# Session 6: Eigenvalues, eigenvectors & Markov chains

Optimization and Computational Linear Algebra for Data Science

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- 1. Orthogonal matrices
- 2. Eigenvalues & eigenvectors
- 3. Properties of eigenvalues and eigenvectors
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# **Orthogonal matrices**

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## **Orthogonal matrices**

Orthogonal matrices

# Eigenvalues & eigenvectors

### Introduction

### **Definition**

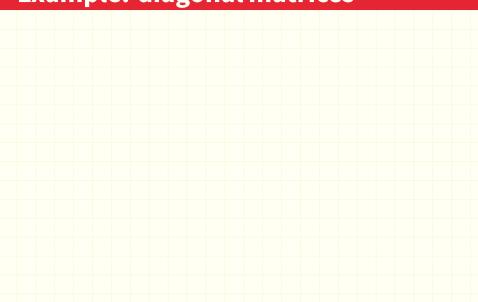
#### Definition

Let  $A\in\mathbb{R}^{n\times n}$ . A **non-zero** vector  $v\in\mathbb{R}^n$  is said to be an *eigenvector* of A is there exists  $\lambda\in\mathbb{R}$  such that

$$Av = \lambda v$$
.

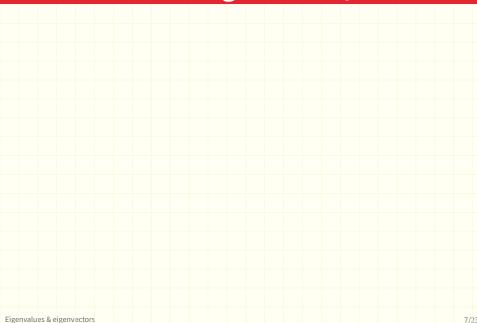
The scalar  $\lambda$  is called the eigenvalue (of A) associated to v.

# **Example: diagonal matrices**

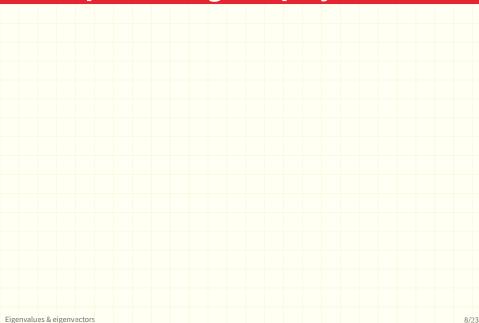


Eigenvalues & eigenvectors

### Matrix with no eigenvalues/vectors



# **Example: orthogonal projection**



### **Eigenspaces**

#### Definition

If  $\lambda \in \mathbb{R}$  is an eigenvalue of  $A \in \mathbb{R}^{n \times n}$ , the set

$$E_{\lambda}(A) = \{ x \in \mathbb{R}^n \, | \, Ax = \lambda x \}$$

is called the eigenspace of A associated to  $\lambda$ . The dimension of  $E_{\lambda}(A)$  is called the multiplicity of the eigenvalue  $\lambda$ .

# **Properties**

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Let  $A \in \mathbb{R}^{n \times n}$ . Suppose that A has an eigenvalue  $\lambda \in \mathbb{R}$  and let  $x \in \mathbb{R}^n$  be an eigenvector associated to  $\lambda$ .

#### Fact #1

For all  $\alpha\in\mathbb{R}$ ,  $\alpha\lambda$  is an eigenvalue of the matrix  $\alpha A$  and x is an associated eigenvector.

Let  $A \in \mathbb{R}^{n \times n}$ . Suppose that A has an eigenvalue  $\lambda \in \mathbb{R}$  and let  $x \in \mathbb{R}^n$  be an eigenvector associated to  $\lambda$ .

#### Fact #2

For all  $\alpha \in \mathbb{R}$ ,  $\lambda + \alpha$  is an eigenvalue of the matrix  $A + \alpha \mathrm{Id}$  and x is an associated eigenvector.

Let  $A \in \mathbb{R}^{n \times n}$ . Suppose that A has an eigenvalue  $\lambda \in \mathbb{R}$  and let  $x \in \mathbb{R}^n$  be an eigenvector associated to  $\lambda$ .

#### Fact #3

For all  $k \in \mathbb{N}$ ,  $\lambda^k$  is an eigenvalue of the matrix  $A^k$  and x is an associated eigenvector.

Let  $A \in \mathbb{R}^{n \times n}$ . Suppose that A has an eigenvalue  $\lambda \in \mathbb{R}$  and let  $x \in \mathbb{R}^n$  be an eigenvector associated to  $\lambda$ .

#### Fact #4

If A is invertible then  $1/\lambda$  is an eigenvalue of the matrix inverse  $A^{-1}$  and x is an associated eigenvector.

### **Spectrum**

#### Definition

The set of all eigenvalues of A is called the *spectrum* of A and denoted by  $\mathrm{Sp}(A)$ .

#### Theorem

A  $n \times n$  matrix A admits at most n different eigenvalues:

 $\#\operatorname{Sp}(A) \le n$ .

### **Proof that** $\#\mathrm{Sp}(A) \leq n$

#### Proposition

Let  $v_1, \ldots, v_k$  be eigenvectors of A corresponding (respectively) to the eigenvalues  $\lambda_1, \ldots, \lambda_k$ .

If the  $\lambda_i$  are all distinct  $(\lambda_i \neq \lambda_j \text{ for all } i \neq j)$  then the vectors  $v_1, \ldots, v_k$  are linearly independent.

Properties 13/23

### **Proof of the proposition**

Properties

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Properties

### **Even better!**

#### Theorem

A  $n \times n$  matrix A admits at most n different eigenvalues:  $\#\mathrm{Sp}(A) \leq n$ .

#### Theorem

Let  $A \in \mathbb{R}^{n \times n}$ . If  $\lambda_1, \dots, \lambda_k$  are distinct eigenvalues of A of multiplicities  $m_1, \dots, m_k$  respectively, then

$$m_1 + \cdots + m_k \le n$$
.

### **Example of application**

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## **Markov chains**

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### An example

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Markov chains

### **Stochastic matrices**

#### Definition

A matrix  $P \in \mathbb{R}^{n \times n}$  is said to be *stochastic* if:

- 1.  $P_{i,j} \ge 0$  for all  $1 \le i, j \le n$ .
- 2.  $\sum_{i=1}^{n} P_{i,j} = 1, \text{ for all } 1 \leq j \leq n.$

# **Probability vectors**

Markov chains

### The key equation

#### Proposition

For all 
$$t > 0$$

Markov chains

$$t \ge 0$$

$$x^{(t+1)} = Px^{(t)}$$
 and consequently,  $x^{(t)} = P^t x^{(0)}$ .

$$x \sim$$

# **Long-term behavior**

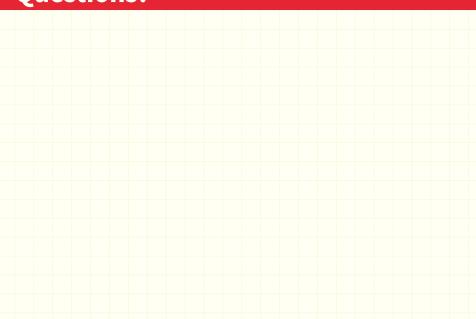
Markov chains

### Next week

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# **Questions?**



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