

# Statistical Learning

Bùi Tiến Lên

2021



KHOA CÔNG NGHỆ THÔNG TIN  
TRƯỜNG ĐẠI HỌC KHOA HỌC TỰ NHIÊN

# Contents

---



1. Probability And Statistics
2. Statistical Learning
3. Fitting Probability Models
4. Linear Regression Revisited
5. Naive Bayes
6. Bayesian Networks

# Notation



symbol	meaning
$a, b, c, N \dots$	scalar number
$\mathbf{w}, \mathbf{v}, \mathbf{x}, \mathbf{y} \dots$	column vector
$\mathbf{X}, \mathbf{Y} \dots$	matrix
$\mathbb{R}$	set of real numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{N}$	set of natural numbers
$\mathbb{R}^D$	set of vectors
$\mathcal{D}, \mathcal{X}, \mathcal{Y}, \dots$	set
$\mathcal{A}$	algorithm

symbol	meaning
$X, Y \dots$	random variable
$\mathbf{X}, \mathbf{Y} \dots$	multivariate random variable
$x, y \dots$	value
$\mathbf{x}, \mathbf{y} \dots$	vector
$p, pr, P, Pr$	probability



# Probability And Statistics

# Bayes Theorem



$$P(h | \mathcal{D}) = \frac{P(\mathcal{D} | h)P(h)}{P(\mathcal{D})} \quad (1)$$

- $P(h)$  = **prior probability** of hypothesis  $h$
- $P(\mathcal{D})$  = prior probability of training data  $\mathcal{D}$
- $P(h | \mathcal{D})$  = probability of  $h$  given  $\mathcal{D}$  (called **posterior probability**)
- $P(\mathcal{D} | h)$  = probability of  $\mathcal{D}$  given  $h$  (called **likelihood**)

# Basic Formulas for Probabilities



Given two random variables  $X$  and  $Y$ , we say that

- The quantity  $p(X, Y)$  is a *joint probability*
- The quantity  $p(Y | X)$  is a *conditional probability*
- The quantity  $p(X)$  is a *marginal probability*

- **Sum rule:**

$$p(X) = \sum_y p(X, Y) \quad (2)$$

or

$$p(X) = \int p(X, Y) dy \quad (3)$$

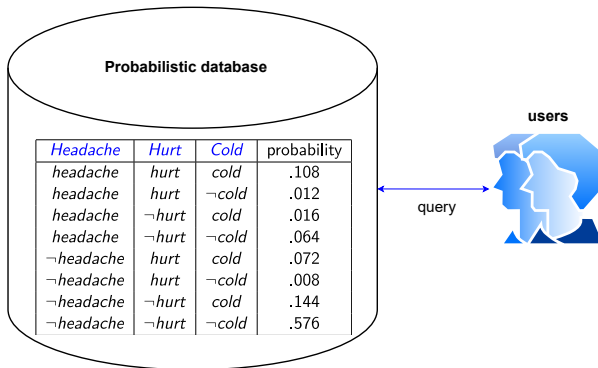
- **Product rule:**

$$p(X, Y) = p(Y | X)p(X) \quad (4)$$

# Example 1



- We have the joint distribution of three random variables  
 $P(\text{Headache}, \text{Hurt}, \text{Cold})$





# Example 1 (cont.)

## Statistical Learning

### Fitting Probability Models

One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

### Linear Regression Revisited

### Naive Bayes

### Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

Examples

More Representation

Compact distribution

Continuous variable

<i>Headache</i>	<i>Hurt</i>	<i>Cold</i>	probability
<i>headache</i>	<i>hurt</i>	<i>cold</i>	.108
<i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.012
<i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.016
<i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.064
$\neg$ <i>headache</i>	<i>hurt</i>	<i>cold</i>	.072
$\neg$ <i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.008
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.144
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.576

$$\begin{aligned}
 P(\text{headache}) &= 0.108 + 0.012 + 0.016 + 0.064 \\
 &= 0.2
 \end{aligned}$$





# Example 1 (cont.)

## Statistical Learning

## Fitting Probability Models

One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

## Linear Regression Revisited

## Naive Bayes

## Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

Examples

More Representation

Compact distribution

Continuous variable

<i>Headache</i>	<i>Hurt</i>	<i>Cold</i>	probability
<i>headache</i>	<i>hurt</i>	<i>cold</i>	.108
<i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.012
<i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.016
<i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.064
$\neg$ <i>headache</i>	<i>hurt</i>	<i>cold</i>	.072
$\neg$ <i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.008
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.144
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.576

$$\begin{aligned}
 P(\text{hurt} \vee \text{headache}) &= 0.108 + 0.012 + 0.072 + 0.008 + 0.016 + 0.064 \\
 &= 0.28
 \end{aligned}$$



# Example 1 (cont.)

## Statistical Learning

## Fitting Probability Models

One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

## Linear

## Regression Revisited

## Naive Bayes

## Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

Examples

More Representation

Compact distribution

Continuous variable

<i>Headache</i>	<i>Hurt</i>	<i>Cold</i>	probability
<i>headache</i>	<i>hurt</i>	<i>cold</i>	.108
<i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.012
<i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.016
<i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.064
$\neg$ <i>headache</i>	<i>hurt</i>	<i>cold</i>	.072
$\neg$ <i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.008
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.144
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.576

$$\begin{aligned}
 P(\neg \text{hurt} \mid \text{headache}) &= \frac{P(\neg \text{hurt} \wedge \text{headache})}{P(\text{headache})} \\
 &= \frac{0.016 + 0.064}{0.108 + 0.012 + 0.016 + 0.064} \\
 &= 0.4
 \end{aligned}$$



# Example 1 (cont.)

## Problem

Let  $\mathbf{X}$  be all the variables. We want the posterior joint distribution of the **query variables**  $\mathbf{Y}$  given specific values  $\mathbf{e}$  for the **evidence variables**  $\mathbf{E}$

## Summing solution

- General idea: compute distribution on query variable by fixing **evidence variables** and summing over **hidden variables**
- Let the **hidden variables** be  $\mathbf{H} = \mathbf{X} - \mathbf{Y} - \mathbf{E}$  and denominator can be viewed as a **normalization constant**  $\alpha$

$$P(\mathbf{Y} \mid \mathbf{E} = \mathbf{e}) = \alpha P(\mathbf{Y}, \mathbf{E} = \mathbf{e}) = \alpha \sum_{\mathbf{h}} P(\mathbf{Y}, \mathbf{E} = \mathbf{e}, \mathbf{H} = \mathbf{h}) \quad (5)$$





# Example 1 (cont.)

## Statistical Learning

### Fitting Probability Models

One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

### Linear

### Regression Revisited

### Naive Bayes

### Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

Examples

More Representation

Compact distribution

Continuous variable

<i>Headache</i>	<i>Hurt</i>	<i>Cold</i>	probability
<i>headache</i>	<i>hurt</i>	<i>cold</i>	.108
<i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.012
<i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.016
<i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.064
$\neg$ <i>headache</i>	<i>hurt</i>	<i>cold</i>	.072
$\neg$ <i>headache</i>	<i>hurt</i>	$\neg$ <i>cold</i>	.008
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	<i>cold</i>	.144
$\neg$ <i>headache</i>	$\neg$ <i>hurt</i>	$\neg$ <i>cold</i>	.576

$$\begin{aligned}
 & P(\textit{Hurt} \mid \textit{headache}) \\
 = & \alpha P(\textit{Hurt}, \textit{headache}) \\
 = & \alpha [P(\textit{Hurt}, \textit{headache}, \textit{cold}) + P(\textit{Hurt}, \textit{headache}, \neg \textit{cold})] \\
 = & \alpha [\langle 0.108, 0.016 \rangle + \langle 0.012, 0.064 \rangle] \\
 = & \alpha \langle 0.12, 0.08 \rangle \\
 = & \langle 0.6, 0.4 \rangle
 \end{aligned}$$



# Statistical Learning



# Statistical Model

## Statistical Learning

### Fitting Probability Models

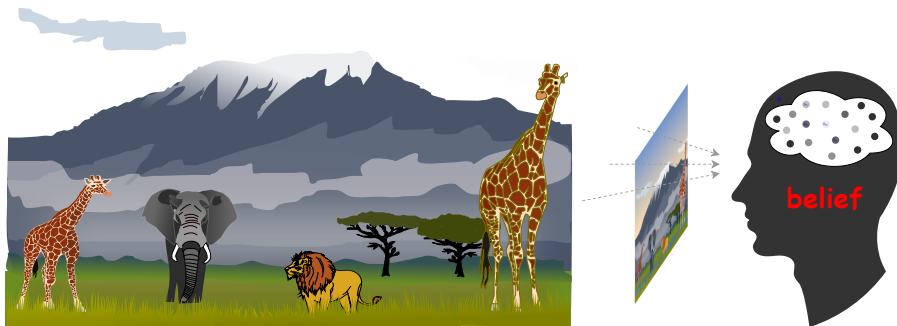
One categorical  
variable  
Two categorical  
variables  
One continuous  
variable  
Joint Probability  
Distributions

### Linear Regression Revisited

### Naive Bayes

### Bayesian Networks

Introduction  
Representation  
Learning  
Parameter Learning  
Structure Learning  
Examples  
More Representation  
Compact distribution  
Continuous variable



The world  $\longrightarrow$  The image  $\longrightarrow$  Model

- All models are wrong, but some are useful (**statistical model**) - George Box

# Probabilistic Approach



## Concept 1

**Learning** is an estimation of *joint probability density* function  $p(\mathbf{x}, y)$  given observed data  $\mathcal{D}$ . **Inductive bias** is expressed as prior assumptions about these joint distributions.

