

# Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

# Contents

1. Recap of the videos
2. More about the dimension
3. Coordinates
4. Why do we care about all these things ?  
Application to data science: image compression

# Logistics

# The teaching team

✚ **Lecturer:** Léo Miolane – *lm4271nyu.edu*  
`leomiolane.github.io/linalg-for-ds.html`

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[leomiolane.github.io/linalg-for-ds.html](https://leomiolane.github.io/linalg-for-ds.html)

❖ **Sections leaders:**

Alex



In person

Irina



Remote

Carles



Remote

# Course components

Three main components:

1. Videos

2-3 short videos to watch **before** each lecture

2. Lectures

Deepens the concepts introduced in the videos

3. Recitations

Practice!

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## Grades:

1. Weekly quizzes (5%)

2. Weekly homeworks (40%)

3. Exams: Midterm (20%) + Final (35%)

# Weekly timeline

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	15	16	17	18	19	20
21	22	23	24	25	26	27



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- ❖ Quizzes have to be answered on **Gradescope**.
- ❖ Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.

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- ❖ Midterm (~ mid-October) and Final will be «take-home exams».
- ❖ Limited time: after downloading the Midterm/Final questions, you will have to upload your work within few hours.

Check out the syllabus on the course webpage!

# Questions on logistics ?

# Vector spaces and subspaces

# Quick recap of video 1.2

A **vector space** is a set  $V$  endowed with two 'nice and compatible' operations  $+$  and  $\cdot$  that verify:

- ❖ For all  $u, v \in V$ ,  $u + v \in V$ .
- ❖ For all  $u \in V$  and all  $\lambda \in \mathbb{R}$ ,  $\lambda \cdot u \in V$ .

**Example:**  $V = \mathbb{R}^n$ , with the usual vector addition  $+$  and scalar multiplication  $\cdot$  is a vector space.

# Quick recap of video 1.2

A subset  $S$  of a vector space  $V$  is called a **subspace** if it is closed under addition and multiplication by a scalar.

**Example:** For all  $v \in \mathbb{R}^n$ ,

$$\text{Span}(v) = \{\lambda v \mid \lambda \in \mathbb{R}\}$$

is a subspace of  $\mathbb{R}^n$ .

# Question ?



# Review of Span and linear dependency

# Span

The *linear span* of vectors  $x_1, \dots, x_k$  as the set of all linear combinations of these vectors.

# Linear dependency

- ❖ Vectors  $x_1, \dots, x_k$  are *linearly dependent* if one of them can be expressed as a linear combination of the others.
- ❖ They are said to be *linearly independent* otherwise.

**Abuse of language:** Instead of saying « $x_1, \dots, x_k$  are linearly dependent», we should say «the family  $(x_1, \dots, x_k)$  is linearly dependent».

# Basis

A family  $(x_1, \dots, x_n)$  of vectors of  $V$  is a basis of  $V$  if

1.  $x_1, \dots, x_n$  are linearly independent,
2.  $\text{Span}(x_1, \dots, x_n) = V$ .

# The dimension

# A useful lemma

## Lemma

Let  $v_1, \dots, v_n \in V$  and let  $x_1, \dots, x_k \in \text{Span}(v_1, \dots, v_n)$ .  
Then, if  $k > n$ ,  $x_1, \dots, x_k$  are linearly dependent.

# Definition of the dimension

## Definition

We say that a vector space  $V$  has dimension  $n$  if it admits a basis  $(v_1, \dots, v_n)$  with  $n$  vectors.

# The dimension is well defined!

## Theorem

If  $V$  admits a basis  $(v_1, \dots, v_n)$ , then every basis of  $V$  has also  $n$  vectors. We say that  $V$  has dimension  $n$  and write  $\dim(V) = n$ .

**Proof.**





# Properties of the dimension

## Proposition

Let  $V$  be a vector space that has dimension  $\dim(V) = n$ . Then

- Any family of vectors of  $V$  that are linearly independent contains at most  $n$  vectors.

i.e. if  $x_1, \dots, x_k \in V$  are linearly independent, then  $k \leq n$ .

- Any family of vectors of  $V$  that spans  $V$  contains at least  $n$  vectors.

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# Properties of the dimension

## Proposition

Let  $V$  be a vector space of dimension  $n$  and let  $x_1, \dots, x_n \in V$ .

1. If  $x_1, \dots, x_n$  are linearly independent, then  $(x_1, \dots, x_n)$  is a basis of  $V$ .
2. If  $\text{Span}(x_1, \dots, x_n) = V$ , then  $(x_1, \dots, x_n)$  is a basis of  $V$ .

