



Algorithms: COMP3121/9101

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DYNAMIC PROGRAMMING

Dynamic Programming

- **The main idea of Dynamic Programming:** build an optimal solution to the problem from optimal solutions for (carefully chosen) smaller size subproblems.
- Subproblems are chosen in a way which allows recursive construction of optimal solutions to such subproblems from optimal solutions to smaller size subproblems.
- Efficiency of DP comes from the fact that the sets of subproblems needed to solve larger problems heavily overlap; each subproblem is solved only once and its solution is stored in a table for multiple use for solving larger problems.

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- Efficiency of DP comes from the fact that the sets of subproblems needed to solve larger problems heavily overlap; each subproblem is solved only once and its solution is stored in a table for multiple use for solving larger problems.

Dynamic Programming: Activity Selection

- **Instance:** A list of activities a_i , $1 \leq i \leq n$ with starting times s_i and finishing times f_i . No two activities can take place simultaneously.
- **Task:** Find a subset of compatible activities of **maximal total duration**.
- Remember, we used the Greedy Method to solve a somewhat similar problem of finding a subset with the **largest possible number** of compatible activities, but the Greedy Method **does not** work for the present problem.
- We start by sorting these activities by their finishing time into a non-decreasing sequence, so will assume that $f_1 \leq f_2 \leq \dots \leq f_n$.
- For every $i \leq n$ we solve the following subproblems:
Subproblem $P(i)$: find a subsequence σ_i of the sequence of activities $S_i = \langle a_1, a_2, \dots, a_i \rangle$ such that:
 - 1 σ_i consists of non-overlapping activities;
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 - 3 σ_i is of maximal total duration among all subsequences of S_i which satisfy 1 and 2.
- Note: the role of Condition 2 is to simplify recursion.

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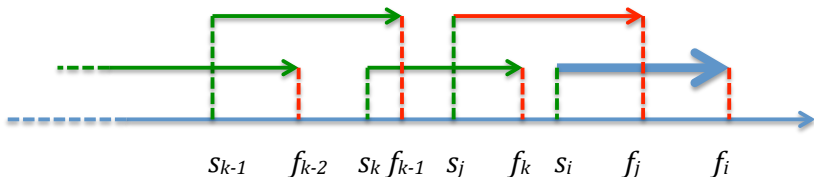
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- Let $T(i)$ be the total duration of the optimal solution $S(i)$ of the subproblem $P(i)$.
- For $S(1)$ we choose a_1 ; thus $T(1) = f_1 - s_1$;
- Recursion:** assuming that we have solved subproblems for all $j < i$ and stored them in a table, we let

$$T(i) = \max\{T(j) : j < i \text{ \& } f_j \leq s_i\} + f_i - s_i$$



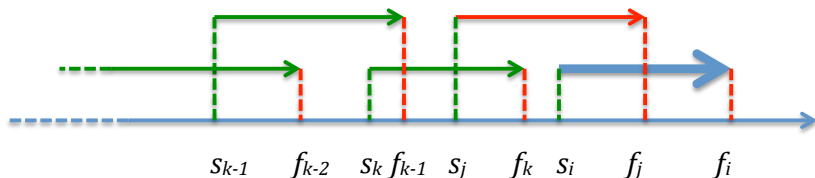
- In the table, for every i , besides $T(i)$, we also store $\pi(i) = j$ for which the above max is achieved:

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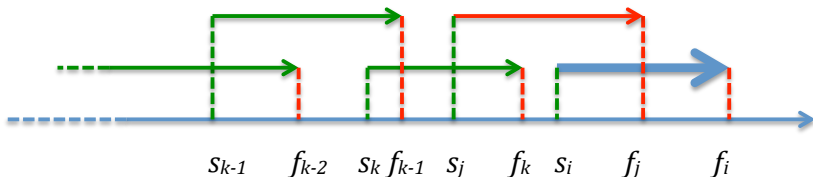
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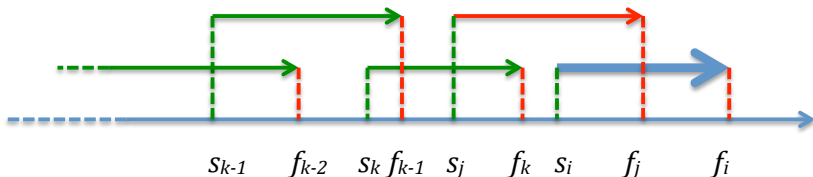
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- Why does such a recursion produce optimal solutions to subproblems $P(i)$?
- Let the optimal solution of subproblem $P(i)$ be the sequence $S = (a_{k_1}, a_{k_2}, \dots, a_{k_{m-1}}, a_{k_m})$ where $k_m = i$;
- We claim: the truncated subsequence $S' = (a_{k_1}, a_{k_2}, \dots, a_{k_{m-1}})$ is an optimal solution to subproblem $P(k_{m-1})$, where $k_{m-1} < i$.
- Why? We apply the same “cut and paste” argument which we used to prove the optimality of the greedy solutions!
- If there were a sequence S^* of a larger total duration than the duration of sequence S' and also ending with activity $a_{k_{m-1}}$, we could obtain a sequence \hat{S} by extending the sequence S^* with activity a_{k_m} and obtain a solution for subproblem $P(i)$ with a longer total duration than the total duration of sequence S , contradicting the optimality of S .
- Thus, the optimal solution $S = (a_{k_1}, a_{k_2}, \dots, a_{k_{m-1}}, a_{k_m})$ for problem $P(i)$ ($= P(a_{k_m})$) is obtained from the optimal solution $S' = (a_{k_1}, a_{k_2}, \dots, a_{k_{m-1}})$ for problem $P(a_{k_{m-1}})$ by extending it with a_{k_m}

