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PORTFOLIO OPTIMIZATION

Sebastian Ceria and Kartik Sivaramakrishnan

a) INTRODUCTION

Every portfolio manager faces the challenge of building portfolios that achieve an optimal tradeoff between risk and return. Harry Markowitz [1], [2] was the first to develop a mathematical framework to solve this problem in the 1950s. The Markowitz model, as it is known in the literature, considers the first two moments of the asset returns, namely the mean and the variance, to measure the return and the risk of the portfolio, respectively. The model is known as Markowitz's MVO (Mean-Variance Optimization) in the financial world.

Let the weight of an asset in a portfolio be the proportion of the total funds invested in this asset. The portfolio return is modeled as a linear function in the weights, representing the portfolio's expected return; and the portfolio risk is modeled as a quadratic function in the weights, representing the variance of the portfolio. The tradeoff between risk and return is obtained by solving a simple quadratic program (QP) of the form

$$\max_w \alpha^T w - \lambda w^T Q w$$

where α is the vector of expected returns, Q is the covariance matrix of returns, and $\lambda > 0$ is the portfolio manager's risk aversion parameter that represents the investor's preference as how to tradeoff risk and return. The solution to the QP determines the asset weights in an *efficient* portfolio – the portfolio with the minimum risk for a given level of expected return or (equivalently) the one with the largest expected return for a given level of allowed risk. The Markowitz model has had a profound impact on the investing world and is now widely used for asset allocation, tactical portfolio construction, hedging, and other

PORTFOLIO OPTIMIZATION

portfolio construction problems. From a computational perspective, MVO is an *easily solvable* optimization problem; it is a convex quadratic program that can be efficiently handled by state-of-the-art optimizers even in the case where there are tens of thousands of assets in the investable universe.

When λ is small, the contribution of the portfolio risk to the overall objective is small, leading to higher risk portfolios with larger returns. Conversely, when λ is large, less risky portfolios with lower returns are generated. In fact, one can solve the MVO model with different values of λ starting from zero to design portfolios with different risk and return profiles. The set of all these portfolios determines the *efficient frontier*. Investors can then choose a portfolio from the efficient frontier based on their return or risk mandates and appetites. It is worth reiterating that the MVO approach does not return a single optimal portfolio but rather a family of them that lie on the efficient frontier. For a given target return, a portfolio on the efficient frontier gives the least risky portfolio. Similarly, for a given target risk, a portfolio on the efficient frontier gives the portfolio with the greatest return.

Consider a simple MVO model where we have a budget constraint and non-negativity requirements on the asset weights. We will highlight the importance of these constraints in Section 16.b. The efficient frontier for this model is shown in Figure 1. Note that this frontier is a hyperbola. Let us briefly explain how this frontier was obtained. First, we minimize the risk of the portfolio over these constraints to find the minimum risk value. We then choose 100 risk values between the minimum risk value and the maximum risk threshold (50%). For each risk value, we solve an MVO model where we maximize the expected return of the portfolio subject to the portfolio constraints and the requirement that the risk of the portfolio be less than or equal to the aforementioned risk value. We then plot these 100 portfolios in the (standard deviation of portfolio return, expected return) space to generate the efficient frontier. Note that portfolios that are below the efficient frontier can be improved, i.e., for a given risk threshold one can get a larger return by choosing the portfolio on the efficient frontier instead. Moreover, since the frontier is efficient, there cannot be any portfolios satisfying the constraints that are above the frontier.

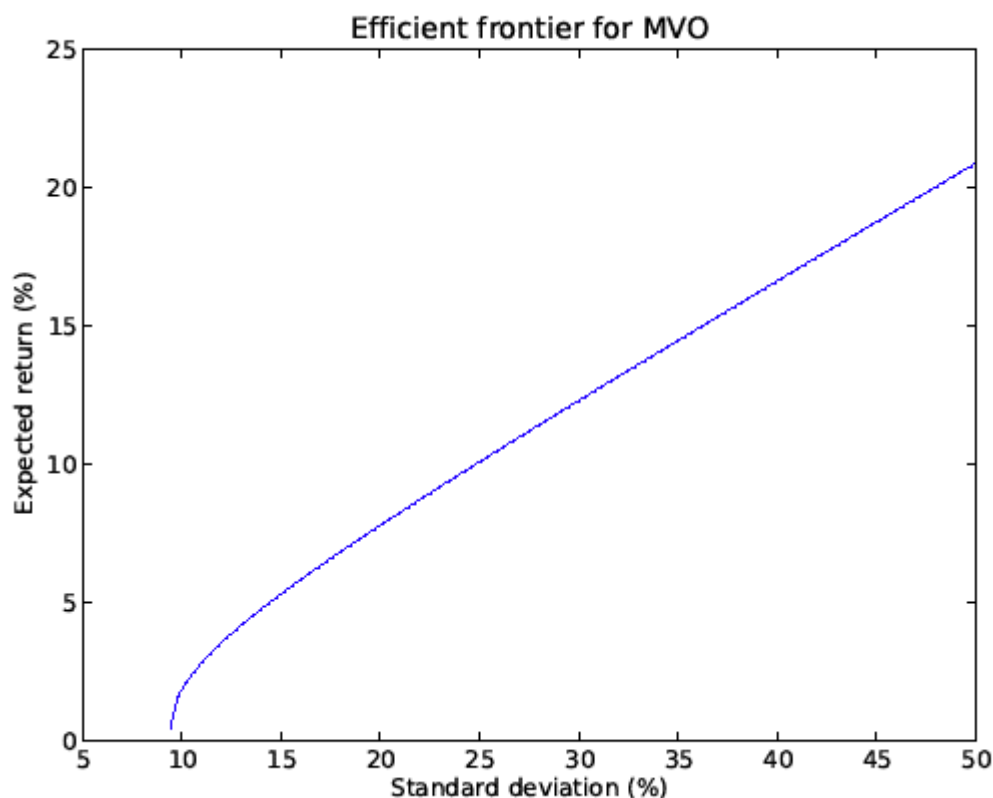


Figure 1: Efficient frontier

The Markowitz MVO model contains two critical parameters that must be estimated: (a) The expected return vector α , and (b) the covariance matrix Q . Both these parameters are estimated with a variety of statistical techniques that rely on historical and forward-looking information, such as analysts' estimates of future earnings. Estimating expected returns is by far a more difficult task, and is what most believe differentiates portfolio managers. While estimating the covariance matrix Q is also non-trivial, the structure of the asset covariances and volatilities is more stable over time, somewhat simplifying the task. Q is commonly estimated by *dimensionality reduction techniques* that decompose this matrix into a *factor model* (see Grinold & Kahn [3] and Zangari [4]).

This chapter is organized as follows: Section 16.b describes the *Enhanced MVO* (EMVO) model with objectives and constraints. Section 16.c describes the commonly used objectives and constraints in the

PORTFOLIO OPTIMIZATION

EMVO model along with a classification that indicates the complexity of the mathematical functions that are used to model them. Section 16.d describes new techniques based on *robust optimization* and *Bayesian approaches* that improve the EMVO model by addressing the issue of estimation errors in the expected returns. Section 16.e describes *Factor Alignment Problems* (FAPs) that arise due to natural disparities in the information that is used to construct EMVO models together with techniques that address them. Section 16.f describes *constraint attribution* that is commonly used by portfolio managers to quantify the impact of each individual constraint in the EMVO model. Section 16.g describes specially structured EMVO models arising in *multi-portfolio* and *trade scheduling* problems. Section 16.h describes *Extreme Tail Loss* (ETL) models and *downside risk* measures, where the return distributions have long left tails. Section 16.i describes how one can incorporate *non-linear* instruments, such as *options* in the EMVO model. Section 16.j describes the common algorithms that are used to solve EMVO models. Finally, Section 16.k describes the important features that portfolio managers should consider when they shop for a *portfolio optimizer* that best meets their needs.

b) THE ENHANCED MVO MODEL

In addition to trading off risk and return, portfolio managers must often make investment decisions that satisfy a set of constraints imposed by asset owners, regulators, risk managers, and trading desks alike. In addition, portfolio managers may implement some of their insights and investment views in the form of additional constraints. Constraints have also been used historically to address some of the well-known deficiencies of the MVO model, especially those concerning the impact of estimation errors in the optimal portfolio; see DeMiguel et al. [5] and Jagannathan & Ma [6].

