Optimization and Computational Linear Algebra for Data Science
Midterm review problems

Problem 0.1. *Let $A, B \in \mathbb{R}^{n \times n}$. For each the following subset of $\mathbb{R}^n$ below, say whether it is a subspace of $\mathbb{R}^n$ and justify your answer:*

1. $E_1 = \{x \in \mathbb{R}^n \,|\, Ax = 0\}$.
2. $E_2 = \{x \in \mathbb{R}^n \,|\, Ax = Bx\}$.
3. $E_3 = \{x \in \mathbb{R}^n \,|\, Ax = e_1\}$.
4. $E_4 = \{x \in \mathbb{R}^n \,|\, Ax \in \mathrm{Span}(e_1)\}$.

Problem 0.2. ***True or False****: There exists matrices $M \in \mathbb{R}^{2 \times 3}$ such that* $\dim(\mathrm{Ker}(M)) = 1$ *and* $\mathrm{rank}(M) = 2$.

Problem 0.3. *Let $n > m$ and $A \in \mathbb{R}^{n \times m}$. Assume that A has "full rank", meaning that* $\mathrm{rank}(A) = \min(n, m) = m$.

1. *Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example)*
2. *Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).*

Problem 0.4. *Let $n < m$ and $A \in \mathbb{R}^{n \times m}$. Assume that A has "full rank", meaning that* $\mathrm{rank}(A) = \min(n, m) = n$.

1. *Does $Ax = b$ has a solution for all $b \in \mathbb{R}^n$? (Prove or give a counter example)*
2. *Can there exists two vectors $x \neq x'$ such that $Ax = Ax'$? (Prove or give a counter example).*

Problem 0.5. ***True or False****: There exists a family of k non-zero orthogonal vectors of $\mathbb{R}^n$, for some $k > n$.*

Problem 0.6. *Let $A \in \mathbb{R}^{n \times m}$.*

1. *Prove that* $\mathrm{Ker}(A^\mathsf{T})$ *and* $\mathrm{Im}(A)$ *are orthogonal to each other, i.e. that for all* $x \in \mathrm{Ker}(A^\mathsf{T})$ *and* $y \in \mathrm{Im}(A)$ *we have $x \perp y$.*
2. *Show that* $\mathrm{Ker}(A^\mathsf{T}) = \mathrm{Im}(A)^\perp$.

Problem 0.7. ***True or False****: The matrix of an orthogonal projection is symmetric.*

Problem 0.8. ***True or False****: The matrix of an orthogonal projection is orthogonal.*

Problem 0.9. *Let S be a subspace of $\mathbb{R}^n$ and let P_S be the orthogonal projection onto S. Show that* $\dim(S) = \mathrm{Tr}(P_S)$.

Problem 0.10. ***True or False***: *Let $A, B \in \mathbb{R}^{n \times n}$. Assume that $v \in \mathbb{R}^n$ is an eigenvector of A and B.*

1. *Is v an eigenvector of $A + B$?*
2. *Is v an eigenvector of AB ?*

Problem 0.11. *Let $A \in \mathbb{R}^{n \times n}$ and let $v_1, v_2 \in \mathbb{R}^n$ be two eigenvectors of A, associated with the same eigenvalue λ.*
Show that any non-zero eigenvector in $\mathrm{Span}(v_1, v_2)$ is an eigenvector of A, associated with λ.

Problem 0.12. *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Let $(v_1, v_2, \dots, v_n)$ be an orthonormal family of eigenvectors of A, associated to the eigenvalues $\lambda_1, \dots, \lambda_n$. Give an orthonormal basis of $\mathrm{Ker}(A)$ and $\mathrm{Im}(A)$ in terms of the v_i's.*

Problem 0.13. *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix, that satisfies $A^2 = \mathrm{Id}$. Show that the matrix*

$$M = \frac{1}{2}(A + \mathrm{Id})$$

is the matrix of an orthogonal projection.

Problem 0.14. *Let $\rho \in (0, 1)$. Let $v_1, \dots, v_k \in \mathbb{R}^n$ such that*

$$\|v_i\| = 1 \qquad \text{and} \qquad \langle v_i, v_j \rangle = \rho \qquad \text{for all} \quad i \neq j.$$

Show that $k \leq n$.