# Optimization and Computational Linear Algebra for Data Science
# Homework 4: Norm and dot product

Due on October 1st, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will note affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

**Problem 4.1** (2 points)**.** *Let $\|\cdot\|$ be the usual Euclidean norm on $\mathbb{R}^n$. For $x \in \mathbb{R}^n$ compute (and justify your result):*

$$\max\left\{ v^{\mathsf{T}} x \,\middle|\, v \in \mathbb{R}^n, \|v\| = 1 \right\}.$$

**Problem 4.2** (2 points)**.** *Show that for all $x \in \mathbb{R}^n$,*

$$\frac{1}{\sqrt{n}}\|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

**Problem 4.3** (2 points)**.** *Let $\langle \cdot, \cdot \rangle$ be a dot product on $\mathbb{R}^n$, and let $\|\cdot\|$ be the induced norm by $\langle \cdot, \cdot \rangle$.*

**(a)** *Show that for all $x, y \in \mathbb{R}^n$ we have*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

**(b)** *Deduce from the previous question that the $\ell_1$ norm $\|\cdot\|_1$ and the infinity norm $\|\cdot\|_\infty$ are* ***not*** *induced by a dot product.*

**Problem 4.4** (4 points)**.** *Let $S$ be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of $S$ by*

$$S^\perp \overset{\text{def}}{=} \{x \in \mathbb{R}^n \,|\, x \perp S\} = \{x \in \mathbb{R}^n \,|\, \forall y \in S,\ \langle x, y \rangle = 0\}.$$

**(a)** *Show that $S^\perp$ is a subspace of $\mathbb{R}^n$.*

**(b)** *Show that $\dim(S^\perp) = n - \dim(S)$. Hint: use the rank-nullity theorem.*

*Let $v = (1,1,1) \in \mathbb{R}^3$ and define*

$$H = \{x \in \mathbb{R}^n \,|\, x \perp v\} = \mathrm{Span}(v)^\perp.$$

**(c)** *Find an orthonormal basis of $H$ and an orthonormal basis of $H^\perp$.*

**(d)** *Write the matrix of $P_H$, the orthogonal projection on $H$.*

**Problem 4.5** ($\star$)**.** *Let $P$ be an $n \times n$ matrix such that*

$$\begin{cases} P^2 = P \\ P^\mathsf{T} = P. \end{cases}$$

*Show that $P$ is the matrix of the orthogonal projection on some subspace $V$ of $\mathbb{R}^n$.*