

Chapter 7

Network Flow



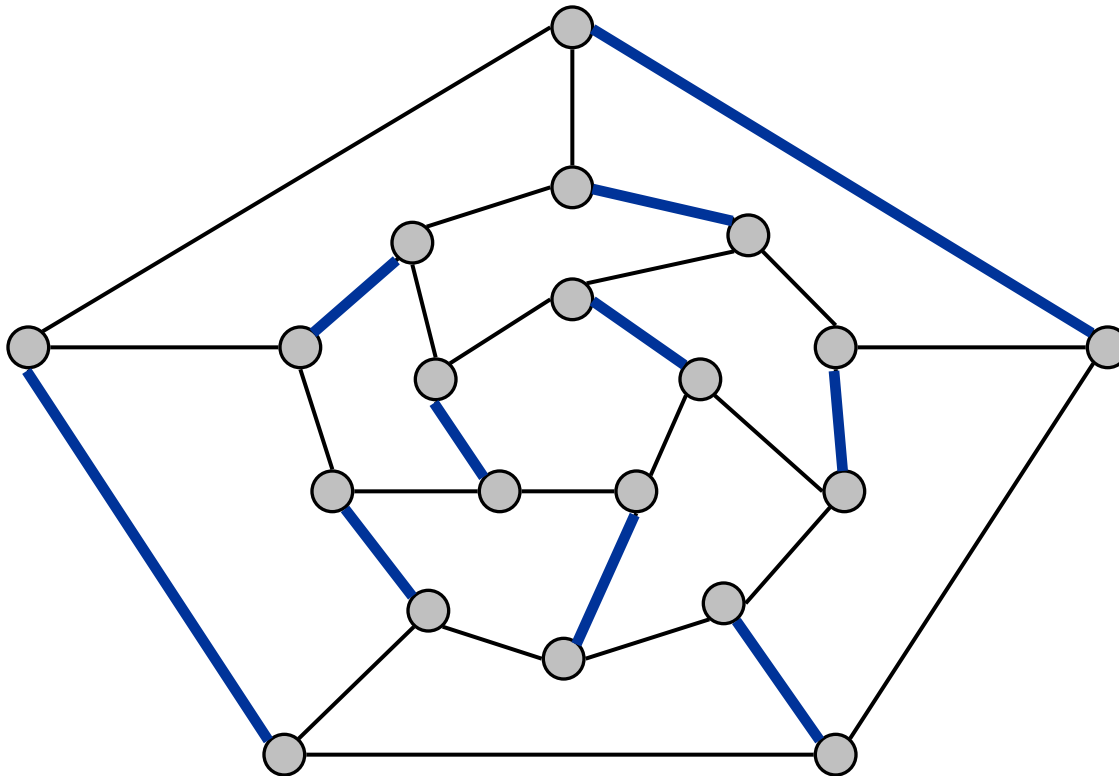
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7.5 Bipartite Matching

Matching

Matching.

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- Max matching: find a max cardinality matching.

