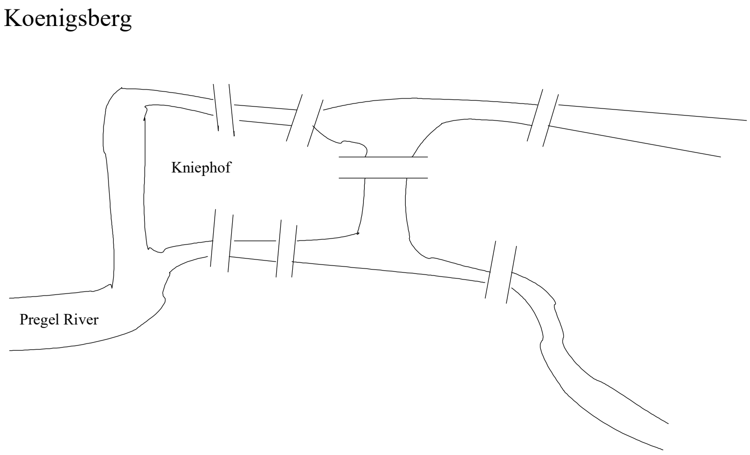
Graph Theory, Part 1

1 The seven bridges of Königsberg

1.1 The Problem

The city of Königsberg (formerly in Prussia, now a part of Russia and called Kaliningrad) is split by the River Pregel into various parts (including the island Kniephof), and back in the day there were seven bridges connecting the various parts, as you can see in the map below:

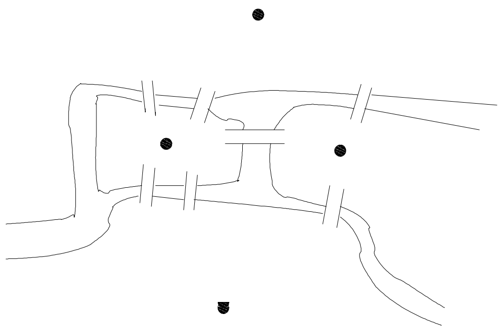


The city-folk wanted to know if it was possible to make a walking tour of the city, starting and ending at the same place, in such a way that the walkers crossed each bridge exactly once. This was considered a difficult problem at the time. After all, there are many different routes you might consider, and it would be tedious to check all of them. Of course, if you find something that works, you are done. But people couldn't seem to make anything work, and so the question arose whether there was not something inherent in the problem that made it insoluble.

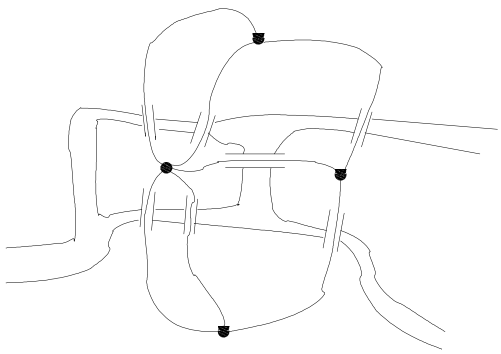
In a 1736 paper which is considered the birth of a branch of math known as *Graph Theory*, the great mathematician Leonhard Euler showed decisively that it is *impossible*. His idea is so stunningly clear that it sets the whole matter to rest.

1.2 Abstraction

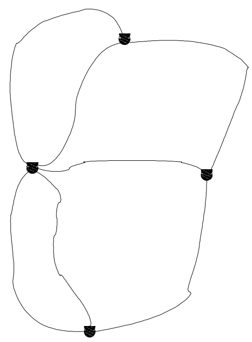
To help us deal with the problem more readily, let us *abstract* from the problem only the relevant information. Basically there are four pieces of land and various bridges connecting them. We don't care what shops you visit or things you go to see on the way when you are not crossing bridges—for us each of the four pieces of land is just a place with bridges to some of the other pieces of land. Let's draw a dot for each of the four pieces of land. Visiting the each piece of land is for us the same as visiting the dot in that land.



