategy. The hedging strategy (including the premium of the derivative) generates the cash needed to repurchase the sellers' own bond.
13. Multiple Assets and Netting Extensions: Although all the results presented in Burgard and Kjaer (2012a) are for one derivative on one underlying asset following the specified dynamics, extension to the situation of a netted derivatives portfolio on several underlying following generalized diffusion dynamics is straightforward.

Notation, Symbology, and Key PDEs

1. The Nature of the Derivative: Consider a derivative contract $\hat{V}$ between a seller B and a counterparty C that may both default. The asset S is not affected by the default of either B or C and is assumed to follow a Markov process with a generator $\mathcal{A}_t$. Similarly we let V denote the same derivative between parties that cannot default.
2. Payout on a Default: At the default of either the counterparty or the seller the value of the derivative to the seller $\hat{V}$ is determine by using an MTM rule M which may equal $\hat{V}$ or V. Throughout the convention used is that the positive derivative values correspond to seller assets and counterparty liabilities.
3. Notations, Parameter Definitions, and Caveats:
    - **a.** r => Risk-free rate
    - **b.** r_B => Yield on a Recovery-less Bond of Seller B
    - **c.** r_C => Yield on a Recovery-less Bond of Counterparty C
    - **d.** λ_B =>

$$\lambda_B = r_B - r$$

    - **e.** λ_C =>

$$\lambda_C = r_C - r$$

f. r_F => Seller funding rate for the borrowed cash on the seller's derivative replication cash account.

$$r_F = r$$

if the derivative can be used as a collateral.

$$r_F = r + (1 - R_B)\lambda_B$$

if the derivative cannot be used as a collateral.

g. s_F =>

$$s_F = r_F - r$$

h. R_B => Recovery on the derivative MTM value in case the seller B defaults

i. R_C => Recovery on the derivative MTM value in case the counterparty C defaults

4. PDE for $\hat{V}$ when $M = \hat{V}$: When the MTM at default is given by

$$M = \hat{V}$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

5. PDE for $\hat{V}$ when $M = V$: When the MTM at default is given by

$$M = V$$

then $\hat{V}$ satisfies the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)\hat{V}^- - (\lambda_B + R_C \lambda_C)\lambda_C \hat{V}^+ + s_F \hat{V}^+$$

6. Credit Funding Adjustment when $M = V$: Let

$$M = V$$

and

$$r_F = r + s_F$$

Then

$$\hat{V} = V + U$$

and the credit funding adjustment U is given by

$$\begin{aligned} U(t,S) = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V^-\big(u, S(u)\big)\right] du \\ - (1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V^-\big(u, S(u)\big)\right] du \\ - \int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V^+\big(u, S(u)\big)\right] du \end{aligned}$$

where

$$D_k(t,u) = e^{-\int_t^u k(v) dv}$$

is the discount factor between times t and u using the rate k. If

$$s_F = 0$$

then U is identical to the regular bilateral CVA derived in many of the papers.

7. Justification of the Buy-Back Strategy: Another important result of Burgard and Kjaer (2012a) is the justification on which the seller's own-credit risk can be taken into account. In the hedging strategy considered the risk is hedged out by the seller buying back its own bonds. It is shown that the cash needed for doing so is generated through the replication strategy.

Model Setup and the Derivation of the Bilateral Risky PDE

1. Underlying Traded Assets and Securities: Consider an economy with the following four traded assets:
   a. P_R => Default-risk free zero-coupon bond
   b. P_B => Default-risky, zero-recovery, zero-coupon bond of party B
   c. P_C => Default-risky, zero-recovery, zero-coupon bond of counterparty C
   d. S => The Spot Asset with no Default Risk
2. Zero-Recovery Bonds Building Blocks: Both the risky bonds P_B and P_C pay $1 at some future date T if the issuing counterparty has not defaulted, and 0 otherwise. These simplistic bonds are useful for modeling and can be used as building blocks for more complex bonds, including the ones with zero recovery.
3. Dynamics of the Underlying Assets: The processes for he assets P_B, P_C, P_R, and S under their corresponding probability measures are specified by

$$\frac{\Delta P_B}{P_B} = r_B(t)\Delta t - \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C(t)\Delta t - \Delta J_C$$

$$\frac{\Delta P_R}{P_R} = r(t)\Delta t$$

$$\frac{\Delta S}{S} = \mu(t)\Delta t - \sigma(t)\Delta W(t)$$

where $W(t)$ is a Weiner process,

$$r(t) > 0$$

$$r_B(t) > 0$$

$$r_C(t) > 0$$

and

$$\sigma(t) > 0$$

are deterministic functions of t and J_B and J_C are two point processes that jump from 0 to 1 on the default of B and C respectively.

4. Asset Independence of the Seller/Counterparty: The assumptions above indicate that the hedging can be done by P_B and P_C alone, and later this assumption is relaxed. Further the spot asset price S is assumed to be unaffected by a default of either B or C. Finally B is taken to be the *seller* and C the *counterparty* respectively.
5. Terminal Derivative Payout at Maturity: It is assumed that the parties B and C enter a derivative on the spot asset S that pays the seller B the amount

$$H(S) \in \mathbb{R}$$

at maturity T. Thus in this convention the payout scenario

$$H(S) \geq 0$$

means that the seller receives cash or asset from the counterparty.

6. Credit Risky Derivative Present Value: The value of the derivative to the seller at time t is denoted $\hat{V}(t, S, J_B, J_C)$ and depends on the spot asset S of the underlying and the default states J_B and J_C respectively of the seller B and the counterparty C. Analogously $V(t, S)$ denotes the value to the seller of the same derivative as if it were a transaction between two default-free counterparties.
7. Default Derivative Close-out Claim: When party B or C default in general the mark-to-market on the derivative determines the close-out or the claim on the position. However the precise nature of this depends on the contractual details and the mechanism by which the mark-market is determined.
8. 2002 ISDA Master Agreement Specification: The 2002 ISDA Master Agreement specifies that the derivative contract will return to the surviving party the recovery value of its positive mark-to-market value (from the point of view of the surviving party) just prior to default, whereas the full mark-to-market has to be paid to the defaulting party if the mark-to-market value is negative (from the view of the surviving party). The master agreement specifies a dealer poll mechanism to establish the mark-to-market to the seller at default $M(t, S)$ without referring to the names of the counterparties involved in the derivative transaction.
9. Handling Imprecise ISDA Close-outs: From the above one would expect $M(t, S)$ to be close to $V(t, S)$ even though it is unclear if the dealers in the poll may or may not include their funding costs in the derivative prices. In case the ISDA Master Agreement is not followed there may be other mechanisms involved. Therefore Burgard and Kjaer (2012a) derive the PDE for the general case $M(t, S)$ and consider the two special cases

$$M(t, S) = \hat{V}(t, S, 0, 0)$$

and

$$M(t,S) = V(t,S)$$

10. Default Close-out Boundary Conditions: Let

$$R_B \in [0,1]$$

and

$$R_C \in [0,1]$$

denote the recovery rates on the derivatives positions for parties B and C respectively – for now they are taken to be deterministic. The above discussions result in the following boundary conditions:

$$\hat{V}(t,S,1,0) = M^+(t,S) + R_B M^-(t,S)$$

when the seller defaults first and

$$\hat{V}(t,S,0,1) = R_C M^+(t,S) + M^-(t,S)$$

when the counterparty defaults first. Li and Tang (2009), Gregory (2009), and the vast majority of the papers on the valuation of the counterparty risk use

$$M(t,S) = V(t,S)$$

11. The Black Scholes Replicating Portfolio: As in the usual Black-Scholes framework the derivative s hedged with a self-financing portfolio that covers all the risk factors of the model. In this case the portfolio Π the seller sets up consists of $\alpha_S(t)$ units of S, $\alpha_B(t)$ units

of P_B, $\alpha_C(t)$ units of P_C, and $\beta(t)$ units of cash such that the portfolio value at time t replicates the value of the derivative to the seller, i.e.,

$$\hat{V}(t) + \Pi(t) = 0$$

Thus

$$-\hat{V}(t) = \Pi(t) = \alpha_S(t)S(t) + \alpha_B(t)P_B(t) + \alpha_C(t)P_C(t) + \beta(t)$$

12. Counterparty Credit Hedge Ratio: Before proceeding note that when

$$\hat{V} \geq 0$$

the seller will incur a loss at the counterparty default. To hedge this loss P_C needs to be shorted, so we expect that

$$\alpha_C \leq 0$$

Assuming that the seller can borrow the bond P_C at a rate close to the risk-free rate r through a repurchase agreement, the spread λ_C between the rate r_C on the bond and the cost of financing the hedge position in C can be approximated to

$$\lambda_C = r_C - r$$

Since P_C is a bond with zero recovery this spread corresponds to the default intensity of C.

13. Own Credit Risk Hedge Ratio: On the other hand, if

$$\hat{V} < 0$$

the seller will gain at own default, which can be hedged by buying back P_B bonds, so one expects that

$$\alpha_C \geq 0$$

in this case. For this to work, enough cash must be generated, and any remaining cash after the purchase of P_B bonds is invested in such a way that it does not generate additional credit risk to the seller, i.e., any remaining positive cash generates a yield at the risk free rate of r.

14. Growth in the Cash Account: Imposing that the portfolio $\Pi(t)$ is self-financing implies that

$$-\Delta\hat{V}(t) = \alpha_S(t)\Delta S(t) + \alpha_B(t)\Delta P_B(t) + \alpha_C(t)\Delta P_C(t) + \Delta\bar{\beta}(t)$$

where the growth in the cash $\Delta\bar{\beta}(t)$ may be decomposed into

$$\Delta\bar{\beta}(t) = \Delta\bar{\beta}_S(t) + \Delta\bar{\beta}_F(t) + \Delta\bar{\beta}_C(t)$$

i.e., it is composed of three parts – the asset cash growth, the growth from the counterparty cash flow, and the growth from a unified (bond + cash) account.

15. Instantaneous Growth vs. Portfolio Rebalancing: The growth above is the growth in the cash account before re-balancing of the portfolio. The self-financing condition ensures that after Δt the rebalancing can happen at zero overall cost. This distinction is clarified in detail by Brigo, Buescu, Pallavicini, and Liu (2012).

16. Growth From the Asset Position $\Delta\bar{\beta}_S(t)$: The asset position provides a dividend income of $\alpha_S(t)\gamma_S(t)S(t)$ at a financing cost of $-\alpha_S(t)q_S(t)S(t)$ so

$$\Delta\bar{\beta}_S(t) = \alpha_S(t)[\gamma_S(t) - q_S(t)]S(t)\Delta t$$

The value of $q_S(t)$ may depend on the risk-free rate $r(t)$ and the repo rate of $S(t)$.

17. Growth From the Counterparty Position $\Delta\bar{\beta}_C(t)$: Using the arguments above, the seller will short the counterparty bonds using a repurchase agreement and incur financing costs of

$$\Delta\bar{\beta}_C(t) = -\alpha_C(t)r(t)P_C(t)\Delta t$$

Note the use of $r(t)$ instead of $r_C(t)$.

18. Positive/Negative Cash Account Asymmetry on $\Delta\bar{\beta}_F(t)$: From the above analysis any surplus cash held by the user after the own-bonds have been purchased must earn the risk-free rate $r(t)$ in order to not introduce any further credit risk to the seller. If borrowing money the seller needs to pay the rate $r_F(t)$. Thus the own bonds/cash account uses a funding/hedging scheme that is not symmetric.
19. Unsecured Funding vs. Derivative as Collateral: For this funding rate, the following two cases are distinct; where the derivative itself can be used as a collateral for the required funding, on no haircut $r_F(t)$ is set to $r(t)$. If however the derivative cannot be used as a collateral, the funding rate is set to the yield of the unsecured seller bond with recovery R_B, i.e.,

$$r_F(t) = r(t) + (1 - R_B)\lambda_B$$

20. Own Bond Cash Position Growth: In practice the second instance above is the more realistic case. Keeping $r_F(t)$ general for now,

$$\Delta\bar{\beta}_F(t) = \left[r(t)\left(-\hat{V} - \alpha_B P_B\right)^+ + r_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\right]\Delta t$$
$$= r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^-\Delta t$$

where the funding spread

$$s_F = r_F - r$$

i.e.,

$$s_F = 0$$

if the derivative cannot be used as a collateral, and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

21. Cumulative Cross Cash Growth Account: From the above analysis it follows that the total change in the cash account (here t is dropped from the notation where applicable to improve clarity) is given by

$$\Delta\bar{\beta} = (\gamma_S - q_S)\alpha_S S\Delta t + \left[r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-}\right]\Delta t - r\alpha_C P_C \Delta t$$

22. Change of the Replication Portfolio Value: Using the $\Delta\bar{\beta}$ computed above in

$$-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B \Delta P_B + \alpha_C \Delta P_C + \Delta\bar{\beta}$$

one gets

$$\begin{aligned}
-\Delta\hat{V} = \alpha_S \Delta S + \alpha_B P_B (r_B \Delta t - \Delta J_B) + \alpha_C P_C (r_C \Delta t - \Delta J_C) + \Delta\bar{\beta} \\
+ \left[(\gamma_S - q_S)\alpha_S S + r\left(-\hat{V} - \alpha_B P_B\right) + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} - r\alpha_C P_C\right]\Delta t \\
= \left[-r\hat{V} + s_F\left(-\hat{V} - \alpha_B P_B\right)^{-} + (\gamma_S - q_S)\alpha_S S + (r_B - r)\alpha_B P_B \right. \\
\left. + (r_C - r)\alpha_C P_C\right]\Delta t - \alpha_B P_B \Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S
\end{aligned}$$

23. Ito's Lemma for Jump Diffusion: On the other hand, by Ito's lemma for jump diffusion and the assumption that a simultaneous jump of both the seller and the counterparty is a zero-probability event, the derivative value moves by

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial S^2}\Delta t + \mathcal{D}\hat{V}_B \Delta J_B + \mathcal{D}\hat{V}_C \Delta J_C$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

can be computed from

$$\hat{V}(t, S, 1, 0) = M^{+}(t, S) + R_B M^{-}(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^{+}(t, S) + M^{-}(t, S)$$

24. <u>Asset and Bond Hedge Ratios</u>: Equating the $\Delta\hat{V}$ from the cash growth and the Ito's lemmas, the following choice of α_S, α_B, and α_C eliminates all risks in the portfolio:

$$\alpha_S = \frac{\partial\hat{V}}{\partial S}$$

$$\alpha_B = \frac{\mathcal{D}\hat{V}_B}{P_B} = -\frac{\hat{V} - (M^{+} + R_B M^{-})}{P_B}$$

$$\alpha_C = \frac{\mathcal{D}\hat{V}_C}{P_C} = -\frac{\hat{V} - (M^{-} + R_C M^{+})}{P_C}$$

25. <u>Cash Account Evolution Expression</u>: Hence the cash account evolution

$$\Delta\bar{\beta}_F(t) = r(t)\left(-\hat{V} - \alpha_B P_B\right)\Delta t + s_F(t)\left(-\hat{V} - \alpha_B P_B\right)^{-}\Delta t$$

can be rewritten as

$$\Delta\bar{\beta}_F(t) = [-r(t)R_B M^- - r_F(t)M^+]\Delta t$$

so the amount of cash deposited by the seller at the risk-free rate equals $-rR_B M^-$ and the amount borrowed at the funding rate r_F equals $-M^+$

26. <u>Derivative as Collateral Black Scholes</u>: Introducing the parabolic differential operator $\mathcal{A}_t$ as

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial S^2} + (\gamma_S - q_S)S\frac{\partial\hat{V}}{\partial S}$$

it follows that $\hat{V}$ is the solution to the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B\mathcal{D}\hat{V}_B - \lambda_C\mathcal{D}\hat{V}_C$$

$$\hat{V}(T, S) = H(S)$$

where

$$\lambda_B = r_B - r$$

and

$$\lambda_C = r_C - r$$

27. <u>Incorporation of the Boundary Condition</u>: Inserting

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with the boundary conditions

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ - \lambda_B \mathcal{D}\hat{V}_B - \lambda_C \mathcal{D}\hat{V}_C$$

finally gives

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B (R_B M^- + M^+) - \lambda_C (R_C M^+ + M^-)$$

$$\hat{V}(T, S) = H(S)$$

where the relation

$$\left(\hat{V} + \mathcal{D}\hat{V}_B\right)^+ = (R_B M^- + M^+)^+ = M^+$$

has been used.

28. Fair Derivative Black Scholes Value: In contrast the risk-free value V satisfies the regular Black-Scholes PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = 0$$

$$V(T, S) = H(S)$$

Thus if one interprets λ_B and λ_C as effective default rates then the difference between $\hat{V}$ and V may be interpreted as follows.

29. Self/Counterparty Default Impact: The term $(\lambda_B + \lambda_C)\hat{V}$ is the additional growth rate the seller B requires on the risky asset $\hat{V}$ to compensate for the risk that default of either the seller the seller or the counterparty will terminate the derivative contract.
30. Receivables Funding Impact Hedge Strategy: The term $s_F M^+$ is the additional funding cost for negative values of the cash account for the hedging strategy.
31. Own Default Close-out: The term $-\lambda_B(R_B M^- + M^+)$ is the adjustment in the growth rate that the seller can accept because of the cash flows occurring at own default.
32. Counterparty Default Close-out: The term $-\lambda_C(R_C M^+ + M^-)$ is the adjustment in the growth rate that the seller can accept because of the cash flow occurring at the counterparty default.
33. Modeling of the Extinguisher Trade: The terms $(\lambda_B + \lambda_C)\hat{V}$, $-\lambda_B(R_B M^- + M^+)$, and $-\lambda_C(R_C M^+ + M^-)$ are related to counterparty risk whereas the term $s_F M^+$ represents the funding cost. From this interpretation it follows that the PDE for a so-called *extinguisher trade*, whereby it is agreed that no party gets anything at default, is obtained by removing the terms $-\lambda_B(R_B M^- + M^+)$ and $-\lambda_C(R_C M^+ + M^-)$ from

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B(R_B M^- + M^+) - \lambda_C(R_C M^+ + M^-)$$

Using $\hat{V}(T, S)$ As Mark-to-Market at Default

1. Conceptually Simple Payout Condition: This section considers the case where the payments in default are based on $\hat{V}(T, S)$ so that

$$M(T, S) = \hat{V}(T, S)$$

in the boundary condition

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^+(t, S) + M^-(t, S)$$

Conceptually this is the simpler case since if the defaulting party is in the money with respect to a derivative contract, then there is no additional impact on the profit and the loss at the point of default.

2. Simplification of the PDE for $\hat{V}$: Similarly if the surviving party is in the money with respect to the derivative contract, then its loss is simply $(1 - R)\hat{V}$. In this case

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^+ - \lambda_B (R_B M^- + M^+) - \lambda_C (R_C M^+ + M^-)$$

reduces to

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ - (1 - R_B)\lambda_B \hat{V}^- - (1 - R_C)\lambda_C \hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

where

$$s_F = 0$$

if the derivative can be posted as collateral and

$$s_F = (1 - R_B)\lambda_B$$

if it cannot.

3. Own Credit And Counterparty Hedges: Moreover the hedge ratios α_B and α_C are given by

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

and

$$\alpha_C = -\frac{(1 - R_C)\hat{V}^+}{P_C}$$

so that

$$\alpha_B \geq 0$$

and

$$\alpha_C \geq 0$$

and the replication strategy generates enough cash $-\hat{V}^-$ for the seller to purchase back its own bonds.

4. Interpretation of the Own Credit Hedge: The cash available to the seller is $-\hat{V}^-$, of which the fraction $1 - R_B$ is invested in buying back the recovery-less bond B and the fraction R_B is invested risk-free. This is equivalent to investing a total amount of $-\hat{V}^-$ into purchasing a seller bond $\bar{B}$ with recovery R_B.
5. Credit Valuation Adjustment (CVA) Formulation: In the counterparty risk literature it is customary to write

$$\hat{V} = V + U$$

where U is called the credit valuation adjustment or the CVA. Inserting this representation into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

$$\hat{V}(T, S) = H(S)$$

$$\frac{\partial V}{\partial t} + \mathcal{A}_t V - rV = 0$$

$$V(T, S) = H(S)$$

yields

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - rU = s_F (V + U)^+ + (1 - R_B)\lambda_B (V + U)^- + (1 - R_C)\lambda_C (V + U)^+$$

$$U(T, S) = 0$$

where V is known and acts as the source term.

6. Funding + CVA Feynman-Kac Integral: Furthermore one may apply the Feynman-Kac integral to the PDE for U, which with the additional assumption of deterministic interest rates produces the following non-linear integral equation:

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^-\right]du$$

$$-(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^+\right]du$$

$$-\int_t^T s_F(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^+\right]du$$

It follows that one can compute U first by computing V and then solving either the non-linear PDE above or the integral equation.

7. Receivable/Payable + Funding Numeraire: Before proceeding with the study of the two cases

$$s_F = 0$$

and

$$s_F = (1-R_B)\lambda_B$$

it is worthwhile to examine a few instances where $\hat{V}$ corresponds to 1 UNIT seller receivable/payable, where those bonds may be with and without recovery.

8. The Seller sells 1 Unit Payable to the Counterparty:
   a. Zero-recovery Payable Numeraire => The first case considered is one unit sold by the seller to the counterparty C. In this situation

$$\hat{V} = \hat{V}^- = -1$$

and

$$R_B = 0$$

Since only deterministic rates and spreads are considered, there is no risk with respect to the underlying market factors, and the term $\mathcal{A}_t\hat{V}$ vanishes, so

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} = (r + \lambda_B)\hat{V} = r_B\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T r_B(s)ds}$$

as expected for

$$\hat{V}(T, S) = -1$$

b. Non-zero Recovery Payable Numeraire => If on the other hand the recovery

$$R_B \neq 0$$

then

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

becomes

$$\frac{\partial\hat{V}}{\partial t} = \{r + (1 - R_B)\lambda_B\}\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

with the solution

$$\hat{V}(t) = -e^{-\int_t^T\{r(s)+(1-R_B)\lambda_B(s)\}ds}$$

As expected the rate $r + (1 - R_B)\lambda_B$ payable on a bond with recovery is equal to the unsecured funding rate r_F that the seller has to pay on negative cash balances when the derivative cannot be posted as a collateral.

9. <u>The Seller Buys 1 Unit Receivable From C</u>:
    a. Zero-Recovery Receivable Funding Numeraire => If $\hat{V}$ describes the purchase of 1 Unit receivable by the seller from the counterparty (i.e., the seller lends to the counterparty without recovery) then

$$\hat{V} = \hat{V}^- = +1$$

and

$$R_C = 0$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} = (r_F + \lambda_C)\hat{V} = [r_F + (r_C - r)]\hat{V}$$

and

$$\hat{V}(T, S) = -1$$

b. Zero-Recovery Receivable Collateralized Numeraire => In this case, if the seller can use the derivative (i.e., the loan asset) as collateral for the funding of its short position within its replication strategy, then (neglecting haircuts)

$$r_F = r$$

the risk-free rate. Then the net result in this case is then

$$\frac{\partial \hat{V}}{\partial t} = r_C \hat{V}$$

so

$$\hat{V}(t) = -e^{-\int_t^T r_C(s)ds}$$

as expected for

$$\hat{V}_C(t) = P_C(t)$$

c. Non-zero Recovery Numeraire => If on the other hand

$$R_C \neq 0$$

then

$$\hat{V}(t) = -e^{-\int_t^T \{r(s)+(1-R_C)\lambda_C(s)\}ds}$$

as expected.

10. <u>The Case $r_F = r$</u>:

a. PDE With Derivatives as Collateral => If the derivative can be posted as collateral the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T,S) = H(S)$$

which is a non-linear PDE that needs to be solved numerically unless

$$\hat{V} \geq 0$$

or

$$\hat{V} \leq 0$$

b. Feynman-Kac Integral for $\hat{V}(t,S) \leq 0$ => Assuming that

$$\hat{V} \leq 0$$

i.e., the seller sold an option to the counterparty so

$$H(S) \leq 0$$

and that all rates are deterministic the Feynman-Kac representation of $\hat{V}$ is given by

$$\hat{V}(t,S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t,T)H\big(S(T)\big)\right]$$

where

$$D_k(t,T) = -e^{-\int_t^T k(s)ds}$$

is the discount factor over $[t,T]$ given that rate k.

c. Own-Credit Adjustment for Payables => Alternatively if for

$$\hat{V} \leq 0$$

we insert the ansatz

$$\hat{V} = V + U_0$$

where the zero subscript in U_0 indicates that the CVA U_0 is computed at

$$s_F = 0$$

into

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = s_F \hat{V}^+ + (1 - R_B)\lambda_B \hat{V}^- + (1 - R_C)\lambda_C \hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

and apply the Feynman-Kac theorem, and finally use that

$$V(t, S) = D_r(t, u)\mathbb{E}_t\left[V\big(u, S(u)\big)\right]$$

one then gets

$$U_0(t, S) = -V(t, S)\left[\int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B(u)}(t, u) du\right]$$

d. Counterparty Adjustment for Receivables => When

$$\hat{V} \geq 0$$

i.e., the “seller” bought an option, symmetry yields that

$$U_0(t,S) = -V(t,S)\left[\int_t^T (1 - R_C)\lambda_C(u) D_{(1-R_C)\lambda_C(u)}(t,u)du\right]$$

Thus one concludes that if

$$\hat{V} \leq 0$$

then U_0 depends only on the credit of the seller, whereas if

$$\hat{V} \geq 0$$

it depends only on the counterparty credit.

11. <u>The Case $r_F = r + (1 - R_B)\lambda_B$</u>:

a. The Derivative Cannot Serve as Collateral => If the derivative cannot be posted as collateral then

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_F\hat{V}^+ + (1 - R_B)\lambda_B\hat{V}^- + (1 - R_C)\lambda_C\hat{V}^+$$

and

$$\hat{V}(T,S) = H(S)$$

become

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (1 - R_B)\lambda_B \hat{V}^- + \{(1 - R_B)\lambda_B + (1 - R_C)\lambda_C\}\hat{V}^+$$

and

$$\hat{V}(T, S) = H(S)$$

which again is a non-linear PDE.

b. Feynman-Kac Integral for $\hat{V}(t, S) \leq 0$ => For

$$\hat{V} \leq 0$$

one writes

$$\hat{V} = V + U$$

and it is easy to see that

$$U = U_0$$

given in

$$U_0(t, S) = -V(t, S)\left[\int_t^T (1 - R_B)\lambda_B(u) D_{(1-R_B)\lambda_B(u)}(t, u) du\right]$$

so $\hat{V}$ is given by

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+(1-R_B)\lambda_B}(t, T) H\big(S(T)\big)\right]$$

c. Feynman-Kac Integral for $\hat{V}(t, S) \geq 0$ => If

$$\hat{V}(t, S) \geq 0$$

then

$$\hat{V}(t, S) = \mathbb{E}_t\left[D_{r+k}(t, T)H\big(S(T)\big)\right]$$

where

$$k = (1 - R_B)\lambda_B + (1 - R_C)\lambda_C$$

d. Own/Counterparty Credit Adjustment => Analogous to the case

$$r_F = r$$

one may make the ansatz

$$\hat{V} = V + U$$

and show that

$$U(t, S) = -V(t, S)\int_t^T k(u)D_k(t, u)du$$

Comparing this $U(t, S)$ with

$$U_0(t, S) = -V(t, S)\int_t^T (1 - R_C)\lambda_C(u)D_{(1-R_C)\lambda_C}(t, u)du$$

shows that when the "seller" buys an option from the counterparty it encounters an additional funding spread

$$s_F = (1 - R_B)\lambda_B$$

Using $V(T, S)$ As Mark-to-Market at Default

1. $M(t, S) = V(t, S)$ Case for the PDE for $\hat{V}$: This section considers the scenario where the payments in the case of default are based on V and hence use

$$M(t, S) = V(t, S)$$

in the boundary conditions

$$\hat{V}(t, S, 1, 0) = M^{+}(t, S) + R_B M^{-}(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = R_C M^{+}(t, S) + M^{-}(t, S)$$

The PDE for $\hat{V}$

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r\hat{V} = (\lambda_B + \lambda_C)\hat{V} + s_F M^{+} - \lambda_B(R_B M^{-} + M^{+}) - \lambda_C(R_C M^{+} + M^{-})$$

then becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B \lambda_B + \lambda_C)V^{-} - (R_C \lambda_C + \lambda_B)V^{+} + s_F V^{+}$$

$$\hat{V}(T, S) = H(S)$$

2. Own-Credit and Counterparty Hedges: The above PDE is linear and has a source term on the right hand side. On writing

$$\hat{V} = V + U$$

the hedge ratios become

$$\alpha_B = \frac{U + (1 - R_B)V^-}{P_B}$$

and

$$\alpha_C = \frac{U + (1 - R_C)V^+}{P_C}$$

Comparing α_B with

$$\alpha_B = -\frac{(1 - R_B)\hat{V}^-}{P_B}$$

shows that in the current case default triggers a windfall cash flow of U that needs to be taken into account in the hedging strategy.

3. Valuation Adjustment Feynman Kac Integrals: Writing

$$\hat{V} = V + U$$

also gives the following PDE for U:

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_F V^+ + (1 - R_B)\lambda_B V^- + (1 - R_C)\lambda_C V^+$$

$$U(T,S)=0$$

Thus application of the Feynman-Kac theorem yields

$$\begin{aligned}U(t,S) = &-(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\big(u,S(u)\big)\right]du \\ &-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du \\ &+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du\end{aligned}$$

The CVA U can be calculated using $V(t,S)$ as a known source term when solving the PDE in the integral or in the differential form.

4. Derivatives as Cash Account Collateral: In the case where we can use the derivative as a collateral for the funding of our cash account, i.e.,

$$s_F=0$$

the last term in the expression for $U(t,S)$ above vanishes and the equation reduces to the regular bilateral CVA derived in many papers and books cited earlier, for example Gregory (2009).

5. Zero Funding Cost Credit Valuation Adjustment: The bilateral benefits in this case does not come from any own default, but from being able to use the cash generated from the hedging strategy and buy back own bonds, thus generating an excess return of $(1-R_B)\lambda_B$. This CVA that corresponds to the case

$$M=V$$

and

$$s_F = 0$$

is denoted U_0.

6. Unsecured Funding Rate Valuation Adjustment: In practice, however, the derivative cannot normally be used as a collateral and

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\big(u,S(u)\big)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$
$$+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

gives a consistent adjustment of the derivative prices for bilateral counterparty risk and funding costs. In the specific case where the funding spread corresponds to that of the unsecured B bond (with recovery R_B), i.e.,

$$s_F = (1-R_B)\lambda_B$$

the first and the third terms may be merged and U may be re-written as

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V\big(u,S(u)\big)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

7. Payable and Receivable Funding Impact: The first and the last term of the previous equation now not only contain the bilateral asset described above, but also the funding liability arising from the fact that the higher rate

$$r_F = r + (1 - R_B)\lambda_B$$

is paid when borrowing for the hedging strategy's cash account.

Funding and Default Payoff Examples

1. Funding and Default Payoff Scenarios: In this section the total derivative value $\hat{V}$ for a call option bought by the "seller" is computed in the following 4 cases:
    a. Case I =>

    $$M = \hat{V}$$

    and

    $$s_F = 0$$

    b. Case II =>

    $$M = \hat{V}$$

    and

    $$s_F = (1 - R_B)\lambda_B$$

    c. Case III =>

$$M = V$$

and

$$s_F = 0$$

d. Case IV =>

$$M = V$$

and

$$s_F = (1 - R_B)\lambda_B$$

2. One-Sided "Bought" Call CVA: A bought call is a one-sided trade that satisfies

$$V \geq 0$$

and

$$\hat{V} \geq 0$$

and furthermore if the rates are constant the CVAs U/U_0 for the four cases above simplifies to

a. Case I =>

$$U_0 = -V(t,S)\left[1 - e^{-(1-R_C)\lambda_C(T-t)}\right]$$

b. Case II =>

$$U = -V(t,S)\left[1 - e^{-\{(1-R_B)\lambda_B + (1-R_C)\lambda_C\}(T-t)}\right]$$

c. Case III =>

$$U_0 = -V(t,S)\frac{(1-R_C)\lambda_C}{\lambda_B + \lambda_C}\left[1 - e^{-(\lambda_B+\lambda_C)(T-t)}\right]$$

d. Case IV =>

$$U = -V(t,S)\frac{(1-R_B)\lambda_B + (1-R_C)\lambda_C}{\lambda_B + \lambda_C}\left[1 - e^{-(\lambda_B+\lambda_C)(T-t)}\right]$$

3. CVA and Funding Impact Analysis: Thus all the four CVA's are linear in $V(t,S)$. All are negative since the seller faces counterparty risks and funding costs when

$$s_F = 0$$

but does not have any bilateral asset because of the one-sidedness of the option payoff. Further the effect of the funding cost is significantly larger than choosing

$$M = \hat{V}$$

or

$$M = V$$

for a bought option. For a sold option the impact of the funding cost does not have any effect.

