Recitation 6

Carles Domingo

Markov Chains

Definition (Markov chain)

A sequence of random variables $(X_0, X_1, \dots)$ is a Markov chain with state space E and "transition matrix" P if for all $t \geq 0$,

$$\mathbb{P}(X_{t+1} = y \,|\, X_0 = x_0, \dots, X_t = x_t) = P(x_t, y)$$

for all $x_0, \dots, x_t$ such that $\mathbb{P}(X_0 = x_0, \dots, X_t = x_t) > 0$.

Stochastic matrix: $P_{ij} \geq 0, \sum_{i=1}^{n} P_{ij} = 1$ for all $1 \leq j \leq n$.

Definition (Invariant measure)

A vector $\mu \in \Delta_n$ is called an invariant measure for the transition matrix P if $\mu = P\mu$, i.e. if μ is an eigenvector of P associated with the eigenvalue 1.

Perron-Frobenius theorem

Theorem (Perron-Frobenius, stochastic case)

Let P be a stochastic matrix such that there exists $k \geq 1$ such that all the entries of P^k are strictly positive. Then the following holds:

1. *1 is an eigenvalue of P and there exists an eigenvector $\mu \in \Delta_n$ associated to 1.*
2. *The eigenvectors associated to 1 are unique up to scalar multiple (i.e. $\mathrm{Ker}(P - \mathrm{Id}) = \mathrm{Span}(\mu)$).*
3. *For all $x \in \Delta_n$, $P^t x \xrightarrow[t \to \infty]{} \mu$.*

Is the condition "there exists $k \geq 1$ such that all the entries of P^k are strictly positive" necessary? Let's see!

Questions: Counterexamples

Definition (Irreducible Markov chain)

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

Definition (Aperiodic Markov chain)

If for all $1 \leq i \leq n$, we have gcd$(\{k | P_{ii}^k > 0\}) = 1$, we say that the Markov chain is aperiodic.

1. Show that if "there exists $k \geq 1$ such that all the entries of P^k are strictly positive", then the Markov chain is irreducible and aperiodic. The converse is also true but harder to prove (come to office hours if you want to know!).
2. Show that irreducible non-aperiodic Markov chains have no invariant measure.
3. Show that non-irreducible aperiodic Markov chains have several invariant measures.

Questions: Counterexamples

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Questions: Counterexamples

2. Show that irreducible non-aperiodic Markov chains have no invariant measure.

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3. Show an example of a non-irreducible aperiodic Markov chains that has several invariant measures.

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Questions: Counterexamples

4. Remember from the lecture that the PageRank algorithm actually computes the invariant measure of the transition matrix

$$G = \alpha P + \frac{1-\alpha}{N} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

with $\alpha \approx 0.85$. Given the previous questions, what would be the problems in taking $\alpha = 1$?

Questions: Counterexamples

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Spectral theorem

Theorem (Spectral theorem)

Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Then, A has n orthogonal eigenvectors $q_1, \ldots, q_n$ and we can write $A = Q\Lambda Q^\top$, where $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ and Λ is diagonal.

Remember that a matrix A is diagonalizable iff it has n linearly independent eigenvectors (equivalently $A = V\Lambda V^{-1}$). Thus, the spectral theorem says that symmetric matrices are diagonalizable in an orthogonal basis.

Questions: Spectral theorem

1. Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices. Show that $AB = BA$ iff A and B diagonalize in the same basis. Does the same hold if we just assume that A, B are diagonalizable?

Questions: Spectral theorem

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Questions: Spectral theorem

Theorem (Courant-Fischer principle)

The eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ of a symmetric matrix A are given by

$$\lambda_k = \max_{\substack{S \subset \mathbb{R}^n \\ \textit{dim}(S)=k}} \min_{\substack{x \in S \\ x \neq 0}} \frac{x^\top A x}{x^\top x} = \min_{\substack{S' \subset \mathbb{R}^n \\ \textit{dim}(S')=n-k+1}} \max_{\substack{x \in S' \\ x \neq 0}} \frac{x^\top A x}{x^\top x}$$

We will show this theorem. Seems like a lot, but we'll go step by step! We will only show the first equality as the argument for the second one is analogous.

Questions: Spectral theorem

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Let $v_1, \cdots, v_n$ be the orthogonal basis of eigenvectors (resp.).

(i) Show that for $S_k = \text{span}(v_1, \ldots, v_k)$, $\min_{x \in S, x \neq 0} \frac{x^\top A x}{x^\top x} \geq \lambda_k$.

(ii) Show that for all $x \in S'_k = \text{span}(v_k, \ldots, v_n)$ such that $x \neq 0$, we have $\frac{x^\top A x}{x^\top x} \leq \lambda_k$.

(iii) Check that for any subspace S with $\dim(S) = k$, $S \cup S'_k \neq \{0\}$.

(iv) Conclude.

Questions: Spectral theorem

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(iv) Conclude.

Extra question: Spectral theorem

For a symmetric matrix, show how you can use a modification of Gaussian elimination to find a decomposition $A = V\Lambda V^\top$ for some diagonal matrix Λ.