UNIVERSITY of HOUSTON | ECE

# ECE 3337: Test #1

# Spring 2024

| **Last name of student** | **Student ID number** |
| --- | --- |
| POSTED SOLUTIONS | |
| **First name of student** | **Email address** |
| | |

## Do This First:

1. Make sure that you have all the pages of the exam.
2. Fill in the above information.
3. Turn off and put away <u>all</u> electronic gadgets (laptop, tablet, smartphone, calculator, smart watch, …)

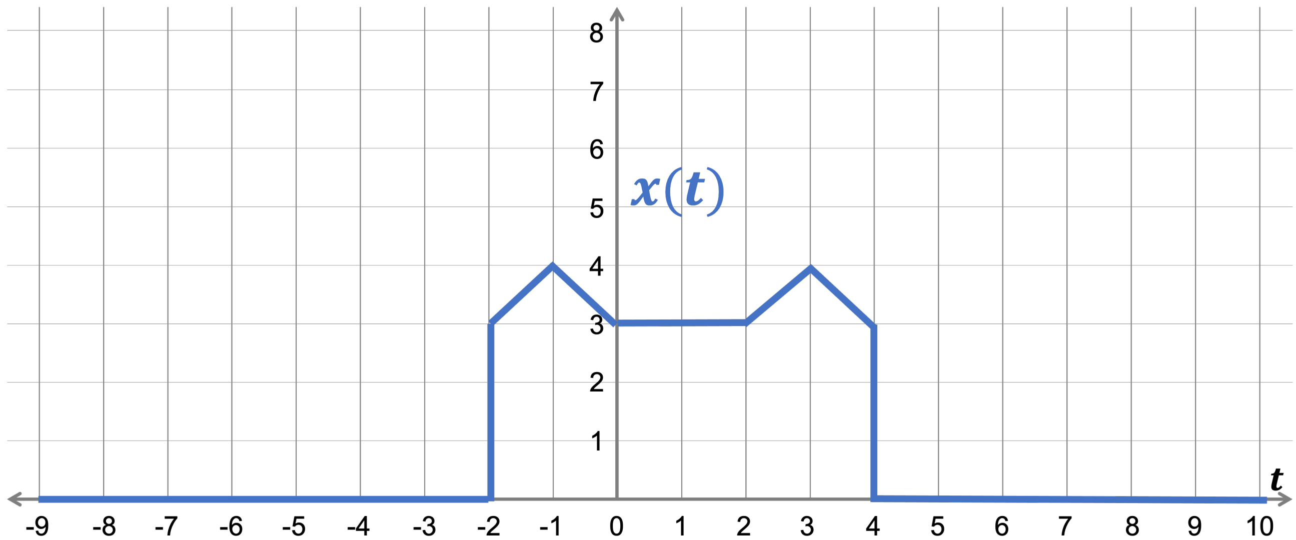
## Exam Rules:

1. You are allowed to bring one 8.5” x 11” handwritten crib sheet (double sided).
2. Indicate your answers in the spaces provided. Use the back of the page to carry out calculations.

| **Q1** | **Q2** | **Q3** | **Q4** | **Q5** | **Q6** | **Total** |
| --- | --- | --- | --- | --- | --- | --- |
| 15 | 15 | 20 | 15 | 15 | 20 | 100 |
| Waveforms & Transformations | Signal Properties | System Properties | Convolution Integral | System Response | Frequency Response | |
| | | | | | | |

# 1. (15 points) Waveforms & Signal Transformations

a. (5 points) Write down a formula in terms of steps and ramps describing the signal sketched below.



