Recitation 8

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Singular Value Decomposition

- $A = U\Sigma V^T$
  - $A, \Sigma \in \mathbb{R}^{n\times m}, \quad U \in \mathbb{R}^{n\times n}, \quad V \in \mathbb{R}^{m\times m}$
  - U, V are orthogonal
    - (U, V not necessarily the same dimension)
  - Σ is "almost diagonal".
    - All entries off the main diagonal are 0.
- (!!) *Every* matrix has a SVD.
- SVD provides a framework for understanding the properties of A.
- Can use the SVD to create matrices (Psuedo-Inverse, Lec 10).

Questions: SVD

1. Explain what the SVD says about the action of any matrix A.
2. Recall from Recitation 3 the proof we used to show
   $$rank(A) = rank(A^T A).$$
   Use the SVD to show that:
   $$rank(A) = rank(A^T A) = rank(AA^T) = rank(A^T).$$

Solutions 1: SVD

1. Explain what the SVD says about the action of any matrix A.

Solution

Let A have SVD $A = U\Sigma V^T$.
Transforming any vector x by A is equivalent to transforming it by V^T, Σ and then U.
Applying V^T is applying a rotation/flipping.
Applying Σ scales/squashes each (standard basis) axis, and also transforms $V^T x$ to a different space.
Applying U also applies a rotation/flipping.

Solutions 2: SVD

2. Recall from Recitation 3 the proof we used to show
   $$rank(A) = rank(A^T A).$$
   Use the SVD to show that:
   $$rank(A) = rank(A^T A) = rank(AA^T) = rank(A^T).$$

Solution

Recall that if B,C are invertible, then rank(BAC) = rank(A).

$A = U\Sigma V^T$

$A^T A = V\Sigma^T \Sigma V^T$

$AA^T = U\Sigma\Sigma^T U^T$

$A^T = V\Sigma^T U^T$

Based on the definition of Σ*, we can easily do the matrix multiplication, and see that:*

$$rank(\Sigma) = rank(\Sigma^T \Sigma) = rank(\Sigma\Sigma^T) = rank(\Sigma^T).$$

We can then deduce that:

$$rank(A) = rank(A^T A) = rank(AA^T) = rank(A^T).$$

Questions: More SVD

Let $A \in \mathbb{R}^{n\times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.
Let $e_{i,m}, e_{i,n}$ denote the ith standard basis vector in $\mathbb{R}^m$, and $\mathbb{R}^n$.

1. Give basis transformations for U, Σ, V^T.
   (Your answer should look like $T(a) = b$. where you select convenient a's that form a basis for the origin space of T. Be careful about the dimensions w/ Σ!)

2. Let $x \in \mathbb{R}^m$ s.t $x = \sum_{i=1}^{m} \alpha_i v_i$, where v_i are the columns of V. and write the expressions for $V^T x$, $\Sigma V^T x$, and $U\Sigma V^T x$.

3. Which vectors span the $Im(A)$? Which vectors span the $Ker(A)$? Which vectors span $Im(A^T)$? Which vectors span $Ker(A^T)$?

Solutions 1: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.
Let $e_{i,m}, e_{i,n}$ denote the ith standard basis vector in $\mathbb{R}^m$, and $\mathbb{R}^n$.

1. Give basis transformations for U, Σ, V^T.
   (Your answer should look like $T(a) = b$. where you select convenient a's that form a basis for the origin space of T. Be careful about the dimensions w/ Σ!)

Solution

$V^T(v_i) = e_{i,m} \quad \textit{for } i \in \{1, ..., m\}$
$\Sigma(e_{i,m}) = \sigma_i e_{i,n} \quad \textit{for } i \in \{1, ..., m\}$ *(Note the* m *here)*
$U(e_{i,n}) = u_i \quad \textit{for } i \in \{1, ..., n\}$

Solutions 2: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.
Let $e_{i,m}, e_{i,n}$ denote the ith standard basis vector in $\mathbb{R}^m$, and $\mathbb{R}^n$.

2. Let $x \in \mathbb{R}^m$ s.t $x = \sum_{i=1}^{m} \alpha_i v_i$, where v_i are the columns of V. Write the expressions for $V^T x$ $\Sigma V^T x$, and $U\Sigma V^T x$.

Solution

Let $e_{i,m}$ denote the ith standard basis vector in $\mathbb{R}^m$.

$x = \sum_{i=1}^{m} \alpha_i v_i$

$V^T x = \sum_{i=1}^{m} \alpha_i e_{i,m}$

$\Sigma V^T x = \sum_{i=1}^{min(m,n)} \sigma_i \alpha_i e_{i,n}$

$U\Sigma V^T x = \sum_{i=1}^{min(m,n)} \sigma_i \alpha_i u_i$

Solutions 3: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.
Let $e_{i,m}, e_{i,n}$ denote the ith standard basis vector in $\mathbb{R}^m$, and $\mathbb{R}^n$.