Counterparty Funding and PDE Extensions

1. Liquid Markets CVA/Funding Impact: The valuation adjustments presented here are particularly relevant when pricing interest rate swaps and vanilla options since these markets are very liquid and having an analytical model that does not take fully all costs into account may consume all profits from a deal depending upon the funding and the credit spreads.
2. Derivatives with more General Payouts: Though the examinations above were conducted on a simple one-asset, one-derivative Black Scholes framework, the results can be immediately extended to derivatives with more general payments than $H\big(S(T)\big)$. These could be Asian options or interest rate swaps.
3. Netted Portfolios with Multiple Trades: In this case the vales V and $\hat{V}$ represent the net derivatives portfolio value rather than the value of a single derivative.
4. Generalized Multi-asset Diffusion Dynamics for Multiple Underlyings: The only restriction is that the asset price SDE's satisfy technical conditions such that the option pricing PDE (now multi-dimensional) admits a unique solution given by the Feynman-Kac representation. Note that if the number of assets exceeds two or three it is computationally more efficient to compute the CVA using Monte-Carlo simulation combined with numerical integration rather than solving the high-dimensional PDE.
5. Stochastic Interest Rates: This is essential for interest rate derivatives, and the effect would be that the discounting in the CVA expression would happen inside the expectation operator.
6. Stochastic Hazard Rates: One way of introducing default time dependence and right/wrong way risk would be to make λ_B and λ_C stochastic and correlate them with each other and with other market factors. This would, again, simply imply that we do not move the discount factors outside of the expectation operator in

$$
\begin{aligned}
U(t,S) = -(1-R_B)\int_t^T \lambda_B(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^-\right]du \\
-(1-R_C)\int_t^T \lambda_C(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^+\right]du \\
-\int_t^T s_F(u)D_r(t,u)\mathbb{E}_t\left[\left\{V\big(u,S(u)\big)+U\big(u,S(u)\big)\right\}^+\right]du
\end{aligned}
$$

or in

$$U(t,S) = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\left(u,S(u)\right)\right]du$$
$$-(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u,S(u)\right)\right]du$$
$$+\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u,S(u)\right)\right]du$$

Also the generator $\mathcal{A}_t$ would incorporate terms corresponding to the new stochastic state variables.

7. Explicitly Specified Default Time Dependence: Another way of specifying default time dependence is by specifying simultaneous defaults. This could be done by setting J_0, J_1, and J_2 be independent point processes and then setting

$$J_B = J_0 + J_1$$

and

$$J_C = J_0 + J_2$$

This approach is the well-known Marshall-Olkin copula and would require some basket default instrument for perfect replication. The hazard rates λ_0, λ_1, and λ_2 of J_0, J_1, and J_2 could be made stochastic in which case the right and the wrong way risk can be modeled as well.

Balance Sheet and Funding Cost Management

1. Funding Position Dependent Cost Adjustment: In both Piterbarg (2010) and Burgard and Kjaer (2012a) the size of the funding cost adjustment is dependent on the specific way the funding is achieved and thus gives rise to prices that are dependent on the funding position of the issuer. The counterparty would clear the price with the best funding position.
2. Funding Cost as Seller Windfall: Burgard and Kjaer (2012b) show that the funding cost term is related to the windfall to the issuer's bondholders upon default of the issuer. This leads them to examine the impact of the derivative asset and the funding positions on the balance sheet from within the confines of a simple balance sheet model.
3. Funding Strategy Balance Sheet Impact: They demonstrate that this impact on the balance sheet and the overall funding position of the issuer reduces the effective marginal funding spread for new positions to zero.
4. Strategies for Balance Sheet Impact Neutralization: Burgard and Kjaer (2012b) discuss two strategies for how the balance sheet impact can directly be neutralized, mitigating the need for a funding cost adjustment to a derivatives price. If such strategies can be put into practice, they lead back to a state where the symmetric prices between the issuer and the counterparty can be achieved.

Unified Framework for Bilateral Counterparty Risk and Funding Adjustments

1. CVA, DVA, and FVA Expressions: Burgard and Kjaer (2012b) list the explicit integral representations for the CVA, the DVA, and the FCA.

$$U(t,S) = CVA + DVA + FCA$$

$$CVA = -(1-R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\left(u,S(u)\right)\right]du$$

$$DVA = -(1-R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^-\big(u,S(u)\big)\right]du$$

$$FVA = -\int_t^T s_F(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t\left[V^+\big(u,S(u)\big)\right]du$$

2. <u>Analysis of the XVA Expressions</u>: It is clear from the above expression that while both the CVA and the FVA are related to the credit position of the issuer, they do not double-count the issuer's credit, but capture the exposures of the mark-to-market value of the derivatives of the opposite sign and as such are opposite sides of the same coin. The DVA term itself can be seen as a funding benefit as it arises from the issuer using a positive cash account to buy back its own bonds, earning the spread on it while at the same time hedging out own credit risk on the derivatives position.
3. <u>Own Credit Hedge Using Bonds</u>: It should be noted that the hedging strategy leading to the set of expressions above involves the issuer re-purchasing its own bonds. It does not involve any dealings in the CDS. The term λ_B is the spread of a zero-recovery bond over the risk-free rate. It is not the hazard rate derived from the CDS market. Thus if there is a basis between the bonds and the CDS for the issuer B, it is the bond market that counts in determining the λ_B used in the DVA and the FVA terms.
4. <u>Contribution of U to the Hedges</u>: The contributions of U to the hedge ratios

$$\alpha_B = \frac{U + (1-R_B)V^-}{P_B}$$

and

$$\alpha_C = \frac{U + (1-R_C)V^+}{P_C}$$

on the other hand comes from the fact that, upon default the close-out amount of V differs from the risky value of the derivative just prior to the default by the amount U because the credit and the funding adjustments for the trade disappear on default of the counterparty or the issuer. Thus a credit risk on the full amount of U needs to be hedged out and this is achieved by means of taking positions in zero-recovery bonds B and C corresponding to the full value of U.

5. Symmetry between the CVA and the DVA: Without the FVA term the risky value $\hat{V}$ would be symmetric in B and C, i.e., the counterparty and the issuer, following the same methodology, would agree on the same price. If however the funding costs for the replication strategy are included, and the funding spread is non-zero, then the two parties hedging out their risks and pricing in their funding costs would not agree on the price. A counterparty that wants to buy this derivative would buy it from the seller with the smallest FVA value, i.e., the lowest funding cost.
6. Balance Sheet Funding Costs Mitigation: Burgard and Kjaer (2012b) also clarify the origins of the FVA term within the ambit of their framework in more detail, and outline different ways of how the funding costs can be mitigated. Doing so successfully can reduce the FVA cost to zero and produce prices that are independent of the funding costs of the issuer, and therefore symmetric.

Simple Model for the Impact of Derivative Asset on Balance Sheet and Funding

1. Balance Sheet and Funding Model: As discussed earlier, derivatives and their funding positions contribute to the issuer asset and liability positions on the issuer default. Therefore they themselves should impact the funding costs of then issuer. Burgard and Kjaer (2012b) quantify this feedback effect using a simple balance sheet and funding model.
2. Pre-trade Asset and Liability: Assuming that, as in a reduced form credit model, the default of the issuer is driven by an instantaneous default process with default intensity λ. Prior to

entering into the derivative contract, let A_0 be the expected assets on default of the issuer and L_0 be the liabilities, so that the expected recovery on default is

$$R_0 = \frac{A_0}{L_0}$$

3. Pre-trade Instantaneous Incremental Funding Cost: Within this simple setup, the funding spread s_F of the issuer over the risk-free rate that compensates for the expected loss upon its default is

$$s_F = (1 - R_0)\lambda$$

Thus the instantaneous funding cost f_0 over the time Δt for a total liability L_0 is

$$f_0 \Delta t = [r + (1 - R_0)\lambda] L_0 \Delta t$$

4. Addition of a Derivative Transaction: Let the seller now add a derivative with a positive value d as an asset, resulting in total assets of

$$A_1 = A_0 + d$$

The positive value d corresponds to $-V^-$ in the previous analysis. The corresponding negative cash is funded by adding a corresponding liability giving a total new liability of

$$L_1 = L_0 + d$$

Thus the new expected recovery is now

$$R_1 = \frac{A_1}{L_1} = \frac{A_0 + d}{L_0 + d}$$

5. Post-Trade Incremental Funding Cost: Assuming that the seller has hedged the market and the counterparty risk, the addition of the derivative asset does not change the default intensity of the seller. Thus the instantaneous funding cost after adding the derivative is

$$
\begin{aligned}
f_1\Delta t = [r + (1 - R_1)\lambda]L_1\Delta t &= r(L_0 + d)\Delta t + (L_1 - A_1)\lambda\Delta t \\
&= rL_0\Delta t + rd\Delta t + (L_0 - A_0)\lambda\Delta t = rd\Delta t + rL_0\Delta t + (1 - R_0)\lambda L_0\Delta t \\
&= rd\Delta t + f_0\Delta t
\end{aligned}
$$

6. Effective Incremental Post-Trade Funding Cost: Thus the effective funding cost for the additional liability d is $rd\Delta t$. While the new liability d draws a new funding spread $(1 - R_1)\lambda$ the change on the recovery and its effect on the funding of the total liabilities results in an effective funding rate for d that is the risk-free rate. Thus within this balance sheet model the spread s_F is zero.
7. Balance Sheet Funding Cost Mitigation: While this balance sheet model is somewhat simplistic, it shows that the proper accounting for the effects of the derivative assets on the balance sheet can mitigate the funding costs and bring the FVA terms down to zero. With a vanishing FVA term, the equation

$$U = CVA + DVA$$

yields an adjustment U and a risky value $\hat{V}$ that are symmetric between the issuer B and the counterparty C.
8. Operational Challenges with the above Approach: Practically, however, the challenge is an operational one in that the benefit of the balance sheet impact is difficult to pin down at the moment of trading and hedging the derivative contract and therefore difficult to allocate as a benefit to the derivatives trading desk.

Balance Sheet Management to Mitigate Funding Costs

1. Issuer Windfall and Liability Matchup: Another way to shield the balance sheet from the impact of both the derivative asset and the funding liability is to actively manage the balance sheet in such a way that the windfall from the derivative asset and the funding position upon default of the issuer is balanced out by a corresponding liability.
2. Multiple Classes of Issuer Bonds: The above can be achieved if the issuer can freely trade two of its own bonds P_1 and P_2 with different recovery rates R_1 and R_2, i.e., different seniority.
3. Dynamics of the Bonds and the Assets: The setup is changed from before in that there are now 4 hedging instruments - P_1, P_2, P_C, and S. All the positive and negative cash in the cash account is invested/raised by buying back and/or issuing P_1/P_2 bonds. The assets follow the dynamics

$$\frac{\Delta P_1}{P_1} = r_1 \Delta t - (1 - R_1) \Delta J_B$$

$$\frac{\Delta P_2}{P_2} = r_2 \Delta t - (1 - R_2) \Delta J_B$$

$$\frac{\Delta P_C}{P_C} = r_C \Delta t - \Delta J_C$$

$$\frac{\Delta S}{S} = \mu \Delta t - \sigma \Delta W$$

where

$$R_1 \in [0, 1)$$

and

$$R_2 \in [0, 1)$$

and

$$R_1 < R_2$$

Neither of the recoveries R_1 or R_2 need equal the derivative recovery rate R_B.

4. Payoff Replication Hedge Portfolio: As before the replicating hedge portfolio Π is setup and given by

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \alpha_S S + \beta_S + \beta_C$$

where

$$\beta_S = -\alpha_S S$$

is the funding account for the asset position and

$$\beta_C = -\alpha_C P_C$$

is the funding position for the P_C bonds.

5. Application of the Self-Financing Criterion: The fact that Π is meant to be a replicating self-financing hedge portfolio implies that

$$\Pi = \alpha_1 P_1 + \alpha_2 P_2 = -\hat{V}$$

$$\Delta\Pi = -\Delta\hat{V}$$

so repeating the delta-hedging arguments of Burgard and Kjaer (2012a) and defining

$$s_1 = r_1 - r$$

and

$$s_2 = r_2 - r$$

yields

$$\alpha_S = -\frac{\partial \hat{V}}{\partial S}$$

$$\alpha_1(1 - R_1)P_1 + \alpha_2(1 - R_2)P_2 = \mathcal{D}\hat{V}_B$$

$$\alpha_C P_C = \mathcal{D}\hat{V}_C$$

$$\alpha_1 s_1 P_1 + \alpha_2 s_2 P_2 + \alpha_C \lambda_C P_C = -\frac{\partial \hat{V}}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial t^2} + r\hat{V} - (q_S - \gamma_S)S\frac{\partial \hat{V}}{\partial S}$$

where

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0)$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0)$$

with $\hat{V}(t, S, 1, 0)$ and $\hat{V}(t, S, 0, 1)$ given by

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = M^-(t, S) + R_C M^+(t, S)$$

with

$$M = V$$

6. Hedge Ratios and Consolidated PDEs: From the above equations one can determine α_1 and α_2 to be

$$\alpha_1 = -\frac{R_2\hat{V} - V^+ - R_B V^-}{(R_2 - R_1)P_1}$$

$$\alpha_2 = -\frac{-R_1\hat{V} + V^+ + R_B V^-}{(R_2 - R_1)P_2}$$

which implies the following pricing PDE:

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} = s_1\frac{R_2\hat{V} - V^+ - R_B V^-}{R_2 - R_1} + s_2\frac{-R_1\hat{V} + V^+ + R_B V^-}{R_2 - R_1} - \lambda_C\left(V^- + R_C V^+ - \hat{V}\right)$$

7. Zero Bond Funding Spread Basis: If furthermore one assumes zero differential between the bond basis and the funding spread, i.e.,

$$s_1 = (1 - R_1)\lambda_B$$

and

$$s_2 = (1 - R_2)\lambda_B$$

then the previous equation simplifies to

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+$$

8. Issue Senior and Repurchase Junior: Comparing this PDE with

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = -(R_B\lambda_B + \lambda_C)V^- - (\lambda_B + R_C\lambda_C)V^+ + s_F V^+$$

implies financing of the negative cash account at vanishing spread

$$s_F = 0$$

In practice the strategy involves issuing the senior P_2 bonds and using some of the proceeds to re-purchase the junior and hence higher-yields P_1 bonds.

9. Excess Return Balancing the Funding Costs: The excess return generated by this strategy exactly offsets the funding costs so the net financing rate becomes r. At the same time the combined positions of the P_1 and the P_2 bonds ensures there is no windfall to the bondholders in the case of default of the issuer while V is positive (and the cash account negative).
10. Junior/Senior Positions at Default: These are shown in the table below which summarizes the total bond positions at the issuers own default and which partially offsets the value of the derivative defined in

$$\hat{V}(t, S, 1, 0) = M^+(t, S) + R_B M^-(t, S)$$

and

$$\hat{V}(t, S, 0, 1) = M^-(t, S) + R_C M^+(t, S)$$

P_1 Position Value	$\frac{R_1 R_2 \hat{V} - R_1 V^+ - R_1 R_B V^-}{R_2 - R_1}$
P_2 Position Value	$-\frac{-R_2 R_1 \hat{V} + R_2 V^+ + R_2 R_B V^-}{R_2 - R_1}$

Total Position Value	$-(V^+ + R_B V^-)$

11. <u>Default Windfall Balancing out Funding Costs</u>: Thus if the issuer is able to offset the impact of the derivative and its funding on the balance with combination of going long senior bonds and shot junior bonds, then the windfall is effectively monetized while the issuer is alive, and by doing so the funding cost term is reduced to zero.

Funding Strategies and Costs Impact

1. <u>Abstract</u>: The economic values of derivatives depends on their funding costs, because they can result in windfalls or shortfalls to bondholders on their firm's default. But this depends on not just who is funding them, but how – so the resulting adjustments depend on the funding strategy deployed. It is another layer of complexity to derivatives pricing, as argued by Burgard and Kjaer (2013).
2. <u>Derivative ITM vs OTM Funding</u>: Assuming that the issuer can hedge its own default when the derivative position is in the money and so provides funding is more realistic than assuming that the issuer can freely dynamically trade spread positions on its own bonds. When the derivative is out of the money and the issuer requires funding, a post-default windfall to the issuer's estate is generated. In that case a funding cost adjustment (FCA) is added in to compensate. There have been a flurry of papers prosing alternative approaches from different authors, including Morini and Prampolini (2012), Brigo, Pallavicini, and Perini (2012), Crepey (2013a, 2013b), and perhaps most famously Hull and White (2012a) that use risk-neutral valuation principles to examine the question.
3. <u>Classical Price with Bilateral CVA</u>: Such an approach, discounting all expected cash flows at the risk free rate results in the classical price with the bilateral CVA. But this disregards the preferences of the different stake holders regarding the value of the pre- and post- own-default cash flows. This is justified if all the risks – including own default – are hedgeable so that net post-default cash flows are zero. But, as mentioned, they are not.

4. Impact on the Shareholders: Shareholders are primarily interested in the pre-default cash flows of derivatives and their hedges, but post-default cash flows matter for bondholders, as they contribute to the recovery realized. Shareholders only care about the latter through the balance sheet effects that are in practice hard to realize and account for.
5. Selective Disregarding of Default Cash flows: Some authors have considered cases where the post-default cash flows on the funding leg are disregarded, but not the ones of the derivative. But it is not clear why some post-default cash flows should be disregarded but not others, and without specifying the funding strategies, the resulting recursive relations cannot be solved.
6. Crepey's Generalization Approach: Crepey's generalization of the original Markovian framework to non-Markovian processes using backward stochastic equations is elegant, but difficult to solve explicitly.
7. Selective Hedging of Default Cash Flows: Burgard and Kjaer (2013) look at funding strategies in terms of holding or issuing own bonds. The strategies hedge out some but not all cash flows at own default. The economic value of a derivative to the shareholders is then given by assuming that they disregard any remaining post-default cash flows and pre-default balance sheet effects.
8. Implementation of the Custom Funding Strategies: The funding cost adjustment is then given by the discounted expected value of the post-default cash flows. The strategies previously considered are then special cases of those considered by Burgard and Kjaer (2013), thus generalizing the previous work. Dealers can consider their own strategies and decide which adjustments represent the economic funding costs they expect to change while in business.
9. CSA Variants and Set-offs: Burgard and Kjaer (2013) also consider boundary conditions covering practical cases such as one-way or two-way credit support annexes (CSAs) governing collateral agreements and the so called set-offs. Set-offs are particularly interesting as they mitigate the need for funding cost adjustments. Their explicit calculations and numerical results show that different funding strategies can yield quiet different funding adjustments and asymmetries in the absence of set-of provisions.

Generalized Semi-Replication and Pricing PDE

1. Imperfect Own Credit Hedge: Consider a derivative contract, possibly collateralized, between an issuer B and a counterparty C with an economic value $\hat{V}$ that incorporates the risk of the counterparty and the issuer and any net funding costs the issuer may encounter prior to own default. This section describes a general semi-replication strategy that the issuer can deploy to perfectly hedge out any market factors and counterparty default, but which may provide a perfect hedge in the event of the issuer's "own" default.
2. Portfolio of Tradeable Instruments: The tradeable instruments used in this strategy are a counterparty zero-coupon zero-recovery bond P_C, two issuer "own" bonds P_1 and P_2 of different seniorities, i.e., different recoveries R_1 and R_2 respectively, and a market instrument S that can be used to hedge out the market factor for the derivative contract (e.g., stock).
3. Underlying Assets and Market Factors Dynamics: The setup can be easily extended to many market factors. The following standard dynamics for these instruments are assumed

$$i = 1,2$$

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C {P_C}^{-} \Delta t - {P_C}^{-} \Delta J_C$$

$$\Delta P_i = r_i {P_i}^{-} \Delta t - (1 - R_i) {P_i}^{-} \Delta J_B$$

   where J_B and J_C are the default indicators for B and C respectively, and

$$P_{i/C}{}^{-} = P_{i/C}(t^{-})$$

   are the pre-default bond prices.
4. Risk-Neutral Bond-Funding Spread: Without loss of generality, P_1 is the junior bond, i.e.,

$$R_1 < R_2$$

and

$$r_1 > r_2$$

In case of zero basis between bonds of different seniority, it is trivial to show that

$$r_i - r = (1 - R_B)\lambda_B$$

where r is the risk-free rate, λ_B corresponds to the spread of a potentially hypothetical zero-recovery zero-coupon bond of the issuer.

5. <u>Generalized Boundary Conditions at Default</u>: Let $\hat{V}(t, S, J_B, J_C)$ be the total economic value of the derivative to the issuer. Using Kjaer (2011) the general boundary conditions at default of the issuer or the counterparty are given by

$$\hat{V}(t, S, 1, 0) = g_B(M_B, X)$$

if B defaults first, and

$$\hat{V}(t, S, 0, 1) = g_C(M_C, X)$$

if C defaults first with general close-out amounts M_B and M_C, and collateral X.

6. <u>Collateral Extensions to Boundary Conditions</u>: If

$$M_B = M_C = V$$

these boundary conditions are called “regular”, where V is the classic Black-Scholes price of the derivative, i.e., without counterparty and own-default risks and no funding costs. For example the regular bilateral boundary conditions with collateral are defined as

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V-X)^+ + (V-X)^- + X$$

7. Extensions to Alternate Close-outs: Burgard and Kjaer (2012a) consider alternate close-out cases for

$$M_B = M_C$$

and Brigo and Morini (2011) extend this to the cases where

$$M_B \neq M_C$$

and include the cost of funding in the close-out amounts. Separately Burgard and Kjaer (2012c) apply the present framework to such funding aware close-outs.

8. Regular Bilateral Uncollateralized Close-outs: An example for the close-out functions are the regular bilateral close-outs with collateral, which are described by

$$g_B = V^+ + R_B V^-$$

and

$$g_C = R_C V^+ + V^-$$

Later other examples such as one-way CSA and set-offs are examined.

Semi-Replication

1. The Semi-replication Hedge Portfolio: For the semi-replication the hedge portfolio Π is set up as

$$\Pi(t) = \alpha_S(t)S(t) + \alpha_1(t)P_1(t) + \alpha_2(t)P_2(t) + \alpha_C(t)P_C(t) + \beta_S(t) + \beta_C(t) - X(t)$$

with $\alpha_S(t)$ units of $S(t)$, $\alpha_{1/2}(t)$, and $\alpha_C(t)$ units of own and counterparty bonds respectively, cash accounts $\beta_S(t)$ and $\beta_C(t)$, and a collateral account $X(t)$.

2. The Asset Financing Cash Accounts: The cash accounts β_S and β_C are used to finance the S and the P_C positions, i.e.,

$$\alpha_C P_C + \beta_C = 0$$

and

$$\alpha_S S + \beta_S = 0$$

and assumed to pay net rates of $(q_S - \gamma_S)$ and q_C respectively, where γ_S may be the dividend income. The hedge positions may be collateralized or repo'ed, so q_S and q_C maybe the collateral or the repos rates respectively.

3. The Replication Portfolio Funding Constraint: The derivative collateral balances X are assumed to be fully re-hypothecable and pay the amount r_X and

$$X > 0$$

corresponds to the counterparty having posted the amount X with the issuer. The strategy shall be designed such that

$$\hat{V} + \Pi = 0$$

except possibly at the issuer default. The issuer bond positions $\alpha_1 P_1$ and $\alpha_2 P_2$ are used to finance/invest any remaining cash that is not funded via the collateral, which yields the following *funding constraint*

$$\hat{V} - \Pi + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

4. Evolution of the Derivative Hedge Portfolio: The evolution of the hedge portfolio Π defined in

$$\Pi = \alpha_S S + \alpha_1 P_1 + \alpha_2 P_2 + \alpha_C P_C + \beta_S + \beta_C - X$$

is given by

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

where ΔS, ΔP_1, ΔP_2, and ΔP_C are given earlier and $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ are changes to the cash and the collateral accounts, excluding rebalancing.

5. Funding Costs for Cash Accounts: As in Burgard and Kjaer (2012a) the hedge account β_S is collateralized with the financing rate q_S and income via dividend γ_S.
6. Growth in the Cash Accounts: Likewise the counterparty bond position is assumed to be setup via a repo transaction costing a repo rate q_C. The derivative collateral account is assumed to cost a collateral rate r_X. Excluding rebalancing these yield the following increments in the accounts:

$$\Delta\bar{\beta}_S = \alpha_S S(\gamma_S - q_S)\Delta t$$

$$\Delta\bar{\beta}_C = -\alpha_C q_C P_C \Delta t$$

$$\Delta\bar{X} = -r_X X \Delta t$$

7. Incremental Change in the Derivative Portfolio: With the pre- and the post-default values of the issuer bond position given by

$$P = \alpha_1 P_1 + \alpha_2 P_2$$

and

$$P_D = \bar{R}_1 \alpha_1 P_1 + \bar{R}_2 \alpha_2 P_2$$

respectively, inserting

$$\Delta S = \mu S \Delta t + \sigma S \Delta W$$

$$\Delta P_C = r_C P_C{}^{-} \Delta t - P_C{}^{-} \Delta J_C$$

and

$$\Delta P_i = r_i P_i{}^{-} \Delta t - (1 - R_i) P_i{}^{-} \Delta J_B$$

and the expressions for $\Delta\bar{\beta}_S$, $\Delta\bar{\beta}_C$, and $\Delta\bar{X}$ into

$$\Delta\bar{\Pi} = \alpha_S \Delta S + \alpha_1 \Delta P_1 + \alpha_2 \Delta P_2 + \alpha_C \Delta P_C + \Delta\bar{\beta}_S + \Delta\bar{\beta}_C - \Delta\bar{X}$$

results in

$$\Delta\bar{\Pi} = [r_1 \alpha_1 P_1 + r_2 \alpha_2 P_2 + \lambda_C \alpha_C P_C + \alpha_S S(\gamma_S - q_S) - r_X X]\Delta t + (P_D - P)\Delta J_B - \alpha_C P_C \Delta J_C + \alpha_S \Delta S$$

where

$$\lambda_C \equiv r_C - q_C$$

is the spread of the zero-coupon bond price P_C yield over its repo rate, i.e., the financing rate of the counterparty default position.

8. Ito's Lemma for the Derivative Contract: The evolution of the derivative $\Delta\hat{V}$ on the other hand is given by Ito's lemma for jump diffusions as

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B\Delta J_B + \mathcal{D}\hat{V}_C\Delta J_C$$

with

$$\mathcal{D}\hat{V}_B = \hat{V}(t, S, 1, 0) - \hat{V}(t, S, 0, 0) = g_B - \hat{V}$$

and

$$\mathcal{D}\hat{V}_C = \hat{V}(t, S, 0, 1) - \hat{V}(t, S, 0, 0) = g_C - \hat{V}$$

9. The Consolidated Derivative Portfolio Increment: Combining the evolution of the derivative in

$$\Delta\hat{V} = \frac{\partial\hat{V}}{\partial t}\Delta t + \frac{\partial\hat{V}}{\partial S}\Delta S + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2\hat{V}}{\partial t^2}\Delta t + \mathcal{D}\hat{V}_B\Delta J_B + \mathcal{D}\hat{V}_C\Delta J_C$$

with the hedge portfolio

$$\Delta\overline{\Pi} = [r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C\alpha_C P_C + \alpha_S S(\gamma_S - q_S) - r_X X]\Delta t + (P_D - P)\Delta J_B - \alpha_C P_C\Delta J_C + \alpha_S\Delta S$$

gives

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2\hat{V}}{\partial t^2} + r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C\alpha_C P_C + \alpha_S S\frac{\partial\hat{V}}{\partial S}(\gamma_S - q_S) - r_X X\right]\Delta t + (g_B + P_D - X)\Delta J_B + \left(\mathcal{D}\hat{V}_C - \alpha_C P_C\right)\Delta J_C + \left(\frac{\partial\hat{V}}{\partial S} + \alpha_S\right)\Delta S$$

where the term in front of ΔJ_B follows from the fact that

$$\hat{V} - P + X = 0$$

from the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

10. Asset and Counterparty Hedges: From the derivatives portfolio increment we can eliminate the stock price and the counterparty risks by choosing

$$\mathcal{D}\hat{V}_C = \alpha_C P_C$$

and

$$\alpha_S = -\frac{\partial \hat{V}}{\partial S}$$

which yields

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1 P_1 + r_2\alpha_2 P_2 + \lambda_C \mathcal{D}\hat{V}_C - r_X X\right]\Delta t + (g_B + P_D - X)\Delta J_B$$

where

$$\mathcal{A}_t\hat{V} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \hat{V}}{\partial t^2} + \alpha_S S \frac{\partial \hat{V}}{\partial S}(\gamma_S - q_S)$$

11. Incorporating the Hedge Ratios: On using the zero bond basis relation

$$r_i - r = (1 - R_B)\lambda_B$$

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

and

$$\mathcal{D}\hat{V}_C = \alpha_C - \hat{V}$$

the above becomes

$$\Delta\hat{V} + \Delta\overline{\Pi} = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} - s_X X + \lambda_B g_B + \lambda_C g_C - \epsilon_h\lambda_B\right]\Delta t + \epsilon_h \Delta J_B$$

where

$$s_X = r_X - r$$

and

$$\epsilon_h = g_B + P_D - X$$

12. Own-Credit Default Hedging Error: From the jump term above it follows that upon issuer default there is a hedge error of size ϵ_h. While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h\lambda_B$ per unit time.
13. Hedge Error Windfall and Shortfall: It can thus be seen that the combination of the derivative $\hat{V}$ and the hedge portfolio Π is risk free as long as the issuer is alive. At issuer default the jump term $\epsilon_h\Delta J_B$ gives rise to a hedge error of size ϵ_h. The hedge error can be a windfall or a shortfall and its size depends on the post-default value of the own bond portfolio, and thus the funding strategy employed.

14. Windfall/Shortfall Accrual Gain/Bleed: While alive, on the other hand, the issuer correspondingly incurs a cost/gain of size $-\epsilon_h \lambda_B$ per unit time. Tis can be seen as the running spread to pay for the potential windfall/shortfall upon issuer default.
15. Collateral/Hedge Error Derivative PDE: Since the issuer wants the strategy to evolve in a self-financing fashion while he is alive the total drift term above should become zero. This produces the following PDE for the risky economic value $\hat{V}$ for the derivative:

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \epsilon_h \lambda_B$$

$$\hat{V}(T, S) = H(S)$$

where $H(S)$ is the payout of the derivative at maturity.
16. PDE for the Valuation Adjustment: To estimate the correction

$$U = \hat{V} - V$$

to the risk-free Black Scholes price V the Black-Scholes PDE for V is used to get the PDE for U to be

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_B(g_B - V) - \lambda_C(g_C - V) + \epsilon_h \lambda_B$$

$$U(T, S) = H(S)$$

17. Decomposition of U into Components: Applying the Feynman-Kac theorem to this PDE gives

$$U = CVA + DVA + FVA + COLVA$$

with

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_C\big(V(u), X(u)\big)\big] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\big[V(u) - g_B\big(V(u), X(u)\big)\big] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

$$D_k(t,u) = e^{-\int_t^u k(v) dv}$$

is the discount factor between t and u for a rate k. The measure of the expectations in these equations is such that S drifts at the rate $q_S - \gamma_S$. The sum of DVA and FCA is sometimes referred to as FVA.

18. Symmetry of CVA, DVA, and $COLVA$: Here the sum of the CVA, the DVA, and the $COLVA$ is symmetric in that is identical – with sign flipped – when computed by the issuer and the counterparty, respectively.
19. Lack of Symmetry in FCA: The $FCA - \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$ on the other hand is the discounted survival probability weighted expected value of the hedge error ϵ_h implied by the semi-replication strategy chosen. Because the hedge error on own default is different for the issuer and for the counterparty, the FCA is not symmetric. This is a generalization of the result presented in Burgard and Kjaer (2012b) for regular bilateral close-outs and a particular choice of bonds that states that the FCA is the cost of generating a windfall to the issuer bondholders in the case of a default. If the issuer wants to break-even

while being alive, the cost has to be included in the derivative price charged to the counterparty.

20. Asset Addition Funding Rate Impact: The analysis above assumes that the funding rates r_1 and r_2 remain unaffected by the addition of the derivative asset and the funding positions. Burgard and Kjaer (2012b) have noted that the presence of potential windfall has a positive balance sheet effect; it improves the recovery rate to the bondholders and therefore the funding spread of the issuer should go down. Hull and White (2012a) have used a similar argument.
21. Balance Sheet Funding Cost Mitigation: For a simple balance sheet model with floating funding costs Burgard and Kjaer (2012b) have demonstrated that this effect can result in an effective marginal funding rate that corresponds to the risk-free rate.
22. Practical Challenges with the Mitigation: However as discussed there, in practice the balance sheet effect on the funding costs is rather indirect, fraught with accounting issues, and in general only feed through over time. Therefore the current treatment assumes that the issuer disregards this effect.

Examples of Different Bond Portfolios

1. Strategies Generating Different Hedge Errors: Using the general framework developed above Burgard and Kjaer (2013) provide three different examples of semi-replication strategies that generate different hedge errors ϵ_h and therefore different valuation adjustments.
2. Zero FCA and Windfall Only: The first strategy, if employed, allows for perfect replication and generates zero FCA. The second one is equivalent to the setup used in Burgard and Kjaer (2012a) for the bilateral close-outs and ensures that there is never a shortfall at issuer default and only a windfall (potentially).
3. Windfall or Shortfall Generation Strategy: The third strategy assumes hedging with a single issuer bond. It generates both potential windfall and potential loses post-default and is an extension of the model derived in Piterbarg (2010).

4. Strategy Economic Value and Adjustments: The different strategies generate different economic values – and therefore different adjustments – to the issuer while he is alive. They demonstrate the assumptions implicitly being made when using different adjustment formulas in practice.
5. Bilateral and Funding Curve Close-outs: Throughout it is assumed that the close-out value is V, i.e., that

$$M_B = M_C = V$$

Funding aware close-outs are discussed separately in Burgard and Kjaer (2012c).

