Session 4: Norms and inner-products

Optimization and Computational Linear Algebra for Data Science

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Orthogonality

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Definition

Definition

- We say that vectors x and y are orthogonal if $\langle x,y\rangle=0$. We write then $x\perp y$.
- We say that a vector x is orthogonal to a set of vectors A if x is orthogonal to all the vectors in A. We write then $x \perp A$.

Exercise: If x is orthogonal to v_1, \ldots, v_k then x is orthogonal to any linear combination of these vectors i.e. $x \perp \operatorname{Span}(v_1, \ldots, v_k)$.

Orthogonality 4/17

Pythagorean Theorem

Theorem (Pythagorean theorem)

Let $x, y \in V$. Then

Proof.

Orthogonality

 $x \perp y \iff ||x + y||^2 = ||x||^2 + ||y||^2.$





| Appl | icatio | n to ran | dom variables | |
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Orthogonality

Orthogonal & orthonormal families

Definition

Let v_1,\ldots,v_k be vectors of V. We say that the family of vectors (v_1,\ldots,v_k) is

- orthogonal if the vectors v_1, \ldots, v_n are pairwise orthogonal, i.e. $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- orthonormal if it is orthogonal and if all the v_i have unit norm: $||v_1|| = \cdots = ||v_k|| = 1$.

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Coordinates in an orthonormal basis

Proposition

Assume that $\dim(V) = n$ and let (v_1, \ldots, v_n) be an **orthonormal** basis of V. Then the coordinates of a vector $x \in V$ in the basis (v_1, \ldots, v_n) are $(\langle v_1, x \rangle, \ldots, \langle v_n, x \rangle)$:

$$x = \langle v_1, x \rangle v_1 + \dots + \langle v_n, x \rangle v_n.$$

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Orthogonality

Orthogonal projection

Picture

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Orthogonal projection

Orthogonal projection and distance

Definition

Let S be a subspace of \mathbb{R}^n . The *orthogonal projection* of a vector x onto S is defined as the vector $P_S(x)$ in S that minimizes the distance to x:

$$P_S(x) \stackrel{\text{def}}{=} \underset{y \in S}{\operatorname{arg\,min}} \|x - y\|.$$

The distance of x to the subspace S is then defined as

$$d(x, S) \stackrel{\text{def}}{=} \min_{y \in S} ||x - y|| = ||x - P_S(x)||.$$

Orthogonal projection 12/17

Important proposition

Proposition

Let S be a subspace of \mathbb{R}^n and let (v_1, \dots, v_k) be an **orthonormal** basis of S. Then for all $x \in \mathbb{R}^n$,

$$P_S(x) = \langle v_1, x \rangle v_1 + \dots + \langle v_k, x \rangle v_k.$$

Consequence

Corollary

For all $x \in \mathbb{R}^n$,

- $x P_S(x)$ is orthogonal to S.
- $||P_S(x)|| \le ||x||.$



Cauchy-Schwarz inequality

Theorem

Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x,y\in V$:

$$|\langle x, y \rangle| \le ||x|| \, ||y||. \tag{1}$$

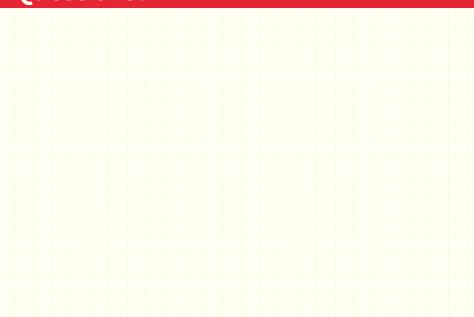
Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

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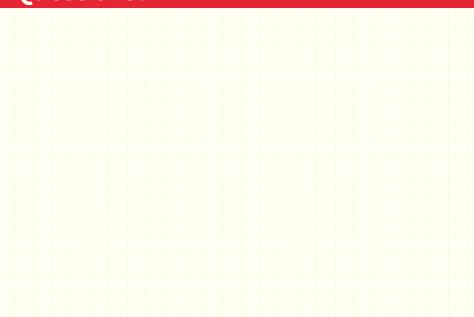
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Questions?



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