Introduction to Vector

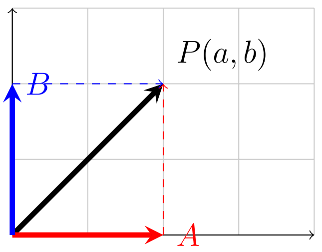
- Definition

  Vector is the quantity that has a **magnitude** and a **direction**.

- Notation

  $\overrightarrow{AB} = \overrightarrow{a} = \mathbf{a}$

  $\mathbf{v} = (v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$



Zero Vector and Equal Vector

Zero Vector

- A zero vector, denoted $\mathbf{0}$, is a vector of length 0, and thus has all components equal to zero.

  $\mathbf{0} = (0,\ 0)$ (vector in R2)

  $\mathbf{0} = (0,\ 0,\ 0)$ (vector in R3)

- Zero vector are parallel with every vector

- Zero vector are orthogonal with every vector

Unit Vector

- Unit vector is a vector of length 1

  $\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$

Equal Vector

- Equal vectors are vectors that have the **same magnitude** and the **same direction**, but the coordinates can be different.
- Two vectors are equal if and only if their corresponding components are equal.

  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, if and only if $a_1 = b_1$ and $a_2 = b_2$

Vector Arithmetic

Addition and Subtraction

- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the sum $\mathbf{v} + \mathbf{w}$ is the vector determined as follows: Position the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. The vector is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$.
- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the difference of $\mathbf{w}$ from $\mathbf{v}$ is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- If $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then $\mathbf{v} \pm \mathbf{w} = (v_1 \pm w_1, v_2 \pm w_2)$
- In set of vectors S in the plane, there is closed under vector addition if:
  - The result of the sum of two defined (any) and single vectors
  - The result of the sum is included in the set H.

Scalar Multiplication

- If $\mathbf{v}$ is a nonzero vector and k is a scalar number, then the product $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$. We define $k\mathbf{v} = 0$ if $k = 0$ or $\mathbf{v} = 0$.
- If $\mathbf{v} = (v_1, v_2)$ and k is any scalar, then $k\mathbf{v} = (kv_1, kv_2)$
- If a vector $\mathbf{v}$ is the product of a vector $\mathbf{w}$ by a scalar, then the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel.
- In set of vectors S in the plane, there is closed under scalar multiplication if:

- The result of the scalar and vector defined (any) and single vectors
- The result of the multiplication is included in the set H.

Properties of Vector Arithmetic

With the explanation above, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in 2- or 3-space and k and l are scalars, then the following relationships hold.

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + 0 = 0 + \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = 0$
- $k(l\mathbf{u}) = kl(\mathbf{u})$
- $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- $(k + l)\mathbf{u} = k\mathbf{u} + k\mathbf{v}$
- $l\mathbf{u} = \mathbf{u}$

Norm of A Vector

The length of a vector $\mathbf{u}$ is often called the norm of $\mathbf{u}$ and is denoted by $||\mathbf{u}||$.

- For vector in 2-space, $||\mathbf{v}|| = \sqrt{{v_1}^2 + {v_2}^2}$
- For vector in 3-space, $||\mathbf{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

With this, one can find distance d between two vectors as :

- For vector in 2-space, $d = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}$
- For vector in 3-space, $d = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2}$

Dot Product

Definition

1. If α is angle between $\mathbf{a}$ and $\mathbf{b}$, then:

$$\mathbf{a}.\mathbf{b} = \begin{cases} 0 \text{ if } \mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \\ ||\mathbf{a}||||\mathbf{b}|| \cos\alpha, \text{ with } 0 \geq \alpha \geq \pi \end{cases}$$

2. If $\mathbf{a}$ and $\mathbf{b}$ are vector in R^2, then $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$

   Similar to above, in R^3, $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Note that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, then $(\mathbf{a}.\mathbf{b}).\mathbf{c} \neq \mathbf{a}.(\mathbf{b}.\mathbf{c})$ because dot product output a scalar number, and scalar numbers cannot be dot product with vector

Properties

- $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$
- $(k\mathbf{u}).\mathbf{v} = k(\mathbf{u}.\mathbf{v})$
- $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$
- $\mathbf{v}.\mathbf{v} = \begin{cases} ||\mathbf{v}||||\mathbf{v}|| \\ 0 & \text{if } \mathbf{v} = 0 \end{cases}$

Angle and Result of Dot Product

- Pay attention to the equation $||\mathbf{a}||||\mathbf{b}|| \cos\alpha$.
- Norm of a vector will always be greater than or equal to 0, but $\cos\alpha$ can be positive, negative or 0 depending on the value of α.

  $$\mathbf{u}.\mathbf{v} = \begin{cases} < 0 & \text{if } \alpha > \frac{\pi}{2} \\ = 0 & \text{if } \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \\ > 0 & \text{if } \alpha < \frac{\pi}{2} \end{cases}$$

