

Session 3: The rank

Optimization and Computational Linear Algebra for Data Science

Contents

1. The rank
2. The rank-nullity Theorem
3. More on the inverse of a matrix
4. Transpose of a matrix
5. Why do we care about all these things ?

Is the rank useful in practice?

The rank

Recap of the videos

Definition

We define the rank of a family x_1, \dots, x_k of vectors of \mathbb{R}^n as the dimension of its span:

$$\text{rank}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \dim(\text{Span}(x_1, \dots, x_k)).$$

Definition

Let $M \in \mathbb{R}^{n \times m}$. Let $c_1, \dots, c_m \in \mathbb{R}^n$ be its columns. We define

$$\text{rank}(M) \stackrel{\text{def}}{=} \text{rank}(c_1, \dots, c_m) = \dim(\text{Im}(M)).$$

Proposition

Let $M \in \mathbb{R}^{n \times m}$. Let $r_1, \dots, r_n \in \mathbb{R}^m$ be the rows of M and $c_1, \dots, c_m \in \mathbb{R}^n$ be its columns. Then we have

$$\text{rank}(r_1, \dots, r_n) = \text{rank}(c_1, \dots, c_m) = \text{rank}(M).$$

How do we compute the rank ?

For $v_1, \dots, v_k \in \mathbb{R}^n$, and $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$ we have

$$\begin{aligned}\text{rank}(v_1, \dots, v_k) &= \text{rank}(v_1, \dots, v_{i-1}, \alpha v_i, v_{i+1}, \dots, v_k) \\ &= \text{rank}(v_1, \dots, v_{i-1}, v_i + \beta v_j, v_{i+1}, \dots, v_k)\end{aligned}$$

As a consequence, the Gaussian elimination method keeps the rank of a matrix unchanged!

Example

Let's compute the rank of $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix}$

Example

The rank-nullity Theorem

Rank-nullity Theorem

Theorem

Let $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation. Then

$$\text{rank}(L) + \dim(\text{Ker}(L)) = m.$$

Proof sketch on an example

Let us solve the linear system $Ax = 0$.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -1 & 5 & 2 & 0 & 0 \end{array} \right) \quad \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 4 & 2 & 1 & 0 \end{array} \right) \begin{array}{l} (R_1) \\ (R_2) - 2(R_1) \\ (R_3) + (R_1) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 0 & 0 & 7 & 0 \end{array} \right) \begin{array}{l} (R_1) \\ (R_2) \\ (R_3) - 2(R_2) \end{array}$$

Proof of the rank-nullity Theorem

Proof of the rank-nullity Theorem

Proof of the rank-nullity Theorem

Proof of the rank-nullity Theorem

Invertible matrices

Invertible matrices

Definition (Matrix inverse)

A **square** matrix $M \in \mathbb{R}^{n \times n}$ is called *invertible* if there exists a matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$MM^{-1} = M^{-1}M = \text{Id}_n.$$

Such matrix M^{-1} is unique and is called the *inverse* of M .

Exercise: Let $A, B \in \mathbb{R}^{n \times n}$. Show that if $AB = \text{Id}_n$ then $BA = \text{Id}_n$.

Invertible matrices

Theorem

Let $M \in \mathbb{R}^{n \times n}$. The following points are equivalent:

1. M is invertible.
2. For all $y \in \mathbb{R}^n$, there exists a unique $x \in \mathbb{R}^n$ such that $Mx = y$.
3. $\text{rank}(M) = n$.
4. $\text{Ker}(M) = \{0\}$.

Proof

Proof

Proof

Proof

Transpose of a matrix

Transpose of a matrix

Definition

Let $M \in \mathbb{R}^{n \times m}$. We define its *transpose* $M^T \in \mathbb{R}^{m \times n}$ by

$$(M^T)_{i,j} = M_{j,i}$$

for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Remark:

- ❖ We have $(M^T)^T = M$.
- ❖ The mapping $M \mapsto M^T$ is linear.

Properties of the transpose

Proposition

For all $A \in \mathbb{R}^{n \times m}$, $\text{rank}(A) = \text{rank}(A^T)$.

Proposition

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$. Then

$$(AB)^T = B^T A^T.$$

Proof.



Symmetric matrices

Definition

A square matrix $A \in \mathbb{R}^{n \times n}$ is said to be *symmetric* if

$$\forall i, j \in \{1, \dots, n\}, A_{i,j} = A_{j,i}$$

or, equivalently if $A = A^T$.

Remark: For all $M \in \mathbb{R}^{n \times m}$ the matrix MM^T is symmetric.

Is the rank useful in practice?

Back to the movies ratings example

Assume that you are given the matrix of movies ratings:

$$\begin{pmatrix} 1 & 1 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1.001 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0.0001 & 0 \\ 2.0001 & 2 & 2 & 0 & 0 \end{pmatrix}$$

Goal: how many different « user profiles » do we have ?

Conclusions

- ❖ The rank is not «robust» !
- ❖ We need to have a way to check if a matrix is «approximately low-rank».
- ❖ Equivalently, given m vectors, one would like to be able to see if there exists a subspace of dimension $k \ll m$ from which the vectors are « not far ».
- ❖ The singular value decomposition (lecture 6-7) will solves our problems !

Questions?

Questions?

Questions?