Section 3.2 Partial Order

Purpose of Section To introduce a partial order relation (or ordering) on a set. We also see how order relations can be illustrated graphically by means of Hasse diagrams and directed graphs.

Order Theory

The British philosopher Edmund Burke once said, "Order is the foundation of all that is good." He probably wasn't referring to the order relation in mathematics, although order in mathematics is as important as order in civilized society. The reader is familiar with the inequality relation $\le$, which imposes an order on numbers, and the relation $\subseteq$ which imposes "order" on sets. Other objects can be "ordered" as well, such as functions, matrices, points in the plane, and so on. Ordering objects according to some rule brings structure to an area of study. In computer science, order not only brings understanding, but efficiency. Imagine trying to find information on the internet if Google didn't have clever "ordering" strategies for storing information.

The theory of order is an area of mathematics which deals with various types of binary relations which capture the essence of ordering and provides one to say when something is "less than" or "preceeds" another.

Definition Order Relations:

1. Partial Order: A partial order, denoted[1] $\preceq$, is a *binary relation* on a set A (i.e. a subset of $A \times A$) such that for all x, y, z of A, the following conditions hold:

Reflexive property:	$x \preceq x$
Antisymmetric property:	$(x \preceq y \text{ and } y \preceq x) \Rightarrow x = y$
Transitive property:	$(x \preceq y \text{ and } y \preceq z) \Rightarrow x \preceq z$

A set A with a partial order relation defined on it is called a partially ordered set (or poset). When $x \preceq y$ we say x precedes y.

[1] We use the notion $a \preceq b$ which looks similar to the common inequality $a \le b$ to remind us it has similar properties.

2. **Strict Order:** : A **strict partial order,** denoted $\prec$, is a *binary relation* on a set A (i.e. a subset of $A \times A$) such that for all x, y, z of A, the following conditions hold:

Irreflexive property:	$x \not\prec x$
Transitive property:	$(x \prec y \text{ and } y \prec z) \Rightarrow x \prec z$

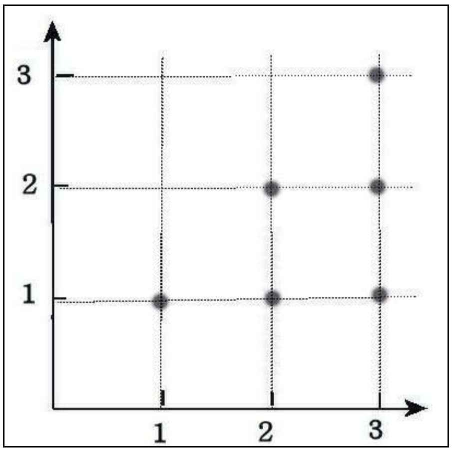
The symbolism $x \not\prec x$ means x does *not* belong to the relation $\prec$.

Note: A common partial order relation is the "less than or equal" relation "$\le$" on the real numbers, and a common strict order relation is the strict "less than" order relation "$<$" on the real numbers.

Example 1 If $A = \{1, 2, 3\}$ then

$$R = \{(1,1), (2,2), (3,3), (2,1), (3,1), (3,2)\}$$

is a partial order on A. Do you recognize the order? It is the greater than or equal to relation "$\ge$". Note that the members in R are equivalent to the inequalities $1 \ge 1,\ 2 \ge 2,\ 3 \ge 3,\ 2 \ge 1,\ 3 \ge 1,\ 3 \ge 2$. The graph of $R \subseteq A \times A$ is shown in Figure 1.



Graph of a Partial Order
Figure 1

Example 2 (The Divide Ordering) Let D denote the relation "divides" on the set of natural numbers $\mathbb{N}$. For example, $1D7,\ 2D7,\ 3D9,\ 7D21$ and so on. Show that D defines a partial order on the natural numbers.

Solution:

We leave the proof to the reader to verify that D satisfies the following properties, where we replace the notation "D" by "$\mid$".

- *Reflexive*: $n \mid n$
- *Antisymmmtric* $(m \mid n) \wedge (n \mid m) \Rightarrow m = n$
- *Transitive*: $(m \mid n) \wedge (n \mid p) \Rightarrow m \mid p$

Example 3 (Checking $\le$)

Check to see if $\le$ is a partial order on the real numbers.