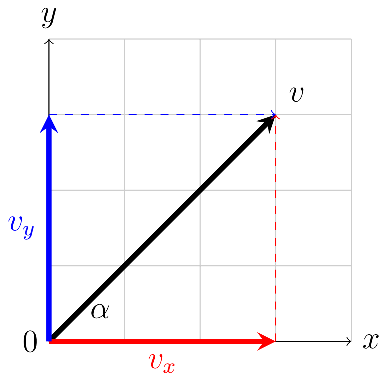
Dot Product and Matrix Multiplication

- Pay attention to the equation $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$
- If $\mathbf{a}$ and $\mathbf{b}$ seen as matrices A and B respectively, then:

  $$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 = \begin{bmatrix} a_1, a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = A.B^T$$

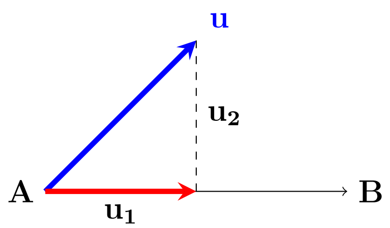
Orthogonal Projection

Orthogonal Projection on The Axis



- v_x is the orthogonal projection of $\mathbf{v}$ on the x-axis
- v_y is the orthogonal projection of $\mathbf{v}$ on the y-axis
- So by definition:
  - $\mathbf{v} = v_x + v_y$
  - $|v_x| = |\mathbf{v}| \cos \alpha$
  - $|v_y| = |\mathbf{v}| \sin \alpha$
- $\mathbf{v}$ can be decomposed into the sum of two vectors, v_x and v_y

Orthogonal Projection and Decomposition



- The vector $\mathbf{u_1}$ is called the **orthogonal projection of u** on A or sometimes the vector component of **a** along A. It is denoted by

$$\mathrm{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u.b}}{||\mathbf{b}||^2}.\mathbf{b}$$

- The vector $\mathbf{u_2}$ is called the **vector component of u orthogonal to a**. Since $\mathbf{u_2} = \mathbf{u} - \mathbf{u_1}$, this vector can be written as

$$\mathbf{u} - \mathrm{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u.b}}{||\mathbf{b}||^2}.\mathbf{b}$$

Cross Product

- Given two vectors **a** and **b**, the cross product of **a** and **b**, denoted by $\mathbf{a} \times \mathbf{b}$, is another vector that is perpendicular to both **a** and **b**.
- Suppose there are two vectors, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{u} \times \mathbf{v} = ((u_2v_3 - u_3v_2), (u_3v_1 - u_1v_3), (u_1v_2 - u_2v_1))$$

  or in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\det \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\det \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

Properties

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- If **u** parallel with **v**, then $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = 0$
- $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}).\mathbf{w}$

Vector Spaces

Definition and Axiom

- Let V be an nonempty set of objects on which two operations are defined: addition and multiplication by scalars, denoted by $+$ and $.$ respectively. If the following axioms are satisfied by all objects **u**, **v**, **w** in V and all scalars k and m, then V is called a vector space and the objects in V is **vectors**.

  1. If **u** and **v** are objects in V, then $\mathbf{u} + \mathbf{v}$ is in V (Closed under Addition)
  2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
  3. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$
  4. There is object 0 in V, called a **zero vector** for V, such that $0 + \mathbf{u} = \mathbf{u} + 0$ and $\mathbf{u} \in V$

5. For each $\mathbf{u}$ in V, there is an object $-\mathbf{u}$ in V, called **negative value** of $\mathbf{u}$, such that $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = 0$
6. If k is any scalar and $\mathbf{u}$ is any object in V, then $k\mathbf{u}$ is in V (Closed under Scalar Multiplication)
7. $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
8. $(k + m)\mathbf{u} = k\mathbf{u} + m\mathbf{u}$
9. $k(m\mathbf{u}) = (km)(\mathbf{u})$
10. $1\mathbf{u} = \mathbf{u}$

- Notation: V or $(V, +, .)$

R^n is a Vector Space

There are several sets of numbers that can be vector spaces

- $\mathbb{R}$ and the vector space $(R, +, .)$
- R^2 are set of vector in R^2, composed by $\mathbb{R}$ and defined by $R^2 = \{\mathbf{u} : \mathbf{u} = (u_1, u_2) | u_1, u_2 \in R\}$
  - Addition operation (+) defined as $\mathbf{u} + \mathbf{v} = (u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$
  - Scalar multiplication (.) defined as $k\mathbf{u} = k(u_1, u_2) = (ku_1, ku_2)$
  - Zero element in $R^2 : \mathbf{0} = (0, 0)$
- $R^n = \{\mathbf{u} : \mathbf{u} = (u_1, u_2, \ldots u_n) | u_1, u_2, \ldots u_n \in R\}$, the operations are similar with R^2
- $M^{2\times 3}$ is set of all matrices of order 2×3 and defined by $M^{2\times 3} = \left\{\mathbf{a} : \mathbf{a} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \middle| a_{ij} \in R\right\}$
  - Addition operation (+) defined as $\mathbf{u} + \mathbf{v} = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix} + \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{pmatrix} = \begin{pmatrix} u_{11} + v_{11} & u_{12}v_{12} + v_{12} & u_{13} + v_{13} \\ u_{21} + v_{21} & u_{22} + v_{22} & u_{23} + v_{23} \end{pmatrix}$
  - Scalar multiplication (.) defined as $k\mathbf{u} = k \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ u_{21} & u_{22} & u_{23} \end{pmatrix} = \begin{pmatrix} ku_{11} & ku_{12} & ku_{13} \\ ku_{21} & ku_{22} & ku_{23} \end{pmatrix}$