- **Classification and Regression:** conditional density estimation

$$p(y \mid \mathbf{x}) \quad (6)$$

- **Unsupervised Learning:** density estimation

$$p(\mathbf{x}) \quad (7)$$



One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

Introduction

Representation

Learning

Parameter Learning

Structure Learning

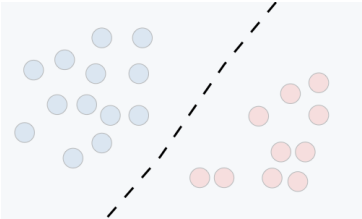
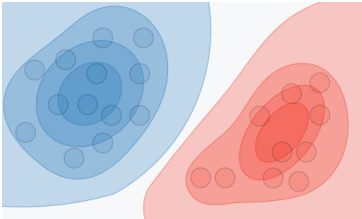
Examples

More Representation

Compact distribution

Continuous variable

# Type of Supervised Model

	Discriminative model	Generative model
<i>Goal</i>	<ul style="list-style-type: none"> <li>Directly estimate <math>P(y   \mathbf{x})</math></li> </ul>	<ul style="list-style-type: none"> <li>Estimate <math>P(\mathbf{x}   y)</math> to then deduce <math>P(y   \mathbf{x})</math></li> </ul>
<i>What's learned</i>	<ul style="list-style-type: none"> <li>Decision boundary</li> </ul> 	<ul style="list-style-type: none"> <li>Probability distributions of the data</li> </ul> 



# Bayesian Learning



## Concept 2

**Bayesian learning** is a process that updates of a probability distribution (**belief**) over the **hypothesis space**  $\mathcal{H} = \{h_1, h_2, \dots\}$  given samples  $\mathcal{D}$ .

- Prior probability of each hypothesis  $h_i$

$$P(h_i) \quad (8)$$

- Given the data  $\mathcal{D}$ , each hypothesis has a posterior probability (update)

$$P(h_i | \mathcal{D}) = \alpha P(\mathcal{D} | h_i) P(h_i) \quad (9)$$

- Predictions use **an average** over the hypotheses

$$P(y) = \sum_i P(y | h_i) P(h_i) \quad (10)$$



# Components of Learning

## Statistical Learning

### Fitting Probability Models

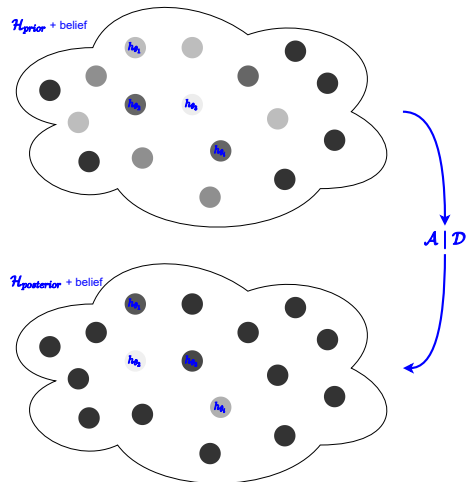
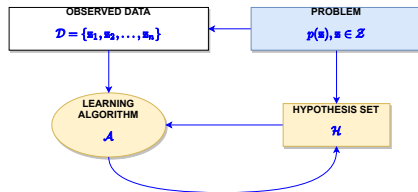
One categorical  
variable  
Two categorical  
variables  
One continuous  
variable  
Joint Probability  
Distributions

### Linear Regression Revisited

### Naive Bayes

### Bayesian Networks

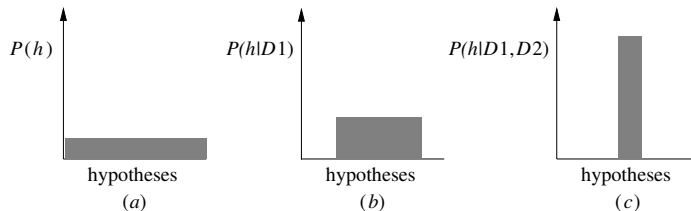
Introduction  
Representation  
Learning  
Parameter Learning  
Structure Learning  
Examples  
More Representation  
Compact distribution  
Continuous variable



# Evolution of Posterior Probabilities



- Changes of a probability distribution  $P(h)$  after observing the data  $D_1$  and  $D_2$





## Example 2

- Does patient have COVID or not?

*"A patient takes a lab test and the result comes back positive. The test returns a correct positive result in only 98% of the cases in which the disease is actually present, and a correct negative result in only 97% of the cases in which the disease is not present. Furthermore, .008 of the entire population have this COVID."*

### Solution

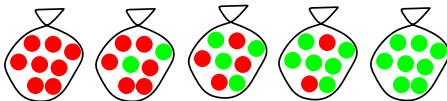
$P(\text{COVID})$	=	_____	$P(\neg \text{COVID})$	=	_____
$P(\oplus \mid \text{COVID})$	=	_____	$P(\ominus \mid \text{COVID})$	=	_____
$P(\oplus \mid \neg \text{COVID})$	=	_____	$P(\ominus \mid \neg \text{COVID})$	=	_____





## Example 3

- Suppose there are five kinds of bags of candies:
  - 10% are  $h_1$ : 100% cherry candies
  - 20% are  $h_2$ : 75% cherry candies + 25% lime candies
  - 40% are  $h_3$ : 50% cherry candies + 50% lime candies
  - 20% are  $h_4$ : 25% cherry candies + 75% lime candies
  - 10% are  $h_5$ : 100% lime candies



### Experiment

- Select one bag
- Candies drawn from the bag:  $\mathcal{D} = \{\text{8 green candies}\}$

### Question

- What kind of bag is it?
- What flavour will the next candy be?



# Example 3 (cont.)

## Statistical Learning

### Fitting Probability Models

One categorical  
variable

Two categorical  
variables

One continuous  
variable

Joint Probability  
Distributions

### Linear

Regression  
Revisited

### Naive Bayes

### Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

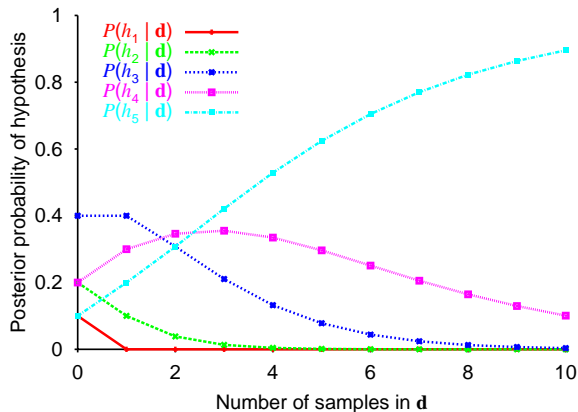
Examples

More Representation

Compact distribution

Continuous variable

1. What kind of bag is it?  
Posterior probability of hypotheses





# Example 3 (cont.)

## Statistical Learning

### Fitting Probability Models

One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

### Linear Regression Revisited

### Naive Bayes

### Bayesian Networks

Introduction

Representation

Learning

Parameter Learning

Structure Learning

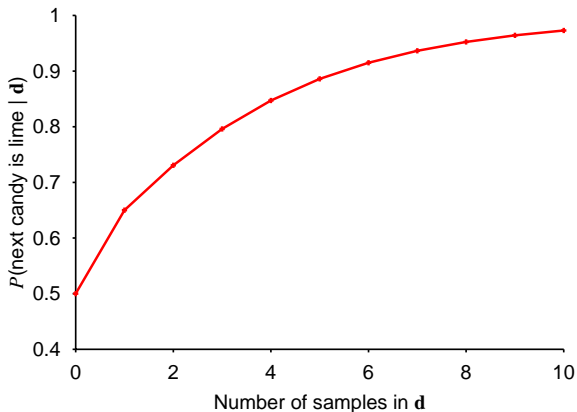
Examples

More Representation

Compact distribution

Continuous variable

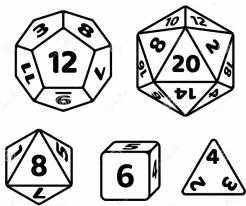
1. What flavour will the next candy be?  
Prediction probability





# Exercise

- Suppose we have a box of dice that contains a 4-sided die, a 6-sided die, an 8-sided die, a 12-sided die, and a 20-sided die



## Experiment

- We select one die
- We roll the die a few more times and get  $\mathcal{D} = \{6, 8, 7, 7, 5, 4\}$

## Question

- What die is selected?





