Recitation 2

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Review: Linear Transformations

Definition: Linear Transformation

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

Example: $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ -x + 4y \end{bmatrix} = x \begin{bmatrix} 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Questions 1: Linear Transformations

Which of the following functions are linear?

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solution

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Review: Matrices

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a $m \times n$ matrix which is an element of $\mathbb{R}^{m \times n}$. (Note the order!)

$$\mathbf{T} = \begin{array}{cc} & n \\ m & \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{pmatrix} \end{array}$$

- What does this mean? If $u_1, u_2, \dots, u_n$ is a basis of $\mathbb{R}^n$ and $v_1, v_2, \dots, v_m$ is a basis of $\mathbb{R}^m$, we have

$$\begin{aligned} T(u_1) &= T_{1,1}v_1 + T_{2,1}v_2 + \cdots + T_{m,1}v_m, \\ T(u_2) &= T_{1,2}v_1 + T_{2,2}v_2 + \cdots + T_{m,2}v_m, \\ &\dots \\ T(u_n) &= T_{1,n}v_1 + T_{2,n}v_2 + \cdots + T_{m,n}v_m. \end{aligned}$$

- Important: The matrix representation depends on the basis!

Review: Matrices

Example: $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x + y \\ -x + 4y \end{bmatrix} = x \begin{bmatrix} 3 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

We choose the canonical basis for both vector spaces:
$e_1 = (1, 0), e_2 = (0, 1)$.

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ -1 \end{bmatrix} = 3e_1 - 1e_2,$$
$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix} = 1e_1 + 4e_2.$$

$$\implies \mathbf{L} = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$$

Review: Matrix products

- The product of $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\mathbf{B} \in \mathbb{R}^{n \times p}$ is a matrix $\mathbf{AB} \in \mathbb{R}^{m \times p}$.
- **Example:** If

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \\ a_{31}b_{11} + a_{32}b_{21} & a_{31}b_{12} + a_{32}b_{22} \end{bmatrix}$$

- The matrix product is associative: $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. It is not commutative: in general $\mathbf{AB} \neq \mathbf{BA}$
- As we will see in the next exercise, matrix products are very useful to compose and evaluate linear transformation.

Questions 2: Matrix products

Let $x = (-2, 0, 3, 1) \in \mathbb{R}^4$. Let $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with matrix $\mathbf{A}$ in the canonical basis and let $B : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation with matrix $\mathbf{B}$ in the canonical, where

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

1. The matrix representation of the composition $A \circ B$ is $\mathbf{AB}$. Compute $\mathbf{AB}$.
2. Compute $(A \circ B)(x)$.

Questions 2: Matrix products

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Review: Kernel, Inverse

Definition: Kernel of a linear transformation

The kernel Ker(L) of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is the subset of $\mathbb{R}^m$ of vectors v such that $L(v) = 0$. Similarly, the kernel of a matrix $\mathbf{L} \in \mathbb{R}^{m \times n}$ is the subset of $\mathbb{R}^n$ of points x such that $\mathbf{L}x = 0$.

Definition: Invertible matrix, inverse

A matrix $M \in \mathbb{R}^{n \times n}$ is called *invertible* if there exists a matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix M^{-1} is unique and is called the *inverse* of M.

Questions 3: Kernel, Inverse

1. Prove that if $M \in \mathbb{R}^{n\times n}$ the matrix M^{-1} is indeed unique.
2. Prove that if $M \in \mathbb{R}^{n\times n}$ is invertible, then Ker$(M) = \{0\}$

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