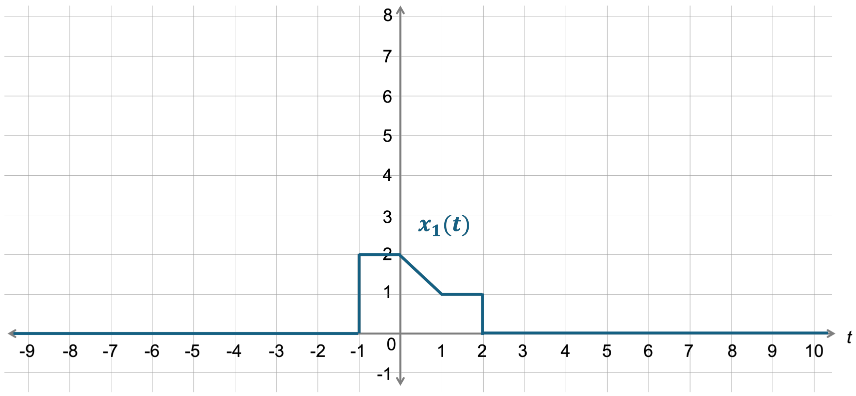
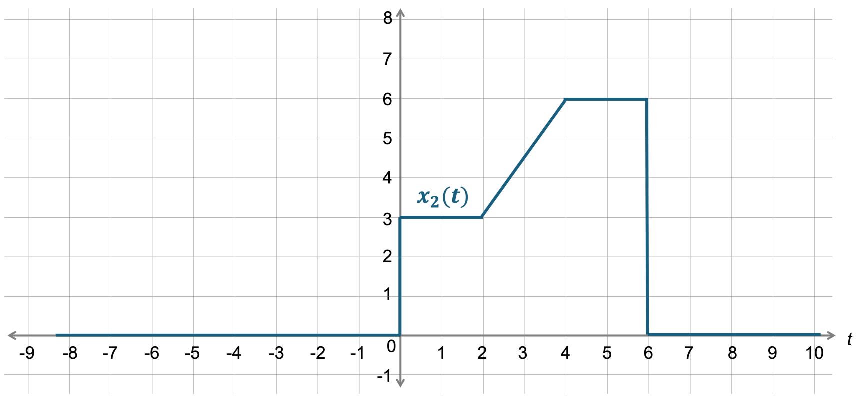
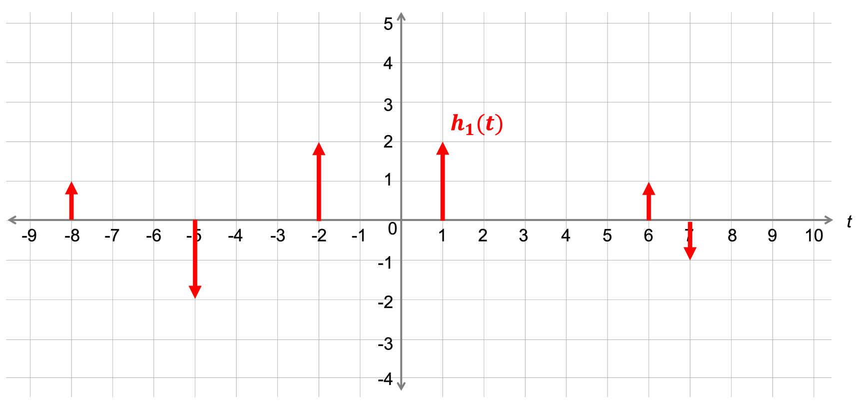
Spring 2024

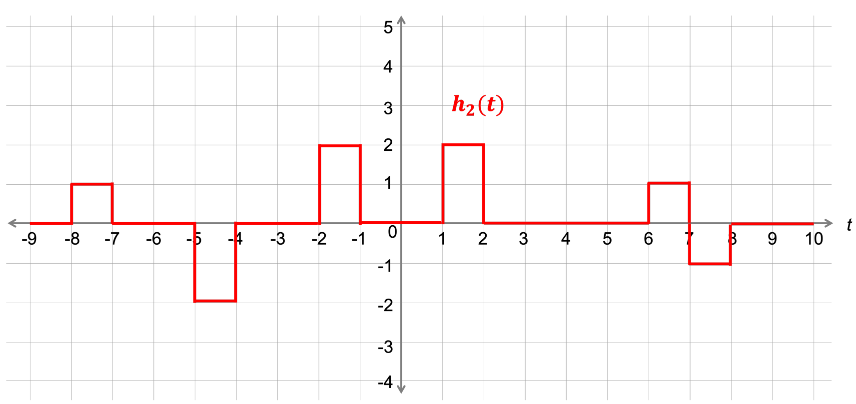
1. (15 points) Waveforms and Convolution

The signals $x_1(t), x_2(t), h_1(t),$ and $h_2(t)$ are as graphed below.









a. (4 points) Write the signal $x_2(t)$ in terms of $x_1(t)$. Hint: $x_2(t)$ is a transformed version of $x_1(t)$.

