第六章习题

1.求不定积分$\int \max\{1,x^2\}dx$

解:$\max\{1,x^2\}=\begin{cases} x^2, & x<-1或x>1 \\ 1, & -1<x<1 \end{cases}$,故原式=$\begin{cases} \frac{1}{3}x^3+C_1 & x<-1 \\ x+C_2 & -1\le x\le 1 \\ \frac{1}{3}x^3+C_3 & x>1 \end{cases}$

由于$y=\max\{1,x^2\}$在-1 和 1 连续,故有$\int \max\{1,x^2\}dx$在-1 和 1 也连续

得到$\begin{cases} -\frac{1}{3}+C_1=-1+C_2 \\ 1+C_2=\frac{1}{3}+C_3 \end{cases}$ 即$\begin{cases} C_1=-\frac{2}{3}+C_2 \\ C_3=\frac{2}{3}+C_2 \end{cases}$,令$C_2=C$

则原式=$\begin{cases} \frac{1}{3}x^3+C-\frac{2}{3} & x<-1 \\ x+C & -1\le x\le 1 \\ \frac{1}{3}x^3+C+\frac{2}{3} & x>1 \end{cases}$

2. 已知$F(x)$为$f(x)$的一个原函数,且当$x\ge 0$时$f(x)F(x)=\sin^2 2x$,已知$F(0)=1.F(x)>0$,试求$F(x)$.

解:$\int f(x)F(x)dx=\int \sin^2 2xdx$,故$\int F(x)dF(x)=\int \frac{1-\cos 4x}{2}dx,$

即$\frac{F^2(x)}{2}=\frac{1}{2}x-\frac{\sin 4x}{8}+C,$ 又$F(x)>0$,故$F(x)=\sqrt{x-\frac{\sin 4x}{4}+2C}$,由$F(0)=1$

得到$C=\frac{1}{2}$,故$F(x)=\sqrt{x-\frac{\sin 4x}{4}+1}$.

3.求下列不定积分.

(1) $\int \frac{x^4+1}{x^6+1}dx$ (2) $\int \sqrt{\frac{\ln(x+\sqrt{1+x^2})}{1+x^2}}dx$ (3) $\int \frac{e^{2x}}{\sqrt{e^x-1}}dx$

(4)$\int \frac{\sin x}{1+\sin x}dx$ (5)$\int \csc^3 x\sec xdx$ (6)$\int \frac{1}{x(x^{10}+1)}dx$

(7) $\int \frac{1}{\sin 2x-2\sin x}dx$ (8)$\int \frac{x^3}{1+\sqrt{1+x^4}}dx$ (9)$\int \frac{dx}{3\sin^2 x+5\cos^2 x}$

(10) $\int \frac{x^8}{(1-x^3)^3}dx$

解:(1)原式=$\int \frac{x^4-x^2+1}{x^6+1}dx+\int \frac{x^2}{x^6+1}dx=\int \frac{1}{x^2+1}dx+\frac{1}{3}\int \frac{1}{x^6+1}dx^3$

$$=\arctan x+\frac{1}{3}\arctan x^3+C$$

(2)原式=$\int \sqrt{\ln(x+\sqrt{1+x^2})}d(\ln(x+\sqrt{1+x^2}))=\frac{2}{3}(\ln(x+\sqrt{1+x^2}))^{3/2}+C$

(3)令 $t=\sqrt{e^x-1}$,则 $x=\ln(t^2+1), e^{2x}=(t^2+1)^2, dx=\frac{2t}{t^2+1}dt$

原式=$\int \frac{(t^2+1)^2}{t}\frac{2t}{t^2+1}dt=2\int (t^2+1)dt=\frac{2}{3}t^3+2t+C$

$$=\frac{2}{3}(e^x-1)^{3/2}+2(e^x-1)^{1/2}+C$$

(4)原式=$\int \frac{\sin x(1-\sin x)}{\cos^2 x}dx=\int \frac{\sin x-\sin^2 x}{\cos^2 x}x$

$$=\frac{1}{\cos x}-\int \frac{1-\cos^2 x}{\cos^2 x}dx=\frac{1}{\cos x}-\tan x+x+C$$