# Fitting Probability Models

- One categorical variable
- Two categorical variables
- One continuous variable
- Joint Probability Distributions

# Learning Strategy



## Concept 3

Fitting probability models to data  $\mathcal{D} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  is referred to as *learning* because we learn about the parameters  $\theta$  of the model

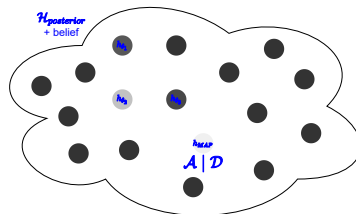
- **Updating** or **summing** over the hypothesis space is often intractable (e.g., 18,446,744,073,709,551,616 Boolean functions of 6 attributes)
- Alternative strategies:
  - **Maximum a posteriori** (MAP) learning
  - **Maximum likelihood** (ML) learning

# Maximum a posteriori



Given data  $\mathcal{D}$

- choose hypothesis  $h$  (called  $h_{MAP}$ ) maximizing  $P(h | \mathcal{D})$
- i.e., maximize  $P(\mathcal{D} | h)P(h)$  or  $\log P(\mathcal{D} | h) + \log P(h)$

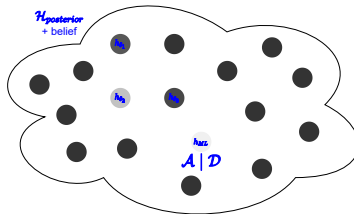


# Maximum likelihood



- For large data sets, prior becomes weak or irrelevant, **maximum likelihood learning**

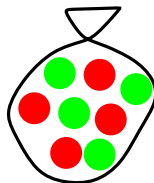
Given data  $\mathcal{D}$ , choose hypothesis  $h$  (called  $h_{ML}$ ) maximizing  $P(\mathcal{D} | h)$  or  $\log P(\mathcal{D} | h)$





## Example 4

- A bag containing two kinds of candies, lime and cherry, has a fraction  $\theta$  of cherry candies?



### Experiment

- Candies drawn from the bag:  $\mathcal{D} = \{\text{lime, cherry, lime, cherry, lime, lime, cherry, lime, lime, lime}\}$

### Question

- What the value of  $\theta$  is?



## Example 4 (cont.)

### ML learning solution

- The model (Bayes net) has one variable *flavor* with one parameter  $\theta$



- Any  $\theta \in [0, 1]$  is possible: continuum of hypotheses  $h_\theta$
- $\theta$  is a **parameter** for this simple (**binomial**) family of models
- Suppose we unwrap  $N$  candies,  $c$  cherries and  $\ell = N - c$  limes. These are **i.i.d.** (independent, identically distributed) observations, so

$$p(\mathcal{D} \mid h_\theta) = \prod_{j=1}^N p(d_j \mid h_\theta) = \theta^c \cdot (1 - \theta)^\ell$$



## Example 4 (cont.)

- Maximize this w.r.t.  $\theta$ —which is easier for the **log-likelihood**:

$$\begin{aligned}
 L(\mathcal{D} \mid h_{\theta}) &= \log p(\mathcal{D} \mid h_{\theta}) \\
 &= \sum_{j=1}^N \log p(d_j \mid h_{\theta}) \\
 &= c \log \theta + \ell \log(1 - \theta) \\
 \frac{dL(\mathcal{D} \mid h_{\theta})}{d\theta} &= \frac{c}{\theta} - \frac{\ell}{1 - \theta} = 0 \\
 \Rightarrow \theta_{ML} &= \frac{c}{c + \ell} = \frac{c}{N}
 \end{aligned} \tag{11}$$

■

- ML seems sensible, but causes problems with **0 counts**!

## Example 4 (cont.)



### MAP learning solution

- Prior for  $h_{\theta}$

$$p(h_{\theta}) = \text{Beta}(a, b) \text{ where } a, b > 0,$$

- Posteriori for  $h_{\theta}$  given  $\mathcal{D}$

$$\begin{aligned} p(h_{\theta} \mid \mathcal{D}) &\propto p(\mathcal{D} \mid h_{\theta})p(h_{\theta}) \\ &= \text{Beta}(a + c, b + \ell) \end{aligned}$$

- MAP estimate

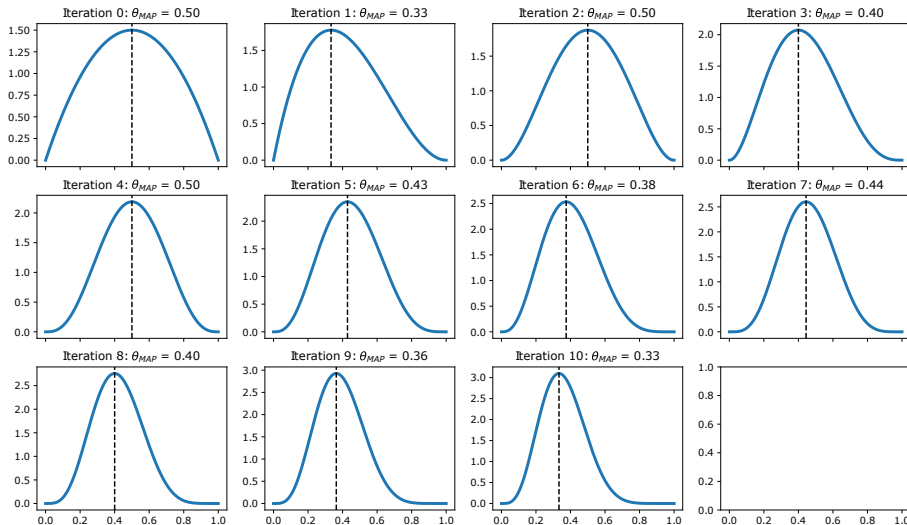
$$\theta_{MAP} = \frac{c + a - 1}{c + \ell + a + b - 2} \quad (12)$$



# Example 4 (cont.)



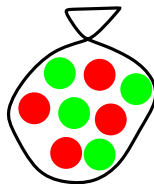
Probability distribution  $P(h_\theta | \mathcal{D})$  with  $a = 2, b = 2$





# Example 5

- A bag has a fraction  $\theta$  of cherry candies, red/green *wrapper* depends probabilistically  $\theta_1, \theta_2$  on *flavor*?



## Experiment

- Candies drawn from some bag  $\mathcal{D}$



## Question

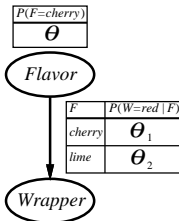
- What the values of  $\theta, \theta_1, \theta_2$  are?