Perfect Replication – The FCA Vanishes

1. Hedging Windfalls and Shortfalls: The first case considered is the one where the issuer is able to perfectly hedge out the windfall/shortfall at own default. This corresponds to the case discussed in the Section *Balance-Sheet Management to Mitigate Funding Costs* in Burgard and Kjaer (2012b) and also covers the risk-neutral approach outlined in Hull and White (2012a).
2. PDE Corresponding to the Perfect Hedge: Perfect hedge is equivalent to the hedge error ϵ_h being zero, i.e.,

$$g_B + P_D - X = 0$$

The valuation PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

becomes

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

Correspondingly the FCA given by

$$FCA = \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\epsilon_h(u)] du$$

vanishes.

3. Corresponding Own Portfolio Hedge Ratios: The hedge ratios α_1 and α_2 that achieve this perfect replication are determined by the no-windfall condition

$$g_B + P_D - X = 0$$

and the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

These two conditions provide two equations that can be solved to find

$$\alpha_1 = \frac{R_2 \hat{V} - g_B + (1 - R_2)X}{(R_1 - R_2)P_1}$$

$$\alpha_2 = \frac{R_1 \hat{V} - g_B + (1 - R_1)X}{(R_2 - R_1)P_2}$$

4. Valuation Adjustment Feynman-Kac Integrals: As an example, for the regular bilateral conditions

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

these adjustments specialize to

$$CVA = -(1 - R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^+]du$$

$$DVA = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u)\mathbb{E}_t[\{V(u) - X(u)\}^-]du$$

with

$$FCA = 0$$

and

$$COLVA = 0$$

if

$$r_X = r$$

5. Case of Classical Bilateral CVA: If the derivatives are not collateralized, i.e.,

$$X = 0$$

these adjustments correspond to the classical bilateral CVA. Thus the classical bilateral CVA can be achieved when perfect replication is possible. As mentioned, in practice such a dynamic balance sheet management via actively traded spread options between junior and senior bonds is in general not a viable option.

Semi-Replication with No Shortfall at Own-Default

1. Collateralized without Shortfall at Default: Burgard and Kjaer (2013) demonstrate a bond portfolio and hedging strategy that constitute an equivalent to the one presented in Burgard and Kjaer (2012b) extended to more general conditions, including the possibility of collateral.
2. Dynamic Trading of Two Bonds: It still involves dynamic trading of two bonds, but is more conservative than the dynamic trading strategy of the previous section as it does not aim at monetizing the potential windfall upon own default by entering into an offsetting position between the two bonds.
3. Bilateral CVA Plus a DVA: While this generates potential windfalls at own default, it does not generate shortfalls, and has the additional advantage that for regular bilateral close-outs without collateral it results in the usual bilateral CVA adjustment plus a funding cost adjustment, so presents a simple extension to the existing framework, where the derivatives dealer does not think he can monetize the windfall.
4. P_1/P_2 Buy-Sell Strategy: The strategy involves a zero recovery bond P_1 with

$$R_1 = 0$$

and a recovery bond

$$R_2 = R_B$$

The issuer runs the following bond positions:

    a. Invest of fund the difference between $\hat{V}$ and V by buying or issuing P_1 bonds.

b. Hold the number of P_2 bonds given the following funding constraint:

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

5. Corresponding Own-Portfolio Hedge Ratios: The strategy is this defined by the following values of α_1 and α_2:

$$\alpha_1 P_1 = -\left(\hat{V} - V\right) = U$$

$$\alpha_2 P_2 = -\alpha_1 P_1 - \hat{V} + X = -(V - X)$$

6. Collateral Adjusted Junior Bond Hedge: The strategy is thus symmetric between positive and negative funding, and the risk-free value V not covered by the collateral X is funded/invested via own unsecured bonds with recovery R_B. Only the adjustment U, which falls away on own-default, is funded/invested via a zero-recovery bond. As such this strategy looks more palatable from a regulatory and accounting perspective than the perfect replication strategy which attempts to actively extract the funding spread from the balance sheet by issuing own senior bonds to buy back own junior bonds.
7. Regular Bilateral Close-out Case: This bond portfolio is equivalent to the one described in Burgard and Kjaer (2012a, 2012b) for the bilateral close-outs considered there. This section analyzes the setup in more detail for the case of regular bilateral close-out given in

$$g_B = (V - X)^+ + R_B (V - X)^- + X$$

and

$$g_C = R_C (V - X)^+ + (V - X)^- + X$$

For this case the hedge error ϵ_h specializes to

$$\epsilon_h = (1 - R_B)(V - X)^+$$

which is always a windfall – possibly zero – to the bondholders of the issuer.

8. Zero-Recovery Own Bond Off-setter: Therefore this strategy is characterized by the ability of the issuer to perfectly hedge out the difference U between the risky value of the derivative before own default and the close-out amount after own default by means of trading own bonds at zero-recovery.
9. R_B Recovery Own Bond Balance: The remainder, i.e., the difference between the close-out and the collateral is invested/funded using own bonds with recovery R_B.This part generates the windfall ϵ_h to the issuer's bondholders when $V - X$ is in the money and the issuer defaults.
10. CVA, DVA, FCA, and $COLVA$: The CVA, the DVA, the FCA, and the $COLVA$ specialize to

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

$$FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[X(u)] du$$

11. Consolidation of the DVA and the FCA: It is possible to combine the DVA (a funding benefit) and FCA (a funding cost) into a funding value FVA as

$$FVA = DVA + FCA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[V(u) - X(u)]du$$

Note that Hull and White (2012a, 2012b) refer to this FCA as FVA. Burgard and Kjaer (2013) use the notation that is consistent with their previous papers and with Gregory (2012).

12. <u>Uncollateralized Classical Bilateral CVA/DVA</u>: For uncollateralized derivatives

$$X = 0$$

and the $COLVA$ term vanishes. The CVA and the DVA then correspond to the classical bilateral CVA. The FCA term provides the funding cost adjustment on top.

13. <u>Gold-Plated Two-Way CSA</u>: For gold-plated two-way CSAs where

$$X = V$$

the CVA, the DVA, and the FVA terms are all zero. If the collateral rate is the risk-free rate, then the $COLVA$ – which represents the spread earned by the issuer – also vanishes. In this case the risky price $\hat{V}$ of the derivative becomes the risk-free price V as well. The intuition is that the collateral cash is exactly what is needed to fund the hedge and eliminate all counterparty risk and as a consequence

$$\alpha_1 = \alpha_2 = 0$$

This corresponds to the result for fully collateralized trades of Piterbarg (2010).

14. <u>One-Way CSA Issuer Posting</u>: Another special case worth considering is that of a one-way CSA whereby the issuer only posts collateral when the risk-free value of the trade is out-of-the-money, i.e.,

$$X = V^-$$

One-way CSAs are common when the issuer trades with sovereign counterparties or with sovereign-like public entities.

15. <u>One-Way CSA Valuation Adjustments</u>: In this case the adjustments specialize to

$$CVA = -(1 - R_C)\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[V(u)^+]du$$

$$FCA = -(1 - R_B)\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[V(u)^+]du$$

and

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[V(u)^-]du$$

Unsurprisingly the introduction of one-way CSA makes the DVA vanish while leaving the CVA and the FCA terms unchanged.

16. <u>Implication of One-Way CSA</u>: Unlike the uncollateralized case any cash available when the derivative is out-of-the-money must be handed over as collateral and thus cannot be used to generate a funding benefit. And unlike two-way CSA there is no influx of collateral cash that can be used to fund the hedge when the derivative is in-the-money. The issuer is thus faced with the uncollateralized and the 2-way cases, and need to charge a higher price to the counterparty to compensate for that in order to break even.

Set-offs

1. Definition of the Set-off Mechanism: It is also instructive to study the case of the so-called set-offs. A set-off is a legal agreement that allows the surviving party to settle the outstanding derivative claims of the defaulting party by means of supplying the bonds of the defaulting party at nominal value rather than cash.
2. Boundary Conditions for Set-offs: Since post-default these bonds trade at their recoveries, this type of settlement is valuable to the surviving party. Explicitly for regular bilateral set-offs without collateral, the boundary conditions are given by

$$g_B = R_B V$$

and

$$g_C = R_C V$$

Inserting this into

$$\alpha_1 P_1 = -U$$

and

$$\alpha_2 P_2 = -(V - X)$$

implies that the hedge error ϵ_h disappears.
3. Set-offs CVA and DVA: The CVA and the DVA adjustments in this case are given by

$$CVA = -(1 - R_C) \int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)] du$$

$$DVA = -(1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t, u) \mathbb{E}_t[V(u)] du$$

4. Issuer and Counterparty Price Symmetry: Significantly with

$$\epsilon_h = 0$$

the FCA term vanishes resulting in symmetric prices between the issuer and the counterparty. Thus adoption of close-outs would mitigate the need for economic funding for adjustments.

Semi-Replication with a Single Bond

1. Strategy P_1/P_2 Hedge Ratios: This section considers a very simple strategy where the issuer uses a single own bond with recovery R_F, i.e.,

$$\alpha_1 P_1 = 0$$

and

$$\alpha_2 P_2 = -\left(\hat{V} - X\right) = -(V + U - X)$$

where the second line follows from the funding constraint

$$\hat{V} - X + \alpha_1 P_1 + \alpha_2 P_2 = 0$$

For aesthetic reasons the remaining bond P_F and its yield are relabeled

$$r_F = r + s_F$$

The hedge ratios imply that the issuer raises all necessary net cash by issuing P_F-bonds and invests any surplus net cash by repurchasing the same bonds.

2. Insufficient Hedge Degrees of Freedom: With a single bond, once the funding constraint is fulfilled, there are no degrees of freedom left for the issuer to hedge out his own default. This is in contrast to the previous setup of strategy I where the issuer is able to hedge out its own default risk when the trade is out-of-the-money at least.
3. Hedge Error Produced by the Strategy: For the own bond portfolios with the hedge ratios above the hedge error ϵ_h amount to

$$\epsilon_h = g_B + P_D - X = g_B - R_F\hat{V} - (1 - R_F)X$$

where the default value of the own bond portfolio given by

$$P_D = -R_F\left(\hat{V} - X\right)$$

has been used.
4. Recursive Setup Formulation for the FCA: Using this hedge error in the general adjustment expression

$$FCA = -(1 - R_B)\int_t^T \lambda_B(u)D_{r+\lambda_B+\lambda_C}(t,u)\mathbb{E}_t[\epsilon_h(u)]du$$

yields a recursive relationship. This is because

$$\hat{V} = V + U$$

appearing on the RHS of the expression for ϵ_h above includes the contribution from U and thus the FCA itself.
5. Elimination of Recursion with the Original PDE: The best way to deal with this situation is by going back to the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_C)\hat{V} = s_X X - \lambda_B g_B - \lambda_C g_C + \lambda_B \epsilon_h$$

and insert the hedge error above to obtain

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r_F + r_C)\hat{V} = -(r_F - r_C)X - \lambda_C g_C$$

$$\hat{V}(T, S) = H(S)$$

6. Uncollateralized Discounting-with- Funding Approach: The boundary condition g_B does not enter – this is because in this strategy there is no attempt to hedge own default. It is worth noting that for uncollateralized trades, i.e., trades with

$$X = 0$$

and zero counterparty risk –

$$\lambda_C = 0$$

– the PDE specializes to a simple funding-with-discounting approach as in Piterbarg (2010).
7. Special Case – Discounting-with-Funding: Thus discounting-with-funding is a special case of this strategy and assumes that the issuer deals with any funding requirement or surplus by using a single funding instrument and is happy to generate a windfall or shortfall upon own default.
8. PDE for Gross Valuation Adjustment: Similarly inserting the hedging error

$$\epsilon_h = g_B + P_D - X = g_B - R_F\hat{V} - (1 - R_F)X$$

into the PDE

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_C)U = s_X X - \lambda_B(g_B - V) - \lambda_C(g_C - V) + \lambda_B \epsilon_h$$

$$U(T, S) = 0$$

gives the adjustment U for this strategy as

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_C)U = -\lambda_C(g_C - V) + s_F(V - X) + s_X X$$

$$U(T, S) = 0$$

9. Collateralized Regular Bilateral Close-outs: This step carries out an analysis of the regular bilateral close-outs with collateral as given in

$$g_B = (V - X)^+ + R_B(V - X)^- + X$$

and

$$g_C = R_C(V - X)^+ + (V - X)^- + X$$

For these

$$g_C - V = -(1 - R_C)(V - X)^+$$

Applying the Feynman-Kac theorem we obtain

$$U = CVA_F + DVA_F + FCA_F + COLVA_F$$

with

$$CVA_F = -(1 - R_C) \int_t^T \lambda_C(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$DVA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^-] du$$

$$FCA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t[\{V(u) - X(u)\}^+] du$$

$$COLVA_F = -\int_t^T s_X(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t[X(u)] du$$

10. DVA_F and FCA_F as FVA_F: Combining DVA_F and FCA_F into FVA_F results in

$$FVA_F = DVA_F + FCA_F = -\int_t^T s_F(u) D_{r_F + \lambda_C}(t, u) \mathbb{E}_t[V(u) - X(u)] du$$

11. Comparison with the No Shortfall Case: These adjustments are very similar to the ones in the strategy described in *semi-replication with no shortfall at own default* except that the discounting used is $D_{r_F + \lambda_C}(t, u)$ rather than $D_{r_F + \lambda_B + \lambda_C}(t, u)$. There is no reference to λ_B, only to the funding rate r_F of the P_F bond used in the own-bond portfolio of the semi-replication strategy.
12. Inapplicability as Generalized Valuation Adjustment: It should be noted that the adjustments CVA_F, DVA_F, FCA_F, and $COLVA_F$ are not direct specializations of the general adjustments defined in

$$CVA = -\int_t^T \lambda_C(u) D_{r + \lambda_B + \lambda_C}(t, u) \mathbb{E}_t\left[V(u) - g_C\left(V(u), X(u)\right)\right] du$$

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_B\big(V(u), X(u)\big)\right] du$$

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

$$COLVA = -\int_t^T s_X(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[X(u)] du$$

In particular FCA_F does not correspond anymore to the discounted expectation of the hedge error ϵ_h upon default.

13. FCA As Collateralized Adjustment Difference: Obviously the expectation above for FCA_F still equals the difference between $CVA_F + FVA_F + COLVA_F$ and the classical bilateral CVA with collateral. The FCA of

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

can thus be calculated as

$$FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$$

The adjustments of the special cases of uncollateralized, gold-plated 2-way CSA, and 1-way CSA can then be derived easily equivalently to those in the strategy *semi-replication with no shortfall at own default*.

14. Simplicity and Relevance of the Strategy: The strategy specified above is thus very simple to understand and implement. It is also of practical relevance not least because dealers who

simply discount by the funding rate assume this strategy implicitly (for zero counterparty risk), including potential windfalls and shortfalls to their estate upon own default.

Burgard and Kjaer (2013) Case Study

1. Strategy Specific Valuation Asymmetry Estimation: Burgard and Kjaer (2013) provide an illustrative case study for the generalized bilateral CVA – the sum of CVA from

$$CVA = -\int_t^T \lambda_C(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_C\big(V(u), X(u)\big)\right] du$$

DVA from

$$DVA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t\left[V(u) - g_B\big(V(u), X(u)\big)\right] du$$

and the generalized FCA from

$$FCA = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_C}(t,u) \mathbb{E}_t[\epsilon_h(u)] du$$

and the valuation asymmetries for strategies I and II computed from the perspective of the issuer and the counterparty (i.e., with counterparty's FCA) respectively together with the issuer's hedge error $\epsilon_h(t_0)$ in case of immediate default of the issuer for three sample trades. For strategy II the FCA is computed as per

$$FCA = FCA_F + (CVA_F - CVA) + (FVA_F - FVA) + (COLVA_F - COLVA)$$

2. ITM, ATM, and OTM Swaps: The sample trades are 10Y swaps with \$100m notional where the issuer pays fixed and receives 6M LIBOR floating. Burgard and Kjaer (2013) consider different close-out provisions for 3 fixed rates – 3.093% (OTM), 2.693% (ATM), and 2.293% (ITM).
3. Dealer Specific Funding Value Adjustments: The adjustments computed from the perspective of the counterparty show that if both sides include their funding costs their economic values may be far apart and the two parties may not agree on the deal. If the counterparty does not include the funding costs in general it will deal with the issuer with the lowest funding costs.
4. Differences between Strategies I and II: As shown in Burgard and Kjaer (2013) the differences between the adjustments of the strategies I and II are in general not particularly big (they increase with funding rate), but as expected strategy II has potentially significant shortfalls upon issuer default, whereas strategy I only generates windfalls.
5. Impact of using Setoffs: As discussed earlier, when using set-offs the impact of the funding cost is mitigated. When following the strategy I the FCA vanishes completely and symmetric prices are obtained. Even when implementing strategy II the FCA prices are pretty small and the prices are close to being symmetric. Setoff close-outs are an attractive way of mitigating the need for funding cost adjustments.
6. Asymmetric Valuation and Hedge Error: Burgard and Kjaer (2013) show how asymmetric valuation and hedging error can vary across funding strategies I and II outlined above, and how they interact with CSA's and set-offs. Each in turn prices an out-of-the money (OTM), at-the-money (ATM), and in-the-money (ITM) 10Y \$100m swap with the OTM and the ITM swaps 40 bp either side of the ATM level.
7. Impact of the Issuer Bond Spread: Issuer bond spreads are considered at 100 bp and 500 bp respectively, while the counterparty spread is set constant at 300 bp. Total adjustments are calculated for the issuer and the counterparty as well as the bilateral CVA and the hedging error. Uncollateralized one-way CSAs in the counterparty's favor, as well as cases including a set-off are also considered.
8. Adjustment Impact between the Sides: The adjustments differ between the two counterparties creating an asymmetry in the derivative's valuation. The degree of this and the resulting hedging error depends on the funding and the collateralization strategy. In the 100 bp case

the uncollateralized swap has a valuation asymmetry of 50 bp of the notional with the magnitude decreasing from OTM through ATM to ITM.

9. One-Way CSA Adjustment Impact: The introduction of the one-way CSA increases the size of the issuer's adjustment but reduces the asymmetry to 20-30 bp with the amount of reduction skewed from the opposite direction from ITM to OTM. This is because under the one-way CSA the issuer has to post more collateral for an OTC swap, thus reducing the funding benefit.
10. XVA Metrics for Strategy I: Strategy I has zero hedge error for OTM and ITM, but produces roughly 2% notional hedging error in the ITM case regardless of the existence of the CSA. In the presence of the set-off the FCA is eliminated, and so the total adjustment is equal to the bilateral CVA.
11. XVA Metrics for Strategy II: Strategy II is more complex, but the valuation asymmetry is dramatically reduced – by roughly a factor of 10 in the case study. The hedging error is also reduced with the biggest reduction coming for the ITM case. When the issuer is ITM and defaults, the setoff implies that the counterparty can pay back the full present value of the trade using the issuer bond notional rather than cash, which reduces the post-default bond-holder windfall.
12. Directional Dependence on the Issuer Spread: The interesting point is that the quantitative findings of the study are independent of the issuer's bond spread – only the magnitudes change with a greater proportional reduction in the valuation asymmetry and the hedging error of the strategy II in the presence of a set-off, for instance – as can be seen from the results for the 500 bp case.

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Accounting for OTC Derivatives: Funding Adjustments and Re-hypothecation Option

Status of Current FCA/FBA Accounting

1. Motivation: Banks hold and routinely exercise the option of freely re-hypothecating variation margin across counter-parties and trades.
2. FCA/FBA Accounting Standards: However, the emerging FCA/FBA cost accounting metrics for funding costs are mostly formulated in terms of netting set specific metrics that fail to properly account for re-hypothecation benefits to common Equity Tier 1 Capital (CET1).
3. Double Counting in FCA/FBA: Additionally, the FCA/FBA standard introduces a double counting issue between the funding benefits and the DVA which leads ultimately to the violation of the fundamental accounting tenet of asset-liability symmetry.
4. FVA/FDA Accounting Objectives: Albanese and Andersen (2014) propose an alternative accounting framework meant to rectify some of the problems in existing standards. This new accounting method, which they call FVA/FDA, explicitly incorporates the re-hypothecation option into its definition of funding costs, and maintains consistency with the Modigliani-Miller Theorem, with fair-value and asset-liability symmetry principles, and with Basel III rules for DVA and equity capital.
5. Pricing at the CET1 Indifference Level: They argue that derivative pricing necessitates an incremental assessment of capital structure impact on new trades and propose that the entry prices should be struck at the indifference level for CET1.
6. Departure From FCA/FBA Accounting: Unlike the FCA/FBA method, the FVA/FDA accounting does not result in outright net-income write-offs due to funding costs.

Comparison Between FCA/FBA and FDA/FVA

1. Where are they similar: FCA/FBA accounting and FVA/FDA accounting lead to very similar and quantitatively close conclusions in the particular case of a portfolio consisting of a single netting set and a single trade.
2. Portfolio Sets and Portfolio Sizing: However material differences arise in the case of large portfolios. Albanese and Andersen (2014) discuss a case study with a representative portfolio whereby the CET1 adjustments for funding are 3 times as large as the ones required in FVA/FDA accounting.
3. Impact on Prices and VAs: After the portfolio effects are accounted for, incremental entry prices for individual trades differ materially between the FCA/FBA and the FVA/FDA methods, with the FCA/FBA accounting often displaying sizeable and risky pricing biases between derivative payables and receivables.
4. Abbreviations and Expansions:

Abbreviation	**Expansion**
A	Asset Account
BCBS	Basel Committee on Banking Supervision
CA	Contra-Asset
CCP	Central Counter-party (i.e., a Clearing House)
CDS	Credit Default Swap
CET1	Common Equity Tier 1 Capital
CL	Contra-Liability
CFD	Central Funding Desk
CSA	ISDA Credit Support Annex Agreement
CVA	Credit Valuation Adjustment, same as FTDCVA
CVA_{CL}	Contra-Liability entry for Credit Valuation Adjustment

DVA	Debt Valuation Adjustment
DVA2	Funding Debt Adjustment (same as FDA)
EE	Expected Exposure
ENE	Expected Negative Exposure
EPE	Expected Positive Exposure
FBA	Funding Benefit Adjustment
FCA	Funding Cost Adjustment
FDA	Funding Liability Adjustment (same as DVA2)
FTDCVA	First-to-default CVA, same as CVA
FTP	Funding Transfer Pricing
FVA	Funding Valuation Adjustment
PFV	Portfolio Fair Valuation
KVA	Capital Valuation Adjustment
L	Liability Account
OIS Rate	Overnight Index Swap Rate
OTC	Over-the-Counter
RE	Retained Earnings
REPO	Re-purchase Agreement
RHO	Re-hypothecation Option Benefit
SFVA	Symmetric Funding Value Adjustment
UCVA	Unilateral CVA

VM	Variation Margin
XVA	"X" Valuation Adjustment (short-hand for all valuation adjustments, such as CVA, DVA, FVA, etc.)

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Funding and Re-Hypothecation Adjustment - Motivation

OTC vs. Repo Markets

1. Repo Market Infrastructure: One key indicator of trade liquidity is the existence of an efficient REPO market infrastructure in support of market-making activities. In a liquid markets, such as those trading government bonds, security acquisitions may be financed by reverse REPO trades.
2. Funding in OTC Derivatives Market: The OTC derivatives market, on the other hand, is not directly supported by a REPO infrastructure, and uncollateralized derivatives must instead be financed by other means.
3. OTC Hedges Variation Margin Funding: As derivatives receivables are an inefficient form of collateral, the rates banks face when funding variation margin (VM) for their hedges for uncollateralized derivatives are typically close to those for unsecured funding.

Modus Operandi of Funding Desks

1. Funding the Derivatives Hedging Cost: Without an efficient REPO market infrastructure, managers cannot prevent or hedge the wealth transfer with unsecured derivatives trading. Instead they must seek to recoup the loss to shareholders by passing on the cost to the clients.
2. The Funding Cost Transfer Chain: At a more granular level, this cost transfer commonly goes through a chain that starts when a bank treasury issues unsecured debt to raise funds needed for uncollateralized derivatives trading activities.
3. The CFD/CVA Desks: In the standard bank setup, the running spread cost of the bank issuance is subsequently passed by the treasury to a central funding desk (CFD) which

consolidates the management of the funding costs on behalf of the bank's trading functions. In many cases the CFD is merged with the CVA desk in a unified trading operation.

4. FTP Policy and Client Pricing: The CFD, in turn, transfers costs of funding to business line desks on an upfront basis by implementing a funding transfer pricing policy. Finally business line desks charge the costs to clients by embedding them into deal structures.

MMT And Asset-Liability Symmetry

1. Modigliani-Miller Theorem and Indifference Pricing: According to the famous Modigliani-Miller Theorem (MMT – Modigliani and Miller (1958)) the indifference price of new trade should not depend the rate on which the bank funds the VM on the hedges.
2. Wealth Transfer Across the Capital Structure: However, inefficient funding strategies may still give rise to wealth transfer across the capital structure of the bank, directed from the shareholders to the senior creditors (Albanese and Iabichino (2013)).
3. Funding Charges in Accounting Statements: While estimates for funding costs have been used and incorporated into funding costs informally for decades, it is only post-crisis that the banks have attempted to recognize these costs in official accounting statements by adding funding related valuation adjustments (FVA) to the existing CVA and DVA credit risk adjustments.
4. FVAs and Asset Write-downs: Despite unresolved controversies in the literature, in the last quarter of 2013 funding costs were reflected in the accounts at various banks including JP Morgan, Deutsche Bank, Nomura, and others. Not only did these give rise to very material adjustments to CET1, they also led to, due to asymmetries in the fair-value adjustments, asset write-downs that at least in one case exceeded $1 billion (JP Morgan Press Release (2013)).

Rigorous Framework For Funding Costs

1. Impact of Basel Committee Recommendations: The question of how and whether to include funding costs in accounts is complicated by the recent decision of the national regulators to accept the recommendation by the Basel Committee (Basel Committee On Banking Supervision (2012)) and mandate the exclusion of the DVA and other own-credit benefits from CET1.
2. Impact of DVA Capital Exclusion: The decision is relevant in the context of funding strategies as popular accounting methods for funding costs are inter-twined with the definition (or sometimes re-definition) of the DVA. In a realistic case-study example, Albanese and Andersen (2014) show that the impact of the DVA capital exclusion is to triple the CET1 deductions for funding whenever the FCA/FBA accounting method is used.
3. Motivation for the Rigorous Funding Valuation: Further, Albanese and Andersen (2014) show that such adverse impacts are to a large extent due to logical faults in the FCA/FBA accounting, and are not justified in a more rigorous framework for funding costs.

Funding Set VM RHO Computation

1. Rationale Behind the VM RHO Computation: The calculation of the CET1 deductions and the FTP amounts within lines of businesses are essentially always model based and depend strongly on a variety of assumptions and approximations.
2. VM Re-hypothecation Across Netting Sets: Of particular relevance here is how the re-hypothecation option (RHO) for the VM is treated. As banks are allowed to re-hypothecate across netting sets in the same business line portfolio, the RHO is valuable and routinely exercised through the shifting of the cash collateral hedges from receivable hedges to payable hedges.
3. Funding Set Portfolio RHO Valuation: As a consequence, rigorously computed funding cost adjustment is an aggregate portfolio level amount that cannot be linearly decomposed across netting sets. Specifically re-hypothecation dictates that funding costs be calculated through a portfolio level simulation with scenarios that are shared at the *finding set* level, whereby a funding set is defined as a portfolio of unsecured or partly collateralized trades and their corresponding hedges among which the variation margin can be re-hypothecated.

Shortcomings of Traditional CVA Systems

1. Funding Cost Valuation Implementation Challenges: Modeling challenges here get intertwined with technology implementation difficulties. Funding cost calculations that require aggregation of trade and collateral values across netting are rarely a good fit for traditional CVA systems optimized for individual netting sets.
2. CVA Systems Retrofit for RHO: In particular modeling of re-hypothecation using common distributed computing setups is often awkward compared to an in-memory architecture where all counter-party credits are simulated dynamically and all scenarios are shared.
3. Approximations to Funding Cost Calculations: To bypass technical implementation difficulties and to re-use grid-based CVA systems, many market participants have implemented an approximate accounting method for funding costs based on metrics that are additive over netting sets.
4. FCA/FBA Netting Set Additivity: The popular FCA/FBA accounting method was designed with linear aggregation in mind. However the netting set additivity inherent in the FCA/FBA accounting typically overlaps between the funding benefits (as captured by the FBA) and the DVA on the derivative payables.
5. Adjusting for DVA Double Counting: The resulting double-counting is handled normally by an outright replacement of DVA with FBA (Castagna (2011), Caccia (2013)). This replacement inevitably intertwines the DVA on payables (a CET1 deduction) with the RHO (which should instead *add* to CET1).

Addressing the Shortcomings of FCA/FBA Accounting

1. Rigorous Modeling of the RHO: To address the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) examine an alternate accounting method – denoted FVA/FDA – on which the RHO is modeled rigorously and care is taken to make entries of financial statements as meaningful as possible.

2. "Going Concern" Viewpoint of Accounting: Reconciling financial statements is often a delicate task as the accrual principles and the "going concern" viewpoint of financial accounting inevitably clashes with the notion of fair market value, DVA, and balance-sheet wealth transfers.
3. Accounting Theory Consistency of the FVA/FDA: Unlike the FCA/FBA Accounting method, the FVA/FDA accounting method is simultaneously consistent with the MMT, the risk-neutral pricing, and the general accounting principles such as asset-liability symmetry. Further in FVA/FDA accounting there is no double-counting of the DVA, and the funding cost adjustment to CET1 has the correct directional dependence with respect to the bank's own credit spread.
4. Impact on Income, CET1, and Price: Consistently with the MMT, the FVA/FDA funding value adjustments do not impact income, but do affect both CET1 and the entry price levels. This should provide sufficient incentives for the trading personnel to manage their funding costs properly, something that cannot be said for classical accounting principles that ignore funding costs entirely.

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Albanese and Andersen (2014) Results Summary

Valuation Adjustment Estimation Framework Setup

1. Book of Uncollateralized OTC Trades: Albanese and Andersen (2014) consider an OTC book containing trades with multiple unsecured counterparties, alongside back-to-back hedges with dealers or clearing houses. The unsecured counterparties do not post VM in full while hedges are fully collateralized.
2. Rate on the VM Collateral: In case the unsecured book is a net receivable, the hedge book is net payable, and the bank needs to procure VM to hedge counterparties. In this situation the bank receives a rate of OIS on the collateral posted as VM.
3. Definition of Funding Value Adjustment: The funding value adjustment (FVA) is defined as the PV of the carry cost of funding VM, net of the OIS receipts on the posted collateral.

OTC Books Funding Set Decomposition

1. Definition of the Derivative Funding Sets: OTC books are decomposed into *funding sets*, defined as trade sets for which VM for hedges can be re-hypothecated across all trades. Funding sets may span a large number (possibly thousands) of netting sets.
2. FVA Additivity over Funding Sets: The FVA is additive over funding sets, but not over netting sets. Therefore the valuation is difficult to carry out with standard CVA systems, and the industry is currently focused on using simpler alternatives which are linear over netting sets.
3. Funding Costs over Netting Sets: Funding costs would be additive over netting sets if either one of the following two mutually exclusive hypothesis hold:
    a. HY1 => Re-hypothecation is possible only between hedges to trades in individual netting sets;

   b. HY2 => The collateral received from hedges to each payable netting set can be fully re-hypothecated as VM for hedges against other receivable netting sets.

4. Complete Cross netting Set Re-hypothecation: This leads to a symmetric FVA (SFVA) which recognizes a re-hypothecation benefit to all VM received, whereas HY1 leads to the FCA metric that aggregates funding costs linearly over all netting sets. The metric

$$FBA = FCA - SFVA$$

   measures the difference between the funding costs under assumptions HY1 and HY2.

5. Shortcomings of FCA and SFVA: Note that in the above expression the FCA generally overstates the funding costs because it neglects the RHO for hedges to unsecured trades in different netting sets. The SFVA on the other hand has errors of opposite sign since it overvalues the RHO.

Inconsistent Booking Under the FCA/FBA

1. Overlap of FBA and DVA: In the interpretation of the FBA, it is often noted that the FBA overlaps with the DVA on payables – in their case study, Albanese and Andersen (2014) note that the FBA is about 20% larger than the standard DVA on payables.
2. DVA CL Replacement with the FBA: As a result advocates of FCA/FBA accounting remove the regular DVA contra-liability (CL) entry on the financial accounting statement and effectively replace it with the FBA number. In fact to better comply with the accounting laws, the CL entry may be broken into two pieces – the DVA, plus a new "funding" term equal to $FBA - DVA$. The net CL entry, however, is still FBA.
3. New Accounting Rules on DVA: DVA on payables is always recognized as a gain on the income statement and until 2012 could theoretically be considered a contribution to CET1. Under FCA/FBA accounting, this would ultimately lead to a net CET1 reduction relating to funding equal to the SFVA.
4. Payables DVA Contribution to CET1: In 2012, the DVA on payables was de-recognized as contributing to CET1 (Basel Committee On Banking Supervision (2012)). As the FBA is

basically re-classified as DVA, this effectively prevents the FBA from contributing to CET1 and sets the overall CET1 deduction for funding equal to the FCA.

