## 5.8 Planar Graphs

### 5.8.1 Drawing Graphs in the Plane

Suppose there are three dog houses and three human houses, as shown in Figure 5.34. Can you find a route from each dog house to each human house such that no route crosses any other route?

A *quadrapus* is a little-known animal similar to an octopus, but with four arms. Suppose there are five quadrapi resting on the sea floor, as shown in Figure 5.35.

**Figure 5.34** Three dog houses and and three human houses. Is there a route from each dog house to each human house so that no pair of routes cross each other?

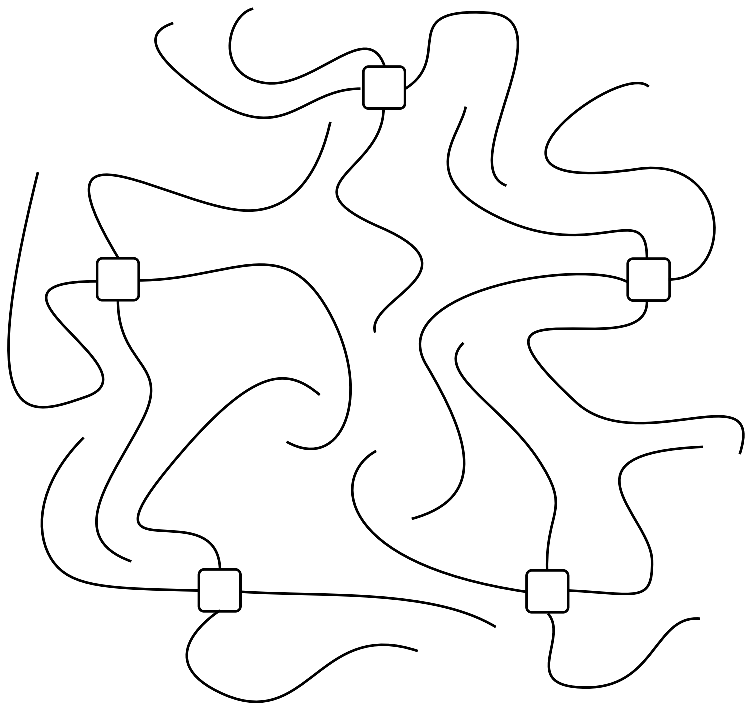

**Figure 5.35** Five quadrapi (4-armed creatures).

Can each quadrapus simultaneously shake hands with every other in such a way that no arms cross?

**Definition 5.8.1.** A *drawing of a graph in the plane* consists of an assignment of vertices to distinct points in the plane and an assignment of edges to smooth, non-self-intersecting curves in the plane (whose endpoints are the nodes incident to the edge). The drawing is *planar* (that is, it is a *planar drawing*) if none of the curves "cross"—that is, if the only points that appear on more than one curve are the vertex points. A *planar graph* is a graph that has a planar drawing.

Thus, these two puzzles are asking whether the graphs in Figure 5.36 are planar; that is, whether they can be redrawn so that no edges cross. The first graph is called the *complete bipartite graph*, $K_{3,3}$, and the second is $K_5$.

In each case, the answer is, "No—but almost!" In fact, if you remove an edge from either of them, then the resulting graphs *can* be redrawn in the plane so that no edges cross. For example, we have illustrated the planar drawings for each resulting graph in Figure 5.37.