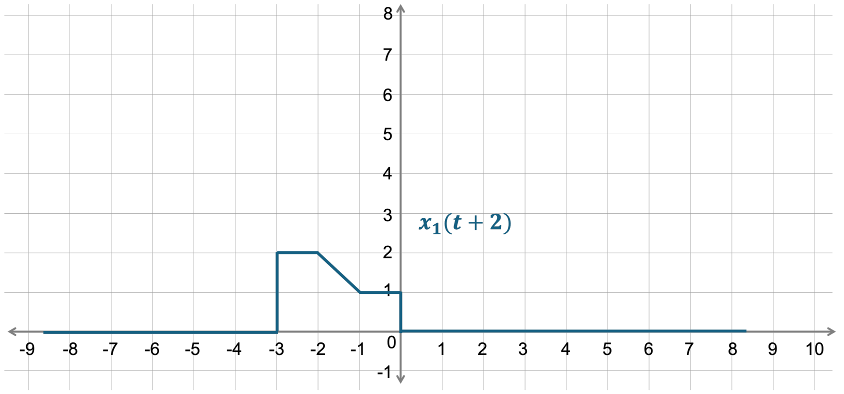
1 point for trying

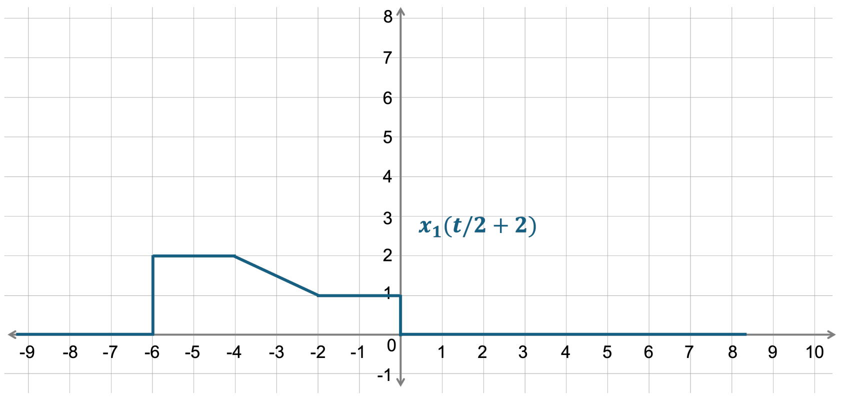
1 point each for A - C

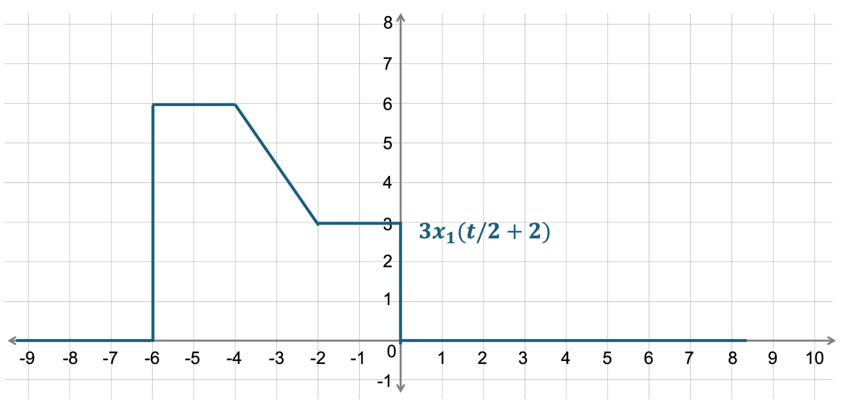
$$x_2(t) = 3x_1\left(-\frac{t}{2} + 2\right)$$

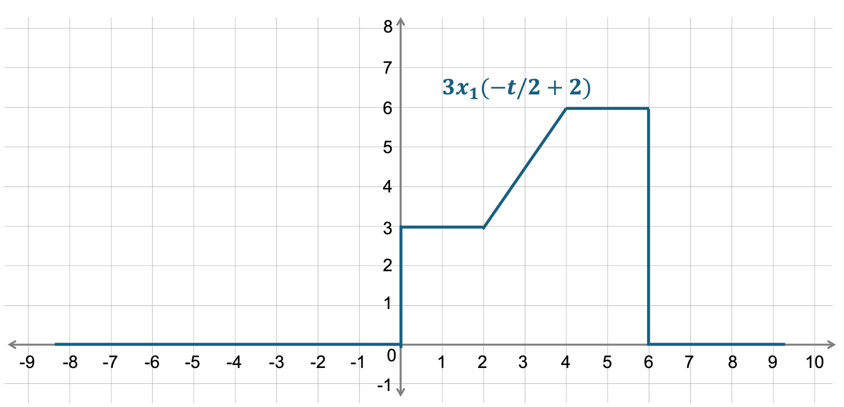
A B C

Solution steps:

