UNIVERSITY of HOUSTON | ECE

ECE 3337: Final Exam

Fall 2024

Last name of student	Student ID number
POSTED SOLUTIONS	—
First name of student	**Email address**
—	—

Do This First:

1. Make sure that you have all the pages of the exam book.
2. Fill in the above information.
3. Turn off and put away all electronic gadgets (laptop, tablet, smartphone, calculator, smartwatch, wearable, …)

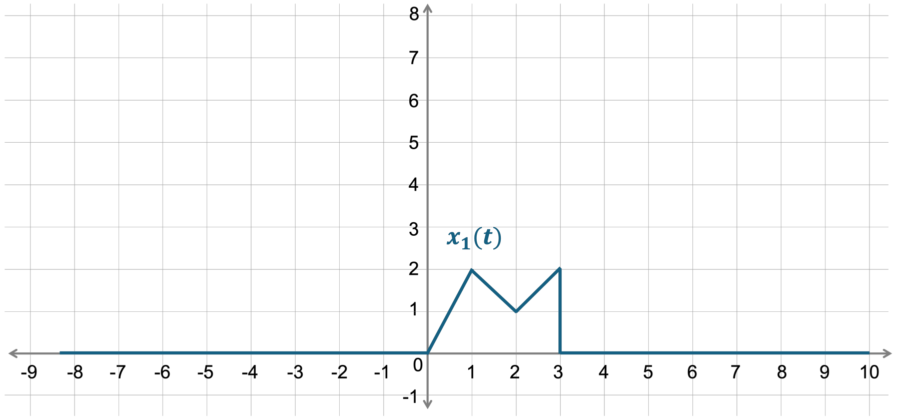
Exam Rules:

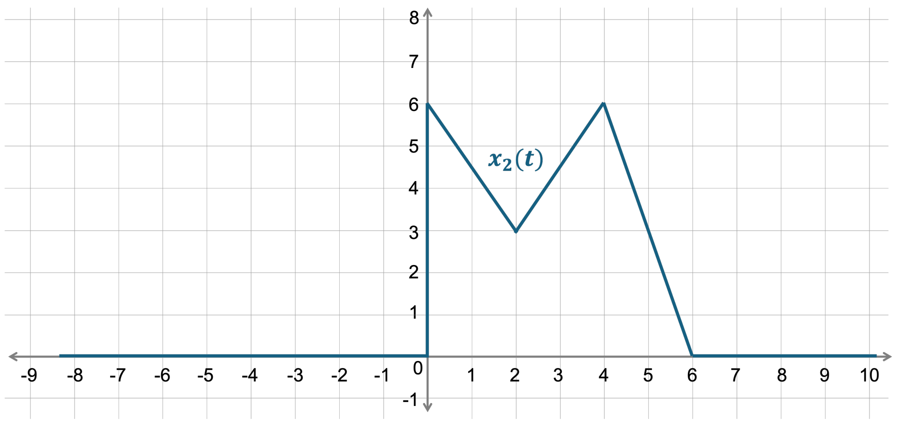
1. This is a 2-hour exam.
2. The syllabus includes all material covered in class.
3. You are allowed to bring two 8.5” x 11” handwritten crib sheets (double-sided).
4. If you need additional space to answer a question, continue on the back page(s).
5. You are not allowed to interact with any other student.

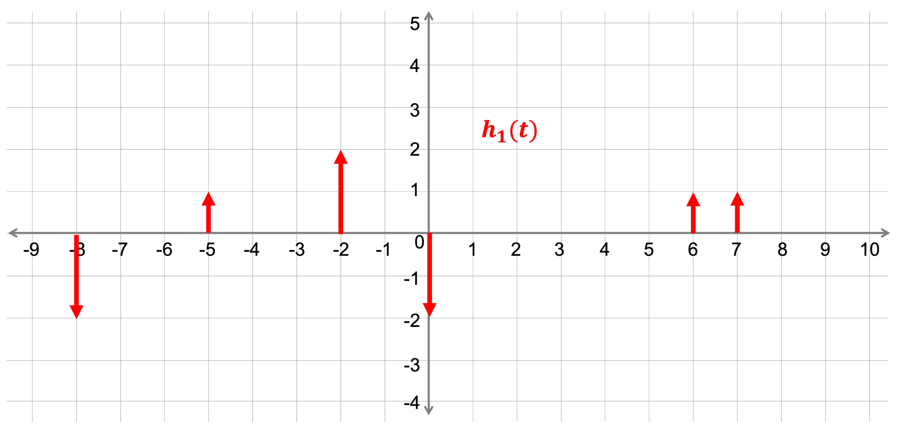
Q1	Q2	Q3	Q4	Q5	Q6	Total
15	10	10	20	25	20	100
—	—	—	—	—	—	—