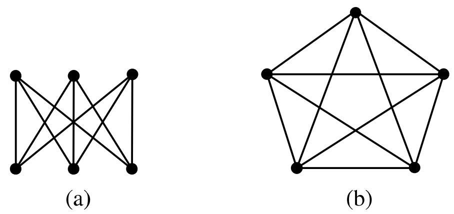

**Figure 5.36** $K_{3,3}$ (a) and $K_5$ (b). Can you redraw these graphs so that no pairs of edges cross?

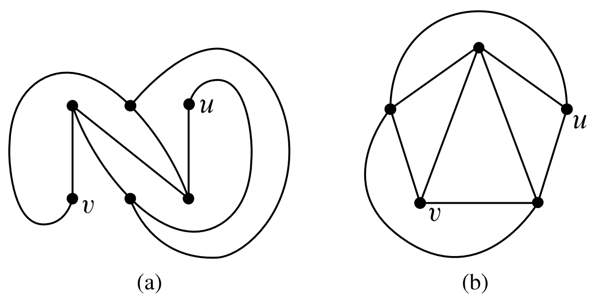


**Figure 5.37** Planar drawings of $K_{3,3} - \{u, v\}$ (a) and $K_5 - \{u, v\}$ (b).

Planar drawings have applications in circuit layout and are helpful in displaying graphical data such as program flow charts, organizational charts, and scheduling conflicts. For these applications, the goal is to draw the graph in the plane with as few edge crossings as possible. (See the box on the following page for one such example.)

### 5.8.2 A Recursive Definition for Planar Graphs

Definition 5.8.1 is perfectly precise but has the challenge that it requires us to work with concepts such as a "smooth curve" when trying to prove results about planar graphs. The trouble is that we have not really laid the groundwork from geometry and topology to be able to reason carefully about such concepts. For example, we haven't really defined what it means for a curve to be smooth—we just drew a simple picture (for example, Figure 5.37) and hoped you would get the idea.

Relying on pictures to convey new concepts is generally not a good idea and can sometimes lead to disaster (or, at least, false proofs). Indeed, it is because of this issue that there have been so many false proofs relating to planar graphs over time.[18] Such proofs usually rely way too heavily on pictures and have way too many statements like,

> As you can see from Figure ABC, it must be that property XYZ holds for all planar graphs.

The good news is that there is another way to define planar graphs that uses only discrete mathematics. In particular, we can define the class of planar graphs as a recursive data type. In order to understand how it works, we first need to understand the concept of a *face* in a planar drawing.

#### Faces

In a planar drawing of a graph. the curves corresponding to the edges divide up the plane into connected regions. These regions are called the *continuous faces*[19] of the drawing. For example, the drawing in Figure 5.38 has four continuous faces. Face IV, which extends off to infinity in all directions, is called the *outside face*.

Notice that the vertices along the boundary of each of the faces in Figure 5.38 form a cycle. For example, labeling the vertices as in Figure 5.39, the cycles for the face boundaries are

$$abca \qquad abda \qquad bcdb \qquad acda. \tag{5.4}$$

[18] The false proof of the 4-Color Theorem for planar graphs is not the only example.

[19] Most texts drop the word *continuous* from the definition of a face. We need it to differentiate the connected region in the plane from the closed walk in the graph that bounds the region, which we will call a *discrete face*.

When wires are arranged on a surface, like a circuit board or microchip, crossings require troublesome three-dimensional structures. When Steve Wozniak designed the disk drive for the early Apple II computer, he struggled mightily to achieve a nearly planar design:

> For two weeks, he worked late each night to make a satisfactory design. When he was finished, he found that if he moved a connector he could cut down on feedthroughs, making the board more reliable. To make that move, however, he had to start over in his design. This time it only took twenty hours. He then saw another feedthrough that could be eliminated, and again started over on his design. "The final design was generally recognized by computer engineers as brilliant and was by engineering aesthetics beautiful. Woz later said, 'It's something you can only do if you're the engineer and the PC board layout person yourself. That was an artistic layout. The board has virtually no feedthroughs.' "[17]

