Lecture 2.1: Linear transformations

Optimization and Computational Linear Algebra for Data Science

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Definition

Examples

You already know some linear transformations from high-school !

Symmetry	Rotation

Definition

Symmetries (about a line passing through the origin) and rotations (about the origin) are mappings

$$\begin{array}{rccc} L: & \mathbb{R}^2 & \to & \mathbb{R}^2 \\ & v & \mapsto & L(v), \end{array}$$

that are “linear”:

Definition

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

Examples of linear maps

- $$L : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x_1, x_2) & \mapsto & (2x_1, x_2) \end{array}$$ is linear

- $$M : \begin{array}{ccc} \mathbb{R}^3 & \to & \mathbb{R}^2 \\ (x_1, x_2, x_3) & \mapsto & (x_1 + x_3, 2x_2) \end{array}$$ is linear

An example of a non-linear map

The function $\begin{array}{rccc} F: & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & x^2 \end{array}$ is **not** linear.

Properties

Composition of linear maps

Proposition

If $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^k$ are both linear, then the composite function

$$\begin{array}{rccc} M \circ L : & \mathbb{R}^m & \to & \mathbb{R}^k \\ & v & \mapsto & M(L(v)) \end{array}$$

is also linear.

Proof.

$\square$

Basic properties

Proposition

If $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then

- $L(0) = 0$.
- $L\Big(\sum_{i=1}^{k} \alpha_i v_i\Big) = \sum_{i=1}^{k} \alpha_i L(v_i)$, for all $\alpha_i \in \mathbb{R}$, $v_i \in \mathbb{R}^m$.

Proof.

$\square$