离散数学
Discrete Mathematics

第3讲 谓词逻辑 Predicate Logic（1）

Outline

- 问题，谓词逻辑
- 谓词公式
- 谓词逻辑推理

苏格拉底三段论

人都是要死的，

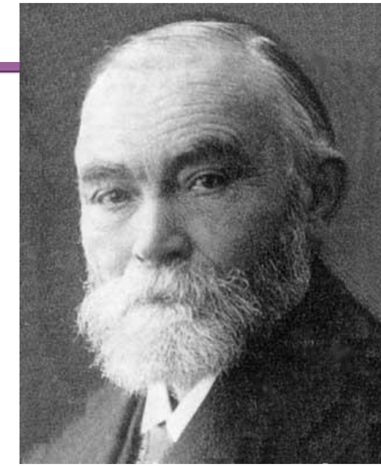
苏格拉底是人，

所以苏格拉底是要死的。

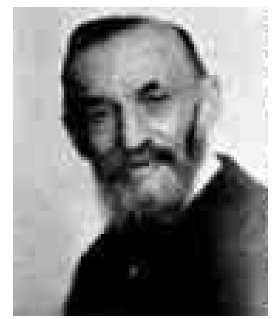
"我们从前谁也不相信这个世界还有比我们的伦理更美满、立身处世之道更进步的民族存在，现在从东方的中国，给我们以一大觉醒！东西双方比较起来，我们觉得在工艺技术上，彼此难分高低；关于思想理论方面，我们稍胜于东方一筹，而在哲学实践方面，实在不能不承认我们相形见绌."

Leibniz

Boole



Frege



Peano

谓词逻辑？

Predicate Logic

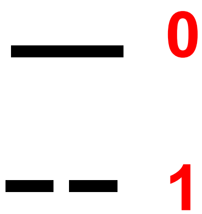
命题逻辑

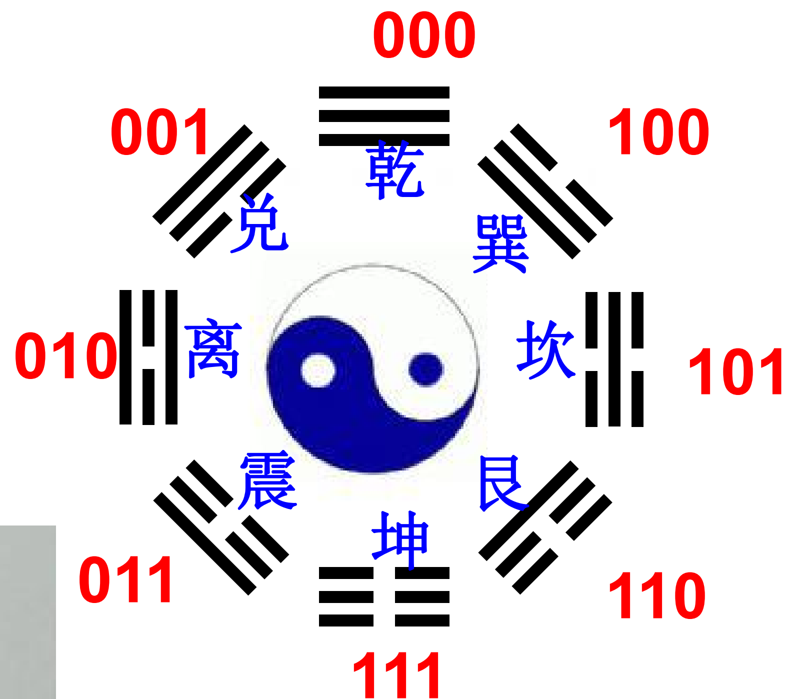
谓词逻辑

（一阶逻辑First-order Logic）

二进制——计算机的语言

“我的这种不可思议的新发明，…… 是因为我发现了一位圣人的古代文字的秘密，这位古代圣人，就是3000年多前，中国早期的君王伏羲氏”。——莱布尼兹





王能超 教授

伏羲八卦图（先天八卦图）
用符号表示的二进制数

Gottfried Leibniz

From Wikipedia, the free encyclopedia

"Leibniz" redirects here. For other uses, see Leibniz (disambiguation).

Gottfried Wilhelm Leibniz (sometimes *von Leibniz*) (German pronunciation: ['gɔtfʁiːt 'vɪlhɛlm fɔn 'laɪbnɪts][1] (July 1, 1646 - November 14, 1716) was a German mathematician and philosopher. He wrote primarily in Latin and French.

Leibniz occupies a prominent place in the history of mathematics and the history of philosophy. Leibniz developed the infinitesimal calculus independently of Isaac Newton, and Leibniz's mathematical notation has been widely used ever since it was published. Leibniz also developed the binary number system, which is at the foundation of virtually all digital computers.

In philosophy, Leibniz is mostly noted for his optimism, *e.g.* his conclusion that our Universe is, in a restricted sense, the best possible one that God could have created. Leibniz, along with René Descartes and Baruch Spinoza, was one of the three great 17th Century advocates of rationalism. The work of Leibniz anticipated modern logic and analytic philosophy, but his philosophy also looks back to the scholastic tradition, in which conclusions are produced by applying reason to first principles or a priori definitions rather than to empirical evidence. Leibniz made major contributions to physics and technology, and anticipated notions that surfaced much later in biology, medicine, geology, probability theory, psychology, linguistics, and information science. He wrote works on politics, law, ethics, theology, history, philosophy, and philology. Leibniz's contributions to this vast array of subjects were scattered in various learned journals, in tens of thousands of letters, and in unpublished manuscripts. As of 2010, there is no complete gathering of the writings of Leibniz.[2]

