离散数学

Discrete Mathematics

第16讲 树 Tree （2）

I think that I shall never see
A graph more lovely than a tree.

by Radia Perlman

3 最小生成树 Minimum Spanning Tree

Otakar Boruvka (1926).

- Electrical Power Company of Western Moravia in Brno.
- Most economical construction of electrical power network.
- Concrete engineering problem is now a cornerstone problem in combinatorial optimization.

3 最小生成树 Minimum Spanning Tree

MST is fundamental problem with diverse applications.

- Network design.
  - telephone, electrical, hydraulic, TV cable, computer, road

- Approximation algorithms for NP-hard problems.
  - traveling salesperson problem, Steiner tree

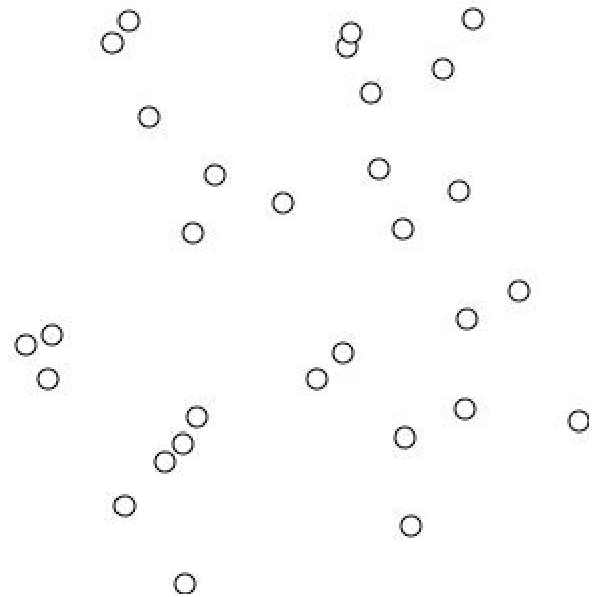
- Indirect applications.
  - max bottleneck paths
  - LDPC codes for error correction
  - image registration with Renyi entropy
  - learning salient features for real-time face verification
  - reducing data storage in sequencing amino acids in a protein
  - model locality of particle interactions in turbulent fluid flows
  - autoconfig protocol for Ethernet bridging to avoid cycles in a network

- Cluster analysis.

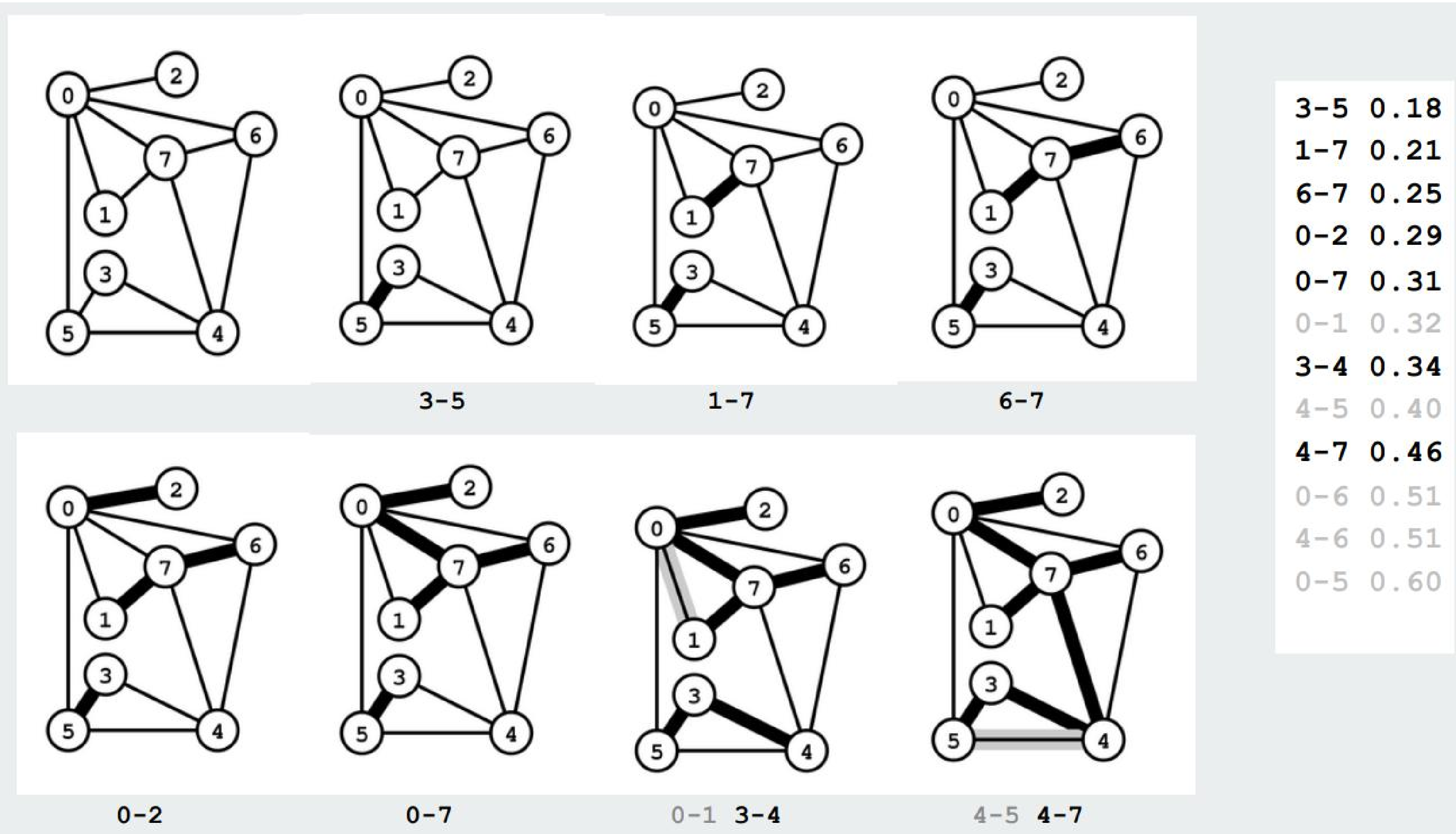
最小生成树算法(MST)