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- Continuing with the solution of the problem, we now let

$$T_{max} = \max\{T(i) : i \leq n\};$$

$$\text{last} = \arg \max\{T(i) : i \leq n\}.$$

- We can now reconstruct the optimal sequence which solves our problem from the table of partial solutions, because in the i^{th} slot of the table, besides $T(i)$, we also store $\pi(i) = j$, ($j < i$) such that the optimal solution of $P(i)$ extends the optimal solution of subproblem $P(j)$.
- Thus, the sequence in the reverse order is given by $\text{last}, \pi(\text{last}), \pi(\pi(\text{last})), \dots$
- Why is such solution optimal, i.e., why looking for optimal solutions of $P(i)$ which must end with a_i did not cause us to miss the optimal solution without such an additional requirement?
- Consider the optimal solution without such additional requirement, and assume it ends with activity a_k ; then it would have been obtained as the optimal solution of problem $P(k)$.
- Time complexity: having sorted the activities by their finishing times in time $O(n \log n)$, we need to solve n subproblems $P(i)$ for solutions ending in a_i ; for each such interval a_i we have to find all preceding compatible intervals and their optimal solutions (to be looked up in a table). Thus, $T(n) = O(n^2)$.

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- Thus, the sequence in the reverse order is given by $\text{last}, \pi(\text{last}), \pi(\pi(\text{last})), \dots$
- Why is such solution optimal, i.e., why looking for optimal solutions of $P(i)$ which must end with a_i did not cause us to miss the optimal solution without such an additional requirement?
- Consider the optimal solution without such additional requirement, and assume it ends with activity a_k ; then it would have been obtained as the optimal solution of problem $P(k)$.
- Time complexity: having sorted the activities by their finishing times in time $O(n \log n)$, we need to solve n subproblems $P(i)$ for solutions ending in a_i ; for each such interval a_i we have to find all preceding compatible intervals and their optimal solutions (to be looked up in a table). Thus, $T(n) = O(n^2)$.

Dynamic Programming: Activity Selection

- Continuing with the solution of the problem, we now let

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More Dynamic Programming Problems

- **Longest Increasing Subsequence:** Given a sequence of n real numbers $A[1..n]$, determine a subsequence (not necessarily contiguous) of maximum length in which the values in the subsequence are strictly increasing.
- Solution: For each $i \leq n$ we solve the following subproblems:
- *Subproblem $P(i)$: Find a subsequence of the sequence $A[1..i]$ of maximum length in which the values are strictly increasing and which ends with $A[i]$.*
- Recursion: Assume we have solved the subproblems for all $j < i$, and that we have put in a table S the values $S[j] = \ell_j$ which are the lengths ℓ_j of maximal increasing sequences which end with $A[j]$
- We now look for all $A[m]$ such that $m < i$ and such that $A[m] < A[i]$.
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- Again, the condition for the sequence to end with $A[i]$ is not restrictive because if the optimal solution ends with some $A[m]$, it would have been constructed as the solution for $P(m)$.
- Time complexity: $O(n^2)$.
- Exercise: (somewhat tough, but very useful) Design an algorithm for solving this problem which runs in time $n \log n$.

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- **Making Change.** You are given n types of coin denominations of values $v(1) < v(2) < \dots < v(n)$ (all integers). Assume $v(1) = 1$, so that you can always make change for any integer amount. Give an algorithm which makes change for any given integer amount C with as few coins as possible, assuming that you have an unlimited supply of coins of each denomination.
- Solution: DP Recursion on the amount C . We will fill a table containing C many slots, so that an optimal solution for an amount i is stored in slot i .
- If $C = 1$ the solution is trivial: just use one coin of denomination $v(1) = 1$;
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- We consider optimal solutions $opt(i - v(k))$ for every amount of the form $i - v(k)$, where k ranges from 1 to n . (Recall $v(1), \dots, v(n)$ are all of the available denominations.)
- Among all of these optimal solutions (which we find in the table we are constructing recursively!) we pick one which uses the fewest number of coins, say this is $opt(i - v(m))$ for some m , $1 \leq m \leq n$.
- We obtain an optimal solution $opt(i)$ for amount i by adding to $opt(i - v(m))$ one coin of denomination $v(m)$.

$$opt(i) = \min\{opt(i - v(k)) : 1 \leq k \leq n\} + 1$$

- Why does this produce an optimal solution for amount $i \leq C$?
- Consider an optimal solution for amount $i \leq C$; and say such solution includes at least one coin of denomination $v(m)$ for some $1 \leq m \leq n$. But then removing such a coin must produce an optimal solution for the amount $i - v(m)$ again by our cut-and-paste argument.