The collection of manuscript papers of Leibniz at the Gottfried Wilhelm Leibniz Bibliothek - Niedersächische Landesbibliothek were inscribed on UNESCO's Memory of the World Register in 2007.[3]

Gottfried Wilhelm Leibniz

Gottfried Wilhelm Leibniz

Full name	Gottfried Wilhelm Leibniz
Born	1 July 1646 Leipzig, Electorate of Saxony
Died	14 November 1716 (aged 70) Hanover, Electorate of Hanover
Era	17th-century philosophy 18th-century philosophy
Region	Western Philosophy
Main interests	Metaphysics, Mathematics, Theodicy
Notable ideas	Infinitesimal calculus, Monadology, Optimism Leibniz formula for pi Leibniz harmonic triangle Leibniz formula for determinants Leibniz integral rule Principle of sufficient reason Diagrammatic reasoning Notation for differentiation Proof of Fermat's little theorem Kinetic energy Entscheidungsproblem AST
Influenced by	[show]
Influenced	[show]

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Symbolic thought

[edit]

Leibniz believed that much of human reasoning could be reduced to calculations of a sort, and that such calculations could resolve many differences of opinion:

> The only way to rectify our reasonings is to make them as tangible as those of the Mathematicians, so that we can find our error at a glance, and when there are disputes among persons, we can simply say: Let us calculate [*calculemus*], without further ado, to see who is right.[22]

Leibniz's calculus ratiocinator, which resembles symbolic logic, can be viewed as a way of making such calculations feasible. Leibniz wrote memoranda[23] that can now be read as groping attempts to get symbolic logic—and thus his *calculus*—off the ground. But Gerhard and Couturat did not publish these writings until modern formal logic had emerged in Frege's *Begriffsschrift* and in writings by Charles Sanders Peirce and his students in the 1880s, and hence well after Boole and De Morgan began that logic in 1847.

Leibniz thought symbols were important for human understanding. He attached so much importance to the invention of good notations that he attributed all his discoveries in mathematics to this. His notation for the infinitesimal calculus is an example of his skill in this regard. C.S. Peirce, a 19th-century pioneer of semiotics, shared Leibniz's passion for symbols and notation, and his belief that these are essential to a well-running logic and mathematics.

But Leibniz took his speculations much further. Defining a character as any written sign, he then defined a "real" character as one that represents an idea directly and not simply as the word embodying the idea. Some real characters, such as the notation of logic, serve only to facilitate reasoning. Many characters well-known in his day, including Egyptian hieroglyphics, Chinese characters, and the symbols of astronomy and chemistry, he deemed not real.[24] Instead, he proposed the creation of a *characteristica universalis* or "universal characteristic", built on an alphabet of human thought in which each fundamental concept would be represented by a unique "real" character:

> It is obvious that if we could find characters or signs suited for expressing all our thoughts as clearly and as exactly as arithmetic expresses numbers or geometry expresses lines, we could do in all matters *insofar as they are subject to reasoning* all that we can do in arithmetic and geometry. For all investigations which depend on reasoning would be carried out by transposing these characters and by a species of calculus.[25]

Complex thoughts would be represented by combining characters for simpler thoughts. Leibniz saw that the uniqueness of prime factorization suggests a central role for prime numbers in the universal characteristic, a striking anticipation of Gödel numbering. Granted, there is no intuitive or mnemonic way to number any set of elementary concepts using the prime numbers. Leibniz's idea of reasoning through a universal language of symbols and calculations however remarkably foreshadows great 20th century developments in formal systems, such as Turing completeness, where computation was used to define equivalent universal languages (see Turing degree).

Because Leibniz was a mathematical novice when he first wrote about the *characteristic*, at first he did not conceive it as an algebra but rather as a universal language or script. Only in 1676 did he conceive of a kind of "algebra of thought", modeled on and including conventional algebra and its notation. The resulting *characteristic* included a logical calculus, some combinatorics, algebra, his *analysis situs* (geometry of situation), a universal concept language, and more.

What Leibniz actually intended by his *characteristica universalis* and calculus ratiocinator, and the extent to which modern formal logic does justice to the calculus, may never be established.[26]

Formal logic

[edit]

Main article: Algebraic logic

Leibniz is the most important logician between Aristotle and 1847, when George Boole and Augustus De Morgan each published books that began modern formal logic. Leibniz enunciated the principal properties of what we now call conjunction, disjunction, negation, identity, set inclusion, and the empty set. The principles of Leibniz's logic and, arguably, of his whole philosophy, reduce to two:

1. All our ideas are compounded from a very small number of simple ideas, which form the alphabet of human thought.
2. Complex ideas proceed from these simple ideas by a uniform and symmetrical combination, analogous to arithmetical multiplication.

With regard to the first point, the number of simple ideas is much greater than Leibniz thought.[citation needed] As for the second, logic can indeed be grounded in a symmetrical combining operation, but that operation is analogous to either of addition or multiplication.[citation needed] The formal logic that emerged early in the 20th century also requires, at minimum, unary negation and quantified variables ranging over some universe of discourse.

Leibniz published nothing on formal logic in his lifetime; most of what he wrote on the subject consists of working drafts. In his book *History of Western Philosophy*, Bertrand Russell went so far as to claim that Leibniz had developed logic in his unpublished writings to a level which was reached only 200 years later.

Mathematician

[edit]

Although the mathematical notion of function was implicit in trigonometric and logarithmic tables, which existed in his day, Leibniz was the first, in 1692 and 1694, to employ it explicitly, to denote any of several geometric concepts derived from a curve, such as abscissa, ordinate, tangent, chord, and the perpendicular.[27] In the 18th century, "function" lost these geometrical associations.