## Example 5 (cont.)



### ML learning solution

- The model (Bayes net) has two variables *flavor* and *wrapper* with three parameters  $\theta$ ,  $\theta_1$ ,  $\theta_2$





## Example 5 (cont.)

- Likelihood for, e.g., cherry candy in green wrapper:

$$\begin{aligned}
 & p(F = \text{cherry}, W = \text{green} \mid h_{\theta, \theta_1, \theta_2}) \\
 &= p(F = \text{cherry} \mid h_{\theta, \theta_1, \theta_2}) p(W = \text{green} \mid F = \text{cherry}; h_{\theta, \theta_1, \theta_2}) \\
 &= \theta \cdot (1 - \theta_1)
 \end{aligned}$$

- $N$  candies,  $r_c$  red-wrapped cherry candies, etc.:

$$p(\mathcal{D} \mid h_{\theta, \theta_1, \theta_2}) = \theta^c (1 - \theta)^\ell \cdot \theta_1^{r_c} (1 - \theta_1)^{g_c} \cdot \theta_2^{r_\ell} (1 - \theta_2)^{g_\ell}$$

$$\begin{aligned}
 L &= [c \log \theta + \ell \log(1 - \theta)] \\
 &\quad + [r_c \log \theta_1 + g_c \log(1 - \theta_1)] \\
 &\quad + [r_\ell \log \theta_2 + g_\ell \log(1 - \theta_2)]
 \end{aligned}$$

## Example 5 (cont.)



- Derivatives of  $L$  contain only the relevant parameter:

$$\frac{\partial L}{\partial \theta} = \frac{c}{\theta} - \frac{\ell}{1 - \theta} = 0 \implies \theta = \frac{c}{c + \ell}$$

$$\frac{\partial L}{\partial \theta_1} = \frac{r_c}{\theta_1} - \frac{g_c}{1 - \theta_1} = 0 \implies \theta_1 = \frac{r_c}{r_c + g_c}$$

$$\frac{\partial L}{\partial \theta_2} = \frac{r_\ell}{\theta_2} - \frac{g_\ell}{1 - \theta_2} = 0 \implies \theta_2 = \frac{r_\ell}{r_\ell + g_\ell}$$

- With **complete data**, *parameters can be learned separately*





## Example 6

- Suppose that the distribution of male height is a normal distribution  $\mathcal{N}(\mu, \sigma^2)$ . Given the following data  $\mathcal{D}$

#	height (m)	#	height (m)
1	1.72	6	1.63
2	1.65	7	1.74
3	1.60	8	1.82
4	1.73	9	1.75
5	1.80	10	1.64

**Question:** what are the values of  $\mu, \sigma$ ?

## Example 6 (cont.)



### ML learning solution

- The model has one variable  $x$  (*height*) with two parameters  $\mu, \sigma^2$

$$p(x \mid \mu, \sigma^2) = \mathcal{N}(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x - \mu)^2}{2\sigma^2} \right] \quad (13)$$

- The likelihood of model with the parameters  $\{\mu, \sigma^2\}$  for observed i.d.d. data  $\mathcal{D} = \{x_1, \dots, x_n\}$  is

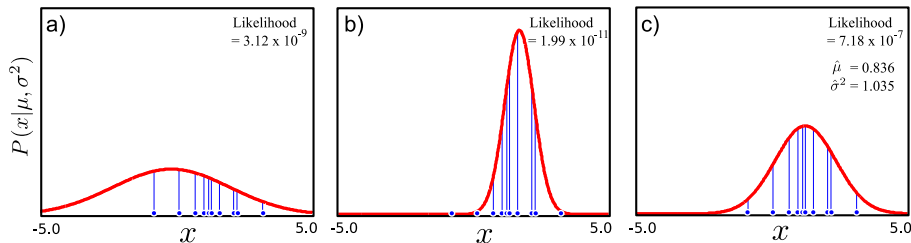
$$\begin{aligned} p(\mathcal{D} \mid \mu, \sigma^2) &= \prod_{i=1}^n \mathcal{N}(x_i \mid \mu, \sigma^2) \\ &= \frac{1}{(2\pi\sigma^2)^{n/2}} \exp \left[ -\frac{1}{2} \sum_{i=1}^n \frac{(x_i - \mu)^2}{\sigma^2} \right] \end{aligned} \quad (14)$$

# Example 6 (cont.)



- The maximum likelihood solution  $\mu_{ML}, \sigma_{ML}^2$

$$\mu_{ML}, \sigma_{ML}^2 = \operatorname{argmax}_{\mu, \sigma^2} [p(\mathcal{D} \mid \mu, \sigma^2)] \quad (15)$$





## Example 6 (cont.)



- Since the logarithm is a monotonic function, the position of the maximum in the transformed function  $L = \log p(\mathcal{D} \mid \mu, \sigma^2)$  remains the same as  $p(\mathcal{D} \mid \mu, \sigma^2)$
- The maximum *log-likelihood* solution

$$\begin{aligned}\mu_{ML}, \sigma_{ML}^2 &= \operatorname{argmax}_{\mu, \sigma^2} [\log p(\mathcal{D} \mid \mu, \sigma^2)] \\ &= \operatorname{argmax}_{\mu, \sigma^2} \left[ -\frac{1}{2} n \log[2\pi] - \frac{1}{2} \log \sigma^2 - \frac{1}{2} \sum_{i=1}^n \frac{(x_i - \mu)^2}{\sigma^2} \right] \quad (16)\end{aligned}$$

## Example 6 (cont.)



- To maximize, we differentiate this *log-likelihood*  $L$  with respect to  $\mu$  and equate the result to zero

$$\begin{aligned}\frac{\partial L}{\partial \mu} &= \sum_{i=1}^n \frac{(x_i - \mu)}{\sigma^2} \\ &= \frac{\sum_{i=1}^n x_i}{\sigma^2} - \frac{n\mu}{\sigma^2} = 0 \\ \mu_{ML} &= \frac{1}{n} \sum_{i=1}^n x_i\end{aligned}\tag{17}$$

# Example 6 (cont.)



- By a similar process, the expression for the variance can be shown to be

$$\sigma_{ML}^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \mu_{ML})^2. \quad (18)$$



# Joint Probability Distributions



- The joint probability distribution is central to probabilistic inference, because once we know the joint distribution we can answer every possible probabilistic question that can be asked about these variables.

Gender	HoursWorked	Wealth	probability
female	$< 40.5$	poor	0.2531
female	$< 40.5$	rich	0.0246
female	$\geq 40.5$	poor	0.0422
female	$\geq 40.5$	rich	0.0116
male	$< 40.5$	poor	0.3313
male	$< 40.5$	rich	0.0972
male	$\geq 40.5$	poor	0.1341
male	$\geq 40.5$	rich	0.1059

# Discussion



- How can we learn joint distributions from observed training data  $\mathcal{D}$ ?

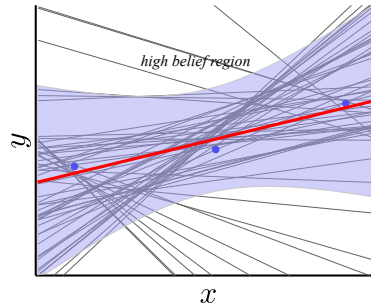
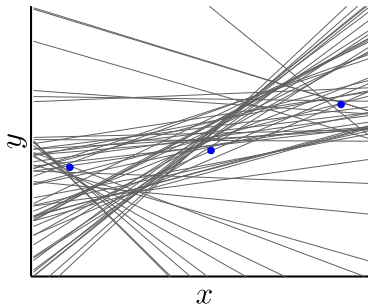


# Linear Regression Revisited

# Linear Regression



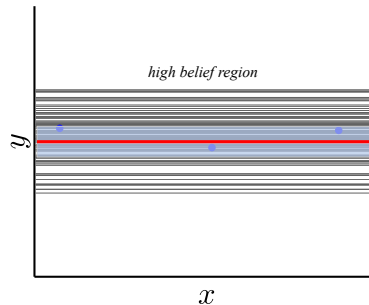
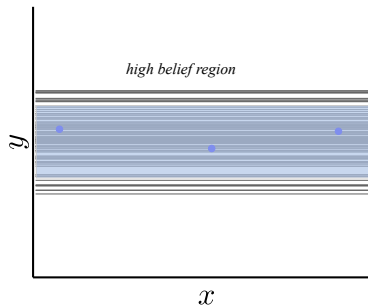
- Linear regression without prior belief



# Linear Regression (cont.)

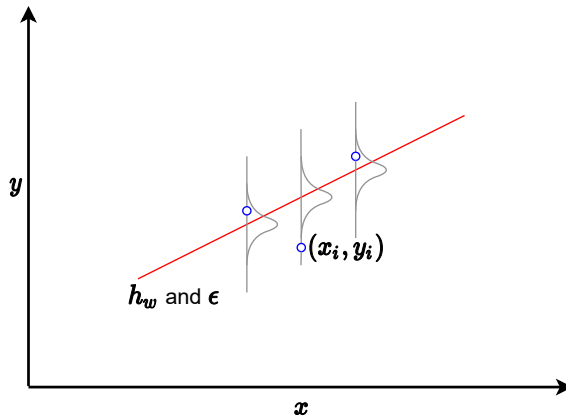


- Linear regression with prior belief





# Linear Regression Revisited



# Linear Regression Revisited (cont.)



- Unknown function  $f$  is modeled by the hypothesis  $h_{\mathbf{w}}$

$$y = h_{\mathbf{w}}(x) = \mathbf{w}^T \mathbf{x} + \epsilon \quad (19)$$

where  $y$  is noisy target value,  $\epsilon$  is *random variable (noise)* drawn independently according to a Gaussian distribution with mean equal to 0 and variance equal to  $\sigma$  ( $\mathcal{N}(0, \sigma^2)$ )

