**Overview and Literature Survey**

We begin with defining the Ornstein-Uhlenbeck process. The *Ornstein-Uhlenbeck* process is a stochastic process with applications in financial mathematics and physical sciences. It was originally applied in physics as a model for the velocity of a massive Brownian particle under the influence of friction, also called a *Damped Random Walk* (Uhlenbeck and Ornstein (1930), MacLeod, Ivezic, Kochanek, Kozlowski, Kelly, Bullock, Kimball, Sesar, Westman, Brooks, Gibson, Becker, and De Vries (2010), Wikipedia (2019)).

The Ornstein-Uhlenbeck process is a mean-reverting stationary Gaussian-Markov process. This means that it is a Gaussian process, it is a Markov process, and it is temporally homogenous. In addition to mean-reversion, the Ornstein-Uhlenbeck process is the only non-trivial process that satisfies these three conditions, up to allowing linear transformations for time and space variables (Doob (1942)). Over time, the process tends to move around its long-term mean; such a process is called *mean-reverting*.

In addition to its mean-reverting nature above, the Ornstein-Uhlenbeck process has also been viewed as an alteration to the Weiner process. It is a modification to the walk random process in continuous time, or Weiner process, in which the properties of the process have been changed such that there is a tendency of the walk to move back to a central location, with a greater attraction when the process is further away from the center. The Ornstein-Uhlenbeck process can also be considered to be a continuous-time analogue of the discrete time AR (1) process.

The *Vasicek model* is regarded as a one-factor interest rate model, in that it was originally developed to describe the evolution of interest rates. The reason it is called a one-factor interest-rate model as it describes the interest rate movements as driven by only one source of market risk. In addition to modeling just the short rate, it is also used to price bonds. Other usage has been in the valuation of interest rate derivatives, as well as adaption to credit markets. Introduced by Vasicek (1977), it may also be viewed as a stochastic investment model.

Similar to the Vasicek model, the *Cox-Ingersoll-Ross model* is also a one-factor model. Thus, it describes the dynamics of the interest rates. It is a particular type of a *one-factor model* – short-rate model – and it too describes the interest rate movements as described by only one source of market risk. The model has been used in the valuation of the interest rate derivatives and the bond prices. It was introduced by Cox, Ingersoll, and Ross (1985) as an extension of the Vasicek model. The next section contains the details of the CIR model’s extension to the Vasicek model.

**Model Definition and Dynamics**

The Ornstein-Uhlenbeck process is defined by the following stochastic difference equation:

where and are parameters and denotes the Weiner process (Gard (1988), Gardiner (2009)). The Vasicek extension to the model is simple - an additional drift term is added.

where is a constant (Bjork (2009)).

Following the literature, we switch to the standard symbology deployed to describe the Vasicek and the CIR processes. The Vasicek model specifies that the instantaneous interest rate follows the stochastic difference equation

where is the Wiener process under the risk-neutral framework modeling the market-risk factor, in that it models the continuous inflow of randomness into the system.

Details of the model parameters are as follows. The standard deviation parameter determines the volatility of the interest rate, and characterizes the amplitude of the instantaneous market randomness inflow. The parameters , , and , together with the initial condition , completely determine the dynamics, and are described as follows, assuming to be non-negative:

* 1. => *Long-Term Mean Level* All future realizations of will revolve around a mean level in the long term.
  2. => *Speed of Reversion* – is the speed at which such realizations will center around in time.
  3. => *Instantaneous Volatility* – measures instant-by-instant the amplitude of market randomness entering the system. Randomness increases with higher .

In addition, the *Long Term Variance* is also of interest. It is computed explicitly using . This is the variance at which future realizations of regroup around the long-term mean.

and are of opposing nature, that is, they tend to move in opposite directions. Increasing increases the amount of randomness entering the system, but at the same time, increasing increases the speed at which the process stochastically stabilizes around the long-term mean with a corridor of variance given by . This can be seen from the long-term variance which increases with but decreases with .