Leibniz was the first to see that the coefficients of a system of linear equations could be arranged into an array, now called a matrix, which can be manipulated to find the solution of the system, if any. This method was later called Gaussian elimination. Leibniz's discoveries of Boolean algebra and of symbolic logic, also relevant to mathematics, are discussed in the preceding section. A detailed[citation needed] treatment of Leibniz's writings on the calculus may be found in Bos (1974).

Calculus

[edit]

Leibniz is credited, along with Sir Isaac Newton, with the inventing of infinitesimal calculus (that comprises differential and integral calculus). According to Leibniz's notebooks, a critical breakthrough occurred on 11 November 1675, when he employed integral calculus for the first time to find the area under the graph of a function $y = f(x)$. He introduced several notations used to this day, for instance the integral sign $\int$ representing an elongated S, from the Latin word *summa* and the *d* used for differentials, from the Latin word *differentia*. This cleverly suggestive notation for the calculus is probably his most enduring mathematical legacy. Leibniz did not publish anything about his calculus until 1684.[28] The product rule of differential calculus is still called "Leibniz's law". In addition, the theorem that tells how and when to differentiate under the integral sign is called the Leibniz integral rule.

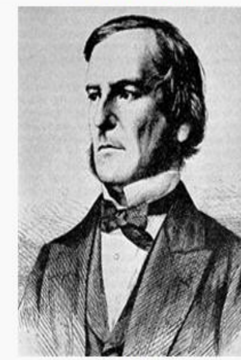
George Boole

From Wikipedia, the free encyclopedia

Not to be confused with George Boolos.

"Boole" redirects here. For other uses, see Boole (disambiguation).

George Boole



George Boole

Full name	George Boole
Born	2 November 1815 Lincoln, Lincolnshire, England
Died	8 December 1864 (aged 49) Ballintemple, County Cork, Ireland
Era	19th-century philosophy
Region	Western Philosophy
School	Mathematical foundations of computer science
Main interests	Mathematics, Logic, Philosophy of mathematics
Notable ideas	Boolean algebra
Influenced by	[show]
Influenced	[show]

George Boole (pronounced /ˈbuːl/) (2 November 1815 – 8 December 1864) was an English mathematician and philosopher.

As the inventor of Boolean logic—the basis of modern digital computer logic—Boole is regarded in hindsight as a founder of the field of computer science. Boole said,

> ... no general method for the solution of questions in the theory of probabilities can be established which does not explicitly recognise ... those universal laws of thought which are the basis of all reasoning ...[1]

Contents [hide]

Biography

[edit]

George Boole's father, John Boole(1779–1848), was a tradesman of limited means, but of "studious character and active mind".[2] Being especially interested in mathematical science and logic, the father gave his son his first lessons; but the extraordinary mathematical talents of George Boole did not manifest themselves in early life. At first, his favorite subject was classics.

It was not until his successful establishment of a school at Lincoln, its removal to Waddington, and later his appointment in 1849 as the first professor of mathematics of then Queen's College, Cork in Ireland (now University College Cork, where the library, underground lecture theatre complex and the Boole Centre for Research in Informatics[3] are named in his honour) that his mathematical skills were fully realized. In 1855 he married Mary Everest (niece of George Everest), who later, as Mrs. Boole, wrote several useful educational works on her husband's principles.

To the broader public Boole was known only as the author of numerous abstruse papers on mathematical topics, and of three or four distinct publications that have become standard works. His earliest published paper was the "Researches in the theory of analytical transformations, with a special application to the reduction of the general equation of the second order." printed in the *Cambridge Mathematical Journal* in February 1840 (Volume 2, no. 8, pp. 64–73), and it led to a friendship between Boole and D.F. Gregory, the editor of the journal, which lasted until the premature death of the latter in 1844. A long list of Boole's memoirs and detached papers, both on logical and mathematical topics, are found in the *Catalogue of Scientific Memoirs* published by the Royal Society, and in the supplementary volume on *Differential Equations*, edited by Isaac Todhunter. To the *Cambridge Mathematical Journal* and its successor, the *Cambridge and Dublin Mathematical Journal*, Boole contributed twenty-two articles in all. In the third and fourth series of the *Philosophical Magazine* are found sixteen papers. The Royal Society printed six important memoirs in the *Philosophical Transactions*, and a few other memoirs are to be found in the *Transactions of the Royal Society of Edinburgh and of the Royal Irish Academy*, in the *Bulletin de l'Académie de St-Pétersbourg* for 1862 (under the name G Boldt, vol. iv. pp. 198–215), and in *Crelle's Journal*. Also included is a paper on the mathematical basis of logic, published in the *Mechanic's Magazine* in 1848. The works of Boole are thus contained in about fifty scattered articles and a few separate publications.

Only two systematic treatises on mathematical subjects were completed by Boole during his lifetime. The well-known *Treatise on Differential Equations* appeared in 1859, and was followed, the next year, by a *Treatise on the Calculus of Finite Differences*, designed to serve as a sequel to the former work. These treatises are valuable contributions to the important branches of mathematics in question. To a certain extent these works embody the more important discoveries of their author. In the sixteenth and seventeenth chapters of the *Differential Equations* we find, for instance, an account of the general symbolic method, the bold and skilful employment of which led to Boole's chief discoveries, and of a general method in analysis, originally described in his famous memoir printed in the *Philosophical Transactions* for 1844. Boole was one of the most eminent of those who perceived that the symbols of operation could be separated from those of quantity and treated as distinct objects of calculation. His principal characteristic was perfect confidence in any result obtained by the treatment of symbols in accordance with their primary laws and conditions, and an almost unrivalled skill and power in tracing out these results.