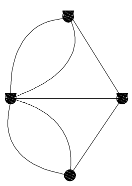
Now let's make connections between the dots, with one connection for each bridge:



We can now remove all the burdensome geographical details:



And draw the dots and connections more neatly. All we are concerned with is the connectivity of the lands by the bridges, not exactly how far or in what direction you need to walk:



Now the problem is to go from dot to dot using each bridge only once. Suppose we start on the rightmost dot. Then when we leave it, we use one of the three bridges, and so there are still two bridges attached to the rightmost dot which we eventually must use. The only way to use another is to return to that dot. When we do, we use another bridge, and are left with one bridge to use. Well then, we must leave that dot again, and use the last of the three bridges attached to it. But then there is no bridge with which to come back, and we are supposed to do a round trip.

OK. Maybe we shouldn't start the trip at the rightmost dot. We will start somewhere else. Nonetheless, we do need to visit the rightmost dot at some point in order to make use of bridges attached to it. On our first visit, we use one bridge coming in. Then we must use another, and so we leave. But there is still one more bridge attached to the right dot that has not been used. So we need to return again, but then we can't leave! This is bad, because now our trip is a round trip which didn't start at the rightmost dot.

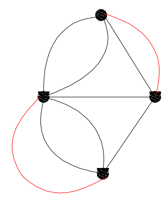
In fact, it is not hard to show that *every* dot (piece of land), whether it is the one you start and end at or not, *must* have an even number of bridges attached, otherwise you will have problems. You use an even number of bridges when you come to a place and then leave again. For dots other than your starting dot, you must always leave after coming, so you always use up a pair of bridges per visit. So there must be an even number of bridges. For you starting point/destination dot, you use a pair of bridges for each passing visit, but also one bridge to leave at the beginning and one bridge to return at the very end. So it should also have an even number of bridges. In Königsberg, *every* land has an odd number of bridges attached, so it is hopeless.

In honor of Euler, who showed that this cannot be done, a path that uses every bridge once is called a *Eulerian circuit*. Königsberg has no Eulerian circuit because it has regions with odd numbers of bridges.

1.3 A Modification

OK. So we are definitely ruined unless *every* dot has an even number of bridges attached. But might we still have problems even if we do have an even number of bridges for every land?

For instance, suppose we build two more two bridges in Königsberg so that each dot has an even number of bridges:

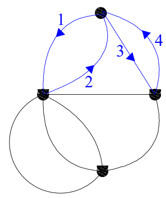


Now is it possible to make an Eulerian circuit? You can work out that it is indeed possible. Is this a general rule for all graphs? It turns out it is.

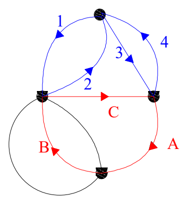
Theorem 1 (Euler, Informal Version). *It is possible to make an Eulerian circuit if and only if all vertices (dots) have an even number of edges (bridges) [and the whole system is connected—if one part is completely cut off from another, it is of course hopeless!].*

We have proved that you cannot make an Eulerian circuit if you do not satisfy the "all-even" condition. How can you prove than you *can* make an Eulerian circuit if you do satisfy the "all-even" condition? The idea is as follows: start off wherever you want and walk over bridges as you please with the only rule being that you never use the same bridge twice. Eventually you will get stuck since bridges keep getting used out. If you (by chance) managed to make an Eulerian circuit, then you are done!

