离散数学
Discrete Mathematics

第17讲 平面图与着色 Planar Graph and Coloring

"Every Planar Map Is Four Colorable", by K. Appel and W. Haken

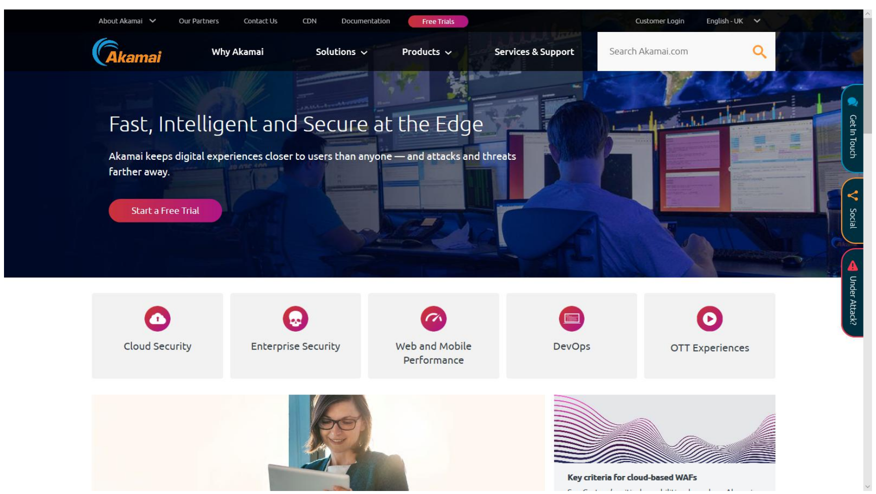
计算机学院科学系 薛思清

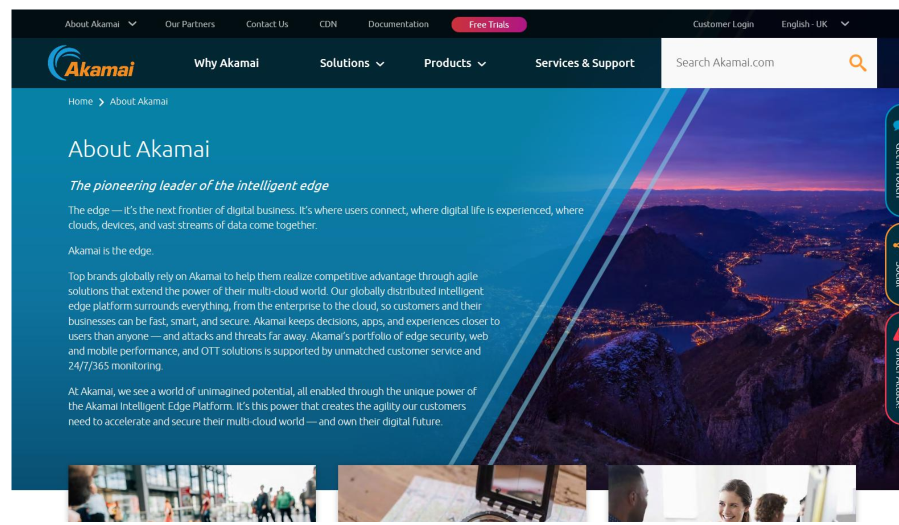
Problems

Wires Arrangement on a Surface(Circuit Board or Microchip)
Fast Register Allocation for Computer Programming;
University Course Scheduling;
Assignment of Radio Frequencies;
Map Coloring Problem.

At Akamai, a new version of software is deployed over each of 20,000(?) servers every few days. The updates cannot be done at the same time. …
——This problem was eventually solved by making a 20,000 node conflict graph and coloring it with 8 colors – so only 8 waves of install are needed!

——by Albert R Meyer

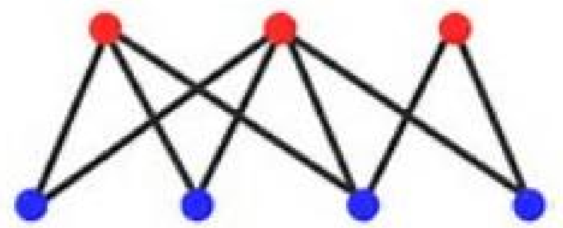


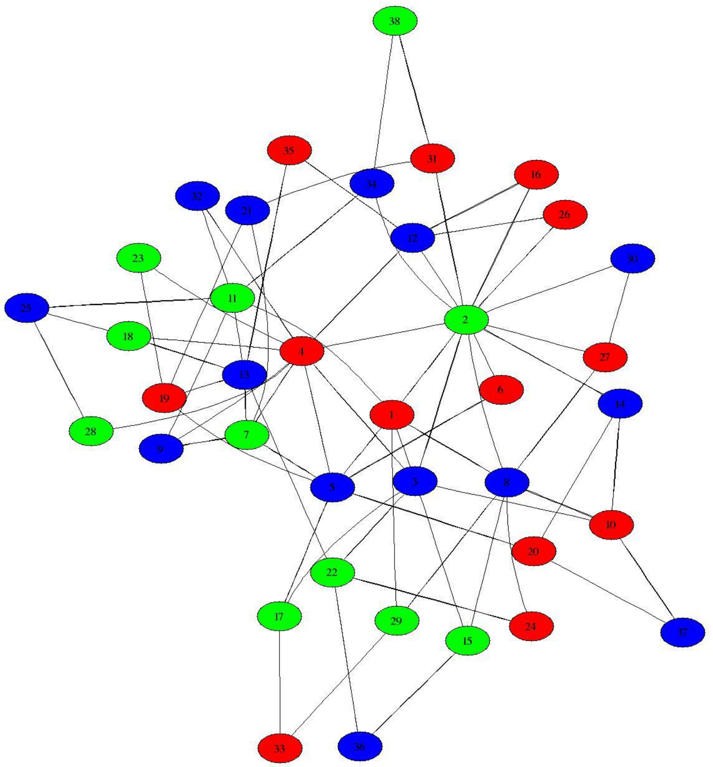


Outline

1 平面图

2 图的着色





How many colors do we need to color the countries of a map in such a way that adjacent countries are colored differently?

The *four-color theorem* states that any map in a plane can be colored using four-colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. This problem is sometimes also called Guthrie's problem after F. Guthrie, who first conjectured the theorem in 1853. The conjecture was then communicated to de Morgan and then into the general community. In 1878, Cayley wrote the first paper on the conjecture.

Fallacious proofs were given independently by Kempe (1879) and Tait (1880). Kempe's proof was accepted for a decade until Heawood showed an error using a map with 18 faces (although a map with nine faces suffices to show the fallacy).

In 1977, Appel and Haken constructed a computer-assisted proof that four colors were sufficient. However, because part of the proof consisted of an exhaustive analysis of many discrete cases by a computer, some mathematicians do not accept it. However, no flaws have yet been found, so the proof appears valid. A potentially independent proof has recently been constructed by N. Robertson, D. P. Sanders, P. D. Seymour, and R. Thomas which also has yet to be verified.

Kempe's attempted proof of the four-color theorem was no a complete failure however. His proof, using the notion of Kempe Chains, was actually a proof of the *five-color theorem*, that any planar map is 5-colorable.

——Matthew R. Wahab

The problem is posed

Unlike many problems in mathematics, the origin of the four-colour problem can be traced precisely – to a letter written in London in 1852. However, for many years it was believed that the problem could be traced back even further – to a lecture given in Germany around 1840. We start our historical narrative by investigating these rival claims and explaining how the confusion arose.

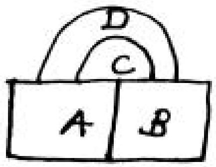
DE MORGAN WRITES A LETTER

On 23 October 1852, Augustus De Morgan, professor of mathematics at University College, London, wrote to his friend Sir William Rowan Hamilton, the distinguished Irish mathematician and physicist. This was nothing unusual. The two men had corresponded for many years, exchanging family news, reporting on the latest scientific gossip in London and Dublin, and sharing bits of mathematical news. Certainly, neither of them could have imagined that the contents of this particular letter would create mathematical history, for it was here that the *four-colour problem* was born.

My dear Hamilton, . . .

A student of mine asked me to day to give him a reason for a fact which I did not know was a fact – and do not yet. He says that if a figure be any how divided and the compartments differently coloured so that figures with any portion of common boundary *line* are differently coloured – four colours may be wanted, but not more – the following is his case in which four *are* wanted

A B C D are names of colours



Query cannot a necessity for five or more be invented . . .

What do you say? And has it, if true been noticed? My pupil says he guessed it in colouring a map of England . . . The more I think of it the more evident it seems. If you retort with some very simple case which makes me out a stupid animal, I think I must do as the Sphynx did . . .

Doing as the Sphynx did would have been rather drastic. The Sphynx of ancient mythology was a legendary figure who leapt to her death after Oedipus had correctly solved a difficult riddle she had set him. The riddle was this: What animal walks on four legs in the morning, two at noon, and three in the evening? The answer is Man (as a baby, as an adult, and as an elderly person with a stick).

