ECE 3337: Test #2

Spring 2024

Last name of student	Student ID number
Solutions	
First name of student	**Email address**

Do This First:

1. Make sure that you have all the pages of the exam.
2. Fill in the above information.
3. Turn off and put away <u>all</u> electronic gadgets (laptop, tablet, smartphone, calculator, smart watch, …)

Exam Rules:

1. You are allowed to bring one 8.5” x 11” handwritten crib sheet (double sided).
2. Indicate your answers in the spaces provided. Use the back of the page to carry out calculations.

Q1	Q2	Q3	Q4	Q5	Q6	Total
20	20	25	15	10	10	100
Laplace Transform	Inverse Laplace Transform	Circuit Analysis	Response Partitions	Op Amp Circuits	Block Diagrams	

1. (20 points) Laplace Transforms

a. (6 points) Find $\mathbf{X}(\mathbf{s})$, the Laplace transform of the signal $x(t) = 2\delta(3t-4) + tu(t-1) + t^2 e^{1-t}u(t)$.

$$x(t) = \frac{2}{3}\delta\left(t-\frac{4}{3}\right) + (t-1)u(t-1) + u(t-1) + e\, t^2 e^{-t} u(t)$$ Rewrite

$$X(s) = \underbrace{\frac{2}{3} e^{-\frac{4}{3}s}}_{A} + \underbrace{e^{-s} \cdot \frac{1}{s^2}}_{B} + \underbrace{\frac{e^{-s}}{s}}_{C} + \underbrace{e \cdot \frac{2}{(s+1)^3}}_{D}$$

Grading:
- 1 pt trying
- +1 pt each A – D
- +1 pt rewrite
- 6 pts for correct answer

(b) (6 points) Given a signal $x_2(t)$, find the Laplace transform of $y(t) = 4e^{-t}x_2(2t-2)$ in terms of $\mathbf{X_2}(\mathbf{s})$

$$\mathcal{L}[x_2(t)] = X_2(s)$$

$$\mathcal{L}[x_2(2t)] = \frac{1}{2} X_2\left(\frac{s}{2}\right) \quad A$$

$$\mathcal{L}[x_2(2(t-1))] = \frac{e^{-s}}{2} X_2\left(\frac{s}{2}\right) \quad B$$

$$\mathcal{L}[4e^{-t} x_2(2t-2)] = 4 \cdot \frac{e^{-(s+1)}}{2} \cdot X_2\left(\frac{s+1}{2}\right) \quad C$$

$$= 2e^{-(s+1)} X_2\left(\frac{s+1}{2}\right)$$

Grading:
- 1 pt trying
- 6 pts correct answer
- 1 pt partial credit for steps A, B, C

c. (8 points) Find the Laplace transform of $z(t) = (t-2)\, e^{-t} \cos(t-2)\, u(t-2)$, and find the values of all its poles and zeros.

$$z(t) = e^{-2}(t-2)\, e^{-(t-2)} \cos(t-2)\, u(t-2) \quad A$$

Grading
- 2 pts trying
- +1 pt each for A – F

$$\mathcal{L}\left[e^{-t}\cos(t)u(t) \right] = \frac{(s+1)}{((s+1)^2+1)} \quad B$$

$$\mathcal{L}\left[t e^{-t}\cos(t)\, u(t) \right] = -\frac{d}{ds}\left\{ \frac{(s+1)}{(s+1)^2+1} \right\} = \frac{s(s+2)}{\left[(s+1)^2+1\right]^2} \quad C$$

$$Z(s) = e^{-2} \cdot e^{-2s} \cdot \frac{s(s+2)}{\left[(s+1)^2+1\right]^2} = \frac{e^{-2(s+1)}\, s(s+2)}{\left[(s+1)^2+1\right]^2} \quad D$$

poles: $\{-1 \pm j, -1 \pm j\}$ E

zeros: $\{0, -2\}$ F

2. (20 points) Inverse Laplace Transform

Find the inverse Laplace transforms for the following systems.