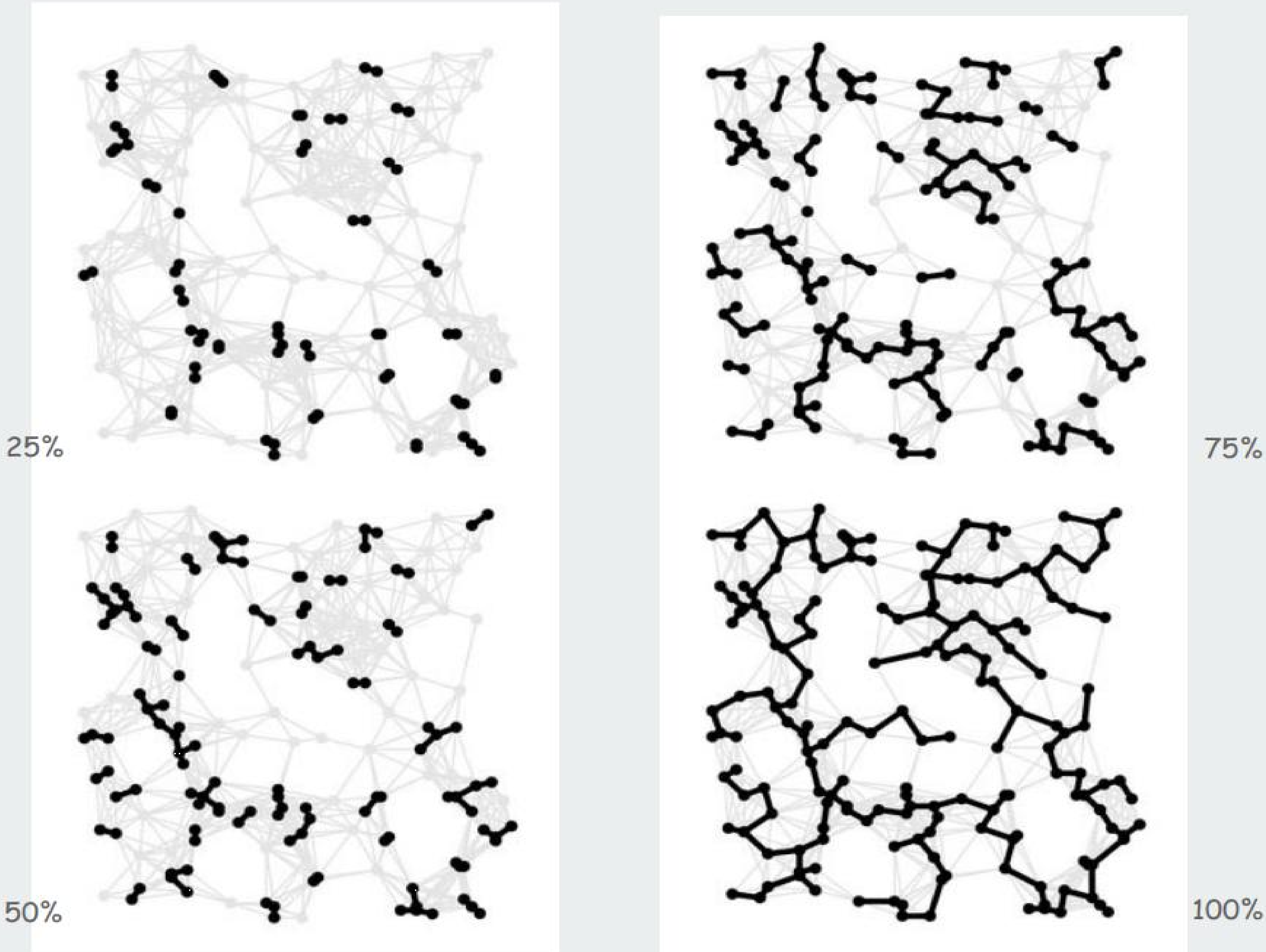
Kruskal算法, Prim算法

Kruskal's Algorithm:
Sort the edges so that: $c(e_1) \le c(e_2) \le \ldots \le c(e_m)$
$T \leftarrow \emptyset$
for $i : 1..m$
if $T \cup \{e_i\}$ has no cycle then
    $T \leftarrow T \cup \{e_i\}$
  end if
end for



Kruskal算法证明?


3-5
1-7
6-7
0-2
0-7
0-1 3-4
4-5 4-7
3-5 0.18
1-7 0.21
6-7 0.25
0-2 0.29
0-7 0.31
0-1 0.32
3-4 0.34
4-5 0.40
4-7 0.46
0-6 0.51
4-6 0.51
0-5 0.60


25%
75%
50%
100%

算法正确性证明

算法正确性证明

设T不是最小生成树，则存在另一棵树T*，为最小生成树。

下面证明T*=T。首先设T的所有边升序排列，假设T中有m条边不在T*中，kruskal算法选择边的顺序不妨假设为$e_1e_2e_3…e_m$：

下面证明可以将e_1加入T*中得到生成树T_1, 且满足条件$w(T_1)<=w(T^*)$:

将e_1加入T*中将形成环，此环中必然然存在边e_1' 在T*中而不在T中,于是,删除e_1' ，则得到生成树T_1，按kruskal算法选择边的顺序知

$w(e_1)<=w(e_1')$，从而$w(T_1)<=w(T^*)$。

依此进行，可以将e_k加入到T_{k-1}中，将形成环，此环中必然存在边e_k'在T*中而不在T中，于是，删除e_k'，则得到生成树T_k。而显然，两边序列$e_1e_2e_3…e_k$与 $e_1e_2e_3…e_k'$均不构成环，而按kruskal算法，必然有 $w(e_k)<=w(e_k')$，

从而 $w(T_k)<= w(T_{k-1})<=w(T^*)$ ……,

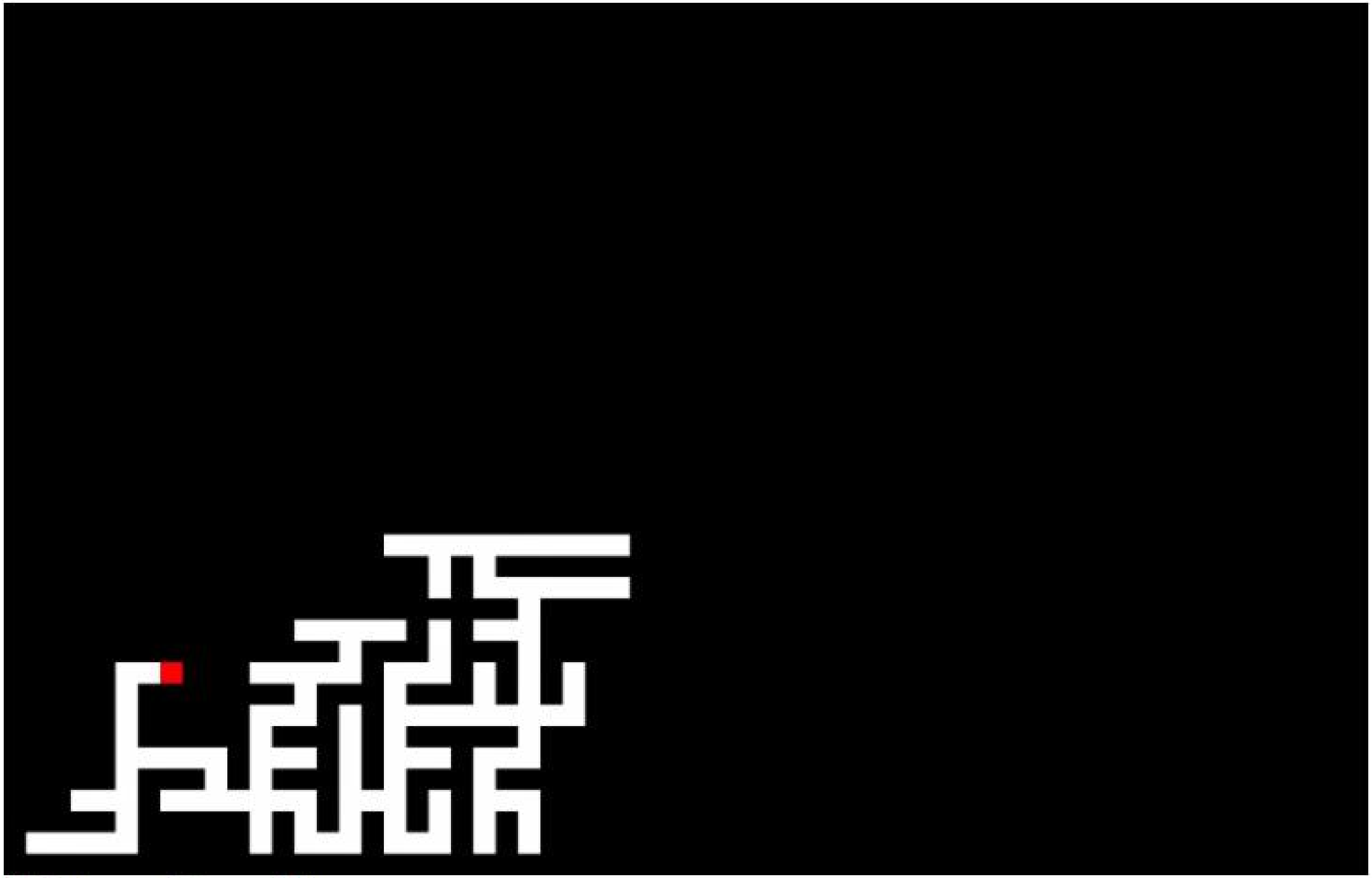
最后可以将e_m加入到T_{m-1}中，得到生成树T_m，且$w(T_m)<=…<=w(T_k)<= w(T_{k-1})<=…$
$<=w(T_1)<=w(T^*)$。

