Financial maths

1. Introduction

1.1. Chapter overview

What would you rather have, £1 today or £1 next week? Intuitively the answer is £1 today. Even without knowing it you are applying the financial mathematics you will learn in this chapter.

You will learn the idea of **compounding**, which means how you calculate interest on interest, so you can work out the value of your £1 today in one week, or in two weeks, or even in two years. You will then learn how to **discount** which is simply the reverse of compounding; that is working out what an amount of money to be received in the future is worth in today's terms.

For these calculations a calculator is essential. This chapter shows you how to do the calculations on a Casio FX83-GT, but any scientific or financial calculator will enable you to do the calculations in a similar way.

Once you can compound and discount then you can begin to work out what many things are worth in today's terms. You will see how to apply the same ideas to an **annuity**, which is just a series of payments to be received over a fixed time. A real world application is with pensions where on retirement the value of your pension fund is used to purchase just such an annuity.

You will also see that the same calculations can enable you to calculate **mortgage** payments.

These techniques are all concerned with calculations of **present value**: what something is worth in today's terms. For an investment adviser, a fund manager, an analyst or a company director, this idea is crucial to evaluating whether an investment is worthwhile.

1.2. Learning outcomes

On completion of this module, you will:

Simple vs. compound interest

- 7.5.1 Distinguish simple interest from compound interest
- 7.5.2 Calculate simple and compound interest over multiple periods
- 7.5.3 Distinguish a simple annual interest rate from a compound annual rate
- 17.1.4 Calculate the reinvestment return on income over a specified investment horizon
- 7.5.6 Define and calculate the effective continuously compounded rate given the nominal rate

Discounting

- 7.6.1 Calculate and interpret future values for: single sums, annuities
- 7.6.2 Calculate and interpret present values for: single sums, annuities, perpetuities
- 7.6.3 Calculate equal instalments on a repayment mortgage given the present value of the borrowings, the fixed mortgage rate and the term of the borrowing

Annualised rates

- 7.5.4 Calculate the annual compound rate given the simple rate and the frequency of compounding
- 7.5.5 Calculate the annual simple rate of interest given the annual compound rate and the frequency of compounding

Investment appraisal techniques

- 7.7.1 Calculate and interpret the net present value (NPV) of a series of investment cash flows
- 7.7.2 Calculate and interpret an internal rate of return (IRR)
- 7.7.3 Explain how NPVs and IRRs can be used in investment decision making and their limitations
- 7.7.4 Explain why decisions using the NPV and IRR techniques in investment decision making may conflict
- 7.7.5 Explain the scenarios in which multiple IRRs may occur

2. Simple vs. compound interest

2.1. Introduction

Interest represents the cost of borrowing money over a period of time. As such, it is often referred to as the 'cost of capital' or the 'time value of money'.

Whether it is paid or received, interest can be calculated on either a simple or compound basis.

Interest is usually paid/received at periodic intervals and is expressed as a percentage of the principal borrowed.

If interest is permitted to accumulate on top of the principal borrowed in order to earn interest itself, it is called 'compound interest'.

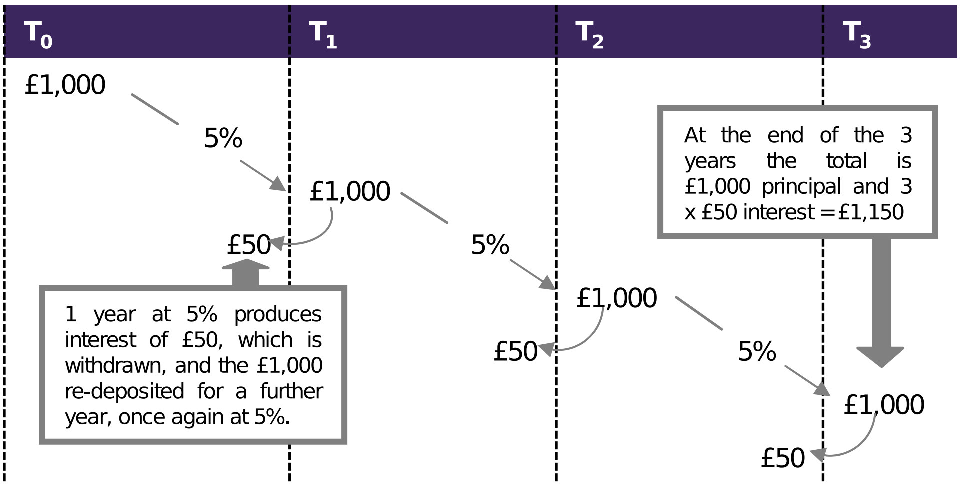
Alternatively, if interest is withdrawn at the end of each period and not aggregated to the principal, it is called 'simple interest'.