Years later, the student who had approached De Morgan that fateful day identified himself as Frederick Guthrie, subsequently a physics professor and founder of the Physical Society in London. But it was not Frederick who had coloured the map of England, as he recalled in 1880:

16

The problem is p

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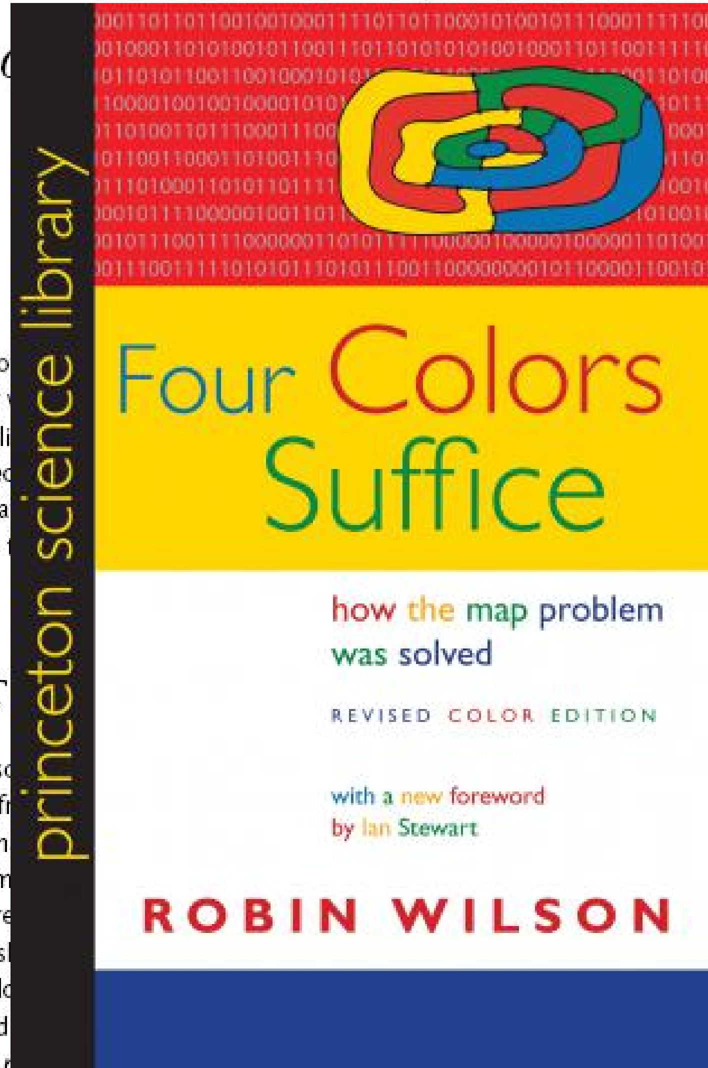

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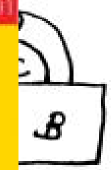

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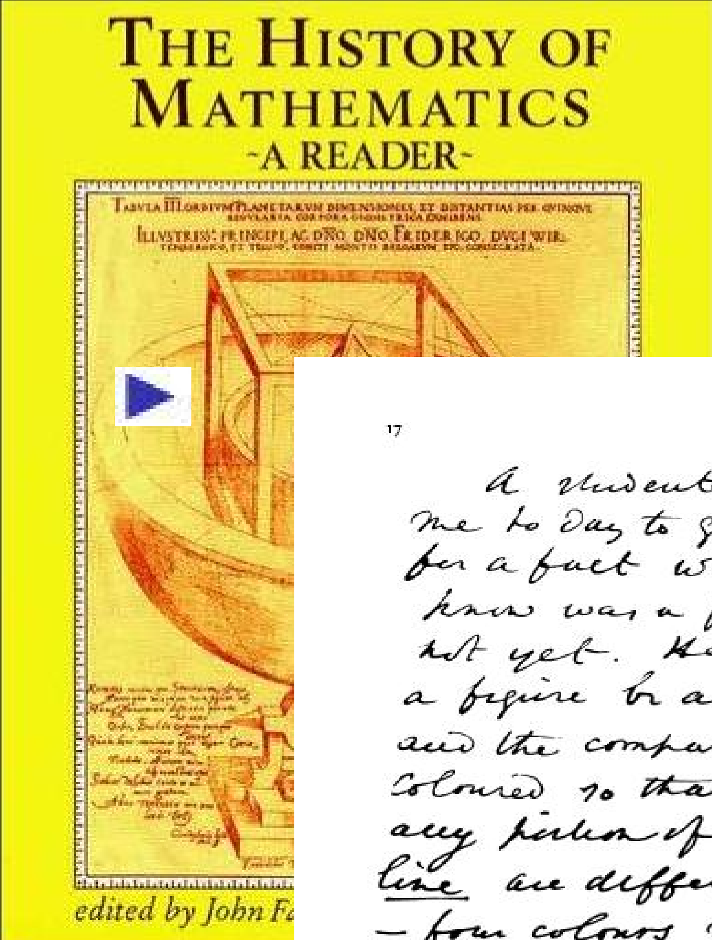

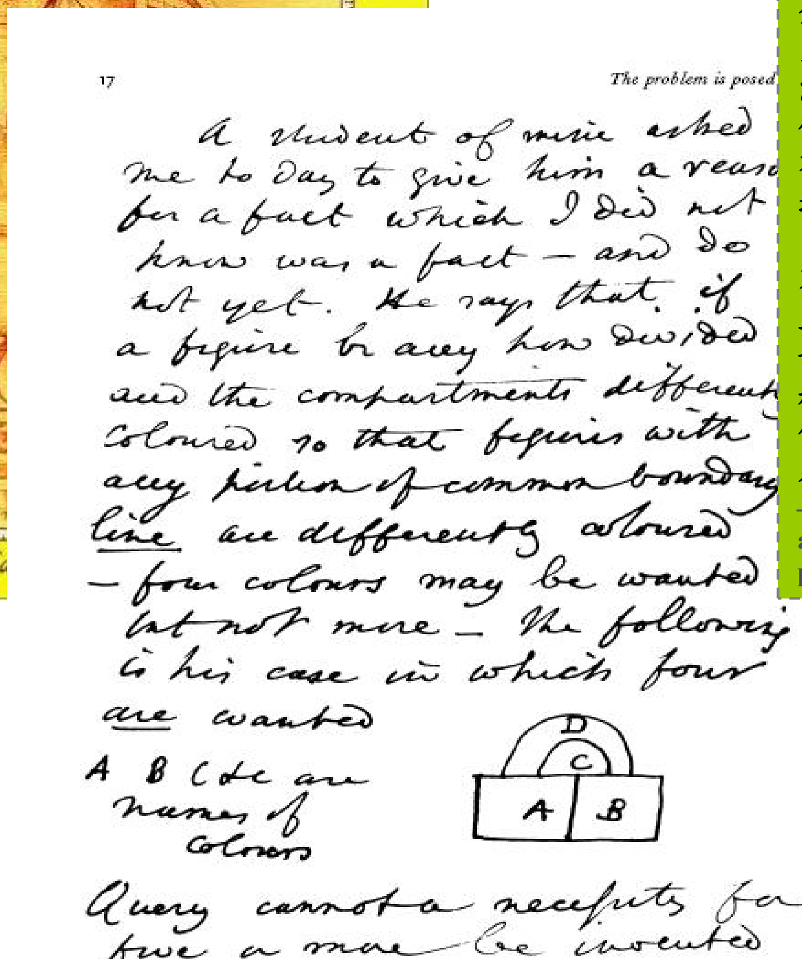
17 *The problem is posed*

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A B C &c are names of colours

Query cannot a necessity for five or more be invented

Part of Augustus De Morgan's letter to Sir William Rowan Hamilton, 23 October 1852.

德·摩尔根致哈密顿的信（1852年10月23日）

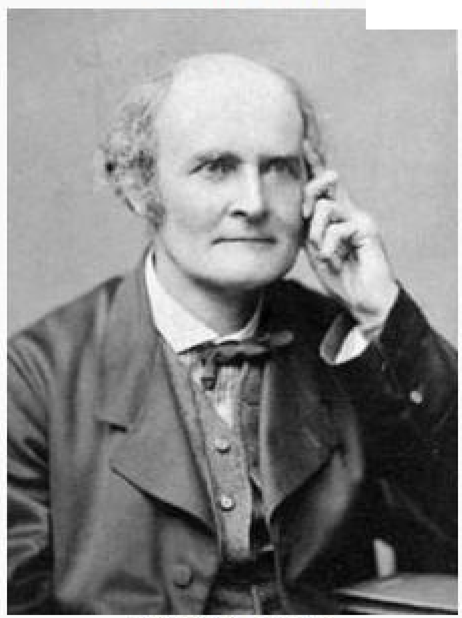
我的一位学生今天请我解释一个我过去不知道，现在仍不甚了了的事实。他说如果任意划分一个图形并给各部分着上颜色，使任何具有公共边界的部分颜色不同，那么需要且仅需要四种颜色就够了。下图是需要四种颜色的例子。现在的问题是是否会出现需要五种或更多种颜色的情形。就我目前的理解，若四个不订分割的区域两两具有公共边界线，则其中三个必包围第四个而使其不与任何第五个区域相毗邻。这事实若能成立，那么用四种颜色即可为任何可能的地图着色，使除了在公共点外同种颜色不会出现画出三个两两具有公共边界的区域ABC，那么似乎不可能再画第四个区域与其他三个区域的每一个都有公共边界，除非它包围了其中一个区域。但要证明这一点却很棘手，我也不能确定问题复杂的程度一对此您的意见如何呢？并且此事如果当真，难道从未有人注意过吗？我的学生说这是在给一幅英国地图着色时提出的猜测。我越想越觉得这是显然的事情。如果您能举出一个简单的反例来，说明我像一头蠢驴，那我只好重蹈史芬克斯的覆辙了……。

——摘录自德•摩尔根致哈密顿信的主要部分，译自J. Fauve1 and J.Gray（eds.），The History of Mathematics： A Reader，pp. 597～598

Francis Guthrie

Augustus De Morgan (1806-187

Arthur Cayley



Portrait in London by
Barraud & Jerrard

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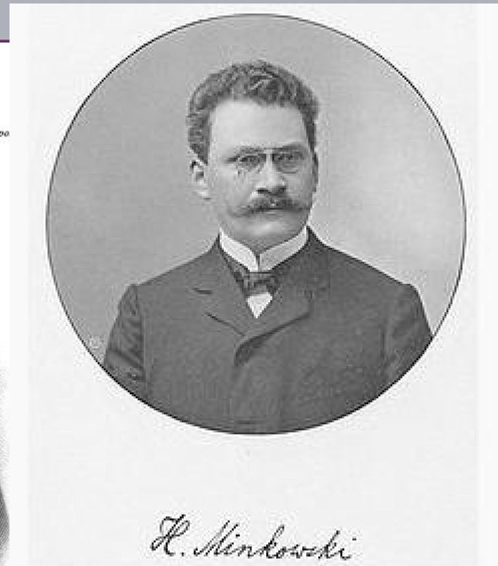
The problem is po

August Ferdinand Möbiu



肯普

赫伍德

EMS December 2002 19

中国政区交通图


序色邻异嵌入染色法生成四色中国地图示例
The Diagram Of Forming Four Color China Map By The Ordinal Color Embedding Coloring
1序色：红 1st Color:Red
2序色：黄 2nd Color:Yellow
3序色：绿 3rd Color:Green
4序色：蓝 4th Color:Blue
作者Author:
我是生命
我是人
序色邻异嵌入四色定理新证
A New Proof Of The Four Color Theorem
By The Ordinal Color
Embedding Coloring
2007.05.05
9 新疆
10 内蒙
27 黑龙江
26 吉林
25 辽宁
31 北京
32 天津
17 河北
12 山西
18 山东
8 青海
11 宁夏
2 甘肃
3 陕西
13 河南
22 江苏
19 安徽
28 上海
7 西藏
1 四川
4 重庆
14 湖北
23 浙江
15 湖南
20 江西
24 福建
5 贵州
6 云南
16 广西
21 广东
33 台湾
29 香港
30 澳门
34 海南
1级优先 1st Coloring
2级优先 2nd Coloring
3级优先 3rd Coloring
4级优先 4th Coloring
5级优先 5th Coloring
主页 Main Page
http://blog.sina.com.cn/4ct