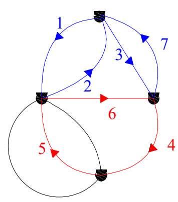
But maybe you didn't. For instance, in the last graph above, you might have just started at the top dot, walked to the left dot and then back again, and then walked to the right dot and back again. In this case, you would have used all four bridges from the top dot, namely:



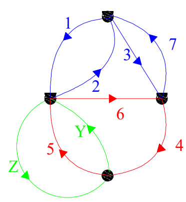
and you can no longer leave! But many bridges are left untraveled, so it is not an Eulerian circuit. What you do in this case is you go to your partly completed circuit, and pick a dot whose bridges are not all used—for instance the right-hand dot. Then start another walk there, where you only use bridges that haven't been used yet:



After traversing A-B-C, we are stuck again at the rightmost dot, because we have used all the bridges from that dot. We combine the two tours, by going on the first one (1-2-3-4) until we get to the departure point of the second tour (the right-hand dot). Then we make the second tour (A-B-C) a side trip, and after we finish the side trip, we complete the original tour. In this case, that means we start the original tour 1-2-3, then take the side trip A-B-C, and then use 4 to finish the original tour. So we now have a bigger tour 1-2-3-A-B-C-4, which we relabel 1-2-3-4-5-6-7.



There are still two bridges unused. between the left and bottom dot. So we make those two bridges into a little tour Y-Z, and then combine it with our existing tour 1-2-3-4-Y-Z-5-6-7.



Each of the mini-trips is a circuit, so it uses an even number of bridges for each dot visited. So what is left of the original graph at each stage after adding the side-trip is a smaller number of unused bridges, with an even number of unused bridges for each dot (because an even number of bridges minus an even number of bridges used is equal to an even number of bridges remaining). So we can always continue adding side-trips until all the bridges are used!

2 The Basics

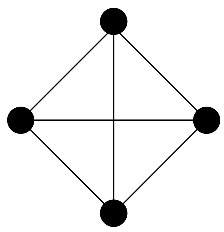
2.1 What is a Graph?

The diagrams which we used to represent the Königsberg problem economically are mathematical objects called *graphs*. Euler's paper on the Königsberg problem is considered the birth of the branch of mathematics now known as *Graph Theory*.

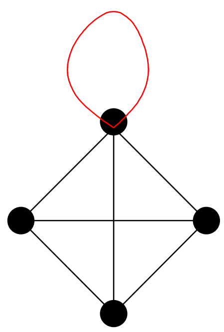
Let's make some definitions.

Definition 1 (Graph). A *graph* is an assembly of two kinds of things, *vertices* and *edges*. The only rule is that each edge starts at a vertex and ends a vertex.

We usually write vertices as dots and edges as lines between the two vertices they join. We do not need to draw the edges as straight lines, and in general we will not be able to avoid drawing pictures where the edges displayed seem to touch each other. In fact, such edges should not be considered to touch each other, but you can imagine that one goes over the other, much like a highway overpass. This is why we always draw the vertices as moderately-sized dots—so that we are not confused into thinking that the crossed edges in



meet at some vertex. Since there is no vertex-dot, they do not. Note that in our Königsberg problem, we had more than one edge connecting a pair of vertices, to represent multiple bridges that accomplish the same thing. It is even possible to have an edge connecting a vertex to itself; this is called a *loop*. Here we have added a loop to the last graph:



Sometimes it is important to assign directions to the edges on a graph, and so we define a graph with this additional information:

Definition 2 (Directed Graph). A *directed graph* is a graph where we have put arrows to assign directions to *all* the edges.

For example, the last graph which designates our Eulerian circuit in the modified Königsberg bridge problem is a directed graph.

2.2 Degree

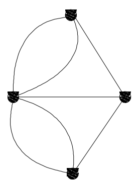
In the Königsberg bridge problem, we were searching for a Eulerian circuit, which can now be defined:

Definition 3 (Circuit, Eulerian Circuit). A *circuit* in a graph is a path which begins and ends at the same vertex. A *Eulerian circuit* is a circuit in a graph which traverses each edge precisely once.

We found that a Eulerian circuit exists if and only if each vertex (representing a piece of land) has an even number of edges (bridges) attached to it. We coin a new term to make it easier to speak of such conditions:

Definition 4 (Degree). The *degree* of a vertex is the number of edges attached to that vertex.