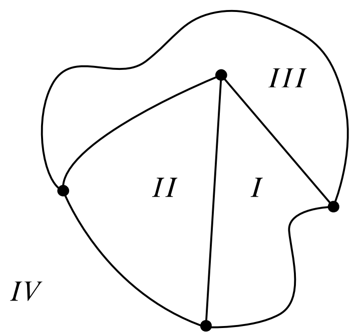


**Figure 5.38** A planar drawing with four faces.

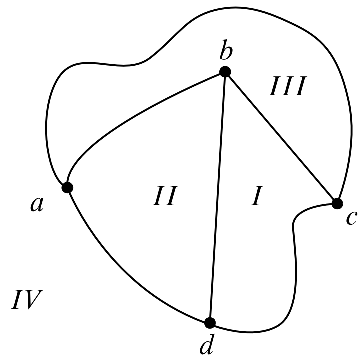


**Figure 5.39** The drawing with labeled vertices.

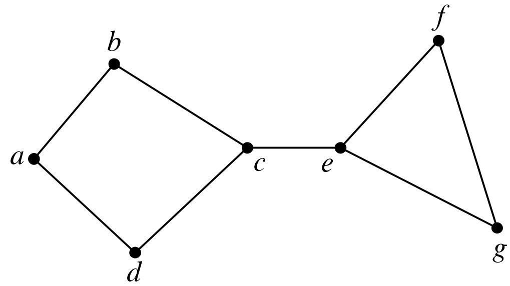


**Figure 5.40** A planar drawing with a *bridge*, namely the edge $\{c, e\}$.

These four cycles correspond nicely to the four continuous faces in Figure 5.39. So nicely, in fact, that we can identify each of the faces in Figure 5.39 by its cycle. For example, the cycle $abca$ identifies face III. Hence, we say that the cycles in Equation 5.4 are the *discrete faces* of the graph in Figure 5.39. We use the term "discrete" since cycles in a graph are a discrete data type (as opposed to a region in the plane, which is a continuous data type).

Unfortunately, continuous faces in planar drawings are not always bounded by cycles in the graph—things can get a little more complicated. For example, consider the planar drawing in Figure 5.40. This graph has what we will call a *bridge* (namely, the edge $\{c, e\}$) and the outer face is

$$abcefgecda.$$

This is not a cycle, since it has to traverse the bridge $\{c, e\}$ twice, but it is a closed walk.

As another example, consider the planar drawing in Figure 5.41. This graph has what we will call a *dongle* (namely, the nodes $v$, $x$, $y$, and $w$, and the edges incident

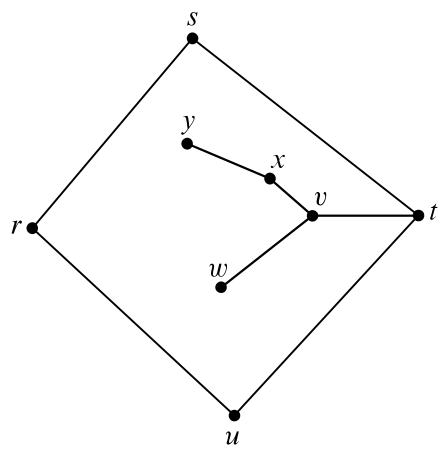


**Figure 5.41** A planar drawing with a *dongle*, namely the subgraph with nodes $v$, $w, x, y$.

to them) and the inner face is

$$rstvxyxvwvtur.$$

This is not a cycle because it has to traverse *every* edge of the dongle twice—once "coming" and once "going," but once again, it is a closed walk.

It turns out that bridges and dongles are the only complications, at least for connected graphs. In particular, every continuous face in a planar drawing corresponds to a closed walk in the graph. We refer to such closed walks as the *discrete faces* of the drawing.

### A Recursive Definition for Planar Embeddings

The association between the continuous faces of a planar drawing and closed walks will allow us to characterize a planar drawing in terms of the closed walks that bound the continuous faces. In particular, it leads us to the discrete data type of *planar embeddings* that we can use in place of continuous planar drawings. Namely, we'll define a planar embedding recursively to be the set of boundary-tracing closed walks that we could get by drawing one edge after another.

**Definition 5.8.2.** A *planar embedding* of a *connected* graph consists of a nonempty set of closed walks of the graph called the *discrete faces* of the embedding. Planar embeddings are defined recursively as follows:

**Base case**: If $G$ is a graph consisting of a single vertex $v$, then a planar embedding of $G$ has one discrete face, namely the length zero closed walk $v$.