- Probability language

$$p(y \mid \mathbf{x}, \mathbf{w}) = \mathcal{N}(y \mid \mathbf{w}^T \mathbf{x}, \sigma^2) \quad (20)$$

- Given data  $\mathcal{D} = \{\mathbf{x}_i, y_i\}_{i=1}^N$ , the likelihood of  $\mathcal{D}$  given  $h_{\mathbf{w}}$  and noise

$$\begin{aligned} p(\mathcal{D} \mid h_{\mathbf{w}}, \epsilon) &= \prod_{i=1}^N p(y_i \mid \mathbf{x}_i, \mathbf{w}) \\ &= \prod_{i=1}^N \mathcal{N}(y_i \mid \mathbf{w}^T \mathbf{x}_i, \sigma^2) \end{aligned} \quad (21)$$

# ML Learning



Choose hypothesis  $h$  maximizing the likelihood

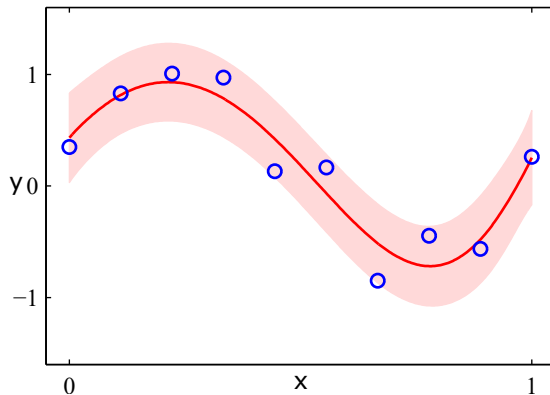
$$\begin{aligned} & \arg \max_{\mathbf{w}, \sigma} p(\mathcal{D} \mid h_{\mathbf{w}}, e) \\ \Leftrightarrow & \arg \max_{\mathbf{w}, \sigma} \prod_{i=1}^N \mathcal{N}(y_i \mid \mathbf{w}^\top \mathbf{x}_i, \sigma^2) \\ \Leftrightarrow & \arg \max_{\mathbf{w}, \sigma} \log \left( \prod_{i=1}^N \mathcal{N}(y_i \mid \mathbf{w}^\top \mathbf{x}_i, \sigma^2) \right) \\ \Leftrightarrow & \arg \min_{\mathbf{w}, \sigma} \frac{1}{2\sigma^2} MSE + \frac{N}{2} \log(\sigma^2) + \frac{N}{2} \log(2\pi) \end{aligned} \quad (22)$$

Solving (20), we have

$$\mathbf{w}_{ML} = (\mathbf{X}^\top \mathbf{X})^{-1} \mathbf{X}^\top \mathbf{y}$$

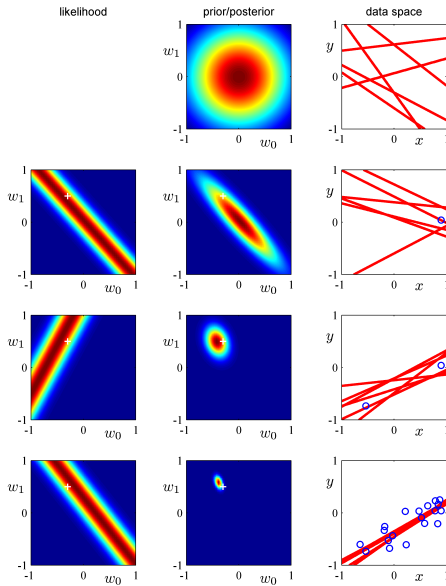
$$\sigma_{ML}^2 = \frac{1}{N} \sum_{i=1}^N (\mathbf{w}_{ML}^\top \mathbf{x}_i - y_i)^2$$

# Example



**Figure 1:** The red curve denotes the regression curve and the red region corresponds to  $\pm\sigma$  standard deviation

# Bayesian Learning



- The figure shows the update process of Bayesian learning where  $w$  are introduced as hypothesis parameters



# Naive Bayes

# When to use



Along with decision trees, neural networks, nearest neighbour, one of the most practical learning methods.

- Moderate or large training set available
- Attributes that describe instances are conditionally independent given classification

Successful applications

- Diagnosis
- Classifying text documents

# Naive Bayes assumption



- Assume target function for classification problem

$$f : \mathcal{X} \rightarrow \mathcal{Y}$$

where each instance  $\mathbf{x}$  described by attributes  $(x_1, x_2 \dots x_D)$  and  $y$  is a corresponding class.

- Naive Bayes assumption**

$$P(x_1, x_2 \dots x_D \mid y) = \prod_{i=1}^D P(x_i \mid y) \quad (23)$$





# Naive Bayes classifiers

- Naive Bayes classifier

$$y_{NB} = \arg \max_y \hat{P}(y) \prod_{i=1}^D \hat{P}(x_i | y) \quad (24)$$

The form of the class-conditional density depends on the type of each feature.

1. In the case of real-valued features, we can use the **Gaussian distribution**:

$$p(\mathbf{x} | y = c) \sim \prod_{j=1}^D \mathcal{N}(x_j | \mu_{jc}, \sigma_{jc}^2)$$

where  $\mu_{jc}$  is the mean of feature  $j$  in objects of class  $c$ , and  $\sigma_{jc}^2$  is its variance.

# Naive Bayes classifiers (cont.)



2. In the case of binary features,  $x_j \in \{0, 1\}$ , we can use the **Bernoulli distribution**:

$$p(\mathbf{x} \mid y = c) \sim \prod_{j=1}^D \text{Ber}(x_j \mid \mu_{jc})$$

where  $\mu_{jc}$  is the probability that feature  $j$  occurs in class  $c$ .

3. In the case of categorical features,  $x_j \in \{v_{j1}, v_{j2}, \dots, v_{jK}\}$ , we can use the **multinoulli distribution**:

$$p(\mathbf{x} \mid y = c) \sim \prod_{j=1}^D \text{Cat}(x_j \mid \mu_{jc})$$

where  $\mu_{jc}$  is a histogram over the  $K$  possible values for  $x_j$  in class  $c$ .

# Naive Bayes Algorithm



LEARNNAIVEBAYES( $\mathcal{D}$ )

**for** each target value (class)  $y_j$

$\hat{P}(y_j) \leftarrow$  estimate  $P(y_j)$  given data  $\mathcal{D}$

**for** each attribute  $x_i$

$\hat{P}(x_i | y_j) \leftarrow$  estimate  $P(x_i | y_j)$  given data  $\mathcal{D}$

CLASSIFYNEWINSTANCE( $x$ )

$y = \arg \max_y \hat{P}(y) \prod_{i=1}^D \hat{P}(x_i | y)$

**return**  $y$

# Naive Bayes Algorithm (cont.)



**Maximum likelihood learning** for  $\hat{P}(y = c)$  and  $\hat{P}(x_i = a \mid y = c)$

$$\hat{P}(y = c) \leftarrow \frac{n_c}{n} \quad (25)$$

$$\hat{P}(x_i = a \mid y = c) \leftarrow \frac{n_a}{n_c} \quad (26)$$

where

- $n$  is number of training examples
- $n_c$  is number of training examples for which  $y = c$
- $n_a$  is number of examples for which  $y = c$  and  $x_i = a$

# Example 1



- Consider *PlayTennis* again

 $\mathcal{D} =$ 

Day	Outlook	Temperature	Humidity	Wind	PlayTennis?
D1	sunny	hot	high	weak	no
D2	sunny	hot	high	strong	no
D3	overcast	hot	high	weak	yes
D4	rain	mild	high	weak	yes
D5	rain	cool	normal	weak	yes
D6	rain	cool	normal	strong	no
D7	overcast	cool	normal	strong	yes
D8	sunny	mild	high	weak	no
D9	sunny	cool	normal	weak	yes
D10	rain	mild	normal	weak	yes
D11	sunny	mild	normal	strong	yes
D12	overcast	mild	high	strong	yes
D13	overcast	hot	normal	weak	yes
D14	rain	mild	high	strong	no