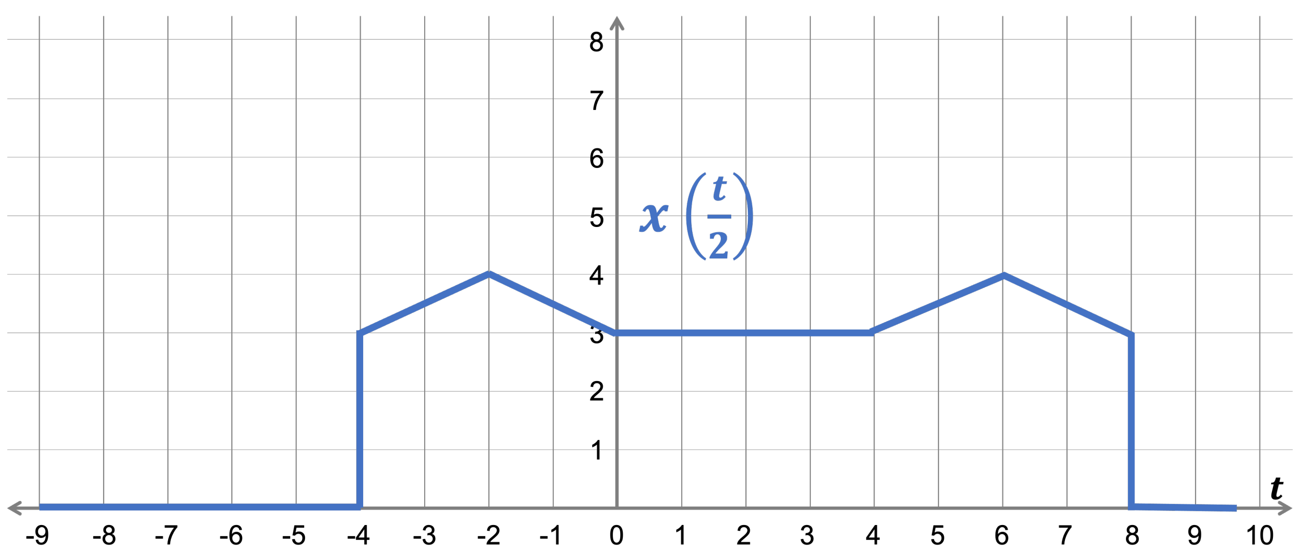

A B C D E F G H

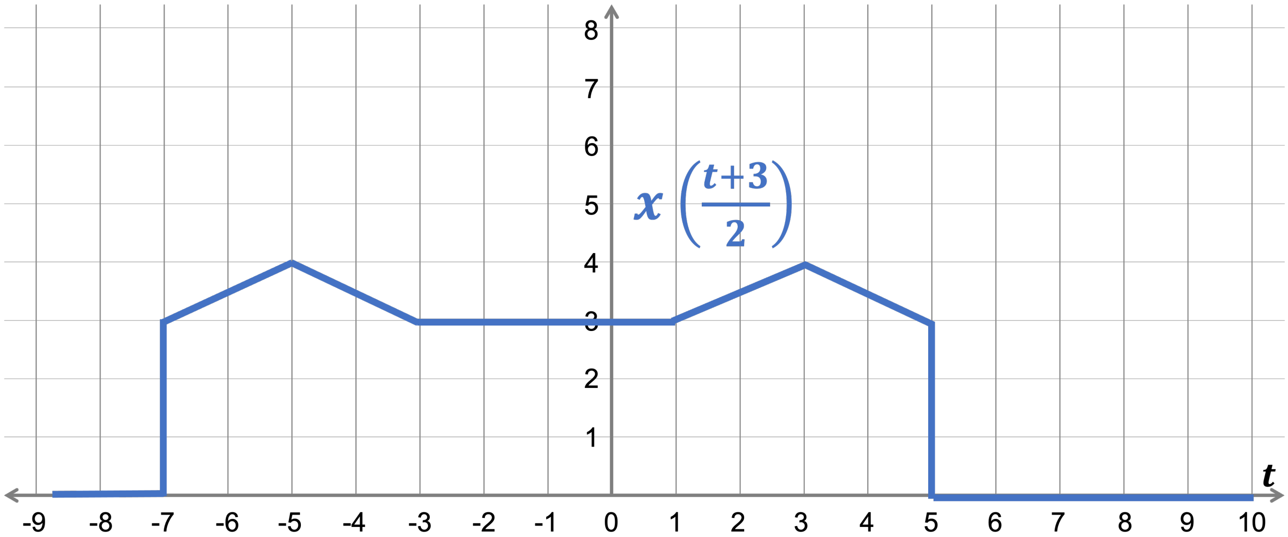
$$3u(t+2) + r(t+2) - 2r(t+1) + r(t) + r(t-2) - 2r(t-3) + r(t-4) - 3u(t-4)$$

