2 Sets, Relations and Functions

Key Topics

Sets
Set Operations
Russell's Paradox
Computer Representation of sets
Relations
Composition of Relations
Reflexive, Symmetric and Transitive Relations
Relational Database Management System
Functions
Partial and Total Functions
Injective, Surjective and Bijective Functions
Functional Programming

2.1 Introduction

This chapter provides an introduction to fundamental building blocks in mathematics such as sets, relations and functions. Sets are collections of well-defined objects; relations indicate relationships between members of two sets *A* and *B*; and

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functions are a special type of relation where there is exactly (or at most)[1] one relationship for each element $a \in A$ with an element in B.

A set is a collection of well-defined objects that contain no duplicates. The term 'well defined' means that for a given value it is possible to determine whether or not it is a member of the set. There are many examples of sets such as the set of natural numbers $\mathbb{N}$, the set of integer numbers $\mathbb{Z}$, and the set of rational numbers $\mathbb{Q}$. The natural numbers $\mathbb{N}$ is an infinite set consisting of the numbers $\{1, 2, \ldots\}$. Venn diagrams may be used to represent sets pictorially.

A binary relation $R\ (A, B)$ where A and B are sets is a subset of the Cartesian product $(A \times B)$ of A and B. The domain of the relation is A and the codomain of the relation is B. The notation aRb signifies that there is a relation between a and b and that $(a, b) \in R$. An n-ary relation $R\ (A_1, A_2, \ldots A_n)$ is a subset of $(A_1 \times A_2 \times \ldots \times A_n)$. However, an n-ary relation may also be regarded as a binary relation $R(A, B)$ with $A = A_1 \times A_2 \times \ldots \times A_{n-1}$ and $B = A_n$.

Functions may be total or partial. A total function $f: A \to B$ is a special relation such that for each element $a \in A$ there is exactly one element $b \in B$. This is written as $f(a) = b$. A partial function differs from a total function in that the function may be undefined for one or more values of A. The domain of a function (denoted by **dom** f) is the set of values in A for which the partial function is defined. The domain of the function is A provided that f is a total function. The codomain of the function is B.

2.2 Set Theory

A set is a fundamental building block in mathematics, and it is defined as a collection of well-defined objects. The elements in a set are of the same kind, and they are distinct with no repetition of the same element in the set.[2] Most sets encountered in computer science are finite, as computers can only deal with finite entities. Venn diagrams[3] are often employed to give a pictorial representation of a set, and they may be used to illustrate various set operations such as set union, intersection and set difference.

There are many well-known examples of sets including the set of natural numbers denoted by $\mathbb{N}$; the set of integers denoted by $\mathbb{Z}$; the set of rational numbers is denoted by $\mathbb{Q}$; the set of real numbers denoted by $\mathbb{R}$; and the set of complex numbers denoted by $\mathbb{C}$.

[1]We distinguish between total and partial functions. A total function $f: A \to B$ is defined for every element in A whereas a partial function may be undefined for one or more values in A.

[2]There are mathematical objects known as multi-sets or bags that allow duplication of elements. For example, a bag of marbles may contain three green marbles, two blue and one red marble.

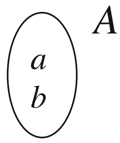
[3]The British logician, John Venn, invented the Venn diagram. It provides a visual representation of a set and the various set theoretical operations. Their use is limited to the representation of two or three sets as they become cumbersome with a larger number of sets.

Example 2.1 The following are examples of sets.

- The books on the shelves in a library
- The books that are currently overdue from the library
- The customers of a bank
- The bank accounts in a bank
- The set of Natural Numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$
- The Integer Numbers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
- The non-negative integers $\mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\}$
- The set of Prime Numbers = $\{2, 3, 5, 7, 11, 13, 17, \ldots\}$
- The Rational Numbers is the set of quotients of integers

$$\mathbb{Q} = \{p/q : p, q \in \mathbb{Z} \, \text{and} \, q \neq 0\}$$

A finite set may be defined by listing all of its elements. For example, the set $A = \{2, 4, 6, 8, 10\}$ is the set of all even natural numbers less than or equal to 10. The order in which the elements are listed is not relevant: i.e. the set $\{2, 4, 6, 8, 10\}$ is the same as the set $\{8, 4, 2, 10, 6\}$.