We briefly describe the enhanced MVO model with various constraints and objectives in this section. First, we must introduce some notation. Consider a portfolio manager with an investment universe of n assets. Let w_i denote the weight (proportion of total funds) invested in the i th asset. Let α_i denote the manager's estimate of the expected return for the i th asset. The portfolio return is given by

PORTFOLIO OPTIMIZATION

$$r_p = \sum_{i=1}^n \alpha_i w_i.$$

The covariance matrix of the asset returns is generally obtained from a factor-risk model. The factors represent macro-economic or fundamental entities or the different industries of the economy. An asset's return is decomposed into a portion driven by these factors (common factor return) and a residual component (specific return). Let us assume that there are k factors in the model. The decomposition of the asset returns can be expressed as

$$r = Bf + u$$

where r is the asset returns vector of size n , f is the factor returns vector of size k , u is the specific returns vector of size n , and B is the $n \times k$ exposure matrix. Each row of the exposure matrix denotes an asset's exposure to the k factors. Each asset-specific return is uncorrelated with the other asset-specific returns; moreover, each specific return is also uncorrelated with the factor returns. The covariance matrix of the asset returns is given by

$$Q = B \Sigma B + \Delta^2$$

where B is the factor covariance matrix of size k and Δ^2 is a diagonal matrix containing the specific variances. The risk of the portfolio is given by

$$Portfolio Risk = w^T (B \Sigma B + \Delta^2) w.$$

The original Markowitz MVO model used the variance of the returns to measure the portfolio risk. This gives rise to a quadratic program (QP) for which an algorithm based on the simplex method for linear programming was available in Markowitz's time. One can also use the standard deviation of the returns to measure the risk of the portfolio; the resulting EMVO is now a convex program that we discuss in Section 16.j. Let Ω denote the set of *admissible portfolios*. The *Enhanced* MVO model is given by

PORTFOLIO OPTIMIZATION

$$\max \alpha^T w - \lambda w^T (B \Sigma B^T + \Delta^2) w$$

$$s. t. \quad w \in \Omega$$

where $\lambda > 0$ determines the tradeoff between return and risk.

Let us illustrate the utility of constraints that define the set of admissible portfolios by considering the following MVO model

$$\max \alpha^T w - \lambda w^T Q w$$

$$s. t. \quad 0 \leq w_i \leq u_i, \quad i = 1, \dots, n$$

with simple bounds on the asset weights. In this case, it is possible to find an equivalent unconstrained portfolio optimization problem. In fact, the portfolio manager is actually solving the following unconstrained MVO model

$$\max_w (\alpha + \gamma - \pi)^T w - \lambda w^T Q w$$

where $\gamma \geq 0$ and $\pi \geq 0$ contain the optimal dual (Lagrangian) multipliers for the lower and upper bounds on the asset variables, respectively. When the lower bound constraint on asset i is binding, i.e., $w_i = 0$, we have $\gamma_i > 0$ and $\pi_i = 0$. This is likely to be the case when α_i is small. In this case, the expected return for the i th asset is increased from α_i to $(\alpha_i + \gamma_i)$. Conversely, when the upper bound on asset i is binding, i.e., $w_i = u_i$, we have $\gamma_i = 0$ and $\pi_i > 0$. This is likely to be the case when α_i is large. In this case, the expected return for the i th asset is reduced from α_i to $(\alpha_i - \pi_i)$. So, imposing these constraints in the optimization problem is equivalent to *shrinking* the expected return vector toward the average of the expected returns in this vector. This, in turn, will prevent the optimizer from taking large bets on the assets with small and large alpha components.

Although constraints change the structure of the MVO model, most of the commonly used constraints can still be modeled within what we term the *Enhanced MVO* (EMVO) model, and the resulting problem can

PORTFOLIO OPTIMIZATION

still be solved efficiently by modern optimizers. In the next section, we will enumerate the most typical constraints and classify them according to how they can be modeled mathematically. In fact, the true test of a good optimizer is in its ability to handle these disparate constraints and to produce accurate solutions in reasonable amounts of solution time, regardless of the type (or number) of constraints used.

c) CONSTRAINTS AND OBJECTIVES IN EMVO

Portfolio managers use constraints to model a variety of business rules that are imposed on the optimal portfolios. We classify constraints in the following categories:

- I. **Holding constraints:** These are constraints that are imposed on the individual portfolio holdings, or in any linearly weighted combination of the holdings:
 - a. **Bounds on individual holdings:** These are general lower and upper bounds on the holdings in an individual asset; or a group of assets, such as an industry, sector, country, or region. The simplest such constraint is the long-only constraint on an asset holding.
 - b. **Bounds on active holdings:** The active holding for an asset is the difference in investment in the asset between the portfolio and the benchmark. Active holding constraints allow the portfolio manager to directly limit the active bet with respect to any asset in the benchmark. One can also impose bounds on the active holdings on a group of assets in the portfolio, such as those in an industry.
 - c. **Budget constraint:** The budget constraint limits the total dollar holding in the portfolio to the budget available to the manager.
 - d. **Bounds on the expected return:** This constraint imposes a lower bound on the expected return of the portfolio.
 - e. **Limits on the portfolio beta:** The beta of the portfolio measures how its returns move with respect to the market. One can impose lower and upper limits on the beta of the portfolio.

PORTFOLIO OPTIMIZATION

- f. **Limiting the factor exposure:** Factor exposure measures how a portfolio is exposed to particular factors in the risk model. Portfolio managers can add factor exposure constraints either to get greater exposure or to limit their exposure to the factors in their risk models.
 - g. **Round lot constraint:** In the real world, assets can be bought or sold in multiples of round lots (for example, 50 or 100 shares). Round lot constraints can be added for the relevant assets in the portfolio to ensure that an asset transaction is a multiple of its round lot size.
 - h. **Issuer holding constraint:** Portfolio managers and ETF providers who track a benchmark often must satisfy regulatory issuer holdings constraints, such as the 5-50 constraint that prescribes that the sum of all the asset weights that are greater than 5% should be less than 50%.
 - i. **Limiting the long/short ratio:** Legal or institutional guidelines often require a portfolio manager to limit the ratio of the long to the short holdings of a portfolio. This constraint is used, for example, to create 130/30 portfolios.
 - j. **Minimum threshold holdings:** This constraint models the restriction that if an asset is held, then the position must be at least some threshold (minimum) amount (either long or short). This constraint is used to ensure that the optimal portfolio does not contain positions that are *too small*.
- II. **Names Constraints:** These are limits that are set on the number of positions held in the optimal portfolio, the number of trades executed, or the maximum number of holdings for any particular group of assets.
- a. **Limit the number of names held:** This constraint limits the number of assets that are held in the portfolio. One can also impose this constraint over a subset of the portfolio;

PORTFOLIO OPTIMIZATION

for example, one can limit the number of assets that are held in a specific industry in the portfolio.

- b. **Limit the number of transactions:** This constraint limits the number of assets traded in a portfolio or a subset of a portfolio, such as an industry in the portfolio. This is another way to control the transaction costs associated with a portfolio.

III. **Risk Constraints:** Although the MVO model allows the portfolio manager to control risk via the risk aversion parameter, an alternative approach is to limit risk directly, by imposing an explicit constraint that the portfolio risk be less than a target value. There are various types of risk that can be controlled in this manner:

- a. **Limit absolute risk:** This constraint limits the standard deviation of the portfolio returns over the rebalancing time horizon.
- b. **Limit active risk:** The active risk of a portfolio is the standard deviation of the difference of the portfolio and benchmark returns over the rebalancing time horizon. This constraint limits the portfolio risk relative to the benchmark. This constraint is commonly known as a *tracking error* constraint, and it is sometimes imposed on the active variance of the portfolio.
- c. **Limit the relative marginal contribution to total risk or active risk:** These constraints impose lower or upper bounds on the total or active risk that can be attributed to a specific asset or group of assets. One special case of the constraint is used in the construction of *risk-parity* portfolios (see Maillard et al. [7], Axioma Advisor [8], and Asness et al. [9]); where the objective is to equalize the total risk over all the assets in a long-only portfolio. This can be done by adding a relative marginal contribution to total risk constraint on each asset in the portfolio with a lower bound of $(1/n)$, where n is the number of assets in the portfolio.