(5)原式=$-\frac{1}{2}\int \sec^2 xd(\csc^2 x)\underline{\underline{t=\csc^2 x}}\ -\frac{1}{2}\int \frac{1}{1-\frac{1}{t}}dt=-\frac{1}{2}\int \frac{t}{t-1}dt$

$$=-\frac{1}{2}\int (1+\frac{1}{t-1})dt=-\frac{1}{2}t-\frac{1}{2}\ln|t-1|+C=-\frac{1}{2}\csc^2 x+\ln|\tan t|+C$$

(6) 原式=$\int \frac{10x^9}{10x^{10}(x^{10}+1)}dx=\frac{1}{10}\int \frac{1}{x^{10}(x^{10}+1)}dx^{10}=\frac{1}{10}\int (\frac{1}{x^{10}}-\frac{1}{x^{10}+1})dx^{10}$

$$=\frac{1}{10}\int \frac{1}{x^{10}}dx^{10}-\frac{1}{10}\int \frac{1}{x^{10}+1}dx^{10}=\ln|x|-\frac{1}{10}\ln(x^{10}+1)+C$$

(7) 原式=$\int \frac{1}{2\sin x(\cos x-1)}dx=\int \frac{1}{4\sin\frac{x}{2}\cos\frac{x}{2}(-2\sin^2\frac{x}{2})}dx=\int \frac{1}{-8\sin^3\frac{x}{2}\cos\frac{x}{2}}dx$

$$=\frac{1}{4}\int \frac{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}d(\cot\frac{x}{2})=\frac{1}{4}\int (\tan\frac{x}{2}+\cot\frac{x}{2})d(\cot\frac{x}{2})$$

$$=\frac{1}{4}\ln|\cot\frac{x}{2}|+\frac{1}{8}\cot^2\frac{x}{2}+C$$

(8)由于 $d(1+\sqrt{1+x^4})=\frac{4x^3}{2\sqrt{1+x^4}}dx=\frac{2x^3}{\sqrt{1+x^4}}dx$

故原式=$\int(\frac{x^3}{1+\sqrt{1+x^4}}\frac{\sqrt{1+x^4}}{2x^3})\frac{2x^3}{\sqrt{1+x^4}}dx=\frac{1}{2}\int\frac{\sqrt{1+x^4}}{1+\sqrt{1+x^4}}d(1+\sqrt{1+x^4})$

$$=\frac{1}{2}\int\frac{(1+\sqrt{1+x^4})-1}{1+\sqrt{1+x^4}}d(1+\sqrt{1+x^4})=\frac{1}{2}\int 1d(1+\sqrt{1+x^4})-\frac{1}{2}\int\frac{1}{1+\sqrt{1+x^4}}d(1+\sqrt{1+x^4})$$

$$=\frac{1}{2}(1+\sqrt{1+x^4})-\frac{1}{2}\ln(1+\sqrt{1+x^4})+C_1=\frac{1}{2}\sqrt{1+x^4}-\frac{1}{2}\ln(1+\sqrt{1+x^4})+C\quad(C=C_1+\frac{1}{2})$$

(9)原式=$\int\frac{1}{3\tan^2 x+5}\frac{1}{\cos^2 x}dx=\int\frac{1}{3\tan^2 x+5}d\tan x\ \underline{\underline{t=\tan x}}\int\frac{1}{3t^2+5}dt$

$$=\frac{1}{5}\int\frac{1}{1+\frac{3}{5}t^2}dt=\frac{\sqrt{\frac{5}{3}}}{5}\int\frac{1}{1+(\sqrt{\frac{3}{5}}t)^2}d(\sqrt{\frac{3}{5}}t)$$

$$=\frac{1}{\sqrt{15}}\arctan(\sqrt{\frac{3}{5}}t)+C=\frac{1}{\sqrt{15}}\arctan(\sqrt{\frac{3}{5}}\tan x)+C$$

(10)令$t=1-x^3$,则$x=(1-t)^{1/3},dx=-\frac{1}{3}(1-t)^{-2/3}$.