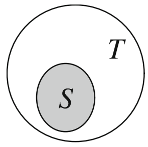
Sets may be defined by using a predicate to constrain set membership. For example, the set $S = \{n: \mathbb{N}: n \leq 10 \wedge n \bmod 2 = 0\}$ also represents the set $\{2, 4, 6, 8, 10\}$. That is, the use of a predicate allows a new set to be created from an existing set by using the predicate to restrict membership of the set. The set of even natural numbers may be defined by a predicate over the set of natural numbers that restricts membership to the even numbers. It is defined by

$$\text{Evens} = \{x | x \in \mathbb{N} \wedge \mathit{even}(x)\}.$$

In this example, $\mathit{even}(x)$ is a predicate that is true if x is even and false otherwise. In general, $A = \{x \in E \mid P(x)\}$ denotes a set A formed from a set E using the predicate P to restrict membership of A to those elements of E for which the predicate is true.

The elements of a finite set S are denoted by $\{x_1, x_2, \ldots x_n\}$. The expression $x \in S$ denotes that the element x is a member of the set S, whereas the expression $x \notin S$ indicates that x is not a member of the set S.

A set S is a subset of a set T (denoted $S \subseteq T$) if whenever $s \in S$ then $s \in T$, and in this case the set T is said to be a superset of S (denoted $T \supseteq S$). Two sets S and T are said to be equal if they contain identical elements: i.e. $S = T$ if and only if $S \subseteq T$ and $T \subseteq S$. A set S is a proper subset of a set T (denoted $S \subset T$) if $S \subseteq T$ and $S \neq T$. That is, every element of S is an element of T and there is at least one element in T that is not an element of S. In this case, T is a proper superset of S (denoted $T \supset S$).



The empty set (denoted by $\varnothing$ or $\{\}$) represents the set that has no elements. Clearly $\varnothing$ is a subset of every set. The singleton set containing just one element x is denoted by $\{x\}$, and clearly $x \in \{x\}$ and $x \neq \{x\}$. Clearly, $y \in \{x\}$ if and only if $x = y$.

Example 2.2

(i) $\{1, 2\} \subseteq \{1, 2, 3\}$
(ii) $\varnothing \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

The cardinality (or size) of a finite set S defines the number of elements present in the set. It is denoted by |S|. The cardinality of an infinite[4] set S is written as $|S| = \infty$.

Example 2.3

(i) Given $A = \{2, 4, 5, 8, 10\}$ then $|A| = 5$.
(ii) Given $A = \{x \in \mathbb{Z}: x^2 = 9\}$ then $|A| = 2$
(iii) Given $A = \{x \in \mathbb{Z}: x^2 = -9\}$ then $|A| = 0$.

2.2.1 Set Theoretical Operations

Several set theoretical operations are considered in this section. These include the Cartesian product operation; the power set of a set; the set union operation; the set intersection operation; the set difference operation; and the symmetric difference operation.

Cartesian Product

The Cartesian product allows a new set to be created from existing sets. The Cartesian[5] product of two sets S and T (denoted $S \times T$) is the set of ordered pairs $\{(s, t) \mid s \in S, t \in T\}$. Clearly, $S \times T \neq T \times S$ and so the Cartesian product of two sets is not commutative. Two ordered pairs (s_1, t_1) and (s_2, t_2) are considered equal if and only if $s_1 = s_2$ and $t_1 = t_2$.

The Cartesian product may be extended to that of n sets $S_1, S_2, \ldots, S_n$. The Cartesian product $S_1 \times S_2 \times \ldots \times S_n$ is the set of ordered tuples $\{(s_1, s_2, \ldots, s_n) \mid s_1$

[4]The natural numbers, integers and rational numbers are countable sets whereas the real and complex numbers are uncountable sets.

[5]Cartesian product is named after René Descartes who was a famous 17th French mathematician and philosopher. He invented the Cartesian coordinates system that links geometry and algebra, and allows geometric shapes to be defined by algebraic equations.

$\in S_1, s_2 \in S_2, \ldots, s_n \in S_n\}$. Two ordered n-tuples $(s_1, s_2, \ldots, s_n)$ and $(s_1', s_2', \ldots, s_n')$ are considered equal if and only if $s_1 = s_1', s_2, = s_2', \ldots, s_n = s_n'$.

The Cartesian product may also be applied to a single set S to create ordered n-tuples of S: i.e. $S^n = S \times S \times \ldots \times S$ (n-times).

Power Set

The power set of a set A (denoted $\mathbb{P}A$) denotes the set of subsets of A. For example, the power set of the set $A = \{1, 2, 3\}$ has 8 elements and is given by

$$\mathbb{P}A = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

There are $2^3 = 8$ elements in the power set of $A = \{1, 2, 3\}$ and the cardinality of A is 3. In general, there are $2^{|A|}$ elements in the power set of A.