Solution

To verify that $\le$ is an order relation on $\mathbb{R}$, we must show $\forall x, y, z \in \mathbb{R}$:

- *Reflexive:* $x \le x$
- *Antisymmetric*: $(x \le y) \wedge (y \le x) \Rightarrow x = y$
- *Transitive:* $(x \le y \text{ and } y \le z) \Rightarrow x \le z$

Hence $\le$ is a partial order on $\mathbb{R}$.

Example 4 Ordered Sets

The power set of $A = \{a, b, c\}$ consists of the family of eight subsets:

$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}.$$

Show that set inclusion relation "$\subseteq$" is a partial order on $P(A)$.

Proof: We show "$\subseteq$" satisfies the following properties:

- *Reflexive:* Clearly any set in $P(A)$ is a subset of itself. Hence $\subseteq$ is reflexive.

- *Antisymmetric:* For any two sets B and C in $P(A)$ satisfying $B \subseteq C$ and $C \subseteq B$ we have $B = C$. Hence $\subseteq$ is anti-symmetric.

- *Transitive:* For any three sets B, C and D in $P(A)$ satisfying $B \subseteq C$ and $C \subseteq D$ we have $B \subseteq D$. Hence $\subseteq$ is transitive.

Hence $\subseteq$ is a partial order on $P(A)$.

Total Order

Although "$\leq$" and "$\subseteq$" are partial orders on $\mathbb{R}$ and $P(A)$, respectively, there is a subtle difference. The partial order "$\leq$" on the real numbers is also a **total order**, meaning that every two real numbers x and y are *comparable*; that is either $x \leq y$ or $y \leq x$. On the other hand "$\subseteq$" is not a total order on $P(A)$ since there exists *incomparable* elements, such as the sets $\{1,2\}$ and $\{3,4\}$ where $\{1,2\} \not\subseteq \{3,4\}$ and $\{3,4\} \not\subseteq \{1,2\}$.

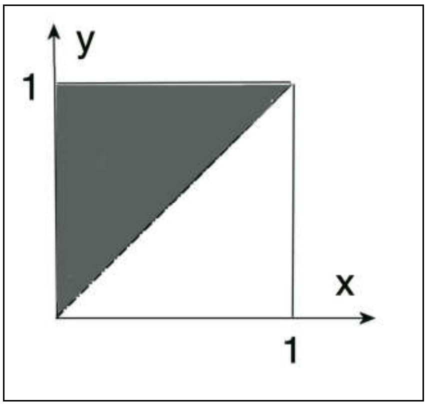
When we write $2 \leq 3$ we don't think of "$\leq$" as a set. We could however as the following example illustrates.

Example 5 (Order Relation as a Set)

Draw the graph of the relation $\leq$ on the set $A = [0,1]$.

Solution

The Cartesian product $A \times A = [0,1] \times [0,1]$ is drawn in Figure 2. The relation $\leq$ defined by $R = \{(x, y) : x \leq y\}$ is the shaded region on and above the line $y = x$. We say $x \leq y \Leftrightarrow (x, y) \in R$.



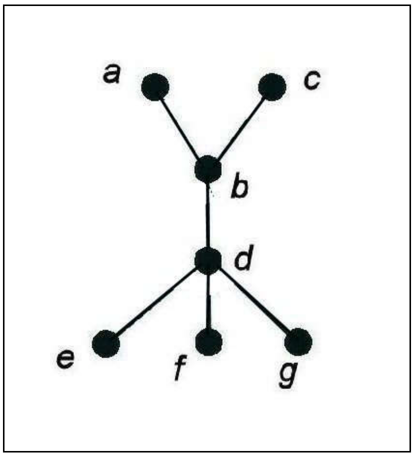
Shaded area representing the order relation $\leq$ on $[0,1]$

Figure 2

Hasse Diagrams and Directed Graphs

Although a partially ordered set can contain an infinite number of elements, many important examples are finite. A useful way to represent a finite partially ordered set is a **Hasse Diagram**,[3] where each element of the ordered set is denoted by a dot (node), where a line segment that goes upward from node x to node y if $x \preceq y$ and there is no z such that $x \preceq z \preceq y$.

A Hasse diagram for a partial order on $A = \{a, b, c, d, e, f, g\}$ is shown in Figure 3, where a few ordering are $e \preceq d, f \preceq d, g \preceq d$. Also $e \preceq c$ since by transitivity one can move upwards from e to c along the lines. On the other hand, $e \not\preceq f$ and $a \not\preceq c$ so not all elements of A are comparable. Hence, the ordering is a partial order and not a total order.