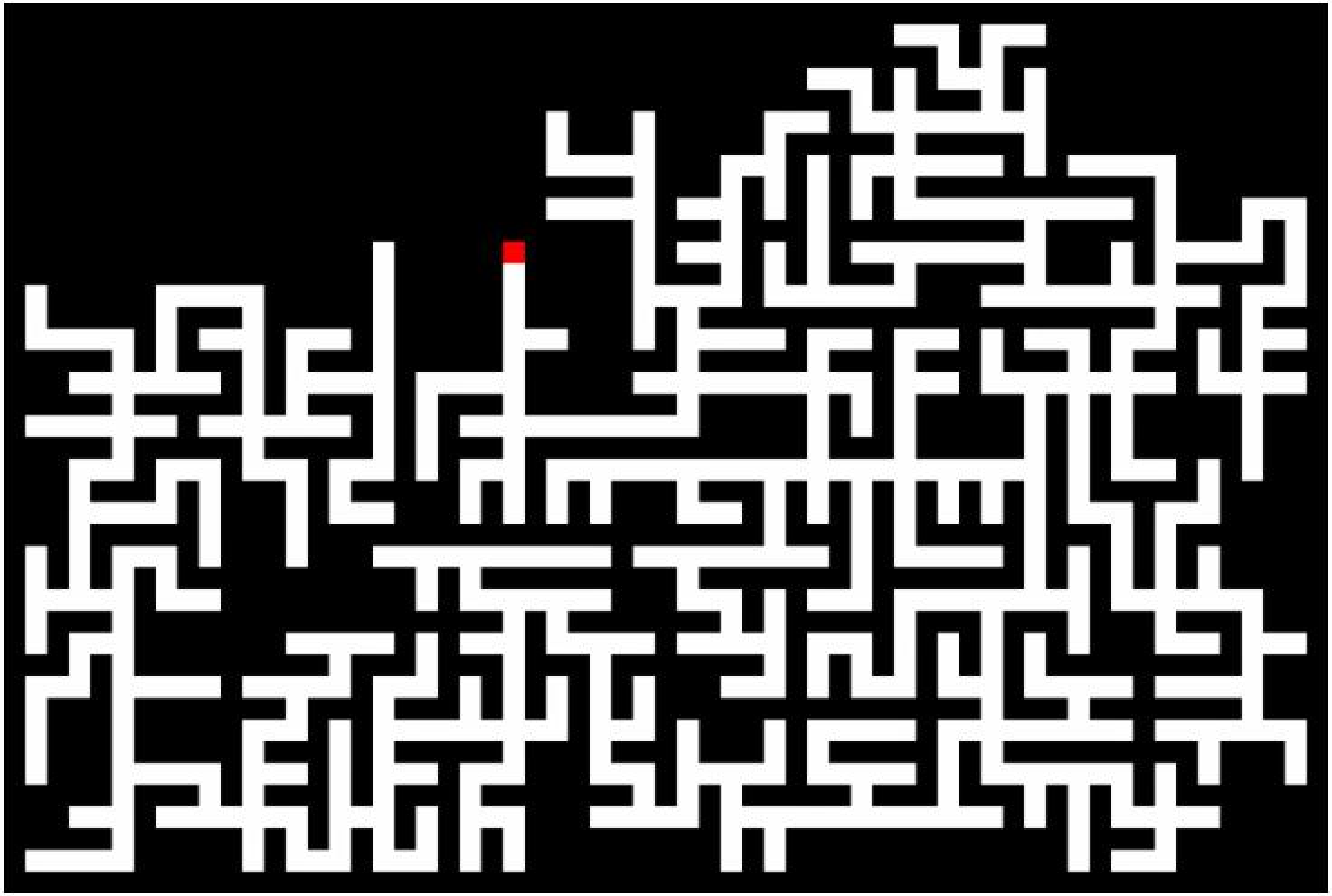
而此时，所T有边（包括与T*中相同的边）都在到T_m中，即$T_m=T$。故$w(T)<=w(T^*)$

因此，T为最小生成树。

Prim's Algorithm

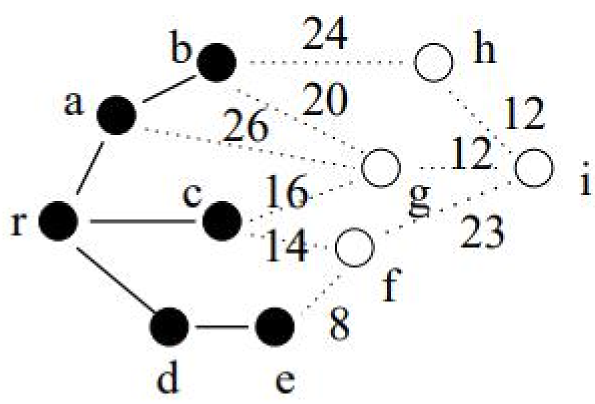
Prim's algorithm has many applications, such as in the generation of this maze, which applies Prim's algorithm to a randomly weighted grid graph.

