Session 6: Eigenvalues, eigenvectors & Markov chains

Optimization and Computational Linear Algebra for Data Science

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Orthogonal matrices

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Orthogonal matrices

Orthogonal matrices

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Eigenvalues & eigenvectors

Introduction

Eigenvalues & eigenvectors

Definition

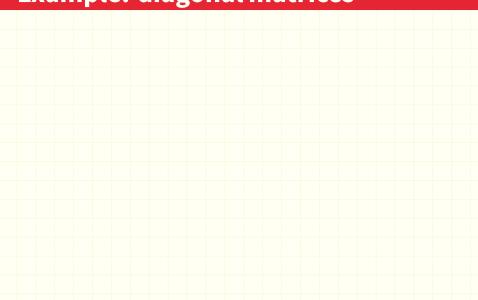
Definition

Let $A\in\mathbb{R}^{n\times n}$. A **non-zero** vector $v\in\mathbb{R}^n$ is said to be an eigenvector of A is there exists $\lambda\in\mathbb{R}$ such that

$$Av = \lambda v$$
.

The scalar λ is called the eigenvalue (of A) associated to v.

Example: diagonal matrices



Eigenvalues & eigenvectors

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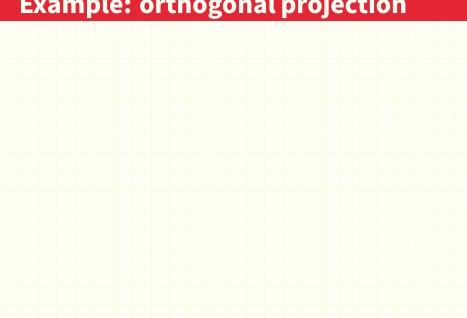
Matrix with no eigenvalues/vectors



Eigenvalues & eigenvectors

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Example: orthogonal projection



Eigenvalues & eigenvectors

Eigenspaces

Definition

If $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, the set

$$E_{\lambda}(A) = \{ x \in \mathbb{R}^n \, | \, Ax = \lambda x \}$$

is called the eigenspace of A associated to λ . The dimension of $E_{\lambda}(A)$ is called the multiplicity of the eigenvalue λ .

Properties

Properties 10/22

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #1

For all $\alpha\in\mathbb{R}$, $\alpha\lambda$ is an eigenvalue of the matrix αA and x is an associated eigenvector.

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #2

For all $\alpha \in \mathbb{R}$, $\lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha \mathrm{Id}$ and x is an associated eigenvector.

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #3

For all $k \in \mathbb{N}$, λ^k is an eigenvalue of the matrix A^k and x is an associated eigenvector.

Properties

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #4

If A is invertible then $1/\lambda$ is an eigenvalue of the matrix inverse A^{-1} and x is an associated eigenvector.

Properties

Spectrum

Definition

The set of all eigenvalues of A is called the *spectrum* of A and denoted by $\operatorname{Sp}(A)$.

Theorem

A $n \times n$ matrix A admits at most n different eigenvalues: $\#\mathrm{Sp}(A) \leq n$.

Properties 12/22

Proof that $\#\mathrm{Sp}(A) \leq n$

Proposition

Let v_1, \ldots, v_k be eigenvectors of A corresponding (respectively) to the eigenvalues $\lambda_1, \ldots, \lambda_k$.

If the λ_i are all distinct ($\lambda_i \neq \lambda_j$ for all $i \neq j$) then the vectors v_1, \ldots, v_k are linearly independent.

Properties 13/22

Proof of the proposition

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Properties

Even better!

Theorem

A $n \times n$ matrix A admits at most n different eigenvalues: $\#\mathrm{Sp}(A) \leq n$.

Theorem

Let $A \in \mathbb{R}^{n \times n}$. If $\lambda_1, \dots, \lambda_k$ are distinct eigenvalues of A of multiplicities m_1, \dots, m_k respectively, then

$$m_1 + \dots + m_k \le n$$
.

Markov chains

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An example

Markov chains

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Stochastic matrices

Definition

A matrix $P \in \mathbb{R}^{n \times n}$ is said to be *stochastic* if:

- 1. $P_{i,j} \ge 0$ for all $1 \le i, j \le n$.
- 2. $\sum_{i=1}^{n} P_{i,j} = 1$, for all $1 \le j \le n$.

Probability vectors

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Markov chains

The key equation

Proposition

For all
$$t \ge 0$$

$$\iota \leq 0$$

$$x^{(t+1)} = Px^{(t)}$$
 and consequently, $x^{(t)} = P^t x^{(0)}$.



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Markov chains

Long-term behavior

Markov chains

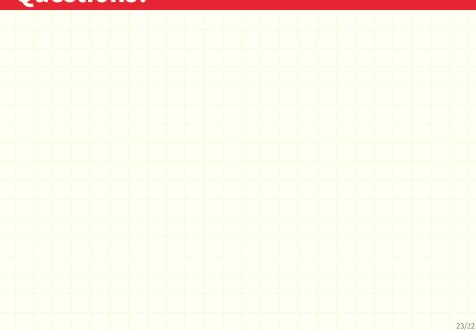
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Markov chains

Questions?



Questions?

