Lecture 15: The Perceptron

COMP90049
Introduction to Machine Learning

Semester 1, 2023

Lea Frermann, CIS

Roadmap

So far... Naive Bayes and Logistic Regression

- Probabilistic models (Naive Bayes and Logistic Regression)
- Maximum likelihood estimation
- Examples and code

Today... The Perceptron

- Geometric motivation
- Error-based optimization
- ...towards neural networks

Recap: Classification algorithms

Naive Bayes

- Generative model of $p(x, y)$
- Find optimal parameter that maximize the log data likelihood
- Unrealistic independence assumption $p(x|y) = \prod_i p(x_i|y)$

Logistic Regression

- Discriminative model of $p(y|x)$
- Find optimal parameters that maximize the conditional log data likelihood
- Allows for more complex features (fewer assumptions)

Recap: Classification algorithms

Naive Bayes

- Generative model of $p(x, y)$
- Find optimal parameter that maximize the log data likelihood
- Unrealistic independence assumption $p(x|y) = \prod_i p(x_i|y)$

Logistic Regression

- Discriminative model of $p(y|x)$
- Find optimal parameters that maximize the conditional log data likelihood
- Allows for more complex features (fewer assumptions)

Perceptron

- Biological motivation: imitating neurons in the brain
- No more probabilities
- Instead: minimize the classification error directly

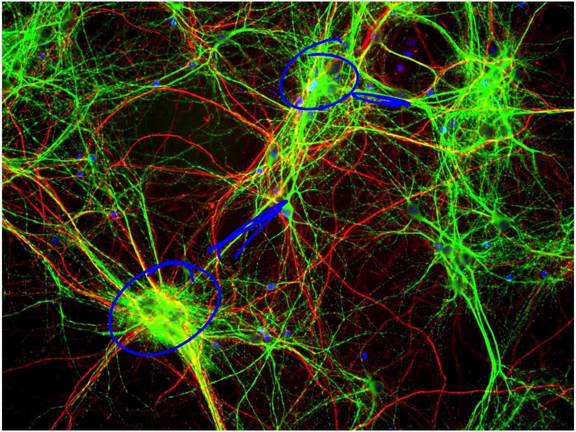
THE UNIVERSITY OF MELBOURNE

Introduction: Neurons in the Brain

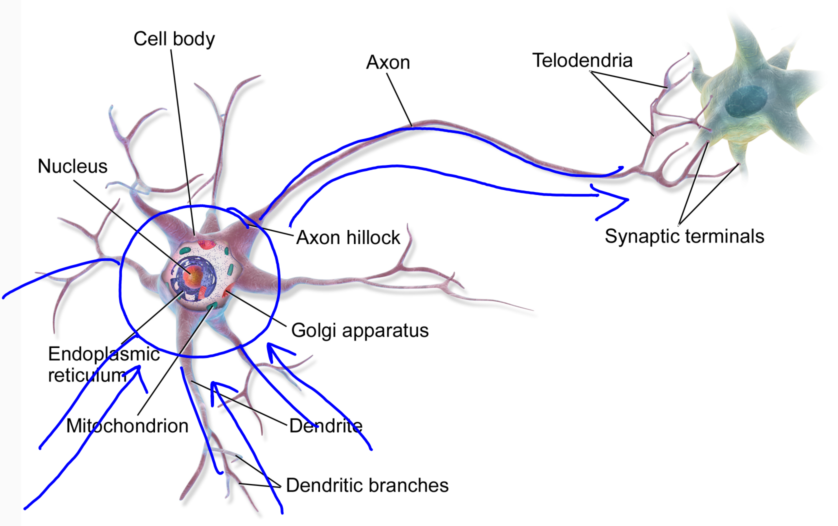
- Humans are the best learners we know
- Can we take inspiration from human learning
- $\rightarrow$ the brain!

Introduction: Neurons in the Brain

https://vimeo.com/227026686



Introduction: Neurons in the Brain



Source: `https://upload.wikimedia.org/wikipedia/commons/1/10/Blausen_0657_MultipolarNeuron.png`

Introduction: Neurons in the Brain

The hype

- 1943 McCulloch and Pitts introduced the first 'artificial neurons'
- If the **weighted sum of inputs** is equal to or greater than a **threshold**, then the **output** is 1. Otherwise the output is 0.
- the **weights** needed to be designed by hand

- In 1958 Rosenblatt invented the **Perceptron**, which can learn the optimal parameters through the **perceptron learning rule**
- The perceptron can be **trained** to learn the correct weights, even if randomly initialized [[for a limited set of problems]].


