# **Recitation 2**

#### **Review: Linear Transformations**

#### **Definition: Linear Transformation**

A function  $L: \mathbb{R}^m \to \mathbb{R}^n$  is a linear transformation if

- 1. for all  $v \in \mathbb{R}^m$  and all  $\alpha \in \mathbb{R}$  we have  $L(\alpha v) = \alpha L(v)$  and
- 2. for all  $v, w \in \mathbb{R}^m$  we have L(v+w) = L(v) + L(w).

**Example**:  $L: \mathbb{R}^2 \to \mathbb{R}^2$  defined as

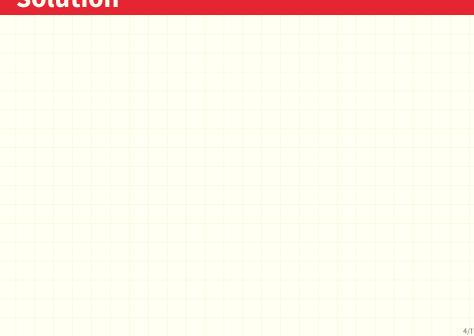
$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ -x + 4y \end{bmatrix} = x \begin{bmatrix} 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

### **Questions 1: Linear Transformations**

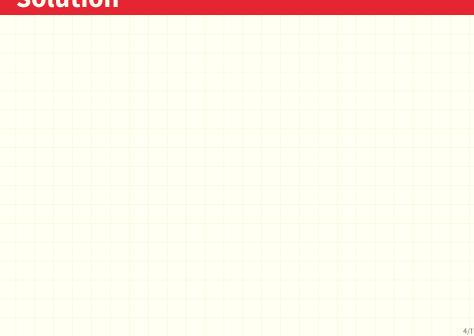
#### Which of the following functions are linear?

- 1.  $f_1:\mathbb{R}^2 o \mathbb{R}^2$  such that  $f_1(a,b)=(2a,a+b)$
- 2.  $f_2: \mathbb{R}^2 \to \mathbb{R}^3$  such that  $f_2(a, b) = (a + b, 2a + 2b, 0)$
- 3.  $f_3: \mathbb{R}^2 \to \mathbb{R}^3$  such that  $f_3(a,b) = (2a,a+b,1)$
- 4.  $f_4: \mathbb{R}^2 \to \mathbb{R}$  such that  $f_4(a,b) = \sqrt{a^2 + b^2}$
- 5.  $f_5: \mathbb{R} \to \mathbb{R}$  such that  $f_5(x) = 5x + 3$

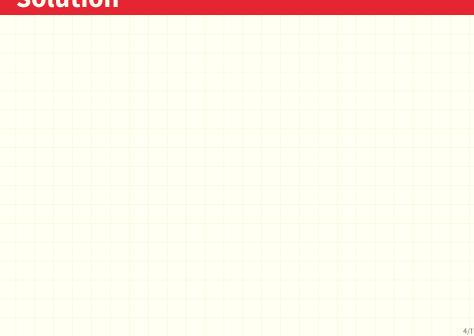
## **Solution**



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#### **Review: Matrices**

A linear transformation  $T: \mathbb{R}^n \to \mathbb{R}^m$  is represented by a  $m \times n$  matrix which is an element of  $\mathbb{R}^{m \times n}$ . (Note the order!)

$$\mathbf{T} = m \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{pmatrix}$$

What does this mean? If  $u_1, u_2, \ldots, u_n$  is a basis of  $\mathbb{R}^n$  and  $v_1, v_2, \ldots, v_m$  is a basis of  $\mathbb{R}^m$ , we have

$$T(u_1) = T_{1,1}v_1 + T_{2,1}v_2 + \dots + T_{m,1}v_m,$$

$$T(u_2) = T_{1,2}v_1 + T_{2,2}v_2 + \dots + T_{m,2}v_m,$$

$$\dots$$

$$T(u_n) = T_{1,n}v_1 + T_{2,n}v_2 + \dots + T_{m,n}v_m.$$

Important: The matrix representation depends on the basis!

#### **Review: Matrices**

**Example**:  $L: \mathbb{R}^2 \to \mathbb{R}^2$  defined as

$$L\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}3x+y\\-x+4y\end{bmatrix} = x\begin{bmatrix}3\\-1\end{bmatrix} + y\begin{bmatrix}1\\4\end{bmatrix}$$

We choose the canonical basis for both vector spaces:  $e_1 = (1, 0), e_2 = (0, 1).$ 

$$L(e_1) = L\begin{pmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3e_1 - 1e_2,$$
  

$$L(e_2) = L\begin{pmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1e_1 + 4e_2.$$

$$\implies \mathbf{L} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$$

#### **Review: Matrix products**

- The product of  $\mathbf{A} \in \mathbb{R}^{m \times n}$  with  $\mathbf{B} \in \mathbb{R}^{n \times p}$  is a matrix  $\mathbf{A}\mathbf{B} \in \mathbb{R}^{m \times p}$ .
- **Example:** If

$$\mathbf{A} = egin{bmatrix} a_{11} & a_{12} \ a_{21} & a_{22} \ a_{31} & a_{32} \end{bmatrix}, \quad \mathbf{B} = egin{bmatrix} b_{11} & b_{12} \ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

- The matrix product is associative: (AB)C = A(BC). It is not commutative: in general  $AB \neq BA$
- As we will see in the next exercise, matrix products are very useful to compose and evaluate linear transformation.

Let  $x=(-2,0,3,1)\in\mathbb{R}^4$ . Let  $A:\mathbb{R}^3\to\mathbb{R}^3$  be a linear transformation with matrix  $\mathbf{A}$  in the canonical basis and let  $B:\mathbb{R}^4\to\mathbb{R}^3$  be a linear transformation with matrix  $\mathbf{B}$  in the canonical, where

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

- 1. The matrix representation of the composition  $A \circ B$  is  $\mathbf{AB}$ . Compute  $\mathbf{AB}$ .
- 2. Compute  $(A \circ B)(x)$ .

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix} \quad x = \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

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#### Review: Kernel, Inverse

#### Definition: Kernel of a linear transformation

The kernel  $\mathrm{Ker}(L)$  of a linear transformation  $L:\mathbb{R}^n\to\mathbb{R}^m$  is the subset of  $\mathbb{R}^m$  of vectors v such that L(v)=0. Similarly, the kernel of a matrix  $\mathbf{L}\in\mathbb{R}^{m\times n}$  is the subset of  $\mathbb{R}^n$  of points x such that  $\mathbf{L}x=0$ .

#### Definition: Invertible matrix, inverse

A matrix  $M\in\mathbb{R}^{n\times n}$  is called *invertible* if there exists a matrix  $M^{-1}\in\mathbb{R}^{n\times n}$  such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix  $M^{-1}$  is unique and is called the *inverse* of M.

#### **Questions 3: Kernel, Inverse**

- 1. Prove that if  $M \in \mathbb{R}^{n \times n}$  the matrix  $M^{-1}$  is indeed unique.
- 2. Prove that if  $M \in \mathbb{R}^{n \times n}$  is invertible, then  $\operatorname{Ker}(M) = \{0\}$

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