16 Equivalence Relations

In this section, we define four types of binary relations. A relation R on a set A is called **reflexive** if $(a, a) \in R$ for all $a \in A$. In this case, the digraph of R has a loop at each vertex.

Example 16.1
(a) Show that the relation $a \leq b$ on the set $A = \{1, 2, 3, 4\}$ is reflexive.
(b) Show that the relation on $\mathbb{R}$ defined by aRb if and only if $a < b$ is not reflexive.

Solution.
(a) Since $1 \leq 1, 2 \leq 2, 3 \leq 3$, and $4 \leq 4$, the given relation is reflexive.
(b) Indeed, for any real number a we have $a - a = 0$ and not $a - a < 0$ ■

A relation R on A is called **symmetric** if whenever $(a, b) \in R$ then we must have $(b, a) \in R$. The digraph of a symmetric relation has the property that whenever there is a directed edge from a to b, there is also a directed edge from b to a.

Example 16.2
(a) Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is symmetric.
(b) Let $\mathbb{R}$ be the set of real numbers and R be the relation aRb if and only if $a < b$. Show that R is not symmetric.

Solution.
(a) bRc and cRb so R is symmetric.
(b) $2 < 4$ but $4 \not< 2$ ■

A relation R on a set A is called **antisymmetric** if whenever $(a, b) \in R$ and $a \neq b$ then $(b, a) \notin R$. The digraph of an antisymmetric relation has the property that between any two vertices there is at most one directed edge.

Example 16.3
(a) Let $\mathbb{N}$ be the set of positive integers and R the relation aRb if and only if a divides b. Show that R is antisymmetric.
(b) Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is not antisymmetric.

Solution.
(a) Suppose that $a|b$ and $b|a$. We must show that $a = b$. Indeed, by the definition of division, there exist positive integers k_1 and k_2 such that $b = k_1a$ and $a = k_2b$. This implies that $a = k_2k_1a$ and hence $k_1k_2 = 1$. Since k_1 and k_2 are positive integers, we must have $k_1 = k_2 = 1$. Hence, $a = b$.
(b) bRc and cRb with $b \neq c$ ■

A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$. The digraph of a transitive relation has the property that whenever there are directed edges from a to b and from b to c then there is also a directed edge from a to c.

Example 16.4
(a) Let $A = \{a, b, c, d\}$ and $R = \{(a, a), (b, c), (c, b), (d, d)\}$. Show that R is not transitive.
(b) Let $\mathbb{Z}$ be the set of integers and R the relation aRb if a divides b. Show that R is transitive.

Solution.
(a) $(b, c) \in R$ and $(c, b) \in R$ but $(b, b) \notin R$.
(b) Suppose that $a|b$ and $b|c$. Then there exist integers k_1 and k_2 such that $b = k_1a$ and $c = k_2b$. Thus, $c = (k_1k_2)a$ which means that $a|c$ ■

Now, let $A_1, A_2, \cdots, A_n$ be a partition of a set A. That is, the $A_i's$ are subsets of A that satisfy
(i) $\cup_{i=1}^n A_i = A$
(ii) $A_i \cap A_j = \emptyset$ for $i \neq j$.
Define on A the binary relation $x\ R\ y$ if and only if x and y belongs to the same set A_i for some $1 \leq i \leq n$.

Theorem 16.1
The relation R defined above is reflexive, symmetric, and transitive.

Proof.
See Problem 16.9 ■

A relation that is reflexive, symmetric, and transitive on a set A is called an **equivalence relation on A.** For example, the relation "=" is an equivalence relation on $\mathbb{R}$.

Example 16.5
Let $\mathbb{Z}$ be the set of integers and $n \in \mathbb{Z}$. Let R be the relation on $\mathbb{Z}$ defined by aRb if $a - b$ is a multiple of n. We denote this relation by $a \equiv b$ $(mod\ n)$ read "a congruent to b modulo n." Show that R is an equivalence relation on $\mathbb{Z}$.