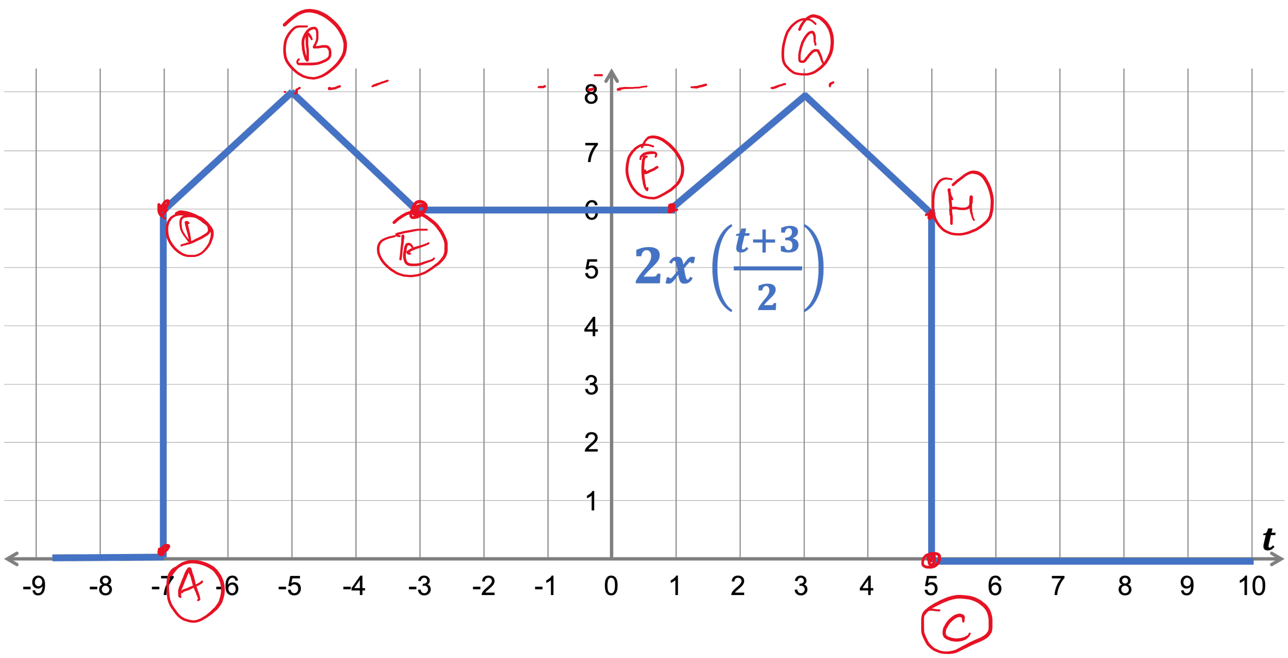
1 point for trying

1/2 point each for A - H

b. (10 points) Plot $2x\left(\frac{t+3}{2}\right)$ on the graph below. Your final answer should be sketched below.






2 points for trying

1 point each for A – H

# 3. (20 points) System Properties

a. (5 points) Is the system $y(t) = x(t)\sin^2(3t)$ linear? Explain your reasons.

1 point for trying

Consider the output of the system for two inputs $x_1(t)$ and $x_2(t)$.

$$y_1(t) = x_1(t)\sin^2(3t)$$

$$y_2(t) = x_2(t)\sin^2(3t)$$

A

If the input is $a\, x_1(t) + b\, x_2(t)$, then the output is

$$y_3(t) = [a\, x_1(t) + b\, x_2(t)]\sin^2(3t)$$ B

$$= a\, x_1(t)\sin^2(3t) + b\, x_2(t)\sin^2(3t)$$

$$= a\, y_1(t) + b\, y_2(t)$$ C

Therefore the system is linear. D

1 point for A – D

b. (5 points) Is the system $y(t) = \frac{d}{dt}(x(t-1))$ time invariant? Explain your reasons.

1 point for trying

For the input $x_1(t)$, the output is $y_1(t) = \frac{d}{dt}(x_1(t-1))$.

A

If the output is delayed by $T$ seconds, $y_2(t) = y_1(t-T) = \frac{d}{dt}\big(x_1(t-1-T)\big) = \frac{d}{dt}(x_1(\{t-T\}-1))$.

B

For the input $x_3(t) = x_1(t-T)$, the output is $y_3(t) = \frac{d}{dt}\big(x_3(t-1)\big) = \frac{d}{dt}\big(x_1(\{t-T\}-1)\big)$. C

The two outputs are identical, and therefore, the system is time invariant.

D

1 point for A – D

c. (5 points Is the system $y(t) = \frac{e^{-t^2}}{t^2}x(t+1)$ BIBO stable? Explain your reasoning.