For example, if we return to the Königsberg bridges graph,



then we see that the degrees of the vertices (starting at the top an proceeding clockwise) are 3, 3, 3, and 5. Now we can state Euler's result succinctly:

Theorem 2 (Euler, Formal Version). *A graph has a Eulerian circuit if and only if all vertices have even degrees.*

Thus the Königsberg bridges graph has no Eulerian circuit.

2.3 Distance and Diameter

Definition 5 (Distance). The *distance* from vertex u to vertex v is the minimum number of edges one must traverse to get from u to v.

In the undirected Königsberg bridges graph vertex u is distance 1 from all other vertices, but vertex v is distance 2 from vertex x. We have a special name for vertices which are at distance 1 from each other.

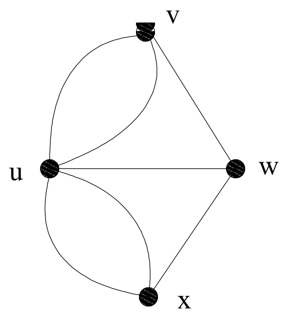
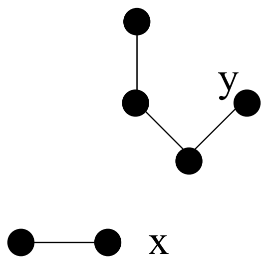


Figure 1: (Undirected) Königsberg Bridges Graph

Definition 6 (Adjacency). Two vertices in a graph are said to be *adjacent* if they are joined by an edge.

If one vertex cannot be reached from another by a path made of edges, then we say that their distance from each other is ∞. If this happens, it means that the graph is not connected. For instance the graph

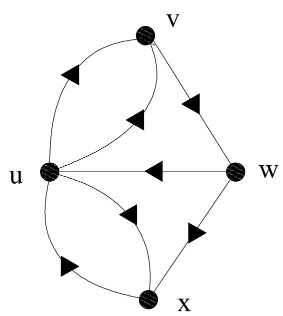


has the distance from vertex x to vertex y equal to ∞, and so the graph is not connected. On the other hand, the Königsberg bridges graph (see Figure 1) is connected.

NOTE: We shall ALWAYS assume our graphs are connected unless explicitly noted otherwise.

When we label the vertices of a graph, as we have done in the graph of Figure 1, we also obtain a compact notation for labeling edges. For example, the edge connecting u to v is called uv. Note that $uv = vu$, since this is an undirected graph, so that the order of the vertices does not matter.

In a directed graph, we only count paths that follow the one-way signs attached to the edges. For example, if we assign directions to edges in the graph of Figure 1:



then the distance from w to x is 1, but the distance from x to w is 3. So distance is not symmetric in a directed graph, as anyone knows who has driven in a city with lots of one-way streets!

In a directed graph, we label edges in a way that indicates which way the arrow goes. For example, in the above graph, we have edge $\overrightarrow{wx}$, also known as $\overleftarrow{xw}$, but we DO NOT have the edge $\overrightarrow{xw} = \overleftarrow{wx}$.

Definition 7 (Diameter). The *diameter* of a graph is the maximum of the distances between pairs of vertices.

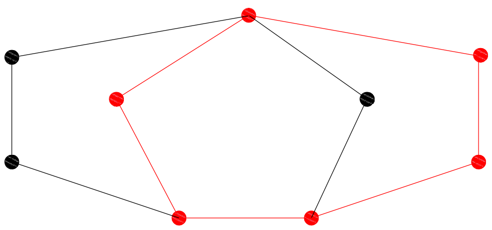
The undirected Königsberg bridges graph has diameter 2 (because every vertex is only distance 1 or 2 from the others), while an octagonal graph has diameter 4 (draw it and find the longest distance).

2.4 Polygons, Girth, and Trees

The octagonal graph we just mentioned is one member of an important family of graphs. Let us consider undirected graphs for the time being.