NEW NAVY DEVICE LEARNS BY DOING

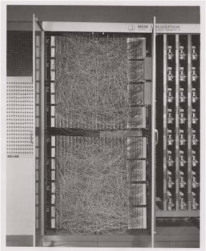
Psychologist Shows Embryo of Computer Designed to Read and Grow Wiser

WASHINGTON, July 7 (UPI)—The Navy revealed the embryo of an electronic computer today that it expects will be able to walk, talk, see, write, reproduce itself and be conscious of its existence.

The embryo—the Weather Bureau's $2,000,000 "704" computer—learned to differentiate between right and left after fifty attempts in the Navy's demonstration for newsmen.

The service said it would use this principle to build the first of its Perceptron thinking machines that will be able to read and write. It is expected to be finished in about a year at a cost of $100,000.

The New York Times, July 8 1958

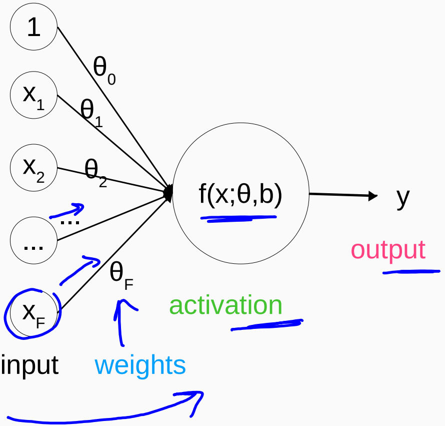


The AI winter

- A few years later Misky and Papert (too?) successfully pointed out the fundamental limitations of the perceptron.
- As a result, research on artificial neural networks stopped until the mid-1980s
- But the limitations can be overcome by combining multiple perceptrons into **Artificial Neural Networks**
- The perceptron is the basic component of today's deep learning success!

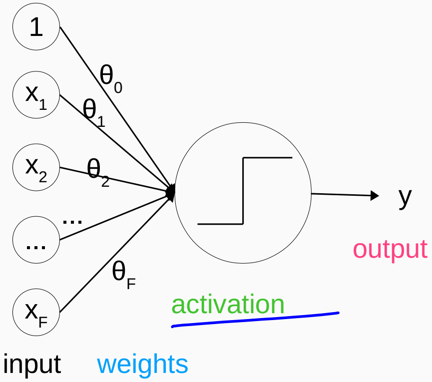
THE UNIVERSITY OF MELBOURNE

Introduction: Artificial Neurons I



$$y = f(\theta^T x)$$

Introduction: Artificial Neurons I



$y = f(\theta x + b)$

$$f: \begin{cases} y = 1 & \text{if} \quad f(\theta^T x) >= 0 \\ y = -1 & \text{if} \quad f(\theta^T x) < \;\; 0 \end{cases}$$

Perceptron: Definition I

- The Perceptron is a **minimal neural network**
- **neural networks** are composed of **neurons**
- A neuron is defined as follows:
    - input = a vector x of numeric inputs $(\langle 1, x_1, x_2, ...x_n \rangle)$
    - output = a scalar $y_i \in \mathbb{R}$
    - hyper-parameter: an **activation function** f
    - parameters: $\theta = \langle \theta_0, \theta_1, \theta_2, ...\theta_n \rangle$
- Mathematically:

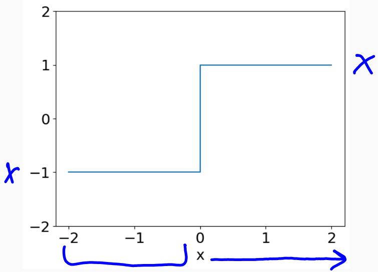
$$y^i = f\left(\left[\sum_j \theta_j x_j^i\right]\right) = f(\theta^T x^i)$$

Perceptron: Definition II

- Task: binary classification of instances into classes 1 and -1
- Model: a single-neuron (aka a "perceptron") :

$$f(\theta^T x) = \begin{cases} 1 & \text{if } \theta^T x \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

- $\theta^T x$ is the decision boundary
- Graphically, f is the **step function**

