Session 2: Linear transformations and matrices

Optimization and Computational Linear Algebra for Data Science

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Linear maps & matrices

Two sides of the same coin

Linear map

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

Matrix

$$L \in \mathbb{R}^{n \times m}$$

Two sides of the same coin

Linear map

$$L : \mathbb{R}^m \to \mathbb{R}^n$$

Matrix

$$L \in \mathbb{R}^{n \times m}$$

Rotations in $\mathbb{R}^2$

Let $\theta \in \mathbb{R}$. The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of angle θ about the origin is linear.

Exercise: what is the canonical matrix of R_θ ?

Operations on matrices

Addition and scalar multiplication

Sum of two matrices of the **same** dimensions:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1}+b_{1,1} & \cdots & a_{1,m}+b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1}+b_{n,1} & \cdots & a_{n,m}+b_{n,m} \end{pmatrix}$$

Multiplication by a scalar λ:

$$\lambda \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} = \begin{pmatrix} \lambda a_{1,1} & \cdots & \lambda a_{1,m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \cdots & \lambda a_{n,m} \end{pmatrix}$$

Product of two matrices

Warning:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} \neq \begin{pmatrix} a_{1,1} \times b_{1,1} & \cdots & a_{1,m} \times b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} \times b_{n,1} & \cdots & a_{n,m} \times b_{n,m} \end{pmatrix}$$

Matrix product

Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{m\times k}$.

Definition (Matrix product)

The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$.

Theorem

The entries matrix product LM are given by

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell} M_{\ell,j}, \qquad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq k.$$

Rotations in $\mathbb{R}^2$

The R_a and R_b denote respectively the matrix of the rotation of angle a and b about the origin, in $\mathbb{R}^2$.

Exercise: Compute the product $R_a R_b$.

Matrix product properties

Kernel and image

Definitions

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation .

Definition (Kernel)

The kernel $\mathrm{Ker}(L)$ (or nullspace) of L is defined as the set of all vectors $v \in \mathbb{R}^m$ such that $L(v) = 0$, i.e.

$$\mathrm{Ker}(L) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^m \,|\, L(v) = 0\}.$$

Definition (Image)

The image $\mathrm{Im}(L)$ (or column space) of L is defined as the set of all vectors $u \in \mathbb{R}^n$ such that there exists $v \in \mathbb{R}^m$ such that $L(v) = u$.

Remark: $\mathrm{Im}(L)$ is also the Span of the columns of the matrix representation of L.

Picture

Example: orthogonal projection

Consider $L : \mathbb{R}^2 \to \mathbb{R}^2$ to be the orthogonal projection onto the x-axis.

Why do we care about this ?

Linear systems

Questions?