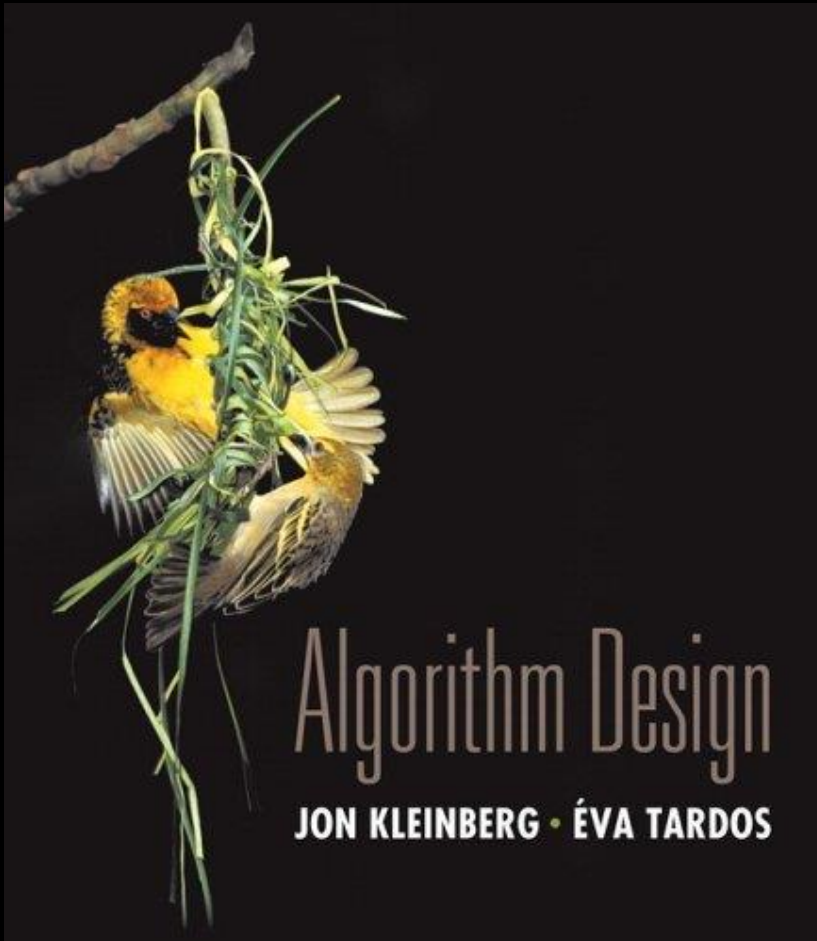


Chapter 1

Introduction: Some Representative Problems



Slides by Kevin Wayne.
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1.1 A First Problem: Stable Matching

Matching Residents to Hospitals

Goal. Given a set of preferences among hospitals and medical school students, design a **self-reinforcing** admissions process.

Unstable pair: applicant x and hospital y are **unstable** if:

- x prefers y to its assigned hospital.
- y prefers x to one of its admitted students.

Stable assignment. Assignment with no unstable pairs.

- Natural and desirable condition.
- Individual self-interest will prevent any applicant/hospital deal from being made.

Stable Matching Problem

Goal. Given n men and n women, find a "suitable" matching.

- Participants rate members of opposite sex.
- Each man lists women in order of preference from best to worst.
- Each woman lists men in order of preference from best to worst.

	favorite ↓ 1 st	2 nd	least favorite ↓ 3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 st	2 nd	least favorite ↓ 3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Perfect matching: everyone is matched monogamously.

- Each man gets exactly one woman.
- Each woman gets exactly one man.

Stability: no incentive for some pair of participants to undermine assignment by joint action.

- In matching M , an unmatched pair m - w is **unstable** if man m and woman w prefer each other to current partners.
- Unstable pair m - w could each improve by eloping.

Stable matching: perfect matching with no unstable pairs.

Stable matching problem. Given the preference lists of n men and n women, find a stable matching if one exists.

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-C, Y-B, Z-A stable?