Applicaitons

Circut boards design

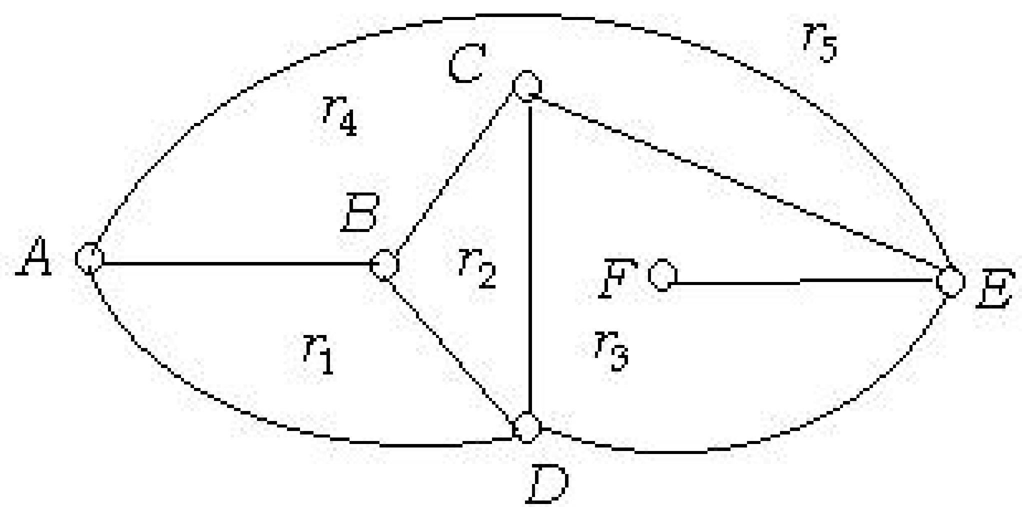
For example, in the design of complex radioeleetronic circuits using printed circuit boards, one problerm is to arrange the elements so that the conductors connecting them do not intersect each other.

Graph data mining

Large amount of data having graph structures, called semi-structured data, such as map data, CAD, biomolecular, chemical molecules, the World Wide Web are stored in databases. For example, Web documents and almost chemical compounds in NCI dataset, which is one of popular graph mining datasets, are known to be expressed by ordered trees and outerplanar graphs, respectively.

Geographic data visualization

- ✓ **平面图（planar graph）：平面嵌入**
- ✓ **面（regions）**
- ✓ **有界面、外部面（无界面）**
- ✓ **边界（boundary）、度/次（degree）**
- ✓ **极大平面图**



平面图 G(6，9，5)

- 平面图（planar graph）：平面嵌入
- 面（regions）
- 有界面、外部面（无界面）
- 边界（boundary）、度/次（degree）
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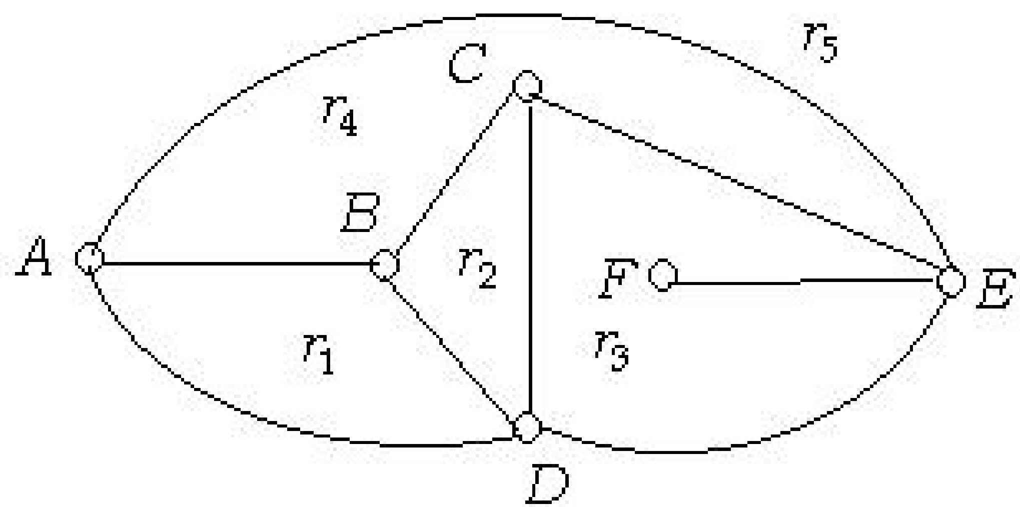

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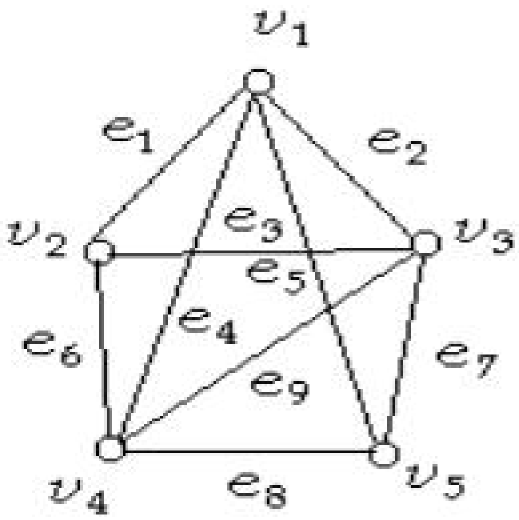
v_1
e_1
e_2
v_2
e_3
v_3
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e_6
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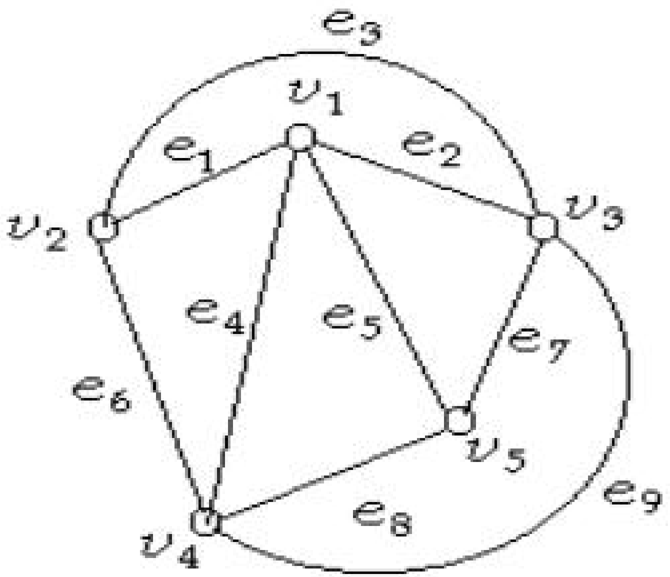
e_8
v_5

(a)


e_3
v_1
e_1
e_2
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v_5
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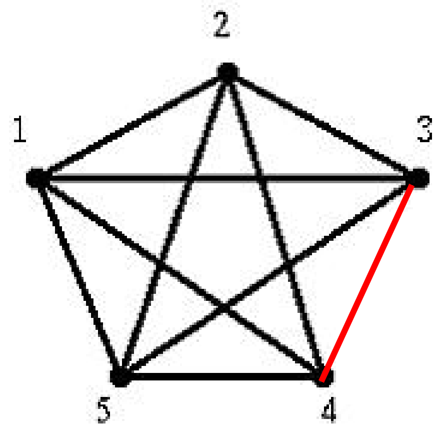
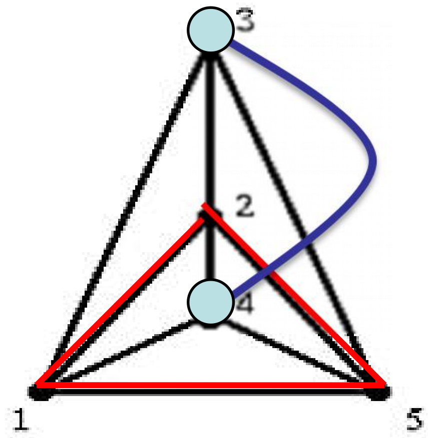
e_8
v_4

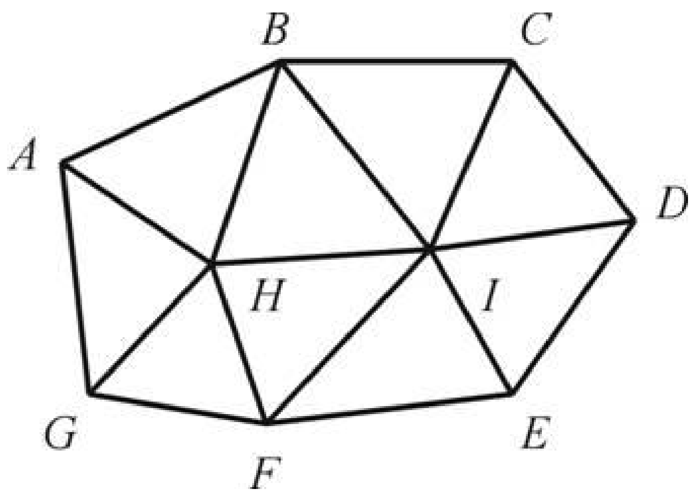
(b)



(a)



(b)



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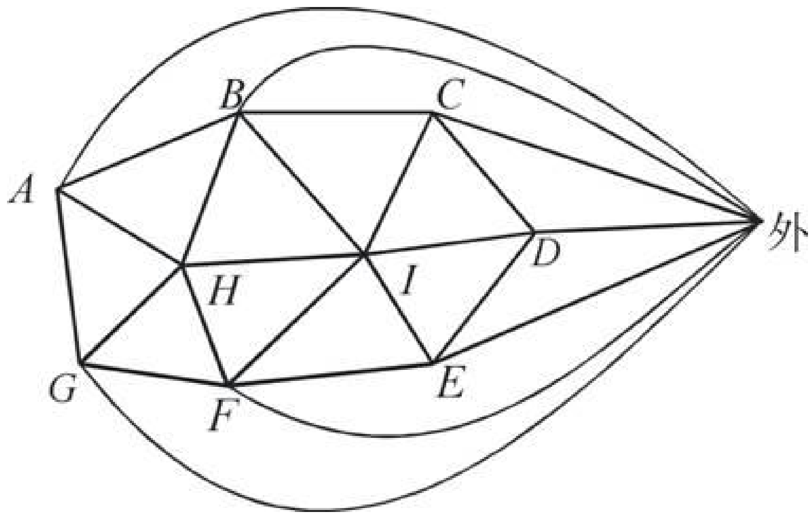