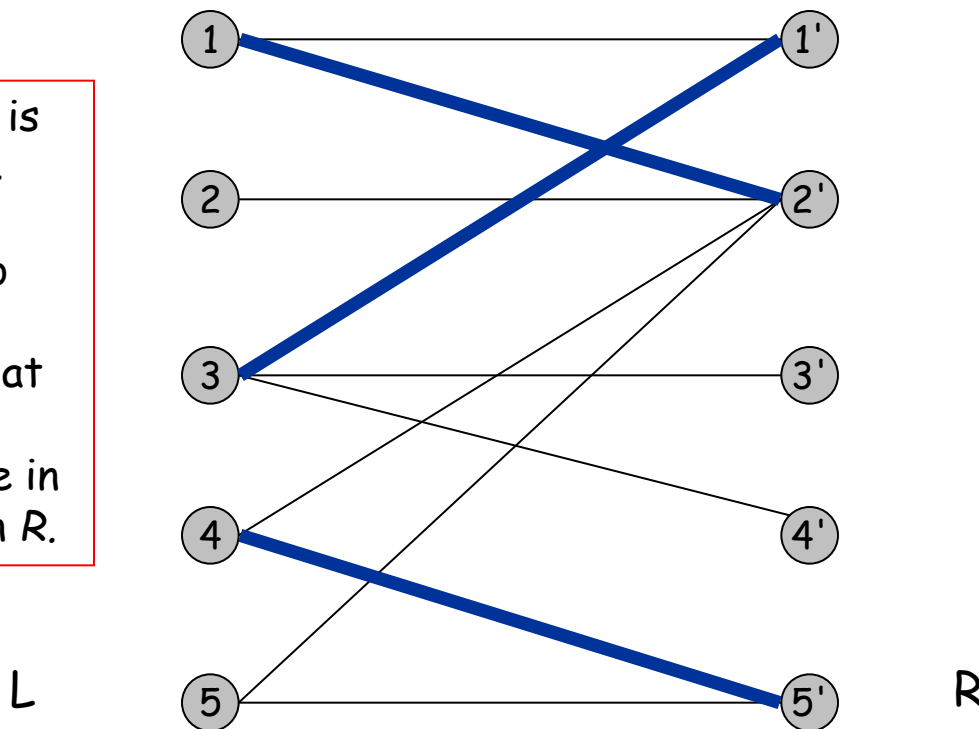


Bipartite Matching

Bipartite matching.

- Input: undirected, **bipartite** graph $G = (L \cup R, E)$.
- $M \subseteq E$ is a **matching** if each node appears in at most one edge in M .
- Max matching: find a max cardinality matching.

Def. A graph G is **bipartite** if the nodes can be partitioned into two subsets L and R such that every edge connects a node in L with a node in R .

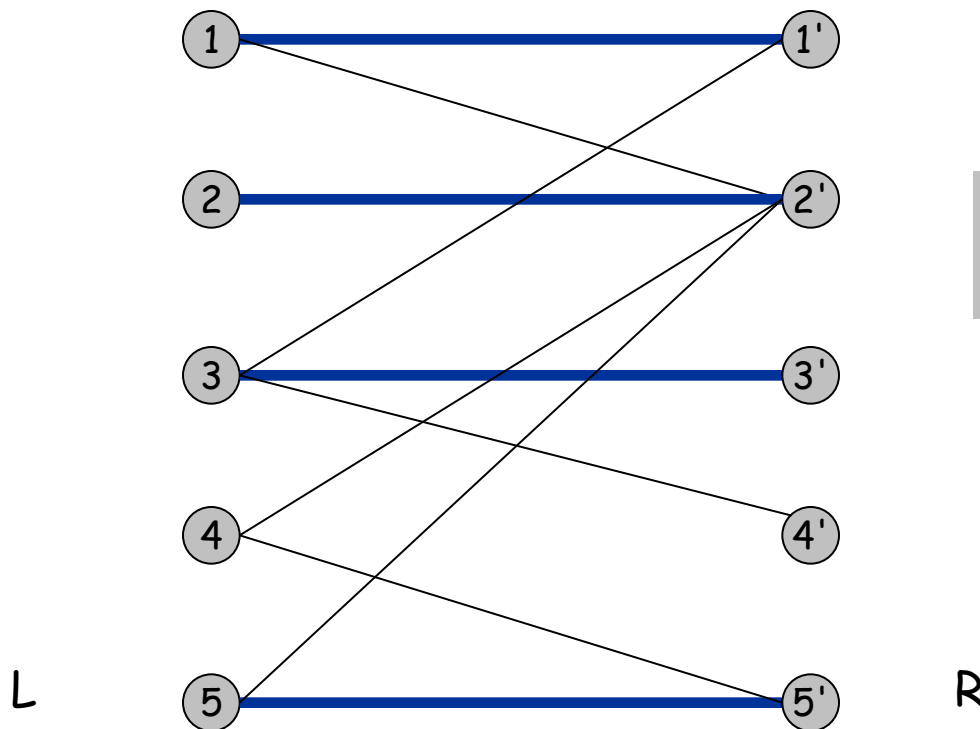


matching
1-2', 3-1', 4-5'