a. (10 points) $X(s) = \frac{e^{-3s}(2s+3)}{(s^2+9s+14)}$

A B C

Consider $X_1(s) = \frac{(2s+3)}{(s^2+9s+14)} = \frac{(2s+3)}{(s^2+7s+2s+14)} = \frac{(2s+3)}{(s+7)(s+2)} = \frac{(2s+4-1)}{(s+7)(s+2)} = \frac{2(s+2)}{(s+7)(s+2)} - \frac{1}{(s+7)(s+2)}$

Computing partial fractions $X_1(s) = \frac{2}{(s+7)} - \frac{1}{(s+7)(s+2)} = \frac{2}{(s+7)} - \frac{1}{5}\left(\frac{1}{s+2} - \frac{1}{s+7}\right) = \frac{11}{5}\frac{1}{(s+7)} - \frac{1}{5}\frac{1}{(s+2)}$

D E

The inverse Laplace transform of each of the terms is

$\frac{11}{5}\frac{1}{(s+7)} \overset{\mathcal{L}}{\leftrightarrow} \frac{11}{5}e^{-7t}u(t)$ and $\frac{1}{5}\frac{1}{(s+5)} \overset{\mathcal{L}}{\leftrightarrow} \frac{1}{5}e^{-2t}u(t)$ F

Combining, $X_1(s) \overset{\mathcal{L}}{\leftrightarrow} x_1(t) = \frac{11}{5}e^{-7t}u(t) - \frac{1}{5}e^{-2t}u(t)$ G

Using the time shifting property to include the effect of e^{-3s}, H

$$x(t) = x_1(t-3) = \frac{11}{5}e^{-7(t-3)}u(t-3) - \frac{1}{5}e^{-2(t-3)}u(t-3)$$

2 points for trying. 1 point for A-H.

b. (10 points) $X(s) = \frac{s^2+1}{(s+2)(s^2+3)}$

Separating into Partial fractions:

$$X(s) = \frac{s^2+1}{(s+2)(s^2+3)} = \frac{A}{(s+2)} + \frac{Bs}{(s^2+3)} + \frac{C}{(s^2+3)}$$ A

Solving,

B C D

$$A = \frac{5}{7}, B = \frac{2}{7}, C = -\frac{4}{7}$$

$$X(s) = \frac{5}{7(s+2)} + \frac{2s}{7(s^2+3)} - \frac{4}{7(s^2+3)}$$

$$= \frac{5}{7}\frac{1}{(s+2)} + \frac{2}{7}\frac{s}{(s^2+(\sqrt{3})^2)} - \frac{4}{7\sqrt{3}}\frac{\sqrt{3}}{(s^2+3)}$$ E

Finding the inverse of each term gives

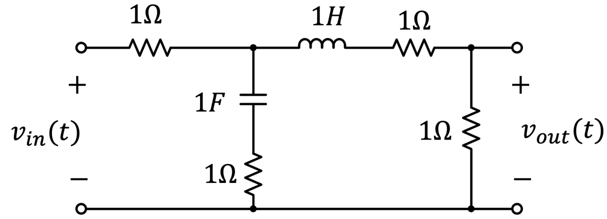
$$x(t) = \frac{5}{7}e^{-2t}u(t) + \frac{2}{7}\cos(\sqrt{3}t)\,u(t) - \frac{4}{7\sqrt{3}}\sin(\sqrt{3}t)\,u(t)u(t)$$

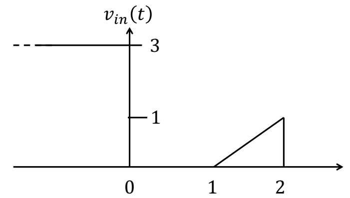
F G H

2 points for trying. 1 point for A-H.

3. (25 points) Circuit Analysis

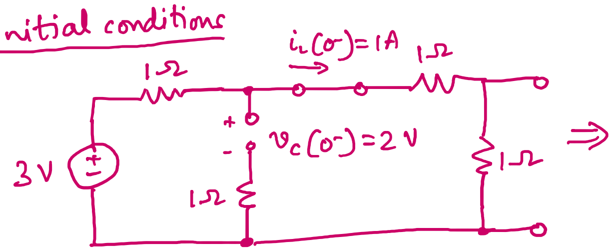
For the circuit below, the input $V_{in}(t) = x(t)$ is given in the graph. The units for $x(t)$ is Volts [V].



