Session 2: Linear transformations and matrices

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

1. Recap of the videos
2. Operation on matrices
3. Kernel and Image
4. Why do we care about all these things ?
    - Solving linear systems

Linear maps & matrices

Two sides of the same coin

Linear map

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

Matrix

$$L \in \mathbb{R}^{n \times m}$$

Two sides of the same coin

Linear map

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

Matrix

$$L \in \mathbb{R}^{n \times m}$$

Rotations in $\mathbb{R}^2$

Let $\theta \in \mathbb{R}$. The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of angle θ about the origin is linear.

Exercise: what is the canonical matrix of R_θ ?

Operations on matrices

Addition and scalar multiplication

- Sum of two matrices of the **same** dimensions:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1}+b_{1,1} & \cdots & a_{1,m}+b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1}+b_{n,1} & \cdots & a_{n,m}+b_{n,m} \end{pmatrix}$$

- Multiplication by a scalar λ:

$$\lambda \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} = \begin{pmatrix} \lambda a_{1,1} & \cdots & \lambda a_{1,m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \cdots & \lambda a_{n,m} \end{pmatrix}$$

A new vector space!

Proposition

- $\mathbb{R}^{n\times m}$ is a vector space.
- $\dim(\mathbb{R}^{n\times m}) =$

Proof.

$\square$

Product of two matrices

Warning:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} \neq \begin{pmatrix} a_{1,1} \times b_{1,1} & \cdots & a_{1,m} \times b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} \times b_{n,1} & \cdots & a_{n,m} \times b_{n,m} \end{pmatrix}$$

Matrix product

Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{m\times k}$.

Definition (Matrix product)

The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$.

Matrix product

Theorem

The entries matrix product LM are given by

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell} M_{\ell,j}, \qquad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq k.$$

Rotations in $\mathbb{R}^2$

The R_a and R_b denote respectively the matrix of the rotation of angle a and b about the origin, in $\mathbb{R}^2$.

Exercise: Compute the product $R_a R_b$.

Matrix product properties

Kernel and image

Definitions

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

Definition (Kernel)

The kernel $\mathrm{Ker}(L)$ (or nullspace) of L is defined as the set of all vectors $v \in \mathbb{R}^m$ such that $L(v) = 0$, i.e.

$$\mathrm{Ker}(L) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^m \mid L(v) = 0\}.$$

Definition (Image)

The image $\mathrm{Im}(L)$ (or column space) of L is defined as the set of all vectors $u \in \mathbb{R}^n$ such that there exists $v \in \mathbb{R}^m$ such that $L(v) = u$.

Picture

Remarks

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

Proposition

- $\mathrm{Ker}(L)$ is a subspace of $\mathbb{R}^m$.
- $\mathrm{Im}(L)$ is a subspace of $\mathbb{R}^n$.

Remark: $\mathrm{Im}(L)$ is also the Span of the columns of the matrix representation of L.

Example: orthogonal projection

Consider $L : \mathbb{R}^2 \to \mathbb{R}^2$ to be the orthogonal projection onto the x-axis.

Why do we care about this ?

Linear systems

Assume that we given a dataset:

$$a_i = (a_{i,1}, \dots, a_{i,m}) \in \mathbb{R}^m, \quad y_i \in \mathbb{R} \qquad \text{for} \quad i = 1, \dots, n.$$

We would like to find $x \in \mathbb{R}^m$ such that

$$x_1 a_{i,1} + \cdots + x_m a_{i,m} = y_i \qquad \text{for all} \quad i \in \{1, \dots, n\}.$$

Matrix notation

Let us write

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \in \mathbb{R}^{n\times m} \qquad \text{and} \qquad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$$

3 possible cases

Gaussian elimination

$$A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{n\times m} \qquad \text{and} \qquad y = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^n.$$

Gaussian elimination

Gaussian elimination

Gaussian elimination

Questions?