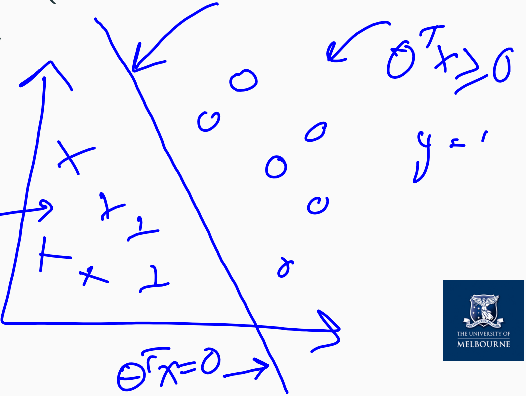


Perceptron: Definition II

- Task: binary classification of instances into classes 1 and -1
- Model: a single-neuron (aka a "perceptron") :

$$f(\theta^T x) = \begin{cases} 1 & \text{if } \theta^T x \geq 0 \\ -1 & \text{otherwise} \end{cases}$$

- $\theta^T x$ is the decision boundary
- Example: 2-d case:



Towards the Perceptron Algorithm I

- As usual, **learning** means to modify the **parameters** (i.e., weights) of the perceptron so that performance is **optimized**
- The perceptron is a **supervised** classification algorithm, so we learn from observations of input-label pairs

$$(x^1, y^1), (x^2, y^2), ...(x^N, y^N)$$

- Simplest way to learn: compare predicted outputs $\hat{y}$ against true outputs y and minimize the number of mis-classifications. Unfortunately, mathematically inconvenient.
- Second simplest idea: Find θ such that gap between the predicted value $\hat{y}^i \leftarrow f(\theta^T x^i)$ and the true class label $y \in \{-1, 1\}$ is minimized

Intuition Iterate over the **training data** and modify weights:

if $y = 1$	and $\hat{y} = 1$	then **do nothing**
if $y = -1$	and $\hat{y} = -1$	then **do nothing**
if $y = 1$	but $\hat{y} = -1$	then **increase** weights
if $y = -1$	but $\hat{y} = 1$	then **decrease** weights

$\theta^T x > 0 \; : f() \rightarrow 1$

THE UNIVERSITY OF MELBOURNE

Intuition Iterate over the **training data** and modify weights:

if $y = 1$	and $\hat{y} = 1$	then **do nothing**
if $y = -1$	and $\hat{y} = -1$	then **do nothing**
if $y = 1$	but $\hat{y} = -1$	then **increase** weights
if $y = -1$	but $\hat{y} = 1$	then **decrease** weights

More formally

Initialize parameters $\theta \leftarrow 0$
for training sample (x, y) **do**
  Calculate the output $\hat{y} = f(\theta^T x)$
  if $y = 1$ and $\hat{y} = -1$ **then**
    $\theta^{(new)} \leftarrow \theta^{(old)} \boxed{+ x}$
  if $y = -1$ and $\hat{y} = 1$ **then**
    $\theta^{(new)} \leftarrow \theta^{(old)} - x$
until tired

$x \; (x_1 \ldots x_n)$

Towards the Perceptron Algorithm II

We can summarize our algorithm into a single learning rule

$$\boxed{\theta_j \leftarrow \theta_j + \eta(y^i - \hat{y}^i)x_j^i}$$

- i iterates over examples (inputs)
- j iterates over dimensions (features) per input

$y = \hat{y}$

- We note that

$$(y^i - \hat{y}^i) = \begin{cases} 0 \text{ if } y^i == \hat{y}^i \\ 2 \text{ if } \underline{y^i = 1 and = \hat{y}^i = -1} \\ -2 \text{ if } \underline{y^i = -1 and = \hat{y}^i = 1} \end{cases} \quad (1)$$

- We set a **learning rate** or **step size** η

$D = \{(\mathbf{x}^i, y^i) | i = 1, 2, ..., N\}$ the set of training instances
Initialise the weight vector $\theta \leftarrow 0$
t $\leftarrow$ 0
repeat
  t $\leftarrow$ t+1
  for each training instance $(x^i, y^i) \in D$ **do**
    compute $\hat{y}^{i,(t)} = f(\theta^T x^i)$
    if $\hat{y}^{i,(t)} \neq y^i$ **then**
      for each each weight θ_j **do**
        update $\boxed{\theta_j^{(t)} \leftarrow \theta_j^{(t-1)} + \eta(y^i - \hat{y}^{i,(t)})x_j^i}$
    else
      $\theta_j^{(t)} \leftarrow \theta_j^{(t-1)}$
until tired
Return $\theta^{(t)}$

An example

Perceptron Example I

Training instances

$\langle x_1^i, x_2^i \rangle$	y^i
$\langle 1, 1, 1 \rangle$	1
$\langle 1, 1, 2 \rangle$	1
$\langle 1, 0, 0 \rangle$	-1
$\langle 1, -1, 0 \rangle$	-1

