Introduction to Vector

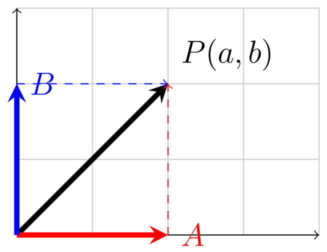
- Definition

  Vector is the quantity that has a **magnitude** and a **direction**.

- Notation

  $\overrightarrow{AB} = \overrightarrow{a} = \mathbf{a}$

  $\mathbf{v} = (v_1, v_2) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$



Zero Vector and Equal Vector

Zero Vector

- A zero vector, denoted $\mathbf{0}$, is a vector of length 0, and thus has all components equal to zero.

  $\mathbf{0}$ = (0, 0) (vector in R2)

  $\mathbf{0}$ = (0, 0, 0) (vector in R3)

- Zero vector parallel with all vector

Unit Vector

- Unit vector is a vector of length 1

  $\hat{u} = \frac{\mathbf{u}}{|\mathbf{u}|}$

Equal Vector

- Equal vectors are vectors that have the **same magnitude** and the **same direction**, but the coordinates can be different.
- Two vectors are equal if and only if their corresponding components are equal.

  $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$, if and only if $a_1 = b_1$ and $a_2 = b_2$

Vector Arithmetic

Addition and Subtraction

- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the sum $\mathbf{v} + \mathbf{w}$ is the vector determined as follows: Position the vector $\mathbf{w}$ so that its initial point coincides with the terminal point of $\mathbf{v}$. The vector is represented by the arrow from the initial point of $\mathbf{v}$ to the terminal point of $\mathbf{w}$.
- If $\mathbf{v}$ and $\mathbf{w}$ are any two vectors, then the difference of $\mathbf{w}$ from $\mathbf{v}$ is defined by $\mathbf{v} - \mathbf{w} = \mathbf{v} + (-\mathbf{w})$
- If $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$, then $\mathbf{v} \pm \mathbf{w} = (v_1 \pm w_1, v_2 \pm w_2)$
- In set of vectors S in the plane, the sum is closed if:
  - The result of the sum of two defined (any) and single vectors
  - The result of the sum is included in the set H.

Scalar Multiplication

- If $\mathbf{v}$ is a nonzero vector and k is a scalar number, then the product $k\mathbf{v}$ is defined to be the vector whose length is $|k|$ times the length of $\mathbf{v}$ and whose direction is the same as that of $\mathbf{v}$ if $k > 0$ and opposite to that of $\mathbf{v}$ if $k < 0$. We define $k\mathbf{v} = 0$ if $k = 0$ or $\mathbf{v} = 0$.
- If $\mathbf{v} = (v_1, v_2)$ and k is any scalar, then $k\mathbf{v} = (kv_1, kv_2)$
- If a vector $\mathbf{v}$ is the product of a vector $\mathbf{w}$ by a scalar, then the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel.

Properties of Vector Arithmetic

With the explanation above, if **u**, **v**, and **w** are vectors in 2- or 3-space and k and l are scalars, then the following relationships hold.

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $\mathbf{u} + 0 = 0 + \mathbf{u}$
- $\mathbf{u} + (-\mathbf{u}) = 0$
- $k(l\mathbf{u}) = kl(\mathbf{u})$
- $k(\mathbf{u} + \mathbf{v}) = k\mathbf{u} + k\mathbf{v}$
- $(k + l)\mathbf{u} = k\mathbf{u} + k\mathbf{v}$
- $l\mathbf{u} = \mathbf{u}$

Norm of A Vector

The length of a vector **u** is often called the norm of **u** and is denoted by $||\mathbf{u}||$.

- For vector in 2-space, $||\mathbf{v}|| = \sqrt{{v_1}^2 + {v_2}^2}$
- For vector in 3-space, $||\mathbf{v}|| = \sqrt{{v_1}^2 + {v_2}^2 + {v_3}^2}$

With this, one can find distance d between two vectors as :

- For vector in 2-space, $d = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2}$
- For vector in 3-space, $d = \sqrt{(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2}$

Dot Product

Definition

1. If α is angle between **a** and **b**, then:

$$\mathbf{a}.\mathbf{b} = \begin{cases} 0 \text{ if } \mathbf{a} = 0 \text{ or } \mathbf{b} = 0 \\ ||\mathbf{a}||||\mathbf{b}|| \cos \alpha, \text{ with } 0 \geq \alpha \geq \pi \end{cases}$$

2. If **a** and **b** are vector in R^2, then $\mathbf{a}.\mathbf{b} = a_1 b_1 + a_2 b_2$

Similar to above, in R^3, $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$