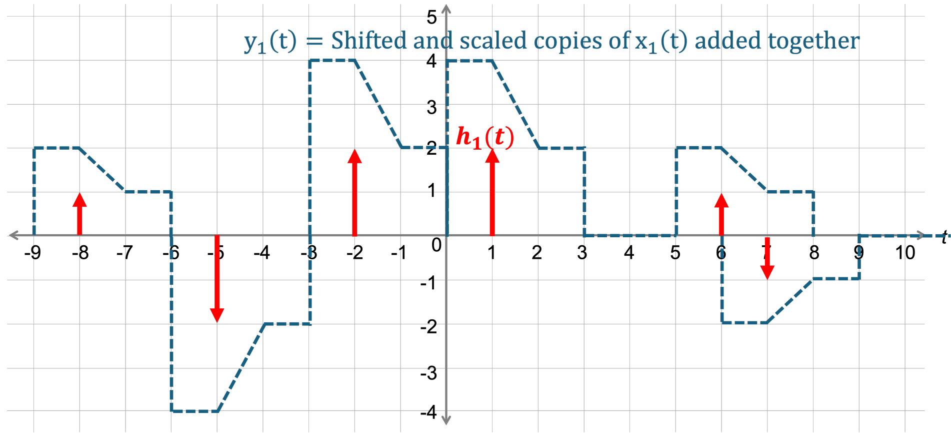


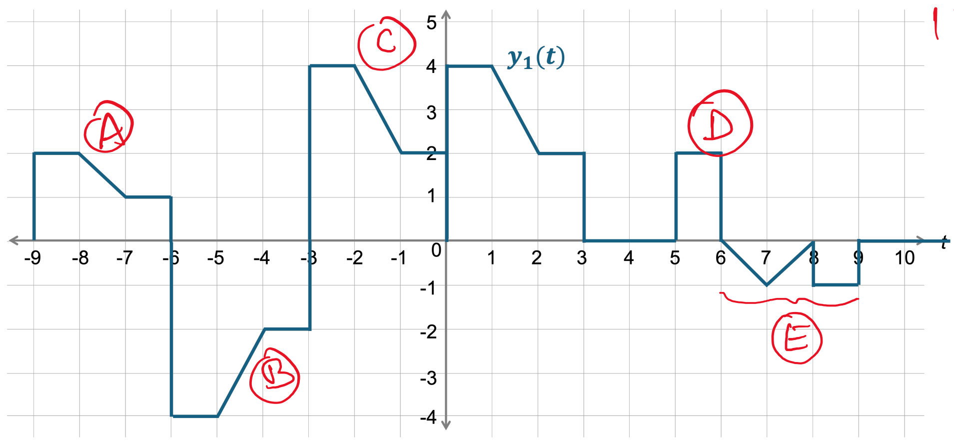






b. (6 points) Graph the output of the convolution: $y_1(t) = x_1(t) * h_1(t)$, where $*$ represents convolution. Note that the heights of the delta function arrows indicate their amplitudes. Your final answer must be presented in the space below, and this will be the only answer graded.





1 point for A-E

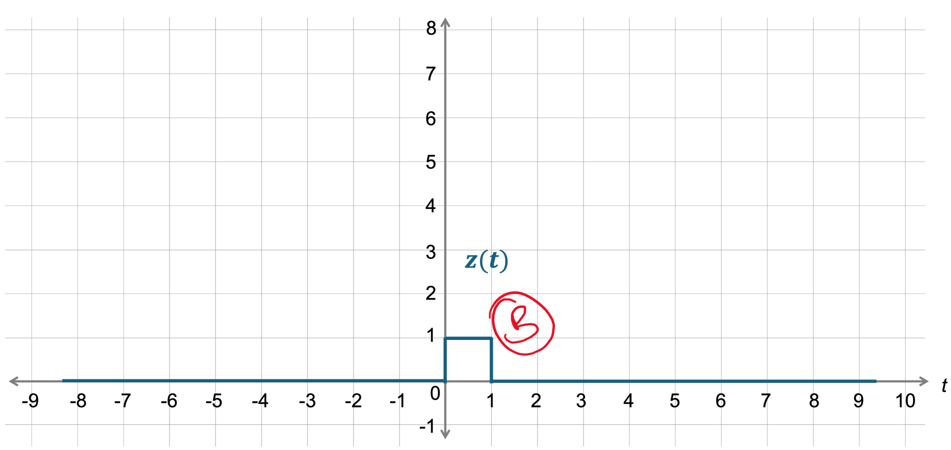
1 point for trying.

c. (5 points) If the signal $h_2(t)$ graphed above satisfies the equation: $h_2(t) = z(t) * h_1(t)$, where $*$ represents convolution, then graph the signal $z(t)$. Your final answer must be presented in the space below, and this will be the only answer graded.