Definition 8 (Polygon). A *polygon* is a sequence of edges and vertices where no edge or vertex is visited more than once, with the exception that the first vertex and the last coincide. [Note that Eulerian circuits need not be polygons: although they never re-use edges, they do not always meet the restriction on multiple visits to vertices.] The *length* of a polygon is the number or edges in the polygon.

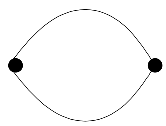
The following graph has a polygon of length 6 in red.



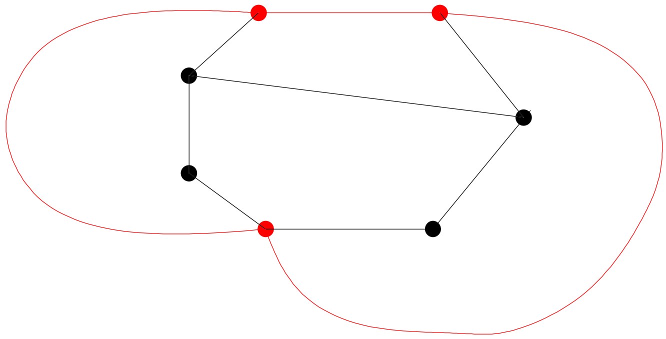
The graph also has polygons of lengths 5 and 7, but none of length less than 5, nor any of length greater than 7. This motivates the next definition:

Definition 9 (Girth). The *girth* of a graph is the length of the shortest polygon in the graph.

A graph of girth 1 must have a loop. A graph of girth 2 must have no loops, but must have two vertices connected thus:



We call this strange object a 2-*gon*. A graph of girth 3 must have no 2-gons, nor loops, but must have three vertices connected like a triangle (although they may be drawn with curvy edges in our drawing of the graph). For the purposes of graph theory, we *do* call such a structure a triangle, however it may be drawn. For example, the following graph has a triangle in red:

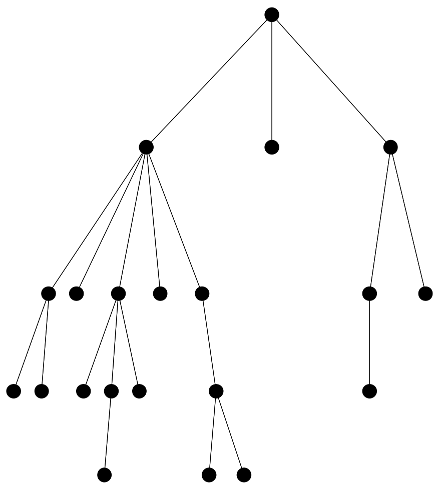


Once we allow this latitude in the use of the terms "triangle," "square," pentagon," and so on, we see that a graph of girth 4 is free of loops, 2-gons, and triangles, but does contain a square.

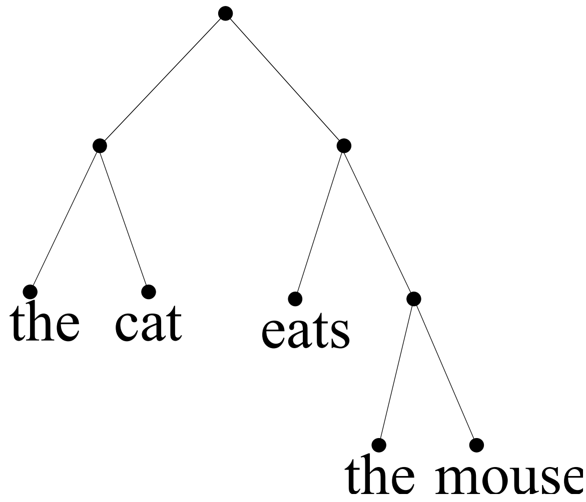
What if the graph has no polygons at all? Then we define the girth to be ∞. We have a special name for such graphs.

