Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

1. Recap of the videos
2. More about the dimension
3. Coordinates
4. Why do we care about all these things ?

   Application to data science: image compression

Logistics

The teaching team

- **Lecturer**: Léo Miolane – *lm4271nyu.edu*

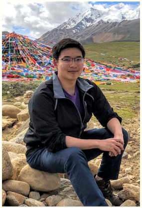
  `leomiolane.github.io/linalg-for-ds.html`

The teaching team

- **Lecturer**: Léo Miolane – *lm4271nyu.edu*
  `leomiolane.github.io/linalg-for-ds.html`

- **Sections leaders**:

Alex



In person

Irina

Remote

Carles

Remote

Course components

Three main components:

1. Videos

   2-3 short videos to watch **before** each lecture

2. Lectures

   Deepens the concepts introduced in the videos

3. Recitations

   Practice!

Course components

Three main components:

1. Videos

   2-3 short videos to watch **before** each lecture

2. Lectures

   Deepens the concepts introduced in the videos

3. Recitations

   Practice!

Grades:

1. Weekly quizzes (5%)
2. Weekly homeworks (40%)
3. Exams: Midterm (20%) + Final (35%)

Weekly timeline

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	15	16	17	18	19	20
21	22	23	24	25	26	27

Grading

- Quizzes have to be answered on **Gradescope**.
- Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.

Grading

- Quizzes have to be answered on **Gradescope**.
- Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.
- I encourage you to type your homeworks using LaTeX.

  Some instructions and template available on the course's webpage.
- Otherwise, you can scan (using dedicated app) your handwritten work. **It has to be legible!!!**

Grading

- Quizzes have to be answered on **Gradescope**.
- Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.
- I encourage you to type your homeworks using LaTeX.

  Some instructions and template available on the course's webpage.
- Otherwise, you can scan (using dedicated app) your handwritten work. **It has to be legible!!!**
- Midterm ($\sim$ mid-October) and Final will be «take-home exams».
- Limited time: after downloading the Midterm/Final questions, you will have to upload your work within few hours.

Check out the syllabus on the course webpage!

Questions on logistics ?

Vector spaces and subspaces

Quick recap of video 1.2

A **vector space** is a set V endowed with two 'nice and compatible' operations $+$ and $\cdot$ that verify:

- For all $u, v \in V$, $u + v \in V$.
- For all $u \in V$ and all $\lambda \in \mathbb{R}$, $\lambda \cdot u \in V$.

Example: $V = \mathbb{R}^n$, with the usual vector addition $+$ and scalar multiplication $\cdot$ is a vector space.

Quick recap of video 1.2

A subset S of a vector space V is called a **subspace** if it is closed under addition and multiplication by a scalar.

Example: For all $v \in \mathbb{R}^n$,

$$\mathrm{Span}(v) = \{\lambda v \,|\, \lambda \in \mathbb{R}\}$$

is a subspace of $\mathbb{R}^n$.

Question ?

Review of Span and linear dependency

Span

The *linear span* of vectors $x_1, \ldots, x_k$ as the set of all linear combinations of these vectors.

Linear dependency

- Vectors $x_1, \dots x_k$ are *linearly dependent* if one of them can be expressed as a linear combination of the others.
- They are said to be *linearly independent* otherwise.

Abuse of language: Instead of saying «$x_1, \dots, x_k$ are linearly dependent», we should say «the family $(x_1, \dots, x_k)$ is linearly dependent».

Basis

A family $(x_1, \ldots, x_n)$ of vectors of V is a basis of V if

1. $x_1, \ldots, x_n$ are linearly independent,
2. $\mathrm{Span}(x_1, \ldots, x_n) = V$.

The dimension

A useful lemma

Lemma

Let $v_1, \ldots, v_n \in V$ and let $x_1, \ldots, x_k \in \mathrm{Span}(v_1, \ldots, v_n)$.
Then, if $k > n$, $x_1, \ldots, x_k$ are linearly dependent.

Definition of the dimension

Definition

We say that a vector space V has dimension n if it admits a basis $(v_1, \ldots, v_n)$ with n vectors.

The dimension is well defined!

Theorem

If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.

Proof.

$\square$

Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$.

- Any family of vectors of V that spans V contains at least n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are such that $\mathrm{Span}(x_1, \dots, x_k) = V$, then $k \geq n$.

Proof.

$\square$

Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$.

- Any family of vectors of V that spans V contains at least n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are such that $\mathrm{Span}(x_1, \dots, x_k) = V$, then $k \geq n$.

Proof.

$\square$

Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If $x_1, \dots, x_n$ are linearly independent, then $(x_1, \dots, x_n)$ is a basis of V.
2. If $\mathrm{Span}(x_1, \dots, x_n) = V$, then $(x_1, \dots, x_n)$ is a basis of V.