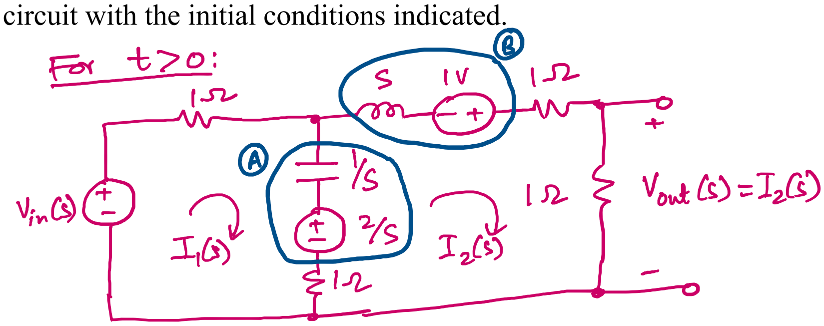


a. (5 points) Draw the **s**-domain representation of the circuit with the initial conditions indicated.

initial conditions



For t>0:



Grading: • 1 pt trying
• +2 pts each for A & B

a. (5 points) Find the **s**-domain representation of the input $v_{in}(t)$ for t > 0.

$$v_{in}(t) = r(t-1) - r(t-2) - u(t-2)$$

$$V_{in}(s) = \underbrace{e^{-s}\cdot\frac{1}{s^2}}_{A} - \underbrace{e^{-2s}\cdot\frac{1}{s^2}}_{B} - \underbrace{e^{-2s}\cdot\frac{1}{s}}_{C}$$

Grading:
• 2 pt trying
• +1 pt each A - C

c. (10 points) Write the mesh current equations in the **s**-domain needed to find the output $\boldsymbol{V}_{out}(\boldsymbol{s})$. Express your answer by filling in the entries in the Cramer's method matrices shown below. Here, $\mathbf{I}_1$ and $\mathbf{I}_2$ represent the mesh currents (left to right), and $\mathbf{V}_1$, $\mathbf{V}_2$ represent mesh voltages.

$$\begin{pmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_3 & \mathbf{A}_4 \end{pmatrix}\begin{pmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{pmatrix}$$

KVL:

$$\overbrace{V_{in}(s) - \frac{2}{s}}^{V_1} = I_1\underbrace{\left(1+\frac{1}{s}+1\right)}_{A_1} - I_2\underbrace{\left(1+\frac{1}{s}\right)}_{A_2}$$

$$\overbrace{\frac{2}{s} + 1}^{V_2} = -I_1\underbrace{\left(1+\frac{1}{s}\right)}_{A_3} + I_2\underbrace{\left(1+\frac{1}{s}+s+1+1\right)}_{A_4}$$

Grading:
• 2 pts trying
• +1 pt each $A_1 - A_4$
• +2 pt each V_1, V_2

d. (5 points) Write the formula for $\boldsymbol{V_{out}(s)}$ in terms of $\boldsymbol{V_{in}(s)}$.

To calculate the output, we only need $I_2 = \frac{\Delta_2}{\Delta}$, where

$$\Delta = \begin{vmatrix} 2+\frac{1}{s} & -\left(1+\frac{1}{s}\right) \\ -\left(1+\frac{1}{s}\right) & \left(3+s+\frac{1}{s}\right) \end{vmatrix} = \frac{(2s^2+6s+3)}{s}$$

Grading:
- 1 pt trying
- 2 pt partial
- 5 pt correct answer

$$\Delta_2 = \begin{vmatrix} 2+\frac{1}{s} & \frac{sV_{in}-2}{s} \\ -\left(1+\frac{1}{s}\right) & \frac{s+2}{s} \end{vmatrix} = \frac{(2s+1)(s+2)+(s+1)(sV_{in}-2)}{s^2}$$

$$\Rightarrow V_{out}(s) = I_2(s) = \frac{(2s+1)(s+2)+(s+1)(sV_{in}-2)}{s(2s^2+6s+3)}$$

4. (15 points) System Response Partitions

A linear time-invariant system is described by the following LCCDE

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

The input is $x(t) = \mathrm{e}^{-t}u(t)$, with the initial conditions $y(0-) = 3,\ y'(0-) = 1$.