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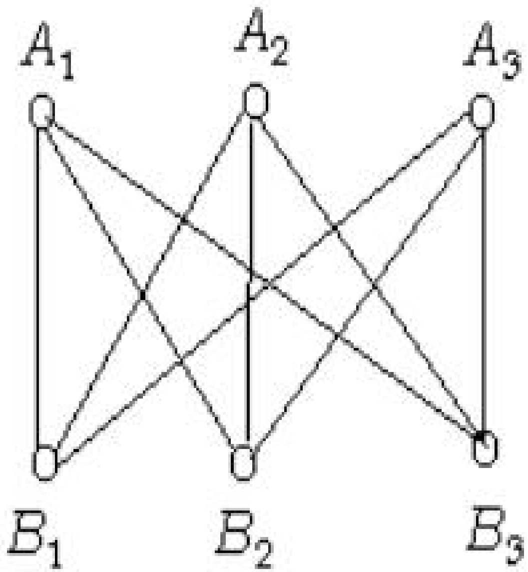
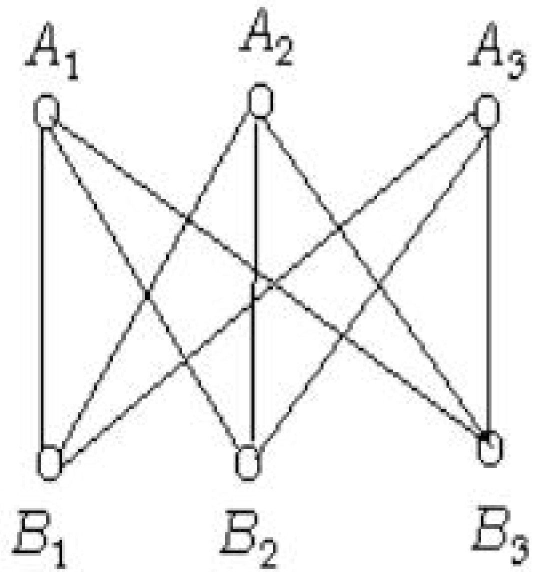


(a)



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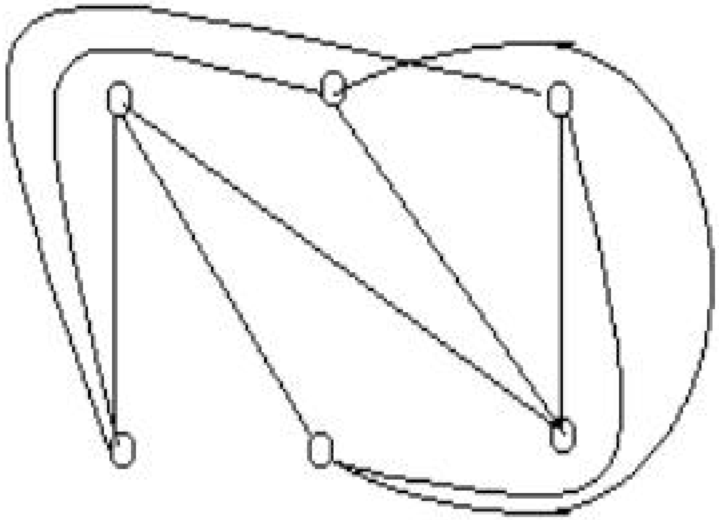

(b)

1 一个有限平面图，面的次数之和等于其边数的两倍。

2 (欧拉定理) 设有一个连通平面图G，共有n个结点e条边f个面，则欧拉公式

$$n-e+f=2$$

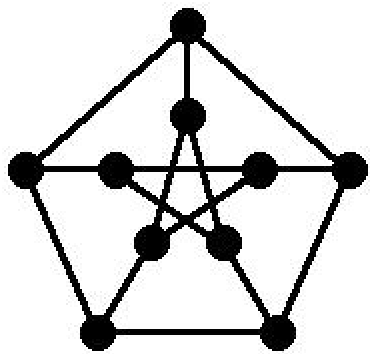
成立。

3 设G为有n个结点e条边的连通平面图，若$n \geq 3$，则 $e \leq 3n-6$。

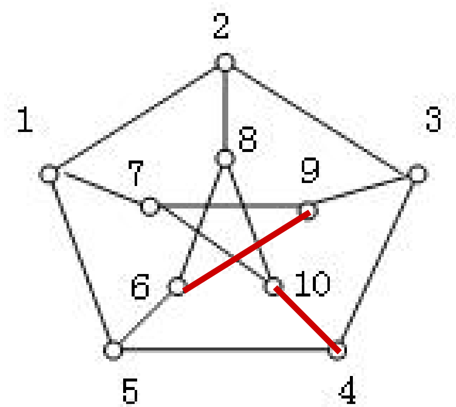
示例5 K_5是平面图吗？ $K_{3,3}$呢？

示例6 n个结点的极大平面图的边数与结点数关系？

示例7 平面连通简单图G，至少有一个结点的度数不大于5。

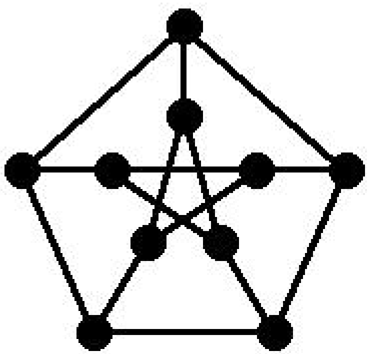


Peterson图

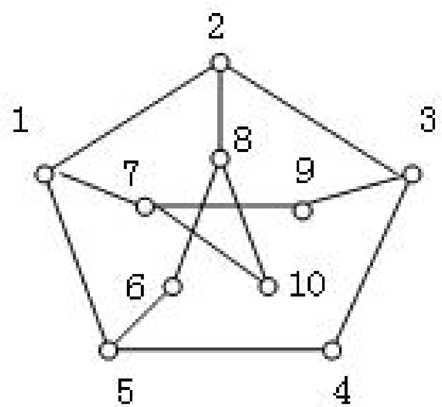


G_1

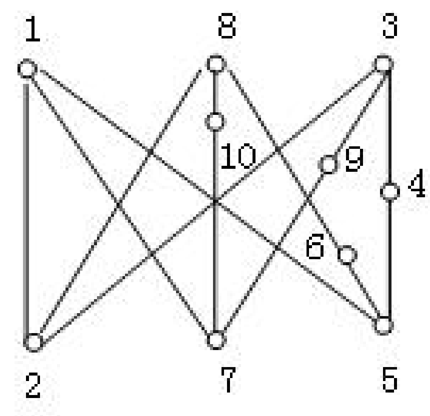
（Kuratowski’s Theorem）图G是平面图，当且仅当K_5与$K_{3,3}$的任何细分图都不是G的子图.



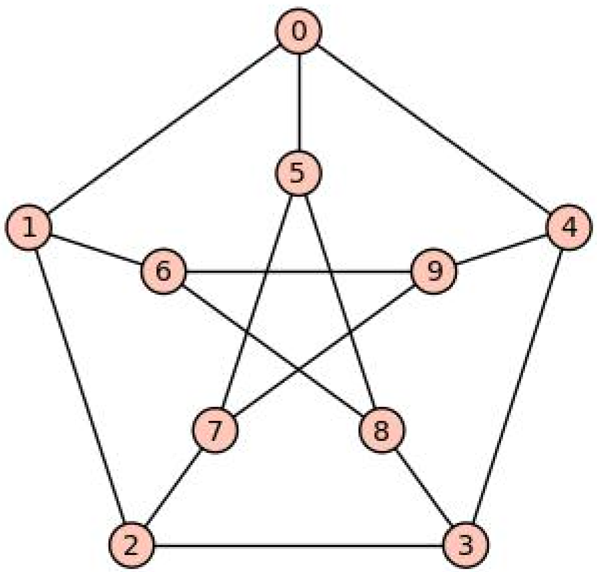
Peterson图



G_1



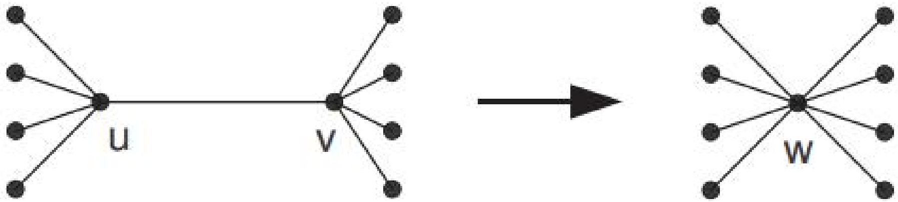
G_2



——from Wiki

Edge Contraction

Definition: Contracting the edge (u, v) in a graph G is the operation of omitting u and v from G and adding a new vertex w whose neighbors are all the neighbors of u and v.



Lemma: A planar graph G remains planar after the contraction of any of its edges.