IV. **Trading Constraints:** Most of the time, a portfolio manager is using the MVO model not to generate a portfolio from scratch (all cash position), but to rebalance an existing portfolio. We

PORTFOLIO OPTIMIZATION

define an asset transaction as the amount of the asset that is either bought or sold in the rebalancing. Here are some of the popular constraints on transactions:

- a. **Bounds on transactions:** These are general lower and upper bounds on the transactions in an individual asset or a set of assets.
- b. **Minimum threshold transactions:** These constraints impose a minimum transaction size for assets; they ensure that if there is a transaction (buy/sell) on an asset, the transaction is at least for the minimum threshold amount.
- c. **Limit turnover:** The turnover of an asset is the total amount of the asset that is either bought or sold. A portfolio manager can impose a turnover constraint on an asset or a group of assets in order to put an upper limit on the turnover of the portfolio, industry, or even an individual asset in the portfolio. This is commonly done to reduce the transaction costs associated with a portfolio rebalancing.

The constraints described above can be mathematically modeled by using either: (a) linear (LIN), (b) quadratic (QUAD), (c) nonlinear-convex (NLC), and (d) combinatorial (COMB) functions of the portfolio weights. Linear constraints can be easily added to the EMVO model without affecting its complexity. In fact, the original Markowitz model had a budget constraint and non-negativity (no-short) constraints on the weights of the portfolio. Adding quadratic constraints to the EMVO model destroys the quadratic programming structure. These constraints are typically added to limit the deviation of the optimized portfolio from one or more benchmarks, such as the Russell 1000. Quadratic constraints can be added to the objective term of the MVO model with weights that quantitatively indicate their importance in the optimization. However, it is a challenging task to calibrate a weight parameter for each constraint that is added to the objective function of the EMVO model. Some of the constraints that have been recently proposed by practitioners can be modeled with non-linear but convex functions, while others require the use of combinatorial and non-convex functions. A popular portfolio constraint is the names constraint that limits the number of assets held in the portfolio; this is very helpful in keeping the

PORTFOLIO OPTIMIZATION

underlying portfolio as simple as possible. Unfortunately, the names constraint is a difficult combinatorial constraint to handle in practice. Constraints play an important role in the Enhanced MVO model; they greatly align the portfolio weights with the desires of the portfolio manager. We will classify constraints by the complexity of the mathematical functions that are required to model them. These are (a) linear, (b) quadratic, (c) non-linear convex, and (d) combinatorial in Table 1.

Some portfolio managers are also allowed to short assets in the portfolio. The short positions allow the manager to achieve leverage, i.e., the ability to finance the portfolio with more than the available budget by investing the proceeds from the assets that are held short. A dollar-neutral portfolio has zero net investment, i.e., the long positions are entirely financed by the short positions. Suppose that we require in addition that the portfolio is fully invested in these long positions. This strategy is said to be leveraged 2:1. The constraints that model this strategy are combinatorial in nature. Contrast this with the equality budget constraint in the long-only case that can be modeled as a simple linear constraint. This discussion highlights that a simple constraint (LIN) can become complicated (COMB) due to the presence of both long and short assets in the portfolio. This, in turn, increases the complexity of an EMVO model.

An important distinction is between *hard* and *soft* constraints in the portfolio. Hard constraints must be exactly satisfied in the model. Soft constraints can be violated by a user-specific amount but there is a penalty imposed for violating these constraints. Sometimes, the portfolio manager is only interested in finding a portfolio that best satisfies a set of constraints and also has a ranking of constraints that are layered in their order of importance. In this case, the EMVO model is solved in a hierarchical fashion, where the constraints are sequentially added in their order of importance. If the resulting problem is infeasible, an auxiliary problem that minimizes the constraint violation is solved. The solution to this auxiliary problem is used to correct the right-hand side of the violated constraint, and the procedure continues through all the levels that the portfolio manager has considered in the hierarchy.

PORTFOLIO OPTIMIZATION

Portfolio managers may also consider other goals in addition to the tradeoff between risk and return.

Some of the commonly used objectives in the MVO model are the following:

- (1) **Minimize active risk:** Minimizing active risk is a common *passive* investment strategy, where the aim is to stay close to a benchmark.
- (2) **Minimize active variance:** One can also minimize the active variance to ensure that the portfolio tracks a benchmark.
- (3) **Minimize transaction costs:** Transaction costs model explicit costs, such as fees and commissions. They are typically modeled by linear or piecewise-linear functions of the asset transactions.
- (4) **Minimize market-impact costs:** Market-impact costs are important for large institutional investors. They model implicit costs associated with trading assets that are difficult to measure in practice. Market-impact costs are typically modeled by quadratic, three-halves, or five-thirds power functions of the asset transactions, and they serve to discourage large transactions.
- (5) **Minimize tax liability:** The U.S. tax code specifies that short term and long term capital gains are taxed at different rates. For a tax-sensitive investor, the difference in the rates can have a significant impact on the after-tax return of the portfolio. One of the primary objectives of the *tax-sensitive* portfolio manager is to minimize the tax liability of the portfolio. The tax liability objective, however, is a combinatorial function of the individual tax-lots of the transactions.
- (6) **Maximize the Sharpe ratio:** The Sharpe ratio is a common performance measure that evaluates the excess portfolio return (portfolio return above the risk-free rate) per unit of risk. In the Sharpe ratio, the risk in the denominator is measured as the standard deviation of the portfolio returns. Although, the Sharpe ratio by itself is a non-convex function of the asset holdings, one can model the problem of maximizing the Sharpe ratio of an EMVO problem with an equality budget constraint as a convex optimization problem (see Cornuejols & Tutuncu [10]). An additional assumption required in this convex reformulation is that the optimal portfolio has an expected

PORTFOLIO OPTIMIZATION

return that is greater than the risk-free rate so that the Sharpe ratio of the optimal portfolio is positive. The reformulation exploits the *scale-invariance* property of the Sharpe ratio.

Strategy	Classification
Bounds on holdings	LIN
Minimum threshold holdings	COMB
Limiting the long/short ratio	NLC
Names constraint	COMB
Turnover constraint	NLC
RMCTR	COMB (risk parity case is NLC)
Budget constraint	LIN
Dollar-neutral (full invest)	COMB
Risk constraint/objective	NLC
Variance constraint/objective	QUAD
Expected return constraint	LIN
Portfolio beta constraint	LIN
Factor exposure constraint	LIN
Round lot constraint	COMB

PORTFOLIO OPTIMIZATION

Issuer holding constraint	COMB
Transaction objective	LIN or NLC
Market-impact objective	QUAD or NLC
Tax liability objective	COMB
Sharpe ratio objective	NLC

Table 16.1: Classification of commonly used constraints and objectives in MVO

When using multiple objectives, the portfolio manager has the additional challenge of assigning appropriate weights (aversion) to the various objectives terms that are used in the EMVO model. On the other hand, the portfolio manager typically has a ranking of objectives that are layered in their order of importance. In addition, each objective has a tolerance that specifies the maximum degradation in the objective value that can be tolerated. In this case, the portfolio manager can solve the EMVO model in a hierarchical manner, where a different (chosen) objective is optimized in each level of the hierarchy, subject to constraints that ensure that the objectives in the preceding levels get no worse than their maximum degradation values. Let us illustrate this approach on a simple example. A portfolio manager wants to track a benchmark as closely as possible and is willing to tolerate a degradation of 1% from the minimum tracking error. The secondary objective is to minimize the tax liability. In the first level of the hierarchical approach, an EMVO model with only the tracking error objective is solved. Suppose that the optimal portfolio to this EMVO model has a 3% tracking error. In the second level, the EMVO model with only the tax objective is solved. This EMVO model has an additional tracking-error constraint that limits the deviation from the benchmark to be 4%. The optimal portfolio to the EMVO model in the second level of the hierarchy is the required portfolio for the manager. The success of this approach

PORTFOLIO OPTIMIZATION

hinges on a portfolio optimizer's ability to interchange objectives with their corresponding constraints and to solve the sequence of EMVO models at the different levels of the hierarchy quickly.