# Example 1 (cont.)



- Naive Bayes model

$$\hat{P}(\text{PlayTennis})$$

yes	9/14
no	5/14

$$\hat{P}(\text{Outlook} \mid \text{PlayTennis})$$

		Outlook		
		overcast	rain	sunny
PlayTennis	yes	4/9	3/9	2/9
	no	0/5	2/5	3/5

$$\hat{P}(\text{Temperature} \mid \text{PlayTennis})$$

		Temperature		
		cool	hot	mild
PlayTennis	yes	3/9	2/9	4/9
	no	1/5	2/5	2/5

$$\hat{P}(\text{Humidity} \mid \text{PlayTennis})$$

		Humidity	
		high	normal
PlayTennis	yes	3/9	6/9
	no	4/5	1/5

$$\hat{P}(\text{Wind} \mid \text{PlayTennis})$$

		Wind	
		strong	weak
PlayTennis	yes	3/9	6/9
	no	3/5	2/5

# Example 1 (cont.)



- Get new instance

$\mathbf{x} = (\text{Outlk} = \text{sun}, \text{Temp} = \text{cool}, \text{Humid} = \text{high}, \text{Wind} = \text{strong})$

- Compute

$$\begin{cases} \hat{P}(\text{yes}) \times \hat{P}(\text{sun} \mid \text{yes}) \times \hat{P}(\text{cool} \mid \text{yes}) \times \hat{P}(\text{high} \mid \text{yes}) \times \hat{P}(\text{strong} \mid \text{yes}) & = .005 \\ \hat{P}(\text{no}) \times \hat{P}(\text{sun} \mid \text{no}) \times \hat{P}(\text{cool} \mid \text{no}) \times \hat{P}(\text{high} \mid \text{no}) \times \hat{P}(\text{strong} \mid \text{no}) & = .021 \end{cases}$$

- Make decision

$y = \text{no}$

## Example 2



- Find Naive Bayes classifier given the following training dataset

#	Vị	Màu	Vỏ	Độc tính
1	Ngọt	Đỏ	Nhẵn	Không
2	Cay	Đỏ	Nhẵn	Có
3	Chua	Vàng	Có gai	Không
4	Cay	Vàng	Có gai	Có
5	Ngọt	Tím	Có gai	Không
6	Chua	Vàng	Nhẵn	Không
7	Ngọt	Tím	Nhẵn	Không
8	Cay	Tím	Có gai	Có
9	Cay	Tím	Có gai	Không
10	Cay	Tím	Có gai	Có
11	Cay	Vàng	Có gai	Có



# Avoiding the zero-probability problem



1. Conditional independence assumption is often violated but it works surprisingly well anyway
2. Suppose that none of the training instances with target value  $y$  have attribute value  $x_i = v$ ? then  $\hat{P}(x_i = v | y) = 0$ , and  $\hat{P}(y) \dots \hat{P}(x_i = v | y) \dots = 0$  (not good in probability language)

# Avoiding the zero-probability problem (cont.)



Typical solution is **Bayesian estimate** for  $\hat{P}(y = c)$  and  $\hat{P}(x_i = a \mid y = c)$

$$\hat{P}(y = c) \leftarrow \frac{n_c + 1}{n + C} \quad (27)$$

$$\hat{P}(x_i = a \mid y = c) \leftarrow \frac{n_a + 1}{n_c + r} \quad (28)$$

where

- $n$  is number of training examples
- $n_c$  is number of training examples for which  $y = c$
- $C$  is the number of classes
- $n_a$  is number of examples for which  $y = c$  and  $x_i = a$
- $r$  is the number of values of attribute  $x_i$

# Example 3



- Find Naive Bayes classifier given the following training dataset

#	height (m)	hair	gender
1	1.72	brown	male
2	1.65	black	female
3	1.60	black	female
4	1.73	black	female
5	1.80	brown	male
6	1.63	black	male
7	1.74	black	female
8	1.82	brown	male
9	1.75	black	male
10	1.64	brown	female



# Bayesian Networks

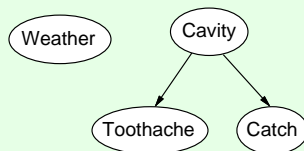
- Introduction
- Representation
- Learning
- Examples
- More Representation

# Bayesian networks



## Concept 4

A **Bayesian network**, or **causal probabilistic network**, is a *directed acyclic graph* (DAG) for conditional independence assertions and hence for compact specification of full joint distributions



# Bayesian networks (cont.)



## Syntax

- Nodes represent **variables**
- Links represent **dependency relations** between variables and quantified by *conditional distributions*
- A **conditional distribution** for each node given its parents:

$$P(X_i \mid \text{parents}(X_i))$$

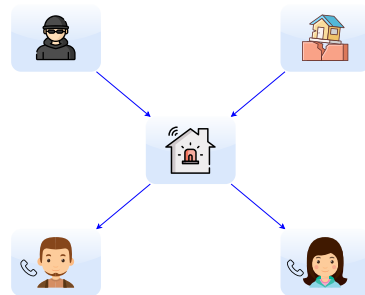
- In the simplest case, conditional distribution represented as a **conditional probability table** (CPT) giving the distribution over  $X_i$  for each combination of parent values

# Example



I'm at work, neighbor John calls to say my alarm is ringing, but neighbor Mary doesn't call. Sometimes it's set off by minor earthquakes. Is there a burglar?

- Variables: **B**urglar, **E**arthquake, **A**larm, **J**ohnCalls, **M**aryCalls
- Network topology reflects “causal” knowledge:
  - A burglar can set the alarm off
  - An earthquake can set the alarm off
  - The alarm can cause Mary to call
  - The alarm can cause John to call



# Queries



$$P(\text{Query} \mid \text{Evidence}) = ?$$

- **Diagnostic** (from effects to causes)

$$P(B \mid J) \rightarrow \text{"If John calls, how likely is the house burglarized?"}$$

- **Causal** (from causes to effects):

$$P(J \mid E) \rightarrow \text{"If earthquake happens, how likely will John make a call?"}$$

- **Intercausal** (between causes of a common effect):

$$P(B \mid A, E) \rightarrow \text{"If earthquake happens and alarm is on, how likely is the house burglarized??"}$$

- **Mixed:**

$$P(A \mid J, E) \rightarrow \text{"..."}$$

$$P(B \mid J, E) \rightarrow \text{"..."}$$





One categorical variable

Two categorical variables

One continuous variable

Joint Probability Distributions

Introduction

Representation

Learning

Parameter Learning

Structure Learning

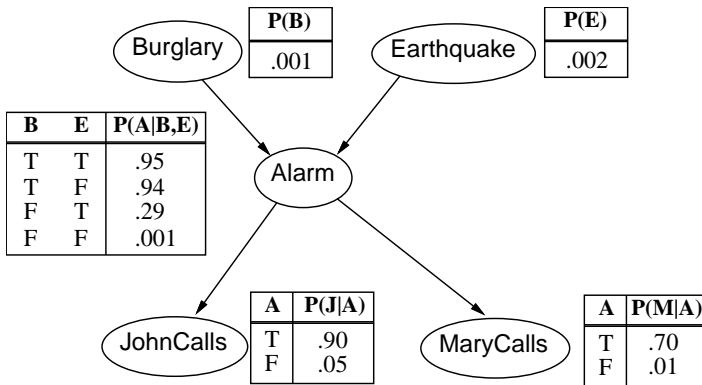
Examples

More Representation

Compact distribution

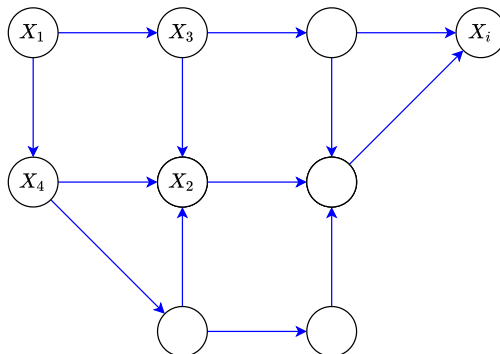
Continuous variable

# Graphical Model





# Type of Variables



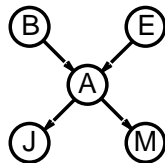
Variables  $\{X_i\}$

- **random** or **deterministic**
- **observed** or **hidden**
- **continuous** or **discrete** (boolean, category)

# CPT Representation



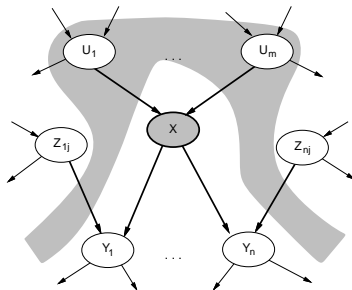
- A CPT for Boolean  $X_i$  with  $k$  Boolean parents has  $2^k$  rows for the combinations of parent values
- Each row requires one number  $p$  for  $X_i = \text{true}$  (the number for  $X_i = \text{false}$  is just  $1 - p$ )
- If each variable has no more than  $k$  parents, the complete network requires  $O(n \times 2^k)$  numbers
  - I.e., grows linearly with  $n$ , vs.  $O(2^n)$  for the full joint distribution
- For **burglary net**,  $1 + 1 + 4 + 2 + 2 = 10$  numbers (vs.  $2^5 - 1 = 31$ )