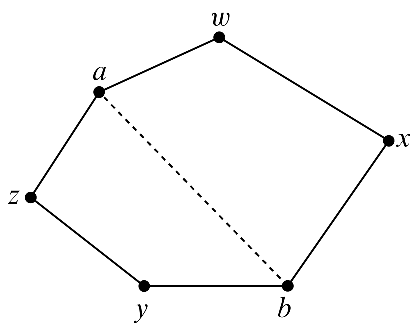


**Figure 5.42** The "split a face" case.

**Constructor Case** (split a face): Suppose $G$ is a connected graph with a planar embedding, and suppose $a$ and $b$ are distinct, nonadjacent vertices of $G$ that appear on some discrete face $\gamma$ of the planar embedding. That is, $\gamma$ is a closed walk of the form

$$a \dots b \dots a.$$

Then the graph obtained by adding the edge $\{a, b\}$ to the edges of $G$ has a planar embedding with the same discrete faces as $G$, except that face $\gamma$ is replaced by the two discrete faces[20]

$$a \dots ba \quad \text{and} \quad ab \dots a,$$

as illustrated in Figure 5.42.

**Constructor Case** (add a bridge): Suppose $G$ and $H$ are connected graphs with planar embeddings and disjoint sets of vertices. Let $a$ be a vertex on a discrete face, $\gamma$, in the embedding of $G$. That is, $\gamma$ is of the form

$$a \dots a.$$

Similarly, let $b$ be a vertex on a discrete face, $\delta$, in the embedding of $H$. So $\delta$ is of the form

$$b \cdots b.$$

Then the graph obtained by connecting $G$ and $H$ with a new edge, $\{a, b\}$, has a planar embedding whose discrete faces are the union of the discrete faces of $G$ and

[20] There is a special case of this rule. If $G$ is a line graph beginning with $a$ and ending with $b$, then the cycles into which $\gamma$ splits are actually the same. That's because adding edge $\{a, b\}$ creates a simple cycle graph, $C_n$, that divides the plane into an "inner" and an "outer" region with the same border. In order to maintain the correspondence between continuous faces and discrete faces, we have to allow two "copies" of this same cycle to count as discrete faces.

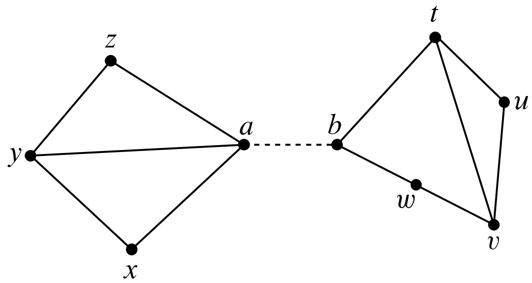


**Figure 5.43** The "add a bridge" case.

$H$, except that faces $\gamma$ and $\delta$ are replaced by one new face

$$a \ldots ab \cdots ba.$$

This is illustrated in Figure 5.43, where the faces of $G$ and $H$ are:

$$G : \{axyza,\ axya,\ ayza\} \qquad H : \{btuvwb,\ btvwb,\ tuvt\},$$

and after adding the bridge $\{a, b\}$, there is a single connected graph with faces

$$\{axyzabtuvwba,\ axya,\ ayza,\ btvwb,\ tuvt\}.$$

### Does It Work?

Yes! In general, a graph is planar if and only if each of its connected components has a planar embedding as defined in Definition 5.8.2. Unfortunately, proving this fact requires a bunch of mathematics that we don't cover in this text—stuff like geometry and topology. Of course, that is why we went to the trouble of including Definition 5.8.2—we don't want to deal with that stuff in this text and now that we have a recursive definition for planar graphs, we won't need to. That's the good news.

The bad news is that Definition 5.8.2 looks a lot more complicated than the intuitively simple notion of a drawing where edges don't cross. It seems like it would be easier to stick to the simple notion and give proofs using pictures. Perhaps so, but your proofs are more likely to be complete and correct if you work from the discrete Definition 5.8.2 instead of the continuous Definition 5.8.1.

### Where Did the Outer Face Go?

Every planar drawing has an immediately-recognizable outer face—its the one that goes to infinity in all directions. But where is the outer face in a planar embedding?