5. DVA Cross Netting Set Re-hypothecation Benefit: As a consequence the re-hypothecation benefits across netting sets are ignored altogether in CET1. As discussed in Albanese and Andersen (2014), this has material impact on both accounting, management, and trading decisions.
6. FCA/FBA Accounting Compromise Solution: The compromise solution in FCA/FBA accounting includes the following:
    a. Enter FCA and unilateral CVA (UCVA) as CET1 deductions.
    b. Eliminate DVA on payables from accounts, replace it with FBA, and enter this amount as a contra-liability (CL) adjustment not contributing to CET1.
    c. Transfer UCVA and SFVA to clients.
    d. As we explain later in the paper, the end result is that the FTP's are struck at the indifference levels to the income.
7. Breakage of the Asset Liability Symmetry: Besides the issues that have already been mentioned, it is clear from the booking rules above that the FCA/FBA accounting implies a loss of asset-liability symmetry, since the DVA on payables is eliminated on favor of the FBA even though CVA is supplemented with (rather eliminated in favor of) FCA. The lack of symmetry is problematic from an accounting standpoint and contradicts FASB 159 adopted in 2007.
8. SFVA as a CET1 Deduction: A possible way around this asymmetry involves deducting SFVA (rather than FCA) from the equity capital. In this case the DVA double counting issue manifests itself by the fact that the SFVA inherits from the DVA a wrong-way sensitivity with respect to the own-credit of the bank, i.e., it may decrease (causing the CET1 to increase) whenever the bank credit deteriorates.
9. FTP Policies in FCA/FBA Accounting: As mentioned above, the funding related FTP policies in FCA/FBA accounting normally pass through the SFVA amount, i.e., include FBA benefits. Prior to 2012 rules, this could be argued to be reasonable from a share-holder perspective (as proxied by CET1, at least). Yet, since FBA has been currently demoted to the status of contra-liability that is not recognized in equity capital considerations, the FCA/FBA FTP policy induces deal-flow volatility to CET1.

10. Interpretation of the FCA/FBA FTP Policy: One way to interpret the effect above is that the FCA/FBA FTP policies are based on indifference pricing to the overall firm (including senior creditors), rather than just share-holders as is normally desired.
11. Inaccuracies of the FCA/FBA Accounting: In addition to the above-mentioned undesirable side-effects, numerical experiments show that the net FTP amounts are often too large in absolute value, despite the inclusion of the FBA benefits. This effect is due to large inaccuracies in modeling VM re-hypothecation and leads to incorrect firm-level hedge ratios for market risk.

Improvements Offered by the FVA/FDA Accounting

1. Proposals to Overcome the FCA/FBA Drawbacks: To overcome the shortcomings of the FCA/FBA accounting, Albanese and Andersen (2014) propose an accounting methodology that:
    a. Reflects and justifies Basel III regulatory requirements regarding counterparty credit risk;
    b. Is consistent with generally accepted accounting principles;
    c. Is consistent with the tenets of classical finance theory, such as the MMT and risk neutral valuation.

    Within this framework they then consistently value cash flow streams for VM funding and re-hypothecation strategies.
2. CET1 as a Shareholder Proxy: Their proposal fundamentally uses CET1 as a proxy for shareholder value and defines FVA as a discounted expectation of future funding costs occurring whenever there is an overall deficit of the VM at the book level. Future scenarios where there is a net excess of OTC collateral do *not* contribute to the FVA.
3. Corporate Finance Interpretation of FVA/FDA: A corporate finance interpretation of this FVA metric equates it to the present value of the wealth transfer from the bank shareholders to the bank senior creditors as a result of the bank entering into OTC trades with unsecured funding.

4. Validity of Asset Liability Symmetry: Importantly the FVA definition is such the funding adjustments are entirely divorced from the DVA on payables, wherefore the asset-liability symmetry still holds and the own-credit metric of the FVA has the correct sign.
5. Embedding of RHO in FVA: Since the RHO is embedded in the valuation of the FVA, the FVA amount is much smaller than the FVA amount in the case of portfolios of realistic size – about one third as large in the case study portfolio of Albanese and Andersen (2014).
6. Consistency of the FVA/FDA with MMT: In FVA accounting, the MMT is satisfied, as the FVA is accompanied by an offsetting CL adjustment which does not overlap with the DVA on payables. This contra-liability is named DVA2 by Hull and White (2014) and is referred to as FDA by Albanese and Andersen (2014). In the case of FCA/FBA accounting the term DVA2 is not meaningful because the approximations involved break the MMT and compromise a rigorous capital structure interpretation.
7. CVA Fair Valuation in FVA/FDA: Within the FVA/FDA framework, the fair valuation of CVA is most naturally a bilateral one (sometimes known as first-to-default CVA, or FTDCVA). As this fair value contains a DVA-like element of self-default benefit, guidelines in the Basel Committee On Banking Supervision (2012) suggest that FTDCVA cannot be directly deducted from CET1.
8. UCVA, FTDCVA, and CVA-CL Metrics: From the FTDCVA number a unilateral CVA (UCVA) number can be split out and it may be recorded as a contra-asset (CA) adjustment that is subtracted from CET1. The remaining “self-CVA” term is listed as a contra-liability (CL) and is to be excluded from CET1.
9. Elements of FVA/FDA Accounting - Summary: In summary the FVA/FDA accounting with rigorous RHO modeling includes the following elements:
    a. The UCVA and the FVA are both entered as CA adjustments and CET1 deductions recognizing the full benefit of the RHO to CET1;
    b. The DVA and the FDA are both entered as CL adjustments, as is the part of the FTDCVA that involves benefits from the bank defaults. None of the CL adjustments are to be counted for the CET1 purposes.
    c. The FTP is designed to immunize the CET1 from deal-flow volatility and to transfer the incremental costs of FVA and UCVA capital deductions to clients.

10. Income, CET1, and MMT Impact: Note that a) and b) preserve the standard CVA-DVA accounting at the net income level, as the FVA and the FDA adjustments cancel against each other. CET1, however, is affected by the funding costs.
11. Conservative Computation of the FTP: The FTP rule in the FVA/FDA accounting aims at preserving the CET1 capital, a principal that is fundamentally more conservative than the one followed in FCA/FBA accounting, where one only insists that new trades not have a negative impact on the income.
12. CET1 Impact from Deal Flow Volatility: In FVA/FDA accounting, deal flow still engenders volatility of the contra-liabilities (such as the DVA) as well as the fair value of the bank itself. However, due to our alignment of CET1 and the shareholder value, mitigating this volatility is irrelevant from the viewpoint of the shareholder.
13. FCA/FBA vs. FVA/FDA FTP Comparison: Notwithstanding the stronger FTP requirement, the fact that the RHO is properly modeled means that the FTP amounts obtained in the FVA/FDA accounting are generally quiet reasonable and often materially smaller than those in the FCA/FBA methodology.
14. FCA/FBA vs. FVA/FDA Derivatives Valuation: Relative to FVA/FDA, FCA/FBA accounting is observed to systematically undervalue the derivatives payables and over-value derivatives receivables, thus potentially giving rise to biased sub-optimal positioning of the OTC book.
15. Non-linear RHO Funding Set Contribution: It should be emphasized that under the FVA/FDA accounting, the notion of an individual unsecured trade price loses its meaning because all trades within the same funding set contribute non linearly to the RHO of the funding set. To a lesser extent, individual unsecured trade price loses its meaning in FCA/FBA accounting as well, since here the smallest possible additive unit is a netting set.
16. Valuation Across Entire Funding Sets: It is therefore possible to conclude that in order to correctly account for funding adjustments, one needs to value derivatives in the context of *entire* funding sets.
17. Funding Set Level Scenario Simulation: In FVA/FDA accounting, in order to account for collateral thresholds and to model re-hypothecation benefits correctly, whenever w new possible trade is priced, one needs to evolve dynamically the full portfolio valuation along with all the CDS curves for all counterparties and compute book level incremental statistics.

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CET1 Capital Deductions in Basel III and Capital Structure Considerations

CET1 Deductions

1. DVA as Full CET1 Deduction: The BCBS recommended in 2012 that the DVA be fully deducted from CET1.
2. BCBS Statement on DVA Impact: The relevant wording from the Basel Committee on Banking Supervision (2012) is:

   Therefore, after considering all the views, the Basel Committee is of the view that all DVA's for derivatives should be fully deducted in the calculation of CET1. The deduction of the DVA's is to occur at each reporting date and requires deducting the spread premium over the risk-free rate for derivative liabilities. In effect, this would require banks to value their derivatives for CET1 purposes as if they (but not their counterparties) were risk free, and to deduct unrealized gains both at the inception of the derivative trade and afterwards, when the creditworthiness of the bank deteriorates.
3. Credit Quality Impact On CET1: The BCBS rule ensures that a bank cannot claim increases in CET1 solely due to the deterioration in its own credit quality, consistently with the spirit of Basel III accord (Basel Committee On Banking Supervision (2011)).
4. Impact on CVA and FVA: While Basel Committee On Banking Supervision (2012) nominally only deals with the derivatives liability and the DVA, it is generally understood that the disallowance of the CET1 increases from deteriorating bank credit is a universal principal that extends to CVA and FVA as well.
5. BCBS Language Relevant to CVA: For instance, for CVA some relevant language is (from Federal Register (2014)):

   CVA equals the credit valuation adjustment that the bank has recognized in its balance sheet valuation of any OTC derivatives contract in its netting set. For the purposes of this *paragraph, CVA does not include any adjustments to CET1 attributable to changes in its own credit risk since the inception of the transaction with the counterparty.*

6. Unilateral CVA vs. Bilateral FTDCVA: From a modeling standpoint, adherence to this particular language can be achieved by deducting from the capital a unilateral UCVA metric as opposed to a smaller bilateral FTDCVA.

"Going Concern" or Defaultable Banks?

1. Regulatory Intent Behind the Deductions: The intent of the regulator regarding both the DVA and the CVA capital deductions is to exclude from CET1 the present value of the cash flows that benefit the bank after the default.
2. Regulatory Notion of "Going Concern": Care must be taken to not extend the regulatory exclusion to a fair value setting. For instance, the regulatory notion of valuation from the viewpoint of "going concern", in which the bank is assumed to be unable to default, is clearly at odds with both the reality and with the objective of consistent market pricing.
3. Unintended Consequence of "Going Concern": If taken literally and applied out the intended context, this no-default assumption has the unwanted and undesirable side effect of increasing the prices on the bank-issued debt and significantly lowering the bank's funding costs.
4. Funding Spread as a Liquidity Spread: We note in particular that if one were to consistently assume that the bank cannot default, then the spread separating the bank funding rates from the OIS rates would need to be interpreted as a liquidity spread. This is one of the possible financial interpretation behind the FCA/FBA accounting rules, along with an alternate approach based on modeling debt buy-back strategies.
5. Trouble with the Liquidity Spread Approach: The liquidity spread assumption is hardly defensible; typical funding spreads are in the range of 50-400 bp while typical liquidity spreads are below 5 bp.

6. Funding Spread without Credit Risk: It would be very difficult to construct a financial interpretation to funding spreads which does not involve the credit risk of the bank. The debt buy-back argument is more subtle and is discussed later.
7. No-default View on Funding Considerations: Besides making fair-value considerations awkward, the no bank default view is not reasonable for funding considerations either. Specifically the FVA is a fair valuation of a cash flow stream resulting from a funding strategy that the bank clearly cannot implement past its own default; once the bank is in a state of default, its funding spread ins infinite, and the bank is unable to borrow funds on an unsecured basis or to conduct most other trading activities. As such any correct measure for funding costs must inescapably reference default by the bank (and its counterparties, for that matter).
8. Wealth Transfer across the Capital Structure: From yet another angle, FVA admits the financial interpretation as an internal wealth transfer across the capital structure of the bank resulting from the implementation of a funding strategy and is not the price of an asset sold to the counterparty on which the bank can default. However, wealth transfers can stop once the default occurs and the equity holders are wiped out.

Categorization of Cash-flow Streams

1. Accounting Merger/Unification Viewpoint Framework: Since a straight "going concern" assumption is inadequate for evolving a comprehensive accounting framework, Albanese and Andersen (2014) put together an alternate framework which reproduces and justifies the regulator mandated CET1 deductions for CVA and DVA, but which is also meaningful from the viewpoints of funding costs, classical Finance Theory, and the generally accepted accounting principles.
2. Fundamental Cash flow Steam Types: For this purpose, Albanese and Andersen (2014) propose that the cash-flow streams be fundamentally classified into the following 5 types:
    a. CF1 => Contractually promised cash flow streams excluding all bank and counterparty credit risk events;

b. CF2 => Trade-related cash flows resulting from counter party defaults, but excluding bank default events;
c. CF3 => Trade related cash flows resulting from the bank default;
d. CF4 => Cash flows streams derived from dynamic trading strategies (such as funding strategies) implemented by the banks and taking place prior to the bank default;
e. CF5 => Cash flow steams deriving from dynamic trading strategies (such as funding strategies) implemented by the bank and taking place at or after the default of the bank.

3. <u>Derivative Contractual Agreement Cash Flows</u>: Any derivatives contract can always be split into separate contractual agreements generating cash flows to types CF1, CF2, and CF3.
4. <u>Dynamic Trading Strategy Cash Flows</u>: Similarly cash flows arising from any dynamic trading strategy can be modeled as a split between types CF4 and CF5.
5. <u>Cash Flows Contributing to CET1</u>: We assume that the splits have been carried out such that each unit of account referring to either a counter party contract or a trading strategy is matched to the relevant cash flow stream. We then designate the units of account whose underlying cash flow streams are of the types CF1, CF2, and CF4 as contributing to CET1, while units of account whose cash flow streams are of type CF3 and CF5 do not contribute to CET1.
6. <u>Counterparty Contract vs. Trading Strategy</u>: A key difference between a counterparty contract and a trading strategy is that the former is settled with a counterparty at the time of the bank default while the latter simply terminates at that point in time. The reason why the 2 cases are treated differently is that a contractually promised cash flow reflects an obligation by the bank that extends to the last maturity, independently of whether a bank defaults or remains a going concern until then.
7. <u>Trading Flows at Bank Default</u>: Cash flows deriving from trading strategies cannot be implemented past the time of the default of the bank, at which time the bank goes into receivership and is unable to carry out normal trading activities. Nevertheless the implementation of the trading strategies prior to default has consequences on a post-default basis which have an impact on the default claim held by senior creditors. These are cash flows of type CF4.
8. <u>Cash Flow Streams</u>:

a. Collateralized Transactions => Collateralized transactions involve the cash flow streams of type CF1 which are immune from counterparty credit risk.
b. UCVA => The UCVA refers to a cash flow stream of type CF2 and is, effectively, the price of a CVA protection contract promised by the bank, excluding the effects of bank default.
c. DVA => The DVA refers to cash flow streams of the type CF3 as it represents the benefit the senior creditors of the bank obtain from the default of the bank on derivative liabilities.
d. CVA CL => The CVA contra-liability is the DVA component of the CVA; like the ordinary DVA on liabilities it is a cash flow of type CF3.
e. FVA => The FVA can be interpreted as the price of the strategy of borrowing VM collateral up to the time the defaults (after which it becomes impossible to borrow any further). This is a cash flow stream of type CF4.
f. FDA => The FDA is the post-default benefit to the senior creditors deriving from owning a title to the portfolio of derivative receivables whose hedges were funded on an unsecured basis. The FDA corresponds to a cash flow steam of type CF5.

9. Corporate Finance Interpretation of CET1: The cash flow rules above are designed to allow for a Corporate Finance Interpretation of CET1 as a proxy for the value of the bank assets to the shareholders – or at least the contributions to CET1 from derivatives trading activities.
10. Cash Flows after Bank Default: In particular, since the shareholders are indifferent to the cash flows occurring at or after the bank default, such cash flow streams should not contribute to CET1 – excluding here the feedback effects of the type discussed in Burgard and Kjaer (2011). Also note that by ensuring that the cash flow types of CF3 have no impact on CET1, we achieve the stated regulatory objective of deducting both FVA and UCVA from CET1.
11. RHO Exercise Impact on CET1: By the same token, funding costs are assessed at the market level, and the benefits resulting from the exercise of the RHO at times prior to the bank default contributes positively to CET1.
12. Self-Default Benefit - Unilateral CVA: One particular rationale for the regulatory mandate to deduct full UCVA from CET1 was to avoid the scenario where an increase in the bank's own credit spread could lead to a higher CET1.

13. Bank Default Continuation Funding Spread: One may ask whether a similar principle applies to FVA, necessitating the definition of a unilateral FVA. However such a remedy would be difficult to justify within a consistent modeling framework, since we would have to introduce the notion of continuation funding spread for the bank at and after its own default.
14. Peer Proxy Continuation Funding Spread: While such a notion could potentially be based on average or minimum peer spreads (post bank default), satisfying the Basel III principles does not depend on it as the FVA defined here normally increases as a function of the bank's own credit spread.
15. Credit Spread Impact on FVA: The FVA is impacted by rising bank credit spreads in 2 different directions; rising spreads tend to decrease FVA because funding costs are cut short by a bank default; yet rising spreads also tend to increase the FVA as the spread paid on unsecured lending increases. Albanese and Andersen (2014) show that in normal circumstances the latter effect dominates.
16. Own-Credit Sensitivity in FCA/FBA: Albanese and Andersen (2014) also show that this *not* the case for SFVA, which is the metric for funding costs that are transferred to the clients in the form of FTP in FCA/FBA accounting. Since the SFVA can be shown to have own-credit sensitivities of the wrong sign when applied to portfolios containing mostly payables, its use as a CET1 deduction would be problematic from a regulatory standpoint. For this reason alone, the CET1 deduction in FCA/FBA accounting must be the full FCA.

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Accounting Principles, Units of Accounts, and Valuation Adjustment Metrics

Accounting Rules

1. Accounting Principles for Derivative Portfolios: We list here the key accounting principles for derivatives portfolios. KPMG (2011) and PWC (2011) contain a discussion on them.
2. Units of Account:
   a. Trading Securities (AP0) => Derivatives contracts are categorized as trading securities and must be listed on the balance sheet on their fair market value. Changes to the fair market value are registered as income or loss on the income statement.
   b. Units of Account and Linearity (AP1) => Financial portfolios are decomposed into elementary *units of account* satisfying a linearity property, in the sense that the fair valuation of the portfolio is the sum of the fair valuations of the units of account.
   c. Fair Value as an Exit Price (AP2) => The *fair value* of a unit of account is defined in IFRS13 as the price that would be received if one were to sell the unit of account in an orderly trade between the market participants at the measurement date. This is called the *exit price* of the unit of account. In the determination of the exit prices neither transaction costs (such as broker commissions) nor entity-specific production costs are considered as part of the fair valuation.
   d. Model Based Valuation (AP3) => When a unit of account is not tradeable in the active market, its fair value may be determined using a model based valuation technique. In this case an income based approach is allowed, where the cash flows of the trades present valued. This includes cash flows associated with the default of one of the counterparties.
   e. Symmetry Principle (AP4) => When (as is often the case) there is no active market for the liability transfer, IASB accounting standards require that the fair value of the liability be determined from the perspective of the asset investor.

f. Non-Performance Risk (AP5) => According to FASB 157, the benefit to senior creditors on the non-performance of the liabilities should be captured. Non-performance risk includes the effects of credit risk, as well as any other factors the influence the likelihood of fulfilling contractual obligations.

3. Application to Derivative Security Accounting: To consider the application of these principles for derivatives securities accounting, Albanese and Andersen (2014) consider the situation where a bank B transacts in partially collateralized OTC derivatives with a set of n credit-risky counterparties.
4. Collateral Posting and CSA Agreements: Each counterparty holds a portfolio of derivatives under an over-arching CSA agreement involving a netting clause, and possibly, but not necessarily, collateral posting obligations for the variation margin (VM).
5. ISDA 2009 Closeout Amount Protocols: In the absence of full collateralization, closeout protocols in place to govern settlements whenever a bank or a counterparty defaults need to be explicitly considered; standard protocols are the ISDA Market Quotation Close-out Convention of 1992 (most common) and the ISDA Close-out Amount Protocol of 2009. Note that closeout protocols are relevant for the fair valuation because of the accounting principles AP3 and AP5 above.
6. Decomposition of the Bank OTC Portfolio: Before discussing any valuation metrics, it needs to be clarified how the bank OTC portfolio is linearly decomposed into units of account.
7. Decomposition of the Unsecured Derivative Contract: The above question is non-trivial as an unsecured derivative contract in isolation generally cannot be considered a proper unit of account= because netting clauses cause valuations to depend on the netting set to which the trade belongs.
8. Funding Sets vs. Netting Sets: The RHO for VM causes valuation to depend on funding sets defined as the largest set of trades among which re-hypothecation is possible. Notice that funding sets can cut across netting sets as it is entirely possible that the VM is re-hypothecated separately across distinct business lines contributing to the same netting sets.
9. Fully Collateralized Counterparty Derivative Trades: Fully collateralized trades done with traders or with CCP's generally have negligible credit risk, and shall be considered here to be default free; fir valuation of such trades is additive at the trade level. The valuation of the

unsecured derivative assumed to be fully collateralized in cash is referred to as its *default-free valuation*.

10. Bank Counterparty Credit Valuation Adjustments: The impact of the counterparty and the bank credit risk on valuations is then captured by other units of accounts called *adjustments*. Adjustments make reference to sub-portfolios as opposed to individual trades and can be interpreted as the valuation of derivatives referring to sub-portfolios as underlying.
11. The Asset Account (A): Albanese and Andersen (2014) construct a reference to stream-lined accounting framework of OTC derivatives based on six balance sheet accounts. The Asset Account (A) refers to the receivable units of account referring to cash-flow streams of the type CF1.
12. The Liabilities Account (L): This balance sheet account includes payable units of account referring to cash flow streams of type CF1.
13. The Contra Asset Account (CA): This balance sheet account includes payables adjustments with underlying cash-flow streams of types CF2 and CF4; these are deducted from CET1.
14. The Contra Liabilities Account (CL): This balance sheet account includes receivable adjustments referring to the cash flow streams of the type CF3 and CF5; these do not contribute to CET1.
15. The Retained Earnings Account (RE): This balance sheet account includes provisions that are set aside to meet future obligations such as funding costs and credit default losses. These entries do contribute to CET1.
16. The Equity Account (PFV): This balance sheet account is defined in such a way that the basic accounting equation

$$A + RE - CA = PFV + L - CL$$

is satisfied. The Equity Account has the meaning of Portfolio Fair Valuation and the variation of the PFV over an accounting period is called the *Income*.

17. The CET1 Capital Measure Components: The CET1 is a capital measure that requires the exclusion of the value of units of accounts referencing cash-flows of the type CF3 and CF5 taking place at or after the time of default of the bank B. As those types of cash flows are captured in the CL account, we exclude this account from the common equity and write

$$CET1 = PFV - CL = A - L - CA + RE$$

18. CET1 as Bank Shareholder Value: Within their framework, Albanese and Andersen (2014) interpret CET1 as the value of the bank to the shareholders, while PFV is the combined value to the shareholders and the senior creditors. Here the term “senior creditors” is used to refer to a class of creditors which is different from collateral lenders, and which either have priority or are at the same level of seniority as the collateral lenders.
19. Computation of the CET1 Deduction: Upon entering into a derivative transaction, the CET1 is subjected to an incremental deduction denoted by ΔCA and is augmented through earnings by the FTP amount received from the clients over and above the default-free valuation. Attributing the FTP to the Retained Earnings (RE) account the CET1 variation is given by

$$\Delta CET1 = -\Delta CA + \Delta RE$$

20. Computation of the Income Increment: A nee trade also affects the CL adjustments by an incremental amount ΔCL, an amount tied to the benefits of the bank default and therefore only having an impact on the senior creditors’ wealth. The ΔCL term is excluded from CET1, but affects the income and the fair valuation of the bank.

$$\Delta PFV = \Delta Income = \Delta CET1 + \Delta CL = -\Delta CA + \Delta CL + \Delta RE$$

Contra-Asset and Contra-Liability Accounting for Credit Risk

1. CVA for the i^{th} Netting Set: To examine the CA and the CL accounts more closely, consider first the fair valuation of the credit risk for the i^{th}netting set, denoted by CVA_i. The quantity can be interpreted as the value of the default protection contract implicitly sold by the bank to the counterparty i, with the notional set to the netting set value at the time of the counterparty default.

2. Decomposition of the CVA Components: As shown in Albanese and Andersen (2014), the precise valuation methodology to be used depends on the closeout rules. The total CVA is computed by summing the CVA_i across n netting sets and is commonly split into 2 components:

$$CVA = FTDCVA = UCVA - CVA_{CL}$$

3. The Unilateral CVA Component – UCVA: The UCVA component is booked as a CA adjustment and is a unilateral CVA metric independent of the closeout rules, i.e., it is the present value of all the counterparty credit risk losses resulting from the default of the counterparty computed under the assumption that the bank does not default.
4. The CL CVA Component CVA_{CL}: The CVA_{CL} is loosely speaking the DVA component of the CVA, i.e., it is the benefit the bank senior creditors receive at the time of the bank default by, in effect, no longer accepting to sustain future counterparty credit losses. It is booked as a CL adjustment. The magnitude of the CVA_{CL} is closely linked to the closeout specifications; Albanese and Andersen (2014) give the relevant equation for CVA_{CL} using the ISDA 1992 closeout rules.
5. Non-Performance Risk Accounting Principle: According to the accounting principle AP5 on non-performance risk, cash flows taking place at or after the default of the bank should be present valued and accounted for.
6. Symmetry Principle Applied towards DVA: By virtue of principle AP4, the value of the default protection sold implicitly by the unsecured counterparty to the bank should be valued as the CVA assessed by the counterparty against the bank. This amount is the DVA for payables, the reporting of which was mandated by FASB 159 in 2007.
7. DVA Impact on CET1 Numbers: DVA enters accounts as a CL adjustment, and as is references cash flows ensuing a bank default, it is excluded from CET1. A split such as the one seen for CVA is not meaningful for DVA, as this quantity only involves post-default cash flows (with no direct relevance to equity holders and to the capital).
8. "Going Concern" Impact on the CVA: It should be noted here in passing that not all banks account for the CVA_{CL} term, as they effectively equate the CVA with the UCVA term. While this "going concern" definition is convenient in a number of ways (e.g., regulatory and

accounting definitions of CVA are better aligned), it is hard to argue that it is correct or consistent with DVA accounting.

Contra-Asset and Contra-Liability Accounting for Funding

1. CA and CL Credit Risk: Accounting for the credit risk through the UCVA (contra-asset entry) and the $DVA + CVA_{CL}$ term (the contra-liability term) as seen before is a fairly well-established practice even if there are minor differences in the way banks sometimes define CVA and DVA.
2. Supplementing the CVA and the DVA Metrics: As discussed before, recently the CVA and the DVA risk metrics have been supplemented by the quantities meant to account for funding cash flows to which the banks are subject to through their postings of the VM on the hedges.
3. Nature of the Posted VM: VM is normally paid in cash (or to a lesser extent) in highly liquid short term government debt and maybe re-hypothecated across trades within the same funding set. The bank posts VM on collateralized derivative liabilities and receives VM on collateralized derivative assets.
4. Funding Adjustment Unit of Account: For funding costs, we work with a unit of account at the funding set level and define FVA as the discounted value of the book level funding costs arising whenever the funding set has a net collateral deficit. Future states of the world whereby the funding set is a net receiver of the VM are modeled as *not* contributing to funding assets.
5. Handling the Different VM States: On the one hand the excess cash deposited as VM can earn a riskless rate of OIS. But on the other hand, derivative counterparties pledging VM are also entitled to interest rate payments at a matching OIS rate. In total, the states of the world where the bank enjoys accumulation of excess VM collateral are modeled as having zero funding costs.
6. Alternate VM Cash Management Strategies: As discussed in Albanese and Andersen (2014), the zero benefit assumption around excess collateral maybe considered a conservative assumption, and some researchers have considered strategies where the bank buys back long-term debt with excess VM (Burgard and Kjaer (2011a, 2011b, 2011c, 2013)).

7. VM Strategies Capital Structure Impact: However these strategies are not very difficult to implement in practice because the VM is very volatile, they also have zero impact on income if one insists on MMT consistency. As a consequence, if the deleveraging transactions occur at fair valuation, wealth is not transferred between share-holders and the senior creditors, i.e., the benefit is not truly there.
8. Origin of the Funding Cost Impact: Neglecting basis spreads, funding costs for the VM procurement are non-zero because the collateral lenders receive only partial recovery upon bank default. If we denote the recovery rate to collateral lenders with ${R_B}^C$, then the case where there is a perfectly functioning REPO market for derivatives would have

$$ {R_B}^C = 1 $$

9. FVA Definition and CET1 Impact: In reality

$$ {R_B}^C < 1 $$

   i.e., recovery is only partial because of market inefficiencies and the spreads for collateralized borrowing are very close the spreads for unsecured borrowing. The FVA is defined as the discounted expectation of the funding costs up until the time of bank default. Hence the FVA is booked as a CA adjustment and a CET1 deduction.
10. Origin of the Funding Benefit Impact: When short term debt is issued to fund the VM collateral, the lenders providing the funds are exposed to the bank default risk. The flip-side of the risk is the DVA-like benefit held by the bank. To account for it, we introduce an FDA entry as the present value of the depreciation of the collateral debt at the time of the bank default, due to incomplete recovery.
11. FDA Definitions and the CET1 Impact: That is the FDA is the present value of $1 - {R_B}^C$ times the notional borrowed for VM funding purposes, received at the time of the bank default. Since the FDA makes reference to cash flows happening at or after the bank default, the FDA is booked as a CL adjustment and excluded from the CET1.
12. Shareholder Debt Holder Transfers: For the MMT to hold, the FVA must equal the FDA. In this case, the funding strategies, and in particular the value of ${R_B}^C$ do not affect the fair

valuation of the bank. However if ${R_B}^C$ is strictly less than 100% then the deal flow induces a wealth transfer from the shareholders to the senior creditors.

13. FVA/FDA Cash Flow Transfer View: In accounting terms if we decrease the value of ${R_B}^C$ from 1 down to 0, a portion of CET1 is gradually demoted to the status of contra-liability, reflecting the wealth transfer from the shareholders to the senior creditors. The FVA may therefore be considered the wealth transfer lost from the shareholders, while the FDA is the amount earned by the senior creditors.

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Accounting Cash Flows

Accounting Cash Flow Setup Framework

1. XVA Metrics Valuation Formulas Setup: Having so far been limited to a largely qualitative accounting discussion, the next step is to follow the treatment set out in Albanese and Andersen (2014) and proceed to provide concrete valuation formulas for the XVA metrics. For this the precise funding and credit related flows taking place in OTC derivatives trading need to be considered, both before and after the default of the counterparties and the bank itself.
2. Modeling Details in the XVA Computation: Since elaborate cash flow details can obscure the main concepts, this section omits certain less essential minutiae, such as flows associated with ratings dependent CSA thresholds and credit-risk sensitive closeout conditions. This is later followed by a more complete cash flow representation, much of which is implemented in the case study carried out by Albanese and Andersen (2014).
3. Transmitting XVA Charges to Clients: Whenever the bank enters into an unsecured trade, the XVA adjustments are accompanied by charges to clients which can be structured in various ways. In their treatment Albanese and Andersen (2014) assume that the XVA related FTP payments are simply upfront and due at the inception of the trade in question.
4. Nature and Type of FTP: As described earlier, business line trading desks, in fact, typically pay FTP charges to the CVA/CFD desks on an upfront basis. However, assuming upfront structure for the client charges is a stylized approximation, since often costs are paid in the instalments embedded in the unsecured derivative structure itself.
5. Default Free Counterparty Portfolio Value: Working within the modeling framework briefly described before, let us introduce the notation

$$V_i^U(t) \; \forall \; i = 1, \cdots, n$$

for the default valuation at the time t of the portfolio held with the counterparty

$$i = 1, \cdots, n$$

computed by neglecting all funding and credit risks including that of the bank itself.

6. Fully Collateralized Netting Set Value: That is, ${V_i}^U(t)$ represents the value of the netting set i in a world where all trades are collateralized in full. ${V_i}^U(t)$ is set to 0 in case the i^{th} counterparty is in a state of default at time t.
7. Counterparty Value Netting Set Additivity: Assuming no closeout risk and no initial margin, the fair valuation of fully collateralized books is additive over individual trades and therefore also additive over netting sets, i.e., we can meaningfully define a total default-free portfolio value of

$$V^U(t) = \sum_i {V_i}^U(t)$$

8. Hedging of the Unsecured Trade: For simplicity's sake for now we assume that the unsecured trades are hedged on a precise back-to-back basis with the hedge trades having a valuation ${V_i}^H(t)$ equal to exactly $-{V_i}^U(t)$. Indeed, one way to interpret the role of the CFD and the CVA desks is that, after the appropriate compensation, they allow the lines of business desks to hedge risk as if there were not funding or credit risk.
9. Value of the Hedge Trades: The common value of the default-free unsecured trades and hedges is denoted as follows:

$$V_i(t) = -{V_i}^H(t) = {V_i}^U(t)$$

It is also assumed for now that neither the bank nor the counterparty post any VM to each other; later these hypotheses are relaxed and extensions including partial VM postings with CSA collateral thresholds are considered.

10. VM Re-hypothecation Across Hedge Trades: Banks typically have a separate funding set for each business line and jurisdiction. Within each such dedicated book, banks always exercise

the RHO for the VM across hedges (albeit not necessarily optimally – collateral systems of many banks tend to be rudimentary).

11. Net VM Across Hedge Trades: The net VM posted at any given time is given by the sum

$$C_{VM}(t) = \sum_i C_{VM,i}(t)$$

where

$$C_{VM,i}(t) = V_i(t)\mathbb{I}_{t<\tau_B}\mathbb{I}_{t<\tau_i}$$

(this is a simplification, as there is typically a material VM from the CVA hedges – this complication is discussed down later).