A. No. Bertha and Xavier will hook up.

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓		least favorite ↓
	1 st	2 nd	3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Matching Problem

Q. Is assignment X-A, Y-B, Z-C stable?

A. Yes.

	favorite ↓ 1 st		least favorite ↓ 3 rd
	1 st	2 nd	3 rd
Xavier	Amy	Bertha	Clare
Yancey	Bertha	Amy	Clare
Zeus	Amy	Bertha	Clare

Men's Preference Profile

	favorite ↓ 1 st		least favorite ↓ 3 rd
	1 st	2 nd	3 rd
Amy	Yancey	Xavier	Zeus
Bertha	Xavier	Yancey	Zeus
Clare	Xavier	Yancey	Zeus

Women's Preference Profile

Stable Roommate Problem

Q. Do stable matchings always exist?

A. Not obvious a priori.

Stable roommate problem.

- $2n$ people; each person ranks others from 1 to $2n-1$.
- Assign roommate pairs so that no unstable pairs.

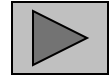
	<i>1st</i>	<i>2nd</i>	<i>3rd</i>
<i>Adam</i>	B	C	D
<i>Bob</i>	C	A	D
<i>Chris</i>	A	B	D
<i>Doofus</i>	A	B	C

A-B, C-D \Rightarrow B-C unstable
A-C, B-D \Rightarrow A-B unstable
A-D, B-C \Rightarrow A-C unstable

Observation. Stable matchings do not always exist for stable roommate problem.

Propose-And-Reject Algorithm

Propose-and-reject algorithm. [Gale-Shapley 1962] Intuitive method that guarantees to find a stable matching.



```
Initialize each person to be free.
while (some man is free and hasn't proposed to every woman) {
    Choose such a man m
    w = 1st woman on m's list to whom m has not yet proposed
    if (w is free)
        assign m and w to be engaged
    else if (w prefers m to her fiancé m')
        assign m and w to be engaged, and m' to be free
    else
        w rejects m
}
```

Proof of Correctness: Termination

Observation 1. Men propose to women in decreasing order of preference.

Observation 2. Once a woman is matched, she never becomes unmatched; she only "trades up."

Claim. Algorithm terminates after at most n^2 iterations of while loop.

Pf. Each time through the while loop a man proposes to a new woman.
There are only n^2 possible proposals. ▀

	1 st	2 nd	3 rd	4 th	5 th
Victor	A	B	C	D	E
Wyatt	B	C	D	A	E
Xavier	C	D	A	B	E
Yancey	D	A	B	C	E
Zeus	A	B	C	D	E

	1 st	2 nd	3 rd	4 th	5 th
Amy	W	X	Y	Z	V
Bertha	X	Y	Z	V	W
Clare	Y	Z	V	W	X
Diane	Z	V	W	X	Y
Erika	V	W	X	Y	Z

Proof of Correctness: Perfection

Claim. All men and women get matched.

Pf. (by contradiction)

- Suppose, for sake of contradiction, that **Zeus** is not matched upon termination of algorithm.
- Then some woman, say **Amy**, is not matched upon termination.
- By Observation 2, **Amy** was never proposed to.
- But, **Zeus** proposes to everyone, since he ends up unmatched. ▪

Proof of Correctness: Stability

Claim. No unstable pairs.

Pf. (by contradiction)

- Suppose **A-Z** is an unstable pair: each prefers each other to partner in Gale-Shapley matching S^* .

- Case 1: Z never proposed to A.
 - \Rightarrow Z prefers his GS partner to A.
 - \Rightarrow A-Z is stable.
- Case 2: Z proposed to A.
 - \Rightarrow A rejected Z (right away or later)
 - \Rightarrow A prefers her GS partner to Z.
 - \Rightarrow A-Z is stable.

men propose in decreasing
order of preference

S^*

Amy-Yancey

Bertha-Zeus

...

← women only trade up

- In either case A-Z is stable, a contradiction. ▪

Summary

Stable matching problem. Given n men and n women, and their preferences, find a stable matching if one exists.

Gale-Shapley algorithm. Guarantees to find a stable matching for **any** problem instance.

Q. How to implement GS algorithm efficiently?

Q. If there are multiple stable matchings, which one does GS find?

Efficient Implementation

Efficient implementation. We describe $O(n^2)$ time implementation.

Representing men and women.

- Assume men are named $1, \dots, n$.
- Assume women are named $1', \dots, n'$.

Engagements.

- Maintain a list of free men, e.g., in a queue.
- Maintain two arrays `wife[m]`, and `husband[w]`.
 - set entry to 0 if unmatched
 - if m matched to w then `wife[m]=w` and `husband[w]=m`

Men proposing.

