# **Session 2: Linear transformations and matrices**

Optimization and Computational Linear Algebra for Data Science

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  Solving linear systems

# Linear maps & matrices

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Two sides of the sa	ame coin
Linear map	Matrix
$L: \mathbb{R}^m \to \mathbb{R}^n$	$L \in \mathbb{R}^{n \times m}$

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### **Rotations in** $\mathbb{R}^2$

Let  $\theta \in \mathbb{R}$ . The rotation  $R_{\theta} : \mathbb{R}^2 \to \mathbb{R}^2$  of angle  $\theta$  about the origin is linear.

**Exercise**: what is the canonical matrix of  $R_{ heta}$  ?

# **Operations on matrices**

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#### Addition and scalar multiplication

Sum of two matrices of the **same** dimensions:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,m} + b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & \cdots & a_{n,m} + b_{n,m} \end{pmatrix}$$

• Multiplication by a scalar  $\lambda$ :

$$\lambda \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} = \begin{pmatrix} \lambda a_{1,1} & \cdots & \lambda a_{1,m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \cdots & \lambda a_{n,m} \end{pmatrix}$$

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### A new vector space!

#### Proposition

- $\mathbb{R}^{n \times m}$  is a vector space.
- $\operatorname{dim}(\mathbb{R}^{n \times m}) =$

#### Proof.

Operations on matrices

#### Product of two matrices

#### Warning:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} \neq \begin{pmatrix} a_{1,1} \times b_{1,1} & \cdots & a_{1,m} \times b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} \times b_{n,1} & \cdots & a_{n,m} \times b_{n,m} \end{pmatrix}$$

Operations on matrices

#### **Matrix product**

Let  $L \in \mathbb{R}^{n \times m}$  and  $M \in \mathbb{R}^{m \times k}$ .

Definition (Matrix product)

The matrix product LM is the  $n \times k$  matrix of the linear map  $L \circ M$ .

### **Matrix product**

#### Theorem

Let  $L \in \mathbb{R}^{n \times m}$  and  $M \in \mathbb{R}^{m \times k}$ .

The entries matrix product LM are given by

$$(LM)_{i,j} = \sum_{i=1}^{m} L_{i,\ell} M_{\ell,j}, \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq k.$$

#### Proof

Operations on matrices

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### **Rotations in** $\mathbb{R}^2$

The  $R_a$  and  $R_b$  denote respectively the matrices of the rotations of angles a and b about the origin, in  $\mathbb{R}^2$ .

**Exercise**: Compute the product  $R_aR_b$ .

## **Matrix product properties**

Operations on matrices

# **Kernel and image**

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#### **Definitions**

Let  $L: \mathbb{R}^m \to \mathbb{R}^n$  be a linear transformation.

#### Definition (Kernel)

The kernel  $\mathrm{Ker}(L)$  (or nullspace) of L is defined as the set of all vectors  $v \in \mathbb{R}^m$  such that L(v) = 0, i.e.

$$\operatorname{Ker}(L) \stackrel{\text{def}}{=} \{ v \in \mathbb{R}^m \, | \, L(v) = 0 \}.$$

#### **Definition (Image)**

The image  $\operatorname{Im}(L)$  (or column space) of L is defined as the set of all vectors  $u \in \mathbb{R}^n$  such that there exists  $v \in \mathbb{R}^m$  such that L(v) = u.

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Kernel and image

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#### Remarks

Let  $L: \mathbb{R}^m \to \mathbb{R}^n$  be a linear transformation.

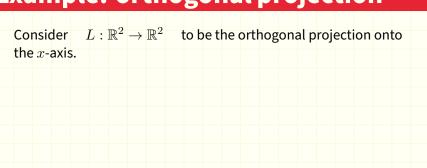
#### Proposition

- $ightharpoonup \operatorname{Ker}(L)$  is a subspace of  $\mathbb{R}^m$ .
- $ightharpoonup \operatorname{Im}(L)$  is a subspace of  $\mathbb{R}^n$ .

**Remark:**  ${\rm Im}(L)$  is also the Span of the columns of the matrix representation of L.

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### **Example: orthogonal projection**





# Why do we care about this?

### **Linear systems**

Why do we care about this?

Assume that we given a dataset:

$$a_i = (a_{i,1}, \dots, a_{i,m}) \in \mathbb{R}^m, \quad y_i \in \mathbb{R} \quad \text{for} \quad i = 1, \dots, n.$$

 $x_1a_{i,1} + \dots + x_ma_{i,m} = y_i$ 

We would like to find  $x \in \mathbb{R}^m$  such that

for all  $i \in \{1, \ldots, n\}$ .

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Why do we care about this?

Let's write

 $A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{ and } \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$ 

Why do we care about this?

Let's find all solutions!

### Conclusion: 3 possible cases

- 1.  $y \notin \operatorname{Im}(A)$ : there is no solution to Ax = y.
- 2.  $y \in \text{Im}(A)$ , then there exists  $x_0 \in \mathbb{R}^m$  such that  $Ax_0 = y$ . The set of solutions in then

$$S = \{x_0 + v \mid v \in \operatorname{Ker}(A)\}.$$

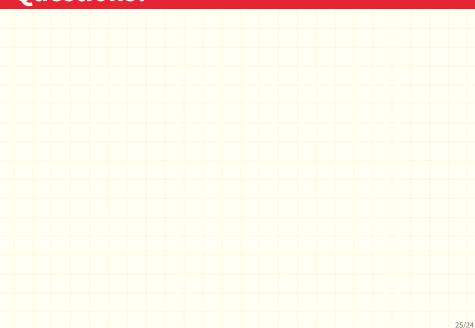
- If  $Ker(A) = \{0\}$ , then  $S = \{x_0\}$ :  $x_0$  is the unique solution.
- If  $Ker(A) \neq \{0\}$ , then Ker(A) contains infinitely many vectors: there are infinitely many solutions.

 $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \in \mathbb{R}^n.$ 

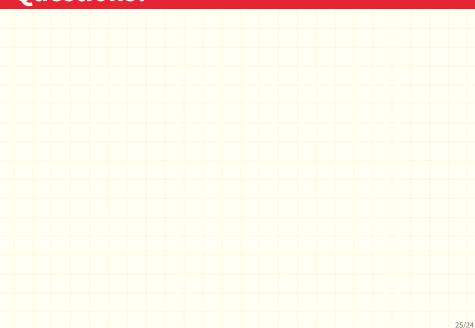
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Why do we care about this?

# **Questions?**



# **Questions?**



# **Questions?**