12. Impact of the Bank/Counterparty Default: Notice the presence of the indicator functions $\mathbb{I}_{t<\tau_i}$ and $\mathbb{I}_{t<\tau_B}$ above, reflecting the fact that the default of either the bank or the counterparty results in an immediate settlement of collateralized hedges. The sign convention is that if $C_{VM}(t)$ is positive (negative) then the bank is a net poster (receiver) of the collateral on the hedges.
13. Assumption of the REPO Rate Value: For simplicity's sake it is assumed that the difference between the OIS rate and the REPO rate for general collateral is quantitatively immaterial and denote both rates with $r_{OIS}(t)$.
14. Recovery Rates Across Debt Classes: Another assumption is that the bank has at least two classes of debt. One is unsecured senior debt with recovery rate R_B used for regulatory costs, initial margin, and administrative costs. The other is debt used to finance VM collateral imperfectly secured by derivative receivables. This second class of debt is modeled as having a recovery rate ${R_B}^C$, a funding rate equal to $r_B(t)$ and a spread over OIS equal to

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

15. Estimation of the Funding Spread: If there existed an efficient REPO market for unsecured OTC derivatives, collateral lenders would be guaranteed a full recovery on the VM, i.e., we would have

$$R_B{}^C = 1$$

and the funding spread would be 0. In general

$$R_B{}^C \leq 1$$

and the risk neutral valuation of the overnight funding spread is given by

$$s_B(t) = \left(1 - R_B{}^C\right)\lambda_B(t)$$

where $\lambda_B(t)$ is the probability of the rate of default at the time t (Lando (1998)). In practice an additional liquidity spread may apply, but this basis is ignored for the present discussion.

    a. FVA Spread vs. FDA Spread => As pointed out by Morini and Prampolini (2011), FVA/FDA indifference assumes that the CDS-bond basis is zero. This is not the case in practice, since FVA is calculated from bond yield spread whereas FDA should, in theory, be calculated from the CDS spread.

16. Hybrid Debt Class Type Assumptions: It needs to be stressed that the assumption that the debt is divided into 2 classes is only formal and does not restrict the generality of the argument. In the general case one can still assume that the traded debt securities are hybrids between the theoretical bonds of the two types considered.

Cash Flows Related to VM Funding

1. Ignore CVA-related Funding Costs: We start by considering the funding flows on the interval $[t, t + \Delta t]$. To simplify the exposition the funding costs for any VM arising from the default hedges that the CVA desk may have entered into are ignored.
2. Borrowing Cost for the VM Funding: Suppose first that the net VM collateral $C_{VM}(t)$ posted by the bank is *positive*, whereby the bank needs to borrow to fund its overall VM position. In this scenario the bank treasury is assumed to issue short-term unsecured debt into the market to raise the necessary funds.
3. Funding Rate on the Collateral: More specifically, if in the time interval $[t, t + \Delta t]$ the bank has to fund a net collateral shortage

$$C_{VM}(t) > 0$$

the treasury issues $C_{VM}(t)$ worth of short-term debt for this purpose, either unsecured or backed by derivative receivables. Then interest charge on the unsecured debt in the time interval $[t, t + \Delta t]$ at the CFD funding rate $r_B(t)$ is $C_{VM}(t) r_B(t) \Delta t$.
4. CFD Role in the Bank Setup: The required VM collateral is then routed through the inter-dealer positions where an interest-rate amount of $C_{VM}(t) r_{OIS}(t) \Delta t$ is received back. As illustrated pictorially in Albanese and Andersen (2014), when

$$C_{VM}(t) > 0$$

the CFD experiences a net negative cash flow in the amount of $-C_{VM}(t) s_B(t) \Delta t$. FVA is defined as the present value of this negative carry over the lifetime of the funding set or until the time of bank default, whichever comes first.
5. Interest Rate on VM Receivables: In case the total collateral requirement for the VM is *negative*, i.e.

$$C_{VM}(t) < 0$$

the treasury OTC funding program is not called upon. In this case we assume that the bank treasury would invest the excess collateral in short-term securities, yielding on average (nearly) OIS levels. As the bank is liable to paying OIS on hedge counterparties on the net collateral received, the bank is therefore assumed to receive no benefits and face no costs due to VM posting obligations when

$$C_{VM}(t) < 0$$

6. Handling the Net Excess Collateral: In principle one can imagine that the net excess collateral would be passed by the CFD to the treasury, which would in turn use the funds to retire outstanding long-term deb. If this were the case on would imagine that the CFD receives from the treasury a benefit based on the interest-savings on the long-term debt, resulting in positive carry.
7. Rapid Fluctuation of the VM Levels: The above assumption is an aggressive one as VM varies greatly in short time scales and generally constitutes an unstable base from which to retire debt. In practice it is far easier to let debt mature than to retire long-dated bonds on the secondary market.
8. Art of Treasury Liquidity Management: Normally collateral is over-provisioned allowing for buffers as the treasury needs to manage liquidity prudently. There are additional costs to collateral over-provisioning and risk management of the liquidity buffers, all of which are ignored here.
9. Funding Rates and Strategies - Assumption: Besides the above "neutral" assumptions about investment returns, the definition of FDA used here makes additional assumptions about treasury funding strategies and funding rates. The specified ones are listed below.
    a. Worst-case Funding Rate Scenario => The worst scenario regarding funding rates is assumed, i.e., that derivative receivables cannot be passed as collateral and that VM borrowing is entirely unsecured.
    b. Best Case Over-provisioning Scenario => The best-case scenario regrading over-provisioning is also assumed, that is the bank procures on an overnight basis only the cash collateral that is strictly needed for VM borrowing, not more.
10. FVA and Funding Strategy Link: Both of the above assumptions are reasonable for accounting purposes, although they admittedly represent idealizations of the actual funding behavior (which no doubt varies from bank to bank). In this view, it is important to note that

FVA calculations are non-unique and tied to the *actual* funding strategy to which a firm commits, a fundamental difference from the ideas underlying fair value pricing of derivatives.

11. Derivative Contract Fair Value Theory: According to classical finance theory, if there exists a replication strategy, then the cost of replication/hedging equals the fair value. Hence if one replication strategy is theoretically possible, then the cost of implementing it ought to equal the fair value *whether or not* the strategy is actually implemented. If more than one replication strategy can potentially be implemented, absence of arbitrage indicates that the cost of implementing each one equals the fair value.
12. MMT on Funding Cost Impact: For funding cost valuation, the MMT tells us that the fair value of funding strategies for trades entered at fair value is zero, as FVA does not represent the fair value of an asset, but instead the wealth transfer amount from the shareholders to the senior creditors.

Cash Flows at Counterparty Default

1. Counterparty Triggered Default Flows: If τ_i is the default time of the counterparty i, let $D_i(\tau_i)$ be the default cash flow received from the bank from the counterparty i as part of the default triggered closeout of the i^{th} portfolio. $D_i(\tau_i)$ is governed by the ISDA agreement between the bank and the i^{th} counterparty.
2. ISDA Standard Closeout Assumption: For simplicity's sake it is assumed that the closeout procedure follows the ISDA Market Quotation Protocol of 1992, and extensions to the other protocols are covered later.
3. Applying the ISDA 1992 Protocol: In the prevailing interpretation of the ISDA 1992 closeout protocol, if either the counterparty or the bank itself default prior to the maturity of the trade portfolio, then the portfolio is settled at default free levels, i.e., XVA adjustments are excluded from the calculation of the settlement amount.
4. Counterparty Default Cash Flow: In other words, the default cash flows at time τ_i received the bank from counterparty i is

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{V_i(\tau_i)\geq 0}R_iV_i(\tau_i) + \mathbb{I}_{V_i(\tau_i)<0}V_i(\tau_i)\right]$$

where τ_B is the default time of the bank and

$$R_i \in [0, 1]$$

is the recovery rate received from the counterparty i. Observe the presence of the default indicator $\mathbb{I}_{\tau_i<\tau_B}$ in this expression, a reflection of the fact that if the bank defaults prior to the counterparty i, then no cash flow takes place at time τ_i since the unsecured derivative portfolio held with counterparty i is assumed to have been unwound at τ_B.

5. Sign of the Counterparty Default Cash Flow: $D_i(\tau_i)$ may be positive or negative, and can be considered an inherent part of the portfolio flows of counterparty i. Notice that if

$$V_i(\tau_i) \geq 0$$

and

$$\tau_B > \tau_i$$

the above equation represents a loss to bank B. The present value of this loss is the CVA_i for portfolio i.

6. Hedging the Counterparty Risk: As indicated earlier, the cash flow formula above assumes that the counterparty credit risk above is left unhedged. In reality the CVA desk seeks to layoff the counterparty credit risk through the purchase of credit hedges such as single names and index CDS.
7. CVA Hedge Collateral VM Pool: This setup does not alter our conclusions about how to account for CVA, but it does mean that CVA hedges are typically entered with dealers/exchanges on a fully collateralized basis, which in itself produces VM postings that can be introduced into the overall VM "pool" generated by the client trades.

Cash Flows at Bank Default

1. Collateral Impact on Bank Default: At the bank default time τ_B, any posted collateral amount stays with the holder, and all the inter-dealer positions are torn up, with no economic impact to any dealer counterparty. Similarly the default of a dealer used as a counterparty for hedging purposes has no impact as all credit risk is covered by collateral.
2. Explicit Cash Flows: The positions with n counterparties are typically settled at ISDA terms, leading to a loss for those counterparties that have a previous exposure to the bank. Under the standard ISDA 1992 terms, the effective default cash flow at time τ_B received by the bank from counterparty i may be written as

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{V_i(\tau_B) \geq 0} V_i(\tau_B) + \mathbb{I}_{V_i(\tau_B) < 0} R_B V_i(\tau_B)\right]$$

where

$$R_B \in [0, 1]$$

is the recovery rate for the bank B.

3. Credit Losses to Counterparty: Notice that the loss to counterparty is therefore $-(1 - R_B)V_i(\tau_B)^-$. Senior bank creditors, who in aggregate have a claim on derivative receivables of value $V_i(\tau_B)^-$ for the i^{th} counterparty, would similarly recover a fraction $R_B V_i(\tau_B)^-$ of their claim.
4. DVA Gain to the Bank: Again for the sake of simplicity it is assumed that the recovery rate on the senior debt and on the derivatives portfolio are identical. Notice that if

$$V_i(\tau_B) < 0$$

then the equation for $D_i(\tau_B)$ represents a gain for the bank relative to a fully collateralized payout of $V_i(\tau_B)$. The present value of this gain is the DVA_i associated with counterparty i.

5. Non additivity of the VM Collateral: Finally note that if is the recovery rate for the VM collateral lenders, upon defaulting the bank does not return (i.e., effectively receives) the following amount in VM cash:

$$D_i(\tau_B) = (1 - {R_B}^C)\left[\sum_i V_i(\tau_B)^-\right]$$

Since the total amount of VM collateral borrowed is reduced by exercising the RHO, the amounts lent and recovered from the collateral lenders cannot be represented as a linear sum over netting sets.

References

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- Lando, D. (1998): On Cox Processes and Credit Risky Securities **eSSRN**.
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Credit and Funding Valuation Adjustments

Introduction

1. Cash Flow Impact On Accounting: Having outlined all relevant cash flows above, the next step is to work out how these cash flows are valued, and how they affect the capital structure of the OTC book. In particular, this section looks at in detail the relevant XVA metrics needed for the accounting treatment outlined earlier.

CVA and DVA

1. Bilateral FTD CVA Formulation: The cash flows triggered by the counterparty defaults were under ISDA 1992 were discussed earlier. The loss in value due to counterparty i is measured by the present value of the protection against the value loss inherent in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{V_i(\tau_i) \geq 0} R_i V_i(\tau_i) + \mathbb{I}_{V_i(\tau_i) < 0} V_i(\tau_i)\right]$$

2. Expression for CVA_i or $FTDCVA_i$:

$$CVA_i = FTDCVA_i = \mathbb{E}\left[e^{-\int_0^{\tau_i} r_{OIS}(u)du}\mathbb{I}_{\tau_i<\tau_B}(1-R_i)V_i(\tau_i)^+\right]$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t r_{OIS}(u)du}\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]\mathbb{Q}(\tau_i \in [t, t+dt])$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)\mathbb{I}_{t<\tau_B}(1-R_i)V_i(t)^+\right]dt$$

$$= \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)+\lambda_B(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

3. Applying a Stochastic Default Intensity: Here $\mathbb{E}[\cdot]$ denotes the expectation in the risk neutral measure $\mathbb{Q}$ and $\lambda_i(t)$ is the (possibly stochastic) default intensity for counterparty i. The concept of default intensity was introduced by Lando (1998) in the context of reduced for models based on Cox processes, but has become meaningful (and quite useful) also for many structural processes.
4. Unilateral CVA Contra-Asset Adjustment: As discussed in the previous section, the CVA entry to be booked as a contra-asset is the unilateral UCVA metric given by the expression

$$UCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

   without the indicator function $\mathbb{I}_{t<\tau_B}$.
5. CVA Contra-Liability Adjustment Expression: This entity is accompanied by a CL adjustment defined as follows:

$$CVA_{CL,i} = UCVA_i - FTDCVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t\geq\tau_B}\lambda_i(t)(1-R_i)V_i(t)^+\right]dt$$

6. DVA Contra-Liability Adjustment Formulation: The $CVA_{CL,i}$ can be interpreted as the "DVA of the CVA", i.e., the benefit the senior creditors have on the default of the bank on the

default protection contract implicitly sold to the counterparties. The more substantial benefit associated with the option of defaulting on the underlying derivatives is instead captured by a DVA term defined as

$$DVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \mathbb{I}_{t<\tau_i} \lambda_B(t)(1-R_B)V_i(t)^-\right] dt$$

7. Valuation Across Netting Sets: CVA and DVA numbers maybe added across counterparties giving rise to

$$FTDCVA = CVA = \sum_i CVA_i$$

$$UCVA = \sum_i UCVA_i$$

$$CVA_{CL} = \sum_i CVA_{CL,i}$$

$$DVA = \sum_i DVA_i$$

8. Accounting of CA/CL Adjustments: The booking of these quantities as CA and CL adjustments was covered earlier and is tabulated below.

FVA and FDA

1. Funding Payments Cash Flow Stream: As explained earlier, to find fair valuation expressions for the FVAs, one needs to value a continuous cash flow stream for funding costs at the rate given by the spread in

$$s_B(t) = r_B(t) - r_{OIS}(t) \geq 0$$

It was argued earlier that the contribution to this cost in the time period $[t, t + \Delta t]$ is $C_{VM}(t)^+ s_B(t)\Delta t$.

2. Funding Cost Valuation Expression: The discounted present value of the future funding costs is defined as follows:

$$FVA = \mathbb{E}\left[\int_0^\infty e^{-\int_0^t r_{OIS}(u)du} C_{VM}(t)^+ s_B(t)dt\right]$$

$$= \mathbb{E}\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ s_B(t)dt\right]$$

where the second equality follows form the definition of the net collateral $C_{VM}(t)$ in

$$C_{VM}(t) = \sum_i V_i(t)\mathbb{I}_{t<\tau_i}\mathbb{I}_{t<\tau_B}$$

As explained earlier we include the cash flows only up to the time of the default of the bank.

3. Accommodating the Timing Element of Default: The are a plethora of competing definitions of FVA, some of which deliberately avoid the complication of a default timing element. Carver (2013) contains details on the "dark art" of FVA definitions.
4. Funding Spreads for VM Receivables: Due to the lack of infrastructure to guarantee water-tight collateralization mechanics with unsecured derivatives receivables as underlying, funding spreads for VM collateral are observed to be near unsecured levels even in the rare cases where derivatives receivables are nominally mentioned as collateral.
5. PV of the Funding Benefits: The present value of the gain that senior creditors of the bank gain due to the inability of the collateral lenders to recover in full on default is referred to as FDA here. To compute this quantity, observe that at the time of a bank default, senior

creditors will gain a benefit equal to the fraction $1 - {R_B}^C$ of the pool of derivative receivables, while a fraction ${R_B}^C$ goes to the collateral lenders.

6. FDA Valuation Adjustment Formulation: From the above we have that

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t r_{OIS}(u)du}\left(1 - {R_B}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$= \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - {R_B}^C\right)\lambda_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

Thanks to the risk neutral valuation relation

$$s_B(t) = \left(1 - {R_B}^C\right)\lambda_B(t)$$

for the bank credit spreads, we find

$$FVA = FDA$$

consistent with the discussion in the earlier section.

7. MMT Consistency of FVA/FDA Accounting: The FVA and the FDA contribute with opposite signs to the bank fair valuation and cancel each other on the bank balance sheet, in agreement with the MMT. In other words, the fair valuation of the bank assets is indifferent to the funding strategy employed, and therefore to the FVA and the FDA. This relation was first noticed in Hull and White (2012), where the FVA is called DVA2 and the equation

$$FVA = FDA$$

was stated as a direct consequence of the MMT.

FCA and FBA

1. Motivation Behind the FCA/FBA Accounting: FCA/FBA accounting is an approximation to the funding value adjustment motivated by a desire to base the methodology upon netting set specific metrics that are computable by means of traditional CVA systems. This methodology is in a sense an extension to large books of the work in Piterbarg (2010) and Burgard and Kjaer (2011) which focused on the case of individual trades treated in isolation and not in a portfolio context.
2. FCA/FBA Methodology - SFVA Computation: To better understand the logic behind FCA/FBA accounting, we note that if hypothetically the bank was never a net receiver of VM collateral and there were no collateral thresholds, then the funding cost of a cash flow $X(T)$ would be represented as

$$SFVA_X = V_X(0) - V_X{}^*(0)$$

where

$$V_X(t) = \mathbb{E}_t\left[e^{-\int_t^T r_{OIS}(u)du} X(T)\right]$$

and

$$V_X{}^*(t) = \mathbb{E}_t\left[e^{-\int_t^T r_B(u)du} X(T)\right]$$

where $\mathbb{E}_t$ represents the time t expectation in the risk-neutral measure. One interpretation of this formula is that the CFD desk has access to borrowing and lending lines at a funding rate of $r_B(t)$ and are not subject to any credit risk.
3. Application of the Feynman-Kac Theorem: By the Feynman-Kac Theorem we have that

$$\mathbb{E}_0\left[e^{-\int_0^T r_B(u)du} X(T)\right]$$

$$= \mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du} X(T) - \int_0^T e^{-\int_0^t r_{OIS}(u)du}\{r_B(t) - r_{OIS}(t)\} V_X^*(t) dt\right]$$

and also by symmetry

$$\mathbb{E}_0\left[e^{-\int_0^T r_{OIS}(u)du} X(T)\right] = \mathbb{E}_0\left[e^{-\int_0^T r_B(u)du} X(T) - \int_0^T e^{-\int_0^t r_B(u)du}\{r_{OIS}(t) - r_B(t)\} V_X(t) dt\right]$$

4. Simplification of the SFVA Calculation: It follows that

$$SFVA_X = V_X(0) - V_X^*(0)$$

may be rewritten as

$$SFVA_X = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du} s_B(t) V_X^*(t) dt\right] = \mathbb{E}_0\left[\int_0^T e^{-\int_0^t r_B(u)du} s_B(t) V_X(t) dt\right]$$

In practice, the dependence on $V_X^*(t)$ in the exposure integral is inconvenient, so the second equality is probably the most useful in applications.

5. Applying SFVA to the Netting Set: Extending the result above to the more general portfolio case with collateralized hedges, we write the SFVA for counterparty i as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t) dt\right]$$

Let us remark that assuming there are no collateral thresholds, the $SFVA_i$ decomposes further into a sum over the SFVA of trades contained in the i^{th} netting set. Collateral thresholds break this property and make the SFVA a netting set specific amount.

6. Decomposition into FCA/FBA Metrics: The netting set specific SFVA can be decomposed into cost and benefit components as follows.

$$SFVA_i = FCA_i - FBA_i$$

where

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

and

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

We also introduce the aggregate amounts

$$SFVA = \sum_i SFVA_i$$

$$FCA = \sum_i FCA_i$$

$$FBA = \sum_i FBA_i$$

CA and CL Adjustments

1. Lack of Consistency in FCA/FBA: While, as described earlier, FVA/FDA accounting cleanly splits the CA and CL adjustments, the lack of consistency in the FCA/FBA accounting requires some effort to strike a reasonable compromise between conflicting assumptions. For this purpose, notice that the FBA in

$$FBA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t)V_i(t)^- dt\right]$$

   is quiet similar to the DVA in

$$DVA_i = \int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\mathbb{I}_{t<\tau_i}\lambda_B(t)(1-R_B)V_i(t)^-\right]dt$$

   yet it differs from it in two ways.

2. Effective Discount Rate in FCA/FBA: First the FBA expression does not contain the indicator function $\mathbb{I}_{t<\tau_i}$. Second, the FBA losses are discounted at the rate

$$r_B(t) = r_{OIS}(t) + \lambda_B(t)(1-R_B)$$

   rather than at (effectively) the rate $r_{OIS}(t) + \lambda_B(t)$. Only in the unlikely case of

$$R_B = 0$$

   does the discounting rule in the two expressions match. In general we have that

$$FBA > DVA$$

3. FBA as a CL Adjustment: The FBA partially accounts for the RHO, but always contains a large overlap with the standard DVA on payables. Since the two components of FBA cannot be disentangled from each other, the full FBA entry is normally configured as a CL adjustment, similar to the way DVA is treated. On the other hand the FCA is clearly a CA deduction from CET1 that adds to the usual CVA deduction.
4. Comparison of FCA/FDA and FCA/FBA:

	CA Adjustment	**CL Adjustment**
FVA/FDA	$UCVA + FVA$	$CVA_{CL} + DVA + FDA$
FCA/FBA	$UCVA + FCA$	$CVA_{CL} + FBA$

5. FCA/FBA vs. FDA/FVA CA Adjustments: To comment on the relative magnitudes of the CA and CL adjustments between the two methods, notice that since the FCA/FBA approximation effectively amounts to recognizing the re-hypothecation benefits only within the individual netting sets, one expects that the FCA is much larger than the FVA, i.e., the FCA/FBA accounting produces much larger CA deductions than the FVA/FDA accounting.
6. FVA/FDA vs FCA/FBA CL Adjustments: As for the CL adjustments, note that the CL adjustment under the FCA/FBA accounting effectively drops the DVA term in order to avoid double-counting. The case study conducted by Albanese and Andersen (2014) confirms these conclusions. They also quantify the RHO amount, the degree of symmetry breaking in the FCA/FBA accounting, and the difference between FBA and DVA.

Own Credit Sensitivities

1. Expected Positive Exposure Valuation Formulation: As discussed earlier CET1 deduction should generally not decrease when the bank spreads increase. To investigate the sensitivity of the FVA to the bank spread, it is necessary to estimate the discounted Expected Positive Exposure (EPE) up to time t as

$$EPE(t) = \mathbb{E}\left[\int_0^T e^{-\int_0^t r_{OIS}(u)du}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where T (the upper integration limit) is the longest maturity on any trade in the funding set in question.

2. <u>FVA Funding Spread Impact Theorem</u>: Assume that $s_B(t)$ is independent of $r_{OIS}(t)$ and $\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+$ and let $s_B(t)$ be subject to a perturbation of the type

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

where ϵ is a scalar and

$$h(t) \geq 0$$

is a bounded deterministic function. Also define the Gateaux derivative

$$\mathcal{D}_h FVA = \left.\frac{\partial FVA}{\partial \epsilon}\right|_{\epsilon=0}$$

Then

$$\mathcal{D}_h FVA \geq 0$$

if and only if

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1-{R_B}^C}s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

3. <u>FVA Funding Spread Impact - Proof</u>: Defining

$$q_B = \frac{1}{1 - {R_B}^C}$$

and re-casting FVA as a function of ϵ in

$$s_B(t) \to s_B(t) + \epsilon h(t)$$

$$FVA = \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s) + q_B[s_B(s) + \epsilon h(s)]\} ds}\{s_B(t) + \epsilon h(t)\}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

where we have used

$$s_B(t) = \left(1 - {R_B}^C\right)\lambda_B(t)$$

Straightforward calculus shows that

$$\begin{aligned}
\mathcal{D}_h FVA &= \left.\frac{\partial FVA}{\partial \epsilon}\right|_{\epsilon=0} \\
&= \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(s) + \lambda_B(s)\} ds}\left\{h(t) \right.\right. \\
&\quad \left.\left. - q_B s_B(t)\int_0^t h(s)\, ds\right\}\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right] \\
&= \int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s) ds}\left\{h(t) - q_B s_B(t)\int_0^t h(s)\, ds\right\}\right] dt
\end{aligned}$$

where the second equality follows from the independence assumption. The criterion

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - q_B s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

then follows.

4. Gateaux Function Lower Bound Corollary: In the setting of the previous theorem, if

$$h(t) \geq \frac{1}{1 - {R_B}^C}\frac{\mathbb{E}\left[s_B(t)e^{-\int_0^t \lambda_B(s)ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\right]}\int_0^t h(s)\,ds \geq 0\ \forall\ t \in [0, T]$$

then

$$\mathcal{D}_h FVA \geq 0$$

5. Gateaux Function Lower Bound Proof: This corollary follows directly from

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t) - \frac{1}{1 - {R_B}^C}s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

and from the fact that

$$EPE(t) \geq 0$$

Further it turns out that it is a sufficient but conservative condition that ensures

$$\mathcal{D}_h FVA \geq 0$$

6. Right-way Regulatory Sensitivity Criterion: Suppose for instance that h and s_B are positive constants, i.e., we have a flat credit curve that is being shifted up in a parallel fashion. "Right-way" regulatory sensitivity is guaranteed by

$$h(t) \geq \frac{1}{1 - {R_B}^C} \frac{\mathbb{E}\left[s_B(t) e^{-\int_0^t \lambda_B(s) ds}\right]}{\mathbb{E}\left[e^{-\int_0^t \lambda_B(s) ds}\right]} \int_0^t h(s)\, ds \geq 0 \ \forall\ t \in [0, T]$$

if

$$h \geq \frac{s_B h T}{1 - {R_B}^C}$$

or

$$s_B \leq \frac{1 - {R_B}^C}{T}$$

7. Right-way Sensitivity Typical Behavior: Some typical values for the constants in $\frac{1-{R_B}^C}{T}$ are

$${R_B}^C = 40\%$$

and

$$T = 10$$

which yields

$$s_B \leq 6\%$$

a condition that is rarely violated (common values for s_B are $1 - 2\%$).

8. Alternate Right-way Regulatory Sensitivity: Alternatively, if we assume

$$EPE(t) \approx EPE$$

is approximately constant in

$$\int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t)-\frac{1}{1-{R_B}^C}s_B(t)\int_0^t h(s)\,ds\right\}\right]dt \geq 0$$

then it can be written (again for constant h and s_B)

$$\mathcal{D}_h FVA \approx EPE \cdot \int_0^T e^{-\lambda_B t}\left(h-\frac{1}{1-{R_B}^C}s_B t\right)dt \approx EPE \cdot hT\left[1-\frac{1}{2}q_B s_B T\right]$$

which leads to a bound that is twice as loose as before

$$s_B \leq \frac{2\left(1-{R_B}^C\right)}{T}$$

9. Normal Situation FVA Regulatory Sensitivity: In most situations the $EPE(t)$ terms in

$$\mathcal{D}_h FVA = \int_0^T EPE(t)\mathbb{E}\left[e^{-\int_0^t \lambda_B(s)ds}\left\{h(t)-q_B s_B(t)\int_0^t h(s)\,ds\right\}\right]dt$$

typically peak long before T, which would widen the bound even further. All in all it is safe to say that in normal conditions FVA increases when s_B is decreased.

10. SFVA Funding Spread Perturbation Impact: If we repeat the arguments behind the FVA funding spread impact theorem for the SFVA defined in

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du}s_B(t)V_i(t)dt\right]$$

the result is easily seen to be

$$\mathcal{D}_h SFVA = \mathbb{E}\left[\int_0^T e^{-\int_0^t \{r_{OIS}(u)+s_B(u)\}du}\left\{h(t) - s_B(t)\int_0^t h(s)\,ds\right\}\left\{\sum_{i=1}^{n} V_i(t)\right\}dt\right]$$

11. Normal Conditions SFVA Regulatory Sensitivity: In case $h(t)$ satisfies the conditions above that ensure the positivity of $\mathcal{D}_h FVA$ and in situations where the funding sets are prevailingly net payables, the SFVA has wrong sign sensitivities and is unsuitable as a CET1 deduction. This issue is expanded later on.

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Triggers and Close-out Adjustments

Introduction

1. Motivation: This section provides a greater detail and a realistic analysis of the cash flows and the close out protocols.
2. Default-Free Fair Valuation: Let

$$V_i^U(t) \ \forall\ i = 1, \cdots, n$$

be the default free valuation at time t of the portfolio held with counterparty i, i.e., the valuation computed by neglecting the upfront XVA payments, and also neglecting credit risks and funding costs. The default free value of the unsecured book is additive over trades, in particular

$$V^U(t) = \sum_i V_i^U(t)$$

3. FTP Transfer via Super-Replication: Hedges are typically struck at par and engender cash flows across time that super-replicate the cash flows of the unsecured trades. Strict super-replication is required in order to ensure a positive return on equity.
4. Value Adjustment on Super-Replication: In the case of swaps adjustments are usually embedded as a fixed spread in addition to either the fixed leg or the floating leg or both. The cash flows are such that typically the default free valuation of the unsecured portfolio is positive at inception.
5. CFD Routing of the Hedge Flows: The hedge typically consists of one or a portfolio of collateralized swaps entered in the inter-dealer market, initially struck at par and whose cash

flows super replicate those of the unsecured trade. The excess cash flows are then routed to the CVA desk and the CFD desk at the time when they are received.

6. Reducing of the VM Posting Impact: In the case of swaptions and FX options premia are typically paid at maturity and struck at the level for which the present value of the option are zero. This structure lessens the amount of VM to be exchanged. Unsecured trade often pay an upfront premium which is added to the book cash account. By transforming a portion of the upfront payment into a payment at the option maturity on achieves super-replication in this case as well. Excess cash flows are given by the XVA adjustments.
7. Hedge Portfolios across Netting Sets: In general hedges are not specific to individual netting sets. Even if hedges are entered initially on a deal-specific basis, the compression cycles of the collateralized swap portfolio reduce the number of hedge trades and obfuscates the attribution of the individual netting sets.
8. Super Replicated Hedge Portfolio Value: Nevertheless by means of a hedge attribution analysis, it is possible to arrive at the concept of the value of the hedge book for a single portfolio i – we call it ${V_i}^H(t)$. Super-replication is achieved when

$${V_i}^H(t) \leq {V_i}^U(t)$$

   The present value of the difference ${V_i}^U(t) - {V_i}^H(t)$ is the FTP.
9. Value of the Hedge Book: The value of the hedge book is denoted as

$${V_i}^H(t) = \sum_i {V_i}^H(t)$$

10. Gap Risk on Collateralized Hedges: As the inter-dealer market and the exchanges require the posting of the collateral in full, the hedge trades are affected by the bank/counterparty credit risk only because of the gap risk exposure. For simplicity's sake the gap risk on collateralized hedges is neglected here since the analysis and the notations become very heavy. However a professional system implementation should account for it as the effect can be quite material.

Collateral Triggers and Close-outs

1. CSA Based Collateral Trigger Levels: Typically CSA collateral agreements include time-dependent collateral trigger levels $\Gamma_i(t)$ that dependent on the counterparty rating level and is such that, if the exposure of the counterparty surpasses $\Gamma_i(t)$ then the counterparty is obliged to post collateral above that threshold. The bank has a similar trigger level $\Gamma_{B,i}(t)$ for each counterparty i. Initial margin on unsecured trade is not a common practice as they would be ineffective unless thresholds are struck at virtually zero level.
2. Collateral Trigger Impact on VM: The equation for the variation margin accounts can be refined for ratings dependent collateral thresholds and also for the fact that variation margins are received on the CVA hedges. The more general expression is

$$C_{VM,i}(t) = \mathbb{I}_{\tau_B > t}\mathbb{I}_{\tau_i > t}\left[-V_i{}^H(t) + \left\{V_i{}^U(t) - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t) - \Gamma_{B,i}(t)\right\}^- - CVA_{i,CA}(t)\right]$$
$$\approx \mathbb{I}_{\tau_B > t}\mathbb{I}_{\tau_i > t}\left[-V_i{}^H(t) + \left\{V_i{}^U(t) - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t) - \Gamma_{B,i}(t)\right\}^-\right]$$

   where again the approximation is implemented in the Albanese and Andersen (2014) case study.
3. Generalized Hedge VM Funding Rate: The equation

$$s_B(t) = r_B(t) - r_{OIS}(t)$$

   for the spread $s_B(t)$ can also be extended. In general $s_B(t)$ should defined as the spread between the funding rate of the bank on the short-term debt instruments used for the purpose of financing variation margin and the rate received on the VM posted. In general the interest rate received on the VM posted may differ from OIS only by a small spread.
4. Trigger Based FVA/FDA Impact: The portfolio FVA is given by (note the VM related alteration)

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

The FDA is sensitive to the value of bank receivables at the time of the bank impact and is given by

$$FDA = \mathbb{E}\left[\int_0^{\tau_B} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du}\left(1 - R_B{}^C\right)\lambda_B(t)\left\{\sum_i \max\left(\min\left(V_i{}^U(t), \Gamma_i(t)\right), -\Gamma_{B,i}(t)\right)\right\}^+ dt\right]$$

5. Composite Trigger-Based FVA Formulation: On the above basis, the expression for asymmetric FVA from the bank perspective in

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i C_{VM,i}(t)\right\}^+ s_B(t)dt\right]$$

needs to be amended to

$$FVA = \mathbb{E}\left[\int_0^{\infty} e^{-\int_0^t \{r_{OIS}(u)+\lambda_B(u)\}du} \left\{\sum_i \left[-V_i{}^H(t)\mathbb{I}_{\tau_i > t} + \left\{V_i{}^U(t)\mathbb{I}_{\tau_i > t} - \Gamma_i(t)\right\}^+ - \left\{V_i{}^U(t)\mathbb{I}_{\tau_i > t} - \Gamma_{B,i}(t)\right\}^-\right]\right\}^+ s_B(t)dt\right]$$

This equation accounts for the VM to be posted to or received from unsecured derivative counterparties depending on the thresholds.