1 point for trying

B

For the input $x(t+1) = 1$, the output at $t = 0$ is

A

$$y(0) = \frac{e^0}{0} \times 1 = \infty$$ C

1 point for A – D

Since the output goes to $\infty$ for some finite input, the system in NOT BIBO stable. D

d. (5 points) Is the system defined by $y(t) = 2x(t-1) + x(t)$ causal? Explain your answer.

1 point for trying

A

The output of the system $y(t)$ at any time $t$ depents on the value of the input $x(t)$ at the present time $t$ and a past time $(t-1)$. It does not depend on values of $x(t)$ in the future. Therefore, the system is causal.

B C D

1 point for A – D

# 4. (15 points) Convolution Integral

a. (6 points) Calculate the output $y(t)$ of an LTI system whose impulse response is $h(t)$ when it is provided the input $x(t)$, where:

$$x(t) = u(t-2)$$
$$h(t) = e^{1-t}u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} u(\tau-2)\, e^{1-(t-\tau)}\, u(t-\tau)\, d\tau \quad \text{A}$$

$$= e \int_{-\infty}^{\infty} \underbrace{u(\tau-2)}_{\tau>2}\, e^{-(t-\tau)}\, \underbrace{u(t-\tau)}_{t>\tau}\, d\tau$$

$$2 < \tau < t \Rightarrow t > 2 \quad \text{Limits \& validity} \quad \text{B}$$

$$= e\, u(t-2) \int_{2}^{t} e^{-(t-\tau)}\, d\tau = e^{1-t}\, u(t-2) \int_{2}^{t} e^{\tau}\, d\tau$$

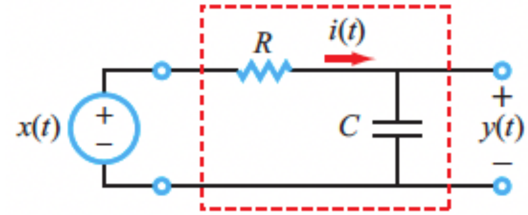
$$= e^{1-t}\, u(t-2) \times \left\{ e^{\tau} \Big|_{2}^{t} \right\}$$

$$= e^{1-t}\, u(t-2)\left(e^{t} - e^{2}\right)$$

$$= \left(e - e^{3-t}\right) u(t-2) \quad \text{C}$$

Grading
- 2 pts trying
- +1 pt A
- +1 pt Limits
- +1 pt validity
- 6 pts correct C

b. (5 points) Calculate the response of the following circuit with $RC = 1$ for the input $x(t) = (t-1)u(t-1)$.




The impulse response is given by $h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$, and the initial conditions are zero.

$h(t) = e^{-t} u(t)$

$x(t) = \delta(t-1) * t u(t)$

It is convenient to calculate $h(t) * t u(t)$ first & delay the result by 1

$y'(t) = t\, u(t) * e^{-t} u(t)$

$$= \int_{-\infty}^{\infty} \underbrace{\tau\, u(\tau)}_{\tau > 0}\, e^{-(t-\tau)}\, \underbrace{u(t-\tau)}_{t-\tau>0}\, d\tau$$

$0 < \tau < t \Rightarrow t > 0$

Limits (A) validity (B)

Grading
- 1 pt for trying
- +1 pt correct limits (A)
- +1 pt correct validity (B)
- 5pts correct answer (C)

$$= u(t) e^{-t} \int_0^t \tau e^{\tau} d\tau$$

$$= e^{-t} u(t) \left\{ \tau e^{\tau} \Big|_0^t - \int_0^t e^{\tau} d\tau \right\}$$

$$= e^{-t} u(t) \left\{ t e^{t} - e^{t} + 1 \right\} = u(t) \left\{ t - 1 + e^{-t} \right\}$$

$$\Rightarrow y(t) = y'(t-1) = u(t-1) \left\{ t - 2 + e^{-(t-1)} \right\} \quad (C)$$

c. (4 points) Compute the convolution $y(t) = u(t-1) * (\delta(t) + 2u(t) + 4r(t-1))$ without computing any integrals

$$y(t) = u(t-1) * \delta(t) + u(t-1) * 2u(t) + u(t-1) * 4r(t-1)$$

$$= \underset{(A)}{u(t-1)} + \underset{(B)}{2r(t-1)} + 4 \times \frac{1}{2} \underset{(C)}{(t-2)^2} u(t-2)$$