Hasse Diagram Representation for a Partially Ordered Set
Figure 3

Example 6 (Ordering the Power Set)

Draw the Hasse diagram for the partial order of the power set

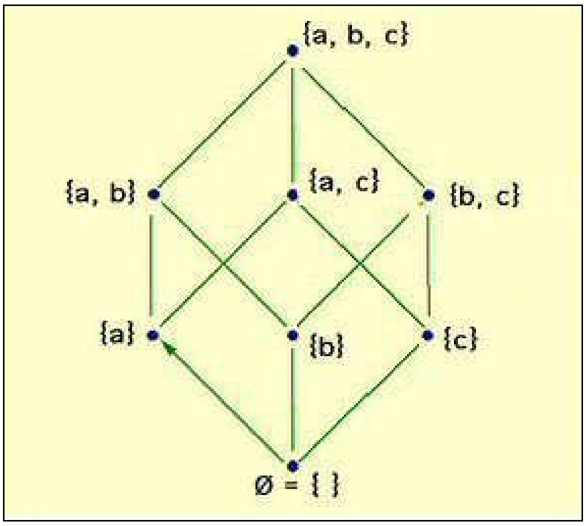
$$P(A) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

ordered by set inclusion.

Solution

[3]A Hasse (pronounced HAHS uh) diagram is named after the German mathematician Helumt Hasse (1898-1979).

The Hasse diagram is shown in Figure 4.



Hasse Diagram for the Inclusion Ordering on a Power Set
Figure 4

The notion of partially ordered sets introduces a whole collection of new ideas and concepts.

Definition:
Let R be a partial order on a set U and set $S \subseteq U$ be a subset of U. :

- **Upper bound:** An element $u \in U$ is called an **upper bound** of S iff $\forall s \in S$, $s \preceq u$.

- **Least Upper bound:** An element $\text{lub}(S) \in U$ is called the **least upper bound** of S (or **supremum** of S) if it is an upper bound of S and if $\text{lub}(S) \preceq u$ for every upper bound u.

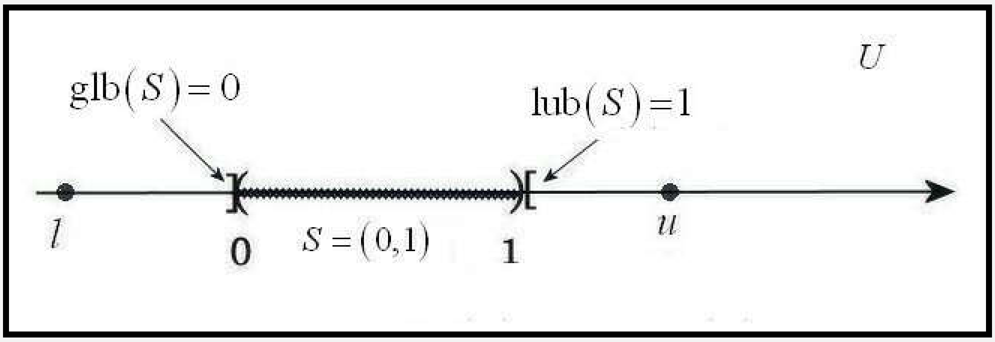
- **Lower Bound** An element $l \in U$ is called a **lower bound** of S iff $\forall s \in S$, $l \preceq s$.

- **Greatest Lower Bound** An element $\text{glb}(S) \in U$ is called the **greatest lower bound** of S (or infimum) if it is a lower bound of S and if $l \preceq \text{glb}(S)$ for every lower bound l of S.

- **Maximum and Minimum** An element $M \in S$ is the **maximum** element of S iff $s \preceq M$, $\forall s \in S$. The **minimum** element $m \in S$ is the minimum element of S iff $m \preceq s$, $\forall s \in S$. In other words, a maximal element of S is an element of S that is not "smaller" than any other element in S, and a minimal element is an element of .

S is an element of S that is not "greater" than any element of S.

- Maximal and Minimal: An element $M \in S$ is a **maximal element** of S iff $\sim(\exists s \in S)(M \preceq s)$. An element $m \in S$ is a **minimal element** of S iff $\sim(\exists s \in S)(s \preceq m)$.



Example 7 The open interval $(0,1) \subseteq \mathbb{R}$ in the partially ordered set of real numbes, ordered with "less than or equal to" $\leq$ has

a) many upper bounds, 1, 3, 5.3, π, and so on.
b) many lower bounds; −1, −5, −10.3, and so on.
c) the least upper bound is 1
d) the greatest lower bound is 0
e) the set $(0,1)$ has no maximum, no maximal, no minimum, and no minimal.