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- However, we do not know which coins the optimal solution includes, so we try all the available coins and then pick m for which the optimal solution for amount $i - v(m)$ uses the fewest number of coins.
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- $opt(C)$ is the solution we need.
- Time complexity of our algorithm is nC .
- **Note:** Our algorithm is **NOT** a polynomial time algorithm in the **length** of the input, because the length of a representation of C is only $\log C$, while the running time is nC .
- But this is the best what we can do...

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- **Note:** Our algorithm is **NOT** a polynomial time algorithm in the **length** of the input, because the length of a representation of C is only $\log C$, while the running time is nC .
- But this is the best what we can do...

More Dynamic Programming Problems

- However, we do not know which coins the optimal solution includes, so we try all the available coins and then pick m for which the optimal solution for amount $i - v(m)$ uses the fewest number of coins.
- It is enough to store in the i^{th} slot of the table such m and $opt(i)$ because this allows us to reconstruct the optimal solution by looking at m_1 stored in the i^{th} slot, then look at m_2 stored in the slot $i - v(m_1)$, then look at m_2 stored in the slot $i - v(m_1) - v(m_2)$, etc.
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Integer Knapsack Problem (Duplicate Items Allowed) You have n types of items; all items of kind i are identical and of weight w_i and value v_i . You also have a knapsack of capacity C . Choose a combination of available items which all fit in the knapsack and whose value is as large as possible. You can take any number of items of each kind.

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- Chose the one for which $opt(i - w_m) + v_m$ is the largest;
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$$\text{opt}(i) = \max\{\text{opt}(i - w_m) + v_m : 1 \leq m \leq n\}.$$

$$\pi(i) = \arg \max\{\text{opt}(i - w_m) + v_m : 1 \leq m \leq n\}.$$

- After C many steps we obtain the optimal (minimal) number of coins $\text{opt}(C)$.
- Which coins are present in the optimal solution can again be obtained by backtracking: if $\pi(C) = k$ then the first object is a_k of weight w_k and value v_k ; if $\pi(C - w_k) = m$ then the second object is a_m and so on.
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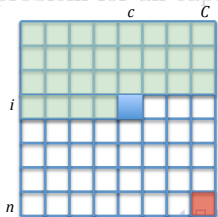
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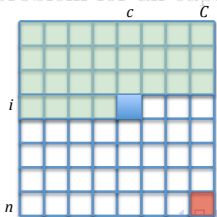


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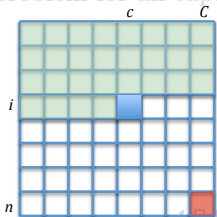


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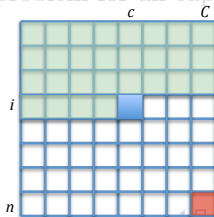


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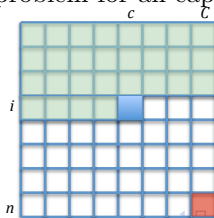


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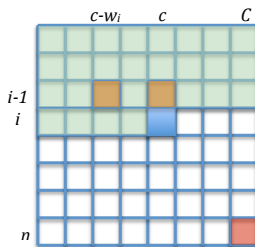
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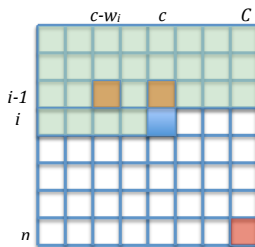
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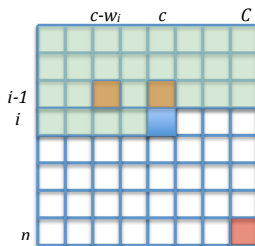
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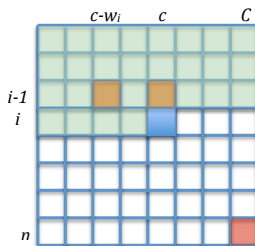
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- **Balanced Partition** You have a set of n integers. Partition these integers into two subsets such that you minimise $|S_1 - S_2|$, where S_1 and S_2 denote the sums of the elements in each of the two subsets.
- **Solution:** Let S be the total sum of all integers in the set; consider the Knapsack problem (with duplicate items not allowed) with the knapsack of size $S/2$ and with each integer x_i of both size and value equal to x_i .
- **Claim:** the best packing of such knapsack produces optimally balanced partition, with S_1 being all the integers in the knapsack and S_2 all the integers left out of the knapsack.
- Why? Since $S = S_1 + S_2$ we obtain

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- Thus, minimising $S/2 - S_1$ will minimise $S_2 - S_1$.
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$$\frac{S}{2} - S_1 = \frac{S_1 + S_2}{2} - S_1 = \frac{S_2 - S_1}{2}$$

i.e. $S_2 - S_1 = 2(S/2 - S_1)$.

- Thus, minimising $S/2 - S_1$ will minimise $S_2 - S_1$.
- So, all we have to do is find a subset of these numbers with the largest possible total sum which fits inside a knapsack of size $S/2$.

More Dynamic Programming Problems

- **Balanced Partition** You have a set of n integers. Partition these integers into two subsets such that you minimise $|S_1 - S_2|$, where S_1 and S_2 denote the sums of the elements in each of the two subsets.
- **Solution:** Let S be the total sum of all integers in the set; consider the Knapsack problem (with duplicate items not allowed) with the knapsack of size $S/2$ and with each integer x_i of both size and value equal to x_i .
- **Claim:** the best packing of such knapsack produces optimally balanced partition, with S_1 being all the integers in the knapsack and S_2 all the integers left out of the knapsack.