a. (5 points) Derive a formula for the output $\mathbf{Y}(s)$. Your answer should be expressed as a ratio of polynomials with a fully factored denominator.

$$y(t) \overset{\mathcal{L}}{\leftrightarrow} Y(s)$$
$$\frac{d}{dt}y(t) \overset{\mathcal{L}}{\leftrightarrow} sY(s) - y(0-)$$
$$\frac{d^2}{dt^2}y(t) \overset{\mathcal{L}}{\leftrightarrow} s^2Y(s) - sy(0-) - y'(0-)$$

Therefore, taking the Laplace transform of the LCCDE yields:

$$s^2Y(s) - sy(0-) - y'(0-) + 5\big(sY(s) - y(0-)\big) + 6Y(s) = X(s)$$
$$s^2Y(s) + 5sY(s) + 6Y(s) - sy(0-) - y'(0-) - 5y(0-) = X(s)$$ A

Using the given initial conditions

$$s^2Y(s) + 5sY(s) + 6Y(s) - 3s - 1 - 15 = X(s)$$
$$s^2Y(s) + 5sY(s) + 6Y(s) = X(s) + 3s + 1 + 15$$

$$Y(s)(s^2 + 5s + 6) = X(s) + 3s + 16$$ B

Since $x(t) = \mathrm{e}^{-t}u(t), X(s) = \frac{1}{(s+1)}$ C

Therefore,

$$Y(s)(s^2 + 5s + 6) = \frac{1}{(s+1)} + 3s + 16 = \frac{1 + (3s+16)(s+1)}{(s+1)}$$
$$= \frac{3s^2 + 16s + 3s + 16 + 1}{(s+1)} = \frac{3s^2 + 19s + 17}{(s+1)}$$

$$Y(s) = \frac{3s^2 + 19s + 17}{(s+1)(s^2 + 5s + 6)} = \frac{3s^2 + 19s + 17}{(s+1)(s+2)(s+3)}$$ D

1 point for trying. 1 point for A-D.

b. (5 points) Find the modes of the system represented by the above LCCDE. Calculate the natural response of this system in the Laplace domain.

The modes of the system correspond to the roots of the characteristic equation. The characteristic equation is: $s^2 + 5s + 6 = 0$ or $(s+2)(s+3) = 0$ or $s = -2, -3$

A B

Calculating the partial fraction of

$$Y(s) = \frac{3s^2 + 19s + 17}{(s+1)(s+2)(s+3)} = \frac{9}{(s+2)} - \frac{13}{2}\frac{1}{(s+2)} + \frac{1}{2}\frac{1}{(s+1)}$$

The natural response corresponds to the terms

$$Y_{natural}(s) = -\frac{13}{2}\frac{1}{(s+3)} + \frac{9}{(s+2)}$$

C D

1 point for trying. 1 point for A-D.

c. (5 points) Calculate the initial value $y(0+)$ of this system's response.

To find the initial value, we can use the initial value theorem.

A

$$y(0+) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} s\frac{3s^2 + 19s + 17}{(s+1)(s+2)(s+3)}$$

B

Dividing Nr and Dr by s^3,

$$= \lim_{s\to\infty} \frac{3 + \frac{19}{s} + \frac{17}{s^2}}{\left(1+\frac{1}{s}\right)\left(1+\frac{2}{s}\right)\left(1+\frac{3}{s}\right)}$$

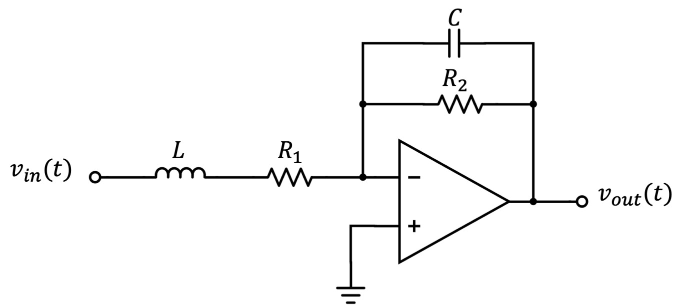
C

$$= \frac{3+0+0}{1\times 1\times 1} = 3$$

D

1 point for trying. 1 point for A-D.

