Optimization and Computational Linear Algebra for Data Science
Homework 5: Orthogonal matrices, eigenvalues and eigenvectors

Due on October 8, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 5.1 (2 points)**.** *Let S be a subspace of $\mathbb{R}^n$ and let P_S be the matrix of the orthogonal projection onto S. Let $M = \mathrm{Id}_n - 2P_S$.*

(a) *Show that the matrix M is orthogonal.*

(b) *Show that if $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda = 1$ or $\lambda = -1$.*

Problem 5.2 (2 points)**.** *Let $v \in \mathbb{R}^n$ be a non-zero vector. What are the eigenvalues of the $n \times n$ matrix*

$$M = vv^{\mathsf{T}}$$

and their multiplicities ? (In the expression vv^{T} we see v as a matrix with 1 column and n rows).

Problem 5.3 (2 points)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix.*
Show that if $v_1, v_2 \in \mathbb{R}^n$ are two eigenvectors of A associated to distinct eigenvalues $\lambda_1 \neq \lambda_2$ ($Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$), then $v_1 \perp v_2$.

Problem 5.4 (4 points)**.** *Let $A \in \mathbb{R}^{n\times n}$. We assume that there exists a basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of A:*

$$Av_i = \lambda_i v_i$$

for all $i \in \{1, \dots, n\}$. We assume that

$$\lambda_1 > |\lambda_i| \qquad \text{for all } i \in \{2, \dots, n\}.$$

We consider the following algorithm:

- *Initialize $x_0 \in \mathbb{R}^n$.*
- *Perform the updates: $x_{t+1} = \frac{Ax_t}{\|Ax_t\|}$.*

(**a**) *Show that for all* $t \geq 1$,

$$x_t = \frac{A^t x_0}{\|A^t x_0\|}.$$

(**b**) *Assume that* x_0 *is a unit vector* ($\|x_0\| = 1$) *whose direction is chosen uniformly at random (this basically means that all the possible directions for* x_0 *are equally likely to be chosen). Let* $(\alpha_1, \dots, \alpha_n)$ *be the coordinates of* x_0 *in the basis* $(v_1, \dots, v_n)$*. Explain we have* $\alpha_1 \neq 0$ *with probability* 1*. You do not have to do a rigorous proof of that, just give an intuitive argument.*

(**c**) *Show that*

$$x_t \xrightarrow[t\to\infty]{} \frac{\alpha_1 v_1}{\|\alpha_1 v_1\|} \qquad \text{and} \qquad \|Ax_t\| \xrightarrow[t\to\infty]{} \lambda_1.$$

Problem 5.5 ($\star$)**.** *Let* $A \in \mathbb{R}^{n \times n}$ *be a symmetric matrix. We define the function*

$$\begin{array}{rccc} f: & \mathbb{R}^n \setminus \{0\} & \to & \mathbb{R} \\ & x & \mapsto & \dfrac{x^\top A x}{x^\top x}. \end{array}$$

We admit that f *has a maximum at some* $x_\star \in \mathbb{R}^n \setminus \{0\}$*. Show that* $x_\star$ *verifies*

$$Ax_\star = \lambda x_\star, \qquad \text{where} \quad \lambda = f(x_\star).$$