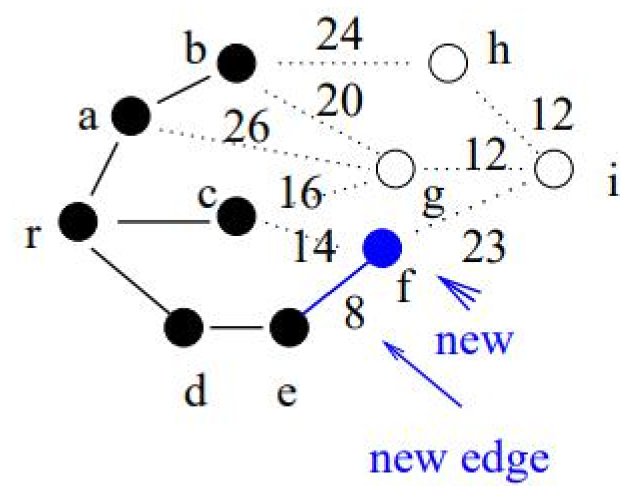




Prim's Algorithm

The idea: expand the current tree by adding the lightest (shortest) edge leaving it and its endpoint.



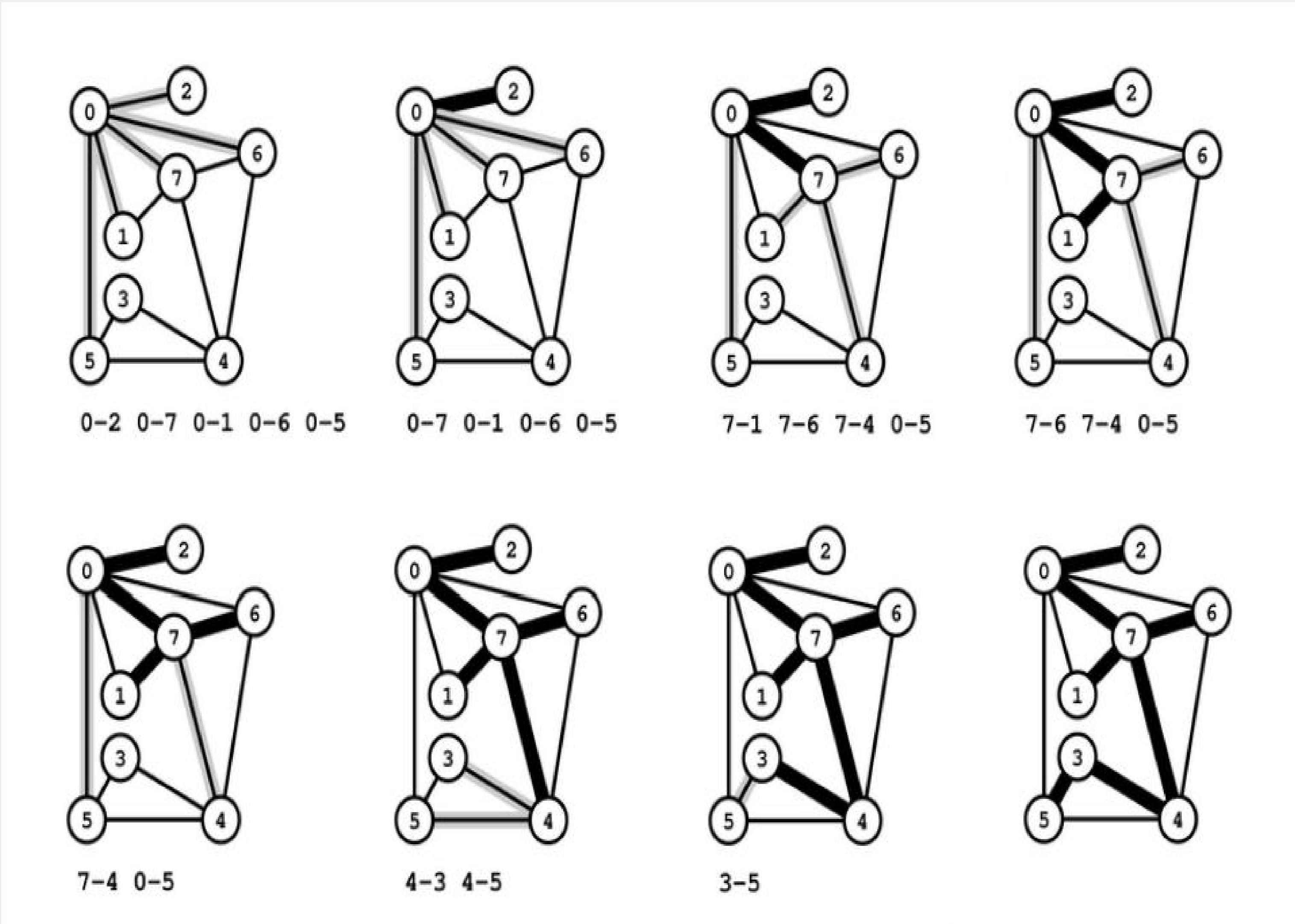


Prim's Algorithm

Step 0: Choose any element r; set $S = \{r\}$ and $A = \emptyset$. (Take r as the root of our spanning tree.)

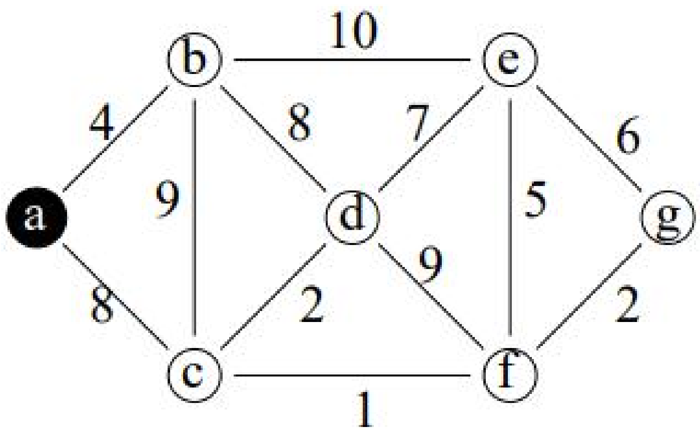
Step 1: Find a lightest edge such that one endpoint is in S and the other is in $V \setminus S$. Add this edge to A and its (other) endpoint to S.

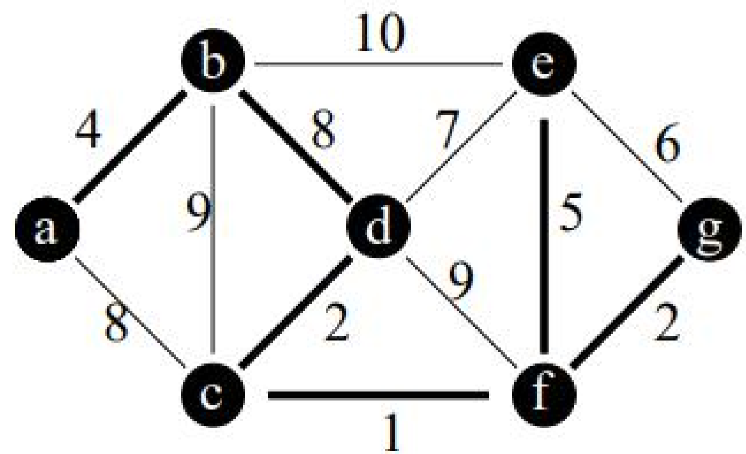
Step 2: If $V \setminus S = \emptyset$, then stop & output (minimum) spanning tree (S, A). Otherwise go to Step 1.

