15.2.3 Proof in Propositional Calculus

Logic enables further truths to be derived from existing truths by rules of inference that are truth preserving. Propositional calculus is both *complete* and *consistent*. The completeness property means that all true propositions are deducible in the calculus, and the consistency property means that there is no formula A such that both A and $\neg A$ are deducible in the calculus.

An argument in propositional logic consists of a sequence of formulae that are the premises of the argument and a further formula that is the conclusion of the argument. One elementary way to see if the argument is valid is to produce a truth table to determine if the conclusion is true whenever all of the premises are true.

Consider a set of premises $P_1, P_2, \ldots P_n$ and conclusion Q. Then to determine if the argument is valid using a truth table involves adding a column in the truth table for each premise $P_1, P_2, \ldots P_n$, and then to identify the rows in the truth table for which these premises are all true. The truth value of the conclusion Q is examined in each of these rows, and if Q is true for each case for which $P_1, P_2, \ldots P_n$ are all true then the argument is valid. This is equivalent to $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a tautology.

An alternate approach to proof with truth tables is to assume the negation of the desired conclusion (i.e. $\neg Q$) and to show that the premises and the negation of the conclusion result in a contradiction (i.e. $P_1 \wedge P_2 \wedge \ldots \wedge P_n \wedge \neg Q$) is a contradiction.

The use of truth tables becomes cumbersome when there are a large number of variables involved, as there are 2^n truth table entries for n propositional variables.

Procedure for Proof by Truth Table

(i) Consider argument $P_1, P_2, \ldots, P_n$ with conclusion Q
(ii) Draw truth table with column in truth table for each premise $P_1, P_2, \ldots, P_n$
(iii) Identify rows in truth table for when these premises are all true.
(iv) Examine truth value of Q for these rows.
(v) If Q is true for each case that $P_1, P_2, \ldots P_n$ are true then the argument is valid.
(vi) That is $P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow Q$ is a tautology

Example 15.3 (**Truth Tables**) Consider the argument adapted from [1] and determine if it is valid.

If the pianist plays the concerto then crowds will come if the prices are not too high.

If the pianist plays the concerto then the prices will not be too high

Therefore, if the pianist plays the concerto then crowds will come.

Solution

We will adopt a common proof technique that involves showing that the negation of the conclusion is incompatible (inconsistent) with the premises, and from this we deduce the conclusion must be true. First, we encode the argument in propositional logic:

Table 15.9 Proof of argument with a truth table

P	C	H	$\neg H$	$\neg H \rightarrow C$	$P \rightarrow (\neg H \rightarrow C)$	$P \rightarrow \neg H$	$P \rightarrow C$	$\neg(P \rightarrow C)$	*
T	T	T	F	T	T	F	T	F	F
T	T	F	T	T	T	T	T	F	F
T	F	T	F	T	T	F	F	T	F
T	F	F	T	F	F	T	F	T	F
F	T	T	F	T	T	T	T	F	F
F	T	F	T	T	T	T	T	F	F
F	F	T	F	T	T	T	T	F	F
F	F	F	T	F	T	T	T	F	F

Let P stand for 'The pianist plays the concerto'; C stands for 'Crowds will come'; and H stands for 'Prices are too high'. Then the argument may be expressed in propositional logic as

$$P \rightarrow (\neg H \rightarrow C)$$
$$P \rightarrow \neg H$$
$$P \rightarrow C$$

Then we negate the conclusion $P \rightarrow C$ and check the consistency of $P \rightarrow (\neg H \rightarrow C) \wedge (P \rightarrow \neg H) \wedge \neg\, (P \rightarrow C)^*$ using a truth table (Table 15.9).

It can be seen from the last column in the truth table that the negation of the conclusion is incompatible with the premises, and therefore it cannot be the case that the premises are true and the conclusion false. Therefore, the conclusion must be true whenever the premises are true, and we conclude that the argument is valid.

Logical Equivalence and Logical Implication

The laws of mathematical reasoning are truth preserving, and are concerned with deriving further truths from existing truths. Logical reasoning is concerned with moving from one line in mathematical argument to another, and involves deducing the truth of another statement Q from the truth of P.