A

Convolution of any signal $p(t)$ with $A\delta(t - T_0)$ results in $Ap(t - T_0)$, i.e. a scaled copy at $t = T_0$. Therefore, $z(t)$ is as shown below. 1 point for trying.

2 points for A and B



2. (10 points) Linear Time-invariant Systems

a. (5 points) An LTI system has the impulse response $h(t) = \cos(t)u(t)$. Calculate the response to the input signal $x(t) = \cos(t)\,u(t)$ and determine whether or not this system is BIBO stable.

Time-domain solution:

$$y(t) = \cos(t)\,u(t) * \cos(t)\,u(t) \quad (A)$$

$$= \frac{(e^{jt}+e^{-jt})}{2}u(t) * \frac{(e^{jt}+e^{-jt})}{2}u(t)$$

Grading
- 1 pt trying
- 2-3 pts partial
- 5 pts correct answer

$$= \frac{1}{4}\left\{e^{jt}u(t)*e^{jt}u(t)+e^{jt}u(t)*e^{-jt}u(t)+e^{-jt}u(t)*e^{jt}u(t)+e^{-jt}u(t)*e^{-jt}u(t)\right\}$$

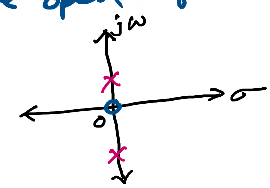
$$= \frac{1}{4}\left\{te^{jt}+2\left(\frac{e^{jt}-e^{-jt}}{2j}\right)+te^{-jt}\right\}u(t) = \underbrace{\frac{1}{2}t\cos(t)\,u(t)}_{\text{unbounded}}+\frac{1}{2}\sin(t)u(t)$$

$\Rightarrow$ NOT BIBO stable

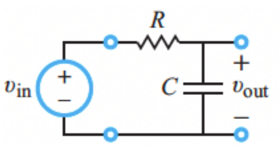
Frequency-domain solution:

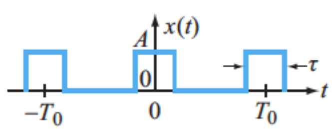
$$Y(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2+1}\right) = \frac{s^2}{(s^2+1)^2}$$

NOT BIBO stable since the poles are on the j axis, so NOT in the open left-half plane.



b. (5 points) Calculate the response of the following system to the input $x(t)$.





$$x(t) = \frac{A\tau}{T_0} + \sum_{n=1}^{\infty}\frac{2A}{n\pi}\sin\left(\frac{n\pi\tau}{T_0}\right)\cos\left(\frac{2n\pi t}{T_0}\right)$$

$$\hookrightarrow H(\omega) = \frac{1/j\omega C}{R+1/j\omega C} = \frac{1}{1+j\omega RC} \quad (A) \qquad |H(\omega)| = \frac{1}{\sqrt{1+(\omega RC)^2}},\ \angle H(\omega) = \tan^{-1}(\omega RC)$$

$$\Rightarrow y(t) = \frac{A\tau}{T_0}|H(0)| + \sum_{n=1}^{\infty}\frac{2A}{n\pi}|H(n\omega_0)|\sin\left(\frac{n\pi\tau}{T_0}\right)\cos\left(\frac{2n\pi t}{T_0}-\angle H(n\omega_0)\right)$$

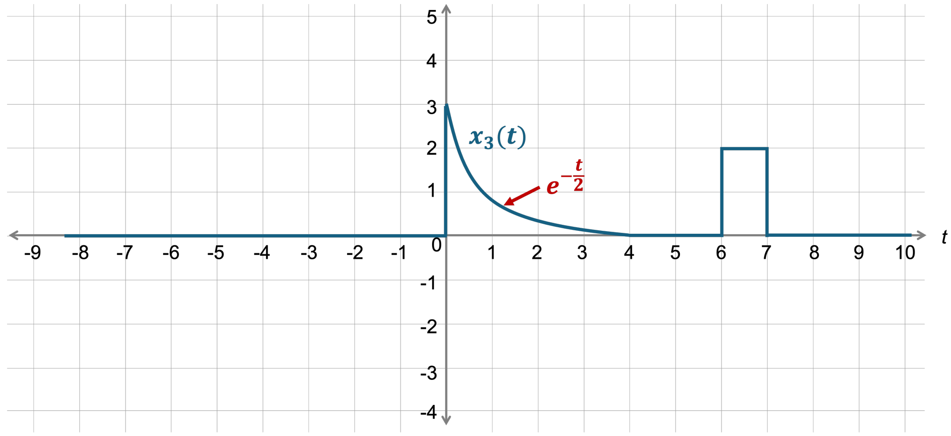
$$= \underbrace{\frac{A\tau}{T_0}}_{(B)} + \sum_{n=1}^{\infty}\underbrace{\frac{2A}{n\pi\sqrt{1+(n\omega_0 RC)^2}}}_{(C)}\sin\left(\frac{n\pi\tau}{T_0}\right)\cos\left(\frac{2n\pi t}{T_0}-\underbrace{\tan^{-1}(n\omega_0 RC)}_{(D)}\right)$$

where $\omega_0 = \frac{2\pi}{T_0}$

Grading:
- 1 pt trying
- +1 pt each (A) – (D)

3. (10 points) Laplace Transform & Its Inverse

a. (5 points) Determine the Laplace transform $\mathbf{X_3(s)}$ of the waveform:



What are the initial and final values of this signal?

