Optimization and Computational Linear Algebra for Data Science
Lecture 3: Rank

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Warning: *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

1 Rank of a matrix

Definition 1.1

We define the rank of a family $x_1, \dots, x_k$ of vectors of $\mathbb{R}^n$ as the dimension of its span:

$$\operatorname{rank}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \dim(\operatorname{Span}(x_1, \dots, x_k)).$$

If the vectors $x_1, \dots x_k$ are linearly independent then $\operatorname{rank}(x_1, \dots x_k) = k$. Indeed, in that case $(x_1, \dots, x_k)$ forms a base of $\operatorname{Span}(x_1, \dots, x_k)$ so $\dim(\operatorname{Span}(x_1, \dots, x_k)) = k$.

Proposition 1.1

Let $x_1, \dots, x_k \in \mathbb{R}^n$ and write $r = \operatorname{rank}(x_1, \dots, x_r)$. Then there exists $i_1, \dots i_r \in \{1 \dots k\}$ such that $(x_{i_1}, \dots, x_{i_r})$ forms a basis of $\operatorname{Span}(x_1, \dots, x_k)$.

Definition 1.2 (*Rank*)

The rank of a matrix $M \in \mathbb{R}^{n \times m}$ is defined as the dimension of the image of M:

$$\operatorname{rank}(M) = \dim(\operatorname{Im}(M)).$$

Let $c_1, \dots, c_m$ be the columns of M, then

Proposition 1.2

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^k$, two linear applications. Then the following holds

(i) $\operatorname{rank}(L) \leq \min(n, m)$.

(ii) $\operatorname{rank}(ML) \leq \min(\operatorname{rank}(L), \operatorname{rank}(M))$.

Theorem 1.1 (*Rank-nullity theorem*)

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Im}(L)) = m.$$