Project 5

Interpolation and integration methods/ cubic splines and surface interpolation

Group 3 - Team 4

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Abstract : This project consists in the implementation a relatively straightforward model that represents the airflow surrounding a cross section of an aircraft's wing. The main objective of this project is to generate a pressure map above and below the wing, thereby approximating the wing's lift (it's ability to keep the plane flying). This has to be done in two steps :

- Two preliminary steps:
    - Usage of the **cubic spline method** in order to interpolate a function from a set of points
    - Implementation of 5 integrations functions, each using different methods to compute an approximated value for the integral of any given function.
- One application:
    - These two methods will be applied to a set of points, from which the curve of an airplane wing will be interpolated. The integral will be calculated to determine the air flow data around the wing and produce a pressure map

1 Introduction

(Luxel Hamouche wrote this part)

This report aims to provide an overview of numerical methods for function approximation and integral calculus, and their application in the field of aerodynamics. The report focuses on the cubic spline method, which is used to interpolate a set of data points into a smooth function, and on numerical integration methods, which are used to approximate the length of a plane curve. We will also attempt to demonstrate the effectiveness of these methods in aerodynamics by using them to interpolate an airfoil and calculate the pressure distribution around the wing of an aircraft. Finally, we wish to give a valuable insight into the practical applications of numerical methods in the field of aerodynamics.

2 Airfoil refinement/ Cubic spline

(Ilyes Bechoual wrote this part)

In this part, the aim is to interpolate data points into a smooth function by using a method called the cubic spline. A mathematical function describing the wing profile is required to model the airflow around the wings. A large number of wing profiles, modeled by coordinates of points, can be accessed through the *UIUC Applied Aerodynamics Group* database. The equations that relate to these points are found by implementing a cubic spline algorithm using a text book on algorithms and numerical analysis called the *Numerical Recipes*.

This method consists of interpolating those coordinates by using a cubic polynomial function to make the first and second derivatives of the function continuous for each segment's extremum. Thus, the cubic spline method is split into 2 parts (or 2 functions). A list of the second derivative values using coordinates and the value of the first derivative at the end points of the airfoil is returned by the first function, called *spline*. The interpolated value of the airfoil data point at a given x position, using coordinates but also the array of the second derivative values calculated by the *spline* function, is returned by the other function, *splint*.

To test those functions, known functions such as $f : x \rightarrow x^3$ and $f : x \rightarrow sin(x^3 * \pi)$ can be implemented, as shown in Figure 1 and 2. An airfoil database with enough points can be chosen, and in this case, `ah79100a.dat` has been chosen. A set of data points can be computed, resulting in a set of data points.

An array of second derivatives of these data points can be obtained using the *spline* method. Then, with a list of n points from 0 to 1, the interpolated value of x can be calculated using *splint*, and the resulting curve can be plotted on the graph. The curve with $n = 1000$ can be seen in Figure 3.

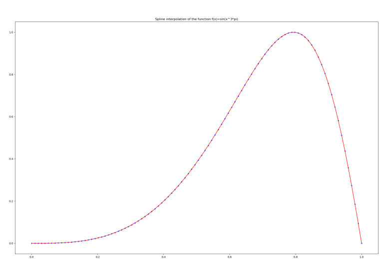

Figure 1: Cubic spline method on f(x) = sin(x^3 * pi) with 100 points.

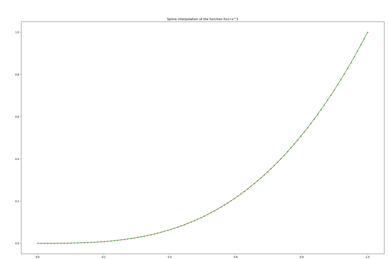

Figure 2: Cubic spline method on f(x) = x^3 with 100 points.