Example 8 (Partially Ordered Set)

Find the following quantities (if they exist) of the partially ordered set

$$S = \{\text{A,B,C,D,E,F,G,H,I,J,L,M,N,O}\}$$

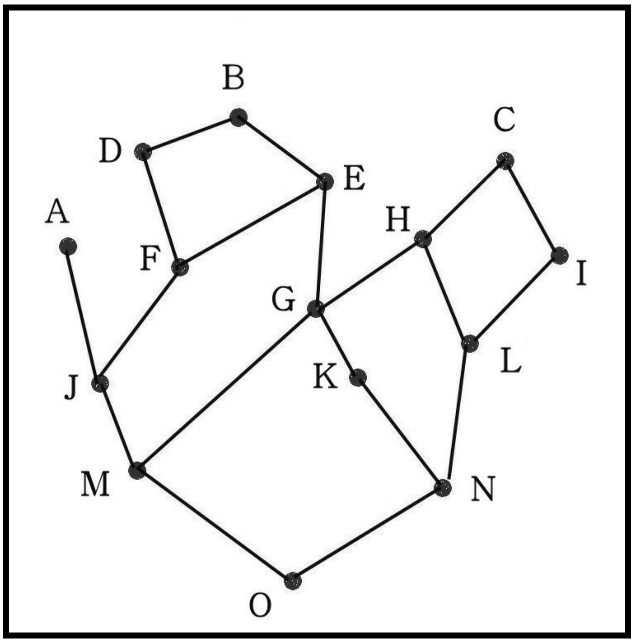
described by the Hasse diagram in Figure 5.

a) the upper bound
b) lower bound
c) the least upper bound, $\textbf{lub}(S)$
d) greatest lower bound, $\textbf{inf}(S)$
e) maximal element(s)
f) minimal element(s)

g) maximum
h) minimum

Solution

a) There is no upper bound of this set.
b) O is a lower bound
c) There is no least upper bound since there is no upper bound.
d) The greatest lower bound is O.
e) The maximal elements are A,B,C.
f) the minimal element is O
g) there is no maximum
h) the minimum is O



Partially Ordered Set
Figure 5

Example 9 (Multiples and Divisors of 24)

Given the set

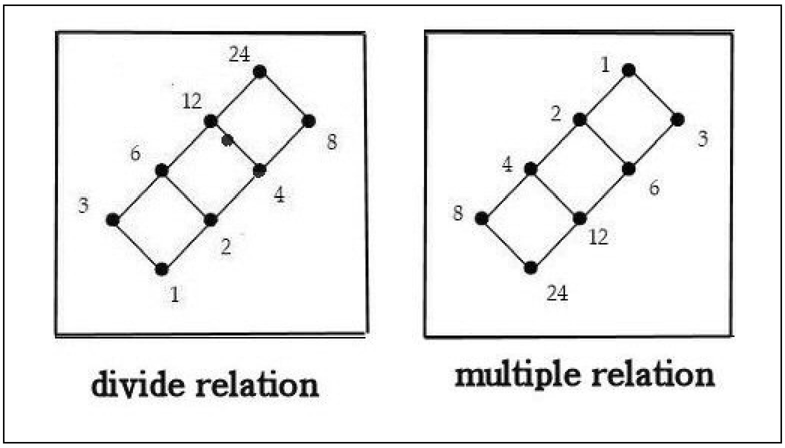
$$A = \{1, 2, 3, 4, 6, 8, 12, 24\}$$

of positive divisors of 24, define two partial orders on A by

$$aMb \Leftrightarrow a \text{ is a multiple of } b$$
$$aDb \Leftrightarrow a \text{ divides } b$$

Note that $24M8$ since 24 is a multiple of 8. Also, that $4D12$ since 4 divides 12. Draw Hasse diagrams for these two partial orders.

Solution



Hasse Diagrams for the Divide and Multiple Relations
Figure 5

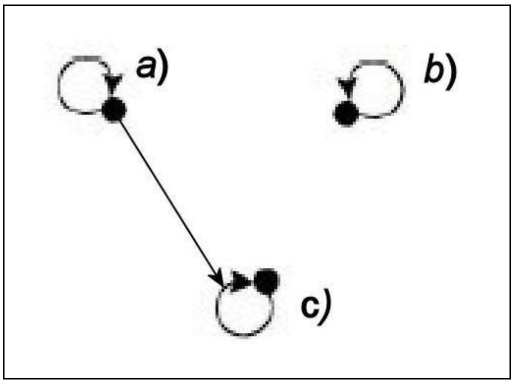
The "24" at the bottom of the Hasse diagram for the "multiple" relation implies that 24 is a multiple of all other divisors. The "24" at the top of the "divide" relation implies that all numbers in the diagram divide 24. Note that both partial orders are not total orders.