The statement Q maybe in some sense be logically equivalent to P and this allows the truth of Q to be immediately deduced. In other cases the truth of P is sufficiently strong to deduce the truth of Q; in other words P logically implies Q. This leads naturally to a discussion of the concepts of logical equivalence ($W_1 \equiv W_2$) and logical implication ($W_1 \vdash W_2$).

Logical Equivalence

Two well-formed formulae W_1 and W_2 with the same propositional variables (P, Q, R …) are logically equivalent ($W_1 \equiv W_2$) if they are always simultaneously true or false for any given truth values of the propositional variables.

If two well-formed formulae are logically equivalent then it does not matter which of W_1 and W_2 is used, and $W_1 \leftrightarrow W_2$ is a tautology. In Table 15.10 above we see that $P \wedge$ Q is logically equivalent to $\neg(\neg P \vee \neg Q)$.

Table 15.10 Logical equivalence of two WFFs

P	Q	P ∧ Q	¬P	¬Q	¬P ∨ ¬Q	¬P ∨ ¬Q
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

Table 15.11 Logical implication of two WFFs

PQR	(P ∧ Q) ∨ (Q ∧ ¬R)	(Q ∨ R)
TTT	T	T
TTF	T	T
TFT	F	T
TFF	F	F
FTT	F	T
FTF	T	T
FFT	F	T
FFF	F	F

Logical Implication

For two well-formed formulae W_1 and W_2 with the same propositional variables (P, Q, R …) W_1 logically implies W_2 ($W_1 \models W_2$) if any assignment to the propositional variables which makes W_1 true also makes W_2 true (Table 15.11). That is, $W_1 \rightarrow W_2$ is a tautology.

Example 15.4 Show by truth tables that $(P \wedge Q) \vee (Q \wedge \neg R) \models (Q \vee R)$.

The formula $(P \wedge Q) \vee (Q \wedge \neg R)$ is true on rows 1, 2 and 6 and formula $(Q \vee R)$ is also true on these rows. Therefore $(P \wedge Q) \vee (Q \wedge \neg R) \models (Q \vee R)$.

15.2.4 Semantic Tableaux in Propositional Logic

We showed in example 15.3 how truth tables may be used to demonstrate the validity of a logical argument. However, the problem with truth tables is that they can get extremely large very quickly (as the size of the table is 2^n where n is the number of propositional variables), and so in this section we will consider an alternate approach known as semantic tableaux.

The basic idea of semantic tableaux is to determine if it is possible for a conclusion to be false when all of the premises are true. If this is not possible, then the conclusion must be true when the premises are true, and so the conclusion is *semantically entailed* by the premises. The method of semantic tableaux is a technique to expose inconsistencies in a set of logical formulae, by identifying conflicting logical expressions.

Table 15.12 Rules of semantic tableaux

Rule No.	Definition	Description
1.	$A \wedge B$ A B	If $A \wedge B$ is true then both A and B are true, and may be added to the branch containing $A \wedge B$
2.	$A \vee B$ A B	If $A \vee B$ is true then either A or B is true, and we add two new branches to the tableaux, one containing A and one containing B
3.	$A \rightarrow B$ $\neg A$ B	If $A \rightarrow B$ is true then either $\neg A$ or B is true, and we add two new branches to the tableaux, one containing $\neg A$ and one containing B
4.	$A \leftrightarrow B$ $A \wedge B$ $\neg A \wedge \neg B$	If $A \leftrightarrow B$ is true then either $A \wedge B$ or $\neg A \wedge \neg B$ is true, and we add two new branches, one containing $A \wedge B$ and one containing $\neg A \wedge \neg B$
5.	$\neg\neg A$ A	If $\neg\neg A$ is true then A may be added to the branch containing $\neg\neg A$
6.	$\neg(A \wedge B)$ $\neg A$ $\neg B$	If $\neg(A \wedge B)$ is true then either $\neg A$ or $\neg B$ is true, and we add two new branches to the tableaux, one containing $\neg A$ and one containing $\neg B$
7.	$\neg(A \vee B)$ $\neg A$ $\neg B$	If $\neg(A \vee B)$ is true then both $\neg A$ and $\neg B$ are true, and may be added to the branch containing $\neg(A \vee B)$
8.	$\neg(A \rightarrow B)$ A $\neg B$	If $\neg(A \rightarrow B)$ is true then both A and $\neg B$ are true, and may be added to the branch containing $\neg(A \rightarrow B)$

We present a short summary of the rules of semantic tableaux in Table 15.12, and we then proceed to provide a proof for Example 15.3 using semantic tableaux instead of a truth table.