During the last few years of his life Boole was constantly engaged in extending his researches with the object of producing a second edition of his *Differential Equations* much more complete than the first edition, and part of his last vacation was spent in the libraries of the Royal Society and the British Museum; but this new edition was never completed. Even the manuscripts left at his death were so incomplete

The Laws of Thought

From Wikipedia, the free encyclopedia

This article is about Boole's book on logic. For overview on the axiomatic rules due to various logicians and philosophers, see Law of thought.

The Laws of Thought, more precisely, *An Investigation of the Laws of Thought on Which are Founded the Mathematical Theories of Logic and Probabilities*, is a very influential 19th century book on logic by George Boole, the second of his two monographs on algebraic logic. It was published in 1854.

George Boole was Professor of Mathematics of then Queen's College, Cork in Ireland (now University College Cork, where the Boole Centre for Research in Informatics [1] is named in his honour)

Boole's work began the discipline of algebraic logic, and is often, but mistakenly, credited as being the source of what we know today as Boolean algebra. In fact, however, Boole's algebra differs from modern Boolean algebra, in that in Boole's algebra A+B cannot be interpreted by set union, due the permissibility of *uninterpretable terms* in Boole's calculus, and so algebras on Boole's account cannot be interpreted by sets under the operations of union, interesection and complement, as is the case with modern Boolean algebra. The task of developing the modern account of Boolean algebra fell to Boole's successors in the tradition of algebraic logic (Jevons 1869, Peirce 1880, Jevons 1890, Schröder 1890, Huntingdon 1904).

Uninterpretable terms

[edit]

In Boole's account of his algebra, terms are reasoned about equationally, without a systematic interpretation being assigned to them. In places, Boole talks of terms being interpreted by sets, but he also recognises terms that cannot always be so interpreted, such as the term 2AB, which arises in equational manipulations. Such terms he classes *uninterpretable terms*; although elsewhere he has some instances of such terms being interpreted by integers.

The coherences of the whole enterprise is justified by Boole in what Stanley Burris has later called the "rule of 0s and 1s", which justifies the claim that uninterpretable terms cannot be the ultimate result of equational manipulations from meaningful starting formulae (Burris 2000). Boole provided no proof of this rule, but the coherence of his system was proved by Theodore Hailperin, who provided an interpretation based on a fairly simple construction of rings from the integers to provide an interpretation of Boole's theory (Hailperin 1976).

Notes

[edit]

1. ^ Boole Centre for Research in Informatics

References

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Begriffsschrift

From Wikipedia, the free encyclopedia

The title page of the original 1879 edition

Begriffsschrift (German for, roughly, "concept-script") is a book on logic by Gottlob Frege, published in 1879, and the formal system set out in that book.

Begriffsschrift is usually translated as *concept writing* or *concept notation*; the full title of the book identifies it as "a formula language, modelled on that of arithmetic, of pure thought." The *Begriffsschrift* was arguably the most important publication in logic since Aristotle founded the subject. Frege's motivation for developing his formal approach to logic resembled Leibniz's motivation for his calculus ratiocinator. Frege went on to employ his logical calculus in his research on the foundations of mathematics, carried out over the next quarter century.

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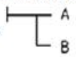
Notation and the system

[edit]

The calculus contains the first appearance of quantified variables, and is essentially classical bivalent second-order logic with identity, albeit presented using a highly idiosyncratic two-dimensional notation: connectives and quantifiers are written using lines connecting formulas, rather than the symbols ¬, ∧, and ∀ in use today. For example, that judgement B materially implies judgement A, i.e. $B \rightarrow A$ is written as



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In the first chapter, Frege defines basic ideas and notation, like proposition ("judgement"), the universal quantifier ("the generality"), the conditional, negation and the "sign for identity of content" $\equiv$; in the second chapter he declares nine formalized propositions as axioms.

Basic concept	Frege's notation	Modern notations
Judging	$\vdash A, \Vdash A$	$p(A) = 1$ $p(A) = i$
Negation	┬ A	$\neg A, \sim A$
Conditional (implication)	A / B (conditional stroke)	$B \Rightarrow A$ $B \subset A$
Universal quantification	⌣x F(x)	$\forall x\colon \Phi(x)$
Existential quantification	┬⌣x┬ F(x)	$\exists x\colon \Phi(x)$
Content identity (equal sign)	$A \equiv B$	$A = B$

In chapter 1, §5, Frege defines the conditional as follows:

"Let A and B refer to judgeable contents, then the four possibilities are:

1. A is asserted, B is asserted;
2. A is asserted, B is negated;

第三讲 谓词逻辑（Predicate Logic）

1.个体、谓词和量词

2.谓词公式

3.等值演算

4.范式

5.推理理论

6.谓词逻辑在计算机科学中的应用

考虑如下3个命题或推理：

1 “有一个整数大于其他每个整数”

2 “任给$\varepsilon > 0$，存在$\delta > 0$，如果$|x-a|<\delta$，则$|f(x)-b|<\varepsilon$”

3 苏格拉底三段论：“人都是要死的，苏格拉底是人，所以苏格拉底是要死的。”

1. 个体、谓词和量词

论域？个体域，全总个体域

量词Quantification？

有一个整数大于其他每个整数

∃ Existential ~
∀Universal ~
(All)

个体Individual？

谓词Predicate？

个体常元(a, b, …)、个体变元(x, y, …)

表示性质、关系。
谓词常元、谓词变元(P, Q, …),
n元谓词,
命题函数(谓词命名式，如G(x, y), x, y仅表示占位），
谓词填式G(x, c) (x个体变元，c个体常量)