Theorem 2.1 (Cardinality of Power Set of A) *There are* $2^{|A|}$ *elements in the power set of A*

Proof Let $|A| = n$ then the cardinality of the subsets of A are subsets of size 0, 1, …, n. There are $\binom{n}{k}$ subsets of A of size k.[6] Therefore, the total number of subsets of A is the total number of subsets of size 0, 1, 2, … up to n. That is

$$|\mathbb{P}\mathrm{A}| = \sum_{k=0}^{n} \binom{n}{k}$$

The Binomial Theorem (we prove it in Example 4.2 in Chap. 4) states that

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Therefore, putting $x = 1$ we get that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^k = |\mathbb{P}\mathrm{A}|$$

Union and Intersection Operations

The union of two sets A and B is denoted by $A \cup B$. It results in a set that contains all of the members of A and of B and is defined by

$$\mathrm{A} \cup B = \{r | r \in A \text{ or } r \in B\}.$$

For example, suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cup B = \{1, 2, 3, 4\}$. Set union is a commutative operation: i.e. $A \cup B = B \cup A$. Venn Diagrams are used to illustrate these operations pictorially.

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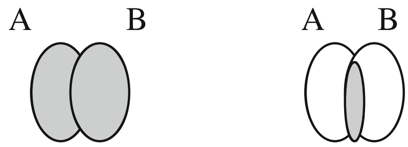
[6]We discuss permutations and combinations in Chap. 5.



$A \cup B$ $A \cap B$

The intersection of two sets A and B is denoted by $A \cap B$. It results in a set containing the elements that A and B have in common and is defined by

$$A \cap B = \{r | r \in A \text{ and } r \in B\}.$$

Suppose $A = \{1, 2, 3\}$ and $B = \{2, 3, 4\}$ then $A \cap B = \{2, 3\}$. Set intersection is a commutative operation: i.e. $A \cap B = B \cap A$.

Union and intersection are binary operations but may be extended to more generalized union and intersection operations. For example

$$\cup_{i=1}^{n} A_i \text{ denotes the union of } n \text{ sets.}$$
$$\cap_{i=1}^{n} A_i \text{ denotes the intersection of } n \text{ sets}$$

Set Difference Operations

The set difference operation $A \backslash B$ yields the elements in A that are not in B. It is defined by

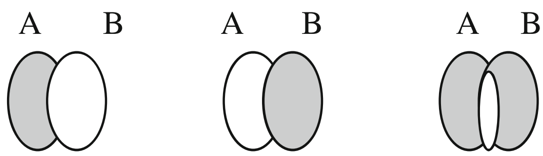
$$A \backslash B = \{a | a \in A \text{ and } a \notin B\}.$$

For A and B defined as A = {1, 2} and B = {2, 3} we have $A \backslash B = \{1\}$ and $B \backslash A = \{3\}$. Clearly, set difference is not commutative: i.e. $A \backslash B \neq B \backslash A$. Clearly, $A \backslash A = \varnothing$ and $A \backslash \varnothing = A$.

The symmetric difference of two sets A and B is denoted by $A \Delta B$ and is given by

$$A \Delta B = A \backslash B \cup B \backslash A$$

The symmetric difference operation is commutative: i.e. $A \Delta B = B \Delta A$. Venn diagrams are used to illustrate these operations pictorially.

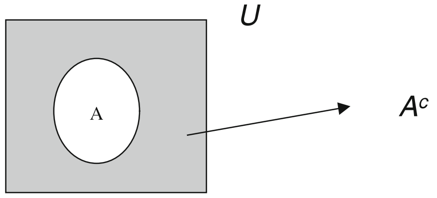


$A \backslash B$ $B \backslash A$ $A \Delta B$

The complement of a set A (with respect to the universal set U) is the elements in the universal set that are not in A. It is denoted by A^c (or A') and is defined as

$$A^c = \{u | u \in U \text{ and } u \notin A\} = U \backslash A$$

The complement of the set A is illustrated by the shaded area below



2.2.2 Properties of Set Theoretical Operations

The set union and set intersection properties are commutative and associative. Their properties are listed in Table 2.1.

These properties may be seen to be true with Venn diagrams, and we give a proof of the distributive property (this proof uses logic which is discussed in Chaps. 14–16).