2.2. Simple interest

Simple interest is calculated on the original principal amount only.

It assumes at the end of each period that earned interest is **not** summed together with the principal.

For example, a £1,000 deposit placed at 5% pa for three years will earn £150 of interest i.e. 3 x £50 pa.



The terminal or final value (TV) of a lump sum invested at a rate of interest (r) over a given number of years (n) on a simple interest basis is calculated as:

TV = original principal amount x [1 + (r x n)].

Using the numbers in the above example:

TV = £1,000 x [1 + (0.05 x 3)] = £1,150

TV = £1,000 x 1.15 =£1,150

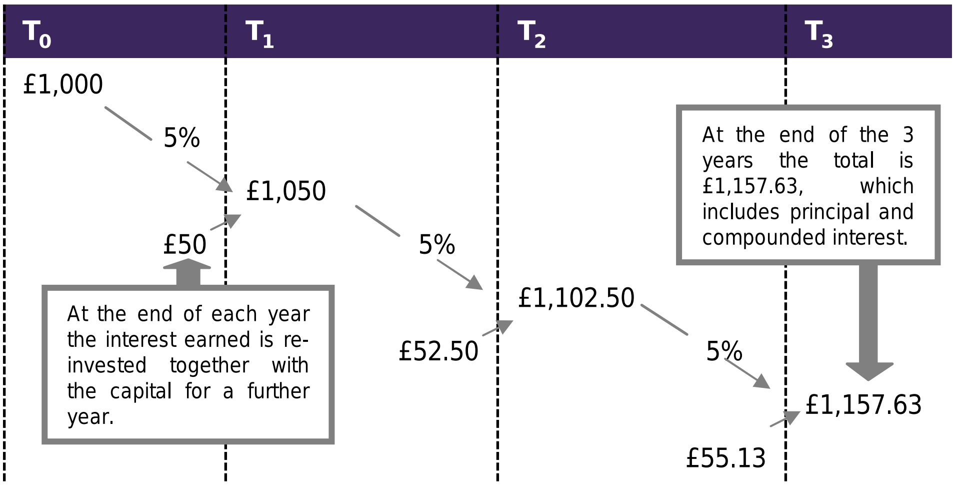
2.3. Compound interest

Compound interest assumes that interest earned for one period is 'rolled-over' into subsequent periods. The interest rate applies to the principal plus accrued interest.

When applying compound interest it is assumed that interest earned is **re-invested** i.e. earning interest on interest. Interest payments will therefore increase exponentially over time.

For example, a three-year deposit of £1,000 at 5% pa would accrue to £1,050 at the start of the **second** year. This amount earns interest of £52.5 (£1,050 x 0.05) which itself is rolled over into the third year.

This process is summarised below:



The terminal or final value (TV) of a lump sum receiving compound interest can be calculated as:

$$TV = PV(1 + r)^n$$

Where:

TV is the terminal value (or future value) of the deposit (how much capital and compounded interest there will be in total).

PV is the amount of money to be deposited, or the present value of the deposit

n is the number of periods the deposit is to run for (the usual period is a year)

r is the rate of interest on the deposit per period.

Using the numbers from the previous example:

Example:
Calculate the terminal value of £1,000 invested for three years at a compound interest rate 5% pa.

PV	= £1,000
r	= 0.05
n	= 3

Therefore:

TV	= £1,000 $(1+0.05)^3$
	= £1,000 x 1.05^3
Answer	= £1,157.625

To do this on the FX83 key in the numbers in the sequence below:

1000	x	1.05	$x^{\bullet}$	3	=

Which will give you the answer:

1000x1.05^3
1157.625

or £1,157.63

Reinvestment return

From the example above we can calculated the reinvestment return. The reinvestment return is simply the total income generated through compounding. Another way to express this is the terminal value minus the initial deposit.

For the example above, this would be:

- £1,157.625 - £1,000 = £157.625

Non-annual compounding

If the frequency of compounding increased - for example to quarterly intervals - we would need to increase the number of periods 'n' by 4 (from 3 years to 12 quarters), but reduce the rate of interest by 4 (from 5% to 1.25%) in return. Using the figures above, but with quarterly compounding, we would get:

- 1000 x 1.0125^{12} = 1160.75

This is a higher value than compounding over annual intervals.

