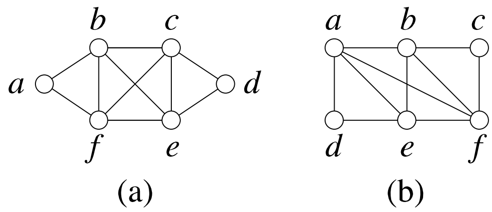

Fig. 7.19. Sample graphs for discussing Hamiltonian cycles

7.6 Hamiltonian Cycles

Modeling Road Systems Again

When we discussed modeling road systems in Section 7.2, the emphasis was on visiting all the roads: crossing the Königsberg bridges, or inspecting a highway. Another viewpoint is that of the traveling salesman or tourist who wants to visit the towns. The salesman travels from one town to another, trying not to pass through any town twice on his trip. In terms of the underlying graph, the salesman plans to follow a cycle.

A cycle that passes through every vertex in a graph is called a *Hamiltonian cycle* and a graph with such a cycle is called *Hamiltonian*. The idea of such a spanning cycle was simultaneously developed by Hamilton in 1859 in a special case, and more generally by Kirkman in 1856.

A *Hamiltonian path* is a path that contains every vertex. If you take a Hamiltonian cycle and delete an edge you obtain a Hamiltonian path, but the reverse is not always true. Some graphs contain a Hamiltonian path but no Hamiltonian cycle; an example is the Petersen graph.

Sample Problem 7.18. *Consider the graph in Figure 7.19*(a)*. Which of the following are Hamiltonian cycles?*

(i) (a, b, e, d, c, f, a).

(ii) (a, b, e, c, d, e, f, a).

(iii) (a, b, c, d, e, f, a).

(iv) (a, b, c, e, f, a).

Solution. (i) and (iii) are Hamiltonian. (ii) is not; it contains a repeat of e. (iv) is not; vertex d is omitted.

Practice Exercise. Repeat this problem for the graph in Figure 7.19(b) and the following cycles.

(i) (a, b, c, f, e, d, a).

(ii) (a, b, f, c, b, e, d, a).

(iii) (a, b, c, d, e, f, a).

(iv) (a, d, e, b, c, f, a).

Which Graphs are Hamiltonian?

At first, the problem of deciding whether a graph is Hamiltonian sounds similar to the problem of Euler circuits. However, the two problems are strikingly different in one regard. We found a very easy test for the Eulerian property, but no nice necessary and sufficient conditions are known for the existence of Hamiltonian cycles.

It is easy to see that the complete graphs with three or more vertices are Hamiltonian, and any ordering of the vertices gives a Hamiltonian cycle. On the other hand, no tree is Hamiltonian because they contain no cycles at all. We can discuss Hamiltonicity in a number of other particular cases, and there are a number of small theorems.

One useful, sufficient condition for a graph to be Hamiltonian is the following, due to Ore. We include the proof, but most students should skip it on a first reading.

Theorem 49. *If G is a graph with v vertices, $v \geq 3$, and $d(x) + d(y) \geq v$ whenever x and y are nonadjacent vertices of G, then G is Hamiltonian.*

Proof. Suppose the theorem is false. There must be at least one counterexample—a graph that satisfies the conditions, but is not Hamiltonian. Suppose there is a counterexample with v vertices for some particular v. There may be more than one counterexample with v vertices; if there are, select one that has the largest possible number of edges among counterexamples and call it G. Choose two nonadjacent vertices p and q of G. $G + pq$ must be Hamiltonian because it has more edges than G. Moreover, pq must be an edge in every Hamiltonian cycle of $G + pq$ because any Hamiltonian cycle that does not contain pq will be Hamiltonian in G. Since G satisfies the conditions, $d(p) + d(q) \geq v$.