Very useful to show that a family of vector forms a basis:

**Example:**  $x_1 = (12, 37)$  and  $x_2 = (-9, 17)$  form a basis of  $\mathbb{R}^2$ .

**Proof of the Proposition.**



# An inequality

## Proposition

Let  $U$  and  $V$  be two subspaces of  $\mathbb{R}^n$ . Assume that  $U \subset V$ . Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover**  $\dim(U) = \dim(V)$ , then  $U = V$ .

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# A bit of vocabulary

## Definition

Let  $S$  be a subspace of  $\mathbb{R}^n$ .

- ❖ We call  $S$  a *line* if  $\dim(S) = 1$ .
- ❖ We call  $S$  a *hyperplane* if  $\dim(S) = n - 1$ .

# Coordinates

# Coordinates of a vector in a basis

## Definition

If  $(v_1, \dots, v_n)$  is a basis of  $V$ , then for every  $x \in V$  there exists a unique vector  $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$  such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

We say that  $(\alpha_1, \dots, \alpha_n)$  are the coordinates of  $x$  in the basis  $(v_1, \dots, v_n)$ .



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We say that  $(\alpha_1, \dots, \alpha_n)$  are the coordinates of  $x$  in the basis  $(v_1, \dots, v_n)$ .

**Proof.**



# Exercise

1. Show that the vectors  $v_1 = (1, 1)$  and  $v_2 = (1, -1)$  form a basis of  $\mathbb{R}^2$ .
2. Express the coordinates of  $u = (x, y)$  in the basis  $(v_1, v_2)$  in terms of  $x$  and  $y$ .

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# **Why do we care about this ?**

# Application to image compression

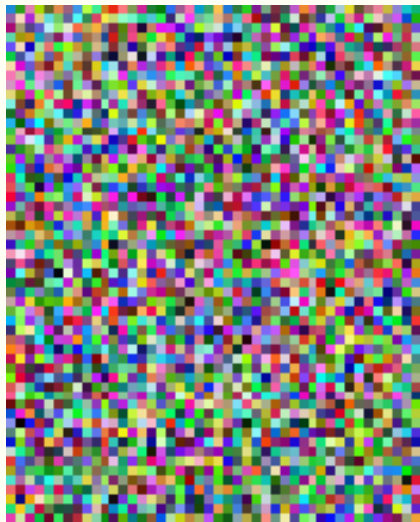
- Image = Grid of pixels
- Represented as a vector  $v \in \mathbb{R}^n$ , for some large  $n$ .
- One need to store  $n$  numbers.



$$n = 44 \times 55 = 2420$$

# Can we do better?

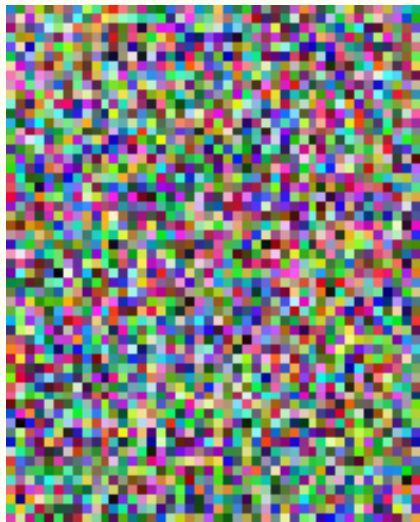
- ❖ If we want to store an arbitrary image, NO!



«Random» image

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- ❖ However, we are mainly storing images coming from the « real world »
- ❖ These images have some *structure*.



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# What do we mean by « structure » ?

Neighboring pixels are very likely to have similar colors.

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- ❖ There exists a basis  $(w_1, \dots, w_n)$  of  $\mathbb{R}^n$  in which «real» images  $v \in \mathbb{R}^n$  are (approximately) **sparse**.
- ❖ This means that the coordinates  $(\alpha_1, \dots, \alpha_n)$  of  $v$  in the basis  $(w_1, \dots, w_n)$  contains a lot of zeros.

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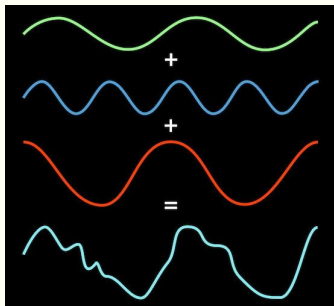
Store only the  $k \ll n$  non-zero coordinates of  $v$  (in the  $w_i$ 's basis' ) !

# A toy example

Consider  $n = 2$ , that is images  $v \in \mathbb{R}^2$  with only 2 pixels.

# Examples of good bases

- Fourier bases (used in .jpeg, .mp3)



- JPEG2000 uses **wavelet bases**, and achieves better performance than JPEG.
- In **Homework 4**, you will use wavelets to compress/denoise images.
- The course **DS-GA 1013** deepens these concepts!

# Questions?