3. Which vectors span the $Im(A)$? Which vectors span the $Ker(A)$? Which vectors span $Im(A^T)$? Which vectors span $Ker(A^T)$?

Solution

Let $\sigma_1, ..., \sigma_k > 0$ *, and* $\sigma_{k+1}, ..., \sigma_{min(m,n)} = 0$.
Let $u_1, ..., u_n$ *be the columns of* U *and* $v_1, ..., v_m$ *be the columns of* V.

$u_1, ..., u_k$ *span the* $Im(A)$.
$v_{k+1}, ..., v_m$ *span* $Ker(A)$.

$v_1, ..., v_k$ *span* $Im(A^T)$.
$u_{k+1}, ..., u_n$ *span* $Ker(A^T)$.

Note the u's and v's!

Questions: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

1. Show that $M^T M$ is invertible.
2. Which vectors span the $Im(M)$? Write the matrix of orthogonal projection onto $Im(M)$ and give a basis transformation for that matrix.
3. Let $w \in \mathbb{R}^n$, and u be the orthgonal projection of w onto $Im(M)$. Show that $M^T u = M^T w$.
4. Show that $M(M^T M)^{-1} M^T$ is the matrix of an orthogonal projection onto $Im(M)$.

Solutions 1: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

Solution

1. *Show that $M^T M$ is invertible.*
   "Questions: SVD" Question 2 earlier in recitation.
   M is full rank in this case.

2. *Which vectors span the $Im(M)$? Write the matrix of orthogonal projection onto $Im(M)$ and give a basis transformation for that matrix.*
   "Questions: More SVD" Question 3 earlier in recitation.
   $u_1, ..., u_m$ *span* $Im(M)$. $P_{Im(M)} = U_r U_r^T$ *where:*
   $$U_r = \begin{bmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{bmatrix}$$
   $U_r U_r^T(u_i) = u_i$ *for* $i \in \{1, ..., m\}$.
   $U_r U_r^T(u_i) = 0$ *for* $i \in \{m+1, ..., n\}$.

Solutions 2: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

3. Let $w \in \mathbb{R}^n$, and u be the orthgonal projection of w onto $Im(M)$. Show that $M^T u = M^T w$.

Solution

Let $w = \sum_{i=1}^{n} \alpha_i u_i$.
$u = P_{Im(M)}(w)$, *so*
  $u = U_r U_r^T w$, *and*
  $u = \sum_{i=1}^{m} \alpha_i u_i$. *(note change in summation).*
Now,
  $M^T u = \sum_{i=1}^{m} \sigma_i \alpha_i v_i$, *and*
  $M^T w = \sum_{i=1}^{min(m,n)} \sigma_i \alpha_i v_i$ *(From More SVD Q2)*
Since $n > m$, $M^T u = M^T w$.

4. *Show that* $M(M^T M)^{-1} M^T$ *is the matrix of an orthogonal projection onto* $Im(M)$.

Solutions 2: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n\times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

4. Show that $M(M^TM)^{-1}M^T$ is the matrix of an orthogonal projection onto $Im(M)$.

Solution

First, note that $(M^TM)^{-1} = (V(\Sigma^T\Sigma)^{-1}V^T)$, and that $\Sigma^T\Sigma$ has rank m and is invertible, where the diagonal entries of $(\Sigma^T\Sigma)^{-1}$ are reciprocals of the entries in $(\Sigma^T\Sigma)$.

Now,

$$M(M^TM)^{-1}M^T = (U\Sigma V^T)(V(\Sigma^T\Sigma)^{-1}V^T)(V\Sigma^TU^T)$$

$$M(M^TM)^{-1}M^T = U\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^TU^T$$

Doing the matrix multiplication of $\Sigma(\Sigma^T\Sigma)^{-1}\Sigma^T$ gives

$$\Sigma(\Sigma^T\Sigma)^{-1})\Sigma^T = I_{m,n},$$

where $I_{m,n}$ is the $n \times n$ matrix where the first m diagonal entries are 1. Finally, its easy to show that $UI_{m,n}U^T = U_rU_r^T$.

Concluding Remarks

- Throughout the course, we've emphasized viewing problems from different points of view
- Matrix multiplication
  - Inner product interpretation
  - Linear combination of columns interpretation
- Linear transformations
  - as matrices (matrix mechanics)
  - as letters (transformations on basis vectors)
- This recitation really emphasizes fluid switching between all of these frameworks
- All the things we've covered are deeply connected to eachother.