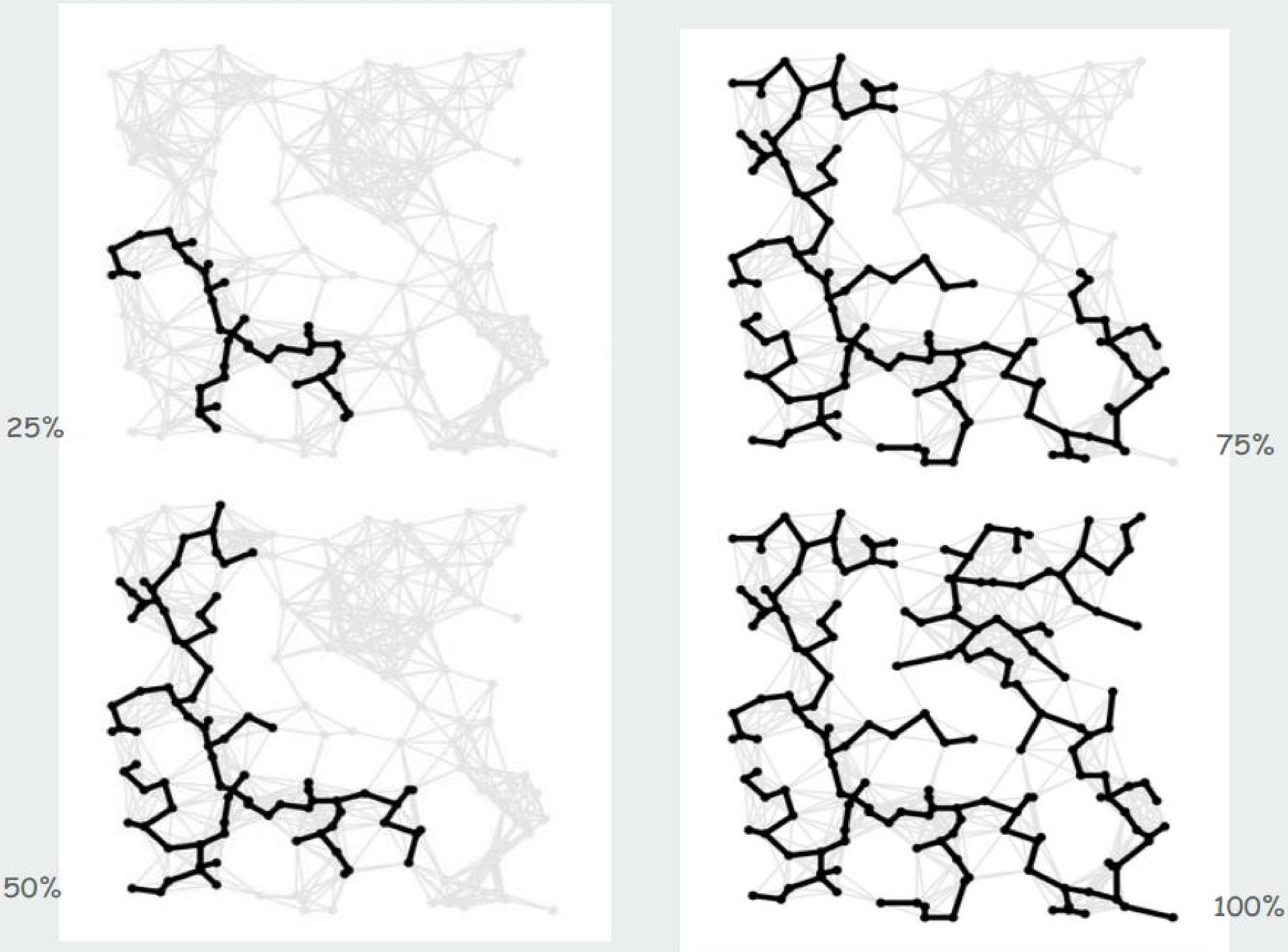


0-1	0.32
0-2	0.29
0-5	0.60
0-6	0.51
0-7	0.31
1-7	0.21
3-4	0.34
3-5	0.18
4-5	0.40
4-6	0.51
4-7	0.46
6-7	0.25

