Matrix

- **Matrix** is a rectangle array of numbers consisting of **rows** and **columns**
- Matrix A can be written as a_{ij}, where i is row and j are column, written as $i \times j$
- A matrix A with the order $n \times n$, or a square matrix, has the entries consisting $a_{11}, a_{22}, ..., a_{nn}$ called **main diagonal** of A

Type of Matrices

- Square Matrix: Matrix with the same number of rows and the number of columns
- Zero Matrix: Matrix all of whose entries are zero
- Identity Matrix: Square matrix where its main diagonal is 1 and other elements are 0
- Column Matrix:
- Row Matrix:

Operations on Matrices

Equality of Matrices

Two matrices are defined to be equal if they have the same size and their corresponding entries are equal.

Addition and Subtraction

Let $A = (a_{ij})$ and $B = (b_{ij})$ and has the same order $m \times n$. Addition and subtraction of $A \pm B$ are defined as :

$$(A \pm B)_{ij} = A_{ij} \pm B_{ij} = a_{ij} \pm b_{ij}$$

The result elements are :

$$A + B = [a_{ij} + b_{ij}]$$

Note:

- Addition and subtraction matrices could be done if and only if the order or size of said matrices is the same

Multiplication

Scalar

Let $A = [a_{ij}]$ and scalar k. Scalar multiplication kA defined as:

$$(kA)_{ij} = k.(A)_{ij} = ka_{ij}$$

Two Matrices

Let $A = (a)_{ij}$ with the order $m \times n$ and $B = (b)_{ij}$ with the order $p \times q$. If $n = p$, then the product of AB is a matrix with the order $m \times q$ and defined as :

$$[AB]_{ij} = \sum_{s=1}^{n} a_{is} b_{sj}$$

Note:

- If $n \neq p$ then product of AB is undefined
- The Cancellation Law does not hold

Exponential

Let $A = [a_{ij}]$ a square matrix of order n, then for $k \geq 0$, defined as $A^k = I$ for $k = 0$, and

$$A^k = \underbrace{A \times A \times A \times ... \times A}_{\text{k times}}$$
$$A^{m+n} = A^n A^m$$

Properties of Matrices Algebra

- Addition matrices are commutative

$$A + B = B + A$$

- Addition matrices are associative

$$(A + B) + C = A + (B + C)$$

- Addition matrices with zero matrix equal the matrix itself

$$A + 0 = A$$

- Each matrix A has negative $-A$,with $-A = -1(A)$ and $A + (-A) = 0$
- Multiplication matrices with scalar are associative

$$s(tA) = st(A)$$

- Addition and Multiplication matrices with scalar are distributive

$$(A + B)C = AC + BC$$
$$r(A + B) = rA + rB$$

Transpose

Let a matrix A with the order $m \times n$, then transpose of matrix A, denoted by A^T, is defined to be matrix with the order $n \times m$ that results from interchanging the rows and columns of A.

Properties

- Transpose of transpose of matrix A is equal to matrix A

$$(A^T)^T = A$$

- Transpose of matrix are distributive

$$(A + B)^T = A^T + B^T$$

- For scalar k, then

$$(kA)^T = k(A)^T$$

- For two matrices A and B, then

$$(AB)^T = B^T A^T$$

Symmetric Matrix

If a matrix is a square matrix and that matrix is equal with its transpose, then it's called a symmetric matrix.

$$A = A^T$$

Inverse Matrix

- A square matrix A is called has an inverse if there is B such that

$$AB = BA = I$$

- Matrix inverse is denoted by A^{-1}
- In the case above, the inverse of A is B, or $A^{-1} = B$

The Inverse of 2×2 Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ab-cd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Orthogonal Matrix

- Matrix A is orthogonal if and only if $A^T = A^{-1}$
- $(A^{-1})^T = (A^T)^{-1}$

Properties of Matrix Inverse

- The inverse of a matrix is **unique**, or there is **only one** inverse for each matrix
- Inverse of inverse matrix is equal the matrix itself

$$(A^{-1})^{-1} = A$$

- If A has an inverse and $n > 0$, then A^n has an inverse and $(A^n)^{-1} = (A^{-1})^n$
- For $k \neq 0$, then $(kA)^{-1} = \frac{1}{k} A^{-1}$
- Matrix $B_{m \times n}$ has an inverse, then AB has an inverse and $(AB)^{-1} = B^{-1}A^{-1}$

Elementary Matrix

An elementary matrix is a matrix that differs from the identity matrix by **exactly one** elementary row operation. For example:

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \leftarrow 7 \times R_2} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Inverse of Elementary Matrix

- Consider this matrix.
  - $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a elementary matrix from I by applying a single row operation $R_2 \leftarrow 3R_2$
  - Inverse of matrix E, denoted by E^{-1}, can be obtained from I by applying the inverse of said single row operation, $R_2 \leftarrow \frac{1}{3}R_2$
- So, the inverse elementary matrix can be obtained by applying the inverse single row operation
- Each elementary matrix has an inverse and its inverse is also an elementary matrix

Multiplication with Elementary Matrix

Let square matrices E and A with the same order $m \times m$, with E is an elementary matrix by applying a single row operation, named R. The product of EA is equal with applying single elementary row operation R to A.

Finding Inverse with Elementary Row Operation

If matrix A has an inverse, then A can be reduced to row echelon form by performing a sequence of elementary row operation to obtained the inverse of A, or A^{-1}

For example: Let $A = \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}$

1. Reduce matrix A to row echelon form $(A|I)$

2. Perform a sequence of elementary row operation:

   (a) $\left[\begin{array}{cc|cc} 4 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{4}R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 2 & 2 & 0 & 1 \end{array}\right]$

   (b) $\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 2 & 2 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$

   (c) $\left[\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right] \xrightarrow{R_1 \leftarrow R_1 - \frac{1}{2}R_2} \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$

3. The result is $(I|A^{-1}) = \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 1 \end{array}\right]$, or $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$

Solving System of Linear Equation with Inverse Matrix

If matrix A has an inverse, then for each matrix B with the order $n \times 1$, system of linear equation $Ax = B$ has unique solution, with $x = A^{-1}B$