Bipartite Matching

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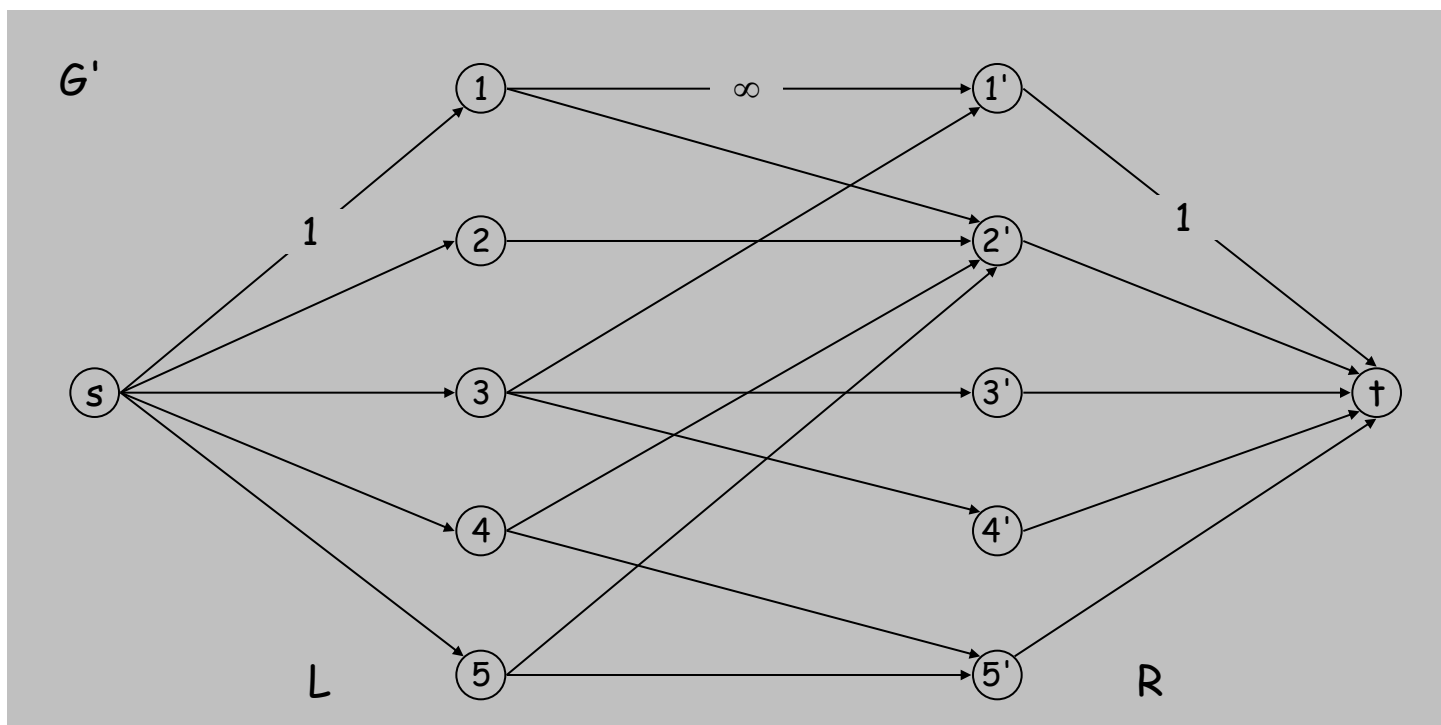


max matching
1-1', 2-2', 3-3' 4-4'

Bipartite Matching

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E')$.
- Direct all edges from L to R , and assign infinite (or unit) capacity.
- Add source s , and unit capacity edges from s to each node in L .
- Add sink t , and unit capacity edges from each node in R to t .