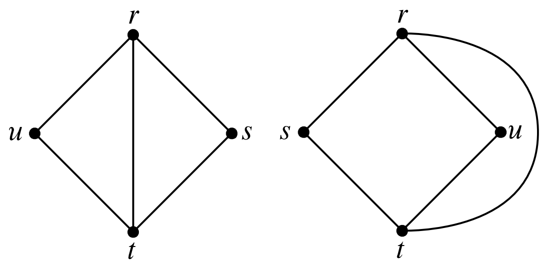


**Figure 5.44** Two illustrations of the same embedding.

There isn't one! That's because there really isn't any need to distinguish one. In fact, a planar embedding could be drawn with any given face on the outside. An intuitive explanation of this is to think of drawing the embedding on a *sphere* instead of the plane. Then any face can be made the outside face by "puncturing" that face of the sphere, stretching the puncture hole to a circle around the rest of the faces, and flattening the circular drawing onto the plane.

So pictures that show different "outside" boundaries may actually be illustrations of the same planar embedding. For example, the two embeddings shown in Figure 5.44 are really the same.

This is what justifies the "add a bridge" case in Definition 5.8.2: whatever face is chosen in the embeddings of each of the disjoint planar graphs, we can draw a bridge between them without needing to cross any other edges in the drawing, because we can assume the bridge connects two "outer" faces.

### 5.8.3 Euler's Formula

The value of the recursive definition is that it provides a powerful technique for proving properties of planar graphs, namely, structural induction. For example, we will now use Definition 5.8.2 and structural induction to establish one of the most basic properties of a connected planar graph; namely, the number of vertices and edges completely determines the number of faces in every possible planar embedding of the graph.

**Theorem 5.8.3** (Euler's Formula)**.** *If a connected graph has a planar embedding, then*

$$v - e + f = 2$$

*where $v$ is the number of vertices, $e$ is the number of edges, and $f$ is the number of faces.*

For example, in Figure 5.38, $|V| = 4$, $|E| = 6$, and $f = 4$. Sure enough, $4 - 6 + 4 = 2$, as Euler's Formula claims.

*Proof.* The proof is by structural induction on the definition of planar embeddings. Let $P(\mathcal{E})$ be the proposition that $v - e + f = 2$ for an embedding, $\mathcal{E}$.

**Base case**: ($\mathcal{E}$ is the one-vertex planar embedding). By definition, $v = 1$, $e = 0$, and $f = 1$, so $P(\mathcal{E})$ indeed holds.

**Constructor case** (split a face): Suppose $G$ is a connected graph with a planar embedding, and suppose $a$ and $b$ are distinct, nonadjacent vertices of $G$ that appear on some discrete face, $\gamma = a \ldots b \cdots a$, of the planar embedding.

Then the graph obtained by adding the edge $\{a, b\}$ to the edges of $G$ has a planar embedding with one more face and one more edge than $G$. So the quantity $v-e+f$ will remain the same for both graphs, and since by structural induction this quantity is 2 for $G$'s embedding, it's also 2 for the embedding of $G$ with the added edge. So $P$ holds for the constructed embedding.

**Constructor case** (add bridge): Suppose $G$ and $H$ are connected graphs with planar embeddings and disjoint sets of vertices. Then connecting these two graphs with a bridge merges the two bridged faces into a single face, and leaves all other faces unchanged. So the bridge operation yields a planar embedding of a connected graph with $v_G + v_H$ vertices, $e_G + e_H + 1$ edges, and $f_G + f_H - 1$ faces. Since

$$
\begin{aligned}
&(v_G + v_H) - (e_G + e_H + 1) + (f_G + f_H - 1) \\
&\quad = (v_G - e_G + f_G) + (v_H - e_H + f_H) - 2 \\
&\quad = (2) + (2) - 2 && \text{(by structural induction hypothesis)} \\
&\quad = 2,
\end{aligned}
$$

$v - e + f$ remains equal to 2 for the constructed embedding. That is, $P(E)$ also holds in this case.

This completes the proof of the constructor cases, and the theorem follows by structural induction. ■

### 5.8.4 Bounding the Number of Edges in a Planar Graph

Like Euler's formula, the following lemmas follow by structural induction from Definition 5.8.2.