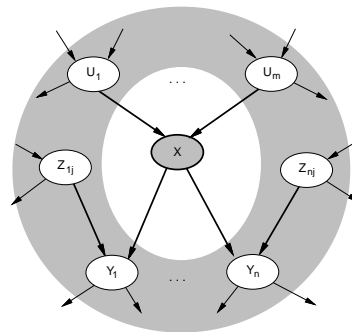
# Conditional Independence



- Each node is conditionally independent of its *nondescendants* given its *parents*



- Each node is conditionally independent of all others given its **Markov blanket** (*parents + children + children's parents*)



# Global semantics and Inference



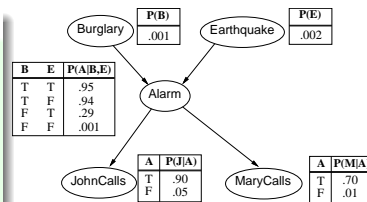
## Concept 5

The **full joint distribution** is defined as the product of the local conditional distributions:

$$P(X_1, \dots, X_n) = \prod_{i=1}^n P(X_i \mid \text{parents}(X_i))$$

- For example,

$$\begin{aligned} P(j \wedge m \wedge a \wedge \neg b \wedge \neg e) &= P(j \mid a)P(m \mid a)P(a \mid \neg b, \neg e)P(\neg b)P(\neg e) \\ &= 0.9 \times 0.7 \times 0.001 \times 0.999 \times 0.998 \\ &\approx 0.00063 \end{aligned}$$



# The Learning Problem



	Known Structure	Unknown Structure
<b>Complete Data</b>	Statistical parametric estimation (closed-form)	Discrete optimization over structures (discrete search)
<b>Incomplete Data</b>	Parametric optimization (EM, gradient descent...)	Combined (Structural EM, mixture models...)

Learning problem includes

- **Parameter learning**
- **Structure learning**



# Known Structure and Complete Data

- Given a training data  $\mathcal{D}$ , find the best parameter  $\theta$ s for multinomial variables

$$P_{\theta}(X_i \mid pa_i) \quad (29)$$

where  $pa_i = \text{parents}(X_i)$  ( $pa_i$  can be  $\emptyset$ )

- Estimate parameter  
Maximum likelihood

$$\hat{\theta}_{ML} = \frac{\text{count}(x_i, pa_i)}{\text{count}(pa_i)} \quad (30)$$

Maximum a posteriori

$$\hat{\theta}_{MAP} = \frac{\alpha(x_i, pa_i) + \text{count}(x_i, pa_i)}{\alpha(pa_i) + \text{count}(pa_i)} \quad (31)$$

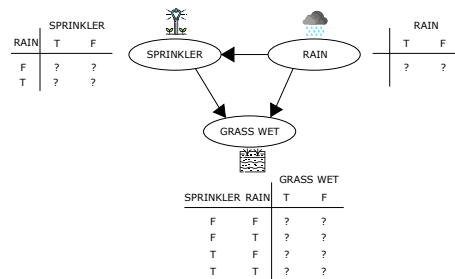
where  $\text{count}(\cdot)$  is the number of instances and  $\alpha(\cdot)$  is the prior parameters.

# Example 4



- Find the best parameter  $\theta$ s given the following training data  $\mathcal{D}$

#	Rain	Sprinkler	Grass Wet
1	T	T	T
2	T	T	F
3	T	F	T
4	T	F	F
5	F	T	T
6	F	T	F
7	F	F	T
8	F	F	F
9	T	T	T
10	F	T	T
11	T	F	T
12	F	F	T
13	F	T	T
14	T	T	T





# Unknown Structure and Complete Data



- Need a method such that a series of locally testable assertions of conditional independence guarantees the required global semantics
1. Choose an ordering of variables  $X_1, \dots, X_n$
  2. For  $i = 1$  to  $n$   
add  $X_i$  to the network  
select parents from  $X_1, \dots, X_{i-1}$  such that

$$P(X_i \mid \text{parents}(X_i)) = P(X_i \mid X_1, \dots, X_{i-1})$$

This choice of parents guarantees the global semantics:

$$\begin{aligned} P(X_1, \dots, X_n) &= \prod_{i=1}^n P(X_i \mid X_1, \dots, X_{i-1}) \quad (\text{chain rule}) \\ &= \prod_{i=1}^n P(X_i \mid \text{parents}(X_i)) \quad (\text{by construction}) \end{aligned}$$

# Example: Burglary alarm



- Suppose we choose the ordering  $M, J, A, B, E$   

$$P(J \mid M) = P(J)?$$

MaryCalls

JohnCalls

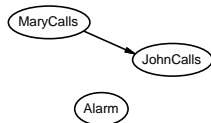
# Example: Burglary alarm (cont.)



- Suppose we choose the ordering  $M, J, A, B, E$

$$P(J | M) = P(J)? \text{ No}$$

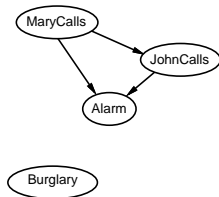
$$P(A | J, M) = P(A | J)? \quad P(A | J, M) = P(A)?$$



# Example: Burglary alarm (cont.)



- Suppose we choose the ordering  $M, J, A, B, E$



$$P(J | M) = P(J)? \text{ No}$$

$$P(A | J, M) = P(A | J)? \quad P(A | J, M) = P(A)? \text{ No}$$

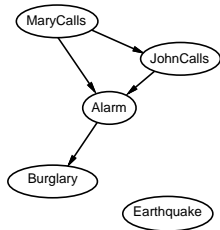
$$P(B | A, J, M) = P(B | A)?$$

$$P(B | A, J, M) = P(B)?$$

# Example: Burglary alarm (cont.)



- Suppose we choose the ordering  $M, J, A, B, E$



$$P(J \mid M) = P(J)? \text{ No}$$

$$P(A \mid J, M) = P(A \mid J)? \quad P(A \mid J, M) = P(A)? \text{ No}$$

$$P(B \mid A, J, M) = P(B \mid A)? \text{ Yes}$$

$$P(B \mid A, J, M) = P(B)? \text{ No}$$

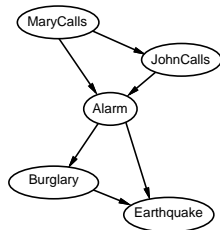
$$P(E \mid B, A, J, M) = P(E \mid A)?$$

$$P(E \mid B, A, J, M) = P(E \mid A, B)?$$

# Example: Burglary alarm (cont.)



- Suppose we choose the ordering  $M, J, A, B, E$



$$P(J | M) = P(J)? \text{ No}$$

$$P(A | J, M) = P(A | J)? \quad P(A | J, M) = P(A)? \text{ No}$$

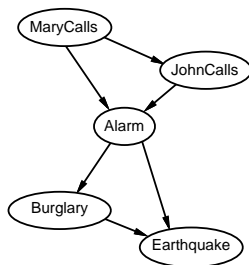
$$P(B | A, J, M) = P(B | A)? \text{ Yes}$$

$$P(B | A, J, M) = P(B)? \text{ No}$$

$$P(E | B, A, J, M) = P(E | A)? \text{ No}$$

$$P(E | B, A, J, M) = P(E | A, B)? \text{ Yes}$$

# Example: Burglary alarm (cont.)

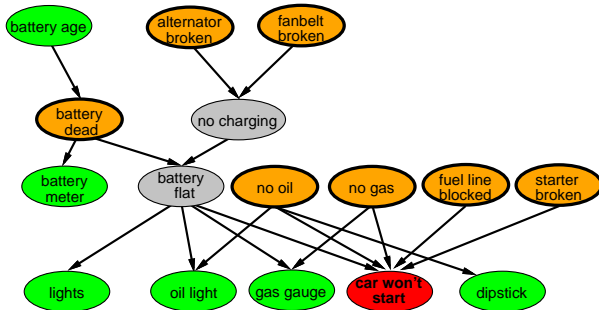


- Deciding conditional independence is hard in noncausal directions (Causal models and conditional independence seem hardwired for humans!)
- Assessing conditional probabilities is hard in noncausal directions
- Network is less compact:  
 $1 + 2 + 4 + 2 + 4 = 13$  numbers needed