Grading:
- 1 pt for each term (A), (B), (C)
- 1 pt for trying

## 5. (15 points) System Response

a. (10 points) Find the step response $y_{step}(t)$ of the system given by the differential equation

$2\frac{dy(t)}{dt} + 4y(t) = 2\frac{dx(t)}{dt} + 3x(t)$ . Assume that $y(0^-) = 2$. 2 points for trying

$2\frac{dy(t)}{dt} + 4y(t) = 2\frac{dx(t)}{dt} + 3x(t)$ can be rewritten in standard form as: $\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + \frac{3}{2}x(t)$ — A

We need to find the step response. Therefore, let $x(t) = u(t)$ and the right hand side is

$\frac{dx(t)}{dt} + \frac{3}{2}x(t) = \frac{du(t)}{dt} + \frac{3}{2}u(t) = \delta(t) + \frac{3}{2}u(t)$ C B

The differential equation simplifies to: $\frac{dy(t)}{dt} + 2y(t) = \delta(t) + \frac{3}{2}u(t)$

The integrating factor is: $e^{\int 2dt} = e^{2t}$ — D

$$\frac{dy(t)}{dt}e^{2t} + 2y(t)e^{2t} = \delta(t)e^{2t} + \frac{3}{2}u(t)e^{2t}$$

$$\frac{d[y(t)e^{2t}]}{dt} = \delta(t)e^{2t} + \frac{3}{2}u(t)e^{2t}$$

$$\int_{0-}^{t} \frac{d[y(t)e^{2t}]}{dt}dt = \int_{0-}^{t} \left[\delta(t)e^{2t} + \frac{3}{2}u(t)e^{2t}\right]dt$$

E

Using the smapling theorem, $\int_{0-}^{t}[\delta(t)e^{2t}]dt = e^{0}u(t) = u(t)$. The $u(t)$ indicates that the integral is 0 for $t < 0$. For the second term $\int_{0-}^{t}\left[\frac{3}{2}u(t)e^{2t}\right]dt = \frac{3}{2}\int_{0-}^{t}[e^{2t}]dt = \frac{3}{4}e^{2t}|_{0-}^{t} = \frac{3}{4}(e^{2t} - 1)u(t)$ F

$$\text{G} \quad y(t)e^{2t} - y(0-) = u(t) + \frac{3}{4}(e^{2t} - 1)u(t)$$

Therefore, for $t > 0$, $y(t)e^{2t} - 2 = u(t) + \frac{3}{4}(e^{2t} - 1)u(t) = \frac{3}{4}e^{2t}u(t) + \frac{1}{4}u(t)$

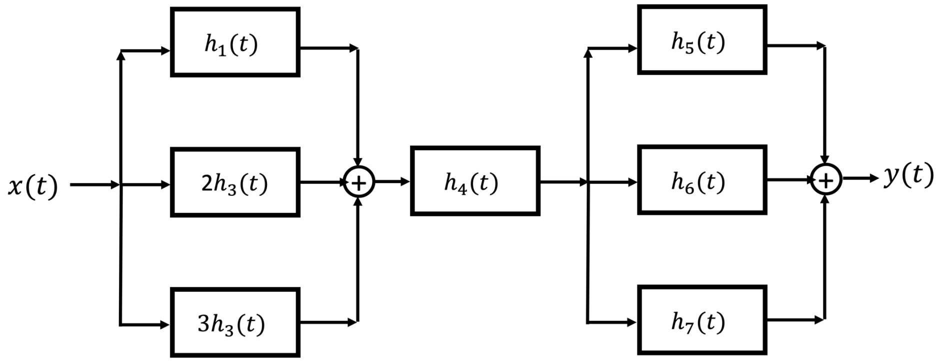
$$y(t) = \left(\frac{3}{4}e^{2t}u(t) + \frac{1}{4}u(t) + 2\right)e^{-2t}$$

$$y(t) = \left(\frac{3}{4}u(t) + \frac{1}{4}e^{-2t}u(t) + 2e^{-2t}\right) \quad \text{— H}$$

Note that this is equivalent to $\frac{9}{4}e^{-2t} + \frac{3}{4}$ for $t > 0$.