As seen above, the Vasicek model is a type of Ornstein-Uhlenbeck stochastic process. Introducing stochastic long term parameters makes it an example of Cointelation SDE. In particular, making the long term mean stochastic to another SDE is a simplified version of the Cointelation SDE (Mahdavi-Damghani (2014)).

We now specify the dynamics of the CIR process. Here the instantaneous interest rate follow the stochastic differential equations, also named the CIR process –

where is Wiener process that models the random market risk factor, and , , and are parameters. As before, the parameter corresponds to the speed of adjustment to the mean , and to the volatility. Further the drift term is exactly the same as in the Vasicek model, and ensures the mean-reversion of the interest rate towards the long-run value , with the speed of adjustment governed by the strictly positive parameters .

The criteria for avoiding negative rates can also be derived. It can be readily seen that the standard deviation factor avoids the possibility of negative interest rates for all positive values of and . In addition, the interest rate of zero is also precluded if the condition

is met. Intuitively, when the rate is close to zero, the standard deviation also becomes very small, which dampens the effect of the random shock on the rate. As a consequence, when the rate gets close to zero, the evolution becomes dominated by the drift factor, which pushes the rate upwards towards equilibrium.

The relation between the CIR process and Ornstein-Uhlenbeck is less intuitive than the Vasicek process. It turns out that the CIR process can also be defined as a sum of squared Ornstein-Uhlenbeck process. The CIR is also an ergodic process in time and space, and possesses a stationary distribution. Among other usage, it is used in the Heston model to model stochastic volatility.

**Mathematical Properties**

An intuitive analogy exists between the Ornstein-Uhlenbeck process and the Brownian motion. The Ornstein-Uhlenbeck process is a scaling limit of a discrete process in the same way that Brownian motion is a scaling limit of random walks. As an example, consider a box containing blue and yellow balls. A ball is chosen at random at each step and replaced by a ball of the opposite color. Let be the count of the balls of any one color after steps. As tends to infinity, the quantity converges to an Ornstein-Uhlenbeck process.

The covariance and the expectation of a Ornstein-Uhlenbeck process is derived using the approach seen below. For a constant the process mean is

and the covariance across different times is

The Ornstein-Uhlenbeck process is an example of a Gaussian process that possesses a bounded variance and admits a stationary probability distribution. This is in contrast to the Wiener process. The Wiener process has a constant drift, whereas for the Ornstein-Uhlenbeck process drift is dependent on the current value of the state variable; if the current value of the process is less than the long-term mean, the drift will be positive; if the current value of the process is greater than the long-term mean, the drift will be negative. The mean acts as the equilibrium level for the process, providing the informative name *mean-reverting*.

The mean-reverting nature is introduced into the Vasicek model by construction. In fact, the Vasicek model was the first one to capture mean-reversion, as essential characteristic of interest rates that sets it apart from other financial quantities. High levels of interest rates hamper economic activity, prompting their decrease. This causes the interest rates move in a limited range, showing a tendency to revert to a long-run value.

The mean reversion is explicitly incorporated into the drift term. The drift factor corresponds to the expected instantaneous change in the interest rate at time . As seen before, is the long-term equilibrium value towards which the interest rate reverts. In the absence of randomness , the interest rate stay constant at . , which determines the speed of reversion, must be positive to ensure the stability around the long-term value. As an example, when is below , becomes positive for positive , resulting in a tendency for the interest rates to move upwards towards equilibrium.

The stochastic integral for the Vasicek process is straightforward to compute. The SDE above may be integrated to obtain

The Vasicek process mean and variance may be computed using similar techniques as applied to the Ornstein-Uhlenbeck stochastic process. Thus, one gets the normally distributed with mean

and variance

The Vasicek model long-term mean and variance may be computed likewise; the results are

and

The Vasicek model suffers from a key limitation, particularly in the pre-crisis era. It is possible for the interest rate to become negative. Improvements by the Cox-Ingersoll-Ross model, the exponential Vasicek model, the Black-Derman-Toy model, and the Black-Karasinski model, and others addresses this. The Vasicek model was further extended by Hull and White. The Vasicek model is also a canonical example of the affine term structure model, along with the Cox-Ingersoll-Ross model, and can be used to produce a simple expression for the bond prices.