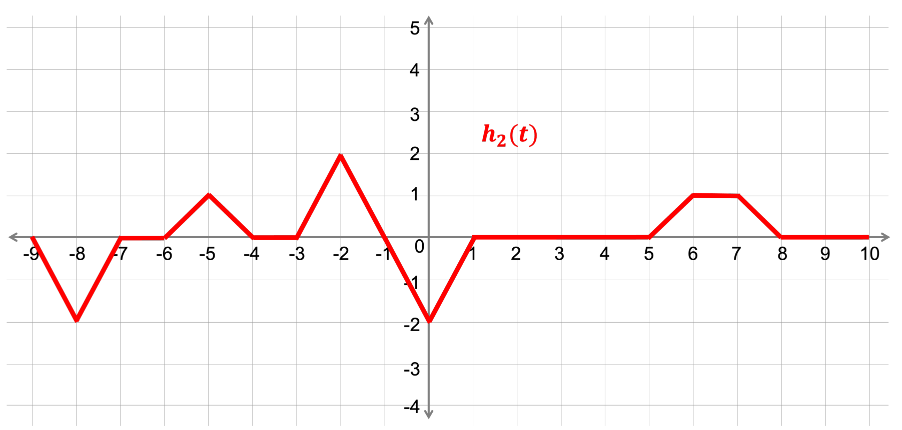
1. (15 points) Waveforms and Convolution

The signals $x_1(t), x_2(t), h_1(t)$, and $h_2(t)$ are as graphed below.









a. (7 points) Write the signal $x_2(t)$ in terms of $x_1(t)$. Hint: $x_2(t)$ is a transformed version of $x_1(t)$.

2 point for trying.

A B D

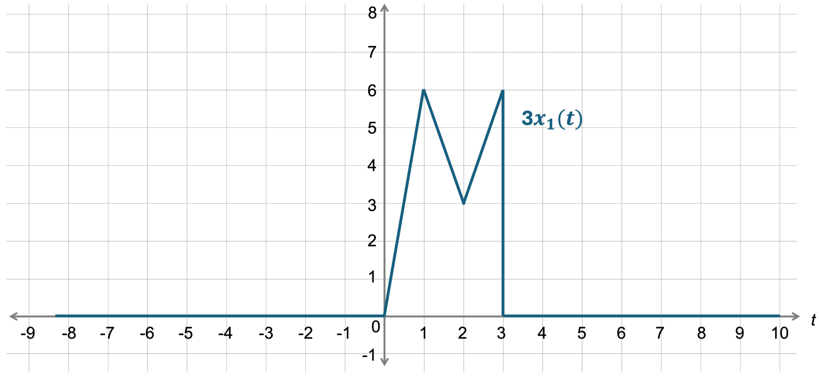
$$x_2(t) = 3x_1\left(-\frac{(t-6)}{2}\right) = 3x_1\left(-\frac{t}{2}+3\right)$$