We conclude this section with the concept of *implied alpha* that we will use later in this chapter. Effectively, constraints can be thought of as distorting the holdings in the unconstrained MVO model. Implied alpha is the vector of expected returns that when used in the unconstrained setting (no constraints in the EMVO model) has the same optimal solution (in terms of portfolio weights) as the constrained EMVO model. Let us illustrate the concept of the implied alpha with the following EMVO model

$$\begin{aligned} \max \quad & \alpha^T w - \frac{\lambda}{2} w^T Q w \\ \text{s.t.} \quad & Aw \leq b \end{aligned}$$

where $Aw \leq b$ includes a set of linear inequality constraints on the portfolio weights.

The first order optimality conditions for this model can be written as

$$\begin{aligned} \alpha - \lambda Qw - A^T \pi &= 0, \\ \pi^T (Aw - b) &= 0, \quad \pi \geq 0, \end{aligned}$$

where π is the vector of dual multipliers for the inequality constraints $Aw \leq b$. Let α^i represent the implied alpha of the portfolio. From the optimality conditions for the unconstrained case, we have $\alpha^i = \lambda Qw$. Plugging this expression in the first order optimality conditions, we obtain

$$\alpha^i = \alpha - A^T \pi.$$

This shows that the implied alpha is obtained by tilting the portfolio manager's alpha in the direction of the constraints as determined by the optimal dual weights. Note that only the active (binding) constraints with positive dual multipliers affect the α vector.

d) FURTHER IMPROVEMENTS TO THE ENHANCED MVO MODEL

A common complaint from practitioners regarding the EMVO (and MVO) model is that the efficient frontiers are often *error maximized* and *investment irrelevant* portfolios. This is due to the fact, they argue, that the solution to the EMVO model is highly sensitive to perturbations in the data parameters. Typically, the mean and the variance estimates are obtained from historical information, and these estimation processes are subject to statistical errors. Furthermore, it is believed that most of the estimation error sensitivity of optimal portfolios is due to errors in the estimated expected returns, rather than in errors in the estimation of the covariance matrix. Michaud [11] also argues that EMVO overweights assets with a large ratio of estimated expected returns to estimated variances, and these are precisely the assets with large estimation errors. Michaud proposes using a Monte Carlo approach by sampling repeatedly from the return distribution and averaging the portfolios that are generated in the EMVO model. Michaud calls his approach *portfolio resampling* and advocates it as a solution to the estimation problem. While the resampling approach may indeed reduce the effect of estimation errors in the (re-sampled) optimal portfolio, the solution to this problem involves the solution of an EMVO problem for each return sample that is drawn, which may be time consuming for large portfolios. In addition, the re-sampled average portfolio generated with this method need not satisfy all the constraints that are imposed in each individual EMVO model, especially the constraints that are non-convex or combinatorial in nature.

One method to address estimation errors in the expected returns in EMVO is to use a technique called *robust optimization* (see Ben-Tal et al. [12]). The robust optimization approach assumes that, although the asset returns are unknown, they are confined to an uncertainty set Ψ . One popular uncertainty set is an ellipsoidal uncertainty set that is constructed from the mean and the covariance of the asset returns. This set is of the form

$$\Psi = \{r: (r - \alpha)^T Q^{-1} (r - \alpha) \leq \kappa^2\}$$

PORTFOLIO OPTIMIZATION

where α is the investor's estimate of the mean return and Q is the covariance matrix. These are also the inputs to the EMVO model. Also, $\kappa > 0$ is the radius of this ellipsoid; a larger value for κ results in a larger uncertainty set. The robust portfolio optimization approach is then set in the following game theoretic setting. The investor has a highly intelligent and rational opponent. The investor chooses the portfolio weights w from the set of all admissible portfolios Ω and the opponent chooses the worst return vector r in the uncertainty set Ψ for this set of portfolio weights. So, the objective of the investor is to choose portfolio weights w in a way that maximizes the worst possible objective value. This problem can be formulated as

$$\max_{\{w \in \Omega\}} \min_{\{\alpha \in \Psi\}} w^T \alpha - \lambda w^T Q w.$$

At the outset, the robust portfolio problem appears daunting since the inner minimization is over an infinite number of returns in the uncertainty set Ψ . However, one can show that for the choice of the ellipsoidal uncertainty set, the robust portfolio problem can be reformulated as

$$\max_{\{w \in \Omega\}} \alpha^T w - \lambda w^T Q w - \kappa \sqrt{w^T Q w}.$$

Note that the robust optimization problem has an additional risk term with the κ multiplier in the objective function. We can interpret this term as the penalty for estimation errors in the returns α . The resulting problem can be formulated as second-order cone program and solved to optimality in about the same time as a regular EMVO model (with the same constraints in Ω) by a state-of-the-art optimizer. Different uncertainty sets are considered in Ceria & Stubbs [13], Goldfarb & Iyengar [14], and Tutuncu & Koenig [15]. Although the robust EMVO model is more conservative in its choice of optimal portfolio weights, the resulting weights are less sensitive to errors in the estimated returns.

Another approach to dealing with estimation errors is to use Bayesian adjustment techniques. For example, it is possible to use stable estimators for the returns. James-Stein estimators take a weighted average of the expected return and another estimator called the *shrinkage target*. Jorion [16] uses the

PORTFOLIO OPTIMIZATION

return from the minimum variance portfolio as the shrinkage target. In some cases (but not always!), optimized portfolios utilizing shrinkage estimates lead to improved realized performance. The Black-Litterman approach ([17], [18]) uses two sources of information to compute the expected return vector in the EMVO model. The first is a *prior* belief that the expected returns can be expressed as

$$\alpha = \pi + \epsilon_{\pi}$$

where π is the market equilibrium return, i.e., the return in the EMVO model that gives the market portfolio, and ϵ_{π} is normally distributed with zero mean and covariance $\tau\Omega$. The τ parameter reflects the level of belief in the equilibrium returns; more weight is given to the equilibrium return if τ is small. The portfolio manager provides a second source of views on the expected return vector. Suppose we have k views that can be expressed in matrix notation as

$$P\alpha = q + \epsilon_q$$

where P is a $k \times n$ views matrix, q is a k dimensional views vector, and ϵ_q is normally distributed with zero mean and covariance matrix Γ . These two sources are combined in the Black-Litterman approach to give the following expression

$$\alpha = (\tau^{-1}\Omega^{-1} + P^T\Gamma^{-1}P)^{-1}(\tau^{-1}\Omega^{-1}\pi + P^T\Gamma^{-1}q)$$

for the expected return. This estimate is then used in the MVO model. An important feature of the Black-Litterman model is that in addition to absolute views on the returns, *relative* views, such as “*The return of Oracle will exceed the return of IBM by 5%*”, can also be incorporated in the model. A good overview of the Black-Litterman model can be found in Litterman [19], Idzorek [20], and Meucci [21].

e) FACTOR ALIGNMENT PROBLEMS

As we discussed in the previous sections, there are three sources of information that are used to build portfolios with the EMVO model, namely, (a) the expected return (alphas), (b) a factor-based risk model

PORTFOLIO OPTIMIZATION

that is used to measure the predicted risk of the admissible portfolios, and (c) constraints or additional objective terms that are used to model portfolio preferences or the set of admissible portfolios. Since (a), (b), and (c) can be generated by disparate and sometimes independent estimation processes, there are natural disparities between these three *entities*. The problems that arise due to these disparities are called *factor alignment problems* (FAPs) in the literature; see Ceria et al. [22]. FAPs result in optimal portfolios that suffer from risk underestimation, undesired exposures to factors with hidden systematic risk, a consistent failure of the portfolio manager to achieve *ex ante* performance targets, and an intrinsic inability to transform a competitive advantage in terms of alphas into outperforming portfolios. FAPs have recently received considerable attention in the financial literature; see [22], Lee & Stefek [23], Bender et al. [24], and Saxena & Stubbs ([25], [26]).