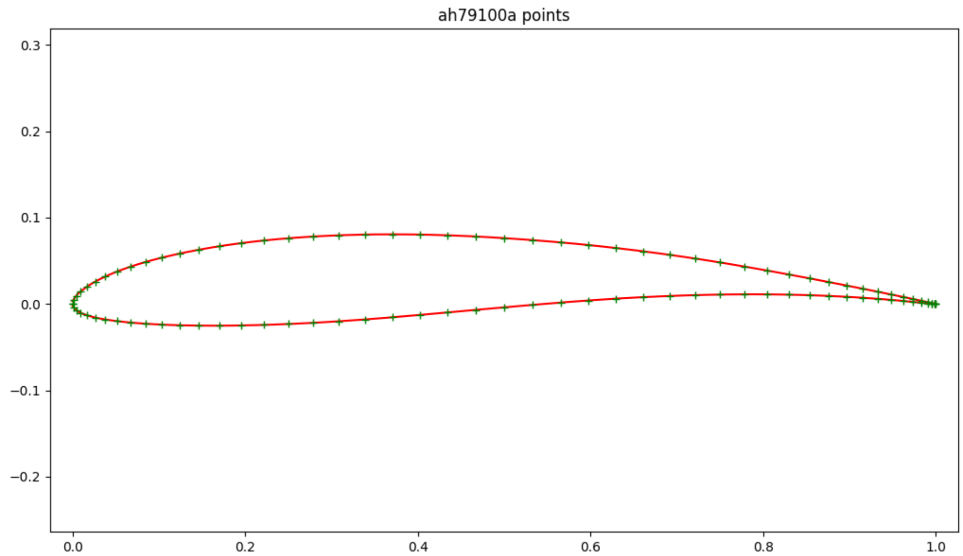

Figure 3: Airfoil Refinement with n = 1000

3 Integration : Computing the length of plane curves

(Arthur Durand wrote this part)

To estimate the pressure distribution around the airfoil, it is necessary to determine the lengths of the splines as a function of $x \to f(x)$. To accomplish this, a specific integral depending on the function f, will be computed. As a reminder, the length of a function's graph, $y = f(x)$, defined and differentiable on a given interval $[0; T]$, can be calculated with the expression :

$$L([0;T]) = \int_0^T \sqrt{1 + f'(x)^2} dx$$

Note that f has been interpolated by a cubic spline in the previous part, so the computing of it's derivative will be quite easy.

In order to compute the length of the airfoil sides, integral computing functions should be implemented, theses functions are going to use five different numerical integration methods :

- **Left/Right Rectangle Method**

  The left/right rectangle method is an easy way to approximate the integral of a function. The process slices the interval in n parts, evaluates the function with the beginning or the end of those intervals and multiplies them by the width of the interval. In order to add every little area of each rectangle (*Figure 4*).

Here is the mathemacal formula for the left rectangle method :

$$I = \int_a^b f(x)dx \approx h(\sum_{i=0}^{n-1} f(a+ih))$$

Here $h = \frac{b-a}{n}$.

So if n (which is the number of intervals) increases, the integral will be more accurate because there will be more thin rectangles and it will represent the real integral(*Figure 5*).

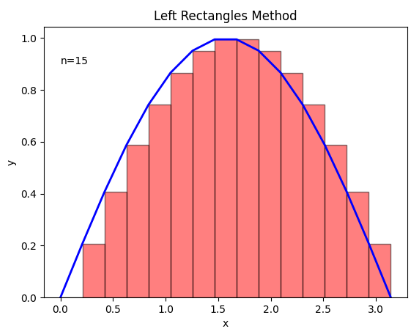

Figure 4: Left rectangle method with n=15.

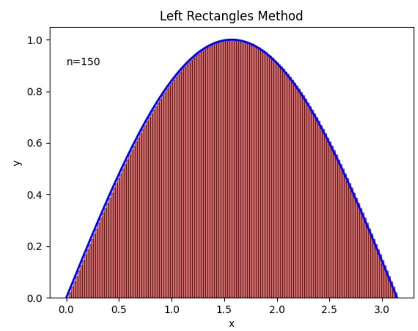

Figure 5: Left rectangle method with n=150.

- **MiddlePoint Method**

The middle point method looks like the left/right rectangular method. In fact, it is approximately the same but, this time the function is evaluated in the middle of the intervals. As we will see in the next part, this method is a little bit more accurate and faster than the left/right rectangular method because it is an average of each points on a small interval. (*Figure 6*). Here is the mathemacal formula for the Middle point method :

$$I = \int_a^b f(x)dx \approx h(\sum_{i=0}^{n-1} f(a+ih+\frac{h}{2}))$$

Here $h = \frac{b-a}{n}$.