原式=$\int\frac{(1-t)^{8/3}(-\frac{1}{3})(1-t)^{-2/3}}{t^3}dt=-\frac{1}{3}\int\frac{(1-t)^2}{t^3}dt=-\frac{1}{3}\int(t^{-3}-2t^{-2}+t^{-1})dt$

$$=\frac{1}{6}t^{-2}-\frac{2}{3}t^{-1}-\frac{1}{3}\ln|t|+C=\frac{1}{6}(1-x^3)^{-2}-\frac{2}{3}(1-x^3)^{-1}-\frac{1}{3}\ln|1-x^3|+C$$

4 求不定积分$\int\frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx$

解:原式=$\frac{1}{2}\int\frac{1}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}d(1+x^2)\ \underline{\underline{t=1+x^2}}\ \frac{1}{2}\int\frac{dt}{\sqrt{t+t^{3/2}}}$

$$=\frac{1}{2}\int\frac{dt}{\sqrt{t}\sqrt{1+\sqrt{t}}}=\int\frac{d(\sqrt{t})}{\sqrt{1+\sqrt{t}}}=\int\frac{d(\sqrt{t}+1)}{\sqrt{1+\sqrt{t}}}$$

$$=2(1+\sqrt{t})^{1/2}+C$$

$$=2(1+\sqrt{1+x^2})^{1/2}+C$$

5 求不定积分$\int\frac{1}{\sin^4 x\cos^4 x}dx$

解:原式=$8\int\frac{d(2x)}{\sin^4 2x}=8\int\csc^4 2xd(2x)$

$$=-8\int(1+\cot^2 2x)d(\cot 2x)$$
$$=-\frac{8}{3}\cot^3 2x-8\cot 2x+C$$

6 求不定积分 $\int\frac{x}{x^8+1}dx$

解:原式= $\frac{1}{2}\int\frac{d(x^2)}{(x^2)^4+1}\overset{t=x^2}{=\!=\!=}\frac{1}{2}\int\frac{1}{1+t^4}dt$

$$=\frac{1}{4}\int(\frac{t^2+1}{1+t^4}+\frac{1-t^2}{1+t^4})dt$$
$$=\frac{1}{4}\int\frac{t^2+1}{1+t^4}dt+\frac{1}{4}\int\frac{1-t^2}{1+t^4}dt$$

$$=\frac{1}{4}\int\frac{t^{-2}+1}{t^{-2}+t^2}dt-\frac{1}{4}\int\frac{1-t^{-2}}{t^{-2}+t^2}dt$$
$$=\frac{1}{4}\int\frac{d(t-t^{-1})}{(t-t^{-1})^2+2}-\frac{1}{4}\int\frac{d(t+t^{-1})}{(t+t^{-1})^2-2}$$
$$=\frac{\sqrt{2}}{8}\arctan(t+t^{-1})-\frac{\sqrt{2}}{16}\ln|\frac{t+t^{-1}-\sqrt{2}}{t+t^{-1}+\sqrt{2}}|+C$$
$$=\frac{\sqrt{2}}{8}\arctan(x^2+x^{-2})-\frac{\sqrt{2}}{16}\ln|\frac{x^2+x^{-2}-\sqrt{2}}{x^2+x^{-2}+\sqrt{2}}|+C$$

7.设 $f'(x\tan\frac{x}{2})=(x+\sin x)\tan\frac{x}{2}+\cos x$,求 $f(x)$

解: $\int f'(x\tan\frac{x}{2})d(x\tan\frac{x}{2})=\int[(x+\sin x)\tan\frac{x}{2}+\cos x]d(x\tan\frac{x}{2})$

上式左边= $f(x\tan\frac{x}{2})$

右边= $\int(x\tan\frac{x}{2}+\sin x\tan\frac{x}{2}+\cos x)d(x\tan\frac{x}{2})$

$$=\int(x\tan\frac{x}{2}+\frac{2\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}+\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}})d(x\tan\frac{x}{2})$$
$$=\int(x\tan\frac{x}{2}+1)d(x\tan\frac{x}{2})=\frac{1}{2}(x\tan\frac{x}{2})^2+x\tan\frac{x}{2}+C$$

即 $f(x\tan\frac{x}{2})=\frac{1}{2}(x\tan\frac{x}{2})^2+x\tan\frac{x}{2}+C$,故 $f(x)=\frac{1}{2}x^2+x+C$

8 求下列函数的积分

(1) $\int e^x \frac{1+\sin x}{1+\cos x}dx$ (2) $\int \frac{x^2 e^x}{(x+2)^2}dx$

(3) $\int \frac{\ln[(x+a)^{x+a}(x+b)^{x+b}]}{(x+a)(x+b)}dx$ (4) $\int \frac{x}{\sqrt{1-x^2}}\ln\frac{x}{\sqrt{1-x^2}}dx$

