Optimization and Computational Linear Algebra for Data Science
Midterm review problems

Problem 0.1. *Let $A, B \in \mathbb{R}^{n\times n}$. For each the following subset of $\mathbb{R}^n$ below, say whether it is a subspace of $\mathbb{R}^n$ and justify your answer:*

1. $E_1 = \{x \in \mathbb{R}^n \,|\, Ax = 0\}$.
2. $E_2 = \{x \in \mathbb{R}^n \,|\, Ax = Bx\}$.
3. $E_3 = \{x \in \mathbb{R}^n \,|\, Ax = e_1\}$.
4. $E_4 = \{x \in \mathbb{R}^n \,|\, Ax \in \mathrm{Span}(e_1)\}$.

Solution:

1. $E_1 = \{x \in \mathbb{R}^n \,|\, Ax = 0\} = \mathrm{Ker}(A)$ is a subspace of $\mathbb{R}^n$.
2. $E_2 = \{x \in \mathbb{R}^n \,|\, Ax = Bx\} = \mathrm{Ker}(A - B)$ is a subspace of $\mathbb{R}^n$.
3. $E_3 = \{x \in \mathbb{R}^n \,|\, Ax = e_1\}$ is not a subspace of $\mathbb{R}^n$ since $0 \notin E_3$.
4. $E_4 = \{x \in \mathbb{R}^n \,|\, Ax \in \mathrm{Span}(e_1)\}$ is a subspace. Indeed,
   - $E_4 \neq \emptyset$, since $A0 = 0 \in \mathrm{Span}(e_1)$: $0 \in E_4$.
   - If $u, v \in E_4$ then $A(u+v) = Au + Av \in \mathrm{Span}(e_1)$ because $Au, Av \in \mathrm{Span}(e_1)$ and $\mathrm{Span}(e_1)$ is a subspace.
   - If $u \in E_4$ and $\lambda \in \mathbb{R}$ then $A(\lambda u) = \lambda Au \in \mathrm{Span}(e_1)$ because $Au \in \mathrm{Span}(e_1)$ and $\mathrm{Span}(e_1)$ is a subspace.

Problem 0.2. ***True or False**: There exists matrices $M \in \mathbb{R}^{2\times 3}$ such that* $\dim(\mathrm{Ker}(M)) = 1$ *and* $\mathrm{rank}(M) = 2$.

Solution: True, take for instance the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Problem 0.3. *Let $n > m$ and $A \in \mathbb{R}^{n\times m}$. Assume that A has "full rank", meaning that* $\mathrm{rank}(A) = \min(n, m) = m$.

1. *Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example)*
2. *Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).*

Solution:

1. $\mathrm{Im}(A) \subset \mathbb{R}^n$ and $\dim(\mathrm{Im}(A)) = m < n$. Hence $\mathrm{Im}(A) \neq \mathbb{R}^n$, so there exists vectors $b \in \mathbb{R}^n$ that does not belong to $\mathrm{Im}(A)$, i.e. for which there exists no x such that $Ax = b$.
2. The rank-nullity theorem gives that $\dim(\mathrm{Ker}(A)) = m - \mathrm{rank}(A) = 0$. Hence $\mathrm{Ker}(A) = \{0\}$. If $Ax = Ax'$ for some $x, x' \in \mathbb{R}^m$, then $x - x' \in \mathrm{Ker}(A)$ which implies that $x - x' = 0$: $x = x'$. Therefore there can not exists two vectors $x \neq x'$ such that $Ax = Ax'$.

Problem 0.4. *Let $n < m$ and $A \in \mathbb{R}^{n \times m}$. Assume that A has "full rank", meaning that* $\mathrm{rank}(A) = \min(n, m) = n$.

1. *Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example)*

2. *Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).*

Solution:

1. $\mathrm{Im}(A) \subset \mathbb{R}^n$ and $\dim(\mathrm{Im}(A)) = n$. Hence $\mathrm{Im}(A) = \mathbb{R}^n$, for all $b \in \mathbb{R}^n$ there exists $x \in \mathbb{R}^m$ such that $Ax = b$.

2. The rank-nullity theorem gives that $\dim(\mathrm{Ker}(A)) = m - \mathrm{rank}(A) = m - n > 0$. Hence there exists $x \neq 0$ such that $Ax = 0 = A0$.

Problem 0.5. ***True or False**: There exists a family of k non-zero orthogonal vectors of $\mathbb{R}^n$, for some $k > n$.*

Solution: An orthogonal family of non-zero vectors is linearly independent. Since there is no linearly independent family of vectors of $\mathbb{R}^n$ that contains strictly more than n vectors, the statement is false.