This is approximalety the same as the left rectangle formula

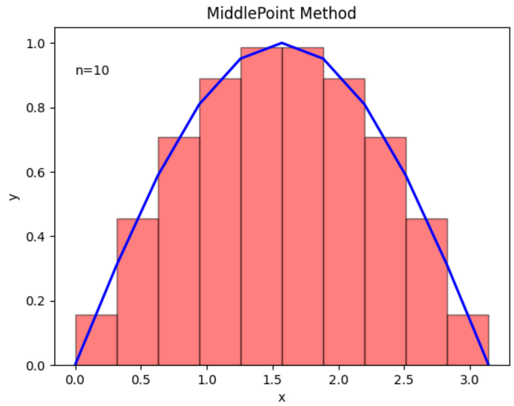

Figure 6: Middle point method with n=10

- **Trapezoids Method**

  The trapezoid method is in the same way of the first two methods, which means it evaluates the integral with polygons. So the main objective is to sum the area of trapezoids to obtain the integral of the function. So the interval is sliced and this time in order to have a trapezoid, it just links both evaluations of the function at the beginning and the end of the new little intervals (*Figure 7*).

$$I = \int_a^b f(x)dx \approx h(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(a+ih))$$

  Here $h = \frac{b-a}{n}$.

  This method is more precise than the previous as Figure 7 shows with only n=6, it fit the curve really well. In addition, it is easy to implement. So it is a good way for a simple approximation of an integral for a first time.

- **Simpson's Method**

  The Simpson's method is trickier because this time to approximate the integral of the function the interpolation polynomials are used, in order to be the most accurate.

  The formula looks like that :

$$h = \frac{b-a}{n} \text{ and } k \in [0, l] \text{ and } \lambda_{i,k} = a_{i-1} + \frac{k}{l}h$$

$$I = \int_{a_{i-1}}^{a_i} f(x)dx \approx \int_{a_{i-1}}^{a_i} (\sum_{i=0}^{l} f(\lambda_{i,k})L_{i,k}(t))dt = \sum_{i=0}^{l} f(\lambda_{i,k})(\int_{a_{i-1}}^{a_i} L_{i,k}(t)dt)$$

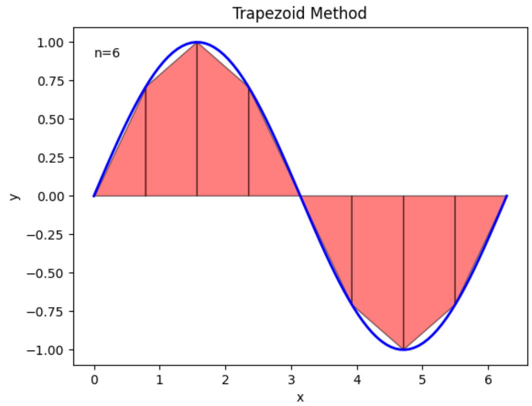

Figure 7: Trapezoid method with n=6.

So for $l = 2$ we can reduce this equation and then :

$$I = \int_a^b f(x)dx \approx h(\frac{f(a)+f(b)}{6} + \frac{1}{3}\sum_{i=1}^{n-1} f(a+ih) + \frac{2}{3}\sum_{i=0}^{n-1} f(a+ih+\frac{h}{2}))$$

This is the best method that can be used to approximate an integral for this project. Due to what the *Figure 8* shows and the next part, it is the fastest way to reach an high precision.

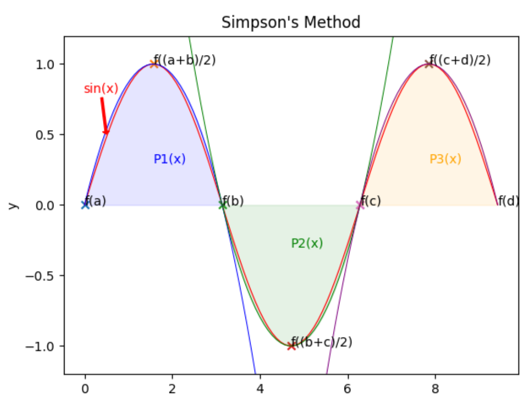

Figure 8: Simpson method

4 Convergence of different methods

(Arthur Durand wrote this part) In order to choose the best method, an algorithm is developed to compare the speed of all the methods to reach a close approximation of the integral.

A test with each method to know how many iterations (n) is needed to validate $I(2*n) - I(n) < \epsilon$ was done. Where I(n), was the value of the integral on an interval sliced in n parts.

First of all a test for $f(x) = \exp(x)$ between 0 and 1 and $\epsilon = 10^{-6}$(*Figure 9*), were the limite of iterations is $n < 10^{3.5}$. The grah shows that the left/right rectangle method reach this limit and also,

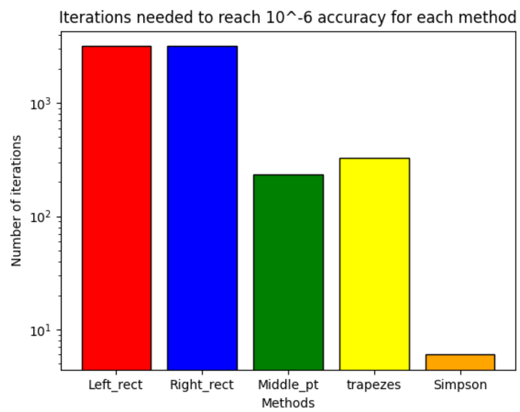

Figure 9: Number of subdivition needed

the Simpson's method is faster than the other by far.

