Propositional and Predicate Logic 15

Key Topics

Propositions
Truth Tables
Semantic Tableaux
Natural Deduction
Proof
Predicates
Universal Quantifiers
Existential Quantifiers

15.1 Introduction

Logic is the study of reasoning and the validity of arguments, and it is concerned with the truth of statements (propositions) and the nature of truth. Formal logic is concerned with the form of arguments and the principles of valid inference. Valid arguments are truth preserving, and for a valid deductive argument the conclusion will always be true if the premises are true.

Propositional logic is the study of propositions, where a proposition is a statement that is either true or false. Propositions may be combined with other propositions (with a logical connective) to form compound propositions. Truth tables are used to give operational definitions of the most important logical connectives, and they provide a mechanism to determine the truth values of more complicated logical expressions.

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Propositional logic may be used to encode simple arguments that are expressed in natural language, and to determine their validity. The validity of an argument may be determined from truth tables, or using the inference rules such as modus ponens to establish the conclusion via deductive steps.

Predicate logic allows complex facts about the world to be represented, and new facts may be determined via deductive reasoning. Predicate calculus includes predicates, variables and quantifiers, and a *predicate* is a characteristic or property that the subject of a statement can have. A predicate may include variables, and statements with variables become propositions once the variables are assigned values.

The universal quantifier is used to express a statement such as that all members of the domain of discourse have property P. This is written as $(\forall x)\ P(x)$, and it expresses the statement that the property $P(x)$ is true for all x. The existential quantifier states that there is at least one member of the domain of discourse that has property P. This is written as $(\exists x)P(x)$.

15.2 Propositional Logic

Propositional logic is the study of propositions where a proposition is a statement that is either true or false. There are many examples of propositions such as '1 + 1 = 2' which is a true proposition, and the statement that 'Today is Wednesday' which is true if today is Wednesday and false otherwise. The statement $x > 0$ is not a proposition as it contains a variable x, and it is only meaningful to consider its truth or falsity only when a value is assigned to x. Once the variable x is assigned a value it becomes a proposition. The statement 'This sentence is false' is not a proposition as it contains a self-reference that contradicts itself. Clearly, if it the statement is true it is false, and if is false it is true.

A propositional variable may be used to stand for a proposition (e.g. let the variable P stand for the proposition '2 + 2 = 4' which is a true proposition). A propositional variable takes the value or false. The negation of a proposition P (denoted $\neg P$) is the proposition that is true if and only if P is false, and is false if and only if P is true.

A well-formed formula (*wff*) in propositional logic is a syntactically correct formula created according to the syntactic rules of the underlying calculus. A well-formed formula is built up from variables, constants, terms and logical connectives such as conjunction (and), disjunction (or), implication (if… then…), equivalence (if and only if) and negation. A distinguished subset of these well formed formulae is the *axioms* of the calculus, and there are *rules of inference* that allow the truth of new formulae to be derived from the axioms and from formulae that have already demonstrated to be true in the calculus.

Table 15.1 Truth table for formula W

A	B	$W(A, B)$
T	T	T
T	F	F
F	T	F
F	F	T

A formula in propositional calculus may contain several propositional variables, and the truth or falsity of the individual variables needs to be known prior to determining the truth or falsity of the logical formula.

Each propositional variable has two possible values, and a formula with n-propositional variables has 2^n values associated with the n-propositional variables. The set of values associated with the n variables may be used derive a truth table with 2^n rows and $n + 1$ columns. Each row gives each of the 2^n truth values that the n variables may take, and column $n + 1$ gives the result of the logical expression for that set of values of the propositional variables. For example, the propositional formula W defined in the truth table above (Table 15.1) has two propositional variables A and B, with $2^2 = 4$ rows for each of the values that the two propositional variables may take. There are $2 + 1 = 3$ columns with W defined in the third column.