Whenever a logical expression A and its negation $\neg A$ appear in a branch of the tableau, then an inconsistency has been identified in that branch, and the branch is said to be *closed*. If all of the branches of the semantic tableaux are closed, then the logical propositions from which the tableau was formed are mutually inconsistent, and cannot be true together.

The method of proof is to negate the conclusion, and to show that all branches in the semantic tableau are closed, and that therefore it is not possible for the premises of the argument to be true and for the conclusion to be false. Therefore, the argument is valid and the conclusion follows from the premises.

Example 15.5 (**Semantic Tableaux**) Perform the proof for Example 15.3 using semantic tableaux.

Solution
We formalized the argument previously as

$$\begin{array}{ll} \text{(Premise 1)} & P \to (\neg H \to C) \\ \text{(Premise 2)} & P \to \neg H \\ \text{(Conclusion)} & P \to C \end{array}$$

We negate the conclusion to get $\neg(P \to C)$ and we show that all branches in the semantic tableau are closed, and that therefore it is not possible for the premises of the argument to be true and for the conclusion false. Therefore, the argument is valid, and the truth of the conclusion follows from the truth of the premises.

```
              P → (¬H → C)
              P → ¬H
              ¬(P → C)
                 |
                 P
                ¬C
              /      \
           ¬P          ¬H
        ------        /    \
        closed      ¬P       (¬H → C)
                  ------     /          \
                  closed   ¬¬H           C
                             |          -----
                             H          closed
                          -------
                          closed
```

We have showed that all branches in the semantic tableau are closed, and that therefore it is not possible for the premises of the argument to be true and for the conclusion false. Therefore, the argument is valid as required.

15.2.5 Natural Deduction

The German mathematician, Gerhard Gentzen (Fig. 15.1), developed a method for logical deduction known as *'Natural Deduction'*, and his formal approach to natural deduction aimed to be as close as possible to natural reasoning. Gentzen worked as an assistant to David Hilbert at the University of Göttingen, and he died of malnutrition in Prague at the end of the Second World War.

Natural deduction includes rules for $\wedge$, $\vee$, $\to$ introduction and elimination and also for *reductio ab absurdum*. There are ten inference rules in the Natural Deduction system, and they include two inference rules for each of the five logical

Fig. 15.1 Gerhard gentzen

Table 15.13 Natural deduction rules

Rule	Definition	Description
$\wedge$ I	$\dfrac{P_1, P_2, \ldots P_n}{P_1 \wedge P_2 \wedge \ldots \wedge P_n}$	Given the truth of propositions P_1, P_2, … P_n then the truth of the conjunction $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ follows. This rule shows how conjunction can be introduced
$\wedge$ E	$\dfrac{P_1 \wedge P_2 \wedge \ldots \wedge P_n}{P_i}$ where $i \in \{1, \ldots, n\}$	Given the truth the conjunction $P_1 \wedge P_2 \wedge \ldots \wedge P_n$ then the truth of proposition P_i follows. This rule shows how a conjunction can be eliminated
$\vee$ I	$\dfrac{P_i}{P_1 \vee P_2 \vee \ldots \vee P_n}$	Given the truth of propositions P_i then the truth of the disjunction $P_1 \vee P_2 \vee \ldots \vee P_n$ follows. This rule shows how a disjunction can be introduced
$\vee$ E	$\dfrac{P_1 \vee \ldots \vee P_n, P_1 \rightarrow E, \ldots P_n \rightarrow E}{E}$	Given the truth of the disjunction $P_1 \vee P_2 \vee \ldots \vee P_n$, and that each disjunct implies E, then the truth of E follows. This rule shows how a disjunction can be eliminated
$\rightarrow$ I	$\dfrac{\text{From } P_1, P_2, \ldots P_n \text{ infer } P}{P_1 \wedge P_2 \wedge \ldots \wedge P_n \rightarrow P}$	This rule states that if we have a theorem that allows P to be inferred from the truth of premises P_1, P_2, … P_n (or previously proved) then we can deduce $(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \rightarrow P$. This is known as the *Deduction Theorem*
$\rightarrow$ E	$\dfrac{P_i \rightarrow P_j, P_i}{P_j}$	This rule is known as *modus ponens*. The consequence of an implication follows if the antecedent is true (or has been previously proved)
$\equiv$ I	$\dfrac{P_i \rightarrow P_j, P_j \rightarrow P_i}{P_i \leftrightarrow P_j}$	If proposition P_i implies proposition P_j and vice versa then they are equivalent (i.e. $P_i \leftrightarrow P_j$)
$\equiv$ E	$\dfrac{P_i \leftrightarrow P_j}{P_i \rightarrow P_j, P_j \rightarrow P_i}$	If proposition P_i is equivalent to proposition P_j then proposition P_i implies proposition P_j and vice versa
$\neg$ I	$\dfrac{\text{From } P \text{ infer } P_1 \wedge \neg P_1}{\neg P}$	If the proposition P allows a contradiction to be derived, then $\neg$P is deduced. This is an example of a *proof by contradiction*
$\neg$ E	$\dfrac{\text{From } \neg P \text{ infer } P_1 \wedge \neg P_1}{\neg P}$	If the proposition $\neg$P allows a contradiction to be derived, then P is deduced. This is an example of a proof by contradiction