1 point for trying.

1 point for A – D

$$x_3(t) = 3e^{-\frac{t}{2}}u(t) + 2u(t-6) - 2u(t-7)$$

$$\mathbf{X_3(s)} = \frac{3}{\left(s+\frac{1}{2}\right)} + \frac{2}{s}e^{-6s} - \frac{2}{s}e^{-7s}$$

A B C

The initial value $x_3(0+) = 3$ (from graph above)

And final value $x_3(\infty) = 0$ (from graph above)

D

b. (5 points) Determine the inverse Laplace Transform of $\mathbf{H(s)} = \frac{s^3+7s^2+3s+4}{s^2+7s+10}$.

1 point for trying.

Order of Nr > Order of Dr. Performing long division, A

$$\mathbf{H(s)} = \frac{s^3+7s^2+3s+4}{s^2+7s+10} = s + \frac{-7s+4}{s^2+7s+10}$$

$$= s + \frac{-7s+4}{(s+5)(s+2)} \quad \text{– B}$$

Performing partial fraction expansion:

$$= s + \frac{6}{s+2} - \frac{13}{s+5} \quad \text{– C}$$

The inverse Laplace transform is

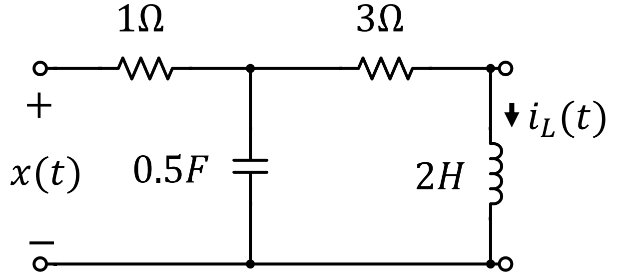
$$\frac{d}{dt}\delta(t) + 6e^{-2t}u(t) - 13e^{-5t}u(t)$$

1 point A– D

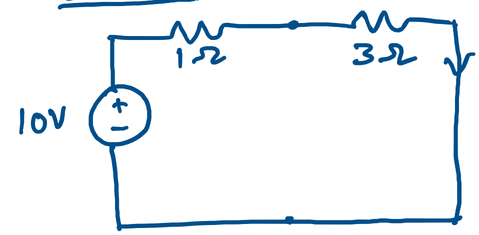
D

4. (20 points) Applications of the Laplace Transform

Consider the following circuit and the input signal $x(t) = 10 - 5u(t)$ V.



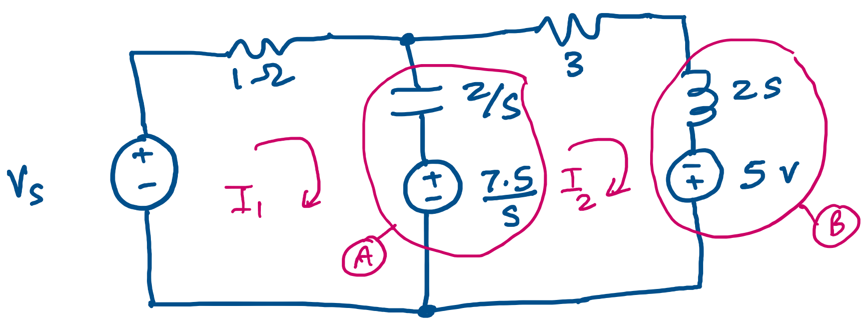
Initial:



$i_L(0^-) = 2.5A$

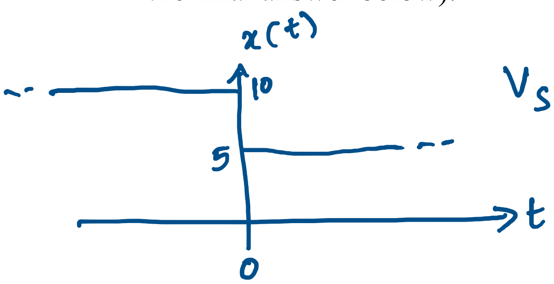
$v_c(0^-) = 7.5V$

a. (2 points) Sketch the Laplace-domain version of the given circuit with the initial conditions indicated (use the back sheet for your work as needed and write the final answer below).



Grading:
- 1 pt each Ⓐ, Ⓑ

b. (3 points) Write the Laplace transform of the input for $t \geq 0$ (use the back sheet for your work and write the final answer below).