- For each man, maintain a list of women, ordered by preference.
- Maintain an array `count[m]` that counts the number of proposals made by man m .

Efficient Implementation

Women rejecting/accepting.

- Does woman w prefer man m to man m' ?
- For each woman, create **inverse** of preference list of men.
- Constant time access for each query after $O(n)$ preprocessing.

Amy	1 st	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th
Pref	8	3	7	1	4	5	6	2

Amy	1	2	3	4	5	6	7	8
Inverse	4 th	8 th	2 nd	5 th	6 th	7 th	3 rd	1 st

```
for i = 1 to n  
    inverse[pref[i]] = i
```

Amy prefers man 3 to 6
since $\text{inverse}[3] < \text{inverse}[6]$
2 7

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of *Gale-Shapley* yield the same stable matching? If so, which one?

An instance with two stable matchings.

- A-X, B-Y, C-Z.
- A-Y, B-X, C-Z.

	1 st	2 nd	3 rd
Xavier	A	B	C
Yancey	B	A	C
Zeus	A	B	C

	1 st	2 nd	3 rd
Amy	Y	X	Z
Bertha	X	Y	Z
Clare	X	Y	Z

Understanding the Solution

Q. For a given problem instance, there may be several stable matchings. Do all executions of Gale-Shapley yield the same stable matching? If so, which one?

Def. Man m is a **valid partner** of woman w if there exists some stable matching in which they are matched.

Man-optimal assignment. Each man receives best valid partner.

Claim. All executions of GS yield **man-optimal** assignment, which is a stable matching!

- No reason a priori to believe that man-optimal assignment is perfect, let alone stable.
- Simultaneously best for each and every man.

Man Optimality

Claim. GS matching S^* is man-optimal.

Pf. (by contradiction)

1. (S^*) Suppose some man is paired with someone other than best partner. Men propose in decreasing order of preference \Rightarrow some man is rejected by valid partner.

2. (S^*) Let Y be **first** such man, and let A be **first** valid woman that rejects him.

3. Let S be a stable matching where A and Y are matched.

4. (S^*) **When Y is rejected, A forms (or reaffirms) engagement with a man, say Z , whom she prefers to Y .**

5. Let B be Z 's partner in S .  Z proposes to A before B .

6. (S^*) ~~Z not rejected by any valid partner at the point when Y is rejected by A .~~ Thus, **Z prefers A to B .** (2,4)

7. (S^*) But **A prefers Z to Y .** (4)

8. Thus A - Z is unstable in S . \cdot

S

Amy-Yancey

Bertha-Zeus

...

\uparrow
since this is first rejection
by a valid partner

Stable Matching Summary

Stable matching problem. Given preference profiles of n men and n women, find a **stable** matching.

↖
no man and woman prefer to be with each other than assigned partner

Gale-Shapley algorithm. Finds a stable matching in $O(n^2)$ time.

Man-optimality. In version of GS where men propose, each man receives best valid partner.

↖
 w is a valid partner of m if there exist some stable matching where m and w are paired

Q. Does man-optimality come at the expense of the women?

Woman Pessimality

Woman-pessimal assignment. Each woman receives worst valid partner.

Claim. GS finds **woman-pessimal** stable matching S^* .

Pf.

1. Suppose A - Z matched in S^* , but Z is not worst valid partner for A .
2. There exists stable matching S in which A is paired with a man, say Y , whom she likes less than $Z \Rightarrow A$ prefers Z to Y .
3. Let B be Z 's partner in S .
4. Z prefers A to B (in S^* , A and B are both valid partners).
5. Thus, A - Z is an unstable in S . \cdot

man-optimality

S

Amy-Yancey

Bertha-Zeus

...

Extensions: Matching Residents to Hospitals

Ex: Men \approx hospitals, Women \approx med school residents.

Variant 1. Some participants declare others as unacceptable.

Variant 2. Unequal number of men and women.

↖
resident A unwilling to
work in Cleveland

Variant 3. Limited polygamy.

↖
hospital X wants to hire 3 residents

Def. Matching S **unstable** if there is a hospital h and resident r (not a pair in S) such that:

- h and r are acceptable to each other; and
- either r is unmatched, or r prefers h to her assigned hospital; and
- either h does not have all its places filled, or h prefers r to at least one of its assigned residents.