Determinant

- Determinant: a **scalar value** that is a function of the entries of a **square matrix**
- Let A be a matrix, then its determinant is denoted by $det(A)$, $detA$, or $|A|$
- The importance of the determinant
  - Indicates that a matrix has an inverse or not
  - Indicates that a linear system has a unique solution or not
  - Plays an important role in determining the values and eigenvectors
  - etc.

Calculate The Determinant of A Matrix

Rule of Sarrus

- 2×2 Matrix

  Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then $det(A) = (a_{11}.a_{22}) - (a_{12}.a_{21})$

- 3×3 Matrix

- Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

$$det(A) = ((a_{11}.a_{22}.a_{23}) + (a_{12}.a_{23}.a_{31}) + (a_{13}.a_{21}.a_{32})) - ((a_{13}.a_{22}.a_{31}) + (a_{11}.a_{23}.a_{32}) + (a_{12}.a_{21}.a_{33}))$$

Note:

- Advantage: Simple for 2×2 and 3×3 matrices
- Disadvantage: For a larger matrix, it would be very troublesome to calculate the determinant

Cofactor Expansion

Minor and Cofactor

- Minor M_{ij} is the determinant of matrix A after removing the i-th row and j-th column
- The cofactor C_{ij} is $(-1)^{i+j}M_{ij}$
- For example, let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

  - $M_{13} = det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}$

    $C_{13} = (-1)^{1+3}M_{13}$
  - $M_{21} = det \begin{bmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{bmatrix}$

    $C_{21} = (-1)^{2+1}M_{21}$

Row and Column Expansion

- Based on the formula derivation from the Rule of Sarrus, there is a pattern
- This derived the cofactor expansion :

$$det(A) = \sum_{j=1}^{n} a_{ij}C_{ij} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

For example, let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then:

  - Row 1 Expansion: $det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$
  - Column 3 Expansion: $det(A) = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

Note: Sometimes, it is important to examine the matrix first to find the easiest row or column to calculate the determinant

Combinatorics

TBA

Elementary Row Operation

Effect of Elementary Row Operations on Determinants

If X' obtained from matrix X by applying an elementary row operation R, then:

Elementary Row Operation	Effect on Determinant
$R_i \leftrightarrow R_j$	$det(X') = -1.det(X)$
$R_i \leftarrow k.R_i, k \neq 0$	$det(X') = k.det(X)$
$R_i \leftarrow k.R_i + l.R_j, k, l \neq 0$	$det(X') = det(X)$

Elementary Row Operation

Let A a matrix, I is reduced row-echelon form of A, r is the times interchange row operations, s is the times multiply the equation with nonzero constant $k_1, k_2, \ldots\ k_s$, and t is the adding a multiple of one equation to another. The determinant of A can be calculate from the elementary row operations that used to change the matrix from A to I as follows:

$$det(A) = \frac{(-1)^r}{(k_1 k_2 \ldots k_s)}$$

Simple Matrices

There are 'cheat' for certain matrices.

- Diagonal Matrix

$$A = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 8 \end{bmatrix}, det(A) = 9 \times 7 \times 8 = 504$$

- Upper Triangular Matrix

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & -4 \\ 0 & 0 & 5 \end{bmatrix}, det(A) = 1 \times 2 \times 5 = 10$$

- Matrix with Row or Column Zero

$$C = \begin{bmatrix} 1 & 6 & 7 \\ 2 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}, det(A) = 0$$

$$D = \begin{bmatrix} 1 & 0 & 7 \\ 2 & 0 & 8 \\ 3 & 0 & 9 \end{bmatrix}, det(A) = 0$$

- Matrix with Identical Row

  $E = \begin{bmatrix} 1 & 4 & 7 \\ 1 & 4 & 7 \\ 3 & 8 & 9 \end{bmatrix}$, $det(A) = 0$

Properties of Determinant

- $det(AB) = det(A).det(B)$
- $det(A + B) \neq det(A) + det(B)$
- $det(A^T) = det(A)$
- $det(A) = \frac{1}{det(A^{-1})}$
- Let A a square matrix of order n, then $det(kA) = k^n det(A)$

Cramer's Rule

Adjoint Matrix

- Let A be a square matrix of order n. The adjoint of matrix A is the <u>transpose</u> of the <u>cofactor matrix</u> of .
- Denoted by $adjA$
- Also called Adjugate Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \Rightarrow [C_{ij}] = \begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix} \Rightarrow$$

$$adjA = [C_{ij}]^T = \begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}$$

Cramer's Rule

- Cramer's Rule: A method that uses determinants to solve systems of equations that have the same number of equations as variables.
- Consider a linear system $Ax = b$ and A has an inverse. Then:

$$
\begin{aligned}
x &= A^{-1}b \\
&= \left(\frac{1}{det(A)}.adjA\right)b \\
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \frac{1}{det(A)}.\begin{bmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= \frac{1}{det(A)}.\begin{bmatrix} b_1C_{11} & b_2C_{21} & \dots & b_nC_{n1} \\ b_1C_{12} & b_2C_{22} & \dots & b_nC_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ b_1C_{1n} & b_2C_{2n} & \dots & b_nC_{nn} \end{bmatrix} \\
&= \frac{1}{det(A)}.\begin{bmatrix} det(A_1) \\ det(A_2) \\ \vdots \\ det(A_n) \end{bmatrix}
\end{aligned}
$$

$\therefore x_j = \frac{det(A_j)}{det(A)}$, for $j = 1, 2, \dots n$

Note: Because determinant of coefficient matrix are used as divisor, then Cramer's Rule can be applied if **the coefficient matrix is square matrix** and its **determinant is nonzero** (or **the coefficient matrix has inverse**).