We illustrate the FAP that arises due to the misalignment between the alpha and the risk factors by considering a simple EMVO model where there are no portfolio constraints, i.e., where the portfolio manager only considers the tradeoff between risk and return. Let's also assume that the covariance matrix Q is given by a factor risk model. For simplicity, we will assume that all the assets in the risk model have the same specific risk σ , i.e., the specific covariance matrix is given by $\Delta^2 = \sigma^2 I$, where I is the identity matrix. The portfolio manager's α can be broken into the following components

$$\alpha = \alpha_B + \alpha_{\{B^\perp\}}$$

where $\alpha_B = B(B^T B)^{-1} B^T \alpha$ is the projection of α that is spanned by the exposures in the risk model and $\alpha_{\{B^\perp\}} = \alpha - \alpha_B$ is the portion of α that is orthogonal to the risk exposures. By definition $B^T \alpha_{\{B^\perp\}} = 0$.

The optimal portfolio is given by

$$w = \frac{1}{\lambda} Q^{\{-1\}} \alpha = \frac{1}{\lambda \sigma^2} \alpha_{\{B^\perp\}} + \frac{1}{\lambda \sigma^2} \left(I - B(B^T B + \sigma^2 \Sigma^{\{-1\}})^{\{-1\}} B^T \right) \alpha_B.$$

PORTFOLIO OPTIMIZATION

The optimal portfolio is given by the sum of two terms that are based on the decomposition of the portfolio manager's alpha. The first term is the $\alpha_{\{B^{\perp}\}}$ component scaled to adjust for the specific risk. The second term is the α_B term that is also scaled to adjust for the specific risk; this component is also attenuated in the optimal portfolio to minimize the systematic risk that it bears. In other words, the optimizer does not perceive any systematic risk in $\alpha_{\{B^{\perp}\}}$ and it over-weights this component relative to α_B . In doing so, the optimizer takes excessive exposure to factors that are missing from the risk model that is used to construct the optimal portfolio. The risk underestimation phenomenon was verified empirically in [26]. The presence of constraints introduces additional sources of misalignment. In this case, it is possible to perform an equivalent analysis if we use the *implied alpha* that we introduced in Section 16.b. The alignment issues are then expressed in terms of the component of the implied alpha that is spanned by the exposures in the risk model and the component that is orthogonal to the risk exposures.

Some novel techniques have been proposed to correctly account for the systematic risk for the *hidden* factors that are not part of the risk model. Axioma's patented *Alpha-Alignment-Factor* (AAF); (see Renshaw et al. [27]) dynamically constructs a new factor during the solution of the EMVO by penalizing the portion of the portfolio that is not spanned by the factors in the risk model. This is done by solving the following optimization problem

$$\max_{\{w \in \Omega\}} \alpha^T w - \lambda (w^T Q w + \gamma w_{\{B^{\perp}\}}^T w_{\{B^{\perp}\}})$$

where $Q = (B^T \Omega B + \Delta^2)$ is a factor risk model, $\gamma > 0$ is a suitable weight, $w_{\{B^{\perp}\}}$ is the portion of w that is orthogonal to the exposures in the risk model, and Ω is the set of admissible portfolios. This optimization problem has several important features: (a) it can be set up as a convex problem and solved to optimality, and (b) it improves the accuracy of risk prediction and also the *ex post* performance of the optimal portfolio.

PORTFOLIO OPTIMIZATION

Saxena & Stubbs ([25], [26]) consider an EMVO model where one maximizes the expected return subject to a risk constraint and other user specified constraints. The risk constraint is of the form

$$w^T(B^T\Omega B + \Delta^2)w \leq \sigma$$

where $\sigma > 0$ is an appropriate risk bound. The EMVO model is solved for different values of the risk bound to generate several portfolios. The realized risk for these portfolios is plotted against the true expected returns for these portfolios to generate the *ex post* traditional risk-return frontier. Saxena & Stubbs then generate the AAF portfolios by solving the EMVO model where the factor risk model is also augmented with the AAF. This model is generated by replacing the earlier risk constraint with the following constraint

$$w^T(B^T\Omega B + \Delta^2)w + \gamma w_{\{B^{\perp}\}}^T w_{\{B^{\perp}\}} \leq \sigma$$

where $\gamma > 0$ can be effectively calibrated. The *ex post* AAF frontier is then generated similarly. They show that the realized risk of the portfolios on the AAF frontier is much closer to the risk predicted by the EMVO model with the AAF. Augmenting the user factor risk model with the AAF addresses the risk under-estimation problem. Moreover, Saxena & Stubbs also show that the *ex post* AAF frontier is above the *ex post* traditional risk-return frontier. In other words, for a given risk bound, the optimal portfolio generated by augmenting the user risk model with the AAF gives a larger return than the optimal portfolio that is generated by the user risk model alone. This, in turn, allows portfolio managers to access portfolios that are above the efficient frontiers generated by their risk models alone.

f) CONSTRAINT ATTRIBUTION

We have discussed the importance of constraints in the EMVO model in Section 16.b. Since constraints are used theoretically to “improve” the qualities of the optimal portfolio, the portfolio manager is often interested in the impact of the individual constraints on the overall performance of the portfolio. In other words, the portfolio manager wants to decompose the optimal portfolio by assigning a portion of the

PORTFOLIO OPTIMIZATION

optimal portfolio holdings to each of the constraints that affect the optimal portfolio, with the understanding that the constraint with a larger portion of the optimal holdings plays an important role in the design of the optimal portfolio. Different decomposition schemes are discussed in Grinold [28], Scherer & Xu [29], and Stubbs & Vandenbussche [30]. One popular decomposition scheme is called *holdings decomposition* and it works as follows: Assume that all the constraints and the objectives in the portfolio problem are convex. We will refer to the problem without any constraints as the MVO model, and the one with constraints as the EMVO model. The holdings decomposition scheme decomposes the difference in the optimal portfolio weights between the EMVO model and the MVO model into components corresponding to the different constraints. This decomposition is based on the first-order optimality conditions for convex optimization problems (see Boyd & Vandenberghe [31]). We illustrate this decomposition on the following EMVO problem

$$\max \alpha^T w - \lambda w^T Q w$$

$$s. t. \quad \sum_{i=1}^n w_i = 1$$

$$w_i \geq 0, \quad i = 1, \dots, n$$

where the constraints include an equality budget constraint and non-negativity (no-shorting) requirements on the asset weights. The optimality constraints to this problem can be written as

$$\alpha - \lambda Q w - \mu e + \sum_{i=1}^n \gamma_i e_i = 0 \quad (OPT)$$

where e is the all-ones vector of size n and e_i is the vector with one in the i th position and zeros elsewhere. In addition, one has the following constraints on the γ_i variables

$$\gamma_i w_i = 0, \quad \gamma_i \geq 0, \quad i = 1, \dots, n.$$

PORTFOLIO OPTIMIZATION

These constraints indicate that dual multipliers on the non-negativity constraints are non-negative and positive for those assets that have zero holding (these represent the active inequality constraints in the model). Since the covariance matrix Q is non-singular, one can solve for w in (OPT) to get

$$w = \frac{1}{\lambda} Q^{\{-1\}} \alpha - \frac{\mu}{\lambda} Q^{\{-1\}} e + \sum_{\{i=1\}}^n \frac{\gamma_i}{\lambda} Q^{\{-1\}} e_i. \quad (HD)$$

Note that $w^{\{MVO\}} = \frac{1}{\lambda} Q^{\{-1\}} \alpha$ contains the optimal weights for the MVO model; this is the first term in the right-hand side of (HD) . The second and the third terms on the right-hand side of (HD) represent the contributions from the equality budget constraint and the various no-shorting constraints. If a no-short constraint is not binding, i.e., $w_i > 0$, then $\gamma_i = 0$ and the constraint does not contribute anything in (HD) . One can regard each of the contributions as a separate portfolio to get the following holdings decomposition

$$w^{\{opt\}} = w^{\{MVO\}} + w^{\{budget\}} + \sum_{\{i=1\}}^n w^{\{ith\ no-short\}}$$

for the managed portfolio. In other words, the EMVO optimal portfolio is the aggregation of a *pure MVO portfolio*, a *budget portfolio* corresponding to the budget constraint, and *no-short portfolios* corresponding to the binding no-short constraints. The portfolio manager requires three sources of information to construct the portfolios corresponding to the different constraints in the holdings decomposition: (1) optimal portfolio weights, (2) the gradient of the constraint evaluated at the optimal portfolio weights if this constraint is differentiable, and (3) the Lagrangian multipliers in the first-order optimality conditions. A good portfolio optimizer should be able to give the manager this information, which can be used to construct the holdings decomposition. The situation is complicated when the underlying constraint is not differentiable. Consider the following turnover constraint

$$\sum_{i=1}^n |x_i - h_i| \leq T$$

where h_i is the initial holding of the i th asset. This constraint is not differentiable at the optimal weights if $x_i = h_i$ for any of the assets in the portfolio. In this case, one has to work with the sub-differential of this constraint that is the combination of intervals (of the form $[-1, 1]$) on the real line. Finding the correct sub-gradient from this sub-differential that satisfies the first optimality conditions is a challenging job, and often requires the solution to an auxiliary problem that resembles an MVO model.