Very useful to show that a family of vector forms a basis:

Example: $x_1 = (12, 37)$ and $x_2 = (-9, 17)$ form a basis of $\mathbb{R}^2$.

Proof of the Proposition. $\square$

An inequality

Proposition

Let U and V be two subspaces of $\mathbb{R}^n$. Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

An inequality

Proposition

Let U and V be two subspaces of $\mathbb{R}^n$. Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

Proof.

$\square$

A bit of vocabulary

Definition

Let S be a subspace of $\mathbb{R}^n$.

- We call S a *line* if $\dim(S) = 1$.
- We call S an *hyperplane* if $\dim(S) = n - 1$.

Coordinates

Coordinates of a vector in a basis

Definition

If $(v_1, \ldots, v_n)$ is a basis of V, then for every $x \in V$ there exists a unique vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \ldots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \ldots, v_n)$.

Coordinates of a vector in a basis

Definition

If $(v_1, \ldots, v_n)$ is a basis of V, then for every $x \in V$ there exists a unique vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \ldots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \ldots, v_n)$.

Proof.

$\square$

Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of $\mathbb{R}^2$.
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y.

Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of $\mathbb{R}^2$.
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y.

Why do we care about this ?

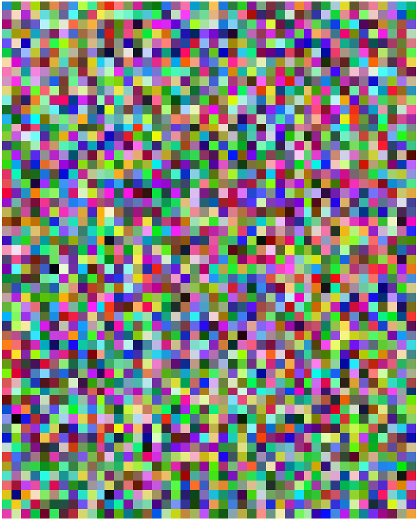
Application to image compression

- Image = Grid of pixels
- Represented as a vector $v \in \mathbb{R}^n$, for some large n.
- One need to store n numbers.

$n = 44 \times 55 = 2420$

Can we do better?

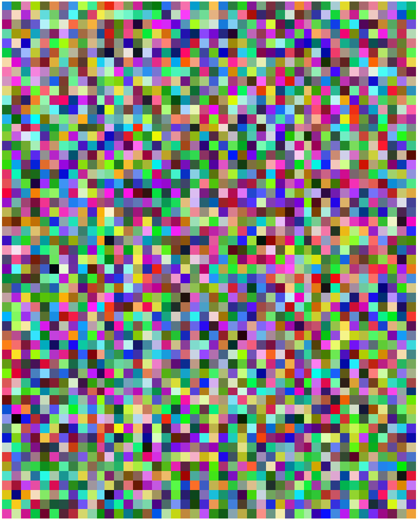
- If we want to store an arbitrary image, NO!



«Random» image

Can we do better?

- If we want to store an arbitrary image, NO!
- However, we are mainly storing images coming from the « real world »
- These images have some *structure*.



«Random» image

Can we do better?

- If we want to store an arbitrary image, NO!
- However, we are mainly storing images coming from the « real world »
- These images have some *structure*.

«Real» image

What do we mean by « structure » ?

Neighboring pixels are very likely to have similar colors.

What do we mean by « structure » ?

Neighboring pixels are very likely to have similar colors.

- There exists a basis $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$ in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- This means that the coordinates $(\alpha_1, \ldots, \alpha_n)$ of v in the basis $(w_1, \ldots, w_n)$ contains a lot of zeros.

What do we mean by « structure » ?

Neighboring pixels are very likely to have similar colors.

- There exists a basis $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$ in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- This means that the coordinates $(\alpha_1, \ldots, \alpha_n)$ of v in the basis $(w_1, \ldots, w_n)$ contains a lot of zeros.

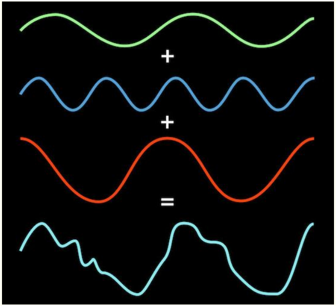
Store only the $k \ll n$ non-zero coordinates of v (in the w_i's basis') !

A toy example

Consider $n = 2$, that is images $v \in \mathbb{R}^2$ with only 2 pixels.

Examples of good bases

- **Fourier bases** (used in `.jpeg`, `.mp3`)



- `JPEG2000` uses **wavelet bases**, and achieves better performance than `JPEG`.
- In **Homework 4**, you will use wavelets to compress/denoise images.
- The course **DS-GA 1013** deepens these concepts!

Questions?