# Example: Car diagnosis

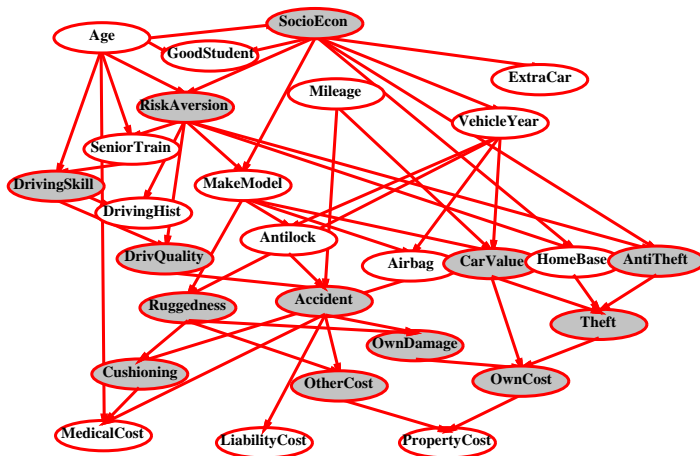


- Initial evidence (red): car won't start
- Testable variables (green), "broken, so fix it" variables (orange)
- Hidden variables (gray) ensure sparse structure, reduce parameters





# Example: Car insurance

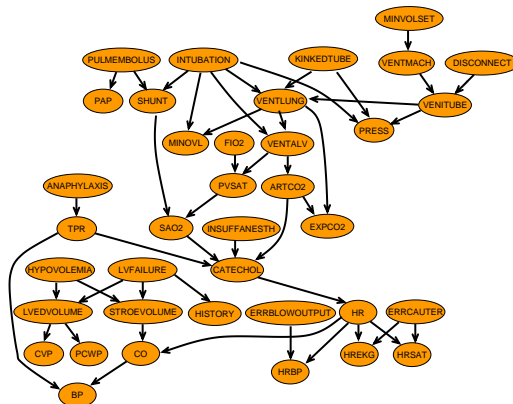


# Example: “ICU Alarm” network



**Domain:** Monitoring Intensive-Care Patients

- 37 variables
- 509 parameters



# Compact conditional distributions



## Problem

- CPT grows exponentially with number of parents
- CPT does not work with continuous-valued parent or child

- **Deterministic nodes  $X$ :**

$$X = f(\text{parents}(X)) \text{ for some function } f \quad (32)$$

- Boolean functions

$$\text{NorthAmerican} = \text{Canadian} \vee \text{US} \vee \text{Mexican}$$

- Numerical relationships among continuous variables

$$\frac{\partial \text{Level}}{\partial t} = \text{inflow} + \text{precipitation} - \text{outflow} - \text{evaporation}$$

# Compact conditional distributions (cont.)



- **Noisy-OR distributions** model multiple noninteracting causes
  - Parents  $U_1 \dots U_k$  include all causes (can add **leak node**)
  - Independent failure probability  $q_i$  for each cause alone

$$P(X \mid U_1 \dots U_j, \neg U_{j+1} \dots \neg U_k) = 1 - \prod_{i=1}^j q_i \quad (33)$$

Number of parameters *linear* in number of parents

# Compact conditional distributions (cont.)



$$q_{cold} = P(\neg fever \mid cold, \neg flu, \neg malaria) = 0.6$$

$$q_{flu} = P(\neg fever \mid \neg cold, flu, \neg malaria) = 0.2$$

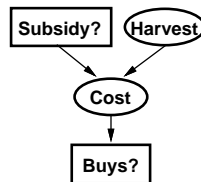
$$q_{malaria} = P(\neg fever \mid \neg cold, \neg flu, malaria) = 0.1$$

<i>Cold</i>	<i>Flu</i>	<i>Malaria</i>	$P(Fever)$	$P(\neg Fever)$
F	F	F	<b>0.0</b>	1.0
F	F	T	0.9	<b>0.1</b>
F	T	F	0.8	<b>0.2</b>
F	T	T	0.98	$0.02 = 0.2 \times 0.1$
T	F	F	0.4	<b>0.6</b>
T	F	T	0.94	$0.06 = 0.6 \times 0.1$
T	T	F	0.88	$0.12 = 0.6 \times 0.2$
T	T	T	0.988	$0.012 = 0.6 \times 0.2 \times 0.1$

# Bayesian nets with continuous variables



Discrete (*Subsidy?* and *Buys?*); continuous (*Harvest* and *Cost*)



- Option 1: discretization → possibly large errors, large CPTs
- Option 2: finitely parameterized canonical families
  1. Continuous variable, discrete+continuous parents (e.g., *Cost*)
  2. Discrete variable, continuous parents (e.g., *Buys?*)

# Continuous child variables



- Need one **conditional density** function for child variable given continuous parents, for each possible assignment to discrete parents
- Most common is the **linear Gaussian** (LG) model, e.g.,:

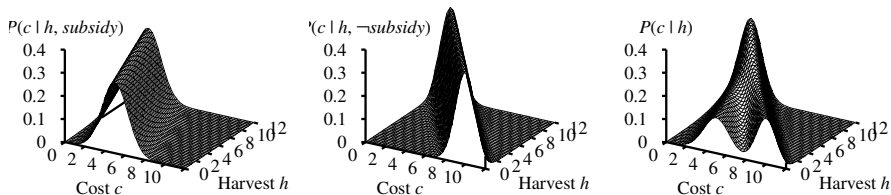
$$P(\text{Cost} = c \mid \text{Harvest} = h, \text{Subsidy?} = \text{true}) \\ = N(a_t h + b_t, \sigma_t)(c) = \frac{1}{\sigma_t \sqrt{2\pi}} \exp \left( -\frac{1}{2} \left( \frac{c - (a_t h + b_t)}{\sigma_t} \right)^2 \right)$$

- Mean *Cost* varies linearly with *Harvest*, variance is fixed
- Linear variation is unreasonable over the full range but works OK if the *likely* range of *Harvest* is narrow

# Continuous child variables (cont.)



- All-continuous network with LG distributions  $\implies$  full joint distribution is a multivariate Gaussian
- Discrete+continuous LG network is a **conditional Gaussian** network i.e., a multivariate Gaussian over all continuous variables for each combination of discrete variable values



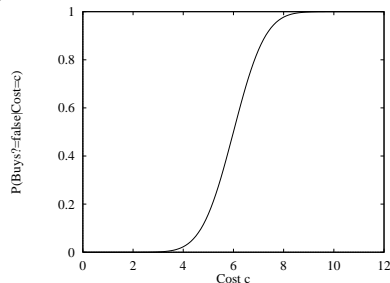
**Figure 2:** The graphs in (1) and (2) show the probability distribution over *Cost* as a function of *Harvest* size, with *Subsidy* true and false, respectively. Graph (3) shows the distribution  $P(\text{Cost} \mid \text{Harvest})$ , obtained by summing over the two subsidy cases.



# Discrete variable given continuous parents



- Probability of *Buys?* given *Cost* should be a “soft” threshold:



- Probit** distribution uses integral of Gaussian:

$$\Phi(x) = \int_{-\infty}^x \mathcal{N}(0, 1)(x) dx$$

$$P(\text{Buys?} = \text{true} \mid \text{Cost} = c) = \Phi\left(\frac{-c + \mu}{\sigma}\right)$$

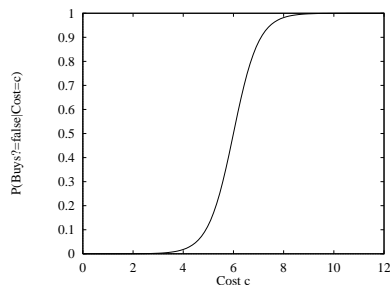
# Discrete variable given continuous parents (cont.)



- **Sigmoid** (or **logit**) distribution also used in neural networks:

$$P(\text{Buys?} = \text{true} \mid \text{Cost} = c) = \frac{1}{1 + \exp(-2\frac{-c+\mu}{\sigma})}$$

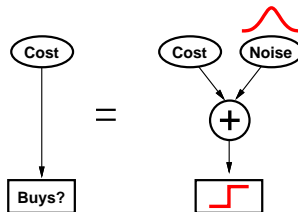
- Sigmoid has similar shape to probit but much longer tails:



# Why the probit/logit?



1. It's sort of the right shape
2. Can view as hard threshold whose location is subject to noise



# References

---



Goodfellow, I., Bengio, Y., and Courville, A. (2016).

*Deep learning.*

MIT press.



Lê, B. and Tô, V. (2014).

*Cở sở trí tuệ nhân tạo.*

Nhà xuất bản Khoa học và Kỹ thuật.



Nguyen, T. (2018).

Artificial intelligence slides.

Technical report, HCMC University of Sciences.



Russell, S. and Norvig, P. (2016).

*Artificial intelligence: a modern approach.*

Pearson Education Limited.