Other decompositions, such as: (a) implied-alpha decomposition, (b) expected-returns decomposition, and (c) utility decomposition, are also considered in the literature and we refer the interested reader to Scherer & Xu [29] and Stubbs & Vandenbussche [30] for more details.

g) SPECIALLY STRUCTURED MVO MODELS

In this section we discuss specially structured EMVO models arising in multi-portfolio and multi-period settings. These models can be interpreted as individual EMVO models that are loosely *coupled* together. Special state-of-the-art parallel algorithms are available to solve these specially structured models.

Let us first consider multi-portfolio optimization. Consider a portfolio manager who handles multiple accounts for various clients. Clients have their own preferences and constraints, which can be expressed in an individual EMVO model and optimized separately. However, assume that the trades for the different accounts are pooled and executed together (which is commonly the case in practice). Moreover, the portfolio manager is generally not allowed to *cross* trades among the various accounts. For a large institutional portfolio manager, the combined market impact of the trading on the various accounts can be very large and scales non-linearly with the size of the trade. This, in turn, affects the portfolio weights obtained by optimizing each account separately. Hence, in order to solve this problem, the portfolio manager actually needs to solve a *multi-portfolio* EMVO model that accounts for the fair allocation of the

PORTFOLIO OPTIMIZATION

transaction costs among the individual accounts; see O’Cinneide et al. [32] and Stubbs & Vandembussche [33]. Different multi-portfolio EMVO models are discussed in [33]. In this section we limit the discussion to the *collusive* multi-portfolio EMVO model. In this model, the total utility maximized by the portfolio manager is the sum of the individual account utilities that is subtracted by the total market-impact cost across all the accounts.

Consider the following collusive multi-portfolio EMVO model

$$\max_{\{w^i \in \Omega_i\}} \sum_{i=1}^k \left((\alpha^i)^T w^i - \lambda_i (w^i)^T Q^i w^i \right) - \theta \left(\sum_{i=1}^k w^i \right)^T \Delta \left(\sum_{i=1}^k w^i \right)$$

over k accounts, where w^i , Ω_i denote the portfolio weights and the set of admissible portfolios for the i th account, respectively. The first term in the objective function represents the usual tradeoff between return and risk across all the individual accounts. For simplicity, we assume that we are starting from an all-cash position and so the asset holding variables also represent the asset transactions. The combined market-impact cost over all the accounts is included in the second term in the objective function, where Δ is a diagonal matrix with positive market-impact costs. Note that the second term links the individual accounts together. This term is not separable across the various accounts; if this were the case, then one could optimize the k accounts separately to find the optimal trades for each account.

To emphasize this point, we rewrite the multi-account model as

$$\max_{\{w^i \in \Omega_i\}} \sum_{i=1}^k \left((\alpha^i)^T w^i - \lambda_i (w^i)^T Q^i w^i \right) - \theta y^T \Delta y$$

$$s. t. \quad \sum_{i=1}^k w^i - y = 0.$$

PORTFOLIO OPTIMIZATION

Note that the only constraints that link the accounts together are the equality constraints in the holding and y variables. One can adopt a Lagrangian dual approach (see Hiriart-Urruty & Lemarechal [34]), where these linking constraints are added to the objective using appropriate Lagrangian multipliers to give a min-max saddle point problem. This saddle point problem is solved in an iterative fashion between a master problem that is over the Lagrangian multipliers and k separable sub-problems that can be solved in parallel. The solution to the i th sub-problem gives the holdings for the i th account. The special structure of these MVO models can also be used to parallelize interior point methods that solve the entire EMVO multi-portfolio model; see Gondzio & Grothey ([35], [36], and [37]).

A second source of specially structured EMVO models arises in a multi-period MVO model called *trade scheduling*, where a portfolio manager wants to liquidate an existing portfolio over k time periods. The objective in this case is to trade off the risk of the portfolio versus the market impact that will occur from trading the portfolio. We will assume a quadratic model for the market-impact costs where Δ^i denotes the market-impact cost matrix for the i th time period. The trade scheduling problem can be modeled as

$$\begin{aligned}
 \min \quad & \sum_{i=1}^k \lambda_i (w^i)^T Q^i w^i + \theta_i (t^i)^T \Delta^i t^i \\
 \text{s.t.} \quad & w^{\{i+1\}} - t^i - w^i = 0, \\
 & (w^i, t^i) \in \Omega_i, \quad i = 1, \dots, k-1, \\
 & w^k = 0,
 \end{aligned}$$

where w^i and t^i denote the portfolio holdings and transactions in the i th period. The first set of constraints expresses the relationship between the holding and transaction variables in two successive time periods. Note that these are the only constraints that link together the time periods. Moreover, each time period is only linked at most to its preceding and successive time periods, and the linking constraints have a *staircase* structure. The second sets of constraints represent the portfolio manager's constraints

PORTFOLIO OPTIMIZATION

over the individual time periods. The last set of constraints models the liquidation requirement, i.e., the holdings of all the assets in the final period should be zero. This EMVO model can also be solved using a decomposition algorithm that can be parallelized in practice. Moreover, one can also parallelize interior point methods to speed up the solution of the trade scheduling problem.

h) EXTREME TAIL LOSS OPTIMIZATION

The MVO model uses the variance of the returns to quantify the risk of the portfolio. Variance considers both positive and negative deviations from the mean and treats both these variations as equally risky. However, in general, the portfolio manager should only be concerned with *underperformance* relative to the mean. Secondly, a portfolio manager may have in her portfolio nonlinear assets, such as options, that have asymmetric return distributions. In this case, the aim of the portfolio manager should be to maximize the probability that the portfolio loss is less than a maximum acceptable level. These requirements have led to the development of various *downside* risk measures in finance.

The most well-known downside risk measure is *Value-at-Risk* (VaR) (see Jorion [38]) first developed by JP Morgan in 1994. The 1998 Basel Accord further popularized VaR by stipulating exposures and minimal capital requirements for banks in terms of the 10-day 95% VaR.

Let $\epsilon > 0$ denote the confidence level. For a given portfolio, confidence level, and time horizon; VaR is defined as the threshold value, such that the probability that the portfolio loss exceeds this threshold value is the given confidence level. For example, a 10-day 95% VaR of \$100 million for a portfolio indicates that there is a 95% probability that the portfolio will not fall in value by more than \$100 million over a ten day period. In other words, there is a 5% probability that the portfolio will lose more than \$100 million over this period. VaR, however, has a number of shortcomings that we list below:

- (1) It is shown in Artzner et al. [39] that VaR is not a sub-additive risk measure, i.e., the VaR of a portfolio can be greater than the sum of the VaR of the individual assets of a portfolio. Using

PORTFOLIO OPTIMIZATION

VaR as a risk measure (say in the MVO model) would lead to concentrated (less-diversified) portfolios that the VaR measure considers less risky.

- (2) It is a challenging job to minimize VaR as a risk measure in the MVO model. The resulting MVO model is a non-convex optimization problem, and one has to resort to mixed-integer programming techniques to solve it to optimality.
- (3) VaR only measures a percentile of the portfolio loss distribution; it does not measure the length of the tail of the portfolio loss distribution. The VaR value can be much smaller than the worst-case loss, which, in turn, could mean absolute ruin for the investor.