Digraphs

Directed graphs can also aid in visualizing order relations. A **directed graph** (or **digraph**) is a collection of dots (called **nodes**), some or all of which are connected by arrows (called **directed edges**). Some dots have arrows from themselves to themselves. For example, the relation

$$R = \{(a,a),(b,b),(c,c),(a,c)\}$$

on $A = \{a,b,c\}$ is illustrated by the digraph in Figure 6.

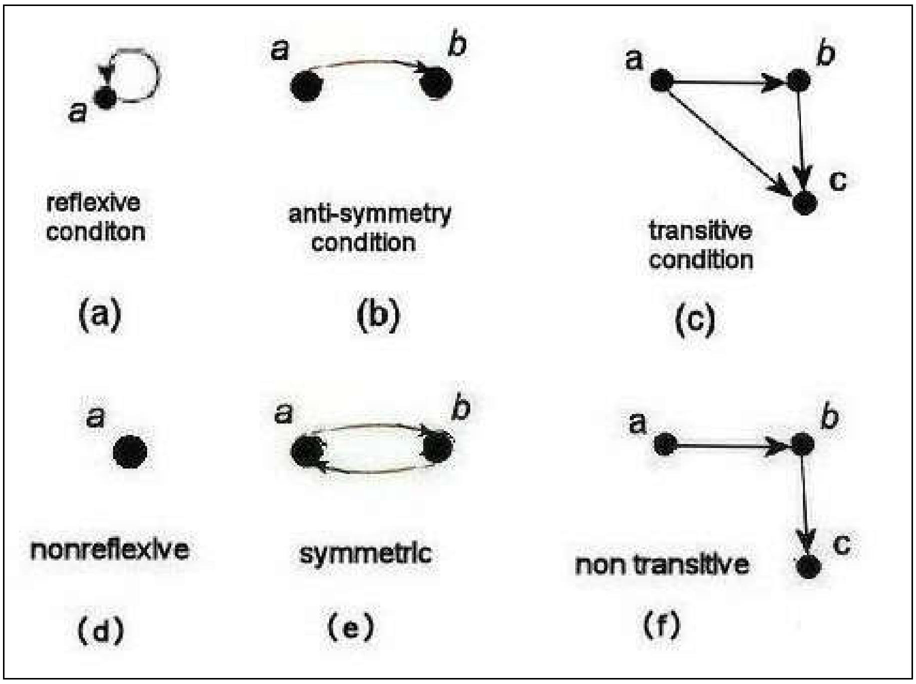


Digraph of an Order Relation
Figure 6

In this diagraph there is no "up or down" as in a Hasse diagram. The arrows do the "talking" here. If $a \preceq b$ one draws an arrow from a to b. If no arrow connects two points, than the points are not related. Since a partial order is reflexive the digraph has an arrow from each point to itself (which are drawn in the form of loops). If the digraph describes a partial order, it must be anti-symmetric , which means if there is an arrow from one point to another, then there *cannot* be an arrow from the second point back to the first. The digraph in Figure 5 passes this test so it describes an antisymmetric relation. Finally, the digraph describes a transitive relation since for any arrow from, say a to b, and another from b to c, then there is an arrow from a to c. Finally, the order is not a total order since not all elements a, b and c are related (a is not related to b for instance).

Figure 7 provides the tests for determining if a relation is an order relation by checking its digraph.