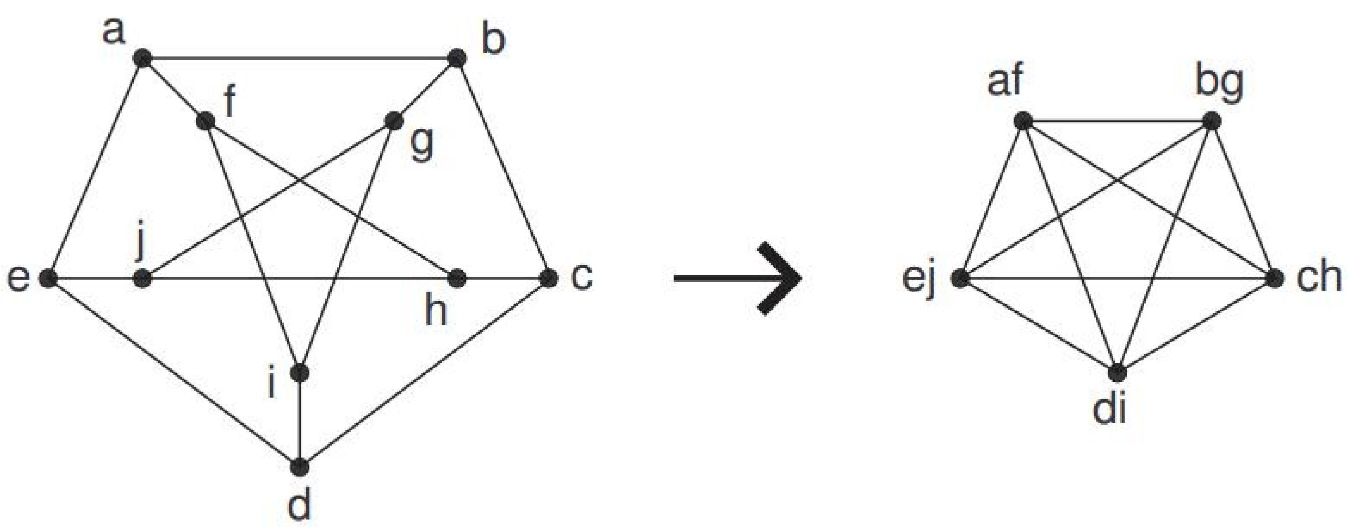
Planar Graphs Characterization

Theorem: A graph is planar iff it does not contain a "subgraph" K_5 or $K_{3,3}$. (Kuratowski's Theorem)

Subgraph: A graph resulting from the original graph after any sequence of omitting edges and contracting edges.

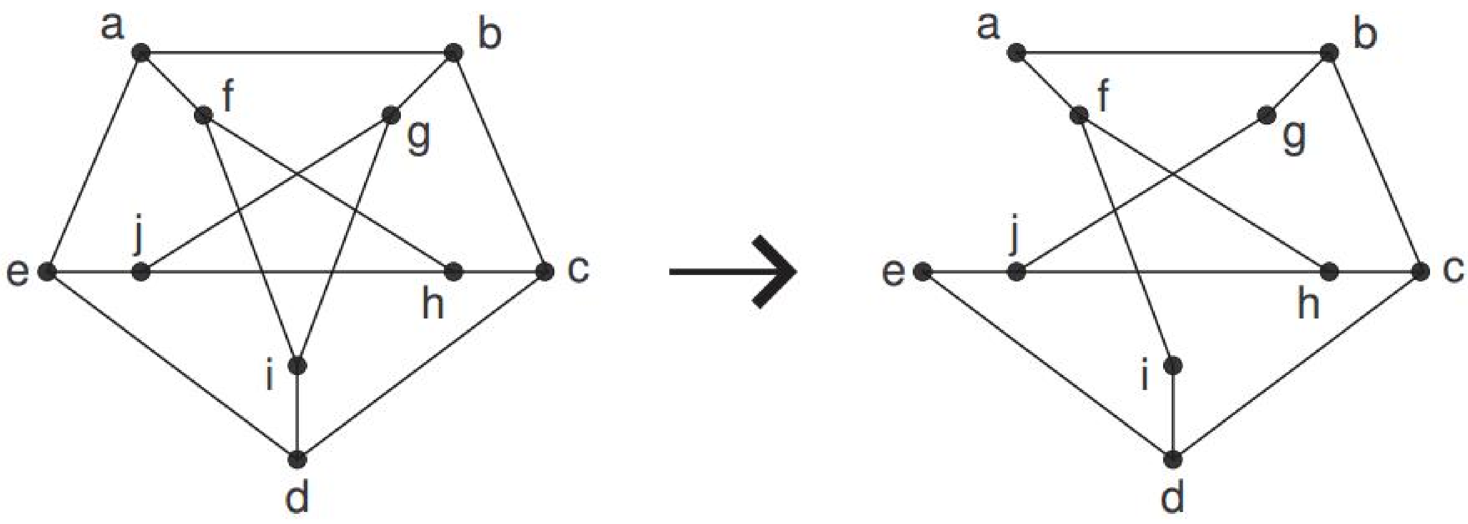
"Easy": If a graph contains K_5 or $K_{3,3}$ as a subgraph then it is not planar.

The Petersen Graph Contains K_5 as a "Subgraph"



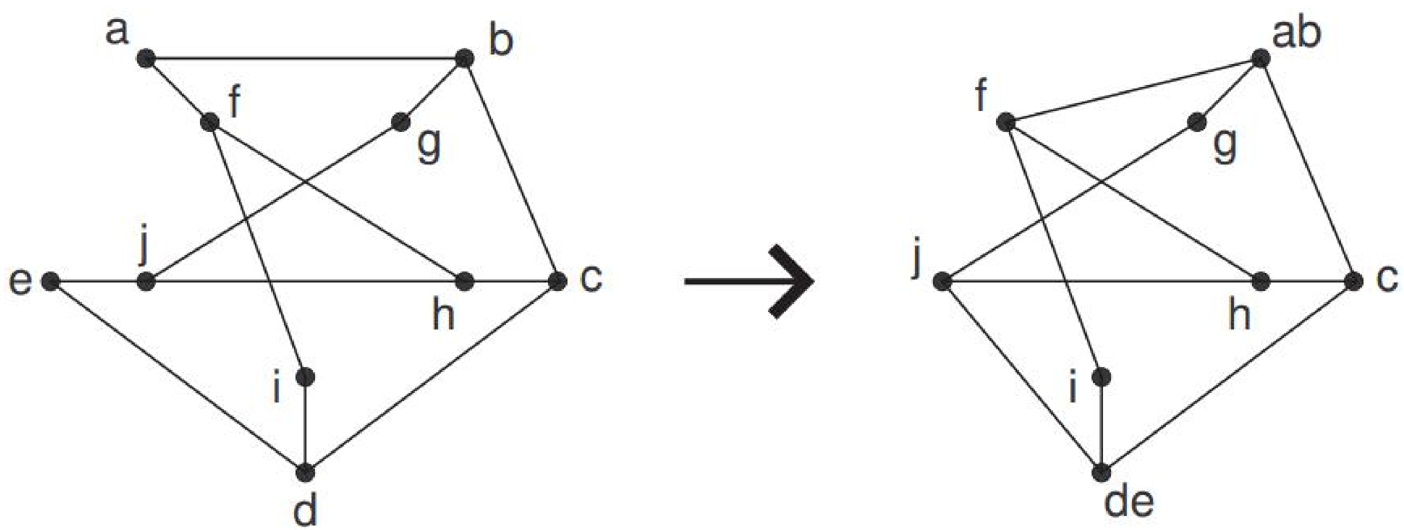
Contract the edges (a, f), (b, g), (c, h), (d, i), and (e, j).

The Petersen Graph Contains $K_{3,3}$ as a "Subgraph"



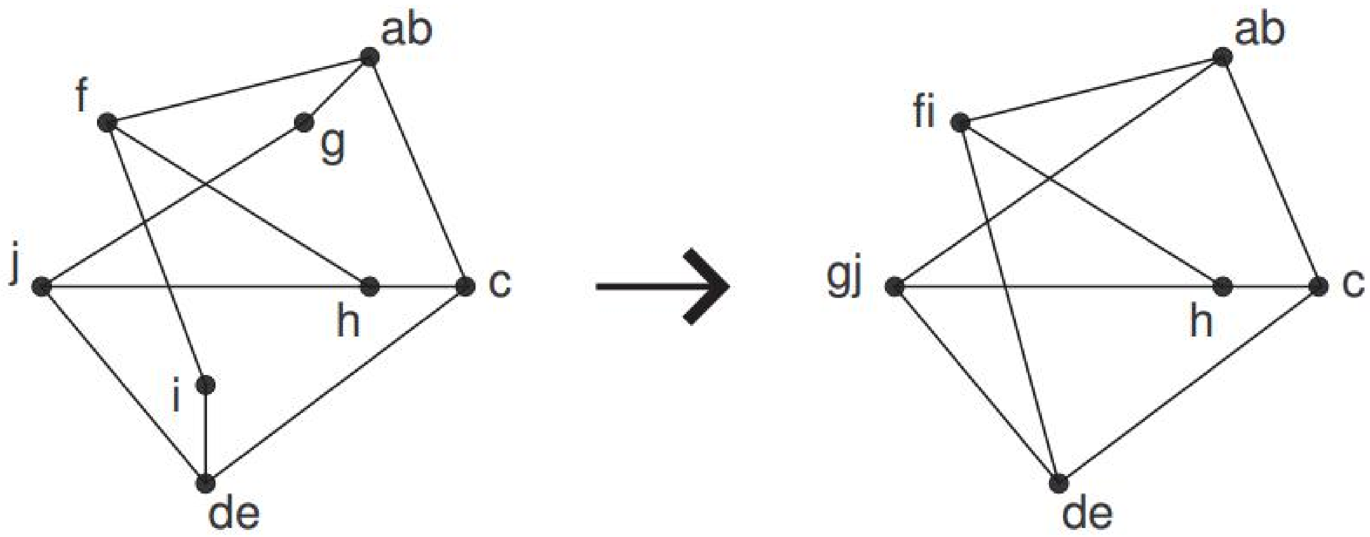
Omit the edges (a, e) and (g, i)

The Petersen Graph Contains $K_{3,3}$ as a "Subgraph"



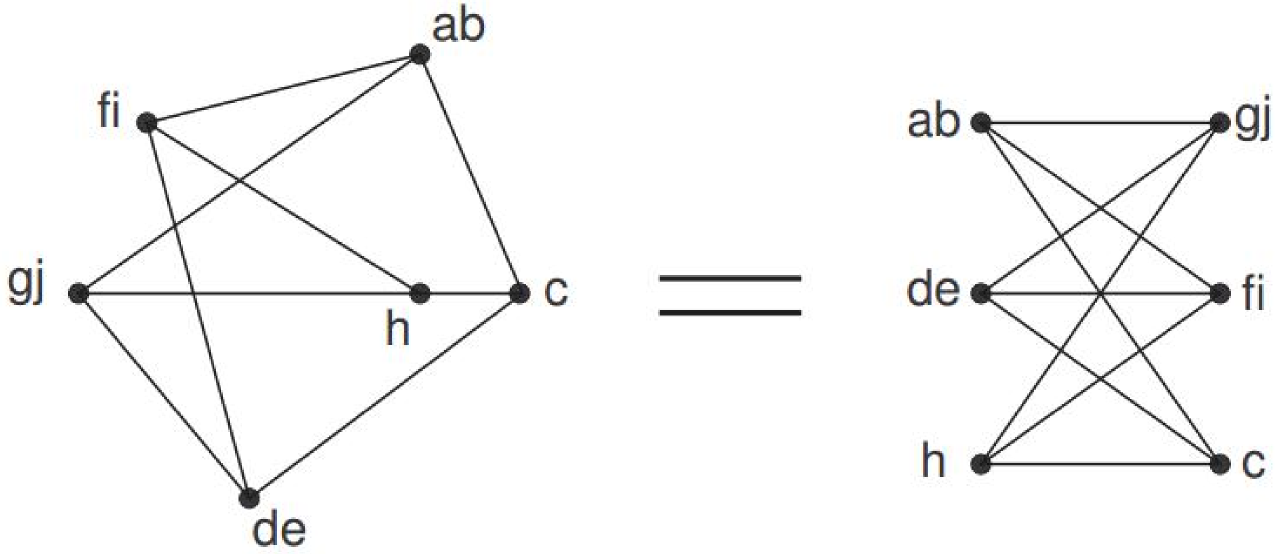
Contract the edges (a, b) and (d, e)