6. Asset-Liability Super Replication Mismatch: Notice that FVA and FBA are not precisely equal but they are very close. Because of the inequality

$$V_i^H(t) \leq V_i^U(t)$$

it turns out

$$FVA \leq FDA$$

This discrepancy is due to the fact that the FDA payments are embedded in the deal structure, and at the time of the bank default, the senior creditors still hold a claim to future FTP payments for funding without the obligation (or even the capability) to continue hedging.

7. Super Replication Impact on MMT: All in all, since banks enact slightly super-replicating strategies (as opposed to precise replication), in order to have a positive return on equity, the FDA tends to be slightly larger than the FVA.
8. Wealth Transfer From Derivative Counterparties: The above can be viewed as a wealth transfer from the derivatives counterparties to the senior creditors. The MMT is still not invalid as the game is still zero sum, as long as one includes in the analysis not only shareholders and senior creditors but also the derivatives counterparties.
9. Trigger Based FCA/FBA Impact: The expression for $C_{VM,i}(t)$ above can be used to extend

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^+ dt\right]$$

as

$$FCA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^+ dt\right]$$

while

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) V_i(t)^- dt\right]$$

becomes

$$FBA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t r_B(u)du} s_B(t) C_{VM,i}(t)^- dt\right]$$

Incorporating ISDA 1992 Close-outs

1. Impact of the ISDA Closeouts: The representation of the cash flows at the time of counterparty defaults under the ISDA 1992 closeout protocol in

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{{V_i}^U(\tau_i) \geq 0} R_i {V_i}^U(\tau_i) + \mathbb{I}_{{V_i}^U(\tau_i) < 0} {V_i}^U(\tau_i)\right]$$

is modified as follows to include collateral thresholds:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i < \tau_B}\left[\mathbb{I}_{{V_i}^U(\tau_i) \geq 0} R_i \min\left({V_i}^U(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{{V_i}^U(\tau_i) < 0} {V_i}^U(\tau_i)\right]$$

Similarly

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{{V_i}^U(\tau_B) \geq 0} {V_i}^U(\tau_B) + \mathbb{I}_{{V_i}^U(\tau_B) < 0} R_B {V_i}^U(\tau_B)\right]$$

becomes

$$D_i(\tau_B) = \mathbb{I}_{\tau_i > \tau_B}\left[\mathbb{I}_{{V_i}^U(\tau_B) \geq 0} {V_i}^U(\tau_B) + \mathbb{I}_{{V_i}^U(\tau_B) < 0} R_B \max\left({V_i}^U(\tau_B), -\Gamma_B(\tau_B)\right)\right]$$

2. Exit Price of the Trade: Under the 2009 ISDA Close-out rules, cash flows differ in that ${V_i}^U(\tau_i)$ is replaced by the exit price of the trade, including the residual DVA_i at the time of the default. On the other hand, in case the bank defaults first, then the i^{th} counterparty has a right to recover its own DVA, which is the CVA. This implies that when valuing CVA from the bank viewpoint, these default losses happen after the default of the bank itself.
3. Closeout Payments after the Default:

$$D_i(\tau_i) = \mathbb{I}_{\tau_i<\tau_B}\left[\mathbb{I}_{{V_i}^{(+)}(\tau_i)\geq 0} R_i \min\left({V_i}^{(+)}(\tau_i), \Gamma_i(\tau_i)\right) + \mathbb{I}_{{V_i}^{(+)}(\tau_i)<0} {V_i}^{(+)}(\tau_i)\right]$$

$$D_i(\tau_B) = \mathbb{I}_{\tau_i>\tau_B}\left[\mathbb{I}_{{V_i}^{(-)}(\tau_B)\geq 0} {V_i}^{(-)}(\tau_B) + \mathbb{I}_{{V_i}^{(-)}(\tau_B)<0} R_B {V_i}^{(-)}(\tau_B)\right]$$

where

$${V_i}^{(+)}(\tau_i) = {V_i}^U(\tau_i) + DVA_i(\tau_i) \approx {V_i}^U(\tau_i)$$

and

$${V_i}^{(-)}(\tau_B) = {V_i}^U(\tau_B) - CVA_i(\tau_B) \approx {V_i}^U(\tau_B)$$

4. SFVA Estimation Under Collateral Trigger: The definition of SFVA in

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) V_i(t) dt\right]$$

is extended as

$$SFVA_i = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t r_B(u)du} s_B(t) \max\left(\min\left({V_i}^U(\tau_i), \Gamma_i(\tau_i)\right), -\Gamma_B(\tau_B)\right) dt\right]$$

5. Trigger Based UCVA and DVA: The valuation formula for the UCVA under ISDA 2009 including CSA thresholds is given by

$$UCVA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t\{r_{OIS}(u)+\lambda_i(u)\}du}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+, \Gamma_i(t)\right)dt\right]$$

The DVA is given by the following extension

$$DVA_i = \mathbb{E}_0\left[\int_0^{\infty} e^{-\int_0^t\{r_{OIS}(u)+\lambda_B(u)\}du}\lambda_B(t)(1-R_B)\left\{-\min\left(\left\{V_i{}^U(t)\right\}^-, \Gamma_B(t)\right)\right\}dt\right]$$

6. MTM ISDA Closeout Impact: Under ISDA 2009, at the time of the bank default, the bank needs to mark-to-market the derivatives by also including the CVA discount. Symmetrically, whenever the counterparty defaults the bank is also entitled to recover a DVA benefit.
7. Modified Bilateral FTD CVA Adjustment: The DVA benefit entitlement, however, typically gives rise to a negligible correction that is currently ignored in order to avoid the need for nested simulations for the calculation of the CVA itself. Hence the fair valuation formula for the CVA is

$$\begin{aligned} FTDCVA_i &= \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t\{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1 \right. \\ &\quad \left. - R_i)\min\left(\left\{V_i{}^U(t)+DVA_i(t)\right\}^+, \Gamma_i(t)\right)\right]dt \\ &\approx \int_0^{\infty} \mathbb{E}\left[e^{-\int_0^t\{r_{OIS}(u)+\lambda_i(u)\}du}\mathbb{I}_{t<\tau_B}\lambda_i(t)(1-R_i)\min\left(\left\{V_i{}^U(t)\right\}^+, \Gamma_i(t)\right)\right]dt \end{aligned}$$

8. Modified CVA CL and CA: In this case we also have an alteration to the contra-liability given by the present value of the default losses occurring after the bank default, i.e.,

$$CVA_{CL,i} = -\int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du} \mathbb{I}_{t\geq\tau_B} \lambda_i(t)(1 - R_i) \min\left(\left\{V_i^U(t) + DVA_i(t)\right\}^+, \Gamma_i(t)\right)\right] dt$$

$$\approx -\int_0^\infty \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(u)+\lambda_i(u)\}du} \mathbb{I}_{t\geq\tau_B} \lambda_i(t)(1 - R_i) \min\left(\left\{V_i^U(t)\right\}^+, \Gamma_i(t)\right)\right] dt$$

The sum

$$FTDCVA_i = UCVA_i + CVA_{CL,i}$$

is the first-to-default unilateral CVA and represents the fair valuation of the counterparty credit.

VM Rehypothecability Across Funding Sets

1. Restrictions on VM Rehypothecation: In their case study, Albanese and Andersen (2014) model rehypothecation by assuming that the VM received can be posted back to meet posting requirements by the hedges in the same funding set. This is a simplifying assumption that may have to be refined in practical implementations, For instance, there may be restrictions in the CSA preventing banks from being able to rehypothecate the VM.
2. Impact of the Rehypothecation Ban: Rehypothecation bans are sometimes triggered by a degradation in the bank credit quality. Whenever this happens, the cost of funding is effectively increased and the bank has a greater interest in "flattening out" its book.
3. Funding Costs Under Stress Conditions: A careful simulation of the funding costs should account for the resulting increase in funding costs that occur whenever such stress conditions manifest themselves.

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Entry Prices, Exit Prices, and Trade FTP

Trade and Portfolio FTP Estimation

1. The Breakeven Trade FTP: We recall that the FTP policy defines the costs or benefits that must at a minimum be passed to the clients in excess of the default-free trade values (FTP can be negative if the trade reduces the credit risk of the funding costs for the bank). In other words, the FTP defines what constitutes a "break-even" transaction and thereby determines the bank's *entry price*, i.e., the price that the bank would bid to acquire a trade or possibly a collection of trades.
2. Marginal FTP Impact on Portfolios: As mentioned earlier XVA-aware accounting systems do not aggregate portfolios linearly over trades, so FTP policies cannot operate only at the trade level but must take into consideration the effect that a new trade has on the risk profiles of existing trade aggregations (at the netting set or at the funding set levels). For this reason it is reasonable to set out FTP policies in terms of *marginal* increments to portfolio level metrics.
3. FCA/FBA vs FVA/FDA Magnitude Differences: As the choice of the accounting methods have material implications on the choice of various accounting quantities, the FTP policies are necessarily different in the FCA/FBA and the FVA/FDA frameworks. This is discussed in the next two sections. The topic of *exit prices* and fair value accounting are subsequently covered.
4. Transaction Cost Incorporation in Pricing: By convention entry prices include transaction costs and exit prices exclude them. For the purposes of this discussion the assumption throughout here is that the transaction costs are zero.

FTP For FCA/FBA Accounting

1. Netting Set Granularity For FTP: In the case of FCA/FBA accounting, the relevant units of account refer either to default-free trades or (for CVA, DVA, FCA, and FBA) to the netting sets. There are no funding set level units of account in the FCA/FBA method.
2. Credit and Funding Cost Components: In the FCA/FBA accounting, the standard FTP policy is to transfer to the clients at a minimum the incremental amount

$$FTP = \Delta UCVA + \Delta SFVA$$

   $\Delta UCVA$ and $\Delta SFVA$ cover the counterparty credit risk and the funding charges respectively, and are paid by the business line trading desk to the CVA and CFD desks.
3. FTP Impact on Equity Capital: From

$$\Delta CET1 = -\Delta CA + \Delta RE$$

   we see that the incremental impact of a new trade on the equity capital is

$$\Delta CET1 = -\Delta CA + FTP = -\Delta UCVA - \Delta FCA + \Delta UCVA + \Delta SFVA = -\Delta FBA$$

   while the change in liabilities is

$$\Delta CL = \Delta FBA$$

   By design the FTP policy in FCA/FBA accounting ensures that the impact on income from adding a new trade is zero, i.e.,

$$\Delta PFV = \Delta CET1 + \Delta CL = 0$$

4. FTP Alignment with the Shareholders' Wealth: By tailoring the FTP in such a way that new trades have no impact on income, the deals flows induces volatility in CET1, as is evidenced from

$$\Delta CET1 = -\Delta FBA$$

As CET1 is a proxy for shareholders wealth, it may be argued that

$$FTP = \Delta UCVA + \Delta SFVA$$

does not fully align executive incentives with shareholders' interests. An FTP policy designed for CET1 indifference would, however, not be viable for FCA/FBA accounting due to the sheer size of the full FCA, as is demonstrated in the case study by Albanese and Andersen (2014).

FTP For FVA/FDA Accounting

1. FTP as a CET1 Enhancer: In FVA/FDA accounting, FVA and UCVA are CET1 deductions and DVA, FVA, and CVA_{CL} are CL adjustments. The upfront charge for a given trade that a given business line desk needs to pay the CFD desk in order to ensure that CET1 stays constant is

   $$FTP = \Delta UCVA + \Delta FVA$$

   With this choice

   $$\Delta CET1 = 0$$

2. Netting and Funding Set Granularity: The calculation of FTP involves assessing the marginal impact on two separate units of account:
   a. The netting set associated with the counterparty to the trade in question.
   b. The overall funding set under which the VM may be re-hypothecated. This calculation in particular is not always trivial – this is treated later.
3. FTP Shareholder Centric View: By focusing on CET1

$$FTP = \Delta UCVA + \Delta FVA$$

effectively expresses a share-holder centric view in the computation of the FTP, where the proxy for "true" deal in the FTP is only the shareholder part of the overall value that a trade brings to the firm.

4. FTP Impact on Net Income: For trades that involve bond holder benefit post bank default, any such benefits are ignored when passing costs to the client. When it comes to net income the FTP policy therefore has an impact given by

$$\Delta PFV = \Delta CL = \Delta DVA + \Delta FDA + \Delta CVA_{CL}$$

Again this is a marginal calculation at the level of both the netting set (DVA, CVA_{CL}) and the funding set (FVA, FDA).

5. Alignment Across Debt and Equity: It needs to be emphasized that while the FTP policy in FVA/FDA accounting is based on principles more conservative than those for FCA/FBA, the inclusion of RHO into FVA causes the funding cost component of the CA account to be materially smaller in the absolute value than FCA/FBA accounting (by about a factor of two in the case study examples of Albanese and Andersen (2014)). As a consequence CET1 immunization is practical in the case of FVA/FDA method and results in an FTP principle that aligns the interests of the bank managers and the shareholders.

Exit Prices and Fair Valuation

1. Deciding on the Unit of Account: Exit prices are important from an accounting point of view as they provide a model-independent method to access fair valuation. As a general approach, the first step is to decide what is to be considered a unit of account.
2. Auctioning for an Account Unit: If a unit of account can be transferred in a market with sufficient liquidity and without altering any of the expected cash flows an auction process may be run and a best bidder chosen.

3. Model Pricing for the Account Unit: Failing the execution of the auction process, a model-based valuation method is used with a model calibrated consistently to all available market pricing information for the relevant risk factors.
4. Consistency with the Accounting Principles: According to accounting principles AP0 and AP1 above, the exit prices discovered through the steps outlined in the general approach above are what should be recorded as fair value for derivative assets and liabilities on the balance sheet.
5. Granularity of a Single Trade: Unless trades are fully collateralized, the notion that a single trade can ever be considered a unit of account is, as discussed before, an incorrect one. So while it may be tempting to ask for the market price of a single trade, the question becomes truly meaningful only when asked about entire netting set portfolios – and sometimes even that may be too granular.
6. Granularity of a Full Portfolio: Further when discussing portfolio prices, not only must the netting set portfolios be held fixed, but so must also the credit quality of the bank and its counterparty – otherwise cash flows induced by the default are not the same. For instance, trying to assign a portfolio from a low-rated bank to a high-rated bank automatically changes the cash flow characteristics of a bilateral portfolio and violates the apples-to-apples provision in the auctioning setup above.
7. Compatibility with the Bidding Bank: Even if the bidding bank has exactly the right credit spread, the bidding bank very likely has trades with the counterparty in question. The bid is therefore not for the original portfolio in question, but for a combined portfolio of old and new trades.
8. Need to invoke Model Pricing: Due to the above effects it is exceedingly rare that an auction of the type discussed is ever practical; instead the model based pricing approach is virtually always called upon.

FVA/FDA Accounting

1. Exit Price Trade Level Granularity: As we have seen, exit price at the level of a netting set operates at too narrow a unit account, as funding costs due to re-hypothecation must still be

modeled at the funding set level. In the FVA/FDA method this certainly is required for the FTP and entry price calculations as just seen above, but is not required for fair value calculations.

2. CA and CL Adjustment Cancellation: The reason for this, of course, is that cash flows associated with the funding operations are modeled as having zero net value for the firm as a whole, consistent with the MMT. This manifests itself in the cancellation of the CA and the CL adjustments for funding, such that funding costs never make it to the level of net income and net fair asset/liability values on the balance sheet.
3. Price Impact on Net Income: DVA and CVA, however, generally do not cancel out, and their difference shows up in net fair values. Specifically for the FVA/FDA accounting, the Portfolio Fair Value (PFV) of the derivatives is, according to the table seen earlier,

$$PFV = A - L - CA + CL = A - L - FTDCVA + DVA$$

an expression that contains in $A - L$ the present values of promised cash flows, and in $-FTDCVA + DVA$, the present values of the lost cash-flows due to default.
4. Trade Level Cash Flow Focus: The above flows are focused on cash flows generated by the trades in isolation and does not reference how the trades are funded. As a consequence, as mentioned above, $A - L - FTDCVA + DVA$ is additive over netting sets. However, this is not the case for FTP expressions in FVA/FDA accounting which are aligned CET1 (a non-additive metric over netting sets).

FCA/FBA Accounting

1. Price Impact on Net Income: In FCA/FBA accounting funding costs are allowed to hit net income as the PFV is given by

$$PFV = A - L - UCVA - FCA + FBA = A - L - UCVA - SFVA$$

This measure is a mixture of partial and incomplete cash flow valuations with a measure of funding costs thrown in. As in the expression for PFV in FVA/FDA accounting, the expression is additive over netting sets.

2. Caveat - Fair Valuation of CVA: UCVA is generally not the fair valuation of the CVA, as it ignores self-survival. Similarly the expression for PFV in FCA/FBA accounting does not contain the DVA and the approximation

$$FBA \approx DVA$$

is a poor one.

3. Violation of the MMT Principles: As mentioned earlier it is clear that

$$PFV = A - L - UCVA - SFVA$$

above violates both the MMT and the principle of asset-liability symmetry. One should also ask the question whether the presence of SFVA in

$$PFV = A - L - UCVA - SFVA$$

amounts to an entity-specific cost adjustment decoupled from trade cash flows, something that violated the principles of exit pricing and normally not allowed in fair value accounting.

4. Popular Justification for the FCA/FBA Accounting: Proponents of the FCA/FBA accounting often attempt to argue away the entity specific nature of the PFV expression by suggesting that their bank's credit spreads are "representative", wherefore the SFVA represents a market average that can be used for exit pricing purposes. This is a questionable line of reasoning for several reasons.
5. Choice of Appropriate Funding Spread: First, even if the bank's funding spread is close to the market average at some point, this may cease to be the case in the future, especially if the bank itself approaches default. And second it is unclear if the market average spread is a metric that can be used for exit pricing and PFV computations – perhaps the "best" spread

(i.e., high spreads for receivables and low spreads for payables) could be argued to be more appropriate.

6. Industry Standard Proxy for Funding: A related line of thought suggest that the SFVA term in

$$PFV = A - L - UCVA - SFVA$$

is not to be computed at the bank's own spread, but at a separately marked industry spread. This removes the entity specific nature of the FCA/FBA PFV, but also decouples FBA entirely from DVA, whereby FCA/FBA accounting would violate one of the accounting principles.

7. Impact of Exogenously Marked Spread: One also wonders where the industry spread curve is supposed to come from, especially since such curves are very hard to detect at the level of individual trades (since FTP entry prices operate on netting or funding set metrics only). The idea of an exogenously marked spread is treated in detail in the next section.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Liquidity Spreads, Asset Liability Symmetry, and Alternative Allocations for Excess Collateral

Motivation

1. Assumptions Underlying the FCA/FBA Scheme: This section reviews the FCA/FBA accounting framework by examining a variety of assumptions that have been put forward to justify at least some of the elements of the FCA/FBA accounting ideas. These assumptions are quiet strong, and not necessarily realistic.
2. Working Capital Management for Derivatives: Derivatives are normally funded on a short-term basis, as the funding needs associated with derivatives trades typically exhibits considerable variation through time. The inherent variability in turn would make it unlikely that the bank's treasury department would commit to systematically using excess collateral from derivative trading to retire general term debt from the bank's liabilities.
3. FCA/FBA Working Capital Assumptions: Yet, as seen earlier, this assumption is essentially one that is required to make sense of some of the FCA/FBA accounting ideas.
4. Returns on the Excess Collateral: Earlier it was suggested that one use a more conservative and reasonable assumption that the excess collateral is simply invested in short-term risk-free investments, earning a rolling rate of $r_{OIS}(t)$. This way no shareholder gain is generated out of variable excess collateral.

Working Capital Management and Operations

1. Shareholder/Creditor Income Share: The sum of the values of the bank to the shareholders and the senior creditors add up in income statements and define the PFV of the bank. The value of the debt from collateral lenders is excluded from this calculation.

2. Handling of Excess Working Capital: Whenever the bank finds itself in the situation where it has excess cash the bank managers have an option to deleverage by buying back the senior debt. According to MMT neither the decision to leverage up with collateral lenders or de-leveraging by debt buy-back have a net impact on income. However the two decision differ in terms of wealth transfer between the shareholders and the senior creditors.
3. Fair Value Bond Buy-back: The interest paid to collateral lenders triggers a wealth transfer between the shareholders and the senior creditors of a bank and this is quantified by the FVA and the FDA. Whenever bonds are bought back/sold to from the senior creditors at their fair value, the wealth of both the shareholders and the senior creditors is not affected by the decision to de-leverage.

Equity Gain and Debt Gain

1. Equity/Debt Gain Overview and Definition: An alternative to the FVA/FDA assumption would be to assume that the short-term collateral excesses can be invested in strategies that lead to time 0 shareholder and debt-holder increases of EG ("Equity Gain") and DG ("Debt Gain") respectively.
2. MMT Consistency Across the Gains: It would generally be a stretch to build into accounting statements any "sure thing" on firm-wide profitably of investment strategies, it seems that one should at least require

$$EG + DG = 0$$

   This is of course the condition required to satisfy MMT.
3. CA and CL Impact of Gains: EG and DG are specific to whatever investment strategy that the treasury commits to, but should one somehow be able to project their values, the FVA/FDA accounting method could be adapted as follows:

$$CA\ Entries := FVA + UCVA - EG$$

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

4. Funding Definitions Impact on Accounting: This is a good time once again to point out that the funding cost definitions are not immutable, but depend strongly on the assumptions made on how the finds are raised and invested. Baking any such assumptions into the accounting numbers obliges the bank to actually follow strategies on which it based its accounting numbers.
5. Maintenance of the Asset-Liability Symmetry: As

$$EG + DG = 0$$

implies

$$EG = -DG$$

it is clear that introducing EG and DG into the FVA/FDA accounting would preserve the asset-liability symmetry and would not have any effects on the Net Income. However if $EG \neq 0$ new terms would arise at the balance sheet accounting level and would potentially affect CET1.
6. Positive EG and Debt Covenants: As bank managers should not engage in trading strategies where

$$EG < 0$$

the introduction of the EG and the DG terms would realistically only involve the cases where

$$EG > 0$$

and therefore

$$DG < 0$$

i.e., it is possible to only consider collateral investment strategies that prevent wealth transfers from shareholders to senior creditors. Such strategies are possible in principle but are non-trivial to setup since bond covenant put serious restrictions on any activities that enrich the shareholders solely at the expense of senior creditors.

7. Counterparty Credit Risk Transfer: Transferring counterparty credit risk to third party investors may prevent wealth transfers from shareholders to senior creditors. However unless such strategies are properly quantified and executed, theoretical equity gains should not be reflected in accounts just because they are possible. As a consequence the FVA/FDA assumption of

$$EG = DG = 0$$

is a more reasonable and a rigorous one.

8. Shareholder Equity Gain Formulation: Pursuing the extensions above a bit further, supposing that the investment benefits are assumed to accrue at the rate of $s_G(t)$ where $s_G(t)$ is interpreted as the spread over $r_{OIS}(t)$ returns. In this case we would write, long the same lines as

$$FVA(t) = \mathbb{E}\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^+ dt\right]$$

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t) \left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

9. Equity Gain as Return Rate: For the special case where

$$s_G = s_B$$

we can observe that

$$FVA - EG(s_B) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t)\left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\} dt\right]$$

where now the max operator has disappeared from the expectation. Approximating the recovery rate of the bank as zero, i.e., setting

$$R_B = 0$$

we find that

$$FVA - EG(s_B) \approx FCA - FBA = SFVA$$

10. Debt Retirement using Equity Gain: Of course setting

$$s_G = s_B$$

in

$$EG = EG(s_G) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_G(t)\left\{\sum_i V_i\,(t)\mathbb{I}_{t<\tau_i}\right\}^- dt\right]$$

basically amounts to debt retirement – a strategy that was questioned, so this cannot be endorsed. Nevertheless

$$FVA - EG(s_B) \approx SFVA$$

does help understand better the underpinning of the FCA, the FBA, and the SFVA terms.

11. Gain Accounting vs FCA/FBA Accounting: It needs to be emphasized, however, that setting

$$s_G = s_B$$

and following the CA and the CL entries above does not reproduce the FCA/FBA method, not even approximately. For instance, note that the FCA/FBA method sets the CA account to $CVA + FCA$ whereas

$$FVA - EG(s_B) \approx SFVA$$

results in a CA account of $CVA + SFVA$

$$CA\ Entries := FVA + UCVA - EG$$

and

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

constitutes an accounting method with asset-liability symmetry, whereas FCA/FBA method does not.

Liquidity Based Analysis and Treatment

1. Asset-Liability Symmetry without MMT: Let us note that it is possible to preserve asset-liability symmetry without satisfying the MMT. Suppose that the entire market decides that a liquidity spread of s_L - unrelated to the compensation of the default risk – applies to all the discounting operations on unsecured derivatives. Incorporation of this spread would only mean a universal redefinition of the default-free security value V_i – something that would result in asset and earnings re-statements across firms, but would not break the asset-liability symmetry.
2. Liquidity Value Adjustment Metric Formulation: More concretely we define a Liquidity Value Adjustment (LVA) as

$$LVA(s_L) = \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t\{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i\,(t)\right\}dt\right]$$

$$= \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t\{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^+\right\}dt\right]$$

$$+ \mathbb{E}_0\left[\int_0^\infty e^{-\int_0^t\{r_{OIS}(s)+\lambda_B(s)\}ds} s_L(t)\left\{\sum_i V_i(t)^-\right\}dt\right] \triangleq LVA_A(s_L) - LVA_L(s_L)$$

3. CA/CL Components of LVA: One obvious way of accounting for liquidity spreads would be to let $LVA_A(s_L)$ and $LVA_L(s_L)$ to be entered as contra-asset and contra-liability, respectively. It is clear here that $LVA_L(s_L)$ - unlike all other terms in

$$CL\ Entries := CVA_{CL} + FDA + DVA + DG$$

– is not associated with self-default or wealth transfers, and therefore should *not* be excluded from CET1. Note also that introducing the LVA would result in an earnings impact equal to the LVA.
4. LVA in FCA/FBA Accounting: A radical idea is to assume that the funding spread s_L is unrelated to default and in actuality is just a friction-type liquidity spread. In that case one may interpret the SFVA as an LVA by noting the identity

$$SFVA = LVA(s_L)$$

as well as

$$FCA = LVA_A(s_L)$$

and

$$FBA = LVA_L(s_L)$$

5. Symmetric Asset-Liability Accounting Rule: In this interpretation we would use the following accounting rule:

$$CA\ Entries := UCVA + LVA$$

$$CL\ Entries := CVA_{CL} + DVA$$

where effectively only $CVA_{CL} + DVA$ (but not FBA) would need exclusion from the regulatory capital.
6. DVA Double Counting and Resolution: Besides stretching the imagination by assuming that the liquidity spreads are not as large as credit spreads, this rule effectively double counts the DVA. Note that this does not preserve the MMT (and results in a funding impact on the earnings), but does preserve the asset-liability symmetry. If as in FCA/FBA accounting, DVA is removed from the CL entries to avoid double counting, the accounting symmetry is broken.

Problems with the Gain Accounting

1. SFVA Wrong-Way Sensitivity Impact: The approach above is problematic not only from an accounting angle, but also due to the regulatory viewpoint due to the wrong-way sensitivity of the SFVA on the bank's own credit.
2. Misplaced Incentives from FCA/FBA Accounting: In case there were a generalized acceptance of the SFVA being interpreted as an LVA and qualified for a CET1 deduction, the FTP's computed using the SFVA would incentivize traders to buy payables and sell receivables, since they overstate both the funding costs for the latter and the funding benefits for the former.
3. Worsening Bank Credit CET1 Impact: Hence portfolios of banks with material credit spreads would drift towards being net payables. One can imagine that the SFVA for the worst funders would become negative and develop a wrong-sign sensitivity with respect to own credit. A

vicious cycle may ensue since the worsening of the credit of the bank would ipso facto increase equity capital.

4. Sudden Accounting Regime Change Impact: If a blow-up occurs and the accounting standards need to be changed for all the market participants to enforce an FCA or an FVA deduction, the system-wide impact on the regulatory capital would be pervasive.

References

- Albanese, C., and L. Andersen (2014): Accounting for OTC Derivatives: Funding Adjustments and the Re-Hypothecation Option **eSSRN**.

Albanese and Andersen (2014) Case Study

Case Study Setting and Purpose

1. Errors Associated with the FCA/FBA Accounting: While market participants are generally aware of the fact that the FCA/FBA accounting is approximate, little is known about the magnitude of the errors involved. This is particularly true of the RHO due to the computational challenges involved, the proper accounting of which involves simulation of the entire funding set with multiple netting sets.
2. FCA/FBA vs. FDA/FVA Accounting Comparison: In their section on case study findings, Albanese and Andersen (2014) attempt to shed some light on the materiality of the errors by using a realistic test-bed to compare the FCA/FBA and the FVA/FDA accounting results. They obtained results using $Global\ Valuation\ Esther^{TM}$, an in memory risk analytics system designed for the simulation of massive OTC portfolios.
3. Global Valuation Esther and Athena: Esther uses an mathematical framework based on operator algebras as discussed in Albanese, Bellaj, Gimonet, and Pietronero (2011) and references therein. Model calibrations were taken from $Global\ Valuation\ Athena^{TM}$ data service and refer to 7 July 2014.
4. The Fixed Income OTC Portfolio: As their test case, they use a realistic portfolio of fixed income derivatives of about 100,000 trades, 1,600 counterparties, and a variety of collateral agreements, some involving thresholds. The portfolio contains trades in 8 different currencies, including swaps, cross-currency swaps, swaptions, FX forwards, and options. The simulation entails 100 time steps at each of which they find scenarios for the default-free valuation of all the netting sets.
5. Types of Netting Sets Considered: Most netting sets in their tests are at least partially unsecured, with the collateral thresholds being either positive finite or infinite. Some large netting sets are of the unilateral "government" type, i.e., the threshold for the bank is zero and the threshold for the counterparty is infinite. Caveat – if all netting sets were of the

government type, then the FCA would equal the FVA, as the government type netting sets do not contribute to the RHO.

6. Fully Collateralized Netting Set Construction: The portfolio contains also a few fully collateralized netting sets with both thresholds at zero, but they do not contribute to funding and contribute to CVA only mildly through the closeout gap risk – which Albanese and Andersen (2014) do capture in their tests, even though they do not discuss in their paper.
7. Custom Scenario Bank CDS Curves: Since funding metrics depend crucially on the bank funding rates, Albanese and Andersen (2014) carry out the calculation for two different funding curves. One corresponds to the 5Y CDS spread of 106 bp and the other to a 5Y CDS spread of 274 bp. Funding costs are simulated dynamically and consistently with the funding curves.
8. Full Market Risk Factors Simulation: CDS curves for all counterparties are simulated dynamically, as needed for ratings dependent collateral policies (of which the test portfolios had a few) and the analysis of credit-correlation effects related to loss-distributions and stress testing. Albanese and Andersen (2014) also simulate all relevant market risk factors and derivative security prices. The various XVA metrics are evaluated by computing the XVA expressions after incorporating thresholds and closeouts.

Scenario Estimation of the XVA Metrics

1. Book Level Incremental and Cumulative Errors: Albanese and Andersen (2014) found that in their simulation trials the funding set calculations were well-behaved. As expected, they found that the incremental FVA is noisier than the book level FVA. In order to the book-level FVA errors to below the 0.5% mark and the incremental metrics within the 2% mark, it was necessary to run about 100,000 scenarios.
2. Netting Set Granularity XVA Caching: Albanese and Andersen (2014) retained the scenario information in memory aggregated at the netting set granularity. By having the scenarios cached in memory the calculation of the incremental XVA metrics was quite efficient. Incremental metrics of interest include the FVA, the symmetric FVA, as well as the UCVA, the FTDCVA, and the DVA.

3. Funding Curve Scenario XVA Metrics: Albanese and Andersen (2014) demonstrate the XVA portfolio metrics for the two funding curve scenarios. Accounting metrics are computed using the accounting rules listed earlier. Accounting entries are computed separately for the following cases:
    a. Only counterparty credit risk is accounted for
    b. Funding is accounted for using FCA/FBA accounting
    c. Funding is accounted for using FVA/FDA accounting
4. CET1 and MMT Accounting Impact: Albanese and Andersen (2014) demonstrate that there are material difference between the FVA/FDA accounting and the FCA/FBA accounting. In particular the CET1 charges for funding are about triple in the FCA/FBA accounting as opposed to the FVA/FDA accounting. Since the FVA/FDA accounting is consistent with the MMT, the write-off one faces by adding funding entries on top of credit adjustments in NILL.
5. FVA/FDA vs. FCA/FBA CL Impact: Contra Liabilities in FVA/FDA accounting are slightly larger than in FCA/FBA accounting. This happens because although the FDA is substantially smaller than FBA, the DVA is preserved as is required by the asset-liability symmetry, and there is no overlap.
6. EE, EPE, and ENE Estimates: For FVA/FDA calculations the key statistics in measuring RHO are the Expected Exposure (EE), the Expected Positive Exposure (EPE), and the Expected Negative Exposure (ENE) of the portfolio as a whole. Using 100,000 simulation trials, Albanese and Andersen (2014) report these metrics for a range of time horizons. The book under consideration starts off as a net receivable and remains as such on average.
7. Cross Netting RHO Impact: The impact of modeling rehypothecation between hedges belonging to different netting sets is sizeable and receives contributions also from the states of the world where the book is a net payable, because even in such situations a fraction of the netting sets are receivables and the corresponding hedges receive collateral.
8. Receivable vs Payable Book Swing: Furthermore the book valuation swings to a net payable status as one can see from the fairly substantial size of the ENE. A large ENE (in this case) is an indication that the FVA is a materially non-linear metric which cannot be represented as a simple sum over the netting sets. In other words the incremental FVA for a netting set or a

trade can be computed accurately only by knowing and accounting for the positions in the entire book.