There is another downside risk measure called Conditional-Value-at-Risk (CVaR) that is closely related to VaR and addresses some of its shortcomings. CVaR measures the expected value of the loss that exceeds VaR. It is larger than VaR since it measures the losses that can occur in the tail of the loss distribution. Let $CVaR(w, \epsilon)$ and $VaR(w, \epsilon)$ denote the CVaR and VaR measures for a portfolio with weights w and confidence level ϵ . We will assume that there are k equally likely scenarios for the asset returns, and let r_i denote the asset return vector for the i th scenario. So, $-r_i^T w$ denotes the loss suffered by the portfolio in the i th scenario. We can express CVaR in terms of VaR as follows:

$$CVaR(w, \epsilon) = VaR(w, \epsilon) + \frac{\frac{1}{k} \sum_{i=1}^k \max[-VaR(w, \epsilon) - r_i^T w, 0]}{\epsilon}.$$

Note that the numerator of the fraction on the right-hand side of the equation measures the average excess loss over VaR and the denominator measures the probability of this loss. Using this expression for CVaR, Rockafellar & Uryasev [40] show that the MVO model where the risk is measured in terms of CVaR (rather than variance) can be written as a linear programming problem. Setting up the linear program, however, requires a scenario-based approach where one has to draw samples from the portfolio return distribution. For large portfolios, more samples have to be drawn, and the solution time grows quickly.

El Ghaoui et al. [41] consider a robust downside risk measure called the *Worst-Case-Value-at-Risk* (WCVaR). Consider the asset returns to be random variables with mean μ and covariance matrix Q . The WCVaR for a confidence level ϵ is the worst VaR obtained over all probability distributions with mean μ and covariance matrix Q (some of these distributions will have long tails). At the outset, this appears to be a hopeless quantity to estimate, since one needs to run through an infinite number of probability distributions, compute the VaR for each distribution, and choose the largest VaR that is obtained. However, [41] in their landmark paper show that for a portfolio with holdings w and confidence level ϵ ,

$$WCVaR_{\{\epsilon\}}(w) = \max_r \{-w^T r: (r - \mu)^T Q^{-1}(r - \mu) \leq \kappa^2\}$$

where $\kappa^2 = \frac{(1-\epsilon)}{\epsilon}$. In other words, the WCVaR for a portfolio can be obtained by solving a convex optimization problem, where WCVaR measures the worst-case loss suffered by the portfolio when the asset returns are unknown and can take any values in an ellipsoidal uncertainty set. This is also the uncertainty set that we considered in Section 16.d. Moreover, one can easily minimize WCVaR (as a downside risk measure) in an MVO model by solving a convex optimization problem as well. This is again a surprising result since minimizing VaR is a difficult non-convex problem. Also, unlike CVaR, minimizing WCVaR does not require a scenario-based optimization approach. One complaint against WCVaR is that it is a conservative downside risk measure. In fact, the WCVaR measure is more conservative for larger values of κ (that determines the radius of the ellipsoidal uncertainty set). If the portfolio manager is optimistic that the return distribution does not have very long tails, one could conceivably choose a smaller value for this parameter.

i) INCORPORATING NON-LINEAR INSTRUMENTS IN THE EMVO MODEL

There has been recent work in incorporating non-linear instruments, such as options, in an EMVO model; see Zymler et al. [42] and Sivaramakrishnan et al. [43]. Our discussion in this section is based on the work in [43]. Consider a portfolio manager who wants to protect the optimal portfolio from adverse

PORTFOLIO OPTIMIZATION

market movements, such as in the financial crisis of 2008. To do so, the portfolio manager can purchase American and European call and put options on the underlying assets in the portfolio. Options are *leveraged* instruments and so they offer better downside protection for a given investment. Note that an option is a *non-linear* instrument, i.e., its return is a non-linear function of the return of the underlying asset. This function is, however, a convex function if we go long on (purchase) the option. The options can have varying times to expiration. The only requirement is that the time to expiration is at least equal to the portfolio manager's rebalancing time horizon, to ensure that all the options have value during the rebalancing period. Sivaramakrishnan et al. derive the following convex return function

$$r_j^O = \max\{-1, f(r_{\{u_j\}})\}$$

for the j th option in the portfolio, where u_j denotes the underlying asset for this option. The convex function is derived from an appropriate option pricing function, such as the Black-Scholes model for a European option and the Cox-Ross-Rubinstein binomial asset pricing model for an American option. This return function is incorporated in the WCVaR risk measure (that we considered in the previous section) to obtain a downside risk measure for the portfolio containing equities as well as options. This risk measure can be represented as

$$WCVaR_{\{\epsilon\}}(w, w^0) = \max_{\{r, r^O\}} -(w^T r + (w^O)^T r^O)$$

$$s.t. \quad (r - \mu)^T Q^{\{-1\}} (r - \mu) \leq \kappa^2$$

$$r_j^O = \max\{-1, f(r_{\{u_j\}})\}, \quad j \in O$$

where O is the set of options in the portfolio. For a portfolio with equity and long option holdings, the WCVaR measure for the portfolio can be obtained by solving a convex optimization problem. The problem of minimizing the downside risk of the portfolio containing equities and options can then be written as

$$\min_{\{w, w^0\}} WCVaR_{\{\epsilon\}}(w, w^0)$$

$$s. t. \quad w^0 \geq 0, (w, w^0) \in \Omega$$

where $w^0 \geq 0$ represents the no-shorting requirements on options and the set Ω contains the portfolio manager's other constraints on the asset and option variables. One specific constraint in Ω is a budget constraint on the option variables. This optimization problem is convex if the set Ω is also convex, i.e., when there are no combinatorial constraints on the asset and option variables. State-of-the-art optimization solvers can solve this problem to optimality in the same time as a typical EMVO problem. Sivaramakrishnan et al. compare this approach with the delta and delta-gamma hedging strategies (see Hull [44]), that are commonly used in practice. They show that the WCVaR-based hedging strategy offers better downside protection for a wider range of option budgets than the delta and the delta-gamma hedging strategies.

j) ALGORITHMS FOR SOLVING MVO MODELS

We briefly describe algorithms for solving MVO models in this section. The basic MVO model that trades off risk versus return can be modeled as a QP. This is no longer the case once constraints (quadratic, NLC, and combinatorial) or objectives (NLC, combinatorial) are added to the EMVO model.

Recently, there has been considerable interest in using second-order cone programming (SOCP) models in portfolio optimization. SOCP deals with a special case of convex optimization problems that can be written as linear conic problems over the second-order (ice-cream) cone; see Boyd & Vandenberghe [31], Alizadeh & Goldfarb [45], and Ben-Tal & Nemirovski [46]. There are two major reasons for this interest:

- (1) A variety of LIN, QUAD, NLC portfolio objectives and constraints can be modeled via SOCP. In fact, the QP model is itself a special case of SOCP. Here are some prominent examples that can be modeled via SOCP but not as QPs:

- (a) **Risk objective or constraint that uses the standard deviation of returns:** Most portfolio managers measure risk using the standard deviation (rather than the variance) of returns, since the standard deviation of the returns has the same unit as the expected returns.
 - (b) **Market-impact terms with the 3/2 and 5/3 powers:** Almgren et al. [47] show that the market-impact cost of trading an asset is best approximated by a function that is a five-thirds power of the trade size of the asset.
 - (c) **Robust portfolio problems:** Most robust EMVO problems where the expected returns lie in well-known uncertainty sets described in Ceria & Stubbs [13], Goldfarb & Iyengar [14], and Tutuncu & Koenig [15].
- (2) Interior point algorithms that were originally developed for solving linear and quadratic programs (LPs and QPs) have recently been extended to solve second-order cone programs; see Boyd & Vandenberghe [31], Alizadeh & Goldfarb [45], and Ben-Tal & Nemirovski [46]. Moreover, with the development of efficient numerical linear algebra techniques and the availability of good software, one can now solve SOCPs with tens of thousands of variables and constraints efficiently, and in times that are comparable to that for LP and QP.