The Petersen Graph Contains $K_{3,3}$ as a "Subgraph"



Contract the edges (f, i) and (g, j)

The Petersen Graph Contains $K_{3,3}$ as a "Subgraph"



The final graph is $K_{3,3}$

3 图的着色(Graph Coloring)

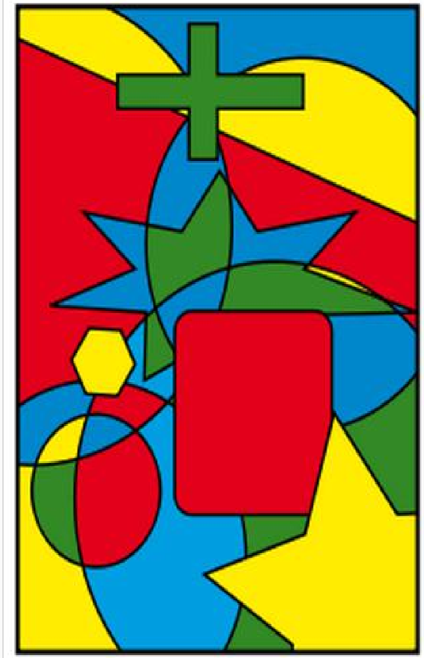
Applicaitons

1) Making Schedule or Time Table
2) Mobile Radio Frequency Assignment
3) Sudoku Game
4) Register Allocation
5) Bipartite Graphs
6) Map Coloring
7) Software Update on a Network of Thousands of Severs

Four color theorem

From Wikipedia, the free encyclopedia

In mathematics, the **four color theorem**, or the **four color map theorem** states that, given any separation of a plane into contiguous regions, producing a figure called a *map*, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color. Two regions are called *adjacent* only if they share a border segment, not just a point. For example, Utah and Arizona are adjacent, but Utah and New Mexico, which only share a point, are not.



Example of a four-colored map

Despite the motivation from coloring political maps of countries, the theorem is not of particular interest to mapmakers. According to an article by the math historian Kenneth May (Wilson 2002, 2), "Maps utilizing only four colours are rare, and those that do usually require only three. Books on cartography and the history of mapmaking do not mention the four-color property."

Three colors are adequate for simpler maps, but an additional fourth color is required for some maps, such as a map in which one region is surrounded by an odd number of other regions that touch each other in a cycle. The five color theorem, which has a short elementary proof, states that five colors suffice to color a map and was proven in the late 19th century (Heawood 1890); however, proving that four colors suffice turned out to be significantly harder. A number of false proofs and false counterexamples have appeared since the first statement of the four color theorem in 1852.

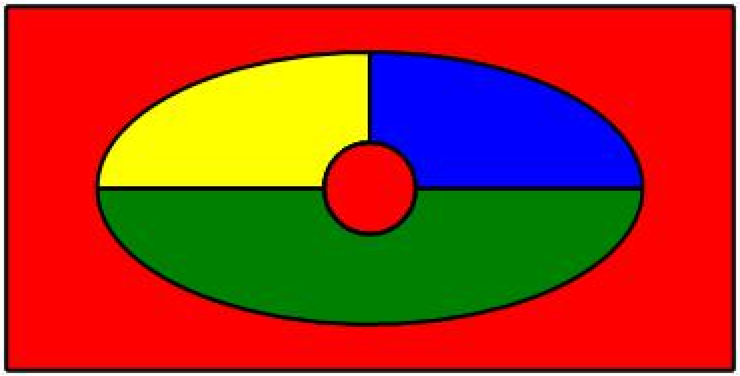
The four color theorem was proven in 1976 by Kenneth Appel and Wolfgang Haken. It was the first major theorem to be proved using a computer. Appel and Haken's approach started by showing that there is a particular set of 1,936 maps, each of which cannot be part of a smallest-sized counterexample to the four color theorem. Appel and Haken used a special-purpose computer program to confirm that each of these maps had this property. Additionally, any map (regardless of whether it is a counterexample or not) must have a portion that looks like one of these 1,936 maps. To show this required hundreds of pages of hand analysis. Appel and Haken concluded that no smallest counterexamples existed because any must contain, yet not contain, one of these 1,936 maps. This contradiction means there are no counterexamples at all and that the theorem is therefore true. Initially, their proof was not accepted by all mathematicians because the computer-assisted proof was infeasible for a human to check by hand (Swart 1980). Since then the proof has gained wider acceptance, although doubts remain (Wilson 2002, 216–222).

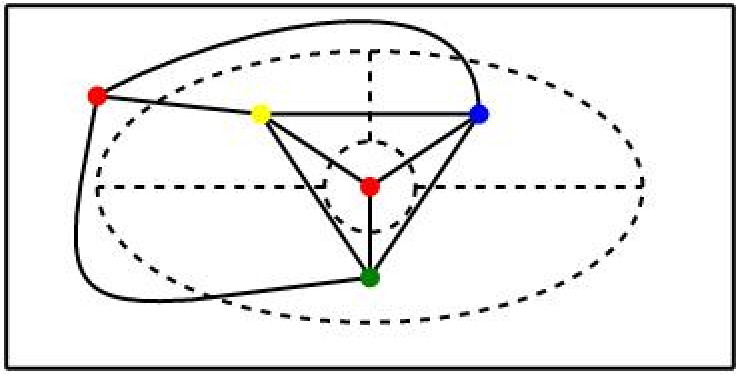
A four-coloring of an actual map of the states of the United States (ignoring water, other countries and text color).

To dispel remaining doubt about the Appel–Haken proof, a simpler proof using the same ideas and still relying on computers was published in 1997 by Robertson, Sanders, Seymour, and Thomas. Additionally in 2005, the theorem was proven by Georges Gonthier with general purpose theorem proving software.

四色猜想→四色定理

对偶图





对于地图的着色问题，可以归纳为对于平面图的结点着色问题，因此四色问题可以归结为要证明对于任何一个平面图，一定可以用四种颜色，对于它的结点进行着色，使得邻接的结点都有不同的颜色。

图G的色数(Chromatic Number)：如果图*G*在着色时最少用*n*种颜色(n-可着色, n-colorable)，称*G*为*n*-色的，用χ(G)表示。

For any simple graph, $\chi(G) \leq \delta_{\max}(G) + 1$.

Let G be a graph that has k mutually adjacent vertices. Then $\chi(G) \geq k$.

Chromatic #s for common graph families

Graph G	$\chi(G)$
empty graph	1
bipartite graph	2
nontrivial path graph P_n	2
nontrivial tree T	2
cube graph Q_n	2
even cycle graph C_{2n}	2
odd cycle graph C_{2n+1}	3
even wheel W_{2n}	3
odd wheel W_{2n+1}	4
complete graph K_n	n

Coloring Algorithm

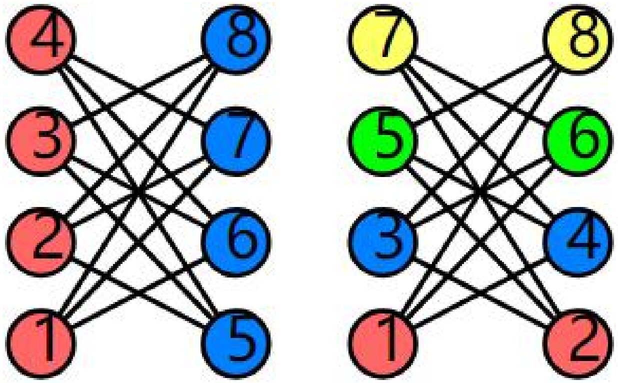
Coloring Algorithm

Basic Greedy Coloring Algorithm

1. Color first vertex with first color.
2. Do following for remaining vertices:

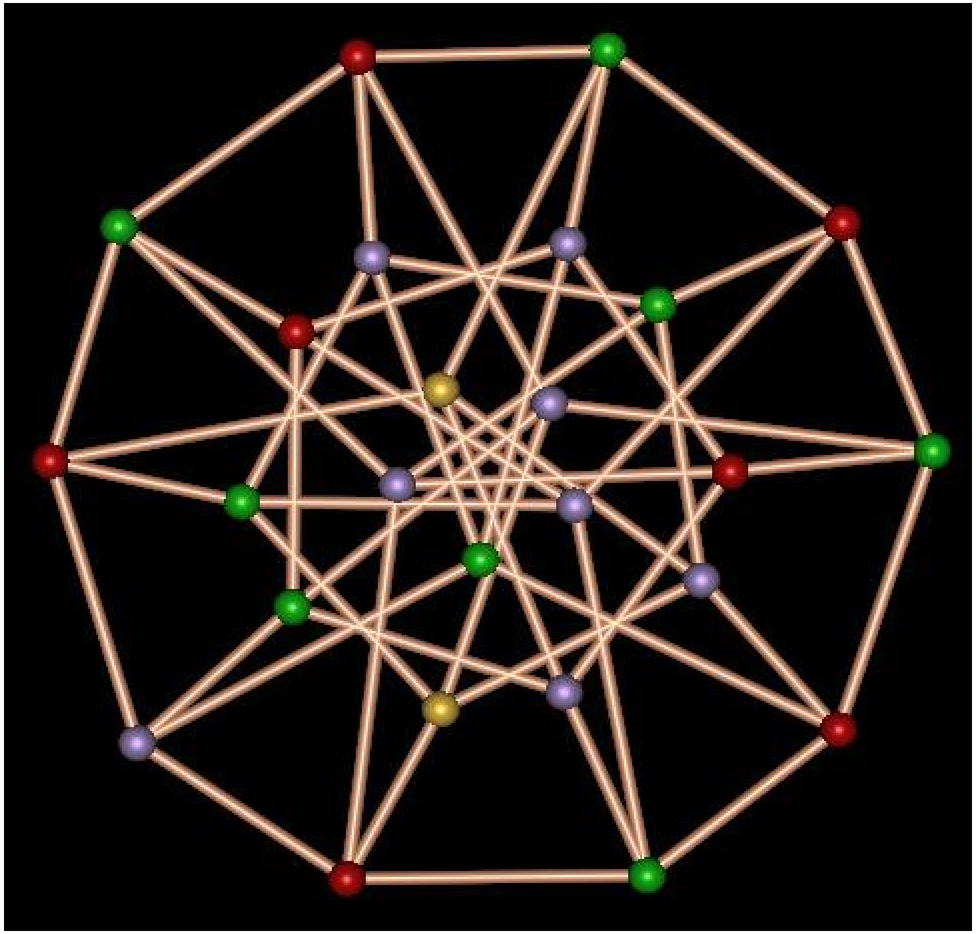
Consider the currently picked vertex and color it with the lowest used numbered color that has not been used on any previously colored vertices adjacent to it. If all previously used colors appear on vertices adjacent to v, assign a new color to it.

red, blue, green, yellow...