Proof of Properties (Distributive Property)
To show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Suppose $x \in A \cap (B \cup C)$ then

$$x \in A \wedge x \in (B \cup C)$$

$$\Rightarrow x \in A \wedge (x \in B \vee x \in C)$$

Table 2.1 Properties of set operations

Property	Description
Commutative	Union and intersection operations are commutative: i.e. $S \cup T = T \cup S$ $S \cap T = T \cap S$
Associative	Union and intersection operations are associative: i.e. $R \cup (S \cup T) = (R \cup S) \cup T$ $R \cap (S \cap T) = (R \cap S) \cap T$
Identity	The identity under set union is the empty set $\varnothing$, and the identity under intersection is the universal set U. $S \cup \varnothing = \varnothing \cup S = S$ $S \cap U = U \cap S = S$
Distributive	The union operator distributes over the intersection operator and vice versa. $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$ $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$.
DeMorgan's[a] Law	The complement of $S \cup T$ is given by $(S \cup T)^c = S^c \cap T^c$ The complement of $S \cap T$ is given by $(S \cap T)^c = S^c \cup T^c$

[a]De Morgan's law is named after Augustus De Morgan, a nineteenth century English mathematician who was a contemporary of George Boole

$$\Rightarrow (x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$$
$$\Rightarrow x \in (A \cap B) \vee x \in (A \cap C)$$
$$\Rightarrow x \in (A \cap B) \cup (A \cap C)$$

Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
Similarly $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
Therefore, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

2.2.3 Russell's Paradox

Bertrand Russell (Fig. 2.1) was a famous British logician, mathematician and philosopher. He was the co-author with Alfred Whitehead of *Principia Mathematica*, which aimed to derive all of the truths of mathematics from logic. Russell's Paradox was discovered by Bertrand Russell in 1901, and showed that the system of logicism being proposed by Frege (discussed in Chap. 14) contained a contradiction.

Question (Posed by Russell to Frege)
Is the set of all sets that do not contain themselves as members a set?

Russell's Paradox
Let $A = \{S \text{ a set and } S \notin S\}$. *Is* $A \in A$? *Then* $A \in A \Rightarrow A \notin A$ *and* vice versa. *Therefore, a contradiction arises in either case and there is no such set A.*

Two ways of avoiding the paradox were developed in 1908, and these were Russell's theory of types and Zermelo set theory. Russell's theory of types was a response to the paradox by arguing that the set of all sets is ill formed. Russell developed a hierarchy with individual elements the lowest level; sets of elements at the next level; sets of sets of elements at the next level; and so on. It is then prohibited for a set to contain members of different types.

A set of elements has a different type from its elements, and one cannot speak of the set of all sets that do not contain themselves as members as these are of different

Fig. 2.1 Bertrand russell

types. The other way of avoiding the paradox was Zermelo's axiomatization of set theory.

Remark Russell's paradox may also be illustrated by the story of a town that has exactly one barber who is male. *The barber shaves all and only those men in town who do not shave themselves*. The question is who shaves the barber.

If the barber does not shave himself then according to the rule he is shaved by the barber (i.e. himself). If he shaves himself then according to the rule he is not shaved by the barber (i.e. himself).

The paradox occurs due to self-reference in the statement and a logical examination shows that the statement is a contradiction.

2.2.4 Computer Representation of Sets

Sets are fundamental building blocks in mathematics, and so the question arises as to how a set is stored and manipulated in a computer. The representation of a set M on a computer requires a change from the normal view that the order of the elements of the set is irrelevant, and we will need to assume a definite order in the underlying universal set $\mathscr{M}$ from which the set M is defined.

That is, a set is always defined in a computer program with respect to an underlying universal set, and the elements in the universal set are listed in a definite order. Any set M arising in the program that is defined with respect to this universal set $\mathscr{M}$ is a subset of $\mathscr{M}$. Next, we show how the set M is stored internally on the computer.

The set M is represented in a computer as a string of binary digits $b_1 b_2 \ldots b_n$ where n is the cardinality of the universal set $\mathscr{M}$. The bits b_i (where i ranges over the values 1, 2, … n) are determined according to the rule

$b_i = 1$ if ith element of $\mathscr{M}$ is in M
$b_i = 0$ if ith element of $\mathscr{M}$ is not in M

For example, if $\mathscr{M} = \{1, 2, \ldots 10\}$ then the representation of $M = \{1, 2, 5, 8\}$ is given by the bit string 1100100100 where this is given by looking at each element of $\mathscr{M}$ in turn and writing down 1 if it is in M and 0 otherwise.

Similarly, the bit string 0100101100 represents the set $M = \{2, 5, 7, 8\}$, and this is determined by writing down the corresponding element in $\mathscr{M}$ that corresponds to a 1 in the bit string.

Clearly, there is a one-to-one correspondence between the subsets of $\mathscr{M}$ and all possible n-bit strings. Further, the set theoretical operations of set union, intersection and complement can be carried out directly with the bit strings (provided that the sets involved are defined with respect to the same universal set). This involves a bitwise 'or' operation for set union; a bitwise 'and' operation for set intersection; and a bitwise 'not' operation for the set complement operation.