

Chapter 6

Dynamic Programming



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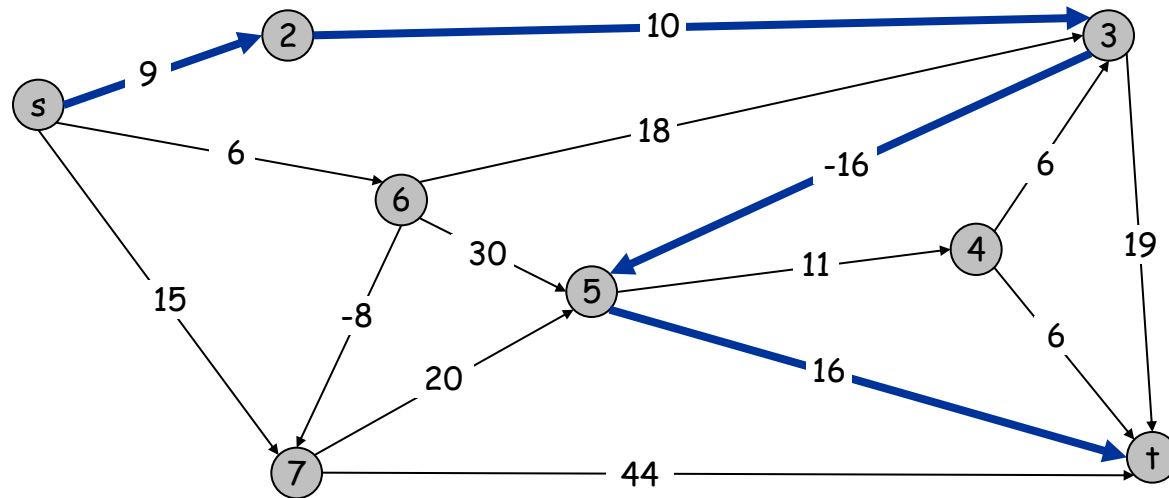
6.8 Shortest Paths

Shortest Paths

Shortest path problem. Given a directed graph $G = (V, E)$, with edge weights c_{vw} , find shortest path from node s to node t .

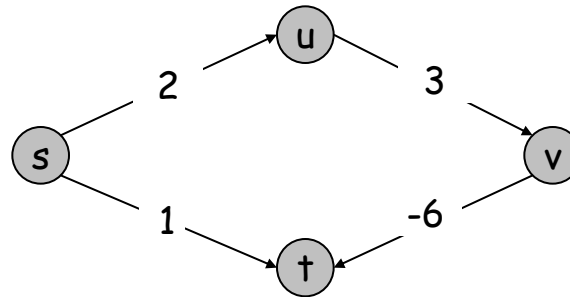
↖ allow negative weights

Ex. Nodes represent agents in a financial setting and c_{vw} is cost of transaction in which we buy from agent v and sell immediately to w .

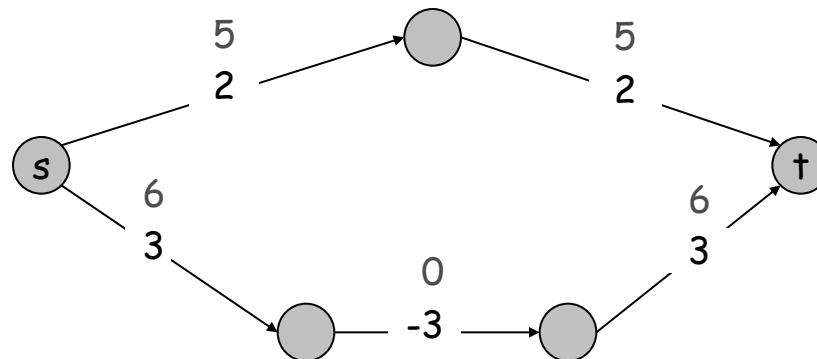


Shortest Paths: Failed Attempts

Dijkstra. Can fail if negative edge costs.

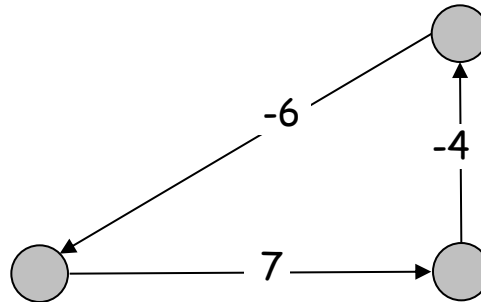


Re-weighting. Adding a constant to every edge weight can fail.