$V_s = \frac{5}{s}$ for $t \geq 0$ part.

Grading
- 1 pt trying
- 3 pts correct answer

c. (5 points) Calculate the Laplace-domain current $\mathbf{I}_L(\mathbf{s})$ for $t \geq 0$. (use the back sheet for your work and write the final answer below). Your answer should be in the form of a ratio of two polynomials.

mesh equations:

$$\left.\begin{aligned} \frac{5}{s} - \frac{7.5}{s} &= I_1\left(1+\frac{2}{s}\right) - I_2\left(\frac{2}{s}\right) \\ \frac{7.5}{s} + 5 &= -I_1\left(\frac{2}{s}\right) + I_2\left(\frac{2}{s} + 3 + 2s\right) \end{aligned}\right\} Ⓐ$$

Grading:
- 1 pt trying
- +2 pts Ⓐ
- +2 pts Ⓑ

$$\Delta = \begin{vmatrix} 1+\frac{2}{s} & -\frac{2}{s} \\ -\frac{2}{s} & 2s+3+\frac{2}{s} \end{vmatrix} = \frac{2s^2+7s+8}{s} \qquad \Delta_2 = \begin{vmatrix} 1+\frac{2}{s} & -\frac{2.5}{s} \\ -\frac{2}{s} & 5+\frac{7.5}{s} \end{vmatrix} = \frac{5s^2+17.5s+10}{s^2}$$

$$\Rightarrow I_L(s) = \frac{\Delta_2}{\Delta} = \left.\frac{5s^2+17.5s+10}{s(2s^2+7s+8)}\right\} Ⓑ$$

d. An LTI system has the following differential equation description, with initial conditions $x(0-) = 1$, $y(0-) = 1$, $x'(0-) = 1$, $y'(0-) = 1$, $y''(0-) = 1$, and input $x(t) = e^{-2t}u(t)$:

$$\frac{d^2y}{dt^2} + 8\frac{dy}{dt} + 15y = \frac{dx}{dt} + 4x \Rightarrow H(s) = \frac{s+4}{(s+3)(s+5)}$$

and

$$X(s) = \frac{1}{(s+2)}$$

(i) (5 points) Identify the poles, zeros, and modes of this system.

$s^2 + 8s + 15 = (s+3)(s+5)$

poles: $\{-3, -5\}$ A

zeros: $\{-4\}$ B

modes: $\{-3, -5\}$ C

Grading:
- 1 pt trying
- + 2 pts A
- +1 pt B
- +1 pt C

(ii) (5 points) Calculate the total response of this system in the Laplace domain. Your answer should be expressed as a ratio of polynomials.

Transform each term of the LCCDE:

A $[s^2Y(s) - s\,y(0^-) - y'(0^-)] + 8[sY(s) - y(0^-)] + 15\,Y(s) = sX(s) - x(0^-) + 4X(s)$

$[s^2Y(s) - s - \cancel{1}] + 8[sY(s) - 1] + 5\,Y(s) = \frac{s}{s+2} - \cancel{1} + \frac{4}{(s+2)}$

$(s^2 + 8s + 15)Y(s) = \frac{s+4}{s+2} + s + 8 = \frac{s^2 + 11s + 20}{(s+2)}$

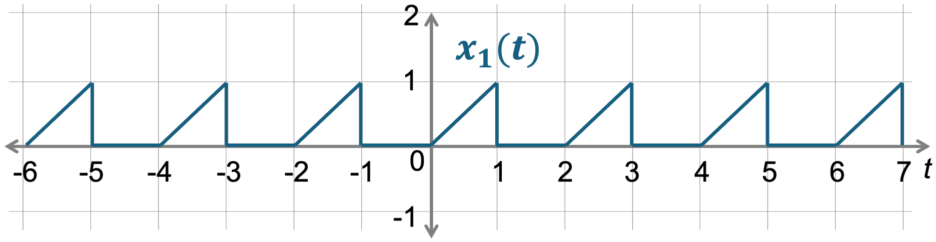
Grading
- 1 pt trying
- +2 pt A
- +2pt B

$\Rightarrow Y(s) = \frac{s^2 + 11s + 20}{(s+2)(s+3)(s+5)} = \frac{s^2 + 11s + 20}{(s^3 + 10s^2 + 31s + 30)}$ B

