Lecture 2.3: Matrix product

Optimization and Computational Linear Algebra for Data Science

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Matrix-vector product

Consider a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ and its associated matrix $\widetilde{L} \in \mathbb{R}^{n \times m}$.

Question: Can we use the matrix $\widetilde{L}$ to compute the image $L(x)$ of a vector $x \in \mathbb{R}^m$?

Proposition

For all $x \in \mathbb{R}^m$ coordinates of $L(x) \in \mathbb{R}^n$ are given by the formula:

$$\big(L(x)\big)_i = \sum_{j=1}^{m} \widetilde{L}_{i,j}\, x_j \qquad \text{for all } \; i \in \{1, \ldots, n\}.$$

Visualizing the formula

$$\big(L(x)\big)_i = \sum_{j=1}^{m} \widetilde{L}_{i,j}\, x_j = \widetilde{L}_{i,1}\, x_1 + \widetilde{L}_{i,2}\, x_2 + \cdots + \widetilde{L}_{i,m}\, x_m$$

Justification of the formula

$$\big(L(x)\big)_i = \sum_{j=1}^{m} \widetilde{L}_{i,j}\, x_j = \widetilde{L}_{i,1}\, x_1 + \widetilde{L}_{i,2}\, x_2 + \cdots + \widetilde{L}_{i,m}\, x_m$$

Matrix-vector product

- We have seen: linear map $\rightarrow$ matrix
- We will see now: matrix $\rightarrow$ linear map

Definition

The linear map associated to a matrix $\widetilde{L} \in \mathbb{R}^{n\times m}$ is the map

$$\begin{array}{rccc} L: & \mathbb{R}^m & \rightarrow & \mathbb{R}^n \\ & x & \mapsto & \widetilde{L}x \end{array}$$

where the "matrix-vector" product $\widetilde{L}x \in \mathbb{R}^n$ is defined by

$$(\widetilde{L}x)_i = \sum_{j=1}^{m} \widetilde{L}_{i,j}\, x_j \qquad \text{for all } i \in \{1,\ldots,n\}.$$

Matrix product

Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{m\times k}$.

Definition - Proposition

- The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$.
- Its coefficients are given by the formula:

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell} M_{\ell,j} \quad \text{for all} \quad 1 \leq i \leq n, \quad 1 \leq j \leq k.$$

Visualizing the formula

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell}\, M_{\ell,j} = L_{i,1}\, M_{1,j} + \cdots + L_{i,m}\, M_{m,j}$$