A rich set of connectives is employed in the calculus to combine propositions and to build up the well-formed formulae. This includes the conjunction of two propositions ($A \wedge B$), the disjunction of two propositions ($A \vee B$) and the implication of two propositions ($A \rightarrow B$). These connectives allow compound propositions to be formed, and the truth of the compound propositions is determined from the truth values of its constituent propositions and the rules associated with the logical connective. The meaning of the logical connectives is given by truth tables.[1]

Mathematical Logic is concerned with inference, and it involves proceeding in a methodical way from the axioms and using the rules of inference to derive further truths. The rules of inference allow new propositions to be deduced from a set of existing propositions. A valid argument (or deduction) is truth preserving: i.e. for a valid logical argument if the set of premises is true then the conclusion (i.e. the deduced proposition) will also be true. The rules of inference include rules such as *modus ponens*, and this rule states that given the truths of the proposition A, and the proposition $A \rightarrow B$, then the truth of proposition B may be deduced.

The propositional calculus is employed in reasoning about propositions, and it may be applied to formalize arguments in natural language. *Boolean algebra* is used in computer science, and it is named after George Boole, who was the first professor of mathematics at Queens College, Cork.[2] His symbolic logic (discussed in Chap. 14) is the foundation for modern computing.

[1]Basic truth tables were first used by Frege, and developed further by Post and Wittgenstein.

[2]This institution is now known as University College Cork and has approximately 18,000 students.

Table 15.2 Conjunction

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

Table 15.3 Disjunction

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

15.2.1 Truth Tables

Truth tables give operational definitions of the most important logical connectives, and they provide a mechanism to determine the truth values of more complicated compound expressions. Compound expressions are formed from propositions and connectives, and the truth values of a compound expression containing several propositional variables is determined from the underlying propositional variables and the logical connectives.

The conjunction of A and B (denoted $A \wedge B$) is true if and only if both A and B are true, and is false in all other cases (Table 15.2). The disjunction of two propositions A and B (denoted $A \vee B$) is true if at least one of A and B are true, and false in all other cases (Table 15.3). The disjunction operator is known as the '*inclusive or*' operator as it is also true when both A and B are true; there is also an *exclusive or* operator that is true exactly when one of A or B is true, and is false otherwise.

Example 15.1 Consider proposition A given by "An orange is a fruit" and the proposition B given by "2 + 2 = 5" then A is true and B is false. Therefore

(i) $A \wedge B$ (i.e. An orange is a fruit and 2 + 2 = 5) is false
(ii) $A \vee B$ (i.e. An orange is a fruit or 2 + 2 = 5) is true

The implication operation ($A \rightarrow B$) is true if whenever A is true means that B is also true; and also whenever A is false (Table 15.4). It is equivalent (as shown by a

Table 15.4 Implication

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

Table 15.5 Equivalence

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

Table 15.6 Not operation

A	$\neg A$
T	F
F	T

truth table) to $\neg A \vee B$. The equivalence operation ($A \leftrightarrow B$) is true whenever both A and B are true, or whenever both A and B are false (Table 15.5).

The not operator ($\neg$) is a unary operator (i.e. it has one argument) and is such that $\neg A$ is true when A is false, and is false when A is true (Table 15.6).

Example 15.2 Consider proposition A given by 'Jaffa cakes are biscuits' and the proposition B given by '2 + 2 = 5' then A is true and B is false. Therefore

(i) $A \rightarrow B$ (i.e. Jaffa cakes are biscuits implies 2 + 2 = 5) is false
(ii) $A \leftrightarrow B$ (i.e. Jaffa cakes are biscuits is equivalent to 2 + 2 = 5) is false
(iii) $\neg B$ (i.e. $2 + 2 \neq 5$) is true.

Creating a Truth Table

The truth table for a well-formed formula $W(P_1, P_2, \ldots, P_n)$ is a table with 2^n rows and $n + 1$ columns. Each row lists a different combination of truth values of the propositions $P_1, P_2, \ldots, P_n$ followed by the corresponding truth value of W.