Now focusing on the speed to converge to the most accurate, the rectangles methods are really slow for high precision and also sometimes they just diverge and never reach the right accuracy (*Figure 11*). On the chart, there are four curve methods because the left and right rectangle has the same speed.

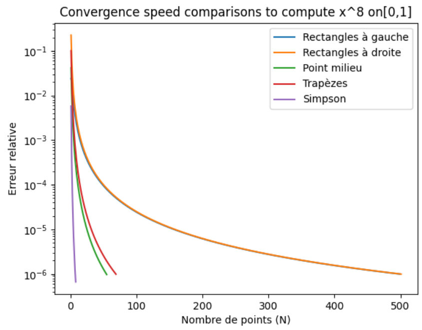

Figure 10: convergence speed $f(x) = x^8$ [0;1] with $\epsilon = 10^{-6}$

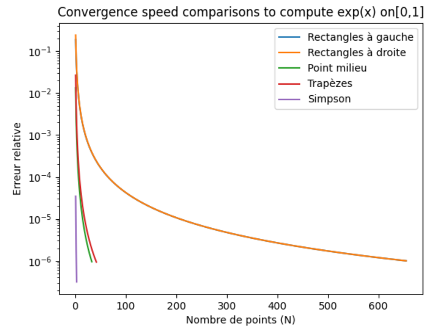

Figure 11: convergence speed f(x) = exp(x) [0;1] with $\epsilon = 10^{-6}$

As (*Figure* 10) shows, the three methods are good for approximate an integral. So after tried with a lower ϵ (*Figure 12*) ,the Simpson's method stayed the best one even if the other were very effective for $\epsilon > 10^{-5}$.

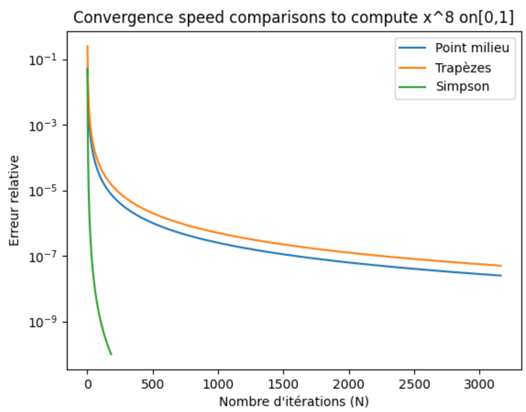

Figure 12: convergence speed $f(x) = x^8$ [0;1] with $\epsilon = 10^{-10}$

5 Modelling the airflow/ Pressure map

(Cesar wrote this part.)

After determining the shape of the wing, we have to determine the shape of the airflow around it; Let us assume that the plane is flying trough a laminar flow (meaning that without any perturbations, air molecules would flow horizontally, following current lines that would be aligned and non-intersecting).

In close proximity to the airfoil, the air currents conform closely to its shape, while further away from the airfoil, the air flows in a horizontal, undisturbed manner. The objective of our study is to ascertain and illustrate the transitional flow behavior of the air as it moves between these two divergent states, in order to compute the length of each flowline and so on the pressure of the air it moves.

This study makes some approximations on fluid dynamics, and could not represent the system in a realistic way.

Let us assume that the only range of disturbed airflow that we will be considering is the interval $[h_{max}; h_{min}]$, where h_{max} (*resp.* h_{min}) is the highest (*resp.* lowest) point of the airfoil. Suppose we have a curve y = f(x) that represents the upper surface of the wing. The airflow above the wing can be described by a family of curves with the following equations:

$$y = f_\lambda(x) = (1 - \lambda)f(x) + \lambda * 3h_{max} \ , \forall \lambda \in [0; 1]$$

Assuming a fixed value of lambda, the equation defines a curve that lies between the upper surface of the wing and the maximum altitude beyond which the airflow remains undisturbed by the wing. After computing, the air flow looks like the **figure 6**.

