

Lecture 1.3: Span, Linear dependency and dimension

Optimization and Computational Linear Algebra for Data Science

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Linear combination & Span

Linear combination

Let V be a vector space (think for instance $V = \mathbb{R}^n$).

Definition

We say that $y \in V$ is a *linear combination* of the vectors $x_1, \dots, x_k \in V$ if there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$y = \sum_{i=1}^k \alpha_i x_i = \alpha_1 x_1 + \dots + \alpha_k x_k.$$

Remarks

- ❖ A linear combination is always a finite sum.
- ❖ If S is a subspace of V , then any linear combination of vectors x_1, \dots, x_k of S is also in S :

$$\alpha_1 x_1 + \dots + \alpha_k x_k \in S, \quad \text{for all } \alpha_1, \dots, \alpha_k \in \mathbb{R}.$$

« Subspaces are closed under linear combinations. »

Exercise: Prove it !

Span

Definition

Let x_1, \dots, x_k be vectors of V . We define the *linear span* of x_1, \dots, x_k as the set of all linear combinations of these vectors:

$$\text{Span}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \left\{ \alpha_1 x_1 + \dots + \alpha_k x_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}.$$

Linear dependency

Linear dependency

Definition

Vectors $x_1, \dots, x_k \in V$ are *linearly dependent* if there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ **that are not all zero** such that

$$\alpha_1 x_1 + \dots + \alpha_k x_k = 0.$$

They are said to be *linearly independent* otherwise.

Key observation: « x_1, \dots, x_k are linearly dependent » is equivalent to « one of the vectors x_1, \dots, x_k can be obtained as a linear combination of the others. »

Why ?

Basis, dimension

Basis

Definition

A family (x_1, \dots, x_n) of vectors of V is a basis of V if

1. x_1, \dots, x_n are linearly independent,
2. $\text{Span}(x_1, \dots, x_n) = V$.

This means that (x_1, \dots, x_n) is a basis of V if

1. None of the x_i is a linear combination of the others $(x_j)_{j \neq i}$.
2. Any vector of V can be expressed as a linear combination of (x_1, \dots, x_n) .

Example: the canonical basis of \mathbb{R}^n

Let us define the vectors $e_1, \dots, e_n \in \mathbb{R}^n$ by

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

$$\vdots$$

$$e_n = (0, 0, 0, \dots, 1).$$

One can verify (exercise!) that the family (e_1, \dots, e_n) is a basis of \mathbb{R}^n . This basis is called the “canonical basis” of \mathbb{R}^n . We conclude that \mathbb{R}^n has dimension n .

Dimension

Theorem

Let V be a vector space.

- ❖ If V admits a basis (v_1, \dots, v_n) , then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.
- ❖ Otherwise, we say that V has infinite dimension: $\dim(V) = +\infty$.

Example:

- ❖ \mathbb{R}^n has dimension n , because the canonical basis (e_1, \dots, e_n) is a basis of \mathbb{R}^n with n vectors.
- ❖ $\{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$ has infinite dimension.