Recitation 5

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Eigenvalues and eigenvectors

Definition

Let $A \in \mathbb{R}^{n \times n}$. A **non-zero** vector $v \in \mathbb{R}^n$ is said to be an *eigenvector* of A is there exists $\lambda \in \mathbb{R}$ such that

$$Av = \lambda v.$$

The scalar λ is called the *eigenvalue* (of A) associated to v. The set

$$E_\lambda(A) = \{x \in \mathbb{R}^n \,|\, Ax = \lambda x\} = \mathrm{Ker}(A - \lambda \mathrm{Id})$$

is called the *eigenspace* of A associated to λ. The dimension of $E_\lambda(A)$ is called the multiplicity of the eigenvalue λ.

Eigenvalues and eigenvectors

Recall:

- If a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_k$ with eigenvectors $v_1, \ldots, v_k$ resp., then $v_1, \ldots, v_k$ are linearly independent. $\implies$ A has at most n different eigenvalues.
- More strongly, if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_k$

$$\sum_{i=1}^{k} \mathrm{dim}(E_{\lambda_i}(A)) \leq n$$

Note: To compute eigenvalues and eigenvectors using determinants and characteristic polynomials, see Léo's video. Recommended but optional and not covered in this recitation.

Questions: Eigendecomposition

1. Let $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$. Show that a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $v_1, \ldots, v_k$ iff $A = V^{-1}\text{diag}((\lambda_i)_{i=1}^n)V$. In this case, we say that A is a *diagonalizable* matrix.
2. Show that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_n$, A is diagonalizable.
3. Write the expression of a matrix in $\mathbb{R}^{2 \times 2}$ for which $[2, -1]$ is an eigenvector of eigenvalue 2 and $[1, 3]$ is an eigenvector of eigenvalue -1.

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Questions: Eigenvalues & trace

1. Show that the trace is invariant by change of basis, i.e. if $X \in \mathbb{R}^{n\times n}$ is invertible and $A \in \mathbb{R}^{n\times n}$, $\text{tr}(A) = \text{tr}(XAX^{-1})$. (Hint: $\text{tr}(BC) = \text{tr}(CB)$).
2. Show that if $A \in \mathbb{R}^{n\times n}$ is diagonalizable and has eigenvalues eigenvalues $\lambda_1, \dots, \lambda_n$, then $\text{tr}(A) = \sum_{i=1}^n \lambda_i$.

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Questions: No (real) eigenvalues

Some matrices do not admit (real) eigenvalues and eigenvectors.

1. Show that if $\theta \in [0, 2\pi)$,

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   does not admit real eigenvalues and eigenvectors in general.

2. Find the matrices, and the real eigenvalues and eigenvectors for the values of θ for which they exist.

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Markov Chains

Definition (Markov chain)

A sequence of random variables $(X_0, X_1, \dots)$ is a Markov chain with state space E and "transition matrix" P if for all $t \geq 0$,

$$\mathbb{P}(X_{t+1} = y \,|\, X_0 = x_0, \dots, X_t = x_t) = P(x_t, y)$$

for all $x_0, \dots, x_t$ such that $\mathbb{P}(X_0 = x_0, \dots, X_t = x_t) > 0$.

Stochastic matrix: $P_{ij} \geq 0, \sum_{i=1}^{n} P_{ij} = 1$ for all $1 \leq j \leq n$.

Definition (Invariant measure)

A vector $\mu \in \Delta_n$ is called an invariant measure for the transition matrix P if $\mu = P\mu$, i.e. if μ is an eigenvector of P associated with the eigenvalue 1.

Perron-Frobenius theorem

Theorem (Perron-Frobenius, stochastic case)

Let P be a stochastic matrix such that there exists $k \geq 1$ such that all the entries of P^k are strictly positive. Then the following holds:

1. *1 is an eigenvalue of P and there exists an eigenvector $\mu \in \Delta_n$ associated to 1.*
2. *The eigenvectors associated to 1 are unique up to scalar multiple (i.e. $\mathrm{Ker}(P - \mathrm{Id}) = \mathrm{Span}(\mu)$).*
3. *For all $x \in \Delta_n$, $P^t x \xrightarrow[t \to \infty]{} \mu$.*

Is the condition "there exists $k \geq 1$ such that all the entries of P^k are strictly positive" necessary? Let's see!

Questions: Counterexamples

Definition (Irreducible Markov chain)

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

Definition (Aperiodic Markov chain)

If for all $1 \leq i \leq n$, we have $\gcd(\{k | P_{ii}^k > 0\}) = 1$, we say that the Markov chain is aperiodic.

1. Show that if "there exists $k \geq 1$ such that all the entries of P^k are strictly positive", then the Markov chain is irreducible and aperiodic. The converse is also true but harder to prove (come to office hours if you want to know!).
2. Show that irreducible non-aperiodic Markov chains have no invariant measure.
3. Show that non-irreducible aperiodic Markov chains have several invariant measures.

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