Problem 0.6. *Let $A \in \mathbb{R}^{n \times m}$.*

1. *Prove that $\mathrm{Ker}(A^\mathsf{T})$ and $\mathrm{Im}(A)$ are orthogonal to each other, i.e. that for all $x \in \mathrm{Ker}(A^\mathsf{T})$ and $y \in \mathrm{Im}(A)$ we have $x \perp y$.*

2. *Show that $\mathrm{Ker}(A^\mathsf{T}) = \mathrm{Im}(A)^\perp$.*

Solution:

1. Let $x \in \mathrm{Ker}(A^\mathsf{T})$ and $y \in \mathrm{Im}(A)$. There exists $v \in \mathbb{R}^m$ such that $y = Av$. Compute now:
$$\langle y, x \rangle = \langle Av, x \rangle = v^\mathsf{T} A^\mathsf{T} x = 0$$
because $x \in \mathrm{Ker}(A^\mathsf{T})$. Hence $x \perp y$.

2. The first question shows that $\mathrm{Ker}(A^\mathsf{T}) \subset \mathrm{Im}(A)^\perp$. Since we know from the homework that
$$\dim(\mathrm{Im}(A)^\perp) = n - \dim(\mathrm{Im}(A)) = n - \dim(\mathrm{Im}(A^\mathsf{T})) = \dim(\mathrm{Ker}(A^\mathsf{T}))$$
where we used the the fact that $\mathrm{rank}(A) = \mathrm{rank}(A^\mathsf{T})$ and the rank-nullity Theorem. We conclude that $\mathrm{Ker}(A^\mathsf{T}) = \mathrm{Im}(A)^\perp$.

Problem 0.7. ***True or False**: The matrix of an orthogonal projection is symmetric.*

Solution: True: Let P_S be the matrix of the orthogonal projection onto a subspace S. We know that if V is a matrix whose columns forms an orthonormal basis of S, then $P_S = VV^\mathsf{T}$, which is symmetric.

Problem 0.8. ***True or False**: The matrix of an orthogonal projection is orthogonal.*

Solution: False. Consider for instance (for $n \geq 1$) the orthogonal projection P onto the subspace $\{0\}$. For all $x \in \mathbb{R}^n$, $Px = 0$. Hence P is the zero matrix which is not orthogonal.

Problem 0.9. *Let S be a subspace of $\mathbb{R}^n$ and let P_S be the orthogonal projection onto S. Show that* $\dim(S) = \mathrm{Tr}(P_S)$.

Solution: Let $k = \dim(S)$ and let $v_1, \dots, v_k$ be an orthonormal basis of S. Let

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times k}.$$

We know from the lectures that then $P_S = VV^\mathsf{T}$. Compute

$$\mathrm{Tr}(P_S) = \mathrm{Tr}(VV^\mathsf{T}) = \mathrm{Tr}(V^\mathsf{T}V) = \mathrm{Tr}(\mathrm{Id}_k) = k = \dim(S),$$

where $V^\mathsf{T}V = \mathrm{Id}_k$ because the columns of V form an orthonormal family.

Problem 0.10. ***True or False****: Let $A, B \in \mathbb{R}^{n\times n}$. Assume that $v \in \mathbb{R}^n$ is an eigenvector of A and B.*

1. *Is v an eigenvector of $A + B$?*
2. *Is v an eigenvector of AB ?*

Solution: Since $v \in \mathbb{R}^n$ is an eigenvector of A and B, there exists $\lambda, \lambda' \in \mathbb{R}$ such that $Av = \lambda v$ and $Bv = \lambda' v$.

1. v an eigenvector of $A + B$ because

$$(A + B)v = Av + Bv = \lambda v + \lambda' v = (\lambda + \lambda')v.$$

2. v an eigenvector of AB because

$$ABv = A(\lambda' v) = \lambda' Av = \lambda\lambda' v.$$

Problem 0.11. *Let $A \in \mathbb{R}^{n\times n}$ and let $v_1, v_2 \in \mathbb{R}^n$ be two eigenvectors of A, associated with the same eigenvalue λ.*
Show that any non-zero eigenvector in $\mathrm{Span}(v_1, v_2)$ *is an eigenvector of A, associated with λ.*

Solution: Let $x \in \mathrm{Span}(v_1, v_2) \setminus \{0\}$. There exists $\alpha, \beta \in \mathbb{R}$ such that $x = \alpha v_1 + \beta v_2$. Compute

$$Ax = A(\alpha v_1 + \beta v_2) = \alpha A v_1 + \beta A v_2 = \alpha\lambda v_1 + \beta\lambda v_2 = \lambda(\alpha v_1 + \beta v_2) = \lambda x.$$

Recall that $x \neq 0$: we conclude that x is an eigenvector of A associated with the eigenvalue λ.