Consider any Hamiltonian cycle in $G + pq$. Since it contains edge pq, it will look like

$$p, x_1, x_2, \ldots, x_{v-2}, q, p.$$

If x_i is any vertex adjacent to p, then x_{i-1} cannot be adjacent to q because if it were, then

$$p, x_1, x_2, \ldots, x_{i-1}, q, x_{v-2}, x_{v-3}, \ldots, x_i, p,$$

will be a Hamiltonian cycle in G. So each of the $d(p)$ vertices adjacent to p in G must be preceded in the cycle by vertices not adjacent to q, and none of these vertices can be q itself. So there are at least $d(p) + 1$ vertices in G that are not adjacent to q. So there are at least $d(q) + d(p) + 1$ vertices in G, whence

$$d(p) + d(q) \leq v - 1,$$

which is a contradiction. □

As a consequence, it follows that if a graph has v vertices ($v \geq 3$) and every vertex has degree at least $\frac{v}{2}$, then the graph is Hamiltonian.

On the other hand, suppose a graph G contains a Hamiltonian cycle

$$x_1, x_2, \ldots, x_v, x_1.$$

Since x_i occurs only once in the cycle, only two of the edges touching x_i can be in the cycle. One can sometimes use this fact to prove that a graph contains no Hamiltonian cycle. For example, consider the graph of Figure 7.20.

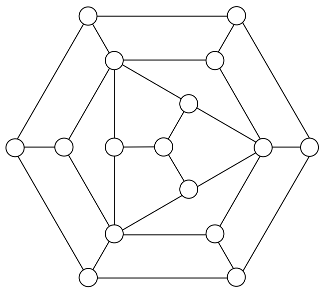

Fig. 7.20. A graph with no Hamiltonian cycle

Suppose the graph contains a Hamiltonian cycle.

The vertices on the outer circuit are each of degree 3, and only two of the edges touching any given vertex can be in a Hamiltonian cycle. Figure 7.21(a) shows as heavy lines all the edges touching three of those vertices; of the nine edges, three are not in the cycle. Similarly, Figure 7.21(b) shows the 15 edges touching the three vertices of degree 5; nine of these are out of the cycle. These sets of edges are disjoint, so there are at least 12 edges not in the cycle. (If the sets were not disjoint, but had k common elements, only $12 - k$ edges would definitely be eliminated.) Similarly, one of the edges touching the central vertex must be deleted in forming the cycle. So 13 edges are barred and the Hamiltonian cycle must be chosen from the remaining 14 edges. Since the graph has 16 vertices, a Hamiltonian cycle in it must contain 16 edges, which is impossible.

A similar argument can be used to prove the impossibility of a Hamiltonian path in this graph.

Another test is applicable only to bipartite graphs. As a bipartite graph is a subgraph of some $K_{m,n}$, its vertices can be partitioned into two subsets, of sizes m and n, such that the graph contains no edge that joins two vertices in the same subset.

Theorem 50. *A bipartite graph with vertex sets of sizes m and n can contain a Hamiltonian cycle only if $m = n$, and can contain a Hamiltonian path only if m and n differ by at most* 1.

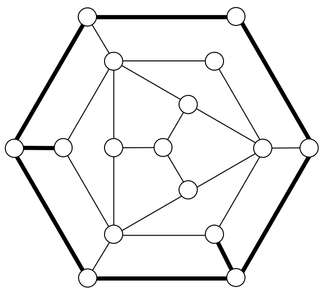

Fig. 7.21. Steps in proving there is no Hamiltonian cycle

Proof. Suppose a bipartite graph G has vertex sets V_1 and V_2, and suppose it contains a Hamiltonian path

$$x_1, x_2, \ldots, x_v.$$

Suppose that x_1 belongs to V_1. Then x_2 must be in V_2, x_3 in V_1, and so on. Since the path contains every vertex, it follows that

$$V_1 = \{x_1, x_3, \ldots\},$$
$$V_2 = \{x_2, x_4, \ldots\}.$$

If v is even, then V_1 and V_2 each contain $v/2$ elements; if v is odd, then $|V_1| = (v+1)/2$ and $|V_2| = (v-1)/2$. In either case, the difference in orders is at most 1. If G contains a Hamiltonian cycle

$$x_1, x_2, \ldots, x_v, x_1,$$

and x_1 is in V_1, then x_v must belong to V_2; so $|V_1| = |V_2| = v/2$. □

It should be realized that neither of these necessary conditions is sufficient; in particular, the following bipartite graph has four vertices in each subset (the vertices in the two sets are colored black and white, respectively). It contains no Hamiltonian cycle; this cannot be proved using the above methods.