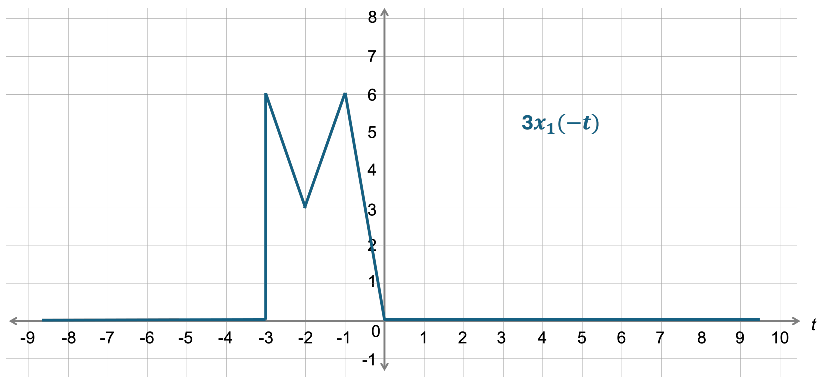
C E

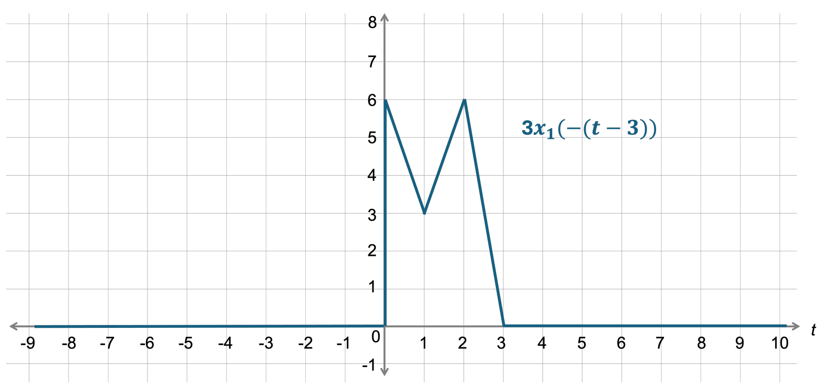
1 point each for A – E

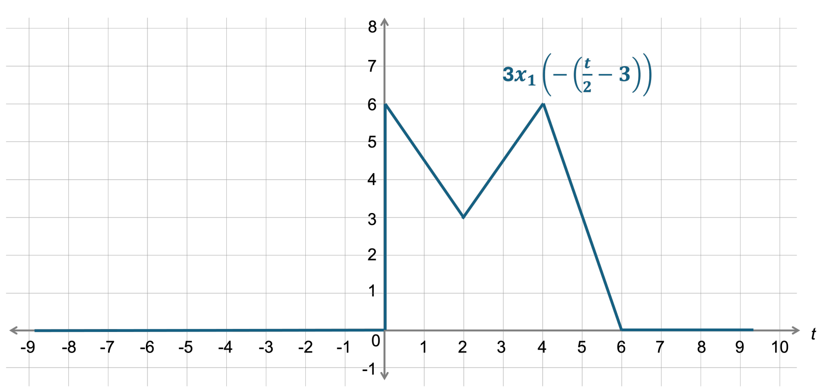
Amp. scaling by 3 – A
Time reversal – B
½ time scaling – C
Time shift 3 – D
Time shift sign – E

Solution Steps:





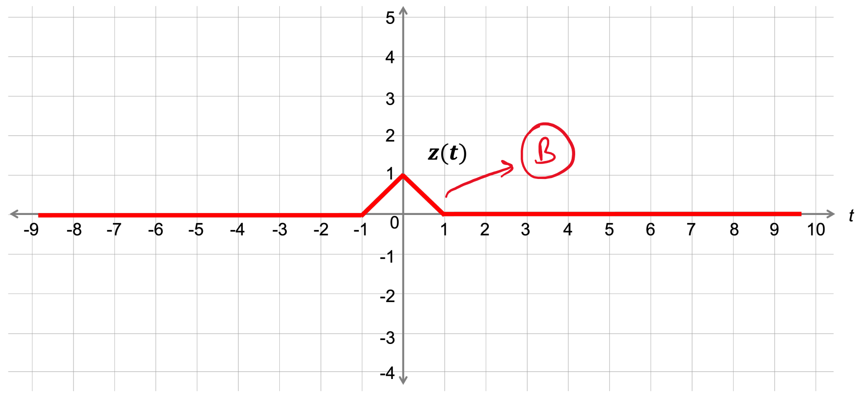




b. (8 points) If the signal $h_2(t)$ graphed above satisfies the equation: $h_2(t) = z(t) * h_1(t)$, where $*$ represents convolution, then graph the signal $z(t)$. Your final answer must be presented in the space below, and this will be the only answer graded.

Convolution of any signal $p(t)$ with $A\delta(t - T_0)$ results in $Ap(t - T_0)$, i.e. a scaled copy at $t = T_0$. A

Therefore, $z(t)$ is as shown below.