**Lemma 5.8.4.** *In a planar embedding of a connected graph, each edge is traversed once by each of two different faces, or is traversed exactly twice by one face.*

**Lemma 5.8.5.** *In a planar embedding of a connected graph with at least three vertices, each face is of length at least three.*

Combining Lemmas 5.8.4 and 5.8.5 with Euler's Formula, we can now prove that planar graphs have a limited number of edges:

**Theorem 5.8.6.** *Suppose a connected planar graph has $v \geq 3$ vertices and $e$ edges. Then*

$$e \leq 3v - 6.$$

*Proof.* By definition, a connected graph is planar iff it has a planar embedding. So suppose a connected graph with $v$ vertices and $e$ edges has a planar embedding with $f$ faces. By Lemma 5.8.4, every edge is traversed exactly twice by the face boundaries. So the sum of the lengths of the face boundaries is exactly $2e$. Also by Lemma 5.8.5, when $v \geq 3$, each face boundary is of length at least three, so this sum is at least $3f$. This implies that

$$3f \leq 2e. \tag{5.5}$$

But $f = e - v + 2$ by Euler's formula, and substituting into (5.5) gives

$$\begin{aligned} 3(e - v + 2) &\leq 2e \\ e - 3v + 6 &\leq 0 \\ e &\leq 3v - 6 \end{aligned}$$

■

### 5.8.5 Returning to $K_5$ and $K_{3,3}$

Theorem 5.8.6 lets us prove that the quadrapi can't all shake hands without crossing. Representing quadrapi by vertices and the necessary handshakes by edges, we get the complete graph, $K_5$. Shaking hands without crossing amounts to showing that $K_5$ is planar. But $K_5$ is connected, has 5 vertices and 10 edges, and $10 > 3 \cdot 5 - 6$. This violates the condition of Theorem 5.8.6 required for $K_5$ to be planar, which proves

**Corollary 5.8.7.** *$K_5$ is not planar.*

We can also use Euler's Formula to show that $K_{3,3}$ is not planar. The proof is similar to that of Theorem 5.8.6 except that we use the additional fact that $K_{3,3}$ is a bipartite graph.

**Theorem 5.8.8.** *$K_{3,3}$ is not planar.*

*Proof.* By contradiction. Assume $K_{3,3}$ is planar and consider any planar embedding of $K_{3,3}$ with $f$ faces. Since $K_{3,3}$ is bipartite, we know by Theorem 5.6.2 that $K_{3,3}$ does not contain any closed walks of odd length. By Lemma 5.8.5, every face has length at least 3. This means that every face in any embedding of $K_{3,3}$ must have length at least 4. Plugging this fact into the proof of Theorem 5.8.6, we find that the sum of the lengths of the face boundaries is exactly $2e$ and at least $4f$. Hence,

$$4f \leq 2e$$

for any bipartite graph.

Plugging in $e = 9$ and $v = 6$ for $K_{3,3}$ in Euler's Formula, we find that

$$f = 2 + e - v = 5.$$

But

$$4 \cdot 5 \not\leq 2 \cdot 9,$$

and so we have a contradiction. Hence $K_{3,3}$ must not be planar. ■

### 5.8.6 Another Characterization for Planar Graphs

We did not choose to pick on $K_5$ and $K_{3,3}$ because of their application to dog houses or quadrapi shaking hands. Rather, we selected these graphs as examples because they provide another way to characterize the set of planar graphs.

**Theorem 5.8.9** (Kuratowski)**.** *A graph is not planar if and only if it contains $K_5$ or $K_{3,3}$ as a minor.*

**Definition 5.8.10.** A *minor* of a graph $G$ is a graph that can be obtained by repeatedly[21] deleting vertices, deleting edges, and merging *adjacent* vertices of $G$. *Merging* two adjacent vertices, $n_1$ and $n_2$ of a graph means deleting the two vertices and then replacing them by a new "merged" vertex, $m$, adjacent to all the vertices that were adjacent to either of $n_1$ or $n_2$, as illustrated in Figure 5.45.

