Recitation 9

Alex Dong

CDS, NYU

Fall 2020

Convexity and Optimization

- We are temporarily stepping away from *Linearity*
- Two types of convexity
  - Convexity for functions (we care about this)
  - Convexity for sets
  - They are related! (see epigraph question)
- Convexity implies if a min exists, it must be a global min
  - Optimization is the process to find the minimum
- Convexity is a *global* property
- Contrast to differentiable, which is a *local* property

Questions: Convexity and Epigraph

Let $f : \mathbb{R}^n \to \mathbb{R}$.
Define the epigraph $epi(f) \subset \mathbb{R}^{n+1}$ to be set of all the points above the graph of f:
$epi(f) = \{[\vec{x}, c] \in \mathbb{R}^{n+1} \mid c \geq f(\vec{x})\}$
where $[\bullet, \bullet]$ denotes a vector concatentation

0. For the function $f : \mathbb{R}^2 \to \mathbb{R}$, s.t $f(x, y) = x^2 + y^2$, draw $epi(f)$
1. Prove that f is convex if and only if $epi(f)$ is convex
(Warning, this problem has *really tricky* notation)
(Hint $\Longleftarrow$ is easier than $\Longrightarrow$)

Solutions 1: Convexity and Epigraph

Solution

epi(f) *is convex* $\implies$ f *is convex.*

Assume epi(f) *is convex. Let* $\vec{x}, \vec{y} \in \mathbb{R}^n, t \in [0, 1]$.
Consider $t(f(\vec{x}) + (1-t)f(\vec{y})$, *convex combo of* $f(\vec{x})$,$f(\vec{y})$
Since $[\vec{x}, f(\vec{x})], [\vec{y}, f(\vec{y})] \in$ *epi*(f), *and epi*(f) *is convex, then,*

$$t[\vec{x}, f(\vec{x})] + (1-t)[\vec{y}, f(\vec{y})] \in epi(f) \qquad \textit{convexity of } epi(f)$$
$$[t\vec{x} + (1-t)\vec{y}\ ,\ tf(\vec{x}) + (1-t)f(\vec{y})] \in epi(f) \qquad \textit{add vectors}$$

Therefore, by definition of epi(f), *we have* $\forall x, y \in \mathbb{R}^n$, $t \in [0, 1]$
$tf(\vec{x}) + (1-t)f(\vec{y}) \geq t\vec{x} + (1-t)\vec{y}$ *So* f *is convex.*

Solutions 2: Convexity and Epigraph

Solution

f is convex $\implies$ *epi(f) is convex.*
Assume f is convex.
Let $[x, c], [y, d] \in epi(f), t \in [0, 1]$.
Since $[\vec{x}, c], [\vec{y}, d)] \in epi(f)$*,then we have*
$\quad c \geq f(\vec{x})$ *and* $d \geq f(\vec{y})$
and this directly implies
$\quad tc \geq tf(\vec{x})$ *and* $(1-t)d \geq (1-t)f(\vec{y}) \qquad\qquad$ *(*)*
Now, consider $t[\vec{x}, c] + (1-t)[\vec{y}, d])$, $\quad$ *convex combo of* $[\vec{x}, c], [\vec{y}, d]$
From (*), we have
$\quad t[\vec{x}, c] + (1-t)[\vec{y}, d]) \geq tf(\vec{x}) + (1-t)f(\vec{y})$
and since f is convex, then
$\quad t[\vec{x}, c] + (1-t)[\vec{y}, d]) \geq tf(\vec{x}) + (1-t)f(\vec{y}) \geq f(t\vec{x} + (1-t)\vec{y})$
$\quad t[\vec{x}, c] + (1-t)[\vec{y}, d]) \geq f(t\vec{x} + (1-t)\vec{y})$
So $t[\vec{x}, c] + (1-t)[\vec{y}, d]) \in epi(f)$.
Since this applies $\forall x, y \in \mathbb{R}^n$, $t \in [0, 1]$, *epi(f) is convex.*

Questions: True and False

1. If f has only 1 global min and no local min, then f is convex
2. Linear combination of two convex functions is convex
3. Convex functions are differentiable at all points
4. Norms are convex functions
5. If f is convex, then g(x) = $f(Ax - b)$ is also convex. ($A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$)
6. Sum of a non-convex function w/ another function can never be convex
7. Union of convex sets is convex
8. Intersection of convex sets is convex
9. Maximum of two convex functions is convex
10. Every subspace is a convex set
11. Every convex set is a subspace

Questions: True and False

1. False, $cos(\theta)$, $\theta \in [0, \pi]$
2. False, negative of convex function is not convex
3. False, f(x) = —x—
4. True, Triangle inequality (Prove it!)
5. True, convexity is a global property, so the $Ax - b$ doesnt matter.
6. False sum of f and -f is 0, which is a convex function.
7. False, Easy counter-example
8. True, True, intersection preserves convexity properties
9. True, Use epigraph proof
10. True, convex combinations are in the subspace by property of subspaces.
11. False, convex sets do not need to contain 0.

Questions: Quadratic Forms

Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = x^T A x$ for some symmetric matrix $A \in \mathbb{R}^{n \times n}$.

1. Give conditions on A so that 0 is the global minimizer of f

Solutions: Quadratic Forms

Let $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = x^T Ax$ for some symmetric matrix $A \in \mathbb{R}^{n\times n}$.

1. Give conditions on A so that 0 is the global minimizer of f

Solution

Let A have spectral decomposition $A = U\Lambda U^T$.
Let $y \in \mathbb{R}^n$ s.t $y = U^T x$.
Then $f(x) = x^T Ax = x^T U\Lambda U^T x = y^T \Lambda y$, and
$y^T \Lambda y = \sum_{i=1}^n \lambda_i y_i^2$
We can see that 0 will be the global minimzer of f if A is positive semi-definite.