2 point for trying. 3 points for A-B

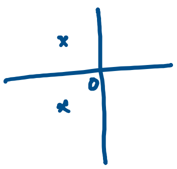
2. (10 points) Linear Time-invariant Systems

a. (5 points) An LTI system has the impulse response $h(t) = e^{-t}\cos(t)u(t)$. Is this system BIBO stable? Calculate the response $y(t)$ to the input signal $x(t) = t\,u(t)$.

$$H(s) = \frac{s+1}{(s+1)^2+1}, \quad X(s) = \frac{1}{s^2} \Rightarrow Y(s) = \frac{(s+1)}{(s+1)^2+1}\cdot\frac{1}{s^2}$$

poles: $\{-1 \pm j\}$

$$= \frac{1}{2}\left\{\frac{1}{s^2} - \frac{1}{(s+1)^2+1}\right\}$$



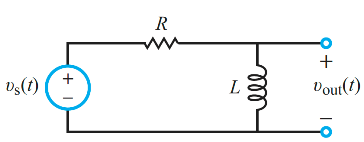
$$\Rightarrow y(t) = \underbrace{\frac{1}{2}t\,u(t)}_{A} - \underbrace{\frac{1}{2}e^{-t}\sin(t)u(t)}_{B}$$

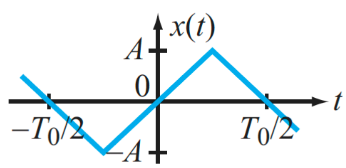
BIBO STABLE C

Grading:

- 2 pts each A, B
- +1 pt C

b. (5 points) Calculate the response $v_{out}(t)$ of the following system to the input $v_s(t) = x(t)$.





$$x(t) = \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{2n\pi t}{T_0}\right)$$

$$\hat{H}(\omega) = \frac{j\omega L}{R+j\omega L}$$

$$|\hat{H}(\omega)| = \frac{\omega L}{\sqrt{R^2+(\omega L)^2}}$$

$$\phi(\omega) = \pi/2 - \tan^{-1}\left(\frac{\omega L}{R}\right)$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$y(t) = \sum_{\substack{n=1\\ n=\text{odd}}}^{\infty} \frac{8A}{n^2\pi^2} \times \underbrace{\frac{n\omega_0}{\sqrt{R^2+n^2\omega_0^2L^2}}}_{A} \sin\left(\frac{n\pi}{2}\right)\sin\left(\underbrace{\frac{2n\pi t}{T_0}+\frac{\pi}{2} - \tan^{-1}\left(\frac{n\omega_0 L}{R}\right)}_{B}\right)$$

Grading:

- 1 pt for trying
- +2 pts each A & B

3. (10 points) Laplace Transform & Its Inverse

a. (5 points) Derive the Laplace transform of $x(t) = A\, e^{-t} \sin(\omega_0 t + \theta)\, u(t)$.

What are the initial and final values of this signal?

$$\sin(\omega_0 t + \theta) = \sin(\omega_0 t) \cos\theta + \cos(\omega_0 t) \sin\theta$$

$$\mathcal{L}\,[\sin \omega_0 t\, u(t)] = \frac{\omega_0}{s^2 + \omega_0^2}$$

$$\mathcal{L}\,[\cos \omega_0 t\, u(t)] = \frac{s}{s^2 + \omega_0^2}$$

Therefore,

$$\sin(\omega_0 t + \theta)\, u(t) \overset{\mathcal{L}}{\leftrightarrow} \frac{\omega_0}{s^2 + \omega_0^2} \cos\theta + \frac{s}{s^2 + \omega_0^2} \sin\theta$$

$$A \sin(\omega_0 t + \theta)\, u(t) \overset{\mathcal{L}}{\leftrightarrow} A \frac{\omega_0}{s^2 + \omega_0^2} \cos\theta + A \frac{s}{s^2 + \omega_0^2} \sin\theta \quad \text{— (A)}$$

Using the Frequency Shifting property of Laplace transforms, replace s by $(s+1)$

$$Ae^{-t} \sin(\omega_0 t + \theta)\, u(t) \overset{\mathcal{L}}{\leftrightarrow} A \frac{\omega_0}{(s+1)^2 + \omega_0^2} \cos\theta + A \frac{(s+1)}{(s+1)^2 + \omega_0^2} \sin\theta \quad \text{— (B)}$$

Initial value is obtained by substituting $t = 0_+$. $x(0_+) = A \sin(\theta)$ — (C)

Final value is obtained as $t \to \infty$. Due to the exponential decay term, $x(\infty) = 0$. — (D)

1 point for trying. 1 point for A-D

b. (5 points) Determine the inverse Laplace Transform of $\mathbf{H(s)} = \frac{s^3+7s^2+4}{s^2+6s+10}$.