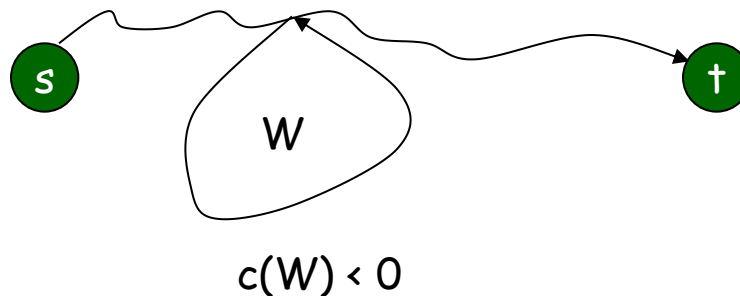


Shortest Paths: Negative Cost Cycles

Negative cost cycle.



Observation. If some path from s to t contains a negative cost cycle, there does not exist a shortest s - t path; otherwise, there exists one that is simple.



Shortest Paths: Dynamic Programming

Def. $OPT(i, v)$ = length of shortest v - t path P using at most i edges.

- Case 1: P uses at most $i-1$ edges.
 - $OPT(i, v) = OPT(i-1, v)$
- Case 2: P uses exactly i edges.
 - if (v, w) is first edge, then OPT uses (v, w) , and then selects best w - t path using at most $i-1$ edges

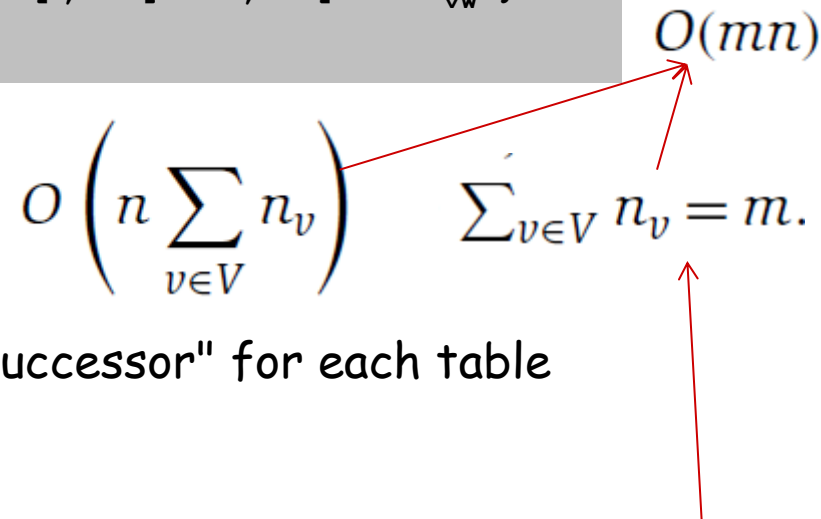
$$OPT(i, v) = \begin{cases} 0 & \text{if } i=0 \\ \min \left\{ OPT(i-1, v), \min_{(v, w) \in E} \{ OPT(i-1, w) + c_{vw} \} \right\} & \text{otherwise} \end{cases}$$

Remark. By previous observation, if no negative cycles, then $OPT(n-1, v)$ = length of shortest v - t path.

Shortest Paths: Implementation

```
Shortest-Path(G, t) {  
  foreach node v ∈ V  
    M[0, v] ← ∞  
  M[0, t] ← 0  
  
  for i = 1 to n-1  
    foreach node v ∈ V  
      M[i, v] ← M[i-1, v]  
      foreach edge (v, w) ∈ E  
        M[i, v] ← min { M[i, v], M[i-1, w] + cvw }  
}
```

Analysis. $\Theta(mn)$ time, $\Theta(n^2)$ space.

$$O\left(n \sum_{v \in V} n_v\right) \quad \sum_{v \in V} n_v = m.$$


Finding the shortest paths. Maintain a "successor" for each table entry.

each edge leaves exactly one of the nodes in V

Shortest Paths: Practical Improvements

Practical improvements.