Product and Scenario Threshold Type Scenarios

1. Impact of the Swap Types Traded: In their analysis, Albanese and Andersen (2014) add their portfolio three swaps – one initially at par, one a payable, and the third initially a receivable. They then compute the XVA metrics under the following 3 scenarios.
2. Thresholded Scenario XVA Impact:
    a. The swaps are added to a netting set with a finite collateral thresholds that already contains other trades
    b. The swaps are added to a netting set with a threshold at infinity
    c. The swaps are added to a netting set which is initially empty, thus neglecting the benefits of netting
3. Netting and Collateral Threshold Impact: The FTPs differ substantially between all the cases above, and are sensitive to both collateral thresholds and netting. Bothe collateral thresholds and netting materially decrease the FTP.
4. FCA/FBA vs. FVA/FDA FTP Estimates: There are also material differences between the FTP's obtained under the FCA/FBA accounting and those using the rules of FVA/FDA accounting. In the latter case the FTPs are smaller by a factor of $1.5 - 2.0$ in absolute value. This is also remarkable from the viewpoint that the FVA/FDA method does not recognize a DVA benefit to clients. This de-recognition is however more than compensated by a correct modeling of the RHO.
5. Accounting Scheme Objective of the FTP: The incremental FTPs in the two methods achieve a different objective. In the case of the FCA/FBA method the incremental fair value of the OTC portfolio ΔPFV is NILL while the Net Equity Capital is systematically depleted, especially in the case of payables. In FVA/FDA accounting instead the Equity Capital is stable while the Bank fair value appreciates systematically.

XVA Metric Errors and Incrementals

1. Error Rates from Simulation Runs: Albanese and Andersen (2014) display the standard errors on the simulation runs – their simulation entails 100,000 scenarios and shows that the FTP for the FVA/FDA method is around 1.2-1.7%, an error of which type would be acceptable in most circumstances. Using 10,000 scenarios would imply relative errors as large as 10% and of the order of 1 bp per annum, which we would consider as being unacceptable large.
2. Tail Loss Distribution Contribution Scenarios: Further 100,000 scenarios allow carrying out of the reverse stress testing analysis by identifying extreme scenarios which either contribute to the tail of the loss distribution or invalidate a collateral procurement strategy.
3. FVA and SFVA Trade Incrementals: Albanese and Andersen (2014) demonstrate the size of the incrementals after adding one trade of the forward FVA and SFVA as follows:

$$FVA(t) = \mathbb{E}\left[e^{-\int_0^t \{r_{OIS}(s)+\lambda_B(s)\}ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$$

and

$$SFVA(t) = \mathbb{E}\left[e^{-\int_0^t r_B(s)ds} s_B(t)\left\{\sum_i V_i(t)\mathbb{I}_{t<\tau_i}\right\}\right]$$

4. Incremental SFVA vs. Incremental FVA: They observed that incremental SFVA is systematically above the incremental FVA in the case of receivables and systematically below the FVA in the case of payables. We conclude that the approximations which are intrinsic to the FCA/FBA accounting over-estimates funding costs for receivables and also over-estimates funding benefits for payables.

Estimation of the FCA/FBA – FVA/FDA Mismatch

1. Payables Derivatives Bias FCA/FBA: The bias in favor of payables in FCA/FBA accounting is a worrisome feature of this method as it induces banks to skew their exposure in favor of the payables. If the traders are incentivized with an FTP computed by this method, they are inclined to sell out of the money options for premium upfront, effectively raising collateral and subjugating themselves to the treasury, thus incurring inefficiencies.
2. Origins of the Payables Bias: The above happens because the FCA/FBA method recognizes a benefit to the excess variation margin at a rate equal to the funding spread of the bank while the FVA/FDA method does not recognize any such benefit).
3. Threshold Impact of FCA/FBA Accounting: The SFVA is a transactional amount only in cases where there are no thresholds or the thresholds are very remote and attained only with a negligible probability. For instance the two SFVAs for the case where the trade is added to an empty netting set is full but the thresholds are neglected are very close. This happens although the FCA and the FBA are very different.
4. FCA/FBA Volatility on Bank PFV: Within FCA/FBA accounting a new trade does not cause volatility in the fair valuation of the bank as a whole given by the PFV. In this case, however, each trade causes a transfer of wealth from CET1 to CL, i.e., from shareholders to senior creditors.
5. Shareholder to Senior Creditor Transfer: On average this transfer is a net loss to CET1. It MMT held this would happen for a zero FTP. The fact that it happens for a non-zero FTP is yet again a signal of the internal inconsistency of the FCA/FBA accounting.
6. Income and CET1 FVA/FDA Impact: In the case of FVA/FDA accounting instead we propose to structure the FTP in seeking to keep CET1 constant. We could also have computed the FTP in such a way to keep the income constant and this would have given rise to zero FTP's.
7. FTP in FCA/FBA vs. FVA/FDA: The FTP in FVA/FDA accounting are lower than those in FCA/FBA accounting for a net receivable book. This signals again the internal inconsistency of the FCA/FBA accounting as the FTP policy should ensure on average that the CET1 is not depleted while it instead appears as though it is.
8. Single Swap on Single Counterparty: It is also useful to consider the case of a single swap transaction with a single counterparty. Albanese and Andersen (2014) demonstrate the CET1 changes which measure the economic value of the transaction are very close between the

FCA/FBA and FDA/FVA accounting schemes. In this case the FVA/FDA methodology can be regarded as an extension of the FCA/FBA method as long as the latter is restricted to portfolios consisting of a single netting set.

9. Two Counterparties with Opposite Swaps: The second interesting particular case in Albanese and Andersen (2014) is the one described whereby there is a portfolio with two counterparties, each with a single swap position. The swaps have identical terms except that one is a payable while the other is a receivable.
10. Collateral Offset from the Swaps: What happens in this case is that the collateral received on the hedge to one swap is nearly always the exact amount the bank is required to post on the hedge to the other swap. Discrepancies arise only in scenarios where one counterparty defaults while the other is still ongoing, in which case one of the hedge positions is closed upon default.
11. Magnitude of the Asymmetric FVA: Assuming that the CDS spread curves of the counterparties are identical, and that they reach 200 bp in 5Y, the resulting asymmetric FVA is small.
12. FCA/FBA vs. FVA/FDA CET1 Deductions: Instead the FCA for each of the two counterparties is large. As a consequence the FCA/FBA accounting method gives rise to capital deductions that are 9 times larger than the deductions resulting from the FCA/FBA method.

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Conclusions with Funding Adjustments with RHO

Traditional Challenges with Derivative Accounting

1. Indifference Pricing of OTC Derivatives: Although funding costs may feel all too real for the derivatives trader that sees his unsecured positions bleed negative carry, well-established finance principles nevertheless insist that fair values of assets are independent of how they are funded.
2. "Going Concern" Accounting Principles Reconciliation: Reconciling the funding carry vs. the Corporate Finance Theory within the confines of the financial accounting statements is not an easy exercise, especially since traditional "going concern" accounting principles were not designed for credit-risky securities.
3. CET1 Equity Capital Regulatory Principles: Complicated matters further are the newly established regulatory principles for the CET1 equity capital that require particular care in the accounting of DVA and DVA-like adjustments.

Problems with FCA/FBA Accounting

1. Not Accounting for the RHO Handling: While some banks have put forward – and into action – the FCA/FBA method for funding cost accounting, Albanese and Andersen (2014) demonstrate that this method is not satisfactory. First FCA/FBA does not properly reflect the re-hypothecation options embedded in variation margin financing and as a result over-estimated funding related deductions from the equity capital.
2. Violation of the Asset-Liability Symmetry: Second the FCA/FBA violates the asset-liability symmetry principles of generally accepted accounting standards while breaking the Modigliani-Miller Theorem dear to financial economists.

3. Wrong-Way Bank Credit Sensitivity: Another popular variation of the FCA/FBA accounting – which involves deducting SFVA from equity capital rather than FCA – is not viable from a regulatory standpoint as the deduction has a wrong-way sensitivity with respect to the bank credit spread for portfolios which are net payables.

FVA/FDA as FCA/FBA Enhancement

1. CET1, Income, and Fair Value: Albanese and Andersen (2014) proposal for funding cost accounting aims to establish some coherence and to clear up a number of holes in the FCA/FBA accounting. In their tests this new method differs significantly from the FCA/FBA method on key accounting numbers (such as CET1, Net Income, and Fair Asset Value), yet the FVA/FDA accounting should resonate with most relevant parties.
2. MMT and Risk-Neutral Pricing: Firstly financial economists and asset pricing experts will appreciate that the accounting rules will satisfy the Modigliani-Miller Theorem and lean heavily on classic risk-neutral pricing principles.
3. Exclusion of Self-Credit Benefits: Secondly regulators will also appreciate that all self-credit benefits are collected cleanly in a Contra-Liability account that can be easily excluded from common equity for capital purposes.
4. Adherence to Accepted Accounting Principles: Thirdly accountants will appreciate the adherence to the accounting principles, and in particular to the asset-liability symmetry principle.

Trading Staff Point of View

1. Explicit Trade-Level Valuation Adjustment: For trading personnel the picture gets more complicated. On the one hand the FVA/FDA adjustment does not trigger the funding related adjustments to income and asset valuations that many prefer to see.

2. Equity Capital Based Incentive Schemes: On the other hand Albanese and Andersen (2014) describe why traders and managers drafting incentive schemes should care about the changes in CET1 than about the Net income.
3. CET1 Based FTP Estimation Schemes: In particular they highlight the link between the shareholder value and CET1 and demonstrate how a rational FTP scheme can be designed around the principle of book-level CET1 indifference pricing.

Challenges with the XVA Metric Estimation

1. Simulation Across Multiple Netting Sets: On the topic of FTP calculation there is no doubt that FVA/FDA method requires a fairly sophisticated calculation engine to support the necessary incremental FVA calculation. Being a book-level quantity the FVA (and therefore the FTP) computation involves simulating through entire books across time, involving a large number of netting sets.
2. Enhancements to Existing CVA Systems: This, in turn, requires modifications to standard CVA calculation engines that normally can only aggregate trades at the level of individual netting sets.
3. Challenges from a Computational Finance Perspective: Given the complexities involved in computing FTP's against the backdrop of an entire book position, there are challenging computational finance questions to be addressed.

Shortfalls of the FVA/FDA Scheme

1. Derivatives Focus of the FVA/FDA Accounting: While FVA/FDA leans on a number of ideas from corporate finance, Albanese and Andersen (2014) make it clear that the method is pragmatic and derivatives focused.
2. Targeted Scoping of the FVA/FDA Accounting: They do not attempt a rigorous, full-blown analysis of the balance sheet that takes into consideration many other assets and operation of a typical bank. Neither do they consider the effects of taxes, bond covenants, dividend

policies, and subtle feedback effects from investment decisions on firm-wide recovery rates (which they assume to be constant) and default probabilities.

3. Inclusive, Wide Ranging Accounting Treatment: It remains an interesting question for future whether a more rigorous and large (an necessarily complex) analysis can provide any insights that can be turned into concrete accounting rules that improve upon what they propose.

Alternate Specialized Value Adjustment Metrics

1. Capital Charge Value Adjustment: While FVA is the most prominent newcomer to the XVA alphabet soup, there are other adjustments just waiting around the block. For instance, it has been suggested that the cost of capital charges should be reflected in the deal pricing through a "capital value adjustment" or KVA (Green, Kenyon, and Dennis (2014)).
2. Trade Scenarios Requiring Initial Margin: Similarly one may consider Margin Valuation Adjustment (MVA) due to the funding cost of the initial margin (IM) for the netting sets that require IM posting. Initial Margin is required when trading with the CCPs, and due to regulatory requirement also becomes far more prevalent in the future for non-cleared products (Basel Committee on Banking Supervision (2013)).
3. MVA and KVA Estimation Complexities: The accounting for – and the associated impact of – MVA, KVA, and other metrics that come up, are topics for future research, as is their practical computation. Albanese and Andersen (2014) note that both the capital and the initial margins are complex quantities that are more involved to calculate dynamically on the path than just portfolio values (as needed for the FVA). Regression-based methods or nested simulations are likely needed here.

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Derivatives Funding, Netting, and Accounting

Introduction, Motivation, Scope, and Synopsis

1. Portfolio Counterparty and Funding: This chapter expands the previous replication results for economic value of derivatives including funding costs and counter party risk into several counter parties and netting sets. It also identifies the specific funding and replication strategy that corresponds to a popular accounting and pricing proposal (Burgard and Kjaer (2015)).
2. Asymmetry of the Accounting Proposal: The resulting strategy is asymmetric in that the negative net cash of the derivatives portfolio is funded at the funding rate, whereas the positive net cash is invested at the risk free rate. This asymmetry makes the funding cost adjustments non-additive across the netting sets.
3. Symmetric Funding Transfer Mechanism Strategies: In contrast, funding transfer mechanisms that recycle positive cash for other funding purposes enable symmetric funding strategies, resulting in additive adjustments across funding sets. These can generate a higher economic value over the life of the trade than the one accounted for under the proposed methodology.

Model Setup and Asset Dynamics

1. Albanese and Andersen (2014) Accounting Approach: In their paper, Albanese and Andersen (2014) discuss how to account for the OTC derivatives with funding costs and counter party risks in a way that is consistent with general accounting principles. This is an important contribution to the debate around funding value adjustments.
2. Contra Asset and Contra Liability: Albanese and Andersen (2014) propose contra-asset and contra-liability terms related to counter party and funding costs that account for a total

enterprise value to the shareholders and the bondholders that is consistent with the Modigliani-Miller Theorem.

3. CET1 Enhancement Policy for the FTP: They also propose a funds transfer policy for the derivatives that keeps the Core Equity Tier 1 (CET1) constant when adding new derivatives to the book, resulting in a funding cost adjustment being applied to the price.
4. Funding Strategy Impact on CET1: Burgard and Kjaer (2013) have shown that different funding strategies correspond to different economic value to the shareholders. So what funding strategy would generate an economic value to the shareholder over the life of the trading book that corresponds to the price that keeps the CET1 constant under the accounting methodology proposed by Albanese and Andersen (2014)? And what happens if a different funding strategy is followed?
5. Funding Strategy Portfolio Effects Impact: To address these questions, this chapter uses the replication framework of Burgard and Kjaer (2011a, 2011b, 2011c, 2013). In order to study the portfolio effects, the framework is extended to two counterparties and allows for several netting sets for each. The portfolio is assumed to be within a single funding set, i.e., the whole portfolio is funded with a single funding strategy.
6. Single Currency Default Free Funding Source: For simplicity, it is assumed that the trade between the issuer and any of the counter parties are uncollateralized, and a single currency economy with a default provider of funding and hedge assets is considered.
7. Risk Free Bond in Portfolio: A third risk-free bond is added to the funding portfolio to allow for the case considered in Albanese and Andersen (2014) where excess cash is invested in risk free assets rather than being used to repurchase own bonds, or being recycled for other funding purposes of the issuer.
8. Neglecting the Feedback Effects: Like Burgard and Kjaer (2013) and Albanese and Andersen (2014), the balance sheet feedback effects are neglected. Such effects were discussed in Burgard and Kjaer (2011b, 2011c) but depend on the funding cost of the existing debt to be directly significant to the shareholders. For simplicity deterministic rates, credit, and funding spread are assumed, although the results can easily be extended in that regard.
9. Notations used in the Formulation:

i	Counter Party Index such that $i = 1, 2$
J_B, J_{Ci}	Independent Poisson Processes during Defaults of the Issuer B and the Counter Party i
S	The Underlying Asset
P_1	Risk Free Zero Coupon Bond with Rate r
P_B	Issuer Risky ZCB with rate r_B and recovery R_B
P_{Ci}	Counter Party i ZCB with rate r_{Ci} and zero recovery
β_S	Asset Cash Account with the Net Rate $\gamma_S - q_S$
β_{Ci}	Bond P_{Ci} Hedge Cash Account

10. Asset/Bank/Counter Party Dynamics: Next the following simple dynamics are assumed.

$$\Delta P_1 = r P_1 \Delta t$$

$$\Delta P_B = r_B P_B^{-} \Delta t - (1 - R_B) P_B^{-} \Delta J_B$$

$$\Delta P_0 = r_0 P_0^{-} \Delta t - P_0^{-} \Delta J_0$$

$$\Delta P_{Ci} = r_{Ci} P_{Ci}^{-} \Delta t - P_{Ci}^{-} \Delta J_{Ci}$$

$$\Delta S = r S \Delta t + \sigma S \Delta W$$

Balance Sheet Dynamics under Semi-Replication

1. Two Counter Parties and Multiple Netting Sets: To derive the replication results the derivation methodology of Burgard and Kjaer (2013) is closely followed, extending it to two counter parties and several netting sets. The issuer has entered into uncollateralized derivatives with both counter parties and for simplicity it is assumed that all trades have the

same maturity T. Let $H_{ij}\big(S(T)\big)$ denote the payoff at the expiry of trades belonging to counter party i and netting set j.

2. Value of Portfolio to Bank: Different trades with a counter party may belong to different netting units. Let $\hat{V}$ be the total economic value of the portfolio including funding costs and counter party risks before the default of any of the counter parties.
3. Definition of the Economic Value: Economic value is defined to be the expected (and discounted) value that will be realized over the life of the trade by the issuer's shareholders.
4. Value on Counter Party Default: Analogously $\hat{V}_i$ is the total value of the trades with counter party i when this name is considered in isolation. This value becomes relevant once the other counter party has defaulted.
5. Value of the Unadjusted Reference Portfolio: Finally the total reference value V on top of which the valuation adjustments are computed is the value of the book with the same payoff but done on a fully collateralized basis with the collateral rate r.
6. Aggregation across Counter Party/Netting Sets: If V_{ij} is the reference value of the trades with counter party i and netting unit j then

$$V = \sum_i V_i = \sum_{i,j} V_{ij}$$

   Analogously

$$V_i = \sum_j V_{ij}$$

   is the reference value of the trades with the counter party i.
7. Portfolio Book Value Post Default: With this notation in place the book value immediately post a first default of issuer B or one of the counter parties C_1 or C_2 is defined by

$$g_B = \sum_{i,j} \left({V_{ij}}^{+} + R_B {V_{ij}}^{-}\right) \equiv g_{B1} + g_{B2}$$

$$g_{C1} = \sum_j \left(R_{C1}V_{1j}^{\ +} + V_{1j}^{\ -}\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

$$g_{C2} = \sum_j \left(R_{C2}V_{2j}^{\ +} + V_{2j}^{\ -}\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

respectively, where for simplicity, regular bilateral close-outs are assumed.

8. Book Value Post Counter Party Default: What these boundary conditions say is that when the counter party k is the first to default, the issuer gets the closeout value $\bar{g}_{Ck}$ based on the close-out of the trades with *that* counter party, *plus* it continues to hold the total value (including counter party risks and funding costs) of the trades with the surviving counter party.
9. Counter Party/Bank Default Boundary Conditions: If i is the surviving counter party, if it then subsequently defaults, the boundary condition is given by $\bar{g}_{Ci}$, whereas if the issuer defaults it is given by g_{Bi}.
10. No Cross Asset Default Impact: Note that it is implicitly assumed here that the default of one counter party does not impact the value of the portfolio with the other counter party. This assumption could be relaxed by considering close-outs g_{Ck} that are more complicated functions of $\hat{V}_i$.
11. The Derivative Master Replication Portfolio: Assuming all parties are alive, a portfolio Π can be setup as

$$\Pi = \delta S + \alpha_1 P_1 + \alpha_B P_B + \alpha_{C1} P_{C1} + \alpha_{C2} P_{C2} + \beta_S + \beta_{C1} + \beta_{C2}$$

12. No Hedging at Bank Default: This portfolio includes the funding instruments and aims to replicate the derivative value $\hat{V}$ in all scenarios except possibly issuer default.
13. Financing/Counter Party Portfolio Positions: The cash accounts β_S and β_{Ci} are used to finance the S and the P_{Ci} positions, i.e.

$$\alpha_{Ci} P_{Ci} + \beta_{Ci} = 0$$

and

$$\delta S + \beta_S = 0$$

and are assumed to pay net rates of $q_S - \gamma_S$ and q_{Ci} respectively, where γ_S may be a dividend income.

14. Definition of the Funding Constraint: The hedge positions maybe collateralized or repo'ed, so q_S or q_{Ci} may be the collateral or the repo rates, respectively. Any other cash is financed/invested via the bond positions $\alpha_1 P_1$, $\alpha_B P_B$, or $\alpha_0 P_0$, which implies that the *funding constraint*

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

must hold at all times until the first default. Later on different funding strategies are specified, but for now this is kept general.

15. Eliminating Market/Counter Party Risks: Next considered is the combination of the derivative book and the hedge portfolio, where $(\delta, \alpha_{C1}, \alpha_{C2})$ have been chosen such that all the market and the counter party risks have been eliminated.

16. Evolution of the Derivative Portfolio: Going through algebra similar to that as in Burgard and Kjaer (2013), it can be shown that subject to the funding constraint

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

the combination of the derivatives and the replicating portfolio (which includes the funding instruments) evolves as

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1 P_1 + r_B\alpha_B P_B + r_0\alpha_0 P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

where

$$\mathcal{A}_t = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + (q_S - \gamma_S) S \frac{\partial}{\partial S}$$

$$P_D = \alpha_1 P_1 + r_B \alpha_B P_B$$

is the post-issuer default value of the issuer bond portfolio, and

$$\lambda_{Ci} = r_{Ci} - q_{Ci}$$

is the spread of the yield of zero coupon bonds P_{Ci} over its repo rate, i.e., the surviving rate of the counter party default hedge position.

17. <u>Absence of Cash/Funding Basis</u>: Under the assumption of zero basis between bonds

$$r_B = r + (1 - R_B)\lambda_B$$

where

$$\lambda_B \equiv r_0 - r$$

Inserting this into

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1 P_1 + r_B\alpha_B P_B + r_0\alpha_0 P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

yields

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

18. The Bank Default Hedge Error: It is thus evident that all risks except own default have been hedged, and as in Burgard and Kjaer (2013), the hedge error upon issuer default ϵ_h can be defined as

$$\epsilon_h = g_B + P_D = \sum_{i,j}\left({V_{ij}}^+ + R_B {V_{ij}}^-\right) + \alpha_1 P_1 + r_B \alpha_B P_B$$

subject to the funding constraint

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

19. Application of the Self Financing Criterion: While the user is still alive, the Δt terms in

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t$$
$$+ (g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$$

should add to zero for the strategy to be self-financing. This gives a PDE for the economic value in this state.

20. Validity of the Funding Constraint: Without issuer default the total value of the portfolio $\hat{V} + \Pi$ remains zero, i.e., yielding perfect replication, while if the issuer defaults first, it jumps to the hedge error ϵ_h.
21. Counter Party Default Close Out: According to

$$g_{C1} = \sum_{j}\left(R_{C1}{V_{1j}}^+ + {V_{1j}}^-\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

and

$$g_{C2} = \sum_j \left(R_{C2}{V_{2j}}^{+} + {V_{2j}}^{-}\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

the closeout amounts g_{C1} and g_{C2} depend upon the value of the portfolio with the surviving party $\hat{V}_2$ and $\hat{V}_1$ respectively. Thus the value of the portfolio with the surviving party $\hat{V}_i$ needs to be determined first.

22. Surviving Counter Party Replication Portfolio: Describing the funding strategy in this state by $(\alpha_{1i}, \alpha_{Bi}, \alpha_{0i})$ the balance sheet dynamics becomes

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

with hedge error

$$\epsilon_{hi} = g_{Bi} + \alpha_{1i}P_1 + \alpha_{Bi}R_BP_B = \sum_j \left({V_{ij}}^{+} + R_B{V_{ij}}^{-}\right) + \alpha_{1i}P_1 + \alpha_{Bi}R_BP_B$$

subject to the funding constraint

$$\hat{\hat{V}}_i + \alpha_{1i}P_1 + \alpha_{Bi}P_B + \alpha_{0i}P_0 = 0$$

Economic Values

1. Lack of Own-Default Benefit: The next step is to solve for the economic values $\hat{V}_i$ and then $\hat{V}$. As per

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

when the counterparty k has defaulted and only counterparty i is alive the shareholders earn or pay the amount $-\lambda_B \epsilon_{hi}$ per unit of time during the life of the issuer while not receiving the benefit ϵ_{hi} upon own default.

2. Economic Value of the Surviving Party: The drift $-\lambda_B \epsilon_{hi}$ but not the windfall $\epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$ should therefore be included in the *economic value* $\hat{V}_i$ to the shareholders, which satisfies the PDE

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

3. Expression for Solution to $\hat{V}_i$: By Burgard and Kjaer (2013) the solution is given by

$$\hat{V}_i = \sum_j \left(V_{ij} + BLCVA_{ij}\right) + FCA_i$$

with

$$BLCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t, u) \mathbb{E}_t\left[{V_{ij}}^{+}(u)\right] du$$
$$- (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t, u) \mathbb{E}_t\left[{V_{ij}}^{-}(u)\right] du \equiv FTDCVA_{ij} + FBA_{ij}$$

$$FCA_i = -\int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t, u) \mathbb{E}_t[\epsilon_{hi}(u)] du$$

4. Backwards Iteration from Counter Party Default: The default of the counter party k is a zero PnL event as the counter party credit risk is hedged, so we can continue backwards across the default event to the state where all the parties are alive in a self-financing fashion.

5. Funding Costs Incurred to Shareholders: Analogous to before

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

tell us that in this state the shareholders earn or pay the amount $-\lambda_B\epsilon_h$ per unit time during the life of the issuer while not receiving the benefit of the hedge error upon own default, and the drift $-\lambda_B\epsilon_h$ should also be included in the economic value $\hat{V}$ to the shareholders.

6. PDE for the Derivative Value: The value is then given as the solution to the PDE

$$\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1}g_{C1} - \lambda_{C2}g_{C2} + \lambda_B\epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

where it is to be recalled that the closeout values g_{C1} and g_{C2} as given by

$$g_{C1} = \sum_j \left(R_{C1}{V_{1j}}^+ + {V_{1j}}^-\right) + \hat{V}_2 \equiv \bar{g}_{C1} + \hat{V}_2$$

$$g_{C2} = \sum_j \left(R_{C2}{V_{2j}}^+ + {V_{2j}}^-\right) + \hat{V}_1 \equiv \bar{g}_{C2} + \hat{V}_1$$

depend on $\hat{V}_2$ and $\hat{V}_1$ respectively.

7. Solving the Coupled System of PDE's: Consequently the PDE's

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci}) \hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

and

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

form a coupled system of PDE's where the to the former feeds into the latter.

8. Funding Strategy Specific Value Adjustments: Based on this observation one creates the Ansatze

$$\hat{V} = V + U$$

and

$$\hat{V}_i = V_i + U_i$$

and after some details (next section) one obtains

$$\hat{V}_\alpha = V + BLCVA + FCA_\alpha$$

with

$$BLCVA = \sum_{i,j} BLCVA_{i,j}$$

and

$$FCA_{\alpha} = -\int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$

$$-\int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$

$$-\int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$

$$--\int_{t}^{T} \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

where the dependence of the economic value $\hat{V}_{\alpha}$, the funding cost adjustment FCA_{α}, and the hedges $\epsilon_{h1,\alpha}$, $\epsilon_{h2,\alpha}$, and $\epsilon_{h,\alpha}$ on the specific funding strategy

$$\alpha = (\alpha_1, \alpha_B, \alpha_0)$$

has been made explicit. Even when there is no strategy superscript for the benefit of clarity, the dependence on the funding strategy is implicit.

9. Correspondence to the Default Scenarios: The four terms of the correspond to the following four default scenarios:
    a. No counter party has defaulted prior to the issuer default
    b. Only counter party 1 has defaulted prior to the issuer default
    c. Only counter party 2 has defaulted prior to the issuer default
    d. Both counter parties have defaulted prior to the issuer default
10. Multiple Counter Parties Valuation Adjustment: As discussed earlier as well as in Burgard and Kjaer (2013), the FCA is the expected value of the windfall or shortfall at the issuer default. The difference in the multi-name setup is that the difference is extended over the four scenarios.

11. Applicability of the Modigliani Miller Theorem: In each of the scenarios, the second additive terms of

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)(\Delta J_B - \lambda_B\Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B\Delta t)$$

are zero sum games in the expectations between the shareholders and the creditors, and hence do not affect the total value of the firm, in line with the Modigliani-Miller Theorem.

12. Terms affecting the Bond Holders: The first terms $\epsilon_h\Delta J_B$ and $\epsilon_h\Delta J_B$ affect the bond holders only, and correspond to the FDA accounting term in Albanese and Andersen (2014) – alias the DVA2 accounting term in Hull and White (2012).
13. Terms affecting the Shareholders: The second terms, the compensating drift $-\epsilon_h\lambda_B\Delta t$ and $-\epsilon_{hi}\lambda_B\Delta t$, affect the shareholders while there are live trades, and give rise to the FCA.
14. FCA Dependence on Funding Strategy: Since ϵ_h and ϵ_{hi} depend on the specific funding strategy, the FCA given by

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

depends on it as well. In particular it may depend on $\hat{V}$ in a non-linear way and thus may not be explicitly computable.

15. Fee Transfer Pricing (FTP) Calculation: From the arguments above and in the last paragraph in particular it follows that the new trades should be charged with the incremental economic value $\Delta\hat{V}$ such that the shareholders do not loose money over the life of the book.
16. FCA Additivity over Netting Sets: The choice of the funding strategy also determines whether the FCA is additive over the counter parties and the netting sets or not, as will be determined in the section on funding strategies.

Derivation of the Coupled Solutions

1. Decomposition of $\hat{V}$ and Re-arrangement: One wishes to solve the coupled PDE system

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T,S) = H(S)$$

and

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

Inserting the Ansatze

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U_{12}$$

into

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

yields

$$\begin{aligned}\frac{\partial \hat{V}_1}{\partial t} + \mathcal{A}_t \hat{V}_1 - (r + \lambda_B + \lambda_{C1})\hat{V}_1 - \lambda_{C1}\hat{V}_2 + \frac{\partial \hat{V}_2}{\partial t} + \mathcal{A}_t \hat{V}_2 - (r + \lambda_B + \lambda_{C2})\hat{V}_2 - \lambda_{C2}\hat{V}_1 \\ + \frac{\partial U_{12}}{\partial t} + \mathcal{A}_t U_{12} - (r + \lambda_B + \lambda_{C1} + \lambda_{C2})U_{12} \\ = -\lambda_B g_{B1} - \lambda_{C1}\bar{g}_{C1} - \lambda_{C1} - \lambda_B g_{B1} - \lambda_{C1}\bar{g}_{C1} - \lambda_{C1}\hat{V}_2 - \lambda_B g_{B2} - \lambda_{C2}\bar{g}_{C2} \\ - \lambda_{C2}\hat{V}_1 + \lambda_B \epsilon_h\end{aligned}$$

2. <u>Substituting the Expressions for $\hat{V}_1/\hat{V}_2$</u>: Identifying the terms of the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

allows the elimination of many of the terms resulting in

$$\frac{\partial U_{12}}{\partial t} + \mathcal{A}_t U_{12} - (r + \lambda_B + \lambda_{C1} + \lambda_{C2}) U_{12} = \lambda_B(\epsilon_h - \epsilon_{h1} - \epsilon_{h2})$$

$$U_{12}(T, S) = 0$$

with the solution

$$U_{12} = -\int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{C1}+\lambda_{C2}}(t,u)\mathbb{E}_t[\epsilon_h(u) - \epsilon_{h1}(u) - \epsilon_{h2}(u)]du$$

3. Replacing the Expression for FCA: With

$$\hat{V}_i = \sum_j \left(V_{ij} + BLCVA_{ij}\right) + FCA_i$$

one can write

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U_{12} = V + U_1 + U_2 + U_{12} = V + \sum_j BLCVA_{ij} + FCA_1 + FCA_2 + U_{12}$$

and can thus define

$$FCA \equiv FCA_1 + FCA_2 + U_{12}$$

to end up with the equations

$$\hat{V}_\alpha = V + BLCVA + FCA_\alpha$$

and

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) \left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

Fair Values

1. Correspondence with Albanese/Andersen Taxonomy: The different parts of the cash flows in

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t\hat{V} - r\hat{V} + \lambda_B\left(g_B - \hat{V}\right) + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t$$
$$+ (g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$$

and

$$\Delta\hat{V}_i + \Delta\Pi_i = \left[\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - r\hat{V}_i + \lambda_B\left(g_{Bi} - \hat{V}_i\right) + \lambda_{Ci}\left(\bar{g}_{Ci} - \hat{V}_i\right)\right]\Delta t + \epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$$

can be identified with the cash flow types CF1 to CF5 introduced in Albanese and Andersen (2014).
2. Counter Party Default Dynamic Hedging: As all the counter party risks are hedged the CF2 cash flows (counter party default) have been transferred into CF4 (dynamic hedging).