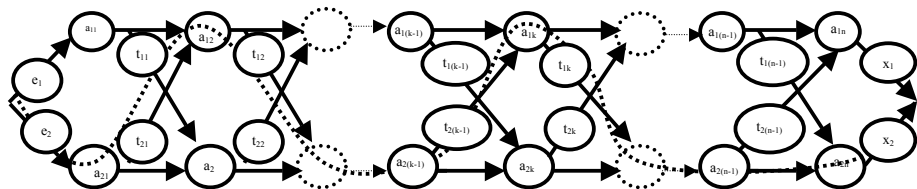
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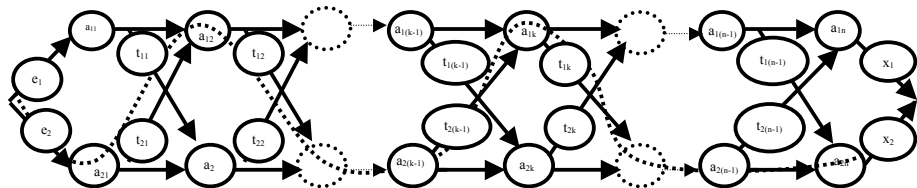
Dynamic Programming: Assembly line scheduling



Instance: Two assembly lines with workstations for n jobs.

- On the first assembly line the k^{th} job takes $a_{1,k}$ ($1 \leq k \leq n$) units of time to complete; on the second assembly line the same job takes $a_{2,k}$ units of time.
- To move the product from station $k - 1$ on the first assembly line to station k on the second line it takes $t_{1,k-1}$ units of time.
- Likewise, to move the product from station $k - 1$ on the second assembly line to station k on the first assembly line it takes $t_{2,k-1}$ units of time.

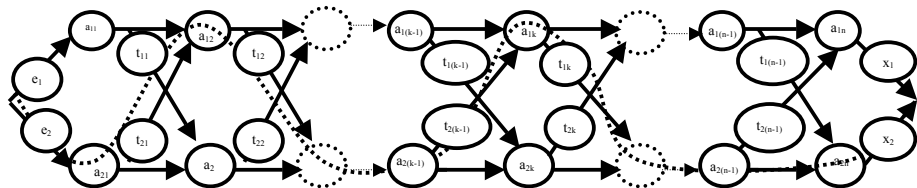
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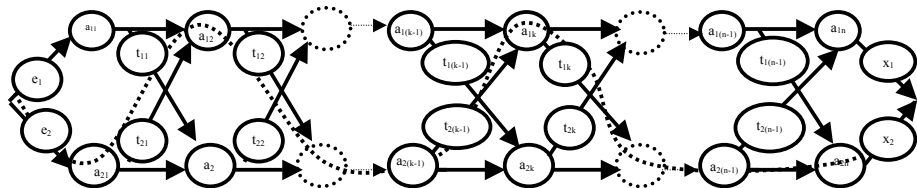
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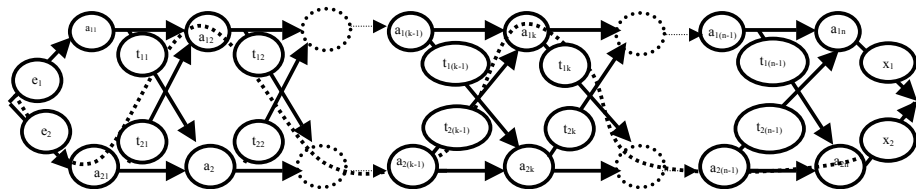
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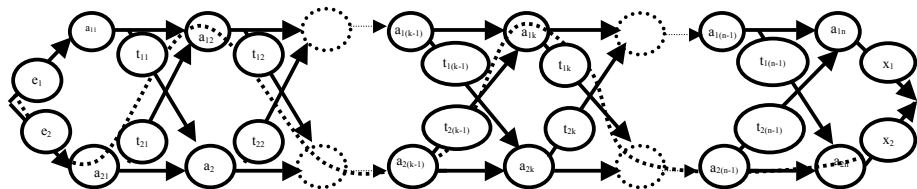
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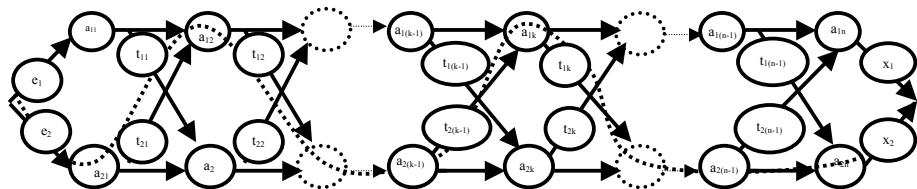
- To bring an unfinished product to the first assembly line it takes e_1 units of time.
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- To get a finished product from the first assembly line to the warehouse it takes x_1 units of time;
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- **Task:** Find a *fastest way* to assemble a product using both lines as necessary.