Note, reinvestment return has increased to £160,75

Continuous compounding

It would be higher still, if 5% pa were compounded monthly, weekly, daily, etc. The highest terminal value would be obtained if we assume interest rates are compounded continuously, millisecond by millisecond.

Example

$$
\begin{aligned}
PV &= £1{,}000 \\
r &= 5\%\ \text{pa} \\
n &= 3 \\
e &= \text{natural exponent}
\end{aligned}
$$

$$
\begin{aligned}
TV &= PV \times e^{(r \times n)} \\
&= £1{,}000 \times e^{(0.05 \times 3)} \\
&= £1{,}161.83
\end{aligned}
$$

Again, reinvestment return has increase further to £161.83

3. Discounting

3.1. Introduction

Discounting is the exact opposite of compounding.

Compounding is concerned with determining the terminal/future value of a principal sum given a rate of interest and frequency of payment.

Discounting is concerned with determining how much to invest **today**, given a rate of interest (the 'discount rate') and frequency of payment, in order to achieve a required terminal value in the future.

3.2. Calculating present values of a single cash flow

The methods used to calculate the **present value** of future cash flows are known as discounted cash flow (DCF) techniques.

The present value of a single lump sum due to be received on a future date at a given level of interest is calculated by re-arranging the compounding equation to make PV (present value) the subject of the formula.

In other words:

$$PV = \frac{TV}{(1+r)^n}$$

Where:

- TV is the amount of money to be received in the future
- PV is the present value of the amount (how much TV is worth now)
- n is the number of periods until the amount is received (the usual period is a year)
- r is the rate of interest on the deposit per period

For example, calculate how much is required to be invested today at an annual interest rate of 5% pa in order to achieve a value of £1,000 in three years’ time:

Example:
Calculate the present value of £1,000 to be received in three years' time with interest rates of 5% pa

TV	= £1,000
r	= 0.05
n	= 3

Therefore:

$$PV = \frac{£1,000}{(1.05)^3}$$

$$= £863.84$$

To do this on the FX83 key in the numbers in the sequence below:

1000	÷	1.05	$X^{\bullet}$	3	=

Which will give you the answer:

1000 ÷ 1.05^3
863.8375985

or £863.84

3.3. Calculating present values of multiple cash flows

Annuities

Annuities refer to a series of:

- Equal cash payments
- Received or made at regular intervals
- Over a specified period of time

The example below is described as a three-year annuity of £5,000 where the payments are paid in arrears (at the end of each year).

T_0	T_1	T_2	T_3
	£5,000	£5,000	£5,000
	Paid at the end of the year		

For a given level of interest rates, the present value of the annuity is calculated by discounting the three annual cash flows to today's value.

If interest rates are equal to 5% pa over the life of the annuity, the present value is calculated as:

A £5,000 three-year annuity with interest rates at 5% pa

T_0	**T_1**	**T_2**	**T_3**
	£5,000	£5,000	£5,000
Present val	$\frac{£5,000}{(1.05)}$	$\frac{£5,000}{(1.05)^2}$	$\frac{£5,000}{(1.05)^3}$
	£4,761.90	£4,535.15	£4,319.19

$$\underbrace{£4,761.90 \quad £4,535.15 \quad £4,319.19}_{£13,616.24}$$

Alternatively, the annuity formula may be used as shown below:

$$\text{PV annuity} = £X \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

Where:

- £X is the annuity payment each year; paid at the end of the year
- r is the interest rate (normally annual) over the life of the annuity
- n is the number of periods (normally years) that the annuity will run for

Using the annuity formula with the Casio fx-83

Taking the information from the previous example gives:

Example:

Calculate the present value of a £5,000 3 year annuity using a discount rate of 5% p.a. Firstly, enter the values into the formula:

$$£5{,}000 \times \frac{1}{0.05}\left[1 - \frac{1}{(1+0.05)^3}\right]$$

Using a scientific calculator, such as the Casio FX83:

1. Start inside the large brackets:

$\boxed{1}$ $\boxed{-}$ $\boxed{1}$ $\boxed{\div}$ $\boxed{1.05}$ $\boxed{X^{\bullet}}$ $\boxed{3}$ $\boxed{=}$

which gives:

1-1 ÷1.05^3
0.136162401

2. Next, multiply this by the values outside the brackets:

$\boxed{ANS}$ $\boxed{x}$ $\boxed{5000}$ $\boxed{x}$ $\boxed{1}$ $\boxed{\div}$ $\boxed{0.05}$ $\boxed{=}$

which gives the present value of the annuity:

Ans x 5000 x 1 / 0.5 =
13616.24015

Answer = £13,616.24

3.4. Mortgages

Mortgages are long-term loans secured on property.