有一个整数大于其他每个整数

论域为整数集合：$\exists x(\forall y G(x,y))$

论域为全总域： $\exists x(Z(x) \land \forall y(Z(y) \rightarrow G(x,y)))$

$$\exists x(Z(x) \land \forall y((Z(y) \land N(x,y)) \rightarrow G(x,y)))$$

2 “任给ε>0，存在δ>0，如果|x-a|<δ，则|f(x)-b|<ε” （实数域）

$(\forall \varepsilon)((\varepsilon > 0) \rightarrow (\exists \delta)((\delta > 0) \wedge ((|x-a| < \delta) \rightarrow (|f(x)-b| < \varepsilon))))$

3 苏格拉底三段论：“人都是要死的，苏格拉底是人，所以苏格拉底是要死的。” （全总个体域）

令F(x)：x是人，G(x)：x是要死的，a:苏格拉底，则可以形式化为：

前提：$\forall x(F(x) \rightarrow G(x))$, $F(a)$

结论：$G(a)$

练习

1 在我们班同学中，并非所有同学都来自湖北。

2 4班有的同学准备周末去郊游。

3 凡是有理数都可写成分数。

4 对于任意的x， y，都存在唯一的z，使得x+y=z。

练习

1 在我们班同学中，并非所有同学都来自湖北（全总域）。

令S(x)：x是同学， C(x)：x在我们班中， E(x)：x来自湖北，则命题可符号化为：$\neg\forall x((S(x)\wedge C(x))\rightarrow E(x))$。$\exists x(S(x)\wedge C(x)\wedge\neg E(x))$。

2 4班有的同学准备周末去郊游。（全总域）

$$\exists x(R(x)\wedge S(x)\wedge T(x))$$

3 凡是有理数都可写成分数。（全总域）

$$\forall x(Q(x)\rightarrow F(x))$$

4 对于任意的x，y，都存在唯一的z，使得x+y=z。（实数域）

$$\forall x(\forall y(\exists z((x+y=z)\wedge\forall u((u=x+y)\rightarrow(u=z))))$$

谓词公式(Predicate Formula)的构造

类于命题公式的归纳构造方式：基于原子公式$P(x_1, x_2, \cdots, x_n)$、联接词、量词定义

→原子公式、子公式

→量词的优先级高于任何联结词

变元？

公式的真值？

等值演算？

范式？

谓词逻辑推理？

2.1 自由变元与约束变元

设α是一个谓词公式，$\forall x\beta(x)$和$\exists x\gamma(x)$是α的子公式，则称$\forall x\beta(x)$与$\exists x\gamma(x)$是α的约束部分（Bound Part），x称为是约束出现（Bound Occurrence）的。约束出现的变元称为约束变元（Bound Variable），不是约束出现的变元称为自由变元（Free Variable）。$\beta(x)$称为是$\forall x$在α中的辖域（Scope）或作用域，$\gamma(x)$称为是$\exists x$在α中的辖域。

1 量词后的用**括号**括起来的子公式就是其辖域，如果子公式是原子公式，则括号可以去掉。

2 当**多个量词连续**出现，它们之间无括号分隔时，后面的量词在前面量词的辖域之中，且量词对变元的约束与**量词的次序**有关，一般不能随意改动。

示例 指出下列公式中，各量词的辖域以及变元的自由出现和约束出现：

1 $\forall x(F(x, y, z) \rightarrow \exists y G(x, y))$

2 $\exists x F(x, y) \wedge G(x, y)$

3 $\forall x \forall y(F(x) \wedge G(y) \rightarrow H(x, y))$

4 $\forall x \exists y(F(x) \wedge G(y) \rightarrow H(x, y))$

练习 指出下列公式中，各量词的辖域以及变元的自由出现和约束出现：

（1）$\forall x(P(x)\rightarrow\exists y(Q(y)\wedge R(x,y)))$

（2）$\forall x(P(x)\rightarrow Q(y))\wedge\exists yR(x,y)$

2.2 变元的改名与代入

为了清晰起见，通常运用改名(换名)规则和代入(替换)规则使得公式A满足下列条件：

- 所有变元在公式A中要么自由出现，要么约束出现，不要既有自由出现，又有约束出现。
- 所有量词后面采用的约束变元互不相同。

关于代入
对变元代入，对谓词/命题变元代入→得到更为复杂的公式，从而表示更复杂的知识。
显然需要限制：
公式置换/代入后不会产生变元混淆。

约束变元改名规则和自由变元代入规则

- **改名规则**：将量词中的作用变元x以及该量词的辖域中相应全部约束变元x都用相同的原公式中不出现的新个体变元y替换，得到公式与原公式等价。
- **代入规则**：将公式所有自由变元x改为不在该公式中出现的新变元y，得到公式与原公式等价。

示例

对公式$\forall x(P(x,y)\wedge\exists yQ(y)\wedge M(x,y))\wedge(\forall xR(x)\rightarrow Q(x))$中的约束变元进行改名。使每个变元在公式中只以一种形式出现（即约束出现或自由出现）。

解 在该公式中，将$P(x,y)$和$M(x,y)$中的约束变元x改名为z，$R(x)$中的x改名为s，$Q(y)$中的y改名为t，改名后为：

$$\forall z(P(z,y)\wedge\exists tQ(t)\wedge M(z,y))\wedge(\forall sR(s)\rightarrow Q(x))$$