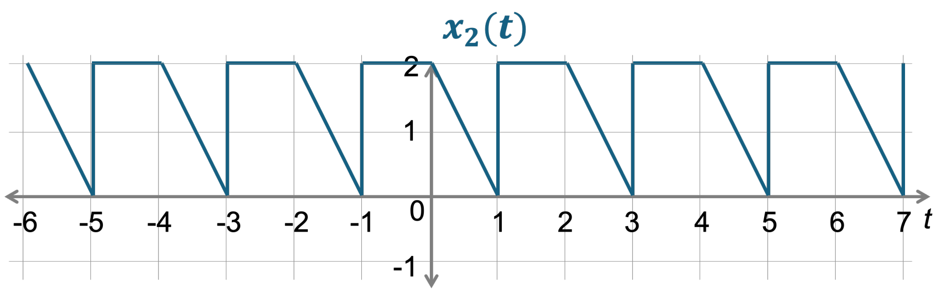
5. (25 points) Fourier Series and Transform

a. (7 points). The Fourier Series of the following signal $x_1(t)$ is

$$x_1(t) = \frac{1}{4} + \sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi^2}[\cos(n\pi) - 1]\cos(n\pi t) - \frac{1}{n\pi}\cos(n\pi)\sin(n\pi t)\right\}$$



Find the Fourier Series of the signal $x_2(t)$ graphed below in cosine/sine form.



2 points for trying.

1 point each A – E

Observe that $x_2(t)$ is a transformed version of $x_1(t)$. Specifically,

$$x_2(t) = 2(1 - x_1(t)) \quad = 2 - 2x_1(t)$$

B

A

Substituting $x_1(t)$, $x_2(t) = 2 - 2\frac{1}{4} - \sum_{n=1}^{\infty}\left\{\frac{2}{n^2\pi^2}[\cos(n\pi) - 1]\cos(n\pi t) - \frac{2}{n\pi}\cos(n\pi)\sin(n\pi t)\right\}$

$$x_2(t) = \frac{3}{2} - \sum_{n=1}^{\infty}\left\{\frac{2}{n^2\pi^2}[\cos(n\pi) - 1]\cos(n\pi t) - \frac{2}{n\pi}\cos(n\pi)\sin(n\pi t)\right\}$$

C D E

b. (6 points) The Fourier series of a signal $x_3(t)$ in complex exponential notation is

$$x_3(t) = \sum_{n=-\infty}^{\infty} e^{-jn3\omega_0}\cos n\pi\omega_0\, e^{jn\omega_0 t}$$

Find the Fourier Series of $x_3(t)$ in cosine/sine form. Please write your final answer below:

1 point for A

2 points each for B, C

$a_0 =$ ________1________________________ A

$a_n =$ ______ $e^{-jn3\omega_0}\cos n\pi\omega_0$ ______________ B

$b_n =$ ______ $j\, e^{-jn3\omega_0}\cos n\pi\omega_0$ ______________ C

$$x_3(t) = \sum_{n=-\infty}^{\infty} e^{-jn3\omega_0} \cos n\pi\omega_0 \, e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} e^{-jn3\omega_0} \cos n\pi\omega_0 \left[\cos(n\omega_0 t) + j\sin(n\omega_0 t)\right]$$

$$= \sum_{n=-\infty}^{\infty} e^{-jn3\omega_0} \cos n\pi\omega_0 \cos(n\omega_0 t) + j\, e^{-jn3\omega_0} \cos n\pi\omega_0 \sin(n\omega_0 t) \,]$$

For $n = 0, x_0 = a_0 = e^{j0} \times \cos(0) = 1 \times 1 = 1$

1 point for trying.

c. (6 points) The Fourier transform of a signal $x(t)$ is $X(\omega)$. Find the Fourier transform of the signal $y(t) = x\left(\frac{t}{2}\right) * x\left(\frac{t-3}{2}\right)$, where $*$ represents convolution.

A - C : 1 point

D : 2 points

1 point for trying.

$$x(t) \overset{\mathcal{F}}{\leftrightarrow} X(\omega)$$

$$x\left(\frac{t}{2}\right) \overset{\mathcal{F}}{\leftrightarrow} 2X(2\omega) \text{ (Table 5-7, \#3).}$$ A

$$x\left(\frac{t-3}{2}\right) \overset{\mathcal{F}}{\leftrightarrow} 2X(2\omega)e^{-j3\omega} \text{ (Table 5-7, \#3, \#4).}$$ B

Convolution in the time domain results in multiplication in the Fourier domain (Table 5-7, #12)

$$Y(\omega) = \frac{1}{2\pi} 2X(2\omega) \cdot 2X(2\omega)e^{-j3\omega} = \frac{2}{\pi}[X(2\omega)]^2 e^{-j3\omega}$$

C D

d. (6 points) Find the Fourier transform of $x_4(t) = 2\,sinc(t + 2\pi) + 3\,sinc(t) + 4\,sinc(t - 2\pi)$. What kind of filter does $x_4(t)$ represent and why?

Hint: $sinc(t) \overset{\mathcal{F}}{\leftrightarrow} \pi\, rect\left(\frac{\omega}{2}\right)$. 1 point for trying.

