Optimization and Computational Linear Algebra for Data Science
Homework 2: Linear transformations & matrices

Due on September 17, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will note affect your grade).
- Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count: from Friday to Monday count 1 day.
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 1.1 (2 points). *Which of the following are linear transformations? Justify.*

(a) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x^2+y^2, x-y) \end{array} \right.$

(b) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+y+1, x-y) \end{array} \right.$

(c) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times m} & \to & \mathbb{R}^{m\times n} \\ A & \mapsto & A^\mathsf{T} \end{array} \right.$ *where* A^T *is transpose of* A*, i.e. the* $m \times n$ *matrix defined by*

$$(A^\mathsf{T})_{i,j} = A_{j,i} \qquad \textit{for all} \quad (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}.$$

(d) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R} \\ A & \mapsto & \mathrm{Tr}(A) \end{array} \right.$ *where* $\mathrm{Tr}(A)$ *is the trace of the matrix* A*, defined by*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{i,i}\,.$$

Problem 1.2 (3 points). *Let* $f : \mathbb{R}^2 \to \mathbb{R}^3$ *be a linear transformation such that*

$$f(1,2) = (1,2,3) \qquad \textit{and} \qquad f(2,2) = (1,0,1).$$

(a) *Compute* $f(1,0)$.

(b) *Give the set* $\{x \in \mathbb{R}^2 \,|\, f(x) = (1,4,5)\}$.

(c) *Give the set* $\{x \in \mathbb{R}^2 \,|\, f(x) = (2,4,5)\}$.

Problem 1.3 (2 points). *Compute*

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 4 \end{pmatrix} \times \begin{pmatrix} -1 & -1 & 0 \\ 1 & 4 & -1 \\ 2 & 1 & 2 \end{pmatrix}$$

Problem 1.4 (3 points).

(**a**) *Let A be a $n \times m$ matrix. Show that the image* $\mathrm{Im}(A)$ *and the kernel* $\mathrm{Ker}(A)$ *of A are subspaces of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$.*

(**b**) *Let*

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Compute a basis of $\mathrm{Ker}(A)$ *and show that* $\mathrm{Im}(A) = \mathbb{R}^3$.

Problem 1.5 ($\star$). *Find an $n \times n$ matrix A such that A, A^2, $\ldots$, A^n are not zero and such that $A^{n+1} = 0$. Is this matrix invertible?*