解(1)解法一:原式=$\int \frac{1+\sin x}{1+\cos x}de^x$

$$=\frac{1+\sin x}{1+\cos x}e^x-\int e^x d(\frac{1+\sin x}{1+\cos x})$$
$$=\frac{1+\sin x}{1+\cos x}e^x-\int e^x\frac{1+\sin x+\cos x}{(1+\cos x)^2}dx$$

$$=\frac{1+\sin x}{1+\cos x}e^x-\int\frac{e^x}{1+\cos x}dx-\int e^x d\frac{1}{1+\cos x}$$
$$=\frac{1+\sin x}{1+\cos x}e^x-\int\frac{e^x}{1+\cos x}dx-\frac{e^x}{1+\cos x}+\int\frac{e^x}{1+\cos x}dx$$
$$=\frac{1+\sin x}{1+\cos x}e^x-\frac{e^x}{1+\cos x}+C=\frac{e^x\sin x}{1+\cos x}+C$$

解法二:原式=$\int\frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}e^x dx=\int(\frac{e^x}{2\cos^2\frac{x}{2}}+\tan\frac{x}{2}e^x)dx$

$$=\int d(\tan\frac{x}{2}e^x)=\tan\frac{x}{2}e^x+C$$

(2)原式=$\int\frac{(x+2-2)^2e^x}{(x+2)^2}dx=\int\frac{[(x+2)^2-4(x+2)+4]e^x}{(x+2)^2}dx$

$$=\int e^x dx-4\int\frac{e^x}{x+2}dx+4\int\frac{e^x}{(x+2)^2}dx$$

$$=e^x-4\int\frac{e^x}{x+2}dx+4\int e^x d(-\frac{1}{x+2})$$

$$=e^x-4\int\frac{e^x}{x+2}dx-\frac{4e^x}{x+2}+4\int\frac{e^x}{x+2}dx$$

$$=e^x-\frac{4e^x}{x+2}+C=\frac{x-2}{x+2}e^x+C$$

【注】:此题中$\int\frac{e^x}{x+2}dx$不易计算,但是利用分部积分可产生一个积分将其抵消,此为分部积分法的一个特点.

(3)原式=$\int\frac{\ln(x+a)}{x+b}dx+\int\frac{\ln(x+b)}{x+a}dx$

$$=\int \ln(x+a)d(\ln(x+b))+\int \frac{\ln(x+b)}{x+a}dx$$
$$=\ln(x+a)\ln(x+b)-\int \frac{\ln(x+b)}{x+a}dx+\int \frac{\ln(x+b)}{x+a}dx$$
$$=\ln(x+a)\ln(x+b)+C$$

(4)原式$=\int \ln\frac{x}{\sqrt{1-x^2}}d(-\sqrt{1-x^2})=-\sqrt{1-x^2}\ln\frac{x}{\sqrt{1-x^2}}+\int \frac{dx}{x\sqrt{1-x^2}}$

现计算$\int \frac{dx}{x\sqrt{1-x^2}}$

$$\int \frac{dx}{x\sqrt{1-x^2}} \xlongequal{x=\cos t} \int \frac{-\sin t}{\cos t \sin t}dt=-\ln|\sec t+\tan t|+C$$
$$=-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C$$

故原式$=-\sqrt{1-x^2}\ln\frac{x}{\sqrt{1-x^2}}-\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|+C$

9 求下列有理函数的积分

(1)$\int \frac{x^4+1}{(x-1)(x^2+1)}dx$　　(2)$\int \frac{1}{x^6(x^2+1)}dx$　　(3)$\int \frac{-x^2-2}{(x^2+x+1)^2}dx$

解:(1) 由于$\frac{x^4+1}{(x-1)(x^2+1)}=\frac{x^4+2x^2+1-2x^2}{(x-1)(x^2+1)}=\frac{(x^2+1)^2}{(x-1)(x^2+1)}-\frac{2x^2}{(x-1)(x^2+1)}$

$$=\frac{x^2+1}{x-1}-\frac{2x^2}{(x-1)(x^2+1)}$$

又$\frac{x^2+1}{x-1}=x+1+\frac{2}{x-1}$

$$\frac{2x^2}{(x-1)(x^2+1)}=\frac{(2x^2+2)-2}{(x-1)(x^2+1)}=\frac{2}{x-1}-\frac{2}{(x-1)(x^2+1)}$$
$$=\frac{2}{x-1}-\frac{1}{x-1}+\frac{x+1}{x^2+1}$$