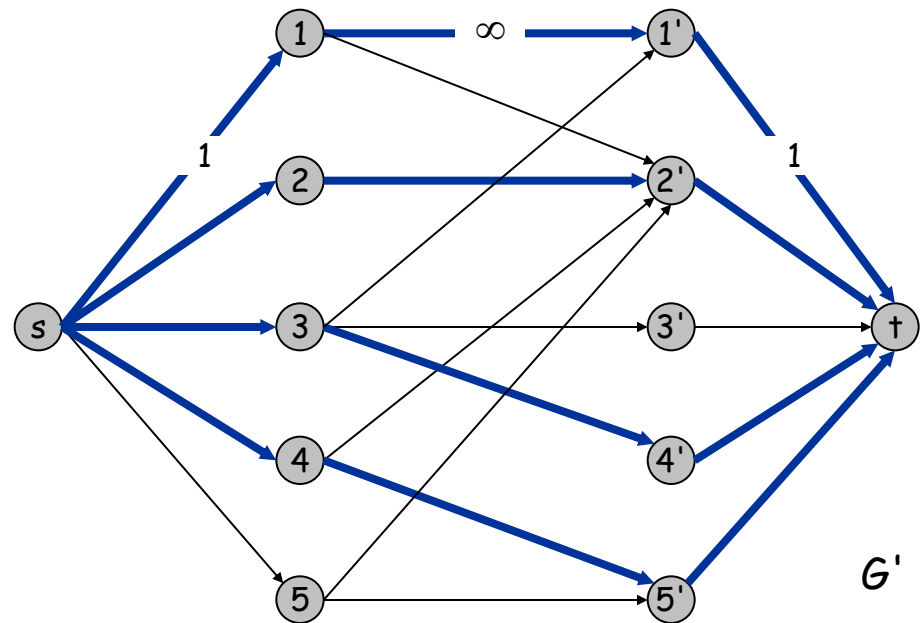
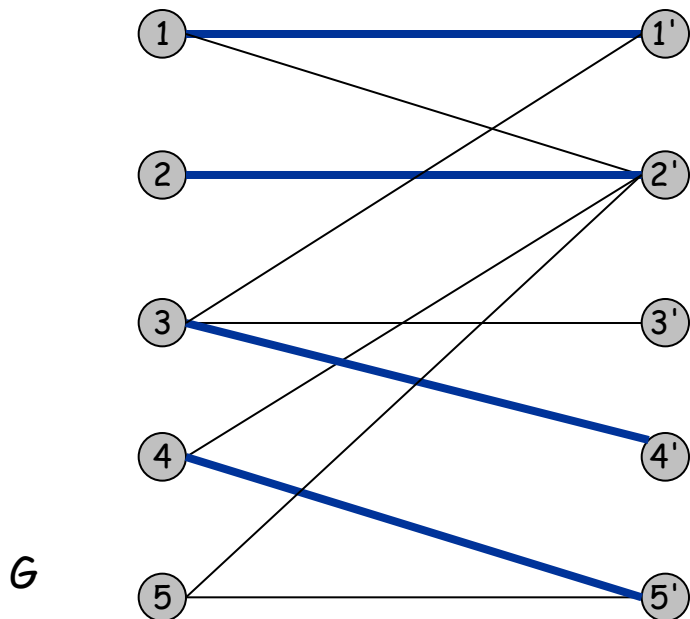


Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G' .