operators—$\wedge$, $\vee$, $\neg$, $\rightarrow$ and $\leftrightarrow$. There are two inference rules per operator (an introduction rule and an elimination rule), and the rules are defined in Table 15.13:

Natural deduction may be employed in logical reasoning and is described in detail in [1, 2].

15.2.6 Sketch of Formalization of Propositional Calculus

Truth tables provide an informal approach to proof and the proof is provided in terms of the meanings of the propositions and logical connectives. The formalization of propositional logic includes the definition of an alphabet of symbols and well-formed formulae of the calculus, the axioms of the calculus and rules of inference for logical deduction.

The deduction of a new formulae Q is via a sequence of well-formed formulae P_1, P_2, … P_n (where $P_n = Q$) such that each P_i is either an axiom, a hypothesis or deducible from an earlier pair of formula P_j, P_k, (where P_k is of the form $P_j \Rightarrow P_i$) and modus ponens. *Modus ponens* is a rule of inference that states that given propositions A, and $A \Rightarrow B$ then proposition B may be deduced. The deduction of a formula Q from a set of hypothesis H is denoted by $H \vdash Q$, and where Q is deducible from the axioms alone this is denoted by $\vdash Q$.

The *deduction theorem* of propositional logic states that if $H \cup \{P\} \vdash Q$, then H $\vdash P \rightarrow Q$, and the converse of the theorem is also true: i.e. if $H \vdash P \rightarrow Q$ then $H \cup \{P\} \vdash Q$. Formalism (this approach was developed by the German mathematician, David Hilbert) allows reasoning about symbols according to rules, and to derive theorems from formulae irrespective of the meanings of the symbols and formulae.

Propositional calculus is *sound*; i.e. any theorem derived using the Hilbert approach is true. Further, the calculus is also *complete*, and every tautology has a proof (i.e. is a theorem in the formal system). The propositional calculus is *consistent*: (i.e. it is not possible that both the well-formed formula A and $\neg A$ are deducible in the calculus).

Propositional calculus is *decidable*: i.e. there is an algorithm (truth table) to determine for any well-formed formula A whether A is a theorem of the formal system. The Hilbert style system is slightly cumbersome in conducting proof and is quite different from the normal use of logic in mathematical deduction.

15.2.7 Applications of Propositional Calculus

Propositional calculus may be employed in reasoning with arguments in natural language. First, the premises and conclusion of the argument are identified and formalized into propositions. Propositional logic is then employed to determine if the conclusion is a valid deduction from the premises.

Consider, for example, the following argument that aims to prove that Superman does not exist.

If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil he would be impotent; if he were unwilling to prevent evil he would be malevolent; Superman does not prevent evil. If superman exists he is neither malevolent nor impotent; therefore Superman does not exist.