练习

使用换名规则和代入规则变换下列公式

1 $\exists x((P(x)\vee R(x))\wedge S(x))\rightarrow\forall x(P(x)\wedge Q(x))$

2 $\forall x(P(x)\leftrightarrow Q(x))\wedge\exists xR(x)\vee S(x)$

3 $\forall xP(x)\wedge\exists xQ(x)\vee(\forall xP(x)\rightarrow Q(x))$

练习

1 $\exists x((P(x)\vee R(x))\wedge S(x))\rightarrow\forall x(P(x)\wedge Q(x))$

$$\exists x((P(x)\vee R(x))\wedge S(x))\rightarrow\forall y(P(y)\wedge Q(y))$$

2 $\forall x(P(x)\leftrightarrow Q(x))\wedge\exists xR(x)\vee S(x)$

$$\forall x(P(x)\leftrightarrow Q(x))\wedge\exists yR(y)\vee S(z)$$

3 $\forall xP(x)\wedge(\exists x)Q(x)\vee(\forall xP(x)\rightarrow Q(x))$

$$\forall xP(x)\wedge(\exists y)Q(y)\vee(\forall zP(z)\rightarrow Q(u))$$

给定一个文字叙述的命题，可以符号化为谓词公式．反之，给定一个谓词公式，它表达怎样的意义？这涉及谓词逻辑的**语义**问题．由于谓词公式仅仅是由一些抽象符号构成，只有对它们**解释**和赋值后，才能讨论公式的意义，公式可能真或可能假。

2.3 公式真值

论域

赋值

个体变元

命题变元

谓词

量化

量词量化谓词

{ 论域，(自由)个体变元，个体常元，命题变元/常元，谓词}

(自由)个体变元，命题变元

解释

指 派

示例 教材P35例题2.9

示例 判定下列公式的类型。其中x,y的个体域为整数集I，Q为命题变元，G(x,y)表示x<y。

（1）$\forall x \exists y G(x-y,x+y) \wedge (Q \vee \neg Q)$

（2）$G(x-y,x+y)$

（3）$\forall x \forall y (G(x,y) \wedge \neg G(x,y)) \wedge Q$

（4）$G(x-y,x+y) \rightarrow G(x-y,x+y)$

（5）$\forall x \forall y ((G(x-y,x+y) \rightarrow G(x+y,x-y)) \leftrightarrow (\neg G(x-y,x+y) \vee G(x+y,x-y)))$

2. 谓词公式-小结

1. 命题是陈述句，在自然语言中，通常陈述句有主语、谓语和宾语，主语和宾语都可能是个体，而谓语及其相连的宾语通常被看成是谓词。(谓词命名式，谓词填式)
2. 在谓词逻辑中还可描述对个体所进行的某种变换，即引入所函词，函词与谓词不同，函词作用在个体上，而产生另一个个体，而谓词作用在个体上之后产生的是一个命题。
3. 个体域不同时，谓词逻辑公式的含义不同。为了使公式有一致的含义，可引入一个全总域，表示宇宙间所有个体所组成的域。在某些情况下，全总域也可指所讨论的问题范围内的所有个体。
4. 量词与个体域总是联系在一起的，因此使用不同的个体域，同一命题可能在一阶逻辑中有不同的符号化形式，但总可使所有的个体变量的个体域为全总域，并通过引入合适的特性谓词来从全总域中分离出可使用量词限制的合适个体域来。
5. 量词是有顺序的，不能将量词的顺序随意改变。若一个谓词公式中所有个体变元都量化了，则该谓词公式就变成了命题（闭公式）。

6. 关于“一阶” 逻辑（FOL）

1）所谓一阶逻辑的“一阶”的含义就是指其中的函数只能作用在个体上，或者说其中的变量只能代表取值于个体，如果允许函数作用在谓词上，则变成二阶或高阶逻辑。

$$\forall P((0 \in P \land \forall i(i \in P \rightarrow i+1 \in P)) \rightarrow \forall n(n \in P))$$

2）在一阶逻辑中，谓词变量和谓词常量的区别并不重要，正如在命题逻辑中，命题常量和命题变项的区别不重要一样，这是因为在一阶逻辑中不能对谓词做某些变换（操作），或者说在一阶逻辑上不能在谓词集合上定义函数。但在个体域上可定义函数，因此个体常量与个体变量的区别显得比较重要。

3）可以说，一阶逻辑一个很重要的特点是在个体域上引入了变量。个体变量既可作为谓词作用的对象也可作为函数作用的对象。一阶逻辑中的阶的含义在某种意义上是指个体处于0阶，而对个体的判断（即命题）处于一阶，一阶逻辑中的函数作用于0阶的个体而得到个体，而谓词作用于个体得到处于一阶的命题。

7 一阶逻辑公式的“解释”

给定一个文字叙述的命题，可以符号化为谓词公式. 反之，给定一个谓词公式，它表达怎样的意义？这涉及谓词逻辑的语义问题。只有对谓词公式解释和赋值后，才能讨论公式的意义，公式可能真或可能假。

一阶逻辑公式的解释显然比命题逻辑公式要复杂得多，因为一阶逻辑公式有非逻辑的符号。对于一阶逻辑公式的解释依赖于一阶逻辑公式所基于的非逻辑符号。

设有非逻辑符号集L，它由三部分组成L = C ∪ F ∪ P：

(1). 个体常量所组成的集合[C] = {$c1$, $c2$, …, cn, …}；

(2). 函数符号所组成的集合[F] = { $f1$, $f2$, …, fn, …}，每个函数f_i有一个元数n，表明它是n元函数；

(3). 谓词符号所组成的集合[P] = {F1, F2, …, Fn, …}，每个谓词F_i有一个元数n，表明它是n元谓词。

由该非逻辑符号集L生成的项可记为Term(L)，生成的公式可记为Form(L)。为了确定Form(L)中公式的真值，先要给出非逻辑符号集L的解释。

非逻辑符号集L的一个解释[L]由四个部分组成：

[1]. 一个非空集合D，D称为解释[L]的论域；

[2]. 对于[C]的个体常量c，其解释为$c \in D$是D中的某个元素；

[3]. 对于[F]的n元函数f，其解释是D上的一个n元函数：$f : D^n \to D$；

[4]. 对于[P]的n元谓词F，其解释是D上的一个n元关系：$F \subseteq D^n$

(1) 上述定义所给出的解释方法是对非逻辑符号集的一种最直观的解释，称为非逻辑符号集L的塔斯基(Tarski)语义，塔斯基是研究语义学的一个最有名的学者，这种语义解释方法在各种自然语言及形式语言的语义研究中也被广泛使用。