Dynamic Programming: Assembly line scheduling



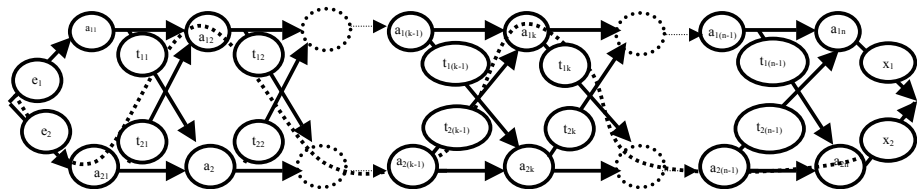
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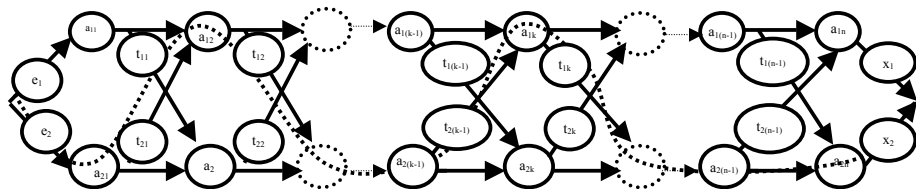
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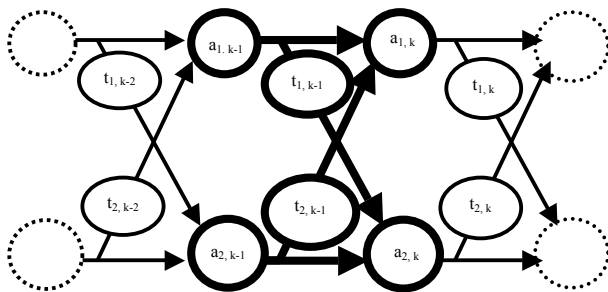
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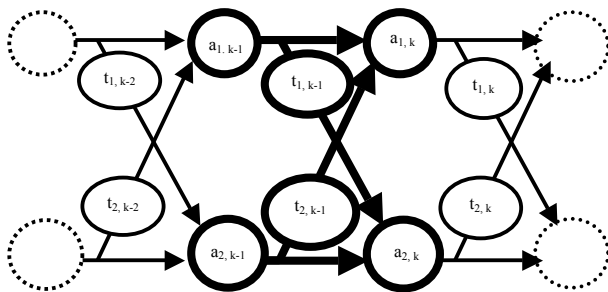
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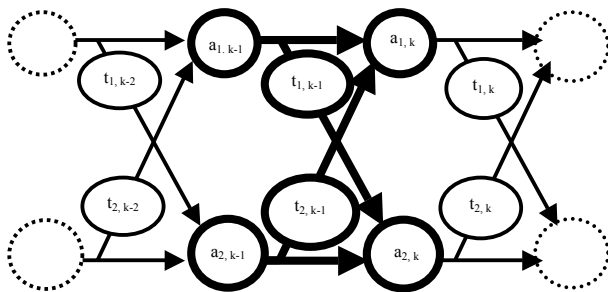
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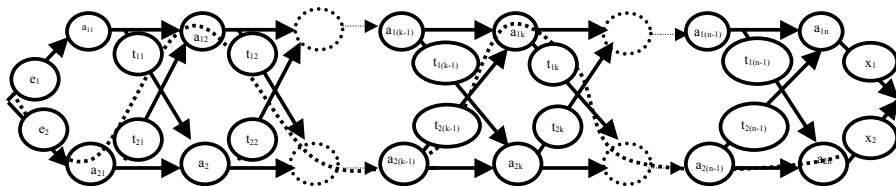
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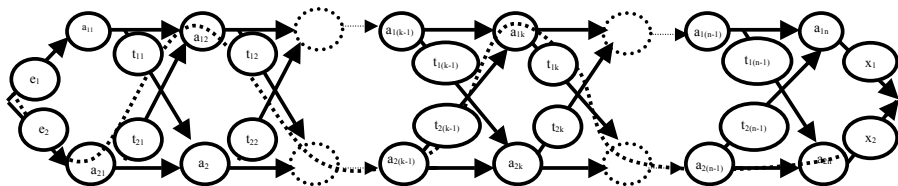
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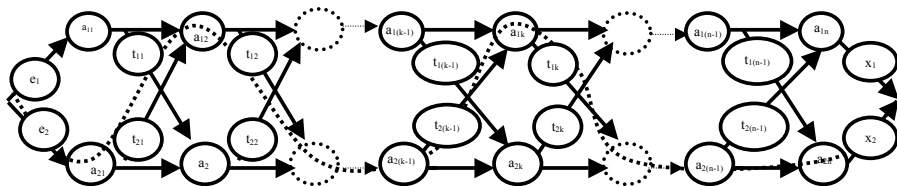
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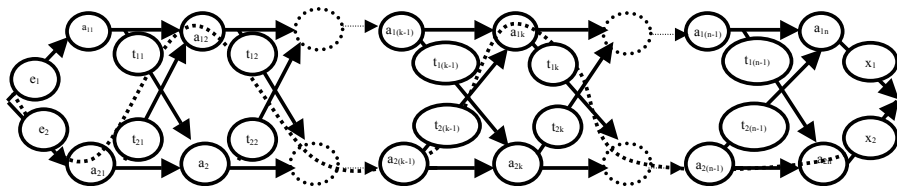
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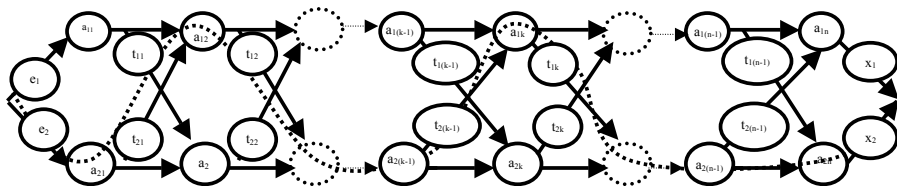
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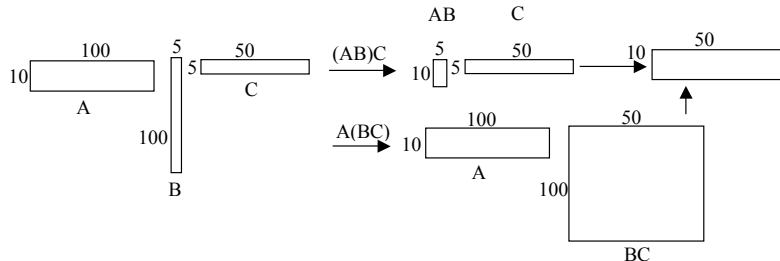
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Dynamic Programming: Matrix chain multiplication