CIR process is an extension to the Vasicek process that maintains the mean-reversion nature, but with a level dependent volatility . For a given positive , the CIR process is strictly positive if ; otherwise it can touch the zero point.

The expected value of in the CIR process is given by

This shows that the long-term mean is indeed . Similarly the variance of is

**The Probability Density Function and its Evolution**

The Fokker-Planck equation may be used to represent the evolution of the Ornstein-Uhlenbeck process. Therefore the Ornstein-Uhlenbeck process can also be described in terms of a probability density function which specifies the probability of finding the process in the state at a time (Risken and Till (1996)).

The Ornstein-Uhlenbeck Fokker-Planck equation may be stated explicitly as follows. satisfies

where . The above is a linear parabolic differential equation that may be solved by a variety of techniques.

Using the Fokker-Planck representation enables determining an explicit expression for the transition probability . It is Gaussian with mean and variance

The transition probability gives the probability of state occurring at time given the initial state at time . Alternatively, may be viewed as the solution of the Fokker-Planck equation with the initial condition

The same technique is used to derive the probability distribution function of a CIR process. The distribution of the future values from a CIR process can be computed in closed form as

where

and is a non chi-squared distribution with degrees of freedom and the non-centrality parameter .

The stationary distribution of the CIR process approaches a gamma distribution. Owing to mean reversion, as time becomes large, the distribution of follows the probability density

where and .

**Ornstein-Uhlenbeck Formal Solution**

The Differences from of Variation of Parameters may be used to solve the Ornstein-Uhlenbeck process. The stochastic differential equation for can be formally solved by the approach specified in Gardner (2009). Starting from

one gets

The integral form is then used to compute the RHS. Integrating from to results in

First moment follows readily from this representation, i.e., the mean, is shown to be

assuming is a constant. Moreover, the Ito isometry can be used to calculate the higher moments. For example, the covariance function is estimated from

**Calibration**

Ordinary Least Squares may be employed in the calibration of these three processes. For instance, the continuous SDE for CIR can be discretized as

which is equivalent to

provided that is i.i.d. . Any linear regression technique may be employed for calibrating the above equation. Other techniques such as martingale estimation and maximum likelihood may also be used.

In addition, finite numerical sampling estimators can also be used to compute uncertainty in the calibrated parameters. For example, by using discretely sampled data at time intervals of width , the maximum likelihood estimators for the parameters of the Ornstein-Uhlenbeck process are asymptotically normal to their true values (Ait Sahalia (2002)). More precisely, the asymptotic limits for the mis-estimation of the parameters is given by

**Simulation**

This section looks at some of the properties of the simulated sample paths. One can represent a temporally homogenous Ornstein-Uhlenbeck process as a scaled, time-transformed Wiener process.

where is the standard Wiener process (Doob (1942)). Using the change of variables this process becomes (for )

The relation above may be used to draw inferences using the known Brownian properties. The mapping is often used to translate the properties of into the corresponding statements for . As an example, the law of iterated logarithm for becomes (Doob (1942))

with probability .

Stochastic simulation of the three processes can be achieved using either of the variants – discretized and exact.

**Application**

Not surprisingly, the Ornstein-Uhlenbeck is one of more widely used models in mathematical finance. It is one of the several approaches used to model – with modifications – interest rates, currency exchange rates, and commodity prices stochastically. The parameter represents the equilibrium mean value supported by fundamentals; the volatility around it caused by market moves, and the rate at which these moves dissipate and the variable reverts towards the mean. One well-known application of the process is a strategy known as pairs trading (Rampertshammer (2007), Skiena (2008), Leung and Li (2015)).