- Learning rate $\eta = 1$
- Initial weights $\theta^{(0)} = \langle 0, 0, 0 \rangle$
- Prediction rule: $\hat{y} = 1$ if $\theta^T x \geq 0$ else -1.
- Update rule: $\theta_j \leftarrow \theta_j + \eta(y^i - \hat{y}^i)x_j^i$

$$\theta_3 \leftarrow 0 + 1(-1-1)0 = 0$$

Epoch 1:

$\langle x_1, x_2 \rangle$	$\theta_1 \cdot 1 + \theta_2 \cdot x_1^i + \theta_3 \cdot x_2^i$	$\hat{y}^{i,(1)}$	y^i
$\langle 1, 1, 1 \rangle$	$0 \times 1 + 0 \times 1 + 0 \times 1 =$	1	$= 1$
$\langle 1, 1, 2 \rangle$	$0 \times 1 + 0 \times 1 + 0 \times 2 =$	1	1
$\langle 1, 0, 0 \rangle$	$0 \times 1 + 0 \times 0 + 0 \times 0 =$	$\boxed{1}$	$\boxed{-1}$
$\langle 1, -1, 0 \rangle$	$-2 \cdot 1 + 0 \cdot -1 + 0 \times 0 =$	$\boxed{-1}$	$\boxed{-1}$

$$\theta^{new} = \boxed{\langle -2, 0, 0 \rangle}$$

We add a bias term.

Perceptron Example III

- $\theta = \langle -2, 0, 0 \rangle$
- learning rate: $\eta = 1$
- Epoch 2:

$\langle x_1, x_2 \rangle$	$\theta_1 \cdot 1 + \theta_2 \cdot x_1^i + \theta_3 \cdot x_2^i$	$\hat{y}^{i,(1)}$	y^i
$\langle 1, 1 \rangle$	$-2 + 1 \times 0 + 1 \times 0 = -2$	-1	1
	Update to $\theta = \langle 0, 2, 2 \rangle$		
$\langle 1, 2 \rangle$	$0 + 1 \times 2 + 2 \times 2 = 6$	1	1
$\langle 0, 0 \rangle$	$0 + 0 \times 2 + 0 \times 2 = 0$	1	-1
	Update to $\theta = \langle -2, 2, 2 \rangle$		
$\langle -1, 0 \rangle$	$-2 + -1 \times 2 + 0 \times 2 = -4$	-1	-1

Perceptron Example IV

- $\theta = \langle -2, 2, 2 \rangle$
- learning rate: $\eta = 1$
- Epoch 3:

$\langle x_1, x_2 \rangle$	$\theta_1 \cdot 1 + \theta_2 \cdot x_1^i + \theta_3 \cdot x_2^i$	$\hat{y}^{i,(1)}$	y^i
$\langle 1, 1 \rangle$	$-2 + 1 \times 2 + 1 \times 2 = 2$	1	1
$\langle 1, 2 \rangle$	$-2 + 1 \times 2 + 2 \times 2 = 4$	1	1
$\langle 0, 0 \rangle$	$-2 + 0 \times 2 + 0 \times 2 = -2$	-1	-1
$\langle -1, 0 \rangle$	$-2 + -1 \times 2 + 0 \times 2 = -4$	-1	-1

We have finished training, because our model has **converged**. There were **no parameter updates for a full epoch**.

Perceptron Convergence

Perceptron Rule:

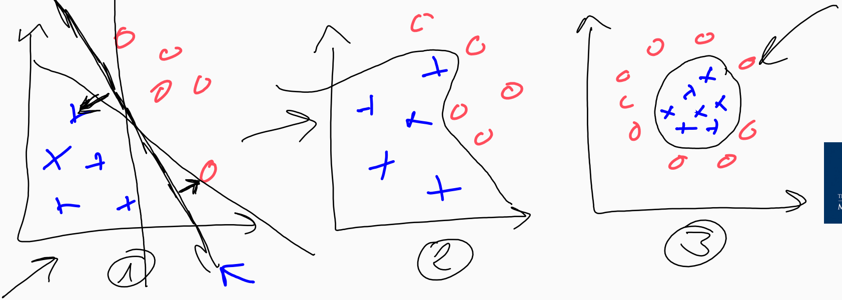
$$\theta_j^{(t+1)} \leftarrow \theta_j^{(t)} + \eta(y_i - \hat{y}^i)x_j^i$$

- So, all we're doing is adding and subtracting constants every time we make a mistake.
- Does this really work!?