Definition 10 (Tree). A *tree* is a graph containing no polygons.

So a tree has no loops nor 2-gons, so that every pair of vertices is either not joined or joined by a single edge. Trees are widely used in such fields as biology, computer science, and linguistics in organizing data. It is common to draw trees to look like upside-down versions of the trees you see in the real world:



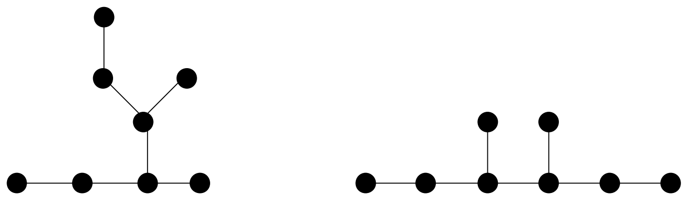
Here is a tree that represents the parsing of an English sentence:



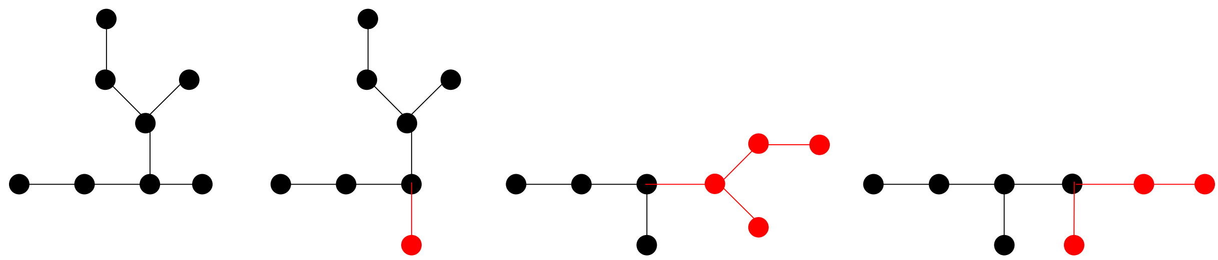
3 Isomorphism of Graphs

Two graphs G and H will be called *isomorphic* if they have the same pattern of connectivity. By this we mean that you can convert G into H by moving the vertices of G around without changing which vertices the edges are connected to. You might also need to bend the edges into new shapes, but without changing which vertices they join. Don't worry if you need edges to move over other edges and vertices when making these changes.

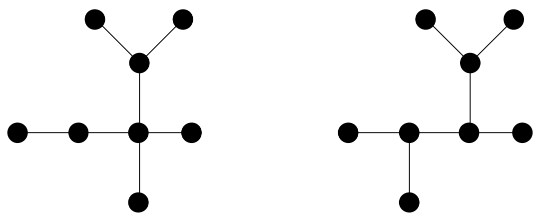
For an example of where isomorphism is important, we turn to chemistry. Hydrocarbons known as alkanes may be represented by trees in which each vertex represents a carbon atom. The edges represent bonds between carbon atoms. A carbon can be bonded to at most four other carbon atoms, so the maximum allowed degree of a vertex is 4. (The hydrogen atoms are not depicted in these graphs, but a chemist automatically knows where to put them if shown the graph of the carbon atoms.) The following two forms of octane (alkanes with 8 carbon atoms) are actually isomorphic:



We show the movements of vertices that make the first graph look like the second. We color red the elements that are moved in each step. We should rotate the final figure by 180 degrees to finish the process.



The following two graphs are not isomorphic, because one has a vertex of degree 4, while the other does not:



4 The Chinese Postman Problem

The Chinese Postman Problem is a generalization of the Königsberg bridges problem. A Eulerian circuit exists only in graphs where all degrees are even. If not, we might ask ourselves, "What is the next closest thing to a Eulerian circuit?" This could be answered in several ways: the one we shall propose is adumbrated in the exploration where we constructed extra bridges in Königsberg in order to make all degrees even, thus enabling a Eulerian circuit to exist.