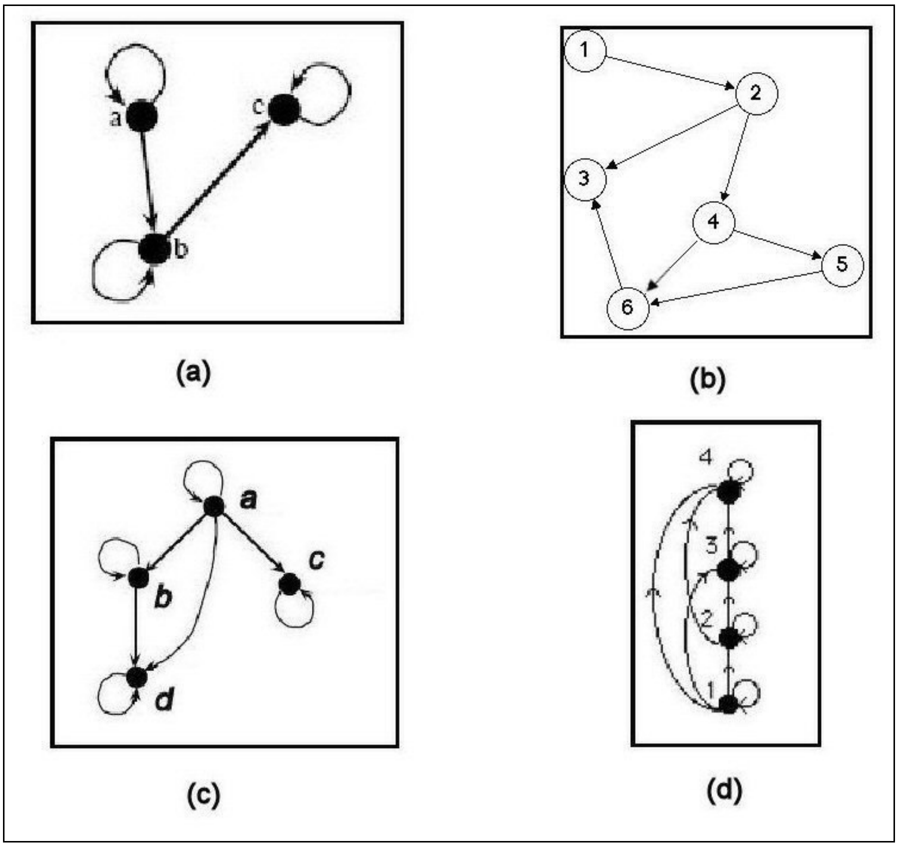
- *Reflexive:* Every node has a loop.
- *Antisymmetry:* If an arrow goes from a to b, there cannot be a second arrow backwards from b to a.
- *Transitivity:* If an arrow goes from a to b, and one from b to c, then an arrow must go from a to c.



Checking for Order Relations in a Digraph
Figure 7

Example 10 Detecting Order Relations on Graphs

Which of the relations illustrated by the graphs in Figure 8 are order relations?



Digraphs
Figure 8

Solution:

a) The relation described in digraph (a) is reflexive (loops), antisymmetric (no 1-cycles), but is *not* transitive (i.e. $a \preceq b, b \preceq c$, but $a \npreceq c$). Hence, the relation is not a partial order.

b) The relation described in digraph (b) is not reflexive (no loops), antisymmetric (no 1-cycles), but *not* transitive (i.e. $2 \preceq 4, 4b \preceq 5$, but $2 \npreceq 5$). Hence, the relation is not an order relation. The digraph does have a name however, it is called an *acyclic (no cycles) directed graph.*

c) The relation described in digraph (c) is reflexive (loops), antisymmetric (no back and forth arrows), and is transitive, hence the relation is a partial order.

d) The relation described in (d) is the digraph for the “less than or equal to” relation "$\leq$" defined on $\{1,2,3,4\}$. It is reflexive (loops), antisymmetric and is transitive, and hence a partial order. The Hasse diagram for this partial order would provide a more compact description of this relation, consisting of four vertical nodes, labeled 4,3,2,1 from top to bottom with a single line between each node. The reader can draw it.

Table 1 lists some common relations and their properties. The set over which the relation is defined is given in parenthesis next to the relation.

Property	Reflexive	Antisymmetric	Transitive	Symmetric
Relation	xRx	$xRy \wedge yRx \Rightarrow x = y$	$xRy \wedge yRz \Rightarrow yRz$	$xRy \Rightarrow yRx$
$\leq (\mathbb{R})$	Y	Y	Y	N
$< (\mathbb{R})$	N	N	Y	N
$\equiv (\mathrm{mod}\, n)$	Y	N	Y	Y
$\approx$ (sets)	Y	N	Y	Y
$\subseteq$ (sets)	Y	Y	Y	N
$\perp$ (lines)	N	N	N	Y
$\Vert$ (lines)	Y	N	Y	Y
$\mid$ on $\mathbb{Z}$	Y	N	Y	N
$\mid$ on $\mathbb{N}$	Y	Y	Y	N
$= (\mathbb{R})$	Y	Y	Y	Y

Properties of Common Relations
Table 1