Optimization and Computational Linear Algebra for Data Science
Homework 4: Norm and inner product

Due on October 4^{th}, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a $(\star)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 4.1 (2 points)**.** *Let $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ compute (and justify your result):*

$$\max \left\{ v^\mathsf{T} x \,\middle|\, v \in \mathbb{R}^n, \|v\| = 1 \right\}.$$

Problem 4.2 (2 points)**.** *Show that for all $x \in \mathbb{R}^n$,*

$$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Problem 4.3 (4 points)**.** *Let S be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of S by*

$$S^\perp \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \,|\, x \perp S\} = \{x \in \mathbb{R}^n \,|\, \forall y \in S, \, \langle x, y \rangle = 0\}.$$

(a) *Show that $S^\perp$ is a subspace of $\mathbb{R}^n$.*

(b) *Show that $\dim(S^\perp) = n - \dim(S)$. Hint: use the rank-nullity theorem.*

Let $v = (1, 1, 1) \in \mathbb{R}^3$ and define

$$H = \{x \in \mathbb{R}^3 \,|\, x \perp v\} = \mathrm{Span}(v)^\perp.$$

(c) *Find an orthonormal basis of H and an orthonormal basis of $H^\perp$.*

(d) *Write the matrix of P_H, the orthogonal projection on H.*

Problem 4.4 (4 points)**.** *In this problem, we will see how to compress (using a method similar to the one used in the `jpeg` standard) and denoise images, by using a particular orthonormal basis called a "wavelet basis".*

All the questions are in the jupyter notebook `wavelets.ipynb` and have to be answered directly in the notebook. (Submit only a pdf export of your notebook: Print $\rightarrow$ Save as pdf)

The notebook may look long, however the questions are all very short: most of them only require to do a matrix product or a plot. You have to use `Python` and its library `numpy`. A useful command: `A @ B` *: performs the matrix product of the matrix* `A` *with the matrix* `B`*.*

Problem 4.5 ($\star$)**.** *Let P be an $n \times n$ matrix such that*

$$\begin{cases} P^2 = P \\ P^{\mathsf{T}} = P. \end{cases}$$

Show that P is the matrix of the orthogonal projection on some subspace V of $\mathbb{R}^n$.