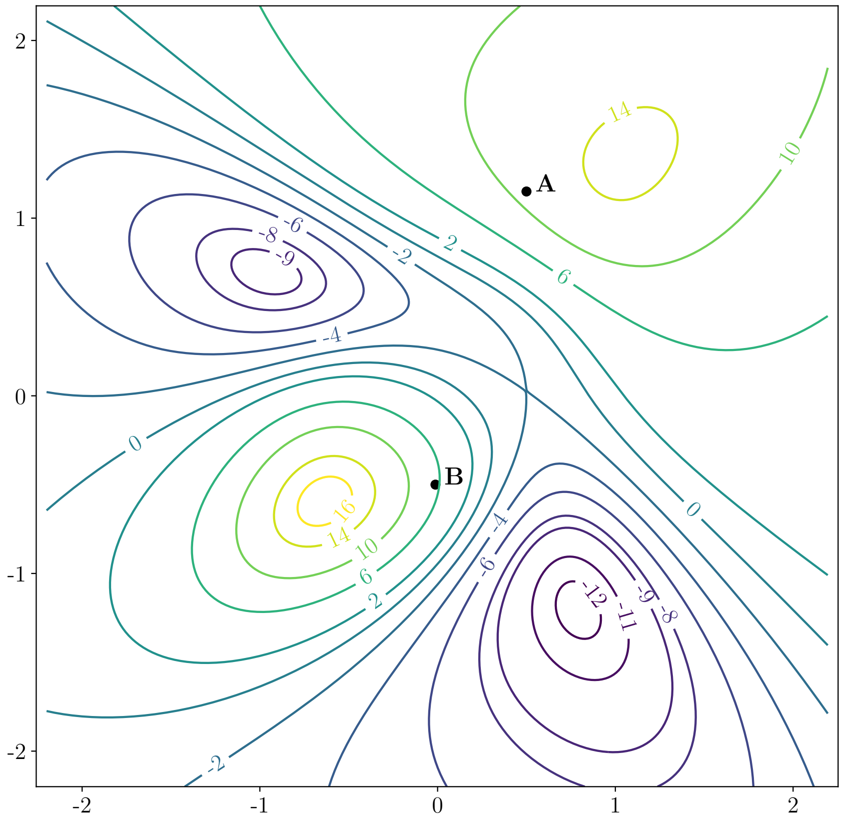
Optimization and Computational Linear Algebra for Data Science
Homework 12: Gradient descent

Due on December 13, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 12.1 (2 points)**.** *The following plot shows the contour lines of a function $f : \mathbb{R}^2 \to \mathbb{R}$.*



(a) *Give (approximately) the coordinates of the global/local minimizers/maximizers, saddle points of f.*

(b) *Assume that we run gradient descent to minimize f. Will gradient descent converge to the global minimizer of f when initialized at point* **A** *? at point* **B** *?*

Problem 12.2 (5 points). *Let $M \in \mathbb{R}^{d\times d}$ be a positive semidefinite matrix, $b \in \mathbb{R}^d$ and $c \in \mathbb{R}$. We aim at minimizing the quadratic function*

$$f(x) = \frac{1}{2} x^\mathsf{T} M x - \langle x, b \rangle + c$$

using gradient descent. We assume that M is positive definite (i.e. all its eigenvalues are positive). We let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d > 0$ be its eigenvalues and let $v_1, \ldots, v_d$ be an orthonormal basis of $\mathbb{R}^d$ consisting of associated eigenvectors ($Mv_i = \lambda_i v_i$ for all i). We write $L = \lambda_1$ and $\mu = \lambda_d$.

We consider standard gradient descent with constant step-size β:

$$x_{t+1} = x_t - \beta \nabla f(x_t).$$

(a) *Show that f is L-smooth, μ-strongly convex and that $x^* = M^{-1}b$ is the unique minimizer of f.*

(b) *We now study the convergence of gradient descent to x^*. Show that for all $t \geq 0$,*

$$x_{t+1} - x^* = (\mathrm{Id} - \beta M)(x_t - x^*).$$

(c) *From now, we set $\beta = 1/L$. Deduce from the previous question that for all $t \geq 0$*

$$\|x_t - x^*\| \leq \left(1 - \frac{\mu}{L}\right)^t \|x_0 - x^*\|.$$

(d) *We would like now to have something more precise than the error bound of the previous question. We define $w_t \stackrel{\text{def}}{=} x_t - x^*$. Let*

$$\alpha_1(t) = \langle v_1, w_t \rangle, \cdots, \alpha_d(t) = \langle v_d, w_t \rangle$$

be the coordinates of w_t in the orthonormal basis $(v_1, \ldots, v_d)$. For $i \in \{1, \ldots, d\}$, express $\alpha_i(t)$ in terms of t, λ_i, L and $\alpha_i(0)$.

(e) *Using the previous question, justify the following sentence:*

> « Gradient descent converges towards the minimizer faster in directions given by the eigenvectors of the Hessian of f corresponding to large eigenvalues than in directions corresponding to eigenvectors with small eigenvalues.»

(f) *Show that for all $t \geq 0$*

$$\|x_t - x^*\| = \sqrt{\sum_{i=1}^{d} \left(1 - \frac{\lambda_i}{L}\right)^{2t} \langle v_i, x_0 - x^* \rangle^2}.$$

Problem 12.3 (3 points). *In this problem, you will implement and compare gradient descent with or without momentum to minimize the Ridge cost function:*

$$f(x) = \frac{1}{2}\|Ax - y\|^2 + \frac{\lambda}{2}\|x\|^2.$$

All the instructions and questions are in the Jupyter notebook `gradient_descent.ipynb`.

It is intended that you code in Python and use the provided Jupyter Notebook. Please only submit a pdf version of your notebook (right-click → 'print' → 'Save as pdf').

Problem 12.4 ($\star$). *We take exactly the same setting of Problem 12.2, but we now consider gradient descent with momentum:*

$$x_{t+1} = x_t - \beta \nabla f(x_t) + \gamma (x_t - x_{t-1}),$$

for $t \geq 1$, where we take

$$\beta = \frac{4}{(\sqrt{L} + \sqrt{\mu})^2} \qquad \text{and} \qquad \gamma = \left(\frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^2.$$

Show now that the $\alpha_i(t) \stackrel{\text{def}}{=} \langle v_i, x_t - x^* \rangle$ satisfy a second order linear recurrence relation (as a sequence indexed by t). Using this relation, show that for all $t \geq 0$

$$|\alpha_i(t)| \leq C_i \left(\frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^t$$

where C_i is a constant that does not depend on t, but that may depend on x_0, x_1, μ and L (a precise expression of C_i is not expected). Deduce that for all $t \geq 0$

$$\|x_t - x^*\| \leq C \left(\frac{\sqrt{L} - \sqrt{\mu}}{\sqrt{L} + \sqrt{\mu}} \right)^t$$

where C is a constant that does not depend on t.