First, letters are employed to represent the propositions as follows:

a: Superman is able to prevent evil
w: Superman is willing to prevent evil
i: Superman is impotent
m: Superman is malevolent
p: Superman prevents evil
e: Superman exists

Then, the argument above is formalized in propositional logic as follows:

Premises	
P_1	$(a \wedge w) \rightarrow p$
P_2	$(\neg a \rightarrow i) \wedge (\neg w \rightarrow m)$
P_3	$\neg p$
P_4	$e \rightarrow \neg i \wedge \neg m$
Conclusion	$P_1 \wedge P_2 \wedge P_3 \wedge P_4 \Rightarrow \neg e$

Proof that Superman does not exist

1.	$a \wedge w \rightarrow p$	*Premise* 1
2.	$(\neg a \rightarrow i) \wedge (\neg w \rightarrow m)$	*Premise* 2
3.	$\neg p$	*Premise* 3
4.	$e \rightarrow (\neg i \wedge \neg m)$	*Premise* 4
5.	$\neg p \rightarrow \neg(a \wedge w)$	1, *Contrapositive*
6.	$\neg(a \wedge w)$	*3, 5 Modus Ponens*
7.	$\neg a \vee \neg w$	*6, De Morgan's Law*
8.	$\neg (\neg i \wedge \neg m) \rightarrow \neg e$	*4, Contrapositive*
9.	$i \vee m \rightarrow \neg e$	*8, De Morgan's Law*
10.	$(\neg a \rightarrow i)$	*2, ∧ Elimination*
11.	$(\neg w \rightarrow m)$	*2, ∧ Elimination*
12.	$\neg \neg a \vee i$	10, A→ B equivalent to ¬A∨ B
13.	$\neg \neg a \vee i \vee m$	11, ∨ Introduction
14.	$\neg \neg a \vee (i \vee m)$	
15.	$\neg a \rightarrow (i \vee m)$	14, A → B equivalent to ¬A∨ B
16.	$\neg \neg w \vee m$	11, A → B equivalent to ¬A∨ B
17.	$\neg \neg w \vee (i \vee m)$	
18.	$\neg w \rightarrow (i \vee m)$	17, A → B equivalent to ¬A∨ B
19.	$(i \vee m)$	7, 15, 18 ∨*Elimination*
20.	$\neg e$	*9, 19 Modus Ponens*

Second Proof

1.	$\neg p$	P_3
2.	$\neg(a \wedge w) \vee p$	P_1 (A → B ≡ ¬A ∨ B)
3.	$\neg(a \wedge w)$	1, 2 A ∨ B, ¬B ├ A
4.	$\neg a \vee \neg w$	3, De Morgan's Law
5.	$(\neg a \to i)$	P_2 (∧-Elimination)
6.	$\neg a \to i \vee m$	5, $x \to y$ ├ $x \to y \vee z$
7.	$(\neg w \to m)$	P_2 (∧-Elimination)
8.	$\neg w \to i \vee m$	7, $x \to y$ ├ $x \to y \vee z$
9.	$(\neg a \vee \neg w) \to (i \vee m)$	8, $x \to z$, $y \to z$ ├ $x \vee y \to z$
10.	$(i \vee m)$	4, 9 Modus Ponens
11.	$e \to \neg(i \vee m)$	P_4 (De Morgan's Law)
12.	$\neg e \vee \neg (i \vee m)$	11, (A → B ≡ ¬A ∨ B)
13.	$\neg e$	10, 12 A ∨ B, ¬B ├ A

Therefore, the conclusion that Superman does not exist is a valid deduction from the given premises.

15.2.8 Limitations of Propositional Calculus

The propositional calculus deals with propositions only. It is incapable of dealing with the syllogism 'All Greeks are mortal; Socrates is a Greek; therefore Socrates is mortal'. This would be expressed in propositional calculus as three propositions *A*, *B* therefore *C*, where *A* stands for 'All Greeks are mortal', *B* stands for 'Socrates is a Greek' and *C* stands for 'Socrates is mortal'. Propositional logic does not allow the conclusion that all Greeks are mortal to be derived from the two premises.

Predicate calculus deals with these limitations by employing variables and terms, and using universal and existential quantification to express that a particular property is true of all (or at least one) values of a variable. Predicate calculus is discussed in the next section.