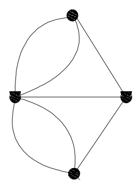
What we seek is a circuit that uses each edge once, and perhaps some edges more than once, but we would like to reuse edges as few times as possible. That is, we want a circuit employing all edges but of minimum total length (as measured in number of edges traversed). This is not a pointless quest: suppose we need to deliver mail, or sweep streets, or pick up garbage. Each segment of street between intersections could be represented as an edge, and the intersections as vertices. Then our insistence that each edge be traversed at least once is the insistence that all houses in town receive services; retracing is tolerated as needed to make a full circuit, but no more than necessary. Of course, in a real-world situation, we would note that different segments of streets are of different length, and we would try to minimize the total *length* (in meters) of street segments retraced, rather than the total *number* of segments of streets retraced. This is more complicated, so in these notes we shall stick to the simpler version, where we just try to minimize the number of edges in our circuit. In the homework, the more general problem will be discussed. This more complicated problem is called the *Chinese Postman Problem*, after the Chinese mathematician Meigu Guan who studied it, and our easier version is called the *Simplified Chinese Postman Problem*.

Using an edge uv of a graph G twice in a circuit is much like having two copies of the edge and using each copy once. In fact, let us make an extra copy of uv for each time we use it after the first traversal. We then use the duplicates instead of the originals, so that no edge is traversed multiple times. And let us do the same for all other edges, add a duplicate for every use beyond the first and traverse duplicates so as to avoid double-use of any edge. What we see is that a circuit in a graph G that traverses all edges at least once and which includes n

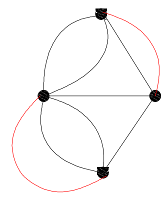
additional traversals of edges already used is equivalent to a Eulerian circuit in a graph obtained by adding n extra edges to G, where the added edges are copies of various edges existing in the original graph G. Thus a minimum-length circuit that uses all edges in G is a Eulerian circuit in graph H obtained from G by duplicating as few edges as possible. We know that a Eulerian circuit will exist in H if and only if all degrees of vertices in H are of even degree. So our goal is to add duplicate edges to G—as few as possible—to make a graph H with all degrees even. This prompts a definition.

Definition 11 (Eulerization, Minimal Eulerization). A *Eulerization* of a graph G is a graph H, all of whose vertices are of even degree, obtained from G by duplicating edges. A *minimal Eulerization* of G is a Eulerization of G with as few extra edges as possible.

So what we are looking for in the Simplified Chinese Postman Problem is a minimal Eulerization of a given graph. For example, the Königsberg bridges graph



has all four vertices of odd degree. With each edge we add, we can at best hope to change two vertices to even degree. So we must add at least two edges. Thus, our modified Königsberg bridges graph

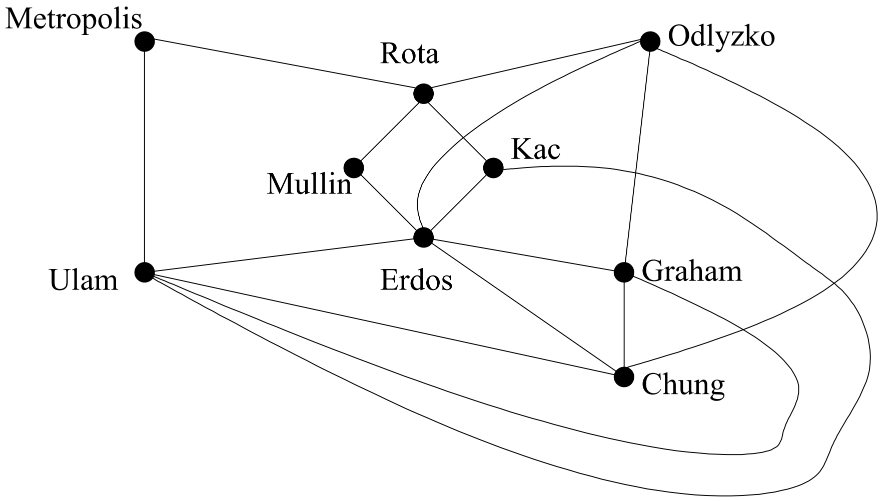


which duplicates two edges, is a minimal Eulerization of the original graph.