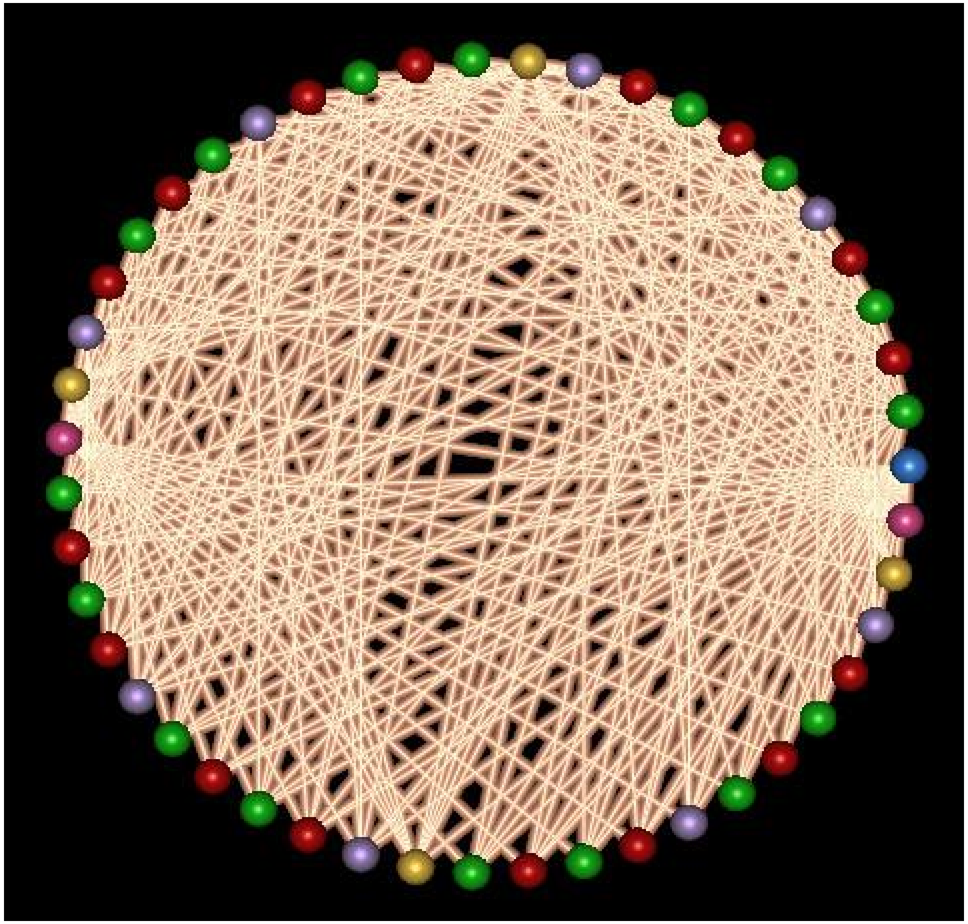


Powell Coloring Algorithm

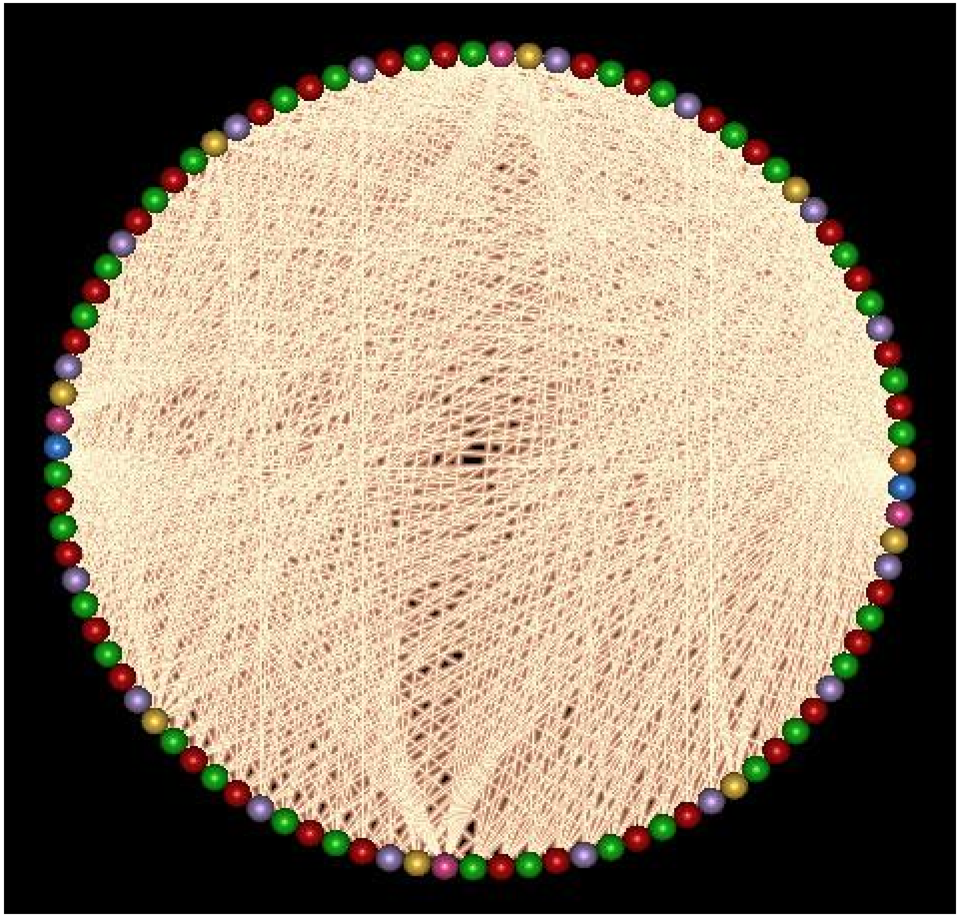
Note that Welsh-Powell is a special case of the basic greedy algorithm, all it does is to suggest a more specific order instead of an arbitrary order in which the vertices are visited.



The Grünbaum graph with a proper m-coloring ($n = 25$, $m = \chi(G) = 4$)..



The Mycielski 6-chromatic graph with a proper m-coloring (n = 95, m = $\chi(G)$ = 6).



The Mycielski 7-chromatic graph with a proper m-coloring (n = 95, m = χ(G) = 7).

平面图

四色定理?

任意平面图至多是4 - 可着色的?

五色定理

任意平面图至多是5 - 可着色的

Every planar graph is five-colorable.

Proof. The proof will be by strong induction on the number, n, of vertices.

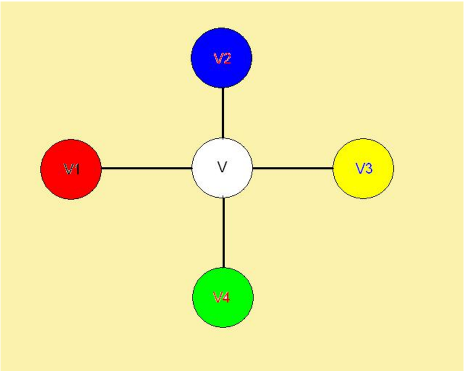
Base cases ($n \leq 5$): immediate.

Induction hypothesis: Every planar graph with $n \leq k$ vertices is five-colorable.

Inductive case: Suppose G is a planar graph with $n = k + 1$ vertices. We will describe a five-coloring of G. First, choose a vertex, g, of G with degree at most 5.(why)