1 point for trying. 1 point for A-D

Order of Nr > Order of Dr. Performing long division,

$$\mathbf{H(s)} = \frac{s^3 + 7s^2 + 4}{s^2 + 6s + 10} = (s+1) - \frac{(16s+6)}{s^2 + 6s + 10} \quad \text{— (A)}$$

$$= (s+1) - \frac{(16s+6)}{(s+3)^2 + 1} \quad \text{— (B)}$$

$$= (s+1) - \frac{(16s + 48 - 48 + 6)}{(s+3)^2 + 1}$$

$$= (s+1) - \frac{(16(s+3) - 42)}{(s+3)^2 + 1}$$

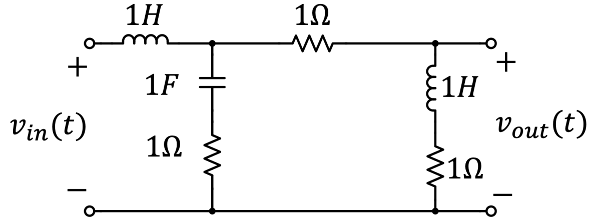
$$= (s+1) - 16 \frac{(s+3)}{(s+3)^2 + 1} + 42 \frac{1}{(s+3)^2 + 1}$$

The inverse Laplace transform is

$$\underbrace{\frac{d}{dt}\delta(t) + \delta(t)}_{(C)} - \underbrace{16\, e^{-3t} \cos(t) + 42\, e^{-3t} \sin(t)}_{(D)}$$

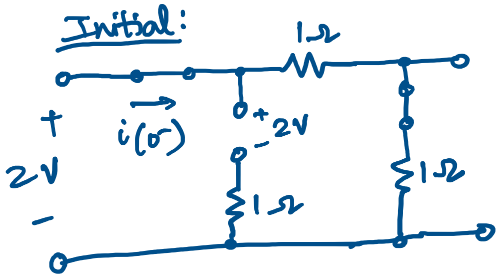
4. (20 points) Applications of the Laplace Transform

Consider the following circuit and the input signal $x(t) = 2 + \sin(t)u(t)$ V.

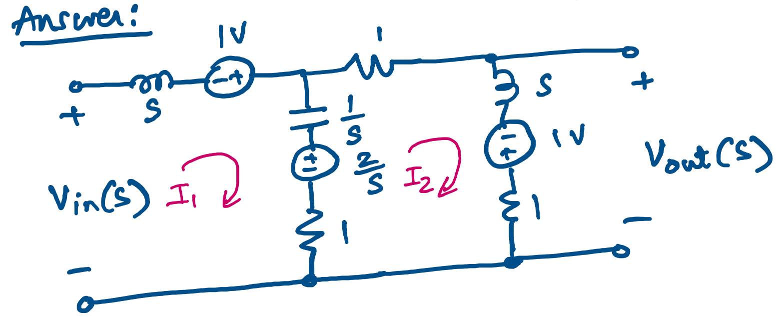


a. (2 points) Sketch the Laplace-domain version of the given circuit with the initial conditions indicated (use the back sheet for your work as needed and write the final answer below).

Initial:



Answer:



$i(0^-) = 1\text{A}$ – 1 pt

$v_c(0^-) = 2\text{V}$ – 1 pt

b. (3 points) Write the Laplace transform of the input for $t \geq 0$ (use the back sheet for your work and write the final answer below).

$$V_{in}(s) = \frac{1}{s^2+1}$$

- 1pt for trying
- 3 pts correct answer

c. (5 points) Calculate the Laplace-domain voltage $\mathbf{V}_{out}(\mathbf{s})$ for $t \geq 0$. (use the back sheet for your work and write the final answer below). Your answer should be in the form of a ratio of two polynomials.

KVL:

$$\left\{\begin{array}{ll} V_{in}(s) + 1 - \frac{2}{s} = I_1\left(s+\frac{1}{s}+1\right) - I_2\left(1+\frac{1}{s}\right) & \text{(A)} \\ \frac{2}{s} + 1 = -I_1\left(1+\frac{1}{s}\right) + I_2\left(1+\frac{1}{s}+1+s+1\right) & \text{(B)} \end{array}\right\}, \quad V_{out} = I_2(s+1) - 1 \quad \text{(C)}$$

