HW 1: Analytical Portion

github.com/Buggy-Virus/SI-ML-Homework/tree/master/assignment-1
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Problem 1

We want to show that for any matrix A it is the case that $\text{rank}(A) = \text{rank}(A'A)$, where A' is the transpose. Consider some matrix A with dimensions $m \times n$, and it's transpose A'. If we can represent A as the following:

$$\begin{bmatrix} a_{00} & a_{01} & \cdots & a_{0n} \\ a_{10} & a_{11} & & \vdots \\ \vdots & & \ddots & \\ a_{m0} & \cdots & & a_{mn} \end{bmatrix}$$

Thus for the product, $A'A$ we have the $n \times n$ matrix:

$$\begin{bmatrix} a_{00}^2 + a_{10}^2 + \cdots + a_{m0}^2 & \cdots & a_{00}a_{0n} + a_{10}a_{1n} + \cdots + a_{m0}a_{mn} \\ \vdots & \ddots & \vdots \\ a_{0n}a_{00} + a_{1n}a_{10} + \cdots + a_{mn}a_{m0} & \cdots & a_{0n}^2 + a_{1n}^2 + \cdots + a_{mn}^2 \end{bmatrix}$$

Note that alternatively we could think of $A'A$ as the collection of its columns concatenated together along the horizontal axis, and this collection of columns defines it's column space or $\text{col}(A'A)$. Then we can represent the first column as the linear combination of each column of A', where the coefficients are a column of A (note that any column of A corresponds to a row of A'). Thus for $0 \leq i \leq n$ we can represent any arbitrary column of $A'A$ as:

$$\begin{bmatrix} a_{00} & \cdots & a_{0m} \\ \vdots & \ddots & \vdots \\ a_{n0} & \cdots & a_{nm} \end{bmatrix} \begin{bmatrix} a_{0i} \\ \vdots \\ a_{mi} \end{bmatrix} = a_{0i} \begin{bmatrix} a_{00} \\ \vdots \\ a_{n0} \end{bmatrix} + \cdots + a_{mi} \begin{bmatrix} a_{0m} \\ \vdots \\ a_{nm} \end{bmatrix}$$

Note that in order for any column of A' to not be used in the linear combination of any column of $A'A$, or in other words it isn't used once, it would require every coefficient to be 0 for every column of $A'A$.

Thus for the jth column of A' to never be used, where $0 \leq j \leq m$, it would require all values $a_{0j} \dots a_{nj}$ to all be zero, as for each column, these are the values of A it has as it's coefficients, which is the jth row of A, which are the same values contained in the column itself.

As such it implies that were a column A' to not be represented in any column of $A'A$ would require that it be a column of zeros itself. Implying that every nonzero column of A' must be a nonzero term in at least one column of $A'A$.

It follows then that the column basis of A' must also be the column basis of $A'A$. Trivially since the column basis of A' spans the columns of A' it must span a collection of columns where each column is a linear combination of A''s columns, and thus spans the columns of $A'A$. Additionally, were you to remove any single vector from the column basis of A' it would no longer span A' as it is the minimum spanning set, and as such it would not be able to form every column of A' as a linear combination, and thus it would no longer be able to form every column of $A'A$ as a linear combination, as $A'A$ requires all columns of A'.
This implies that the column basis of A' is a minimum spanning set of $A'A$'s columns, additionally it is linearly independent by definition, thus it must also be the column basis of $A'A$. The dimension of the column basis of a matrix is the same as the dimensions of the row basis of a matrix, and this value is equal to the rank of the matrix, we have:

$$\text{rank}(A') = \text{rank}(A) = \text{rank}(A'A)$$