Problem 0.12. *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Let $(v_1, v_2, \dots, v_n)$ be an orthonormal family of eigenvectors of A, associated to the eigenvalues $\lambda_1, \dots, \lambda_n$. Give an orthonormal basis of* $\mathrm{Ker}(A)$ *and* $\mathrm{Im}(A)$ *in terms of the v_i's.*

Solution: Let $I = \{i \in \{1, \dots, n\} \,|\, \lambda_i = 0\}$ and $k = \#I$.

For $i \in I$, we have $Av_i = 0$. Hence the familiy $(v_i)_{i\in I}$ is a familiy of k linearly independent vectors (because the v_i's are orthonormal) of $\mathrm{Ker}(A)$. Therefore $\dim(\mathrm{Ker}(A)) \geq k$.

For $i \notin I$, we have $v_i = \frac{1}{\lambda_i} A v_i \in \mathrm{Im}(A)$. Hence the familiy $(v_i)_{i \notin I}$ is a familiy of $n-k$ linearly independent vectors (because the v_i's are orthonormal) of $\mathrm{Im}(A)$. Therefore $\dim(\mathrm{Im}(A)) \geq n-k$.

The rank-nullity Theorem gives that $\dim(\mathrm{Ker}(A)) + \dim(\mathrm{Im}(A)) = n$. This implies (together with the two inequalities above) that $\dim(\mathrm{Ker}(A)) = k$ and $\dim(\mathrm{Im}(A)) = n-k$.

Recall that the familiy $(v_i)_{i \in I}$ is a familiy of k linearly independent vectors of $\mathrm{Ker}(A)$: it is therefore a basis of $\mathrm{Ker}(A)$. Recall that the familiy $(v_i)_{i \notin I}$ is a familiy of $n-k$ linearly independent vectors of $\mathrm{Im}(A)$: it is therefore a basis of $\mathrm{Im}(A)$.

Problem 0.13. *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, that satisfies $A^2 = \mathrm{Id}$. Show that the matrix*

$$M = \frac{1}{2}(A + \mathrm{Id})$$

is the matrix of an orthogonal projection.

Solution: Let λ be an eigenvalue of A and v an associated eigenvector. We have $v = A^2 v = \lambda^2 v$, hence $\lambda^2 = 1$, i.e. $\lambda \in \{-1, 1\}$.

Let k be the multiplicity of the eigenvalue 1. A is symmetric, so the spectral theorem gives that there exists an orthogonal matrix V sucht that

$$A = V \mathrm{Diag}(1, \dots, 1, -1, \dots, -1) V^{\mathsf{T}},$$

with k 1 and $n-k$ -1. Since $VV^{\mathsf{T}} = \mathrm{Id}$, we get that

$$M = \frac{1}{2}(A + \mathrm{Id}) = V \mathrm{Diag}(1, \dots, 1, 0, \dots, 0) V^{\mathsf{T}},$$

with k 1 and $n-k$ 0. Let $V_{(k)}$ be the matrix consisting of the first k colums of V. We have

$$M = V \mathrm{Diag}(1, \dots, 1, 0, \dots, 0) V^{\mathsf{T}} = V_{(k)} V_{(k)}^{\mathsf{T}}.$$

V is orthogonal so its colums forms an orthonormal family. We conclude that M is the orthogonal projection onto the span of the first k colums of V.

Problem 0.14. *Let $\rho \in (0,1)$. Let $v_1, \dots, v_k \in \mathbb{R}^n$ such that*

$$\|v_i\| = 1 \qquad \textit{and} \qquad \langle v_i, v_j \rangle = \rho \qquad \textit{for all} \quad i \neq j.$$

Show that $k \leq n$.

Solution: Let

$$V = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_k \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times k}.$$

We have

$$V^{\mathsf{T}} V = \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & & \vdots \\ \vdots & & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} = (1-\rho)\mathrm{Id}_k + \rho J$$

where $J \in \mathbb{R}^{k \times k}$ is the all-ones matrix. The eigenvalues of J are 0 and k (from the homework) hence the eigenvalues of $V^{\mathsf{T}} V = (1-\rho)\mathrm{Id}_k + \rho J$ are all strictly positive (because $(1-\rho) > 0$).

This gives that $\operatorname{rank}(V^{\mathsf{T}}V) = k$.

Since $\operatorname{rank}(V^{\mathsf{T}}V) \leq \operatorname{rank}(V) \leq k$ (recall that V is $n \times k$), we get $\operatorname{rank}(V) = k$. This means that $v_1, \ldots, v_k$ are k linearly independent vectors of $\mathbb{R}^n$: $k \leq n$.