故原式$=\int[(x+1+\frac{2}{x-1})-(\frac{2}{x-1}-\frac{1}{x-1}+\frac{x+1}{x^2+1})]dx$

$$=\int(x+1+\frac{1}{x-1}-\frac{x+1}{x^2+1})dx=\frac{x^2}{2}+x+\ln|x-1|-\int\frac{x+1}{x^2+1}dx$$

$$=\frac{x^2}{2}+x+\ln|x-1|-\frac{1}{2}\int\frac{d(x^2+1)}{x^2+1}-\int\frac{1}{x^2+1}dx$$

$$=\frac{x^2}{2}+x+\ln|x-1|-\frac{1}{2}\ln(x^2+1)-\arctan x+C$$

(2) $$\frac{1}{x^6(x^2+1)}=\frac{(1+x^2)-x^2}{x^6(x^2+1)}=\frac{1}{x^6}-\frac{1}{x^4(x^2+1)}=\frac{1}{x^6}-\frac{(1+x^2)-x^2}{x^4(x^2+1)}$$

$$=\frac{1}{x^6}-\frac{1}{x^4}+\frac{1}{x^2(x^2+1)}=\frac{1}{x^6}-\frac{1}{x^4}+\frac{1}{x^2}-\frac{1}{x^2+1}$$

故原式=$\int(\frac{1}{x^6}-\frac{1}{x^4}+\frac{1}{x^2}-\frac{1}{x^2+1})dx=-\frac{1}{5}x^{-5}+\frac{1}{3}x^{-3}-x^{-1}-\arctan x+C$

(3)原式=$\int(\frac{x-1}{(x^2+x+1)^2}-\frac{1}{x^2+x+1})dx$

$$=\frac{1}{2}\int\frac{1}{(x^2+x+1)^2}d(x^2+x+1)-\frac{3}{2}\int\frac{1}{(x^2+x+1)^2}dx-\int\frac{1}{x^2+x+1}dx$$

由于

$$\frac{3}{2}\int\frac{1}{(x^2+x+1)^2}dx=\frac{3}{2}\int\frac{1}{((x+\frac{1}{2})^2+\frac{3}{4})^2}dx$$

$$\underline{\underline{令x+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan t}}\ \frac{3}{2}\int\frac{\frac{\sqrt{3}}{2}\sec^2 t}{\frac{9}{16}\sec^4 t}dx$$

$$=\frac{4\sqrt{3}}{3}\int\cos^2 tdt=\frac{4\sqrt{3}}{3}\int\frac{1+\cos 2t}{2}dt$$

$$=\frac{2\sqrt{3}}{3}(t+\frac{\sin 2t}{2})=\frac{2\sqrt{3}}{3}\arctan(\frac{2x+1}{\sqrt{3}})+\frac{1}{2}\frac{2x+1}{x^2+x+1}$$

故原式=$-\frac{1}{2}\frac{1}{x^2+x+1}-\frac{1}{2}\frac{2x+1}{x^2+x+1}-\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}-\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$

$$=-\frac{x+1}{x^2+x+1}-\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$$

10 求下列三角函数的积分

(1) $\int\frac{\sin x}{1+\sin x}dx$　　　　(2) $\int\frac{\sin^2 x}{1+2\sin^2 x}dx$

(3) $\int \frac{(2+\sin x)\cos x}{1+\cos x}dx$ (4) $\int \frac{1}{\sin^4 x+\cos^4 x}dx$

解:(1)原式=$\int \frac{\sin x(1-\sin x)}{(1+\sin x)(1-\sin x)}dx = \int \frac{\sin x-\sin^2 x}{\cos^2 x}dx = \int \frac{\sin x}{\cos^2 x}dx - \int \tan^2 xdx$

$$= -\int \frac{1}{\cos^2 x}d\cos x - \int (\sec^2 x - 1)dx$$

$$= \frac{1}{\cos x} - \tan x + x + C$$

(2)原式=$\frac{1}{2}\int \frac{(2\sin^2 x+1)-1}{1+2\sin^2 x}dx = \frac{1}{2}\int (1-\frac{1}{1+2\sin^2 x})dx$,现计算$\int \frac{1}{1+2\sin^2 x}dx$