Is it always possible to Eulerize a graph? Yes! A Eulerization that is probably not very efficient is one that adds precisely one extra copy of each existing edge. This will definitely make all degrees even. This also shows that the postman certainly does not need to more than double the length of the streets serviced.

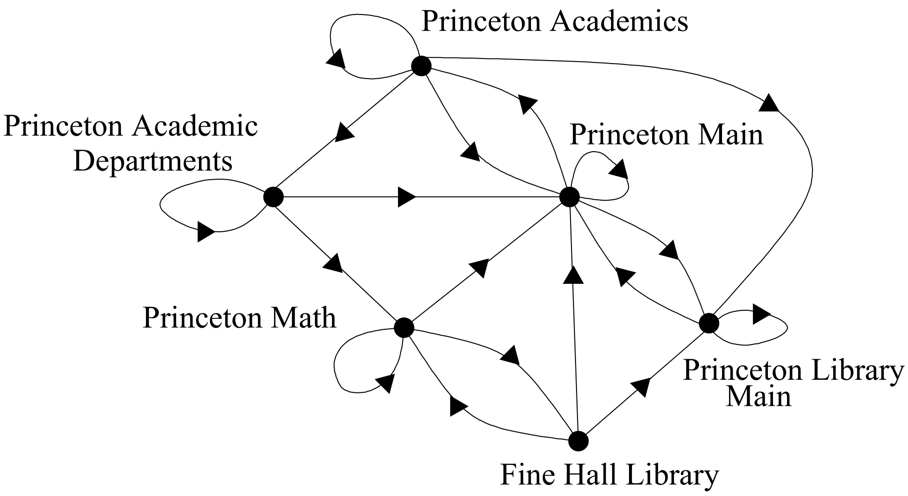
5 Graphs as Models of Adjacency

Graphs are used as a basic mathematical model for showing adjacency or any other relationship that occurs between pairs of individuals. For example, one graph has all the world's mathematicians as vertices. Two vertices (mathematicians) are joined by an edge if they are co-authors of a joint paper. One particular mathematician, Paul Erdös, is famous for having co-written with so many people, that most seasoned mathematicians are not far in distance from Erdös in the co-authorship graph. Here is a tiny portion of the authorship graph



The distance in the graph between a mathematical author and Erdös is given a name, "the Erdös number." Erdös has an Erdös number of zero, and people who coauthored with him are joined to him with an edge in the graph, so they have Erdös number one. People who wrote with someone who wrote with Erdös have Erdös number two. And so forth.

We could also graph the internet by making a site for each vertex. In our graph, an edge between sites indicates a link. But internet links have a direction—Site A can have a link sending you to Site B, but Site B may not have a link back to Site A. Thus, we should use a directed graph to represent the links on the internet.



5.1 Small World

When we start to look at large graphs like the coauthor graph or the internet graph, we often find what is called the *small world* phenomenon. The first issue is one of connectedness. Since people can make personal webpages that are unlinked to anything, we shall not always be able to get from one page to another by following links. However, in these large graphs it is often the case that there are many tiny isolated components, like little islands, along with one gigantic connected component, like a huge continent.

The second issue deals with how closely connected things are in this giant component. This is characterized by the diameter of the graph—the maximum distance you need to go to get from one vertex to another. Around the year 2000, people estimated that there were about 800 million pages in the web. However, researchers from Notre Dame performed an automated survey of a portion of the internet and concluded that the diameter is about 19, i.e., any two pages in the giant connected component are 19 links apart or less.