1 point for A - E

$$sinc(t) \overset{\mathcal{F}}{\leftrightarrow} \pi\, rect\left(\frac{\omega}{2}\right)$$

$$3sinc(t) \overset{\mathcal{F}}{\leftrightarrow} 3\pi\, rect\left(\frac{\omega}{2}\right)$$ A

$$2sinc(t + 2\pi) \overset{\mathcal{F}}{\leftrightarrow} 2\pi\, rect\left(\frac{\omega}{2}\right) e^{j2\pi\omega} \text{ (Table 5-7, \#4).}$$ B

$$4sinc(t - 2\pi) \overset{\mathcal{F}}{\leftrightarrow} 4\pi\, rect\left(\frac{\omega}{2}\right) e^{-j2\pi\omega} \text{ (Table 5-7, \#4).}$$ C

$$X_4(\omega) = 2\pi\, rect\left(\frac{\omega}{2}\right) e^{j2\pi\omega} + 3\pi\, rect\left(\frac{\omega}{2}\right) + 4\pi\, rect\left(\frac{\omega}{2}\right) e^{-j2\pi\omega}$$

$$= \pi\, rect\left(\frac{\omega}{2}\right)\left[2e^{j2\pi\omega} + 3 + 4e^{-j2\pi\omega}\right]$$ D

E

Multiplication with $rect\left(\frac{\omega}{2}\right)$ implies that $X_4(\omega) = 0$ for $|\omega| > 1$. Therefore, $x_4(t)$ is a low pass filter.

6. (20 points) Applications of Fourier Methods

a. (5 points) What type of filter does the following system represent in the frequency domain when $L = 0.01H, C = 0.01F$, and $R = 0.1\Omega$?

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = x(t) + LC\frac{d^2x(t)}{dt^2}$$

$$H(\omega) = \frac{1 + (j\omega)^2 LC}{(j\omega)^2 LC + (j\omega)RC + 1} \quad A$$

$= 0$ when $\omega^2 LC = 1$, at $\omega_0 = \frac{1}{\sqrt{LC}}$

when $L = 0.01\,H$, $C = 0.01F$, $LC = 10^{-4}$

$\Rightarrow \omega_0 = 100$ rad/sec

This is a $\underbrace{\text{band-stop filter}}_{B}$ @ 100 rad/sec

Grading
- 1 pt trying
- +2 pt A
- +2 pt B

b. (5 points) Calculate the center frequency, bandwidth, and quality factor Q for the filter in part (a) above.

$\omega_0 = 100$ rad/sec A

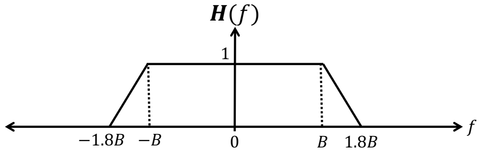
$B = \frac{R}{L} = \frac{0.1}{0.01} = 10$ rad/sec B

$Q = \frac{\omega_0}{B} = \frac{100}{10} = 10$ C

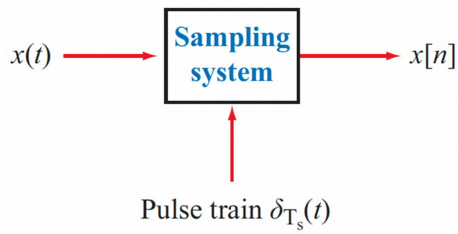
Grading
- 2 pt trying
- +1 pt each A, B, C

c. (10 points) Audio signal frequencies range from 20Hz to 20kHz, and we wish to digitize them using the following available components:

- Low-pass filters with the following frequency response (B is adjustable):

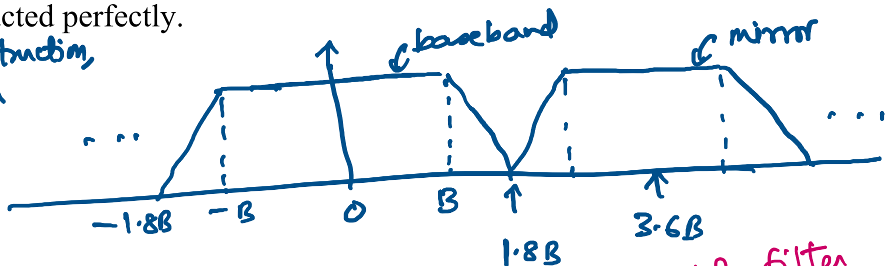


- A signal sampling system with an adjustable sampling rate.



Sketch a diagram showing how these components will be adjusted and interconnected. Find the value of B and the minimum sampling rate that will allow the signal to be sampled and reconstructed perfectly.

For perfect reconstruction, the mirror spectra should not overlap with the baseband spectrum



$\Rightarrow B = 20\text{ kHz}$ Ⓐ setting for LP filter

sampling rate $= 3.6 \times 20 = 72\text{ kHz}$ Ⓑ setting for Sampling system

Grading

- 2 pts trying
- +3 pts each Ⓐ, Ⓑ
- 2 pts diagram Ⓒ

Diagram:

Ⓒ