The processes are often used for pricing the term structure of bonds. For example, under no-arbitrage assumptions, a bond may be priced using the CIR interest rate process. The bond price is exponential affine in the interest rate with the form

where

It may be easily seen that

and

The mean reverting nature of the Ornstein-Uhlenbeck process makes it applicable in physical sciences. For instance, it has been recognized to be a prototype of a noisy relaxation process. As an example, consider a Hookean spring with spring constant , whose dynamics is highly over-damped with friction coefficient .

The Ornstein-Uhlenbeck process can be used to model the oscillations of a spring under thermodynamic noise. In the presence of thermal fluctuations with temperature , the length of the spring fluctuates stochastically around the spring rest-length ; its stochastic dynamic is described by an Ornstein-Uhlenbeck process with , , and where is derived from the Stokes-Einstein equation for effective diffusion constant.

The Langevin equation used in physical sciences is a form of the Ornstein-Uhlenbeck process. There the stochastic differential equation of an Ornstein-Uhlenbeck process is rewritten as a Langevin equation

where is the white Gaussian noise with

The time correlations among the spring fluctuations can then be readily. Fluctuations are correlated as

with the *correlation time* .

Finally, application of standard statistical physics allows the calculation of the other ensemble quantities. For example, at equilibrium the spring stores an average energy

in accordance with the equi-partition theorem.

The Ornstein-Uhlenbeck process has also been proposed as an improvement over the Brownian motion for modeling change in organismal phenotypes over time (Martins (1994)). A Brownian motion model implies that a phenotype can change without a limit, whereas for most phenotypes natural selection imposes a cost for changing too far in either direction.

**Generalizations and Extensions**

It is possible to extend the Ornstein-Uhlenbeck processes to process where the background driving process is a Levy process instead of a simple Brownian motion.

In particular, the Chan-Karolyi-Longstaff-Sanders Enhancement has been used to generalize the exponent in the volatility term. Often in finance, stochastic processes are used where the volatility increase for larger values of the variable. The CKLS process (Chan, Karolyi, Longstaff, and Sanders (1992)), where the volatility term is replaced by , can be solved in closed form for as well as for which corresponds to the conventional Ornstein-Uhlenbeck process. Another special case is which corresponds to the CIR model.

The Ornstein-Uhlenbeck process can be extended to higher dimensions A multi-dimensional version of the Ornstein-Uhlenbeck process, using the multi-dimensional state variable denoted by the dimensional vector , can be defined from

where is an dimensional Weiner process, and are constant matrices (Gardiner (2009)).

The solution to the multi-dimensional process can be developed analogous to the one-dimensional case. It is

and the mean is . These expressions make use of matrix exponential.

The Fokker-Planck equation can be extended to multiple dimensions. Thus, the process can also be described in terms of the probability density function which satisfies the Fokker-Planck equation (Gardiner (2009))

where the matrix with components is defined by

As in the case for one-dimension, the process is a linear transformation of Gaussian random variables, and therefore itself must be a Gaussian. This enables the transition probability , which is a Gaussian, to be written down explicitly.

The explicit solution for the stationary case may be written down as follows. A stationary solution exists if the real part of the eigenvalues of are larger than zero, and is given by

where the matrix is determined from (Risken and Till (1996)).

The CIR process can be readily expanded using the time-dependent functions for mean and volatility. These time-varying functions that replace the coefficients make it more consistent with the pre-assigned term-structure of interest rates and possibly volatilities. The most general – and formal - approach is in Maghsoodi (1996). Brigo and Mercurio (2001) present a more tractable approach where an external time-dependent shift is added to the model for consistency with an input term structure of rates.

Further, the CIR process may also be enhanced using stochastic mean and stochastic volatility. Such an extension has been given by Chen (1996) and is known as the Chen model. Finally it may be noted that the CIR process is a special case of basic affine jump diffusion, which still permits a closed-form expression for bond prices.

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