5. (10 points) Operational Amplifiers



a. (6 points) Calculate the transfer function $\mathbf{H(s)}$ of this circuit.

Grading:
- 2 pts trying
- 4 pts minor error
- 6 pts correct answer

$$H(s) = -\frac{\left(\frac{1}{sC} \parallel R_2\right)}{R_1 + sL}$$

$$= -\frac{R_2 \cdot \frac{1}{sC}}{R_2 + \frac{1}{sC}} \cdot \frac{1}{R_1 + sL}$$

$$= -\frac{R_2}{(1 + sCR_2)(R_1 + sL)}$$

b. (4 points) Identify the poles and zeros of the transfer function $\mathbf{H(s)}$.

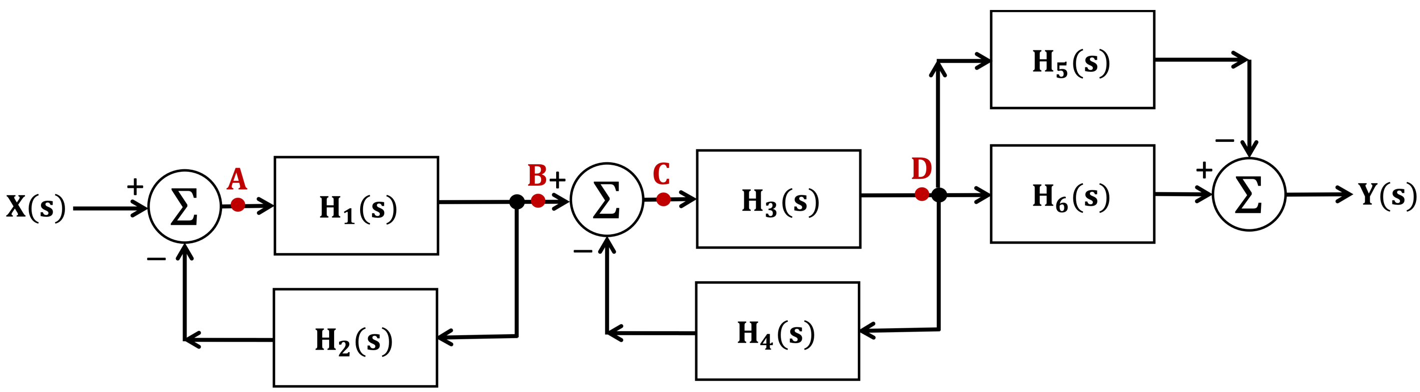
No zeros (A)

poles at $\left(-\frac{1}{CR_2}\right)$ (B) and $\left(-\frac{R_1}{L}\right)$ (C)

Grading
- 1 pt trying
- +1 pt each for (A), (B), (C).

6. (10 points) System Block Diagrams

Calculate the overall transfer function $\mathbf{H(s)}$ for the following connection of LTI systems. Your final answer should be expressed in the Laplace domain.



$$A = X - BH_2 \qquad \text{A}$$
$$B = AH_1 \qquad \text{B}$$
$$C = B - DH_4 \qquad \text{C}$$
$$D = CH_3 \qquad \text{D}$$
$$Y = -DH_5 + DH_6 = (H_6 - H_5)D$$

Solving these equations:

$$A = X - AH_1H_2$$
$$A = \frac{1}{1 + H_1H_2}X \qquad \text{E}$$
$$B = \frac{H_1}{1 + H_1H_2}X$$

Similarly

$$D = \frac{H_3}{1 + H_3H_4}B = \frac{H_3}{(1 + H_3H_4)}\frac{H_1}{(1 + H_1H_2)}X \qquad \text{F}$$

$$Y = (H_6 - H_5)\frac{H_3}{(1 + H_3H_4)}\frac{H_1}{(1 + H_1H_2)}X \qquad \text{G}$$

The transfer function is

$$H = \frac{Y}{X} = (H_6 - H_5)\frac{H_3}{(1 + H_3H_4)}\frac{H_1}{(1 + H_1H_2)} \qquad \text{H}$$

2 points for trying. 1 point for A-H.