(2) 对L的一个解释也可看成是为L构造了一个模型，研究一个形式语言的模型的有关内容构成了数理逻辑的一个重要分支：模型论(Model Theory)。

(3) 给定一阶语言，我们可以构造它的一个解释，我们也可以给定一个解释所需的东西，然后研究公式的真值，这种研究实际上从某种意义说是对解释的形式化研究。

(4) 只给出解释还不能确定Form(L)中的公式的真值，因为公式中可能存在自由变元，必需为这些自由变元指派具体的个体，不指定具体的个体，则带有自由变元的公式还不能成为命题逻辑的公式。

A formula is *valid* if it is true in every interpretation; however, as there may be an uncountable number of interpretations, it may not be possible to check this requirement in practice. M is said to be a model for a set of formulae if and only if every formula is true in M.

There is a distinction between proof theoretic and model theoretic approaches in predicate calculus. *Proof theoretic* is essentially syntactic, and there is a list of axioms with rules of inference. The theorems of the calculus are logically derived (i.e. $\vdash A$) and the logical truths are as a result of the syntax or form of the formulae, rather than the *meaning* of the formulae. *Model theoretical*, in contrast is essentially semantic. The truth derives from the meaning of the symbols and connectives, rather than the logical structure of the formulae. This is written as $\vdash_M A$.

A calculus is *sound* if all of the logically valid theorems are true in the interpretation, i.e. proof theoretic $\Rightarrow$ model theoretic. A calculus is *complete* if all the truths in an interpretation are provable in the calculus, i.e. model theoretic $\Rightarrow$ proof theoretic. A calculus is *consistent* if there is no formula A such that $\vdash A$ and $\vdash \neg A$.

The predicate calculus is sound, complete and consistent. *Predicate calculus is not decidable*: i.e. there is no algorithm to determine for any well-formed formula A whether A is a theorem of the formal system. The undecidability of the predicate

8. 关于“一阶逻辑语言”

① 一阶逻辑语言符号包括：

② 个体常量：通常用排在前面的小写字母表示，a, b, c, …, a_i, b_i, c_i, …

③ 个体变项：通常用排在后面的小写字母表示，x, y, z, …, x_i, y_i, z_i, …

④ 函数符号：通常用排在中间的小写字母表示，f, g, h, …, f_i, g_i, h_i, …

⑤ 谓词符号：通常用排在中间的大写字母表示，F, G, H, …, F_i, G_i, H_i, …

⑥ 量词符号：全称量词∀、存在量词∃

⑦ 联结符号：¬、∧、∨、↔、→

⑧ 辅助符号：（、）、，(逗号)

⑨ 进而可以基于语言符号表定义项(个体单词)，谓词公式(原子公式以及更复杂的公式)

9. 关于谓词公式的书写

一般地，可以用括号表示优先级，但可通过假设联结符号及量词之间的优先级而去掉一些括号，使得公式的书写更为简洁，约定：

(1). 公式的最外层括号可省略；

(2). 联结词¬的优先级高于∧，而∧高于∨，∨高于→，→高于↔，量词的优先级高于任何联结符号，公式：$\neg F(x, y) \wedge Q(y, z) \vee \neg F(y, z) \rightarrow G(y, x) \leftrightarrow Q(x, z) \rightarrow F(y, z)$

表示：$(((((\neg F(x, y)) \wedge Q(y, z)) \vee (\neg F(y, z))) \rightarrow G(y, x)) \leftrightarrow (Q(x, z) \rightarrow F(y, z)))$，

但下面的书写既比较简洁，又比较容易理解：

$$((\neg F(x, y) \wedge Q(y, z) \vee \neg F(y, z)) \rightarrow G(y, x)) \leftrightarrow (Q(x, z) \rightarrow F(y, z))$$

等值公式

公式 α 和 β 在论域D上等值?

1 从命题逻辑中“移植”的等值式

2 与量词有关，一阶逻辑特有的一些等值式

代入定理？置换定理？ ---代入/置换 谓词公式

谓词逻辑中特有的等值式

[1]. 在有限个体域D = {$a_1, a_2, \cdots, a_n$} 中消除量词：

(1). $\forall x A(x) \Leftrightarrow A(a_1) \wedge A(a_2) \wedge \cdots \wedge A(a_n)$

(2). $\exists x A(x) \Leftrightarrow A(a_1) \vee A(a_2) \vee \cdots \vee A(a_n)$

[2]. 量词否定等值式：

(1). $\neg(\forall x A(x)) \Leftrightarrow \exists x(\neg A(x))$

(2). $\neg(\exists x A(x)) \Leftrightarrow \forall x(\neg A(x))$

[3]. 量词分配等值式：

(1). $\forall x(A(x) \land B(x)) \Leftrightarrow (\forall x A(x)) \land (\forall x B(x))$

(2). $\exists x(A(x) \lor B(x)) \Leftrightarrow (\exists x A(x)) \lor (\exists x B(x))$

[4]. 量词顺序变换等值式：

(1). $\forall x \forall y(A(x, y)) \Leftrightarrow \forall y \forall x(A(x, y))$

(2). $\exists x \exists y(A(x, y)) \Leftrightarrow \exists y \exists x(A(x, y))$

示例 证明 $\exists x(A(x)\vee B(x))\Leftrightarrow\exists xA(x)\vee\exists xB(x)$

证明

令个体域为D，设在任一指派π下，若左式⇔T，则在D中存在一个个体c使得$A(c)\vee B(c)\Leftrightarrow T$，从而$A(c)\Leftrightarrow T$或$B(c)\Leftrightarrow T$，因此有$\exists xA(x)\Leftrightarrow T$或$\exists xB(x)\Leftrightarrow T$，所以$\exists xA(x)\vee\exists xB(x)\Leftrightarrow T$。