- Zero element in $M^{2\times3}$: $\mathbf{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- $D^{3\times3}$ is the set of all diagonal matrices of order 3×3 and the operations are similar with matrices above

- $C_{[1,3]}$ is the set of all real-valued functions that are continuous in a closed interval $[1, 3]$
  - Addition operation $(+)$ defined as $\mathbf{f}+\mathbf{g} = (\mathbf{f}+\mathbf{g})(x) = f(x)+g(x)$
  - Scalar multiplication (.) defined as $k\mathbf{f} = (k\mathbf{f})(x) = kf(x)$
  - Zero element in $C_{[1,3]}$: $\mathbf{0} = f_0(x) = 0, \forall x \in R$

    $f_0(x)$ is a function that maps to 0 for any value of x

- $D_{[a,b]}$ is the set of all differentiated functions on a closed interval $[a, b]$
  - A function is differentiable at a point if its derivative is at that point
  - Every differentiable function must be continuous, but not all continuous functions are differentiable functions

- TBA

- TBA

Properties of Vector Spaces

Let V be a vector space, $\mathbf{u}$ a vector in V, and k a scalar; then:

- $0\mathbf{u} = 0$
- $k\mathbf{0} = \mathbf{0}$
- $(-1)\mathbf{u} = -\mathbf{u}$
- If $k\mathbf{u} = 0$, then $k = 0$ or $\mathbf{u} = 0$

Subspace

- A subset W of a vector space V is called a **subspace** of V if W is itself a vector space under the addition and scalar multiplication defined on V.

- Because W is a vector space, W cannot be blank and must hold all the vector space axiom. However, if W is part of a larger set V that is already known to be a vector space, then certain axioms need not be verified for W because they are "inherited" from V.

- Thus, to show that a set W is a subspace of a vector space V, we need only verify Axioms 1 and 6:

  If W is a set of one or more vectors from a vector space V, then W is a subspace of V if and only if the following conditions hold.

  - If $\mathbf{u}$ and $\mathbf{v}$ are vectors in W, then $\mathbf{u} + \mathbf{v}$ is in W. (Axiom 1)
  - If k is any scalar and $\mathbf{u}$ is any vector in W, then $k\mathbf{u}$ is in W. (Axiom 6)

Linear Dependency

Linear Combination

Let $\mathbf{v} = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ be vectors in R^n. A vector $\mathbf{w}$ is a linear combination of $\mathbf{v}$ if $\mathbf{w}$ can be expressed in the form:

$$\mathbf{w} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + \cdots + k_n\mathbf{v}_n$$

where $k_1, k_2, \dots, k_n$ are scalars.

Span of Vector

If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots \mathbf{v}_r\}$ is a set of vectors in a vector space V, then span(S) is all linear combinations of the vectors in S.

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Linearly Dependent and Independent

- If a vector is cannot be expressed as another vector in a set of vectors, then that set of vector is called **linearly independent**

  If the linear combination has no trivial solution, then it is a linearly dependent

- On the other hand, if a vector can be removed withour changing the span, then that set of vector is called **linearly dependent**

  If linear combination has a trivial solution, then it is a linearly independent

Properties

- A finite set of vectors that contains the zero vector is linearly dependent.
- A set with exactly two vectors is linearly independent if and only if neither vector is a scalar multiple of the other.
- Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a set of vectors in R^n. If $r > n$, then S is linearly dependent

Basis and Dimension

Basis

If V is any vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ is a minimal set of vectors in V, then S is called a basis for V if the following two conditions hold:

1. S is linearly independent
2. S spans V

A basis is the vector space generalization of a coordinate system in 2-space and 3-space.

Dimension

The number of vectors in a basis for V is called the dimension of V, denoted by $\dim(V)$.

Row Space, Column Space, and Null Space

Definition

For a A matrix

- The vectors formed from the rows of A are called the row vectors of A
- The vectors formed from the columns of A are called the column vectors of A.

Based on this, If A is an $m \times n$ matrix,

- The subspace of R^n spanned by the row vectors of A is called the **rowspace** of A
- The subspace of R^m spanned by the column vectors of A is called the **column space** of A

- The solution space of the homogeneous system of equations $Ax = 0$, which is a subspace of R^n, is called the **nullspace** of A.

Rank and Nullity

Given any A matrix:

- The common dimension of the row space and column space of a matrix A is called the rank of A and is denoted by $\text{rank}(A)$.
- The dimension of the nullspace of A is called the nullity of A and is denoted by $\text{nullity}(A)$.