Solution.
$\equiv$ is reflexive: For all $a \in \mathbb{Z}$, $a - a = 0 \cdot n$. That is, $a \equiv a$ $(mod\ n)$.
$\equiv$ is symmetric: Let $a, b \in \mathbb{Z}$ such that $a \equiv b$ $(mod\ n)$. Then there is an integer k such that $a - b = kn$. Multiply both sides of this equality by (-1) and letting $k' = -k$ we find that $b - a = k'n$. That is $b \equiv a$ $(mod\ n)$.
$\equiv$ is transitive: Let $a, b, c \in \mathbb{Z}$ be such that $a \equiv b$ $(mod\ n)$ and $b \equiv c$ $(mod\ n)$. Then there exist integers k_1 and k_2 such that $a - b = k_1 n$ and $b - c = k_2 n$. Adding these equalities together we find $a - c = kn$ where $k = k_1 + k_2 \in \mathbb{Z}$ which shows that $a \equiv c$ $(mod\ n)$ ■

Theorem 16.2
Let R be an equivalence relation on A. For each $a \in A$ let

$$[a] = \{x \in A | xRa\}$$

$$A/R = \{[a] | a \in A\}.$$

Then the union of all the elements of A/R is equal to A and the intersection of any two distinct members of A/R is the empty set. That is, A/R forms a partition of A.

Proof.
By the definition of $[a]$ we have that $[a] \subseteq A$. Hence, $\cup_{a \in A}[a] \subseteq A$. We next show that $A \subseteq \cup_{a \in A}[a]$. Indeed, let $a \in A$. Since A is reflexive, $a \in [a]$ and consequently $a \in \cup_{b \in A}[b]$. Hence, $A \subseteq \cup_{b \in A}[b]$. It follows that $A = \cup_{a \in A}[a]$. This establishes (i).
It remains to show that if $[a] \neq [b]$ then $[a] \cap [b] = \emptyset$ for $a, b \in A$. Suppose the contrary. That is, suppose $[a] \cap [b] \neq \emptyset$. Then there is an element $c \in [a] \cap [b]$. This means that $c \in [a]$ and $c \in [b]$. Hence, $a\ R\ c$ and $b\ R\ c$. Since R is symmetric and transitive, $a\ R\ b$. We will show that the conclusion $a\ R\ b$ leads to $[a] = [b]$. The proof is by double inclusions. Let $x \in [a]$. Then $x\ R\ a$. Since $a\ R\ b$ and R is transitive, $x\ R\ b$ which means that $x \in [b]$. Thus, $[a] \subseteq [b]$. Now interchange the letters a and b to show that $[b] \subseteq [a]$. Hence, $[a] = [b]$

which contradicts our assumption that $[a] \neq [b]$. This establishes (ii). Thus, A/R is a partition of A ■

The sets $[a]$ defined in the previous exercise are called the **equivalence classes** of A given by the relation R. The element a in $[a]$ is called a **representative** of the equivalence class $[a]$.

Example 16.6
Let R be an equivalence relation on A. Show that if aRb then $[a] = [b]$.

Solution.
$[a] \subseteq [b]$: Let $c \in [a]$. Then cRa. But aRb so that cRb since R is transitive. Hence, $c \in [b]$.
$[b] \subseteq [a]$: Let $c \in [b]$. Then cRb. Since R is symmetric, bRa. Hence, cRa since R is transitive. Thus, $c \in [a]$ ■

Example 16.7
Find the equivalence classes of the the equivalence relation on $\mathbb{Z}$ defined by $a \equiv b \bmod 4$.

Solution.
For any integer $a \in \mathbb{Z}$, the congruence class of a is

$$[a] = \{n \in \mathbb{Z} | n - a = 4k \text{ for some } k \in \mathbb{Z}\}.$$

Hence,

$$\begin{aligned}
[0] =& \{0, \pm 4, \pm 8, \pm 12, \cdots\} \\
[1] =& \{\cdots, -11, -7, -3, 1, 5, 9, \cdots\} \\
[2] =& \{\cdots, -10, -6, -2, 2, 6, 10, \cdots\} \\
[3] =& \{\cdots, -9, -5, -1, 3, 7, 11, \cdots\}.
\end{aligned}$$

Note that $\{[0], [1], [2], [3]\}$ is a partition of $\mathbb{Z}$. Also, note that $[0] = [\pm 4] = [\pm 8] = \cdots ; [1] = [-11] = [-7] = \cdots$, etc ■