- 1pt trying
- +1 pt each (A), (B), (C), (D)

$$\Delta = \frac{s^3+4s^2+4s+2}{s}$$

$$\Rightarrow I_2 = \frac{s^2+4s+2+(s+1)V_{in}}{s^3+4s^2+4s+2}$$

$$\Delta_2 = \frac{s^2+4s+2+(s+1)V_{in}}{s}$$

$$\Rightarrow V_{out} = \frac{(s^2+4s+2+(s+1)V_{in})(s+1)}{s^3+4s^2+4s+2} - 1 \quad \text{(D)}$$

d. An LTI system has the following differential equation description, with initial conditions $x(0-) = 1$, $y(0-) = 1$, $x'(0-) = 1$, $y'(0-) = 1$, $y''(0-) = 1$, and input $x(t) = te^{-2t}u(t)$:

$$\frac{d^2y}{dt^2} + 12\frac{dy}{dt} + 35y = \frac{dx}{dt} + 3x$$

(i) (5 points) Identify the poles, zeros, and modes of this system.

$s^2 + 12s + 35 = (s+5)(s+7)$ characteristic polynomial

$H(s) = \frac{(s+3)}{(s+5)(s+7)} \Rightarrow$ poles: $\{-5, -7\}$ A

zero: $\{-3\}$ B

modes: $\{-5, -7\}$ C

Grading
- 1 pt trying
- A 2 pts
- B, C, 1 pt each.

(ii) (5 points) Calculate the total response of this system in the Laplace domain.

$$\left.\left[s^2Y - sy(0^-) - y'(0^-)\right] + 12\left[sY - y(0^-)\right] + 35Y = sX \quad x(0^-) + \quad X\right\} \text{ A}$$

$$\left[s^2Y - s - \cancel{1}\right] + 12\left[sY - 1\right] + 35Y = sX - \cancel{1} + 3X$$

$$Y(s^2 + 12s + 35) = (s+3)X + s + 12$$

$$X(s) = \mathcal{L}\left[t e^{-2t} u(t)\right] = \frac{1}{(s+2)^2} \text{ B}$$

$$\Rightarrow Y(s) = \frac{\frac{(s+3)}{(s+2)^2} + s + 12}{s^2 + 12s + 35} = \frac{s^3 + 16s^2 + 53s + 51}{(s+2)^2(s^2 + 12s + 35)} \text{ C}$$

- 1 pt Trying
- A 2 pts
- B, C, 1 pt each

5. (25 points) Fourier Series and Transform

a. (7 points). Find the Fourier Series of the signal $x(t) = |\sin(t)|$ in complex exponential representation where $|\ |$ indicates the absolute value.

$$x_n = \frac{1}{T_0}\int_0^{\pi} \sin t\, e^{-jn\omega_0 t} dt$$

A

The period of $\sin(t)$ is 2π, and the period of $|\sin t|$ is π. Here, $T_0 = \pi$ and $\omega_0 = \frac{2\pi}{T_0} = 2$.

$$x_n = \frac{1}{\pi}\int_0^{\pi} \sin t\, e^{-j2nt} dt \quad \text{— B}$$

Using Euler's identities

$$x_n = \frac{1}{\pi}\int_0^{\pi} \left(\frac{e^{jt} - e^{-jt}}{2j}\right) e^{-j2nt} dt$$

$$x_n = \frac{1}{2j\pi}\int_0^{\pi} \left(e^{jt} - e^{-jt}\right) e^{-j2nt} dt = \frac{1}{2j\pi}\int_0^{\pi} \left(e^{jt}\right) e^{-j2nt} dt - \frac{1}{2j\pi}\int_0^{\pi} \left(e^{-jt}\right) e^{-j2nt} dt$$

$$= \frac{1}{2j\pi}\int_0^{\pi} e^{j(1-2n)t}\, dt - \frac{1}{2j\pi}\int_0^{\pi} e^{-j(1+2n)t}\, dt \quad \text{— C}$$

$$= \frac{1}{2j\pi}\left\{\left[\frac{e^{j(1-2n)t}}{j(1-2n)}\right] - \left[\frac{e^{-j(1+2n)t}}{-j(1+2n)}\right]\right\}_0^{\pi}$$

$$= \frac{1}{2j\pi}\left\{\left[\frac{e^{j(1-2n)\pi} - 1}{j(1-2n)}\right] - \left[\frac{e^{-j(1+2n)\pi} - 1}{-j(1+2n)}\right]\right\}$$

$$= \frac{1}{2j^2\pi}\left\{\left[\frac{e^{j(1-2n)\pi} - 1}{(1-2n)}\right] + \left[\frac{e^{-j(1+2n)\pi} - 1}{(1+2n)}\right]\right\} \quad \text{— D}$$