Note that if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, then $(\mathbf{a}.\mathbf{b}).\mathbf{c} \neq \mathbf{a}.(\mathbf{b}.\mathbf{c})$ because dot product output a scalar number, and scalar numbers cannot be dot product with vector

Properties

- $\mathbf{u}.\mathbf{v} = \mathbf{v}.\mathbf{u}$
- $(k\mathbf{u}).\mathbf{v} = k(\mathbf{u}.\mathbf{v})$
- $\mathbf{u}(\mathbf{v} + \mathbf{w}) = \mathbf{u}.\mathbf{v} + \mathbf{u}.\mathbf{w}$
- $\mathbf{v}.\mathbf{v} = \begin{cases} ||\mathbf{v}||||\mathbf{v}|| \\ 0 & \text{if } \mathbf{v} = 0 \end{cases}$

Angle and Result of Dot Product

- Pay attention to the equation $||\mathbf{a}||||\mathbf{b}|| \cos\alpha$.
- Norm of a vector will always be greater than or equal to 0, but $\cos\alpha$ can be positive, negative or 0 depending on the value of α.

  $$\mathbf{u}.\mathbf{v} = \begin{cases} < 0 & \text{if } \alpha > \frac{\pi}{2} \\ = 0 & \text{if } \mathbf{u} \text{ and } \mathbf{v} \text{ are orthogonal} \\ > 0 & \text{if } \alpha < \frac{\pi}{2} \end{cases}$$

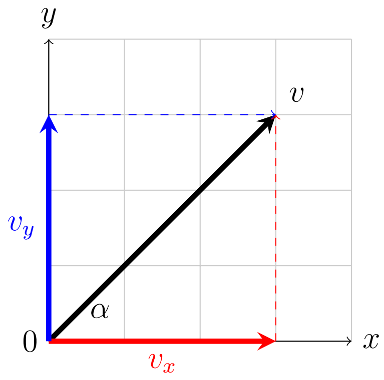
Dot Product and Matrix Multiplication

- Pay attention to the equation $\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2$
- If $\mathbf{a}$ and $\mathbf{b}$ seen as matrices A and B respectively, then:

  $$\mathbf{a}.\mathbf{b} = a_1b_1 + a_2b_2 = \begin{bmatrix} a_1, a_2 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = A.B^T$$

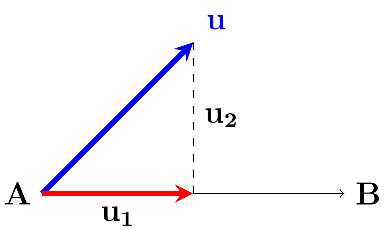
Orthogonal Projection

Orthogonal Projection on The Axis



- v_x is the orthogonal projection of $\mathbf{v}$ on the x-axis
- v_y is the orthogonal projection of $\mathbf{v}$ on the y-axis
- So by definition:
  - $\mathbf{v} = v_x + v_y$
  - $|v_x| = |\mathbf{v}| \cos \alpha$
  - $|v_y| = |\mathbf{v}| \sin \alpha$
- $\mathbf{v}$ can be decomposed into the sum of two vectors, v_x and v_y

Orthogonal Projection and Decomposition



- The vector $\mathbf{u_1}$ is called the **orthogonal projection of u** on A or sometimes the vector component of **a** along A. It is denoted by

$$\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u.b}}{||\mathbf{b}||^2}.\mathbf{b}$$

- The vector $\mathbf{u_2}$ is called the **vector component of u orthogonal to a**. Since $\mathbf{u_2} = \mathbf{u} - \mathbf{u_1}$, this vector can be written as

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = \mathbf{u} - \frac{\mathbf{u.b}}{||\mathbf{b}||^2}.\mathbf{b}$$

Cross Product

- Given two vectors $\mathbf{a}$ and $\mathbf{b}$, the cross product of $\mathbf{a}$ and $\mathbf{b}$, denoted by $\mathbf{a} \times \mathbf{b}$, is another vector that is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

- Suppose there are two vectors, $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, then

$$\mathbf{u} \times \mathbf{v} = ((u_2v_3 - u_3v_2), (u_3v_1 - u_1v_3), (u_1v_2 - u_2v_1))$$

or in determinant notation,

$$\mathbf{u} \times \mathbf{v} = \left(\det \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, -\det \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \det \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\right)$$

Properties

- $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$
- If $\mathbf{u}$ parallel with $\mathbf{v}$, then $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u} = 0$
- $(k\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (k\mathbf{v}) = k(\mathbf{u} \times \mathbf{v})$
- $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$
- $\mathbf{u}.(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}).\mathbf{w}$