- For any three matrices of compatible sizes we have $A(BC) = (AB)C$.
- However, the number of real number multiplications needed to perform in order to obtain the matrix product can be very different:

$$A = 10 \times 100; \quad B = 100 \times 5; \quad C = 5 \times 50$$

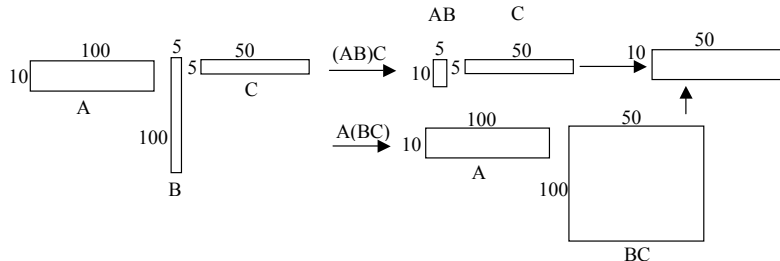


- To evaluate $(AB)C$ we need $(10 \times 5) \times 100 + (10 \times 50) \times 5 = 5000 + 2500 = 7500$ multiplications;
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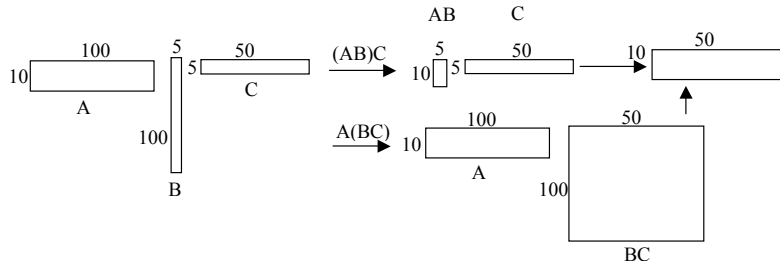


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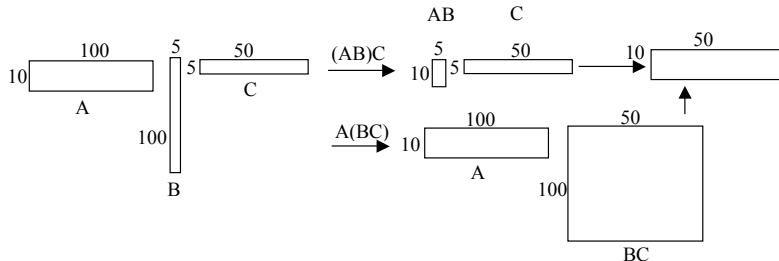


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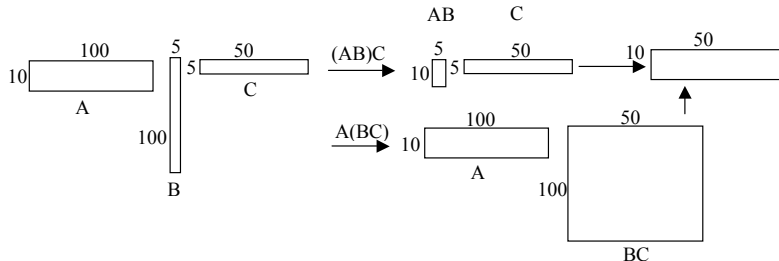


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- **Task:** Group them in such a way as to minimise the total number of multiplications needed to find the product matrix.
- The total number of different distributions of brackets is equal to the number of binary trees with n leaves.
- The total number of different distributions of brackets satisfies the following recursion (why?):

$$T(n) = \sum_{i=1}^{n-1} T(i)T(n-i)$$

- One can show that the solution satisfies $T(n) = \Omega(2^n)$.
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- **Task:** Group them in such a way as to minimise the total number of multiplications needed to find the product matrix.
- The subproblems $P(i, j)$ to be considered are:

“group matrices $A_i A_{i+1} \dots A_{j-1} A_j$ in such a way as to minimise the total number of multiplications needed to find the product matrix”.
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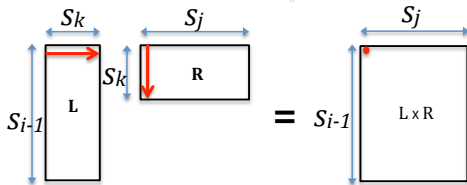
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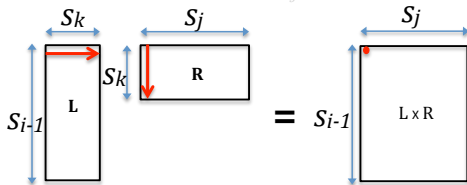
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- To multiply an $s_{i-1} \times s_k$ matrix L and an $s_k \times s_j$ matrix R it takes $s_{i-1} s_k s_j$ many multiplications:



Total number of multiplications: $s_{i-1} s_j s_k$

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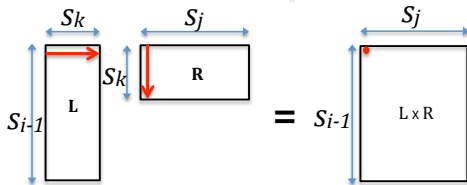
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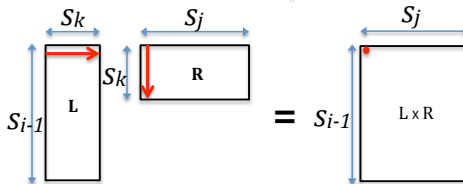
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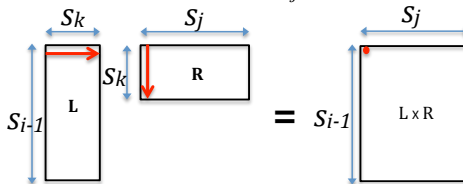
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- The recursion:

$$m(i, j) = \min\{m(i, k) + m(k + 1, j) + s_{i-1}s_js_k : i \leq k \leq j - 1\}$$

- Note that the recursion step is a brute force search but the whole algorithm is not, because all the subproblems are solved only once, and there are only $O(n^2)$ many such subproblems.
- k for which the minimum in the recursive definition of $m(i, j)$ is achieved can be stored to retrieve the optimal placement of brackets for the whole chain $A_1 \dots A_n$.
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- Assume we want to compare how similar two sequences of symbols S and S^* are.
- Example: how similar are the genetic codes of two viruses.
- This can tell us if one is just a genetic mutation of the other.
- A sequence s is a **subsequence** of another sequence S if s can be obtained by deleting some of the symbols of S (while preserving the order of the remaining symbols).
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Retrieving a longest common subsequence:

LCS-LENGTH(X, Y)

```
1   $m \leftarrow \text{length}[X]$ 
2   $n \leftarrow \text{length}[Y]$ 
3  for  $i \leftarrow 1$  to  $m$ 
4      do  $c[i, 0] \leftarrow 0$ 
5  for  $j \leftarrow 0$  to  $n$ 
6      do  $c[0, j] \leftarrow 0$ 
7  for  $i \leftarrow 1$  to  $m$ 
8      do for  $j \leftarrow 1$  to  $n$ 
9          do if  $x_i = y_j$ 
10             then  $c[i, j] \leftarrow c[i - 1, j - 1] + 1$ 
11                  $b[i, j] \leftarrow \nwarrow$ 
12             else if  $c[i - 1, j] \geq c[i, j - 1]$ 
13                 then  $c[i, j] \leftarrow c[i - 1, j]$ 
14                      $b[i, j] \leftarrow \uparrow$ 
15                 else  $c[i, j] \leftarrow c[i, j - 1]$ 
16                      $b[i, j] \leftarrow \leftarrow$ 
17  return  $c$  and  $b$ 
```

		j	0	1	2	3	4	5	6
i	y_j			B	D	C	A	B	A
		x_i							
0			0	0	0	0	0	0	0
1	A		0	0	0	0	1	←1	1
2	B		0	1	←1	←1	1	1	←2
3	C		0	1	1	1	2	←2	2
4	B		0	1	1	2	2	2	3
5	D		0	1	2	2	2	3	3
6	A		0	1	2	2	3	3	4
7	B		0	1	2	2	3	4	4

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- What if we have to find a longest common subsequence of three sequences S_1, S_2, S_3 ?
- Can we do $\text{LCS}(\text{LCS}(S_1, S_2), S_3)$?
- Not necessarily! Consider

$S_1 = ABCDEGG$

$\text{LCS}(S_1, S_2) = ABEG$

$S_2 = ACBEEFG$

$\text{LCS}(S_2, S_3) = ACEF$

$S_3 = ACCEDGF$

$\text{LCS}(S_1, S_3) = ACDG$

$\text{LCS}(\text{LCS}(S_1, S_2), S_3) = \text{LCS}(ABEG, ACCEDGF) = AEG$

$\text{LCS}(\text{LCS}(S_2, S_3), S_1) = \text{LCS}(ACEF, ABCDEGG) = ACE$

$\text{LCS}(\text{LCS}(S_1, S_3), S_2) = \text{LCS}(ACDG, ACBEEFG) = ACG$

But

$\text{LCS}(S_1, S_2, S_3) = ACEG$

- So how would you design an algorithm which computes correctly $\text{LCS}(S_1, S_2, S_3)$?