$e^{j(1-2n)\pi}$ and $e^{-j(1+2n)\pi}$ are always -1 (check by substituting $n = 1, 2, 3, 4 \ldots$). Therefore,

$$x_n = \frac{-1}{2\pi}\left\{\left[\frac{-2}{(1-2n)}\right] + \left[\frac{-2}{(1+2n)}\right]\right\}$$

$$x_n = \frac{2}{2\pi}\left\{\left[\frac{1}{(1-2n)}\right] + \left[\frac{1}{(1+2n)}\right]\right\}$$

$$x_n = \frac{1}{\pi}\left\{\left[\frac{2}{(1-4n^2)}\right]\right\}$$

$$x_n = \frac{2}{\pi}\frac{1}{(1-4n^2)} \quad \text{— E}$$

2 points for trying. 1 points each for A-E

b. (6 points) The signal $x_2(t)$ is defined as follows

$$x_2(t) = \sum_{n=0}^{\infty} e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0)\, e^{jn\omega_0(t)}$$

Find the Fourier Series of $x_2(t)$ in cosine/sine form. Please write your final answer below:

$a_0 =$ ______________________________

$a_n =$ ______________________________

$b_n =$ ______________________________

$$x_2(t) = \sum_{n=0}^{\infty} e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0)\, e^{jn\omega_0 t}$$

Substituting $n = 0$ gives the DC term.

$$a_0 = 0 \quad \text{A}$$

Using Euler's identifies, $e^{jn\omega_0 t} = \cos(n\omega_0 t) + j\sin(n\omega_0 t)$. Therefore,

$$x_2(t) = a_0 + \sum_{n=1}^{\infty} e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0)\, e^{jn\omega_0 t}$$

$$= a_0 + \sum_{n=1}^{\infty} e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0)\, [\cos(n\omega_0 t) + j\sin(n\omega_0 t)]$$

$$= a_0 + \sum_{n=1}^{\infty} e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0) \cos(n\omega_0 t) + e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0)\, j\sin(n\omega_0 t)$$

We can identify a_n and b_n from the above expression.

$$a_n = e^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0) \quad \text{— B}$$

$$b_n = je^{-jn(3\omega_0+2)} \sin(2n\pi\omega_0) \quad \text{— C}$$

1 point for trying. 2 points each for A-C

c. (6 points) The Fourier transform of a signal $x(t)$ is $X(\omega)$. Find the Fourier transform of the signal $y(t) = x(2t) * x(-3t - 3)$, where $*$ represents convolution.

1 point for trying. 1 point for A-E

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega)$$

$$x(2t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2} X\left(\frac{\omega}{2}\right) \quad \text{— A}$$

(Table 5-7, #3 Time scaling property).

$$x(-3t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{3} X\left(-\frac{\omega}{3}\right) \quad - \text{B}$$

(Table 5-7, #3 Time scaling property).

$$x(-3(t+1)) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{3} X\left(-\frac{\omega}{3}\right) e^{+j\omega} \quad - \text{C}$$

(Table 5-7, #4 Time shifting property).

— D

Convolution in the time domain results in multiplication in the Fourier domain (Table 5-7, #12)

$$Y(\omega) = \frac{1}{2\pi}\frac{1}{2} X\left(\frac{\omega}{2}\right) \cdot \frac{1}{3} X\left(-\frac{\omega}{3}\right) e^{j\omega} = \underbrace{\frac{1}{12\pi}\left[X\left(\frac{\omega}{2}\right) X\left(-\frac{\omega}{3}\right)\right] e^{j\omega}} \quad - \text{E}$$

d. (6 points) Find the Fourier transform of $x_4(t) = sinc(t-10)e^{j100t}$. What kind of filter does $x_4(t)$ represent and why?