- Maintain only one array $M[v]$ = shortest v - t path that we have found so far ($i : 1, 2, \dots n-1$).
- No need to check edges of the form (v, w) unless $M[w]$ changed in previous iteration.

Theorem. Throughout the algorithm, $M[v]$ is length of some v - t path, and after i rounds of updates, the value $M[v]$ is no larger than the length of shortest v - t path using $\leq i$ edges.

Overall impact.

- Memory: $O(m + n)$.
- Running time: $O(mn)$ worst case, but substantially faster in practice.

Bellman-Ford: Efficient Implementation

```
Push-Based-Shortest-Path( $G, s, t$ ) {  
    foreach node  $v \in V$  {  
         $M[v] \leftarrow \infty$   
         $\text{successor}[v] \leftarrow \phi$   
    }  
  
     $M[t] = 0$   
    for  $i = 1$  to  $n-1$  {  
        foreach node  $w \in V$  {  
            if ( $M[w]$  has been updated in previous iteration) {  
                foreach node  $v$  such that  $(v, w) \in E$  {  
                    if ( $M[v] > M[w] + c_{vw}$ ) {  
                         $M[v] \leftarrow M[w] + c_{vw}$   
                         $\text{successor}[v] \leftarrow w$   
                    }  
                }  
            }  
        }  
        If no  $M[w]$  value changed in iteration  $i$ , stop.  
    }  
}
```

6.9 Distance Vector Protocol

Distance Vector Protocol


Communication network.

- Node \approx router.
- Edge \approx direct communication link.
- Cost of edge \approx delay on link. \leftarrow naturally nonnegative, but Bellman-Ford used anyway!

Dijkstra's algorithm. Requires global information of network.

Bellman-Ford. Uses only local knowledge of neighboring nodes.

Synchronization. We don't expect routers to run in lockstep. The order in which each `foreach` loop executes is not important. Moreover, algorithm still converges even if updates are **asynchronous**.



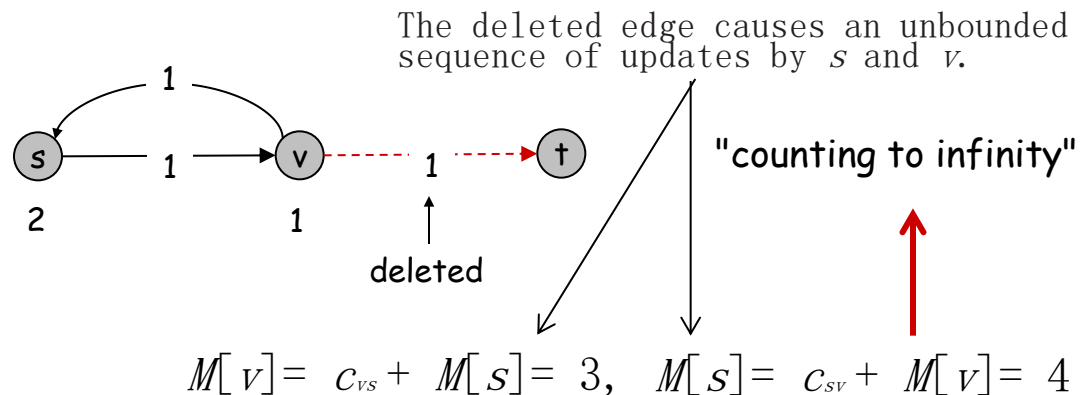
each time a node w experiences an update to its $M[w]$ value, it becomes "active" and eventually notifies its neighbors of the new value

Distance vector protocol.

- Each router maintains a vector of shortest path lengths to every other node (distances) and the first hop on each path (directions).
- Algorithm: each router performs n separate computations, one for each potential destination node.
- "Routing by rumor."

Ex. RIP, Xerox XNS RIP, Novell's IPX RIP, Cisco's IGRP, DEC's DNA Phase IV, AppleTalk's RTMP.

Caveat. Edge costs may **change** during algorithm (or fail completely).



Path Vector Protocols

Link state routing.

- Each router also stores the entire path.
- Based on Dijkstra's algorithm.
- Avoids "counting-to-infinity" problem and related difficulties.
- Requires significantly more storage.

not just the distance and first hop



Ex. Border Gateway Protocol (BGP), Open Shortest Path First (OSPF).