

# Optimization and Computational Linear Algebra for Data Science

## Homework 5: Orthogonal matrices, eigenvalues and eigenvectors

Due on October 11, 2020

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- Unless otherwise stated, all answers must be mathematically justified.
  - Partial answers will be graded.
  - Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
  - You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
  - Problems with a  $(\star)$  are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
  - If you have any questions, feel free to contact me ([lm4271@nyu.edu](mailto:lm4271@nyu.edu)) or to stop at the office hours.
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**Problem 5.1** (2 points). Let  $S$  be a subspace of  $\mathbb{R}^n$  and let  $P_S$  be the matrix of the orthogonal projection onto  $S$ . Let  $M = \text{Id}_n - 2P_S$ .

- (a) Show that the matrix  $M$  is orthogonal.
- (b) Show that if  $\lambda \in \mathbb{R}$  is an eigenvalue of  $M$ , then  $\lambda = 1$  or  $\lambda = -1$ .

**Problem 5.2** (2 points). Let  $v \in \mathbb{R}^n$  be a non-zero vector. What are the eigenvalues of the  $n \times n$  matrix

$$M = vv^\top$$

and their multiplicities? (In the expression  $vv^\top$  we see  $v$  as a matrix with 1 column and  $n$  rows).

**Problem 5.3** (2 points). Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix. Show that if  $v_1, v_2 \in \mathbb{R}^n$  are two eigenvectors of  $A$  associated to distinct eigenvalues  $\lambda_1 \neq \lambda_2$  ( $Av_1 = \lambda_1 v_1$  and  $Av_2 = \lambda_2 v_2$ ), then  $v_1 \perp v_2$ .

**Problem 5.4** (4 points). Let  $A \in \mathbb{R}^{n \times n}$ . We assume that there exists a basis  $(v_1, \dots, v_n)$  of  $\mathbb{R}^n$  consisting of eigenvectors of  $A$ :

$$Av_i = \lambda_i v_i$$

for all  $i \in \{1, \dots, n\}$ . We assume that

$$\lambda_1 > |\lambda_i| \quad \text{for all } i \in \{2, \dots, n\}.$$

We consider the following algorithm:

- Initialize  $x_0 \in \mathbb{R}^n$ .
- Perform the updates:  $x_{t+1} = \frac{Ax_t}{\|Ax_t\|}$ .

(a) Show that for all  $t \geq 1$ ,

$$x_t = \frac{A^t x_0}{\|A^t x_0\|}.$$

(b) Assume that  $x_0$  is a unit vector ( $\|x_0\| = 1$ ) whose direction is chosen uniformly at random (this basically means that all the possible directions for  $x_0$  are equally likely to be chosen). Let  $(\alpha_1, \dots, \alpha_n)$  be the coordinates of  $x_0$  in the basis  $(v_1, \dots, v_n)$ . Explain we have  $\alpha_1 \neq 0$  with probability 1. You do not have to do a rigorous proof of that, just give an intuitive argument.

(c) Show that

$$x_t \xrightarrow[t \rightarrow \infty]{} \frac{\alpha_1 v_1}{\|\alpha_1 v_1\|} \quad \text{and} \quad \|Ax_t\| \xrightarrow[t \rightarrow \infty]{} \lambda_1.$$

**Problem 5.5** ( $\star$ ). Let  $A \in \mathbb{R}^{n \times n}$  be a symmetric matrix. We define the function

$$\begin{aligned} f: \mathbb{R}^n \setminus \{0\} &\rightarrow \mathbb{R} \\ x &\mapsto \frac{x^\top A x}{x^\top x}. \end{aligned}$$

We admit that  $f$  has a maximum at some  $x_\star \in \mathbb{R}^n \setminus \{0\}$ . Show that  $x_\star$  verifies

$$Ax_\star = \lambda x_\star, \quad \text{where} \quad \lambda = f(x_\star).$$