Pf. \leq

- Given max matching M of cardinality k .
- Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k . ▪

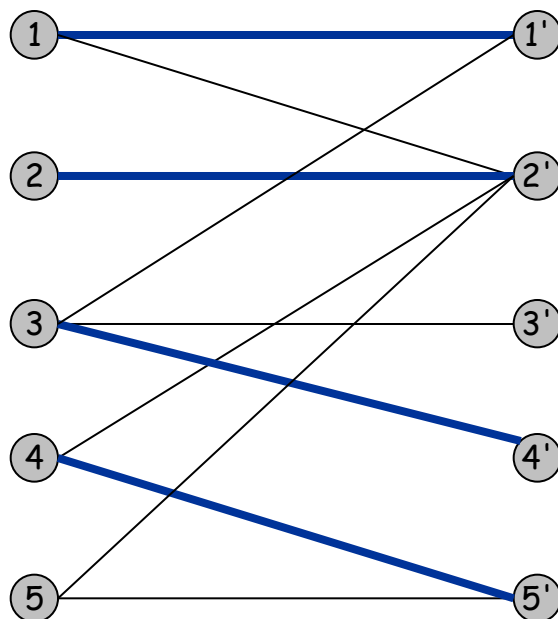
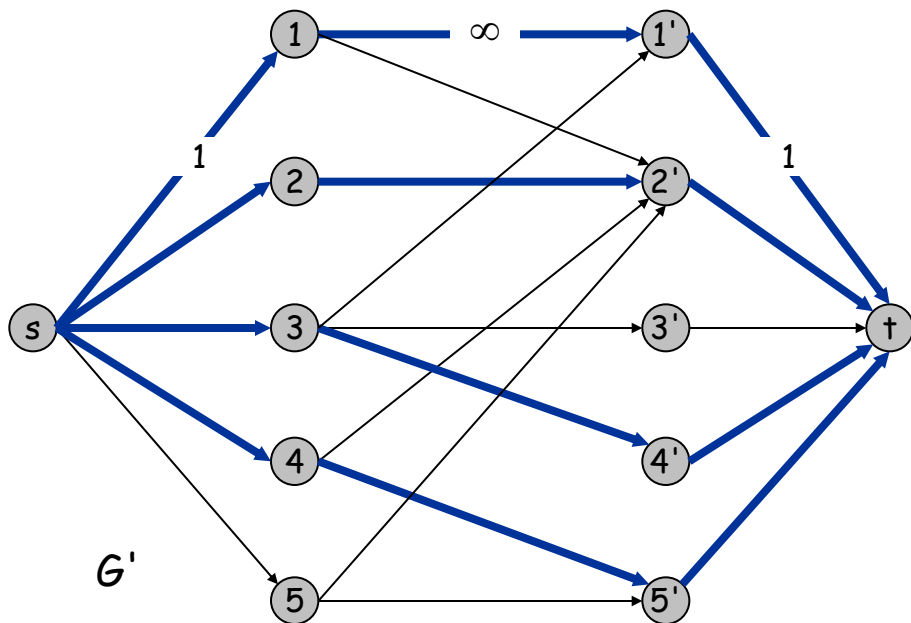


Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G' .

Pf. \geq

- Let f be a max flow in G' of value k .
- Integrality theorem \Rightarrow k is integral and can assume f is 0-1.
- Consider M = set of edges from L to R with $f(e) = 1$.
 - each node in L and R participates in at most one edge in M
 - $|M| = k$: apply flow-value lemma to cut $(L \cup s, R \cup t)$.



Perfect Matching

Def. A matching $M \subseteq E$ is **perfect** if each node appears in exactly one edge in M .

Q. When does a bipartite graph have a perfect matching?

Structure of bipartite graphs with perfect matchings.