则$\int \frac{1}{1+2\sin^2 x}dx = \int \frac{1}{\sec^2 x+2\tan^2 x}\frac{1}{\cos^2 x}dx$

$$= \int \frac{1}{3+\tan^2 x}d(\tan x) \underline{\underline{t=\tan x}} \int \frac{1}{1+3t^2}dt = \frac{1}{\sqrt{3}}\arctan(\sqrt{3}t)+C$$

$$= \frac{1}{\sqrt{3}}\arctan(\sqrt{3}\tan x)+C$$

故原式=$\frac{x}{2} - \frac{1}{2\sqrt{3}}\arctan(\sqrt{3}\tan x)+C$

(3) $\frac{(2+\sin x)\cos x}{1+\cos x} = \frac{(2+\sin x)(1+\cos x-1)}{1+\cos x} = 2+\sin x - \frac{2+\sin x}{1+\cos x}$

原式=$\int (2+\sin x - \frac{2+\sin x}{1+\cos x})dx = 2x - \cos x - \int \frac{2+\sin x}{1+\cos x}dx$

$$= 2x - \cos x - \int \frac{2+\sin x}{2\cos^2 \frac{x}{2}}dx = 2x - \cos x - \int \sec^2 \frac{x}{2}dx - \int \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}dx$$

$$= 2x - \cos x - 2\tan \frac{x}{2} + 2\ln|\cos \frac{x}{2}| + C$$

(4)原式=$\int \frac{1}{\sin^4 x+\cos^4 x}dx = \int \frac{1}{1-2\sin^2 x\cos^2 x}dx = \int \frac{1}{1-\frac{1}{2}\sin^2 2x}dx$

$$= \int \frac{1}{\cos^2 2x+\frac{1}{2}\sin^2 2x}dx = \int \frac{2}{2\cos^2 2x+\sin^2 2x}dx = \int \frac{2}{2+\tan^2 2x}\sec^2 2xdx$$

$$= \int \frac{1}{2+\tan^2 2x}d(\tan 2x) = \frac{1}{\sqrt{2}}\arctan(\frac{\tan 2x}{\sqrt{2}})+C$$

11 求下列无理函数的积分.

(1) $\int \frac{1}{\sqrt{1+a^x}}dx\ (a>0\text{且}a\neq 1)$ (2) $\int \frac{1}{x\sqrt[3]{1+x^2}}dx$ (3) $\int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}dx$

解:(1)解法一:以 $a^{-\frac{x}{2}}$ 为积分变量,原式= $\int \frac{a^{-\frac{x}{2}}}{\sqrt{1+a^{-x}}}dx = -\frac{2}{\ln a}\int \frac{1}{\sqrt{1+(a^{-\frac{x}{2}})^2}}d(a^{-\frac{x}{2}})$

$$= -\frac{2}{\ln a}\ln(a^{-\frac{x}{2}}+\sqrt{1+a^{-x}})+C$$

解法二:先变形后选择积分变量,原式= $\int \frac{\sqrt{1+a^x}+(1-\sqrt{1+a^x})}{\sqrt{1+a^x}}dx$

$$= \int dx + \int \frac{1-\sqrt{1+a^x}}{\sqrt{1+a^x}}dx = x - \int \frac{a^x}{\sqrt{1+a^x}(1+\sqrt{1+a^x})}dx$$

$$= x - \frac{2}{\ln a}\int \frac{1}{1+\sqrt{1+a^x}}d(1+\sqrt{1+a^x}) = x - \frac{2}{\ln a}\ln(1+\sqrt{1+a^x})+C$$

(2)令 $t=\sqrt[3]{1+x^2}$,则 $2xdx=3t^2dt$,原式= $\frac{1}{2}\int \frac{3t^2}{(t^3-1)t}dt = \frac{1}{2}\int (\frac{1}{t-1}+\frac{1-t}{t^2+t+1})dt$

$$= \frac{1}{2}\ln|t-1| + \frac{1}{2}\int \frac{-\frac{1}{2}(2t+1)+\frac{3}{2}}{t^2+t+1}dt$$

$$= \frac{1}{2}\ln|t-1| - \frac{1}{4}\ln(t^2+t+1) + \frac{3}{4}\frac{2}{\sqrt{3}}\arctan(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}})+C$$

$$= \frac{1}{2}\ln|\sqrt[3]{1+x^2}-1| - \frac{1}{4}\ln(\sqrt[3]{(1+x^2)^2}+\sqrt[3]{1+x^2}+1) + \frac{\sqrt{3}}{2}\arctan(\frac{2\sqrt[3]{1+x^2}+1}{\sqrt{3}})+C$$