Perceptron Convergence

Perceptron Convergence

- The Perceptron algorithm is guaranteed to **converge** for linearly-separable data
    - the convergence point will depend on the initialisation
    - the convergence point will depend on the learning rate
    - (no guarantee of the margin being maximised)
- No guarantee of convergence over non-linearly separable data



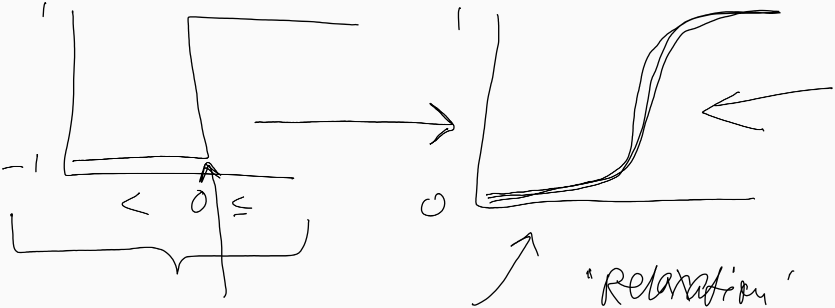
Back to Logistic Regression and Gradient Descent

Perceptron Rule

$$\theta_j^{(t+1)} \leftarrow \theta_j^{(t)} + \eta(y_i - \hat{y}^i)x_j^i$$

Gradient Descent

$$\theta^{(t+1)} \leftarrow \theta^{(t)} - \eta \frac{\partial f}{\partial \theta^{(t)}}$$



Back to Logistic Regression and Gradient Descent

Perceptron Rule

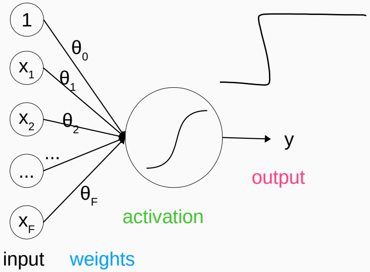
$$\theta_j^{(t+1)} \leftarrow \theta_j^{(t)} + \eta(y_i - \hat{y}^i)x_j^i$$

Gradient Descent

$$\theta^{(t+1)} \leftarrow \theta^{(t)} - \eta \frac{\partial f}{\partial \theta^{(t)}}$$

Activation Functions

A single ‘neuron’ with a **sigmoid activation** which optimizes the **cross-entropy** loss (negative log likelihood) is equivalent to **logistic regression**



f: y = 1 / (1 + exp(-θ^Tx))

Online learning vs. Batch learning

- The perceptron algorithm is an **online algorithm**: update weights after **each** training example
- In contrast, Naive Bayes and logistic regression (with Gradient Descent) are **batch** algorithms:
  - compute statistics of the *whole* training data set
  - update all parameters at once
- Online learning can be more efficient for large data sets
- Gradient Descent can be converted into an online version: **stochastic gradient descent**

Multi-Class Perceptron

We can generalize the perceptron to more than 2 classes

- create a weight vector for each class $k \in Y$, θ^k
- score input wrt each class: $\theta_k^T x$ for all k
- predict the class with maximum output $\hat{y} = \text{argmax}_{k \in Y}\, \theta_k^T x$
- learning works as before: if for some (x^i, y^i) we make a wrong prediction $\hat{y}^i \neq y^i$ such that $\theta_{y^i}^T x^i < \theta_{\hat{y}^i}^T x^i$,

$$
\begin{aligned}
\theta_{y^i} &\leftarrow \theta_{y^i} + \eta x^i && \text{move towards predicting } y^i \text{ for } x^i \\
\theta_{\hat{y}^i} &\leftarrow \theta_{\hat{y}^i} - \eta x^i && \text{move away from predicting } \hat{y}^i \text{ for } x^i
\end{aligned}
$$

Summary

This lecture: The Perceptron

- Biological motivation
- Error-based classifier
- The Perceptron Rule
- Relation to Logistic Regression
- Multi-class perceptron

Next

- More powerful machine learning through combining perceptrons
- More on activation functions
- Learning with backpropagation

References

- Rosenblatt, Frank. ”The perceptron: a probabilistic model for information storage and organization in the brain.” Psychological review 65.6 (1958): 386.
- Minsky, Marvin, and Seymour Papert. ”Perceptrons: An essay in computational geometry.” MIT Press. (1969).
- Bishop, Christopher M. Pattern recognition and machine learning. Springer, 2006. Chapter 4.1.7

THE UNIVERSITY OF MELBOURNE