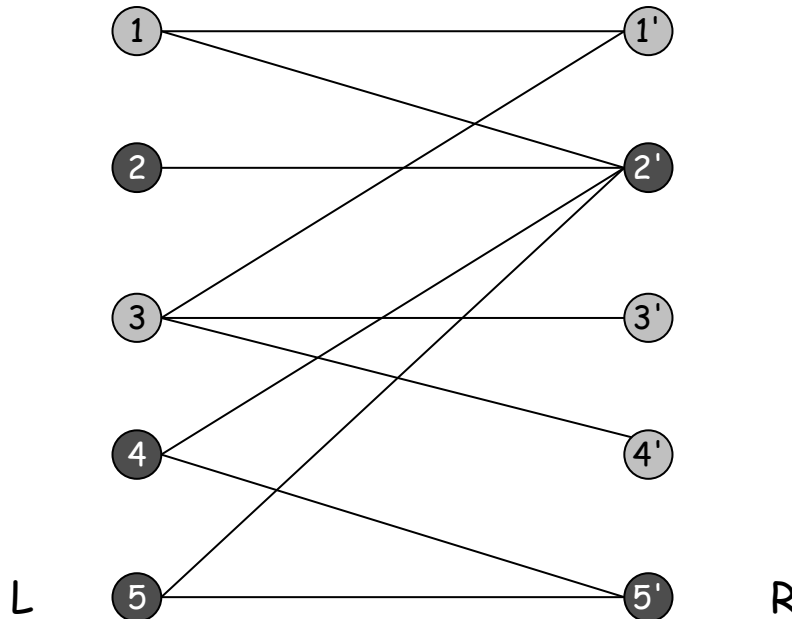
- Clearly we must have $|L| = |R|$.
- What other conditions are necessary?
- What conditions are sufficient?

Perfect Matching

Notation. Let S be a subset of nodes, and let $N(S)$ be the set of nodes adjacent to nodes in S .

Observation. If a bipartite graph $G = (L \cup R, E)$, has a perfect matching, then $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. Each node in S has to be matched to a different node in $N(S)$.



No perfect matching:

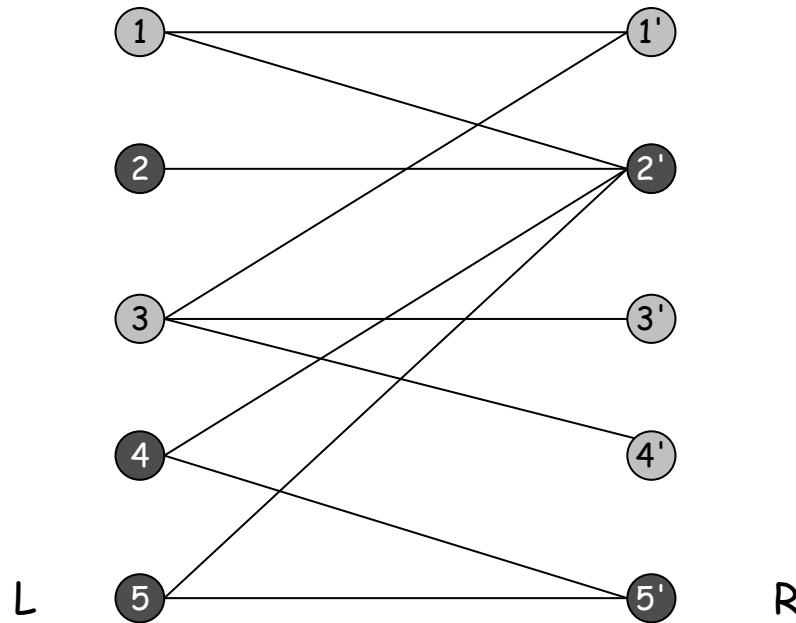
$S = \{ 2, 4, 5 \}$

$N(S) = \{ 2', 5' \}.$

Hall's marriage theorem

Theorem. [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, graph G has a perfect matching iff $|N(S)| \geq |S|$ for all subsets $S \subseteq L$.

Pf. \Rightarrow This was the previous observation.



No perfect matching:

$S = \{ 2, 4, 5 \}$

$N(S) = \{ 2', 5' \}.$

Bipartite Matching: Running Time

Which max flow algorithm to use for bipartite matching?

- Generic augmenting path: $O(m \text{ val}(f^*)) = O(mn)$.
- Capacity scaling: $O(m^2 \log C) = O(m^2)$.

Non-bipartite matching.

- Structure of non-bipartite graphs is more complicated, but well-understood. [Tutte-Berge, Edmonds-Galai]
- Blossom algorithm: $O(n^4)$. [Edmonds 1965]
- Best known: $O(m n^{1/2})$. [Micali-Vazirani 1980]