The example above (Table 15.7) gives the truth table for a formula W with three propositional variables (meaning that there are $2^3 = 8$ rows in the truth table).

Table 15.7 Truth table for W(P, Q, R)

P	Q	R	W(P, Q, R)
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	F

15.2.2 Properties of Propositional Calculus

There are many well-known properties of the propositional calculus such as the commutative, associative and distributive properties. These ease the evaluation of complex expressions, and allow logical expressions to be simplified.

The *commutative property* holds for the conjunction and disjunction operators, and it states that the order of evaluation of the two propositions may be reversed without affecting the resulting truth value: i.e.

$$\mathrm{A} \wedge \mathrm{B} = \mathrm{B} \wedge \mathrm{A}$$
$$\mathrm{A} \vee \mathrm{B} = \mathrm{B} \vee \mathrm{A}$$

The *associative property* holds for the conjunction and disjunction operators. This means that order of evaluation of a sub-expression does not affect the resulting truth value, i.e.

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$
$$(A \vee B) \vee C = A \vee (B \vee C)$$

The conjunction operator *distributes* over the disjunction operator and vice versa.

$$A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$$

The result of the logical conjunction of two propositions is false if one of the propositions is false (irrespective of the value of the other proposition).

$$A \wedge \mathrm{F} = \mathrm{F} \wedge A = \mathrm{F}$$

The result of the logical disjunction of two propositions is true if one of the propositions is true (irrespective of the value of the other proposition).

$$A \vee \mathrm{T} = \mathrm{T} \vee A = \mathrm{T}$$

The result of the logical disjunction of two propositions, where one of the propositions is known to be false is given by the truth value of the other proposition. That is, the Boolean value ‘F’ acts as the identity for the disjunction operation.

$$A \vee \mathrm{F} = A = \mathrm{F} \vee A$$

The result of the logical conjunction of two propositions, where one of the propositions is known to be true, is given by the truth value of the other proposition. That is, the Boolean value ‘T’ acts as the identity for the conjunction operation.

$$A \wedge \mathrm{T} = A = \mathrm{T} \wedge A$$

The $\wedge$ and $\vee$ operators are *idempotent*. That is, when the arguments of the conjunction or disjunction operator are the same proposition A the result is A. The idempotent property allows expressions to be simplified.

$$A \wedge A = A$$

$$A \vee A = A$$

The *law of the excluded middle* is a fundamental property of the propositional calculus. It states that a proposition A is either true or false: i.e. there is no third logical value.

$$A \vee \neg A$$

We mentioned earlier that $A \rightarrow B$ is logically equivalent to $\neg A \vee B$ (same truth table), and clearly $\neg A \vee B$ is the same as $\neg A \vee \neg\neg B = \neg\neg B \vee \neg A$ which is logically equivalent to $\neg B \rightarrow \neg A$. Another words, $A \rightarrow B$ is logically equivalent to $\neg B \rightarrow \neg A$, and this is known as the *contrapositive*.

De Morgan was a contemporary of Boole in the nineteenth century, and the following law is known as De Morgan’s law.

$$\neg(A \wedge B) \equiv \neg A \vee \neg B$$
$$\neg(A \vee B) \equiv \neg A \wedge \neg B$$

Certain well-formed formulae are true for all values of their constituent variables. This can be seen from the truth table when the last column of the truth table consists entirely of true values. A proposition that is true for all values of its constituent propositional variables is known as a *tautology*. An example of a tautology is the proposition A $\vee$ $\neg$A (Table 15.8)

A proposition that is false for all values of its constituent propositional variables is known as a *contradiction*. An example of a contradiction is the proposition A $\wedge$ $\neg$A.

Table 15.8 Tautology $B \vee \neg B$

B	$\neg B$	$B \vee \neg B$
T	F	T
F	T	T