The initial amount advanced by the building society or bank represents the present value of all future mortgage payments. A mortgage, or any other long-term loan, is simply an annuity where the borrower makes regular payments to the lender in the form of capital and/or interest payments.

Consider a 25-year repayment mortgage - (i.e. each payment represents part capital and part interest) - of £100,000 at 7.5% interest pa.

In order to calculate the value of each annual mortgage payment, the annuity formula must be used:

Example:

Calculate the annual payments on a £100,000 25-year mortgage using interest rates of 7 ½ % p.a. Begin by populating the annuity formula:

$$\pounds 100{,}000 = \text{Annual Payment} \times \frac{1}{r}\left[1 - \frac{1}{(1+r)^n}\right]$$

$$= \text{Annual Payment} \times \frac{1}{0.075}\left[1 - \frac{1}{(1.075)^{25}}\right]$$

$$= \text{Annual Payment} \times 13.33333 \times 0.836021$$

$$= \text{Annual Payment} \times 11.14694$$

So:

$$\frac{\pounds 100{,}000}{11.14694} = \pounds 8{,}971.07$$

The above illustration shows that, assuming annual compound interest payments, a borrower would make payments of £747.58 per month for 25 years to pay off a £100,000 mortgage/loan at 7.5% pa.

3.5. Perpetuities

A perpetuity is a series of:

- Equal cash payments
- Received or made at regular intervals
- Over an unspecified period of time (into perpetuity)

The formula to calculate the present value of a perpetuity is given as:

$$\text{Present value of a perpetuity} = \frac{\pounds x}{r}$$

Where £x is the annual payments and r is the discount rate.

For example, assuming the first payment is made in one year's time, the present value of a £500 perpetuity at an interest rate of 10% is equal to £5,000 (£500 / 0.1).

Uses of the perpetuity formula

The perpetuity formula can be used to value those investments that have fixed periodic cash flows that are paid indefinitely. One such security is a standard preference share.

Example 1:

ACME, plc has recently issued preference shares which will pay an annual dividend of £2 per share. If investors are expecting a return of 15% pa from such an investment, what must be the fair value of each share?

PV = price

r = 0.15

x = £2

$$PV = \frac{£2}{0.15} = £13.33$$

Example 2:

Bogota, **plc's** preference shares are currently selling for £15 per share. The shares pay an annual dividend of £3. What must be the return that investors are expecting on these shares?

PV = £15

r = expected return (unknown)

x = £3

$$£15 = \frac{£3}{r}$$

We can manipulate this equation using algebra. Multiplying both sides by r and dividing both sides by £15 yields:

$$r = \frac{£3}{£15} = 0.20 = 20\%$$

The answer is 20%.

4. Annualised rates

4.1. Annual percentage rates

Some credit agreements do not state annual interest rates. Instead, they quote the interest charged on the outstanding balance per month or per quarter.

Simple interest into a compound rate

A credit card, for example, might quote an interest rate of, say, 18% per annum (p.a.), but state that this is charged monthly. The per annum rate is a simple rate, and assumes that no compounding has occurred. To work out how much will be charged monthly, we simply divide the per annum rate by 12 months.

- 18 / 12 = 1.5%

This means that at the end of each month 1.5% is charged to the outstanding balance. If we pay off the amount on the credit card, we have nothing else to concern us. If, however, we leave the balance on the card interest is compounding throughout the year.

The impact of this compounding, can be assessed using the following formula:

$$\text{APR} = (1 + \text{Monthly rate})^{12} - 1$$

Using the example of our credit card, the APR would be:

- APR= 1.015^{12}- 1 = 0.1956

Or 19.56%. Greater than the 18% p.a. rate quoted.

A compound rate into a simple rate

If the annual percentage rate (APR) is known and the frequency of charging is known, we can work out the period rate to be charged.

Example

An investor has a bank loan with an APR of 6%. Interest is charged monthly.

The monthly rate would be calculated by using the following formula:

$$\text{Monthly rate} = \sqrt[n]{1 + \text{APR}} - 1$$

$$\text{Monthly rate} = \sqrt[12]{1.06} - 1$$

$$= 0.00487$$

Or 0.487%.

This would give a simple rate (or **flat rate**) of:

- 0.487×12 months $= 5.84\%$

5. Investment appraisal techniques

5.1. Introduction

The principle of discounting may be used to test the viability of a project, such as the construction of a building or the purchase of a financial investment.