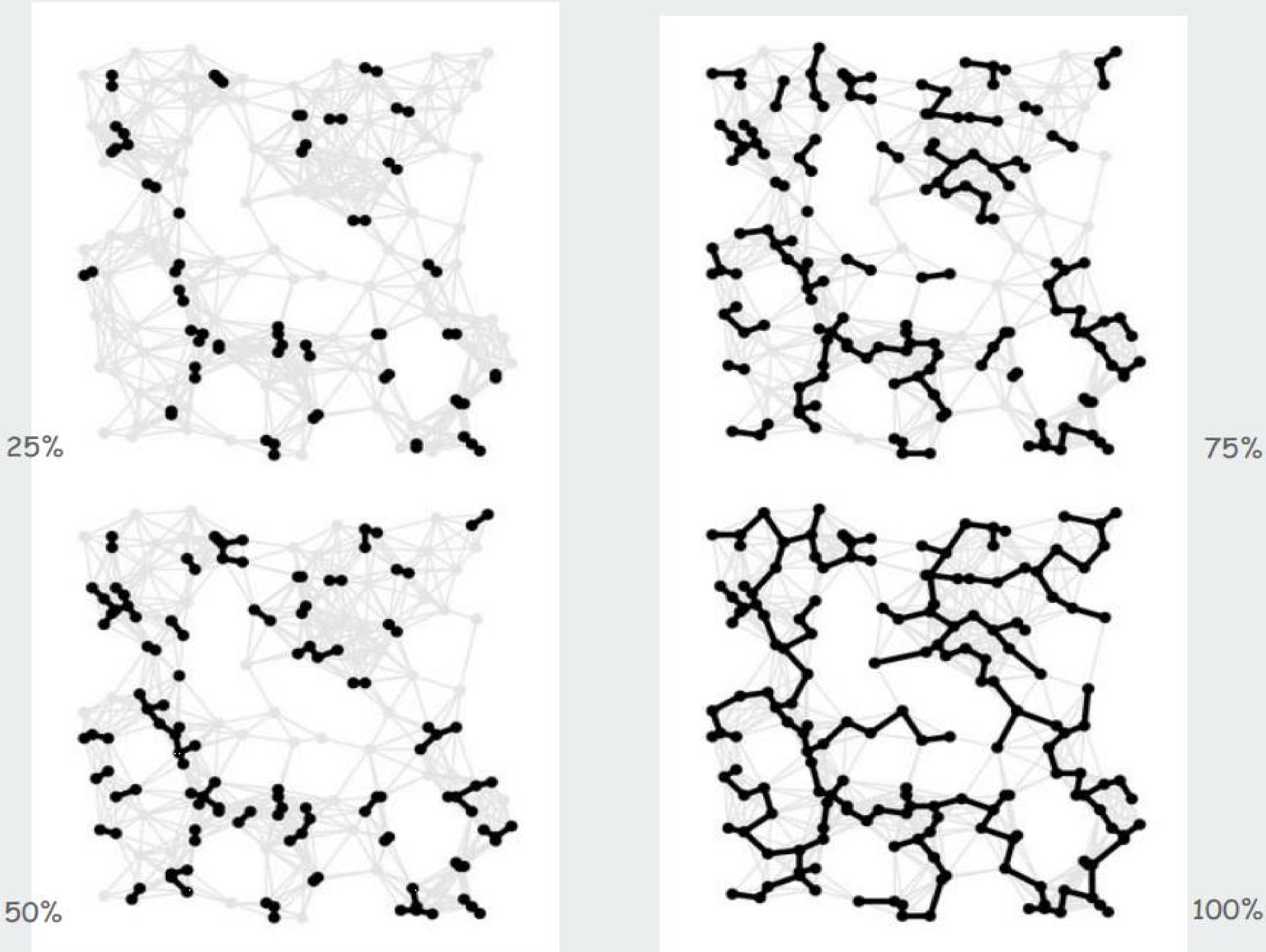
Example



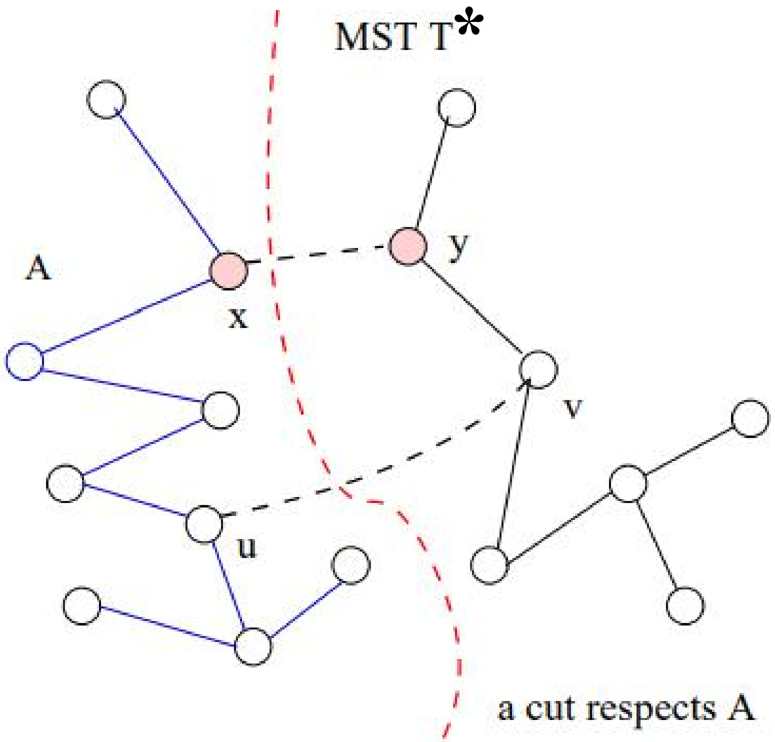




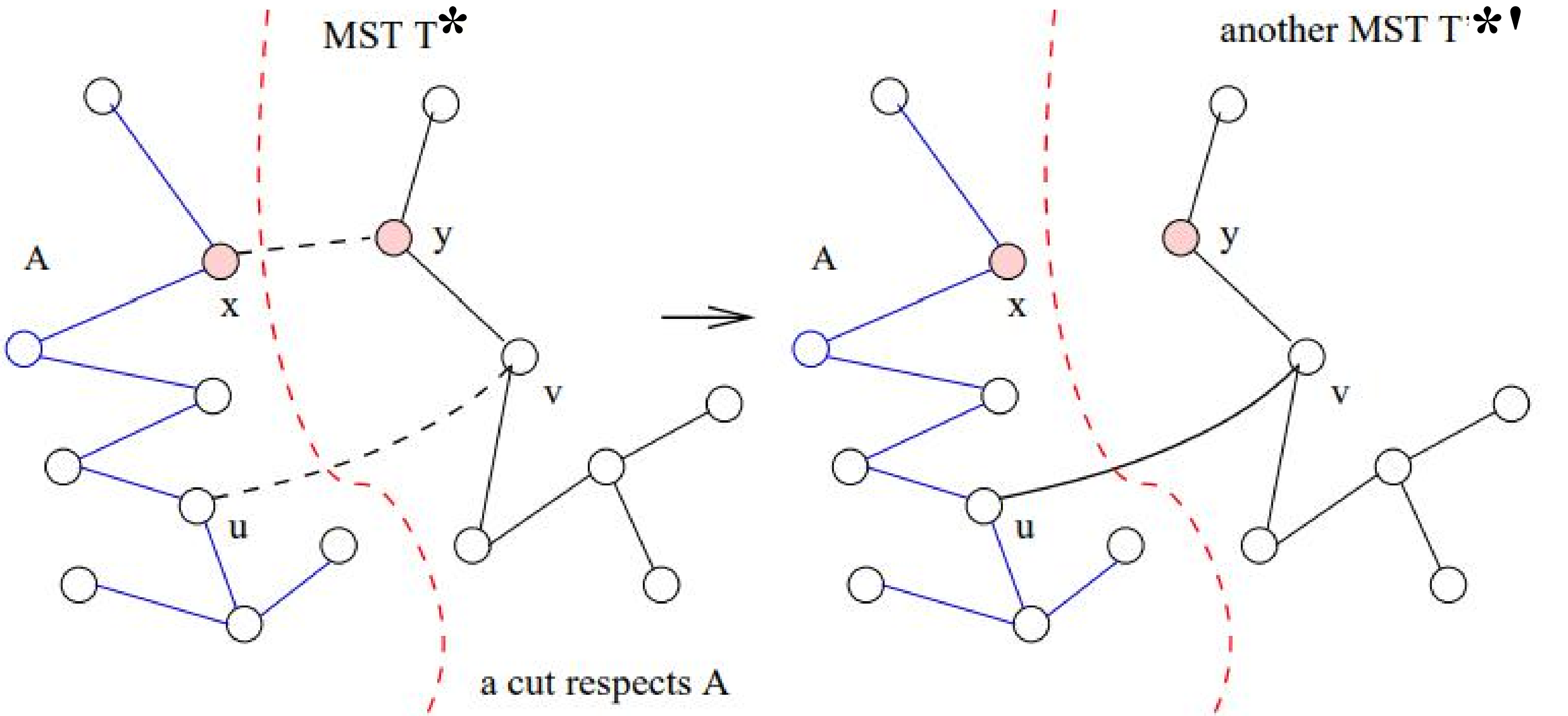
25%
50%
75%
100%


25%
75%
50%
100%

Proof



Proof



Proof idea

- Let X be the object produced by a greedy algorithm and X^* be *any* optimal solution.
- If $X = X^*$, the algorithm is optimal.
- Otherwise, show that you can *exchange* some piece of X^* for some piece of X without deteriorating the quality of X^*.
- Argue that this process can be iterated repeatedly to turn X^* into X without changing its cost.
- Conclude that X is optimal.

Theorem: If G is a connected, weighted graph, Prim's algorithm correctly finds an MST in G.

Proof: Let T be the spanning tree found by Prim's algorithm and T^* be any MST of G. We will prove $c(T) = c(T^*)$. If $T = T^*$, then $c(T) = c(T^*)$ and we are done.

