

Optimization and Computational Linear Algebra for Data Science

Homework 4: Norm and inner product

Due on October 1st, 2019

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- Unless otherwise stated, all answers must be mathematically justified.
 - Partial answers will be graded.
 - You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will not affect your grade).
 - Problems with a (\star) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
 - If you have any questions, feel free to contact me (lm4271@nyu.edu) or to stop at the office hours.
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Problem 4.1 (2 points). Let $\|\cdot\|$ be the usual Euclidean norm on \mathbb{R}^n . For $x \in \mathbb{R}^n$ compute (and justify your result):

$$\max \left\{ v^T x \mid v \in \mathbb{R}^n, \|v\| = 1 \right\}.$$

Problem 4.2 (2 points). Show that for all $x \in \mathbb{R}^n$,

$$\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1.$$

Problem 4.3 (2 points). Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathbb{R}^n , and let $\|\cdot\|$ be the induced norm by $\langle \cdot, \cdot \rangle$.

(a) Show that for all $x, y \in \mathbb{R}^n$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) Deduce from the previous question that the ℓ_1 norm $\|\cdot\|_1$ and the infinity norm $\|\cdot\|_\infty$ are **not** induced by an inner product.

Problem 4.4 (4 points). Let S be a subspace of \mathbb{R}^n . We define the orthogonal complement of S by

$$S^\perp \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid x \perp S\} = \{x \in \mathbb{R}^n \mid \forall y \in S, \langle x, y \rangle = 0\}.$$

(a) Show that S^\perp is a subspace of \mathbb{R}^n .

(b) Show that $\dim(S^\perp) = n - \dim(S)$. Hint: use the rank-nullity theorem.

Let $v = (1, 1, 1) \in \mathbb{R}^3$ and define

$$H = \{x \in \mathbb{R}^3 \mid x \perp v\} = \text{Span}(v)^\perp.$$

(c) Find an orthonormal basis of H and an orthonormal basis of H^\perp .

(d) Write the matrix of P_H , the orthogonal projection on H .

Problem 4.5 (★). *Let P be an $n \times n$ matrix such that*

$$\begin{cases} P^2 = P \\ P^\top = P. \end{cases}$$

Show that P is the matrix of the orthogonal projection on some subspace V of \mathbb{R}^n .