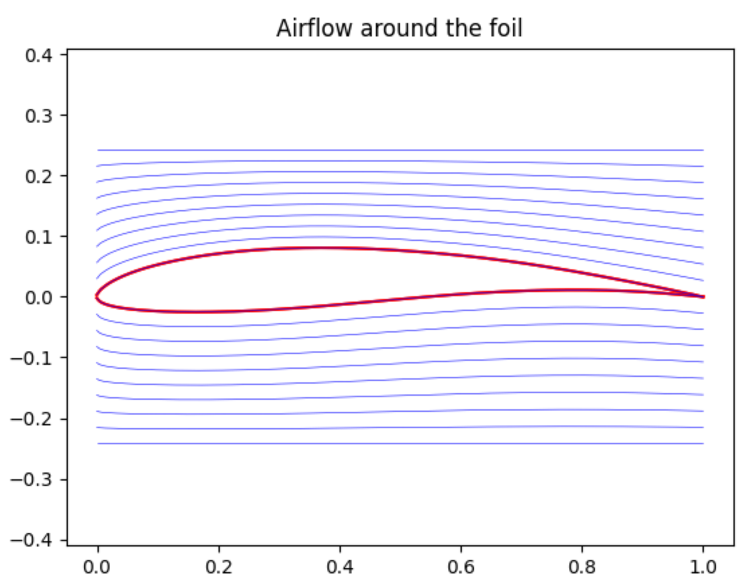

Figure 13: Airflow illustration with 10 flowlines computed on each side of the foil

With the equations of each flow line known, as presented in the second part formula, the lengths of these curves can now be determined by computing each of theses intregrals:

$$L_{f_\lambda}([0;T]) = \int_0^T \sqrt{1 + f'_\lambda(x)^2} dx \ , \forall \lambda \in [0;1]$$

Due to time complexity considerations, the Simpson's method will be used to compute these integrals, as its relative efficiency has been demonstrated in Part 2.

Let us assume that the airfoil is 2 meters wide ($d = 2m$), and that the plane is flying at a 900km/h ($v = 250m/s$), trough air subject at atmospheric pressure ($P_{atm} = 10100Pa$).

To calculate the air pressure along each flow line, the speed of the air flow needs to be determined first, along with the time it takes for the air to cross the width of the wing.

$$time = \frac{d}{v} = 8 * 10^-3s$$

Now that we know the time it takes for the air to cross the width of the wing, we can focus on the relative velocity of the air in the wing's reference frame. The velocity of the air in each flowline λ is given by :

$$v_\lambda = \frac{L_{f_\lambda}([0;T])}{time}$$

The pressure acting on each slice is determined by applying Bernoulli's law to approximate the air pressure in a moving fluid :

$$P_{atm} = P_s + \tfrac{1}{2}\rho v^2 \text{ , where } \rho = 1.341kg/m^3 \text{ is the air's density}$$

$$P_s = P_{atm} - \tfrac{1}{2}\rho v^2$$

With the static pressure along each flow line now determined, the only remaining task is to plot a suitable number of flow lines, using a color scale to represent the pressure along each line.

All of this reasoning is implemented in the function *pressuremap()* in our **part3.py** file. Unfortunately, we were unable to complete the plotting of the pressure map in time for the first submission of this report, but we expect to obtain a similar graph the one from the project topic.

6 Conclusion

(Luxel Hamouche wrote this part)

The first section demonstrated the implementation of the cubic spline method to interpolate a set of data points into a smooth function. The *spline* function was used to calculate the second derivatives of the airfoil data points, while the *splint* function was used to interpolate data points at a given x position. The effectiveness of the method was verified by applying it to known functions and by using it to interpolate an airfoil profile. The resulting curve with $n = 1000$ points showed a smooth and continuous curve, proving the effectiveness of the cubic spline method for this application.

Next, we saw that the length of a plane curve can be approximated using numerical integration methods. Each method has its own advantages and disadvantages in terms of accuracy, speed and ease of implementation. The trapezoidal method is a good choice for simple approximations, while Simpson's method is more accurate and suitable for complex integrals. Gaussian quadrature is the most accurate, but also the most complex and computationally expensive. Overall, numerical integration provides an efficient way of approximating the length of a plane curve, which is essential for applications such as determining the pressure distribution around an airfoil.

Finally, the process of modeling the airflow and pressure map around the airfoil of an aircraft involves approximations and assumptions based on fluid dynamics. By determining the shape of the airflow around the wing and computing the length of each flowline, we can estimate the pressure acting on each slice of the wing. Using Bernoulli's law and the velocity of the air in each flowline, we can approximate the static pressure along each flowline and plot a pressure map. Although we were unable to complete the plotting of the pressure map, we expect to obtain a graph, which will provide insight into the airflow and pressure distribution around the wing of the aircraft.