For example, Figure 5.46 illustrates why $C_3$ is a minor of the graph in Figure 5.46(a). In fact $C_3$ is a minor of a connected graph $G$ if and only if $G$ is not a tree.

We will not prove Theorem 5.8.9 here, nor will we prove the following handy facts, which are obvious given the continuous Definition 5.8.1, and which can be proved using the recursive Definition 5.8.2.

**Lemma 5.8.11.** *Deleting an edge from a planar graph leaves another planar graph.*

**Corollary 5.8.12.** *Deleting a vertex from a planar graph, along with all its incident edges, leaves another planar graph.*

**Theorem 5.8.13.** *Any subgraph of a planar graph is planar.*

**Theorem 5.8.14.** *Merging two adjacent vertices of a planar graph leaves another planar graph.*

[21] The three operations can be performed in any order and in any quantities, or not at all.

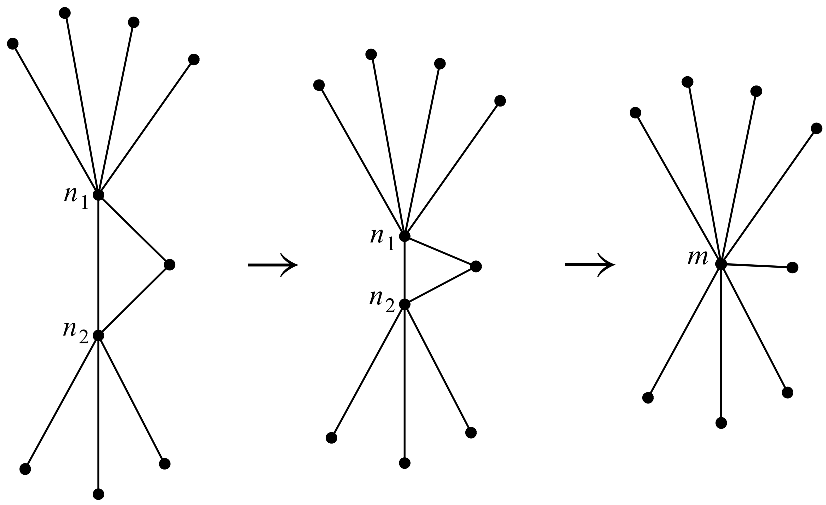


**Figure 5.45** Merging adjacent vertices $n_1$ and $n_2$ into new vertex, $m$.

### 5.8.7 Coloring Planar Graphs

We've covered a lot of ground with planar graphs, but not nearly enough to prove the famous 4-color theorem. But we can get awfully close. Indeed, we have done almost enough work to prove that every planar graph can be colored using only 5 colors. We need only one more lemma:

**Lemma 5.8.15.** *Every planar graph has a vertex of degree at most five.*

*Proof.* By contradiction. If every vertex had degree at least 6, then the sum of the vertex degrees is at least $6v$, but since the sum of the vertex degrees equals $2e$, by the Handshake Lemma (Lemma 5.2.1), we have $e \geq 3v$ contradicting the fact that $e \leq 3v - 6 < 3v$ by Theorem 5.8.6. ∎

**Theorem 5.8.16.** *Every planar graph is five-colorable.*

*Proof.* The proof will be by strong induction on the number, $v$, of vertices, with induction hypothesis:

> Every planar graph with $v$ vertices is five-colorable.