1 point for A - H

b. (5 points) Find the overall impulse response of the following system if each of the individual systems $h_1(t) \ldots h_7(t)$ are LTI.




$$h(t) = \underbrace{[h_1(t) + 2h_3(t) + 3h_3(t)]}_{A} \underset{B}{*} h_4(t) \underset{C}{*} \underbrace{[h_5(t) + h_6(t) + h_7(t)]}_{D}$$

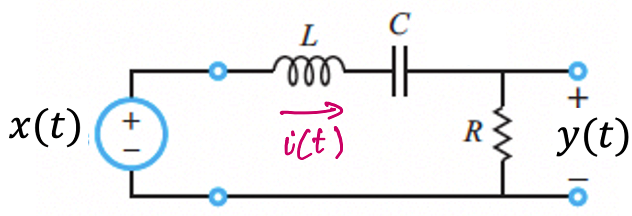
where $*$ represents convolution.

1 point for trying

1 point for A - D

# 6. (20 points) System Frequency Response

Consider the following system with $L = 1H$, $C = 1F$:




a. (5 points) Show that the following linear differential equation describes the above system.

Sorry, this question was in error ☹

$$\frac{d^2y(t)}{dt^2} + \frac{R}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}x(t)$$

KVL: $x(t) = L\frac{di}{dt} + v_c + y(t)$ ; $y(t) = iR \Rightarrow \frac{dy}{dt} = R\frac{di}{dt} \Rightarrow \frac{d^2y}{dt^2} = R\frac{d^2i}{dt^2}$

$\Rightarrow \frac{dx}{dt} = L\frac{d^2i}{dt^2} + \frac{dv_c}{dt} + \frac{dy}{dt}$

$i = C\frac{dv_c}{dt} \Rightarrow \frac{dv_c}{dt} = \frac{i}{C} = \frac{1}{RC}y(t)$

$= \frac{L}{R}\frac{d^2y}{dt^2} + \frac{1}{RC}y(t) + \frac{dy}{dt}$

cast into standard form: $\boxed{\frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{1}{LC}y = \frac{R}{L}\frac{dx}{dt}}$

Grading:
- Full pts for reasonable attempt

b. (5 points) Calculate the frequency response function $\hat{\mathbf{H}}(\omega)$ for this system.

$$\hat{H}(\omega) = \frac{1/LC}{(j\omega)^2 + \frac{R}{L}(j\omega) + \frac{1}{LC}} = \frac{1}{(j\omega)^2 LC + RC(j\omega) + 1}$$

$$= \frac{1}{(j\omega)^2 + R(j\omega) + 1}$$ since $L = C = 1$

Grading:
- 1 pt for trying
- 5 pts for correct $\hat{H}(\omega)$

c. (2 points) For what values of $R$ is this system's response oscillatory?

Characteristic equation

$$s^2 + a_1 s + a_2 = 0$$

Discriminant $\Delta = a_1^2 - 4a_2$

Grading
- 1 pt trying
- 2 pts correct answer

oscillatory when $\Delta < 0 \Rightarrow a_1^2 < 4a_2$

$$\left(\frac{R}{L}\right)^2 < 4\left(\frac{1}{LC}\right) \Rightarrow R^2 < 4$$

or $R < 2\Omega$

d. (2 points) For what values of $R$ is this system critically damped?

Grading
- 1 pt trying
- 2 pts correct answer

$R = 2\Omega$

e. (2 points) For what values of $R$ is this system's response stable?

For stability, $a_1 > 0$ & $a_2 > 0$

This is always true for positive $R$ values.

Grading
- 1 pt trying
- 2 pts correct answer

f. (4 points) The following input-output pair is observed for a different linear time-invariant system:

$$x(t) = 5 + \sin(2t) + \cos(3t) = 5 + \cos(2t - 90^\circ) + \cos(3t)$$

$$y(t) = 10 + \cos(2t) + \sin(3t - 30^o) = 10 + \cos(2t) + \cos(3t - 30^\circ - 90^\circ)$$

Calculate the output for the new input $x_2(t) = 1 + \cos(2t) + \sin(3t)$

For $\omega = 0$, the system doubles the input

$\omega = 2$, the system changes phase by $+90^\circ$

$\omega = 3$, the system changes phase by $-120^\circ$

$$\Rightarrow y_2(t) = \underset{A}{2} + \cos(\underset{B}{2t + 90^\circ}) + \underset{C}{\sin(3t - 120^\circ)}$$

Grading:
- 1 pt trying
- +1 pt for each part A, B, C