Otherwise, $T \neq T^*$, so we have $T - T^* \neq \emptyset$. Let (u, v) be any edge in $T - T^*$. When (u, v) was added to T, it was a least-cost edge crossing some cut $(S, V - S)$. Since T^* is an MST, there must be a path from u to v in T^*. This path begins in S and ends in $V - S$, so there must be some edge (x, y) along that path where $x \in S$ and $y \in V - S$. Since (u, v) is a least-cost edge crossing $(S, V - S)$, we have $c(u, v) \leq c(x, y)$.

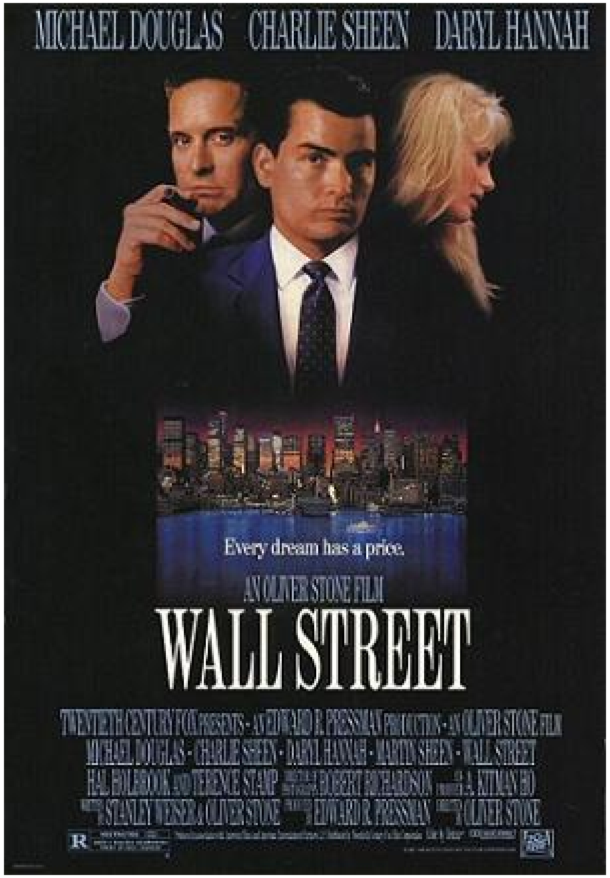
Let $T^{*\prime} = T^* \cup \{(u, v)\} - \{(x, y)\}$. Since (x, y) is on the cycle formed by adding (u, v), this means $T^{*\prime}$ is a spanning tree. Notice $c(T^{*\prime}) = c(T^*) + c(u, v) - c(x, y) \leq c(T^*)$. Since T^* is an MST, this means $c(T^{*\prime}) \geq c(T^*)$, so $c(T^*) = c(T^{*\prime})$.

Note that $|T - T^{*\prime}| = |T - T^*| - 1$. Therefore, if we repeat this process once for each edge in $T - T^*$, we will have converted T^* into T while preserving $c(T^*)$. Thus $c(T) = c(T^*)$. ■

More——

Proposition. Both greedy algorithms compute an MST.

> Greed is good. Greed is right. Greed works. Greed clarifies, cuts through, and captures the essence of the evolutionary spirit." - Gordon Gecko



More——

Greedy algorithms are by far one of the easiest and most well-understood algorithmic techniques. There is a wealth of variations, but at its core the greedy algorithm optimizes something using the natural rule, "pick what looks best" at any step. So a greedy routing algorithm would say to a routing problem: "You want to visit all these locations with minimum travel time? Let's start by going to the closest one.

Can we characterize when greedy algorithms give an optimal solution to a problem?

The answer is yes, and the framework that enables us to do this is called a **matroid（拟阵）**.
That is, if we can phrase the problem we're trying to solve as a matroid, then the greedy algorithm is guaranteed to be optimal.

Matroids(拟阵)

Why study matroids?

- **Matroids are common mathematical structures.**
- **In a matroid, we can always find the minimum-weight maximal independent set using the greedy algorithm.**
- **Algorithm: Apply <u>the Red and Blue rules</u> arbitrarily.**

… …

Theorem. The blue edges form a MST.

Matroids(拟阵)

A matroid is a pair (S, $\mathscr{I}$) where S is a finite set and $\mathscr{I}$ is a family of subsets of S such that

(1) $\mathscr{I}$ is nonempty.

(2) Elements of $\mathscr{I}$ are called the independent sets. If *I* is in $\mathscr{I}$, then every subset of *I* is independent set.

(3) If A, B are in $\mathscr{I}$ with |A| = |B| + 1, then there is an element a in A-B such that $B \cup \{a\}$ is in $\mathscr{I}$.

Matroids(拟阵)

- Graph $G = (V, E)$
  - $\mathscr{I}$ is the set of forests in G (acyclic subgraphs).
- Vector space V
  - $\mathscr{I}$ is the set of all linearly independent subsets of V.
- Columns/rows of a matrix A
  - $\mathscr{I}$ is the set of all bases of A.

在组合数学中，拟阵是一个对向量空间中线性独立概念的概括与归纳的数学结构。拟阵理论广泛地借用了线性代数和图理论的术语，因为它是这些领域的重点概念的抽象。拟阵在几何，拓扑学，组合优化，网络理论和编码理论上都有很多应用。它抽象了很多图的性质, 为组合优化问题和设计多项式算法提供了强有力的工具。

Matroids provide a link between graph theory, linear algebra, transcendence theory, and semimodular lattices, etc..

有关术语

有向树

根树（Rooted Tree），树高度

有序树: (完全)m分树、 (完全)二叉树

父结点/孩子结点;

分支结点/叶子结点, 内部结点/外部结点;

孩子结点/非孩子结点

问题探讨

（1）存在一棵有n片叶的完全两分树;

（2）某完全m分树的叶子结点数为t, 分支结点数为i, 则(m-1)i=t-1;

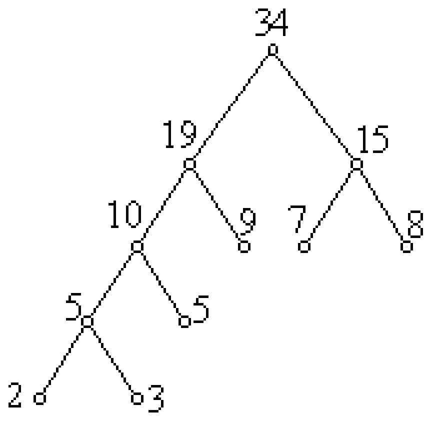
（3）T为有t片叶的完全两分树，则T有2(t-1)条边.