We will briefly highlight how the EMVO model

$$\max \alpha^T w - \lambda \sqrt{w^T Q w}$$

with risk measured as the standard deviation of returns can be modeled as an SOCP. For simplicity, we will consider the case when the covariance matrix Q is positive-definite. Let $t = Q^{\frac{1}{2}} w$, where $Q^{\frac{1}{2}}$ (the square root of Q) is obtained via an eigenvalue decomposition. Note that t is an n dimensional vector with components t_1, \dots, t_n , where n is the number of assets in the EMVO model. The EMVO problem can be equivalently written as

$$\max_{\{w, \bar{t}, t\}} \alpha^T w - \lambda \bar{t}$$

PORTFOLIO OPTIMIZATION

$$s. t. \quad t - Q^{\left\{\frac{1}{2}\right\}} w = 0$$

$$\bar{t} \geq \sqrt{t_1^2 + \dots + t_n^2}$$

where \bar{t} is a new scalar variable. The resulting formulation is an SOCP. Observe that the objective function in the SOCP is a linear function in the w and \bar{t} variables. Moreover, the constraints that express the relationship between the t and w variables are also linear. The only source of nonlinearity in the SOCP is the last constraint that models the requirement that the vector $(\bar{t}, t_1, \dots, t_n)$ lies in a second-order cone of size $(n+1)$.

Combinatorial constraints, such as the names constraint, further complicate the EMVO model. In this case, the resulting EMVO model becomes a mixed-integer problem (MIP); see Wolsey [48]. MIPs are notoriously difficult to solve to optimality. One reason for this difficulty is that MIPs have variables that take discrete rather than continuous values, and the feasible set for the admissible portfolios is no longer convex. In most cases, one can use heuristic approaches to find a good solution but it is hard to verify that the solution found is indeed the best, or even just close to best possible. There exist exact approaches called *branch and cut* techniques (see [48]) to solve MIPs to optimality, but these approaches lead to solution times that are not practical. Therefore, a portfolio manager often has to be satisfied with a good (while not the best) solution. The branch and cut technique described above also provides a provable upper bound that indicates how far this solution is from the best one possible. We will illustrate MIP modeling by modeling the names constraint below. Assume that the portfolio weights have the following lower and upper bounds

$$0 \leq w_i \leq u_i, \quad i = 1, \dots, n.$$

We want to model the requirement that no more than K assets can be held in the portfolio, where K is much smaller than n , the number of assets in the investable universe. Consider the binary vector z in the

PORTFOLIO OPTIMIZATION

portfolio problem, where $z_i = 1$ if asset i is held in the portfolio and 0 otherwise. The names constraint can be modeled using the following constraints

$$w_i \geq 0, \quad w_i - u_i z_i \leq 0,$$

$$\sum_{i=1}^n z_i \leq K,$$

$$z_i \in \{0, 1\}, \quad i = 1, \dots, n.$$

Note that if $z_i = 0$, the first set of constraints ensure that $w_i = 0$. Conversely, if $z_i = 1$, these constraints ensure that the weight of the i th asset is between its permissible bounds. The names constraint is then modeled as a linear inequality constraint in the binary z variables.

k) HOW TO CHOOSE AN OPTIMIZER

Consider a portfolio manager who manages a 130-30 international portfolio with about 200 assets and a large-cap growth mandate. Assume that the portfolio manager's factor risk model uses the fundamental factors in the Fama-French three-factor model (see Fama & French ([49], [50])). Two of the fundamental factors in this model are the *size* and *value* factors. The size factor measures the additional returns that investors have generated by investing in small companies. The value factor measures the additional returns that investors have generated by investing in companies with high book to price ratios. Since the manager wants to invest primarily in large-cap growth companies, the alpha is constructed from negative linear combinations of the asset exposures to the size and value factors. Alternatively, one can short the size and value factor exposures in the constraints in order to get large negative exposures to these factors. We will describe the EMVO model that represents the portfolio manager's strategy below:

- (a) **Maximize expected return:** The manager uses custom *alphas* that are generated from negative linear combinations of the asset exposures to the size and value factors. This ensures that the

PORTFOLIO OPTIMIZATION

optimizer takes larger bets on large-cap growth companies that have small book to price ratios.

This is a linear objective function.

- (b) **Tracking-error constraint:** The benchmark is the MSCI EAFE growth index with a tracking error of 8%. Assume the portfolio manager uses the standard deviation of the returns as a measure of risk. The TE constraint is then an NLC constraint that can be modeled as an SOCP constraint.
- (c) **130-30 requirement:** Short 30% of the portfolio that the manager believes to be overvalued. Invest the proceeds from the short-sale in the long assets. Assume that the long position is fully invested, i.e., there is no cash in the portfolio. This is a combinatorial constraint with the full investment requirement. Moreover, one has to create separate long and short variables for each asset that represent the amount held long and short, respectively.
- (d) **Short the *size* and *value* factor exposures:** This is to get a large negative exposure to these factors, thereby satisfying the large-cap growth mandate. These are linear inequality constraints in the factor exposure variables. There are also two linear equality constraints that link the factor exposure variables with the portfolio weights.
- (e) **Limit on the number of names:** Do not hold more than 100 assets in the portfolio. This is a combinatorial constraint.
- (f) **Threshold holdings on positions:** If an asset is held, hold at least 20 bps. This is a combinatorial constraint.
- (g) **Liquidity constraints:** Limit the holding in each asset to 10% of the ADV. These are linear inequality constraints.
- (h) **Transaction-cost constraint:** Limit the turnover to 5% of the portfolio value. This is a linear inequality constraint.
- (i) **Minimize market impact:** Use the $3/2s$ market-impact function. This is an NLC objective function that can be modeled using SOCP.

PORTFOLIO OPTIMIZATION

We will describe how one chooses a good portfolio optimizer to solve this EMVO model.

- (1) A good portfolio optimizer should be able to handle constraints directly rather than choosing a set of weights (or preferences) for each constraint and then adding it to the objective function. It is a challenging task to calibrate the weights for the constraints that are added to the objective function so that the resulting problem is equivalent to the portfolio manager's original problem. Moreover, since the aforementioned EMVO model has combinatorial constraints, the portfolio manager cannot adopt a dual Lagrangian approach to compute the correct set of weights for each constraint that is added to the objective. The dual problem is no longer equivalent to the original problem. In optimization terminology, there is a *duality gap* (see Boyd & Vandenberghe [31]) between the two problems, and solving the dual problem does not give the best solution to the original problem.
- (2) The $3/2$ market-impact objective and the tracking error constraint in the EMVO model can be modeled exactly via SOCP functions. A good portfolio optimizer should be able to handle SOCP constraints and objectives. Besides, there exist efficient algorithms to solve SOCPs in about the same time as a QP model.
- (3) The EMVO model has three different combinatorial functions: (a) the fully invested 130-30 requirement, (b) the names constraint, and (c) minimum threshold holdings constraint. These constraints model different objectives of the manager that can potentially be in conflict with each other. Some portfolio managers use an ad-hoc manner to handle combinatorial constraints; they simply solve the convex model without these constraints, and in a post-processing phase try to round the resulting solution to satisfy the combinatorial constraints. This is not a good strategy as it is hard to find a heuristic solution in this manner that satisfies the other combinatorial constraints in the problem. Moreover, there is no guarantee that the solution found in this manner will be close to an optimal solution to the original problem. A good portfolio optimizer should

have a good branch and bound algorithm to find provably good solutions, i.e., good solutions that also come with certificates on how close the purported solution is from a global optimal solution.

- (4) A good portfolio optimizer should ideally use a *primal-dual interior point method* (IPM) to solve the EMVO model. Interior point algorithms are extremely versatile; they can be used to solve linear, quadratic, and second-order cone-based EMVO models. This is unlike simplex and active set-based algorithms to solve linear and quadratic programs. An active-set algorithm cannot be used for the EMVO model due to the presence of the nonlinear tracking error constraint. Interior point algorithms can also be easily embedded in a branch and bound algorithm to solve portfolio models with combinatorial constraints. IPMs have proven worst-case complexity estimates; see Ben-Tal & Nemirovski [46]. Moreover, with recent advances in numerical linear algebra algorithms and software, IPMs can routinely solve convex portfolio problems with tens of thousands of assets reliably in a few seconds of computing time.
- (5) Suppose the portfolio manager follows a *value-momentum* mandate instead. Let the risk model continue to be the Fama-French three-factor model, which does not contain the momentum factor. In this case, a factor alignment problem would ensue, since the optimizer would overload on the momentum factor where it does not perceive a systematic risk. The FAP issue can be mitigated by dynamically incorporating the AAF (discussed in Section 16.e) in the tracking error constraint. The resulting portfolio problem can still be formulated as an SOCP and solved efficiently with an interior point optimizer. This shows that a good interior point optimizer handling SOCP objectives and constraints can potentially also address the FAP issue.

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