Hint: $sinc(t) \overset{\mathcal{F}}{\leftrightarrow} \pi\, rect\left(\frac{\omega}{2}\right)$

$$sinc(t) \overset{\mathcal{F}}{\leftrightarrow} \pi\, rect\left(\frac{\omega}{2}\right)$$

Using the time shifting property

$$sinc(t-10) \overset{\mathcal{F}}{\leftrightarrow} \pi\, rect\left(\frac{\omega}{2}\right) e^{-j10\omega} \quad - \text{A}$$

Using the frequency shifting property

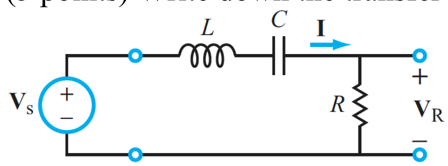
$$sinc(t-10)\, e^{j100t} \overset{\mathcal{F}}{\leftrightarrow} \underbrace{\pi\, rect\left(\frac{\omega - 100}{2}\right)}_{\text{B}} \underbrace{e^{-j10(\omega-100)}}_{\text{C}}$$

This frequency spectrum is a $rect$ function centered around $\omega = 100$ (D) and width 2. Therefore, it represents a bandpass filter. E

1 point for trying. 1 point for A-E

6. (20 points) Applications of Fourier Methods

a. (5 points) Write down the transfer function of the following circuit in the frequency domain.



• 1 pt trying

• 5 pts correct answer

$$H(\omega) = \frac{j\omega RC}{(1-\omega^2 LC) + j\omega RC}$$

b. (5 points) Calculate the center frequency ω_0, bandwidth B, and the frequency ω_{45} at which the phase shift of the filter in part (a) above is 45^o, when $L = 2H, C = 1F$, and $R = 1\Omega$.

Center frequency $\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{2\times 1}} = \frac{1}{\sqrt{2}}$ rad/sec A

B = Bandwidth $= \frac{R}{L} = \frac{1}{2}$ rad/sec B

$$\phi(\omega) = \angle H(\omega) = \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega RC}{1-\omega^2 LC}\right) = \frac{\pi}{4}$$

$$\Rightarrow \omega RC = 1 - \omega^2 LC \Rightarrow \omega^2 LC + \omega RC - 1 = 0$$

Grading:

· 1 pt trying

· +1 pt each A & B

· +2 pts C

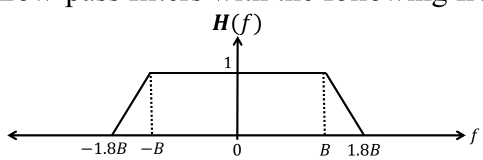
$$\Rightarrow 2\omega^2 + \omega - 1 = 0$$

Roots : $\frac{-1 \pm \sqrt{1^2 + 4\cdot 2}}{4}$ Pick the positive root

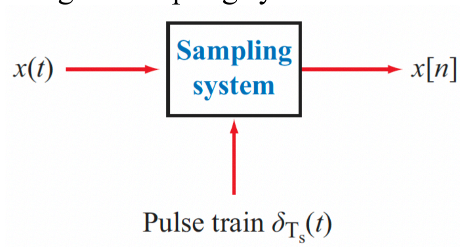
$$\omega_{45} = \frac{\sqrt{9} - 1}{4} = \frac{1}{2} \text{ rad/sec}$$ C

c. (10 points) If the signal $x(t)$ is bandlimited to 10kHz, calculate the minimum sampling rate for the signal $y(t) = x^2(t)$ using the following available components that will ensure the sampled signal can be reconstructed perfectly.

- Low-pass filters with the following frequency response (B is adjustable):



- A signal sampling system with an adjustable sampling rate $f_S = 1/T_S$.

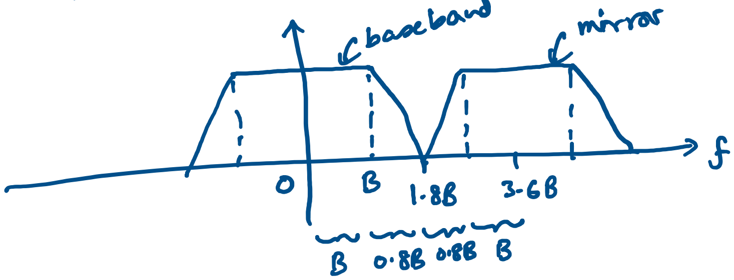


Grading:
- 2pts trying
- (A) – 2 pts
- (B) – 3 pts
- (C) – 3 pts

Sketch a diagram showing how these components will be adjusted and interconnected.

y(t) is bandlimited to 20kHz since multiplication in the time domain is equivalent to convolution in the frequency domain.

$\Rightarrow B = 20\text{kHz}$ (A)



Sampling rate $= 3.6B$

$= 3.6 \times 20\text{kHz} = 72\text{kHz}$ (B)

Diagram