**Base cases** ($v \leq 5$): immediate.

**Inductive case**: Suppose $G$ is a planar graph with $v + 1$ vertices. We will describe a five-coloring of $G$.

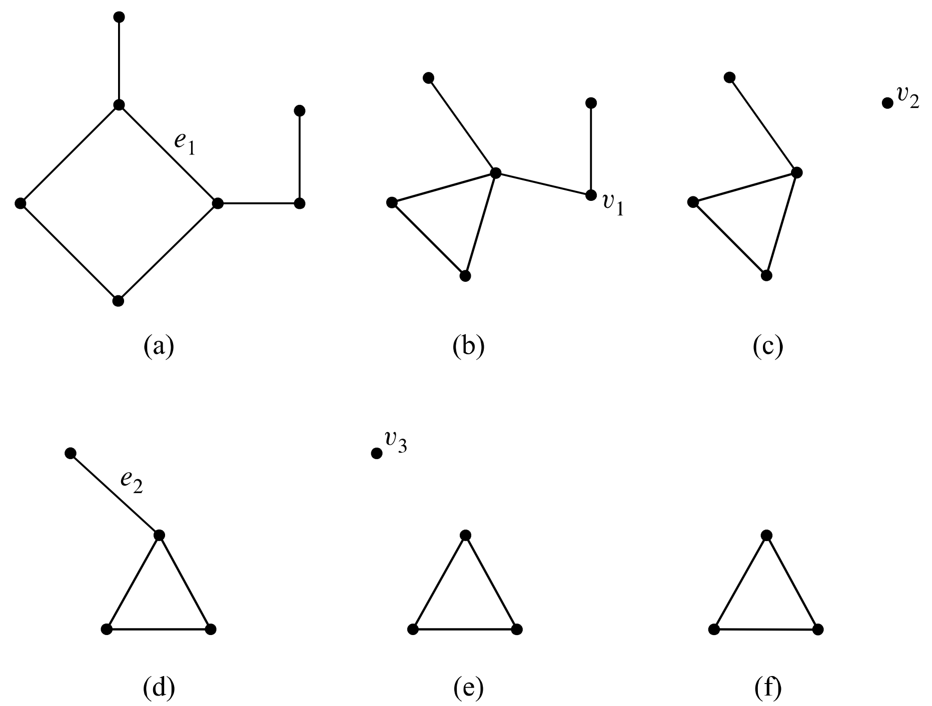


**Figure 5.46** One method by which the graph in (a) can be reduced to $C_3$ (f), thereby showing that $C_3$ is a minor of the graph. The steps are: merging the nodes incident to $e_1$ (b), deleting $v_1$ and all edges incident to it (c), deleting $v_2$ (d), deleting $e_2$, and deleting $v_3$ (f).

First, choose a vertex, $g$, of $G$ with degree at most 5; Lemma guarantees there will be such a vertex.

**Case 1:** ($\deg(g) < 5$): Deleting $g$ from $G$ leaves a graph, $H$, that is planar by Corollary 5.8.12, and, since $H$ has $v$ vertices, it is five-colorable by induction hypothesis. Now define a five coloring of $G$ as follows: use the five-coloring of $H$ for all the vertices besides $g$, and assign one of the five colors to $g$ that is not the same as the color assigned to any of its neighbors. Since there are fewer than 5 neighbors, there will always be such a color available for $g$.

**Case 2:** ($\deg(g) = 5$): If the five neighbors of $g$ in $G$ were all adjacent to each other, then these five vertices would form a nonplanar subgraph isomorphic to $K_5$, contradicting Theorem 5.8.13 (since $K_5$ is not planar). So there must be two neighbors, $n_1$ and $n_2$, of $g$ that are not adjacent. Now merge $n_1$ and $g$ into a new vertex, $m$. In this new graph, $n_2$ is adjacent to $m$, and the graph is planar by Theorem 5.8.14. So we can then merge $m$ and $n_2$ into a another new vertex, $m'$, resulting in a new graph, $G'$, which by Theorem 5.8.14 is also planar. Since $G'$ has $v - 1$ vertices, it is five-colorable by the induction hypothesis.

Define a five coloring of $G$ as follows: use the five-coloring of $G'$ for all the vertices besides $g$, $n_1$ and $n_2$. Next assign the color of $m'$ in $G'$ to be the color of the neighbors $n_1$ and $n_2$. Since $n_1$ and $n_2$ are not adjacent in $G$, this defines a proper five-coloring of $G$ except for vertex $g$. But since these two neighbors of $g$ have the same color, the neighbors of $g$ have been colored using fewer than five colors altogether. So complete the five-coloring of $G$ by assigning one of the five colors to $g$ that is not the same as any of the colors assigned to its neighbors.

■