反之，设在任一指派π下，若右式⇔T，则$\exists xA(x)\Leftrightarrow T$或$\exists xB(x)\Leftrightarrow T$，即在D中存在个体c，使得$A(c)\Leftrightarrow T$，或存在个体d，使得$B(d)\Leftrightarrow T$，从而在D中存在一个个体c或d（不妨设为c）使得$A(c)\vee B(c)\Leftrightarrow T$，

所以$\exists x(A(x)\vee B(x))\Leftrightarrow T$。

由以上两方面知等值式成立。

[5]. 量词辖域的收缩与扩张等值式：下述等值式中，变元x不在B中出现

(1). $\forall x(A(x) \lor B) \Leftrightarrow (\forall x A(x)) \lor B$

(2). $\forall x(A(x) \land B) \Leftrightarrow (\forall x A(x)) \land B$

(3). $\forall x(A(x) \to B) \Leftrightarrow (\exists x A(x)) \to B$

(4). $\forall x(B \to A(x)) \Leftrightarrow B \to (\forall x A(x))$

(5). $\exists x(A(x) \lor B) \Leftrightarrow (\exists x A(x)) \lor B$

(6). $\exists x(A(x) \land B) \Leftrightarrow (\exists x A(x)) \land B$

(7). $\exists x(A(x) \to B) \Leftrightarrow (\forall x A(x)) \to B$

(8). $\exists x(B \to A(x)) \Leftrightarrow B \to (\exists x A(x))$

注意，应用这些公式时，公式中，合取（析取）的两个分支，和蕴涵的前件与后件选取的约束变元不同，当约束变元相同时，且又不能运用∀对∧的分配律和∃对∨的分配律时则可使用换名规则，使得约束变元不同。

示例 证明等值式。

$$\exists x(A(x)\rightarrow B(x))\Leftrightarrow\forall xA(x)\rightarrow\exists xB(x)$$

证明 $\exists x(A(x)\rightarrow B(x))$

$\Leftrightarrow\exists x(\neg A(x)\vee B(x))$

$\Leftrightarrow\exists x\neg A(x)\vee\exists xB(x)$

$\Leftrightarrow\neg\forall xA(x)\vee\exists xB(x)$

$\Leftrightarrow\forall xA(x)\rightarrow\exists xB(x)$

练习 判断以下式子是否成立？

(1) $\exists xA(x) \land \exists xB(x) \Leftrightarrow \exists x(A(x) \land B(x))$

(2) $\forall xA(x) \lor \forall xB(x) \Leftrightarrow \forall x(A(x) \lor B(x))$

(3) $\forall x\exists yP(x, y) \Leftrightarrow \exists y\forall xP(x, y)$

$\underline{\forall x(A(x) \vee B(x))} \Leftrightarrow (A(a) \vee B(a)) \wedge (A(b) \vee B(b)) \wedge (A(c) \vee B(c))$

$\Leftrightarrow (A(a) \wedge A(b) \wedge A(c)) \vee (B(a) \wedge A(b) \wedge A(c)) \vee (A(a) \wedge B(b) \wedge A(c)) \vee (B(a) \wedge B(b) \wedge A(c)) \vee (A(a) \wedge A(b) \wedge B(c)) \vee (B(a) \wedge A(b) \wedge B(c)) \vee (A(a) \wedge B(b) \wedge B(c)) \vee (B(a) \wedge B(b) \wedge B(c))$

而 $\underline{(\forall x A(x)) \vee (\forall x B(x))} \Leftrightarrow (A(a) \wedge A(b) \wedge A(c)) \vee (B(a) \wedge B(b) \wedge B(c))$

设D = {a, b, c}，则：

(i). $\forall x\forall y A(x, y) \Leftrightarrow \forall y A(a, y) \wedge \forall y A(b, y) \wedge \forall y A(c, y)$

$\Leftrightarrow (A(a, a)\wedge A(a, b)\wedge A(a, c)) \wedge (A(b, a)\wedge A(b, b)\wedge A(b, c)) \wedge (A(c, a)\wedge A(c, b)\wedge A(c, c))$

(ii). $\forall x\exists y A(x, y) \Leftrightarrow \exists y A(a, y) \wedge \exists y A(b, y) \wedge \exists y A(c, y)$

$\Leftrightarrow (A(a, a)\vee A(a, b)\vee A(a, c)) \wedge (A(b, a)\vee A(b, b)\vee A(b, c)) \wedge (A(c, a)\vee A(c, b)\vee A(c, c))$

(iii). $\exists x\forall y A(x, y) \Leftrightarrow \forall y A(a, y) \vee \forall y A(b, y) \vee \forall y A(c, y)$

$\Leftrightarrow (A(a, a)\wedge A(a, b)\wedge A(a, c)) \vee (A(b, a)\wedge A(b, b)\wedge A(b, c)) \vee (A(c, a)\wedge A(c, b)\wedge A(c, c))$

(iv). $\exists x\exists y A(x, y) \Leftrightarrow \exists y A(a, y) \vee \exists y A(b, y) \vee \exists y A(c, y)$

$\Leftrightarrow (A(a, a)\vee A(a, b)\vee A(a, c)) \vee (A(b, a)\vee A(b, b)\vee A(b, c)) \vee (A(c, a)\vee A(c, b)\vee A(c, c))$

小结

1 谓词、量词

2 谓词公式

3 等值演算