应用

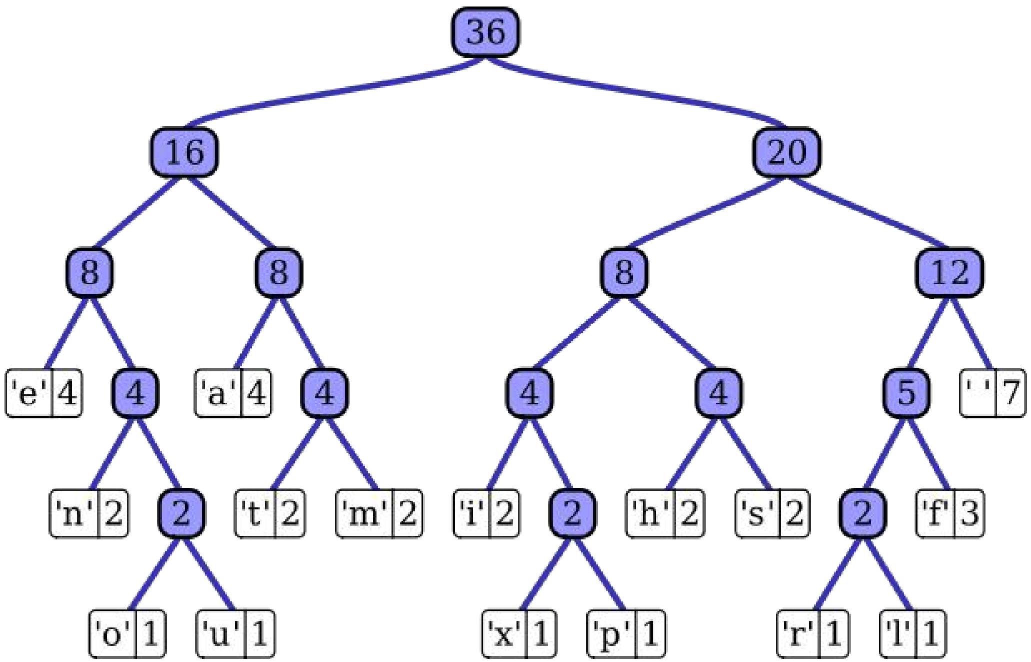
(1) 二叉树，树的遍历，逆波兰式

(2) 前缀码，最优二分树，Huffman算法

示例 求带权为 2,3,5,7,8,9 的最优二分树

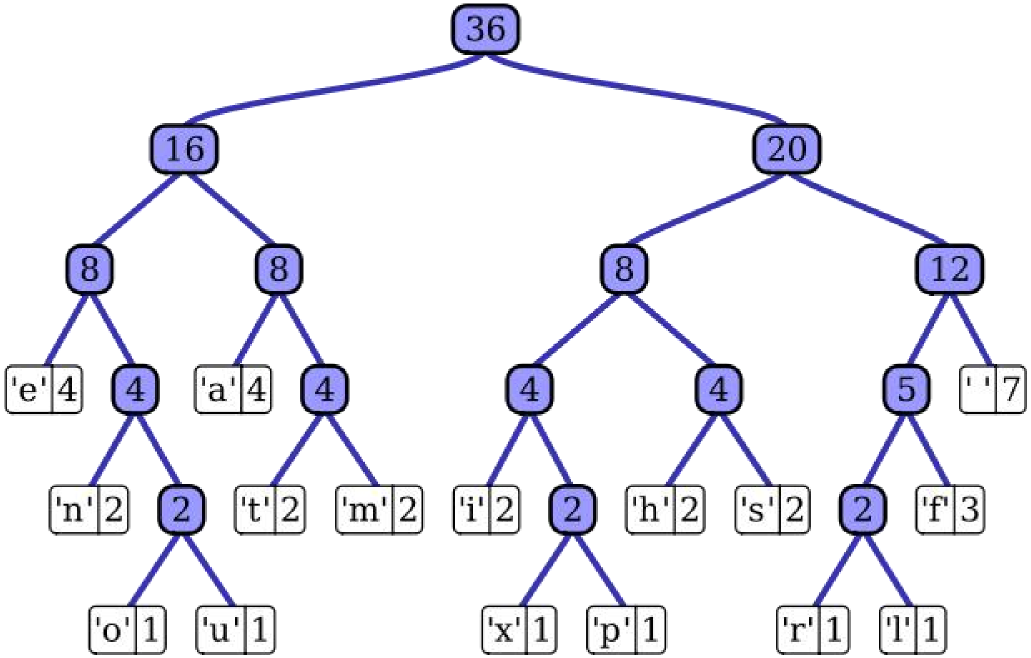


T



Char	Freq	Code
space	7	
a	4	
e	4	
f	3	
h	2	
i	2	
m	2	
n	2	
s	2	
t	2	
l	1	
o	1	
p	1	
r	1	
u	1	
x	1	

Huffman tree generated from the exact frequencies of the text "this is an example of a huffman tree". The frequencies and codes of each character are above. Encoding the sentence with this code requires 135 bits, as opposed to 288 (or 180) bits if 36 characters of 8 (or 5) bits were used. (This assumes that the code tree structure is known to the decoder and thus does not need to be counted as part of the transmitted information.)

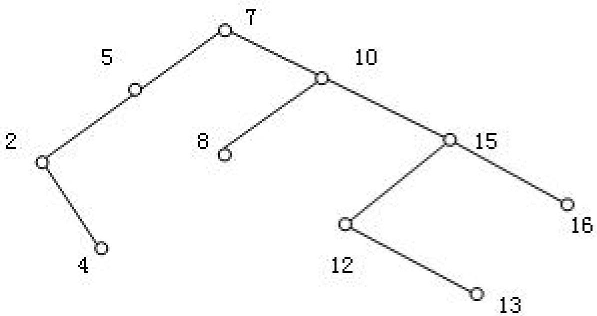


Char	Freq	Code
space	7	111
a	4	010
e	4	000
f	3	1101
h	2	1010
i	2	1000
m	2	0111
n	2	0010
s	2	1011
t	2	0110
l	1	11001
o	1	00110
p	1	10011
r	1	11000
u	1	00111
x	1	10010

Huffman tree generated from the exact frequencies of the text "this is an example of a huffman tree". The frequencies and codes of each character are below. Encoding the sentence with this code requires 135 bits, as opposed to 288 (or 180) bits if 36 characters of 8 (or 5) bits were used. (This assumes that the code tree structure is known to the decoder and thus does not need to be counted as part of the transmitted information.)

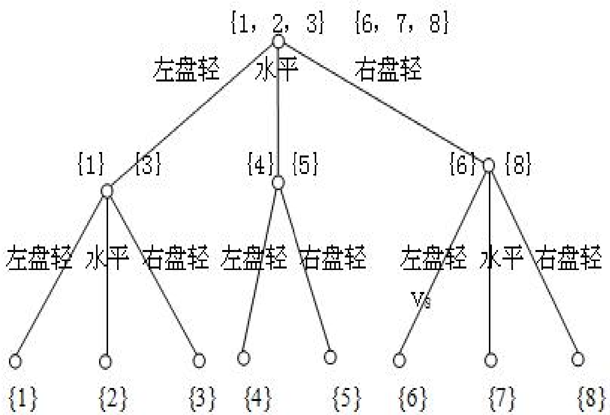
应用

二叉查找树



决策树

平衡树



1 高度为h的m元树：树叶数t<=m^h

2 若高度为h的m元树树叶数为t，

则h>= $\lceil \log_m t \rceil$，

若m元树为完全的平衡的，则h= $\lceil \log_m t \rceil$