(3) $\frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}} = \frac{1}{(x+1)(x-1)}\sqrt[3]{\frac{x+1}{x-1}}$,令 $t=\sqrt[3]{\frac{x+1}{x-1}}$,则 $x=\frac{t^3+1}{t^3-1}=1+\frac{2}{t^3-1}$

$dx = \frac{-6t^2}{(t^3-1)^2}dt$,故原式= $\int \frac{1}{(x+1)(x-1)}\sqrt[3]{\frac{x+1}{x-1}}dx = \int [\frac{t}{(\frac{t^3+1}{t^3-1})^2-1}\frac{-6t^2}{(t^3-1)^2}]dt$

$$= -\frac{3}{2}\int dt = -\frac{3}{2}t+C$$

$$= -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$$

12 求不定积分$\int e^{\sin x}\frac{x\cos^3 x-\sin x}{\cos^2 x}dx$

解:原式=$\int (e^{\sin x}x\cos x-e^{\sin x}\frac{\sin x}{\cos^2 x})dx=\int e^{\sin x}x\cos xdx-\int e^{\sin x}\frac{\sin x}{\cos^2 x}dx$

又 $\int e^{\sin x}x\cos xdx=\int xd(e^{\sin x})=xe^{\sin x}-\int e^{\sin x}dx$

$$\int e^{\sin x}\frac{\sin x}{\cos^2 x}dx=\int e^{\sin x}d(\frac{1}{\cos x})=e^{\sin x}\frac{1}{\cos x}-\int\frac{1}{\cos x}d(e^{\sin x})$$
$$=e^{\sin x}\frac{1}{\cos x}-\int e^{\sin x}dx$$

故原式=$e^{\sin x}(x-\sec x)+C$

13 求不定积分$\int\frac{x^2+1}{x\sqrt{x^4+1}}dx$

解:原式=$\int\frac{x^2+1}{2x^2\sqrt{x^4+1}}d(x^2)\overset{令t=x^2}{=\!=\!=}\int\frac{t+1}{2t\sqrt{t^2+1}}dt$

$$=\frac{1}{2}\int\frac{1}{\sqrt{t^2+1}}dt+\int\frac{1}{2t\sqrt{t^2+1}}dt$$
$$=\frac{1}{2}\ln|t+\sqrt{t^2+1}|+\int\frac{1}{2t\sqrt{t^2+1}}dt$$

现计算$\int\frac{1}{2t\sqrt{t^2+1}}dt$

$$\int\frac{1}{2t\sqrt{t^2+1}}dt\overset{t=1/s}{=\!=\!=}\int\frac{s}{2\sqrt{1+\frac{1}{s^2}}}(-\frac{1}{s^2})ds$$

$$=-\frac{1}{2}\int\frac{1}{\sqrt{s^2+1}}ds=-\frac{1}{2}\ln|s+\sqrt{s^2+1}|+C$$
$$=-\frac{1}{2}\ln|\frac{1}{t}+\sqrt{\frac{1}{t^2}+1}|+C$$

故原式=$\frac{1}{2}\ln|t+\sqrt{t^2+1}|-\frac{1}{2}\ln|\frac{1}{t}+\sqrt{\frac{1}{t^2}+1}|+C$

$$=\frac{1}{2}\ln|x^2+\sqrt{x^4+1}|-\frac{1}{2}\ln|x^{-2}+\sqrt{x^{-4}+1}|+C$$

14 求不定积分$\int\frac{3\sin x+2\cos x}{2\sin x+3\cos x}dx$

解:设 $3\sin x+2\cos x=a(2\sin x+3\cos x)+b(2\sin x+3\cos x)'$

即 $3\sin x+2\cos x=(2a-3b)\sin x+(3a+2b)\cos x$

得到 $\begin{cases}2a-3b=3\\3a+2b=2\end{cases}$,解出 $\begin{cases}a=\dfrac{12}{13}\\b=-\dfrac{5}{13}\end{cases}$

故原式=$\dfrac{12}{13}\int dx-\dfrac{5}{13}\int\dfrac{(2\sin x+3\cos x)'}{2\sin x+3\cos x}dx=\dfrac{12}{13}x-\dfrac{5}{13}\ln|2\sin x+3\cos x|+C$