There are two main discounted cash flow (DCF) techniques used for project appraisal purposes:

- Net present value approach (NPV)
- Internal rate of return approach (IRR)

5.2. Net present value

The NPV technique of investment appraisal measures the present value of the project's cash inflows against the present value of the project's cash outflows in order to determine the viability of a project and/or investment.

The difference between the present value of the inflows and the present value of the outflows is known as the **net present value** (NPV) of the project:

$$NPV = PV_i - PV_o$$

Net present value

Present value of all cash inflows

Present value of all cash outflows

If the **NPV is equal to, or greater than zero**, the project is viable and is worth carrying out. That is the present value of the project's cash inflows are equal to, or greater than, the present value of the project's cash outflows.

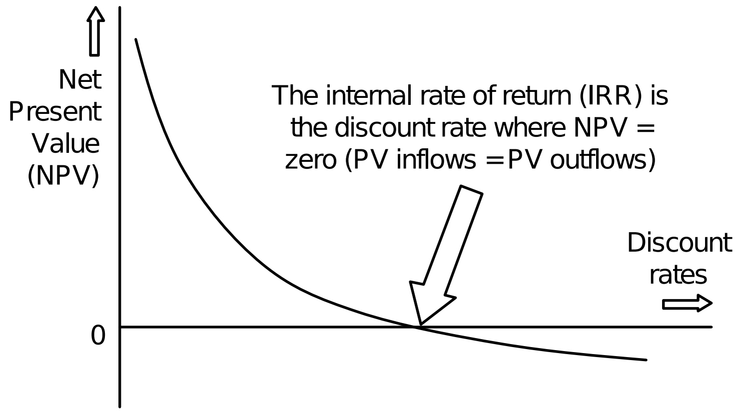
A negative NPV, however, indicates the project should not be attempted as the present value of the project's cash outflows are greater than the present value of the project's cash inflows.

5.3. Internal rate of return

The IRR is defined as the discount rate that, when applied to the cash flows of a project, will equate the present value of the cash inflows with the present values of the cash outflows.

In other words, it is the **discount rate that will calculate the net present value of a project as zero**.

The internal rate of return is therefore the discount rate where the present value of the inflows equals the present value of the outflows.



IRR decision rule

When evaluating a project's viability using the IRR technique, the IRR of the project must be compared to the company's **cost of capital** (the cost of equity and the cost of debt).

- If the company's cost of capital is **less** than or equal to the project's internal rate of return, the project should be accepted.

- If the company's cost of capital is **more** than the project's internal rate of return, the project should be rejected.

For example, if a project has an internal rate of return of 15% and the company's cost of capital was only 10%, then the project would be worth proceeding with.

Problems with the IRR

There are limitations of using the IRR as a method of investment appraisal.

- The IRR ignores the **quantity of earnings**. If the only choices a firm has in a year are Project A which returns 20% on £100,000 and project B which returns 40% of £10,000, the lower IRR project is likely to be more appealing since it generates higher **actual earnings**

- IRR cannot be used when the discount rate is variable. Although it would still be possible to calculate the NPV

- If the project has a number of inflows and outflows over time then it may result in multiple IRRs

- If there is a big difference between the IRR and the project discount rate it may result in conflicting decisions

Because of the problems inherent in using the IRR, the net present value technique of investment appraisal provides a superior method for evaluating the viability of projects and investments.

Estimation of IRR

Unfortunately, there is no closed-form solution for the internal rate of return. We can only estimate the IRR by **trial and error** (or iterative) process. Since the examination for this type of question is multiple choice, we can use the four given choices as the rate at which the cash flows are discounted. The rate that gives us a net present value of zero is the right answer. Advanced financial calculators and some computer software can estimate the IRR given a project’s cash flows to a very high degree of accuracy.

Example of exam question for IRR

Company B makes an up-front investment in a ten-year project of £49,900. It expects to generate £6,000 at end of each year from this project. Calculate the internal rate of return.

1. 3%
2. 3.5%
3. 4%
4. 4.5%

Explanation

The company will generate an income of £6,000 per year for 10 years (an annuity) after investing £49,900.

To work through this question, we will need the annuity formula:

We then substitute in the numbers that we have been given:

$$£49{,}900 = £6{,}000 \times \frac{1}{r} \times \left(1 - \frac{1}{(1+r)^{10}}\right)$$

As mentioned earlier, there is no closed-form solution so we need to use trial and error with the multiple choices we are given. If our choice does not work, we try again.

The correct solution is 3.5%:

6. Financial maths: summary

6.1. Key concepts

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Now you have finished this chapter you should attempt the chapter questions.