Introduction to System of Linear Equations

Linear Equation

- **Linear equation** with n **unknown** $x_1, x_2, ..., x_n$ is an equation of the form $a_1x_1 + a_2x_2 + ... + a_nx_n = b$ with $a_1, a_2, \ldots, a_n$ are **constant** ($\mathbb{R}$)
  For example :
  $$x_1 + 3x_2 + 4x_2 = 5$$
- Components of Linear Equation
  - Unknown
  - Constant
  - Equal Sign (=)

System of Linear Equation

- **System of Linear Equation** or **Linear System** is a **finite set** of **linear equations** that involves the **same unknowns**
- Components of Linear System
  - Finite Set
  - Linear Equations
  - Same Unknowns

Solution

- Linear System is a representation of a problem, so it must have a solution
- The solution in the system of linear equation is a set of numbers that is substituted to the unknowns, then the system is satisfied. Or in another word the solution satisfied all the linear equations in the system
- Types of linear system by its solution
  1. Has exact one solution / the lines intersect at exactly one point
  2. Has infinitely many solution / the lines parallel
  3. Has no solution / inconsistent / has no intersect

Forms of Linear System

- General

$$
\begin{gathered}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{gathered}
$$

- Matrix Equation: $Ax = b$

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\
\vdots & & & & \\
a_{11} & a_{12} & a_{13} & \dots & a_{1n}
\end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix}
=
\begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_n \end{bmatrix}
$$

- Augmented Matrix: $[A|b]$

$$
\left[\begin{array}{ccccc|c}
a_{11} & a_{12} & a_{13} & \dots & a_{1n} & b_1 \\
a_{21} & a_{22} & a_{23} & \dots & a_{2n} & b_2 \\
\vdots & & & & & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} & b_m
\end{array}\right]
$$

If $m =$ number of linear equations and $n =$ number of unknowns, then augmented matrix would have the order $m \times (n + 1)$ and coefficient matrix have the order $m \times n$

Method of Linear System Solution

- There are methods to find solution of a linear system (if any).

  1. Elimination-Substitution
  2. Geometric
  3. Gauss-Jordan Elimination

(Elimination-substitution and geometric methods already discussed in high school, so it will not be discussed too much)

- The disadvantage of elimination-substitution and geometric: A big linear system

  For example, consider this linear system :

$$
\begin{aligned}
a + 2b + 3c + 5d + 8e &= -7 \\
2a - 3b - 5c + 4d + 3e &= 9 \\
-6a - 8b + c + 2d + 7e &= -22 \\
-3a + 5b + c - 9d + 8e &= 4 \\
7a - 4b + 3c + 8d - e &= 12
\end{aligned}
$$

  This linear system has more than 3 unknowns and 3 equations, so elimination-substitition method are not effective and geometric method is hard to use.

- Solution: Gauss-Jordan Elimination

Gauss-Jordan Elimination

Elementary Row Operation

Linear System Equivalence

Two linear systems are equivalence if it has the same solution

Elementary Row Operation

- There are 3 operations that don't change the linear system solution
  1. Multiply an equation through by a nonzero constant
  2. Interchange two equations
  3. Add a multiple of one equation to another
- These operations are called **elementary row operations**
- Matrices that applied elementary row operation are still equivalence or row equivalence

Row-Echelon Form

- An augmented matrix in the form of Row-Echelon and Reduced Row-Echelon makes it easier to find the solution
- Reduced Row-Echelon characteristics:
  1. The first element nonzero in a row is 1, called **leading 1**
  2. The next leading 1 in the lower row occurs farther to the right
  3. If any rows consist entirely of zeros, then they are grouped at the bottom of the matrix
  4. Each column that contains a leading 1 has zeros everywhere else in that column
- If the matrix only fulfilled 1, 2, and 3, then it is called row-echelon

Gauss–Jordan Elimination

Gauss–Jordan Elimination method: convert the <u>augmented matrix</u> into its <u>equivalence</u> using elementary row operation in the <u>reduced row-echelon form</u>.

1. Let a linear system $A =$

$$\begin{gathered}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m
\end{gathered}$$

2. Convert using Gauss–Jordan Elimination with Elementary Row Operation

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right] \xRightarrow{\text{ERO}} \left[\begin{array}{cccc|c} 1 & 0 & \dots & 0 & b'_1 \\ 0 & 1 & \dots & 0 & b'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b'_m \end{array}\right]$$

3. The solutions are in reduced-row echelon form

Note :

- If a row is entirely zero except in the far right, then that system is inconsistent

$$\left[\begin{array}{cccc|c} 0 & 0 & \dots & 0 & 2 \\ 0 & 1 & \dots & 0 & b'_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & b'_m \end{array}\right]$$

- If the number of unknown is more than the number of nonzero rows, then that system has infinitely many solutions
  - If a linear system has infinitely many solutions, then it has free parameter

Homogeneous Linear Systems

- A linear system is said to be homogeneous if the constant terms are all zero, or in the other word, the numbers in the right of the equal sign are all zero

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & 0 \\ a_{21} & a_{22} & \dots & a_{2n} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & 0 \end{array}\right]$$

- Homogeneous Linear Systems is definitely consistent
- If the solution is $(x_1, x_2, x_3, \dots x_n) = (0, 0, 0, \dots, 0)$, then this solution is called **trivial solution**, other than this is called **nontrivial solution**.
- A nontrivial solution occurred when there is at least one free parameter

Under-Determined and Over-Determined Linear System

- Under-determined linear system: Number of unknowns > numbers of the equation
  - If it has a free parameter, then it has infinitely many solutions OR inconsistent
  - Impossible to have a unique solution
- Over-determined system of the linear equation: Number of unknown < number of equations

  - Can has a unique solution, infinitely many solutions, or inconsistent