3. Windfall/Shortfall Cash Flows: As discussed, the components $(g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$ are martingales with short term jump size ϵ_h or ϵ_{hi} which are CF5 terms that represent windfalls or shortfalls to the creditor (aka the funding provider) and can be accounted for through a contra-liability term FDA, and the drift compensators $-\lambda_B \epsilon_h$ and $-\lambda_B \epsilon_{hi}$ per unit time are of CF4 types and generate the FCA (this is called FVA in Albanese and Andersen (2014)) through the life of the trade.
4. Equivalence of the FCA and the FDA: Since $(g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$ have zero expectations it follows that

$$FCA = FDA$$

The total terms of $(g_B + P_D)(\Delta J_B - \lambda_B \Delta t)$ and $\epsilon_{hi}(\Delta J_B - \lambda_B \Delta t)$ therefore do not contribute to the value of shareholders and bondholders.
5. Portfolio Fair Value to Bank: Referring to the combined value as the *fair value* $\hat{V}_{FV}$ it then follows that

$$\hat{V}_{FV} = \sum_{i,j} \left(V_{ij} + BLCVA_{ij}\right) \equiv \sum_{i,j} \hat{V}_{FV,ij}$$

where the bilateral CVA is given in

$$BLCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$
$$- (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^-(u)\right] du \equiv FTDCVA_{ij} + FBA_{ij}$$

$\hat{V}_{FV}$ corresponds to the fair value that keeps PFV constant in Albanese and Andersen (2014).

1. <u>Exploring Alternate Funding Strategies</u>: This section considers four funding strategies. The first three strategies are described in Burgard and Kjaer (2013) and extended to the multi-name setup, so that the additivity of the adjustments may be studied. The last strategy is the one that replicates the adjustments proposed in Albanese and Andersen (2014).
2. <u>Counter Party Default Portfolio Change</u>: For the strategies presented here $(\alpha_1, \alpha_B, \alpha_0)$ represents the holdings prior to the first default, and $(\alpha_{1i}, \alpha_{Bi}, \alpha_{0i})$ represents that holdings when counter party k has defaulted, but counter party i and the issuer are still alive.
3. <u>Perfect Replication Strategy Portfolio Composition</u>: There are many strategies that imply perfect replication, i.e.,

$$\epsilon_{hi} = 0$$

and

$$\epsilon_h = 0$$

in addition to the funding constraints

$$\hat{V} + \alpha_1 P_1 + \alpha_B P_B + \alpha_0 P_0 = 0$$

and

$$\widehat{\hat{V}_i} + \alpha_{1i} P_1 + \alpha_{Bi} P_B + \alpha_{0i} P_0 = 0$$

for example

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -V - \frac{1 - R_B}{R_B} \sum_{i,j} V_{i,j}{}^{+}$$

$$\alpha_0 P_0 = -U + \frac{1 - R_B}{R_B} \sum_{i,j} V_{i,j}{}^{+}$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -V_i - \frac{1 - R_B}{R_B} \sum_{j} V_{i,j}{}^{+}$$

$$\alpha_{0i} P_0 = -U_i + \frac{1 - R_B}{R_B} \sum_{i,j} V_{i,j}{}^{+}$$

which implies that the economic value in this case is equal to $\hat{V}_{FV}$

4. Violation of the Bond Covenants: However, as discussed in Burgard and Kjaer (2013) these funding strategies involve issuing senior bonds to repurchase risker ones in a dynamic fashion, and would, in general, violate bond covenants.
5. Strategy I in Burgard and Kjaer (2013): In a multi-name setup, the strategy I of Burgard and Kjaer (2013) corresponds to the following: example

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -V$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -V_i$$

$$\alpha_{0i} P_0 = -U_i$$

6. <u>No Adjustment on Bank Default</u>: This strategy is deigned to wipe out the adjustments U and U_i upon default of the issuer. As a result the hedge errors $\epsilon_{hi,I}$ and $\epsilon_{h,I}$ for this strategy are given by

$$\epsilon_{hi,I} = g_{Bi} - R_B V_i = \sum_j (1 - R_B) {V_{i,j}}^+$$

$$\epsilon_{h,I} = g_B - R_B V = \sum_{i,j} (1 - R_B) {V_{i,j}}^+ = \epsilon_{h1,I} + \epsilon_{h2,I}$$

which are always positive and thus a windfall to the bondholders.

7. <u>Value of the Derivative to the Bank</u>: It follows from

$$\widehat{V}_\alpha = V + BLCVA + FCA_\alpha$$

and

$$\begin{aligned} FCA_\alpha = &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u) \left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C2}}(t,u)\right] \left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du \end{aligned}$$

that the economic value $\hat{V}_I$ and FCA for this strategy is given by

$$\hat{V}_I = V + BLCVA + FCA_I$$

$$FCA_I = \sum_{i,j} FCA_{ij,I}$$

with

$$FCA_{ij,I} = -\int_t^T s_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

where

$$s_B = (1 - R_B)\lambda_B$$

8. Additivity across the Netting Sets: Hence for this funding strategy $\hat{V}$ and FCA are additive across funding sets.
9. Using Senior Unsecured Bank Bond: Strategy II of Burgard and Kjaer (2013) makes use of only senior unsecured bonds of a single recovery, and does so regardless of the sign.
10. Portfolio Composition of the Strategy: In a multi-name setup this is described by

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -\hat{V}$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -\hat{V}_i$$

$$\alpha_{0i} P_0 = 0$$

11. Hedge Errors of the Strategy: In this case, the hedge errors are given by

$$\epsilon_{hi,II} = g_{Bi} - R_B \hat{V}_i = \sum_j (1 - R_B) {V_{i,j}}^+ - R_B U_i$$

$$\epsilon_{h,II} = g_B - R_B \hat{V} = \sum_{i,j} (1 - R_B) {V_{i,j}}^+ - R_B U$$

and the FCA

$$\begin{aligned} FCA_\alpha = &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du \\ &- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du \end{aligned}$$

becomes recursive due to its dependence on U_i and U.

12. Formulation without using the Adjustment: However, analogous to Burgard and Kjaer (2013) it is possible to rewrite

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci} \bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

as

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - r_B \hat{V}_i = \lambda_{Ci}\left(\hat{V}_i - \bar{g}_{Ci}\right)$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - r_B \hat{V} = \lambda_{C1}\left(\hat{V} - \bar{g}_{C1} - \hat{V}_2\right) + \lambda_{C2}\left(\hat{V} - \bar{g}_{C2} - \hat{V}_1\right)$$

with the boundary conditions

$$\hat{V}_i(T, S) = H_i(S)$$

and

$$\hat{V}(T, S) = H(S)$$

13. <u>Solution for CVA and FVA</u>: Solving this equation via Feynman-Kac gives

$$\hat{V}_{II} = \hat{V}_{1,II} + \hat{V}_{2,II} = V + CVA_F + FVA_F$$

where the modified adjustments

$$CVA_F = \sum_{i,j} CVA_{ij,F}$$

and

$$FVA_F = \sum_{i,j} FVA_{ij,F}$$

are additive across counter parties and netting sets with

$$CVA_{ij,F} = -(1 - R_{Ci}) \int_t^T \lambda_{Ci}(u) D_{\lambda_B + \lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

$$FCA_{ij,F} = -\int_t^T s_B(u) D_{\lambda_B + \lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

14. Unconventional CVA and FVA Terms: Please note that these are not the traditional CVA and FVA terms, in particular the $CVA_{ij,F}$ is not the expected values of the losses due to the counter party default, and the $FCA_{ij,F}$ embedded in the $CVA_{ij,F}$ is not the expected value of the hedge errors upon issuer default anymore.
15. Derivative Value and the FCA_{II}: Thus these terms would not be the ones to be used in contra-asset and the contra-liability accounting entries. However since the total sum of the adjustments U are the same, it is still possible to extract the FCA by means of

$$\hat{V}_{II} = V + BLCVA + FCA_{II} = V + CVA_F + FVA_F$$

and thus

$$FCA_{II} = \sum_{i,j} \left(CVA_{ij,F} + FVA_{ij,F} - BLCVA_{ij}\right) = \sum_{i,j} FCA_{ij,II}$$

16. FCA_{II} Additivity across the Netting Sets: FCA_{II} is therefore additive across netting sets. Strategy II as defined in

$$\alpha_1 P_1 = 0$$

$$\alpha_B P_B = -\hat{V}$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i} P_1 = 0$$

$$\alpha_{Bi} P_B = -\hat{V}_i$$

$$\alpha_{0i} P_0 = 0$$

implies that the replication of the adjustments themselves are funded at the rate r_B.

17. Correspondence with the Piterbarg Methodology: Finally note that if there is no counterparty risk, then this model is equivalent to the one developed in Piterbarg (2010) where everything boils down to discounting with the r_B curve.
18. Albanese and Andersen (2014) Strategy III: In Albanese and Andersen (2014) it is assumed that any excess variation margin of a funding set has to be invested at the risk free rate r whereas any shortfall must be funded at the rate r_B albeit it is let somewhat unclear as to how the adjustments themselves are funded.
19. Differential between Investment and Funding: The simplest such strategy would be to invest the total amount $-\hat{V}$ into risk free bonds, if positive, and to fund via the unsecured bond P_B if negative.
20. The Corresponding Portfolio Allocation Strategy: This approach translates into the strategy

$$\alpha_1 P_1 = -\hat{V}^-$$

$$\alpha_B P_B = -\hat{V}^+$$

$$\alpha_0 P_0 = 0$$

$$\alpha_{1i} P_1 = -\hat{V}_i^{\ -}$$

$$\alpha_{Bi} P_B = -\hat{V}_i^{\ +}$$

$$\alpha_{0i} P_0 = 0$$

21. The PDE Driving the Economic Value: It would follow that the economic value satisfies the non-linear PDE system

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i = r_B \hat{V}_i - s_B \hat{V}_i^{\ -} + \lambda_{Ci}\left(\hat{V}_i - \bar{g}_{Ci}\right)$$

$$\hat{V}_i(T, S) = H_i(S)$$

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} = r_B \hat{V} - s_B \hat{V}^- + \lambda_{C1}\left(\hat{V} - \bar{g}_{C1} - \hat{V}_2\right) + \lambda_{C2}\left(\hat{V} - \bar{g}_{C2} - \hat{V}_1\right)$$

$$\hat{V}(T, S) = H(S)$$

22. Use of Monte Carlo Methods: This non-linear PDE could be solved using Monte-Carlo methods, see e.g., Labordere (2012), but the performance of such methods is unclear for real world large portfolios such as the one studied in Albanese and Andersen (2014).
23. Nonlinear PDE Adjustment Impact: Note that due to non-linearity there may be no natural way of expressing $\hat{V}$ as a sum of V and some valuation adjustment U other than defining

$$U \equiv \hat{V} - V$$

In any case this strategy will not lead to the adjustments described in Albanese and Andersen (2014).

24. A Variation Implied Funding Strategy: The non-linearity is caused by the funding strategy being non-linear function of $\hat{V}$. The following small variation of this strategy – called strategy III henceforth – yields a system of linear PDE's:

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -{V_i}^-$$

$$\alpha_{Bi} P_B = -{V_i}^+$$

$$\alpha_{0i} P_0 = -U_i$$

where the usage of the zero-recovery bonds P_0 for the funding of the adjustments U and U_i is motivated by the fact that for closeouts based on the risk free values of the derivatives the adjustments disappear on default.

25. Hedge Errors of the Strategy: Thus funding the adjustment U via a zero recovery bond ensures that the funding disappears as well when the issuer defaults. The hedge errors in this case are given by

$$\epsilon_{hi,III} = g_{Bi} - {V_i}^- - R_B {V_i}^+ = -\sum_j (1 - R_B) {V_{i,j}}^- - (1 - R_B) {V_i}^+$$

$$\epsilon_{h,III} = g_{Bi} - V^- - R_B V^+ = -\sum_{i,j} (1 - R_B) {V_{i,j}}^- - (1 - R_B) V^+$$

and inserting these into the general formula

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$

$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

yields

$$FCA_{III} = FCA_{AA} - \sum_{i,j} FBA_{ij}$$

where FBA_{ij} is defined from

$$BLCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

$$- (1 - R_B) \int_t^T \lambda_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^-(u)\right] du \equiv FTDCVA_{ij} + FBA_{ij}$$

and

$$
\begin{aligned}
FCA_{AA} = &-\int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t[\{V_1(u) + V_2(u)\}^+] du \\
&- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C_1}}(t,u) \left[1 - D_{\lambda_{C_2}}(t,u)\right] \mathbb{E}_t[{V_1}^+(u)] du \\
&- \int_t^T s_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C_2}}(t,u) \left[1 - D_{\lambda_{C_1}}(t,u)\right] \mathbb{E}_t[{V_2}^+(u)] du \\
&- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) \left[1 - D_{\lambda_{C_2}}(t,u)\right]\left[1 - D_{\lambda_{C_2}}(t,u)\right] \mathbb{E}_t[0] du
\end{aligned}
$$

where the zero bond basis relation

$$
s_B = (1 - R_B)\lambda_B
$$

has been used.

26. Independence of the Default Poisson's: FCA_{AA} corresponds to the funding cost adjustment used in Albanese and Andersen (2014) – where it is referred to as FVA – provided that the default times τ_i are given by the first jump times of independent (of each other and the market) Poisson processes with intensities λ_{Ci}.
27. Non Additivity of the FCA_{AA}: From

$$FCA_{AA} = -\int_t^T s_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C1}+\lambda_{C2}}(t,u)\mathbb{E}_t[\{V_1(u)+V_2(u)\}^+]du$$

$$-\int_t^T s_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C_1}}(t,u)\left[1-D_{\lambda_{C_2}}(t,u)\right]\mathbb{E}_t[V_1{}^+(u)]du$$

$$-\int_t^T s_B(u)D_{r+\lambda_B}(t,u)D_{\lambda_{C_2}}(t,u)\left[1-D_{\lambda_{C_1}}(t,u)\right]\mathbb{E}_t[V_2{}^+(u)]du$$

$$-\int_t^T \lambda_B(u)D_{r+\lambda_B}(t,u)\left[1-D_{\lambda_{C_2}}(t,u)\right]\left[1-D_{\lambda_{C_2}}(t,u)\right]\mathbb{E}_t[0]du$$

it can be seen that FCA_{AA} is non-additive and its calculation requires the simulation of the entire funding set in a path consistent way which would be a challenging task in terms of memory requirements.

28. Relation between FCA_{III} and FCA_{AA} : It is worth noting that the negative FBA component in the FCA_{III} of

$$FCA_{III} = FCA_{AA} - \sum_{i,j} FBA_{ij}$$

cancels the FBA component of the bilateral CVA such that the economic value may be written as

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

29. Expectation of the Hedge Error: Similar to the FCA_F in strategy II it should be noted that FCA_{AA} is not the expected value of the hedge error at issuer default – this is given by FCA_{III}.

30. FTP Policy of Albanese and Andersen (2014): The adjustment obtained by charging the incremental $\hat{V}$ is very similar to the FTP pricing proposed in Albanese and Andersen (2014),

with the only difference being that they propose charging the increment of the slightly different economic value

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

where the unilateral CVA is defined as

$$UCVA = \sum_{i,j} UCVA_{ij}$$

with

$$UCVA_{ij} = -(1 - R_{Ci}) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_{Ci}}(t,u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

and has replaced FTDCVA in

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

31. Incorporating Accounting and Regulatory Principles: Albanese and Andersen (2014) motivate this choice by means of accounting and regulatory principles that stipulate that CVA should be unilateral and not depend on the issuer credit quality.
32. Derivative Value to the Book: In the next section it is shown that a small, albeit more complicated modification of the funding strategy

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -V_i^-$$

$$\alpha_{Bi} P_B = -V_i^+$$

$$\alpha_{0i} P_0 = -U_i$$

- the strategy IV – yields the economic value

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

rather than

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

33. Comparison of the FCA Magnitudes: Intuitively, this strategy is more conservative than the other strategies, in that FCA_{IV} is most negative. The section below proves that the inequality

$$FCA_{IV} < FCA_{III} \leq FCA_I$$

indeed holds.

Derivation of $\hat{V}_{IV}$

1. Funding Strategy that generates $\hat{V}_{IV}$: This section specifies the funding strategy that generates the economic value $\hat{V}_{IV}$ given in

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

2. Methodology for Calculating the Adjustment: So far in this chapter, the methodology has been to specify a funding strategy, derive the resulting economic value $\hat{V}$, and finally derive the valuation adjustment(s) by making the Ansatze

$$\hat{V} = V + U$$

3. Avoidance of Logical Formulation Fallacy: The strategy developed in this section contains the $UCVA$, so it is critical that the arguments are developed carefully to avoid the logical fallacy of *circulus in probando*.
4. Definition of the Unilateral CVA - UCVA: Let i, j be the counter party and the netting unit index, respectively. The unilateral CVA $UCVA_{ij}$ is defined as

$$UCVA_{ij} = -(1 - R_C) \int_t^T \lambda_{Ci}(u) D_{r+\lambda_{Ci}}(t, u) \mathbb{E}_t\left[{V_{ij}}^+(u)\right] du$$

which is the solution to

$$\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci}) UCVA_{ij} = (1 - R_{Ci}) \lambda_{Ci} {V_{ij}}^+$$

$$UCVA_{ij}(T, S) = 0$$

5. Counter Party/Netting Set Additivity: Furthermore the following definitions are set to hold:

$$UCVA_i = \sum_j UCVA_{ij}$$

and

$$UCVA = \sum_i UCVA_i$$

6. Motivation for the UCVA Definition: These definitions are motivated by making the issuer default-free by setting

$$\lambda_B = 0$$

in the PDE's

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B \epsilon_{hi}$$

$$\hat{V}_{ij}(T, S) = H_i(S)$$

and

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

in which case one obtains

$$\hat{V}_i = \sum_j \left(V_{ij} + UCVA_{ij}\right) = V_i + UCVA_i$$

and

$$\hat{V} = \sum_{i,j} \left(V_{ij} + UCVA_{ij}\right) = V + UCVA$$

7. Determination of the Funding Strategy: Before specifying the funding strategy one writes

$$\hat{V}_i = V_i + UCVA_i + U_i$$

and

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U = V + UCVA + U_1 + U_2 + U$$

where $UCVA_i$ and $UCVA$ are defined above and the remaining valuation adjustments U_i and U are unspecified for now.

8. The Funding Strategy Portfolio Components: The funding strategy is then given by

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^- - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^+$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

9. Funding of CVA/Other Adjustments: This strategy is very similar to the strategy

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i}P_1 = -V_i^-$$

$$\alpha_{Bi}P_B = -V_i^+$$

$$\alpha_{0i}P_0 = -U_i$$

apart from that the unilateral CVA is funded at the risk-free rate and the remaining adjustments (denoted U and U_i) at the zero recovery bond rate r_0.

10. The "One Counter Party" Case: From

$$\Delta\hat{V} + \Delta\Pi = \left[\frac{\partial\hat{V}}{\partial t} + \mathcal{A}_t\hat{V} + r_1\alpha_1P_1 + r_B\alpha_BP_B + r_0\alpha_0P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right]\Delta t + (g_B + P_D)\Delta J_B$$

it follows that the PDE

$$\frac{\partial\hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i = -\lambda_B g_{Bi} - \lambda_{Ci}\bar{g}_{Ci} + \lambda_B\epsilon_{hi}$$

$$\hat{V}_i(T, S) = H_i(S)$$

before invoking the zero basis conditions

$$r_B = r + (1 - R_B)\lambda_B$$

and

$$r_0 = r + \lambda_B$$

takes the form

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci} \hat{V}_i + r\alpha_{i1} P_1 + r_B \alpha_{iB} P_i + r_0 \alpha_{i0} P_0 = -\lambda_{Ci} \bar{g}_{Ci}$$

$$\hat{V}_i(T, S) = H_i(S)$$

11. <u>Using the Adjustment Breakdown Ansatz</u>: On inserting the funding strategy

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -{V_i}^- - UCVA_i$$

$$\alpha_{Bi} P_B = -{V_i}^+$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

and the ansatz

$$\hat{V}_i = V_i + UCVA_i + U_i$$

into

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci} \hat{V}_i + r\alpha_{i1} P_1 + r_B \alpha_{iB} P_i + r_0 \alpha_{i0} P_0 = -\lambda_{Ci} \bar{g}_{Ci}$$

$$\hat{V}_i(T, S) = H_i(S)$$

one obtains

$$\begin{aligned}
\frac{\partial V_i}{\partial t} + \mathcal{A}_t V_i - rV_i + rV_i + \frac{\partial UCVA_i}{\partial t} + \mathcal{A}_t UCVA_i + \frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - r{V_i}^- - rUCVA_i \\
- [r + (1 - R_B)\lambda_B]{V_i}^+ - (r + \lambda_B)U_i + \lambda_{Ci}\bar{g}_{Ci} - \lambda_{Ci}V_i - \lambda_{Ci}UCVA_i - \lambda_{Ci}U_i \\
= \frac{\partial V_i}{\partial t} + \mathcal{A}_t V_i - rV_i \\
+ \sum_j \left[\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci})UCVA_{ij} + (1 - R_{Ci})\lambda_{Ci}{V_{ij}}^+\right] \\
+ \frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - (r + \lambda_B + \lambda_{Ci})U_i - s_B{V_i}^+
\end{aligned}$$

with

$$s_B = r_B - r$$

12. <u>Stochastic Integral Expression for U_i</u>: Recognizing that V_i and $UCVA_{ij}$ satisfy the Black-Scholes PDE and the PDE

$$\frac{\partial UCVA_{ij}}{\partial t} + \mathcal{A}_t UCVA_{ij} - (r + \lambda_{Ci})UCVA_{ij} = (1 - R_{Ci})\lambda_{Ci}{V_{ij}}^+$$

$$UCVA_{ij}(T, S) = 0$$

respectively, one obtains

$$\frac{\partial U_i}{\partial t} + \mathcal{A}_t U_i - (r + \lambda_B + \lambda_{Ci})U_i = s_B{V_i}^+$$

$$U_i(T, S) = 0$$

so it follows that

$$U_i = -\int_t^T s_B(u) D_{r+\lambda_B+\lambda_{Ci}}(t, u) \mathbb{E}_t[{V_i}^+(u)] du$$

13. U_i Interpretation as the FCA_i: If one writes

$$FCA_i \equiv U_i$$

then it was shown earlier that

$$\hat{V}_i = V_i + UCVA_i + FCA_i$$

for the case of a single counter party.

14. One Counter Party Formulation Recast: Before proceeding to the two counter party case

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - \lambda_{Ci} \hat{V}_i + r \alpha_{i1} P_1 + r_B \alpha_{iB} P_i + r_0 \alpha_{i0} P_0 = -\lambda_{Ci} \bar{g}_{Ci}$$

$$\hat{V}_i(T, S) = H_i(S)$$

is rewritten as

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t \hat{V}_i - (r + \lambda_B + \lambda_{Ci}) \hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci} \bar{g}_{Ci} - s_B {V_i}^+ = 0$$

which will be helpful for the two counter party case below.

15. The Two Counter Parties Case: From

$$\Delta \hat{V} + \Delta \Pi = \left[\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} + r_1 \alpha_1 P_1 + r_B \alpha_B P_B + r_0 \alpha_0 P_0 + \lambda_{C1}\left(g_{C1} - \hat{V}\right) + \lambda_{C2}\left(g_{C2} - \hat{V}\right)\right] \Delta t + (g_B + P_D) \Delta J_B$$

it follows that the PDE

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (r + \lambda_B + \lambda_{Ci})\hat{V} = -\lambda_B g_B - \lambda_{C1} g_{C1} - \lambda_{C2} g_{C2} + \lambda_B \epsilon_h$$

$$\hat{V}(T, S) = H(S)$$

before invoking the zero basis conditions takes the form

$$\frac{\partial \hat{V}}{\partial t} + \mathcal{A}_t \hat{V} - (\lambda_{C1} + \lambda_{C2})\hat{V} + r\alpha_1 P_1 + r_B \alpha_B P_i + r_0 \alpha_0 P_0 = -\lambda_{C1} g_{C1} - \lambda_{C2} g_{C2}$$

$$\hat{V}(T, S) = H(S)$$

16. Applying the Adjustment Decomposition Ansatz: Analogous to the single counter party case, inserting the ansatz

$$\hat{V} = \hat{V}_1 + \hat{V}_2 + U$$

and the funding strategy

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^- - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^+$$

$$\alpha_{0i}P_0 = -(\hat{V}_i - V_i - UCVA_i) = -U_i$$

one obtains

$$\sum_i \left[\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci}\bar{g}_{Ci} - s_B {V_i}^+\right] + \frac{\partial U}{\partial t} + \mathcal{A}_t U$$
$$- (r + \lambda_B + \lambda_{C1} + \lambda_{C2})U - s_B(V^+ - {V_1}^+ - {V_2}^+) = 0$$

17. <u>Stochastic Integral Expression for U_i</u>: By

$$\frac{\partial \hat{V}_i}{\partial t} + \mathcal{A}_t\hat{V}_i - (r + \lambda_B + \lambda_{Ci})\hat{V}_i + \lambda_B V_i + \lambda_B UCVA_i + \lambda_{Ci}\bar{g}_{Ci} - s_B {V_i}^+ = 0$$

the summation above is zero so U satisfies

$$\frac{\partial U}{\partial t} + \mathcal{A}_t U - (r + \lambda_B + \lambda_{C1} + \lambda_{C2})U - s_B(V^+ - {V_1}^+ - {V_2}^+) = 0$$

$$U(T, S) = 0$$

and is thus given by

$$U = -\int_t^T s_B(u) D_{r+\lambda_B+\lambda_{C1}+\lambda_{C2}}(t, u)\mathbb{E}_t[{V_1}^+(u) + {V_2}^+(u) - V^+(u)]du$$

18. <u>Recovering the $\hat{V}_{IV}/FCA_{AA}$ Relation</u>: Wrapping up, it has been shown that

$$\hat{V}_{IV} = V_1 + V_2 + UCVA_1 + UCVA_2 + FCA_1 + FCA_2 + U$$

and as the arguments in the Albanese and Andersen strategy section (strategy III) yield

$$FCA_1 + FCA_2 + U = FCA_{AA}$$

one in fact has the equality

$$\hat{V}_{IV} = V + UCVA + FCA_{AA}$$

as claimed.

Proof of the Statement $FCA_{IV} \leq FCA_{III} \leq FCA_{I}$

1. FCA_{IV} more conservative than FCA_{I}: In this section it is shown that the FCA from strategy IV is more conservative than the FCA from strategy I.
2. Comparison between FCA_{IV} and FCA_{III}: From

$$\hat{V}_{III} = V + BLCVA + FCA_{III} = V + BLCVA + FCA_{AA}$$

and

$$\hat{V}_{IV} = V + BLCVA + FCA_{IV} = V + UCVA + FCA_{AA}$$

it follows that

$$FCA_{IV} = FCA_{III} + UCVA - FTDCVA \leq FCA_{III}$$

3. Comparison between FCA_{I} and FCA_{III}: This is because

$$UCVA \leq FTDCVA$$

due to the risk-free vs. risky discounting and the positiveness of the expected exposures. So it remains to prove that FCA_{III} is more conservative than FCA_I.

4. Hedge Errors I and III: From Burgard/Kjaer's strategy I and Albanese and Andersen (2014) strategy III, the hedge errors $\epsilon_{hi,I}$, $\epsilon_{hi,III}$, $\epsilon_{h,I}$, and $\epsilon_{h,III}$ for the two strategies are given by

$$\epsilon_{hi,I} = \sum_j (1 - R_B){V_{ij}}^+$$

$$\epsilon_{hi,III} = -\sum_j (1 - R_B){V_{ij}}^- + (1 - R_B){V_i}^+$$

$$\epsilon_{h,I} = \sum_{i,j} (1 - R_B){V_{ij}}^+$$

$$\epsilon_{h,III} = -\sum_{i,j} (1 - R_B){V_{ij}}^- + (1 - R_B)V^+$$

5. Comparison across the Hedge Errors: On dropping $(1 - R_B)$ term for clarity

$$\begin{aligned}\epsilon_{h,III} &= V^+ - \sum_{i,j} {V_{ij}}^- = \max\left(\sum_{i,j} V_{ij}, 0\right) - \sum_{i,j} {V_{ij}}^- \\ &= \max\left(\sum_{i,j} V_{ij} - \sum_{i,j} {V_{ij}}^-, -\sum_{i,j} {V_{ij}}^-\right) = \max\left(\sum_{i,j} {V_{ij}}^+, -\sum_{i,j} {V_{ij}}^-\right) \\ &\geq \sum_{i,j} {V_{ij}}^+ = \epsilon_{h,I}\end{aligned}$$

6. Hedge Error $III > I$: From this proof it follows that

$$\epsilon_{h,III} \geq \epsilon_{h,I}$$

so the bondholder windfalls under strategy III are always greater than or equal to that of strategy I.

7. $\underline{FCA_{III}}$ more conservative than FCA_I: By

$$FCA_\alpha = -\int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}+\lambda_{C2}}(t,u) \mathbb{E}_t\left[\epsilon_{h,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C1}}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h1,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u) D_{\lambda_{C2}}(t,u)\left[1 - D_{\lambda_{C1}}(t,u)\right] \mathbb{E}_t\left[\epsilon_{h2,\alpha}(u)\right] du$$
$$- \int_t^T \lambda_B(u) D_{r+\lambda_B}(t,u)\left[1 - D_{\lambda_{C2}}(t,u)\right]\left[1 - D_{\lambda_{C2}}(t,u)\right] \mathbb{E}_t[0] du$$

this translates into

$$FCA_{III} \leq FCA_I$$

so strategy III has the highest funding costs as expected.

Discussion

1. Albanese/Andersen Consistent Funding Strategy: The aim of this chapter was to find a funding strategy that generates an economic value to the shareholders that corresponds to the FTP price that keeps CET1 constant in the accounting methodology of Albanese and Andersen (2014).
2. Specification of the Funding Strategy: This strategy is to be found in the strategy III described by

$$\alpha_1 P_1 = -V^-$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -U$$

$$\alpha_{1i} P_1 = -V_i^{\ -}$$

$$\alpha_{Bi} P_B = -V_i^{\ +}$$

$$\alpha_{0i} P_0 = -U_i$$

or more precisely its refinement in

$$\alpha_1 P_1 = -V — UCVA$$

$$\alpha_B P_B = -V^+$$

$$\alpha_0 P_0 = -\left(\hat{V} - V - UCVA\right) = -(U_1 + U_2 + U)$$

$$\alpha_{1i} P_1 = -V_i^{\ -} - UCVA_i$$

$$\alpha_{Bi} P_B = -V_i^{\ +}$$

$$\alpha_{0i} P_0 = -\left(\hat{V}_i - V_i - UCVA_i\right) = -U_i$$

3. Characteristics of the Third Strategy: Being able to specify such a funding strategy means that the proposed accounting methodology is consistent with no double counting or missing terms. The main qualitative difference between this strategy and the others discussed in the

section on funding strategies is that in this strategy the positive and the negative cash positions of the funding set are not treated symmetrically.

4. No Rehypothecation of Funding Flows: IN particular, apart from the adjustments themselves, a positive cash position (i.e., when V is negative) is not re-hypothecated across the funding sets, but re-invested at the risk free rate even if the issuer as a whole is funding negative.
5. Rehypothecation across Multiple Funding Sets: By contrast, in strategies I and II net positive cash in a funding set still earns funding rates and this is achieved by re-purchasing the issuer's own bonds, or equivalently, by recycling it for the issuer's other funding needs through an FTP or an economic funding process, typically involving a central treasury department.
6. Operational Challenges with Strategy III: Under strategy III the issuer becomes uncompetitive in selling derivatives that are asset like to the counter party out of a cash rich funding set as it cannot compensate the counter party for the credit risk exposure that it takes with respect to the issuer – this is because with strategy III the issuer cannot monetize DVA as a funding benefit in this funding set.
7. Potential Arbitrage with III/IV: It should also be noted that with the CET1 neutral pricing policy proposed by Albanese and Andersen (2014) an issuer with separate funding sets within the same netting set becomes arbitrageable, i.e., when one funding set is significantly cash positive and the other is significantly cash negative, then a default-free counter party buying a derivative asset from a cash negative funding set gets a price that is lowered by the funding benefit and then could sell it back to the cash positive funding set where it is not charged the corresponding funding cost.
8. Elimination of the Bank Credit Risk: The counter party does not pick up the credit risk to the issuer in this case since the two derivatives are back-to-back and close-outs net in the same netting set.
9. Funding Sets Spanning across Netting Sets: Thus, the funding sets must span at least the netting nets, and, with internal transfer processes in place, could span the whole of the issuer business. In this case strategy III converges to strategy I as long as the issuer as a whole is funding negative, i.e., which is typically the case for a bank.
10. Albanese/Andersen Methodology on I: What happens if the accounting is based on smaller funding units, and the accounting methodology of Albanese and Andersen (2014) is applied

but one of the funding strategies I or II is followed? In this case, if the issuer is able to achieve the CET1 neutral price, and then follows the funding strategy I (as an example), then the issuer achieves an economic value to the shareholders that is higher than what was originally accounted for.

11. The Three Different Economic Values: In this case, three different economic values can be found – the accounting fair value V_{FV} which is the value to the shareholder and the bond holder combined, the accounting value that makes the derivative CET1 neutral in the accounting methodology of Albanese and Andersen (2014), and the economic value $\hat{V}$ that corresponds to the funding strategy employed.
12. The Negative CET1 Impact Scenario: Note that if the issuer charges the economic value $\hat{V}$ then it will see a negative CET1 impact under this accounting methodology, but recover it over the life of the trade.
13. Funding Set Positive Cash Recycling: Ideally the accounting methodology could be modified such that the benefit of recycling positive cash balances of a given funding set to other businesses is accounted for so that the CET1 risk-neutral price is aligned with the economic value that can be achieved by recycling the funding through internal FTP processes.

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