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- **Instance:** Two sequences $s = \langle a_1, a_2, \dots, a_n \rangle$ and $s^* = \langle b_1, b_2, \dots, b_m \rangle$.
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- **Solution:** Find the longest common subsequence $LCS(s, s^*)$ of s and s^* and then add differing elements of the two sequences at the right places, in any order; for example:

$s = abacada$

$s^* = xbycazd$

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Dynamic Programming: Edit Distance

- **Edit Distance** Given two text strings A of length n and B of length m , you want to transform A into B. You are allowed to insert a character, delete a character and to replace a character with another one. An insertion costs c_I , a deletion costs c_D and a replacement costs c_R .
- Task: find the lowest total cost transformation of A into B.
- Note: if all operations have a unit cost, then you are looking for the minimal number of such operations required to transform A into B; this number is called *the edit distance* between A and B.
- If the sequences are sequences of DNA bases and the costs reflect the probabilities of the corresponding mutations, then the minimal cost represents the probability that one sequence mutates into another sequence in the course of DNA copying.
- Subproblems: Let $C(i, j)$ be the minimum cost of transforming the sequence $A[1..i]$ into the sequence $B[1..j]$ for all $i \leq n$ and all $j \leq m$.

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Dynamic Programming: Maximizing an expression

- **Instance:** a sequence of numbers with operations $+$, $-$, \times in between, for example

$$1 + 2 - 3 \times 6 - 1 - 2 \times 3 - 5 \times 7 + 2 - 8 \times 9$$

- **Task:** Place brackets in a way that the resulting expression has the largest possible value.
- What will be the subproblems?
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- **Instance:** A directed weighted graph $G = (V, E)$ with weights which can be negative, but without cycles of negative total weight and a vertex $s \in V$.
- **Goal:** Find the shortest path from vertex s to every other vertex t .
- **Solution:** Since there are no negative weight cycles, the shortest path cannot contain cycles, because a cycle can be excised to produce a shorter path.
- Thus, every shortest path can have at most $|V| - 1$ edges.
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- Let us denote the length of the shortest path from s to v among all paths which contain at most i edges by $\text{opt}(i, v)$, and let $\text{pred}(i, v)$ be the immediate predecessor of vertex v on such shortest path.

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Dynamic Programming: Floyd Warshall algorithm

- Let again $G = (V, E)$ be a directed weighted graph where $V = \{v_1, v_2, \dots, v_n\}$ and where weights $w(e(v_p, v_q))$ of edges $e(v_p, v_q)$ can be negative, but there are no negative weight cycles.
- We can use a somewhat similar idea to obtain the shortest paths from **every** vertex v_p to **every** vertex v_q (including back to v_p).
- Let $\text{opt}(k, v_p, v_q)$ be the length of the shortest path from a vertex v_p to a vertex v_q such that all intermediate vertices are among vertices $\{v_1, v_2, \dots, v_k\}$, ($1 \leq k \leq n$).
- Then

$$\text{opt}(k, v_p, v_q) = \min\{\text{opt}(k-1, v_p, v_q), \text{opt}(k-1, v_p, v_k) + \text{opt}(k-1, v_k, v_q)\}$$

- Thus, we gradually **relax** the constraint that the intermediary vertices have to belong to $\{v_1, v_2, \dots, v_k\}$.
- Algorithm runs in time $|V|^3$.

Dynamic Programming: Floyd Warshall algorithm

- Let again $G = (V, E)$ be a directed weighted graph where $V = \{v_1, v_2, \dots, v_n\}$ and where weights $w(e(v_p, v_q))$ of edges $e(v_p, v_q)$ can be negative, but there are no negative weight cycles.
- We can use a somewhat similar idea to obtain the shortest paths from **every** vertex v_p to **every** vertex v_q (including back to v_p).
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Another example of relaxation:

- Compute the number of partitions of a positive integer n . That is to say the number of distinct multi-sets of positive integers $\{n_1, \dots, n_k\}$ which sum up to n , i.e., such that $n_1 + \dots + n_k = n$.

Note: multi-sets means that the set can contain several copies of the same number, but all permutations of elements count as a single multi-set.

Hint: Let $\text{nump}(i, j)$ denotes the number of partitions $\{j_1, \dots, j_p\}$ of j , i.e., the number of sets such that $j_1 + \dots + j_p = j$ which, in addition, have the property that every element j_q of each partition satisfies $j_q \leq i$. We are looking for $\text{nump}(n, n)$ but the recursion is based on relaxation of the allowed size i of the parts of j for all $i, j \leq n$. To get a recursive definition of $\text{nump}(i, j)$ distinguish the case of partitions where all components are $\leq i - 1$ and the case where at least one component is of size i .

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PUZZLE

You have 2 lengths of fuse that are guaranteed to burn for precisely 1 minute each. Other than that fact, you know nothing; they may burn at different (indeed, at variable) rates, they may be of different lengths, thick nesses, materials, etc. How can you use these two fuses to time a 45 second interval?