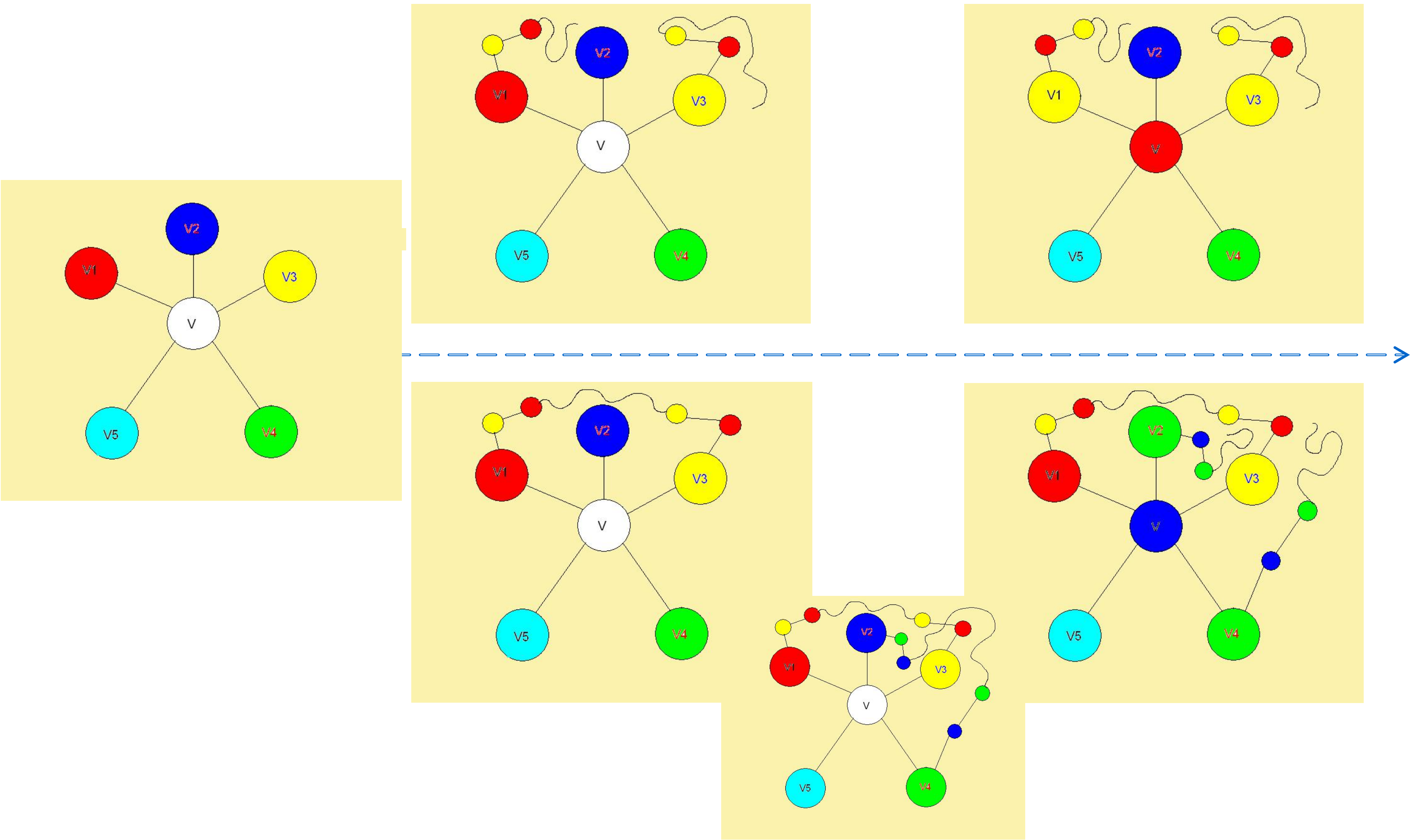
Case 1 ($\deg(g) < 5$): Deleting g from G leaves a graph, H, that is planar. ...

Case 2 ($\deg(g) = 5$):



Every planar graph is five-colorable.

Textbook: (Kempe Chain)



Every planar graph is five-colorable.

(Edge contraction)

Case 2 (deg (g) = 5):

There must be two neighbors, v_1 and v_2, of g that are not adjacent.(why)

Now merge v_1,v_2 and g into a new vertex, v', resulting in a new graph, G', which is planar. Now G' has n=k − 1 vertices and so is five-colorable by the induction hypothesis.

Now define a five coloring of G as follows: use the five-coloring of G' for all the vertices besides g, v_1 and v_2.

Next assign the color of v' in G' to be the color of the neighbors v_1 and v_2.

Since v_1 and v_2 are not adjacent in G, this defines a proper five-coloring of G except for vertex g.

But since these two neighbors of g have the same color, the neighbors of g have been colored using fewer than five colors altogether. So complete the five-coloring of G by assigning one of the five colors to g that is not the same as any of the colors assigned to its neighbors.

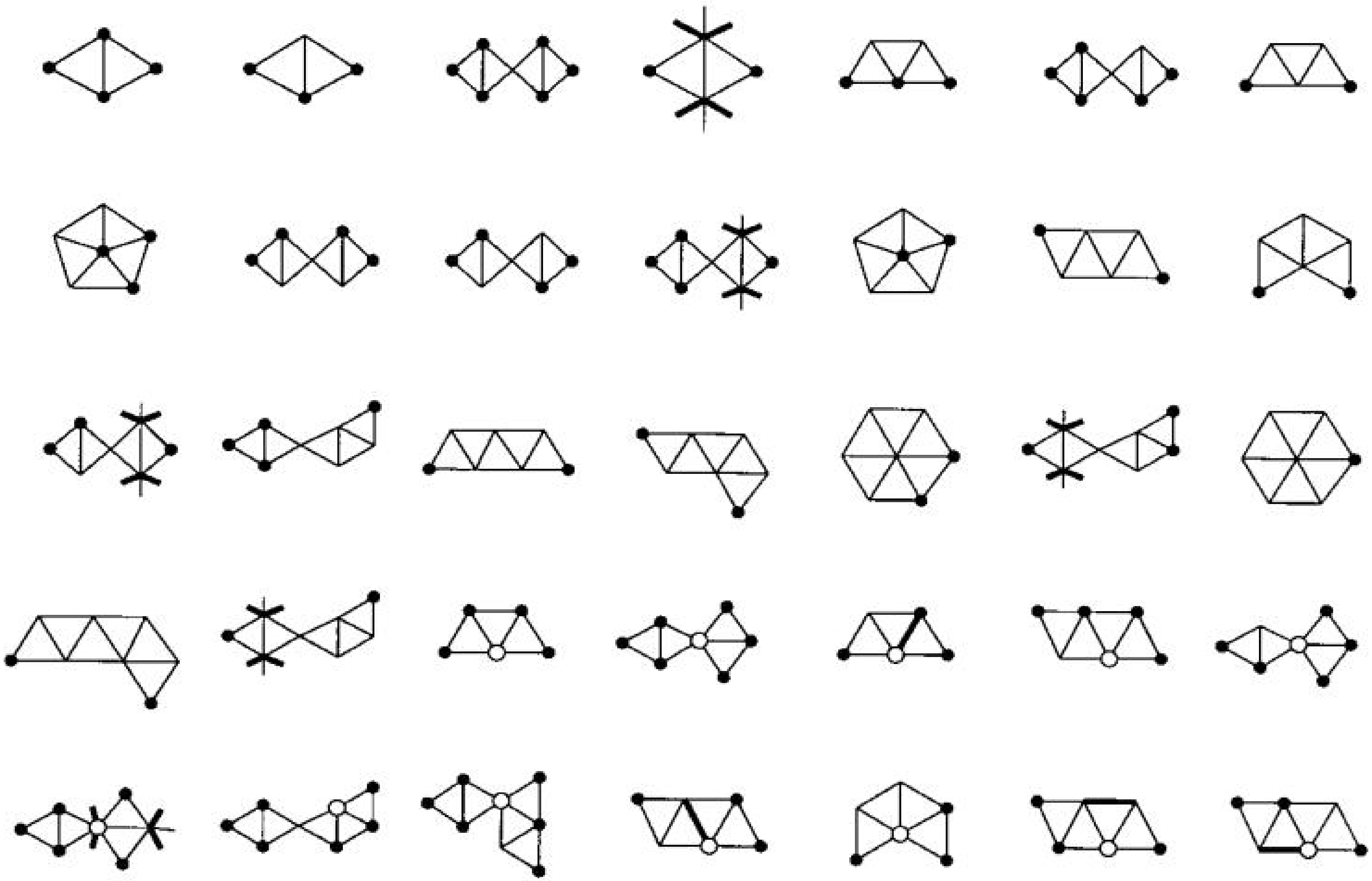
More- Four-colorable.

The Tale of a Brainteaser

Francis Guthrie certainly did it, when he coined his innocent little coloring puzzle in 1852. He managed to embarrass successively his mathematician brother, his brother's professor, Augustus de Morgan, and all of de Morgan's visitors, who couldn't solve it; the Royal Society, who only realized ten years later that Alfred Kempe's 1879 solution was wrong; and the three following generations of mathematicians who couldn't fix it [19].

by Georges Gonthier, *Formal Proof—The FourColor Theorem, 2005*

THE UNAVOIDABLE SET OF REDUCIBLE CONFIGURATIONS



by Robertson,Saunders, Seymour, and Thomas, *The Four-Colour Theorem, 1995*

Unfortunately, the proof by Appel and Haken (briefly, A&H) has not been fully accepted. There has remained a certain amount of doubt about its validity, basically for two reasons:

(i) part of the A&H proof uses a computer and cannot be verified by hand, and

(ii) even the part of the proof that is supposed to be checked by hand is extraordinarily complicated and tedious, and as far as we know, no one has made a complete independent check of it.

The basic idea of the proof is the same as that of A&H. We exhibit a set of "configurations"; in our case there are 633 of them. We prove that none of these configurations can appear in a minimal counterexample to the 4CT.

by Robertson,Saunders, Seymour, and Thomas, *The Four-Colour Theorem, 1995*

Even Appel and Haken's 1976 triumph [2] had a hint of defeat: they'd had a computer do the proof for them! Perhaps the mathematical controversy around the proof died down with their book [3] and with the elegant 1995 revision [13] by Robertson, Saunders, Seymour, and Thomas. However something was still amiss: both proofs combined a textual argument, which could reasonably be checked by inspection, with computer code that could not. Worse, the empirical evidence provided by running code several times with the *same* input is weak, as it is blind to the most common cause of "computer" error: programmer error.

For some thirty years, computer science has been working out a solution to this problem: formal program proofs. The idea is to write code that describes not only *what* the machine should do, but also *why* it should be doing it—a formal proof of correctness. The validity of the proof is an objective mathematical fact that can be checked by a *different* program, whose own validity can be ascertained empirically because it does run on *many* inputs. The main technical difficulty is that formal proofs are very difficult to produce, even with a language rich enough to express all mathematics.

In 2000 we tried to produce such a proof for part of code from [13], just to evaluate how the field had progressed. We succeeded, but now a new question emerged: was the statement of the correctness proof (the *specification*) itself correct? The only solution to that conundrum was to formalize the *entire* proof of the Four-Color Theorem, not just its code. This we finally achieved in 2005.

by Georges Gonthier, *Formal Proof—The FourColor Theorem, 2005*

Problems

1 n个结点的完全图K_n及K_n的补图的色数是多少?

2 n个结点的环图、树、二分图的色数分别是多少?

3 请给出一个平面图，使它是可4-着色的，但不是可3-着色的。

4 A graph with maximum degree at most k is (k + 1)-colorable。

5 证明：少于30条边的平面连通简单图至少有一个结点的度不大于4。

6 证明：6个结点和12条边的连通平面简单图的每个面的度均为3。

2018
2019

Szczęśliwego Nowego Roku!

Happy New Year!

Καλή Χρονιά!

¡Feliz Año Nuevo!

عام سعيد

Feliz Ano Novo!

新年快乐

Felice Anno Nuovo!

С Новым Годом!

2019

Godt Nytår!

Yeni yılınız kutlu olsun!

새해복많이받으세요

Bonne année!

Frohes neues Jahr!

Gott nytt år!

Gelukkig Nieuwjaar!

Hyvää uutta vuotta

新年おめでとう