Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

1. Subspaces
2. Linear dependency
3. Properties of the dimension
4. Coordinates
5. Why do we care about all these things ?

   Application to data science: image compression

Logistics

The teaching team

- **Lecturer**: Léo Miolane – *lm4271nyu.edu*
  `leomiolane.github.io/linalg-for-ds.html`
- **Course assistant**: Chen Li

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- **Sections leaders**:

Alex

Irina

Carles

Course components

Three main components:

1. Videos

   2-3 short videos to watch **before** each lecture

2. Lectures

   Deepens the concepts introduced in the videos

3. Recitations

   Practice!

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Grades:

1. Weekly quizzes (5%)
2. Weekly homeworks (40%)
3. Exams: Midterm (20%) + Final (35%)

Weekly timeline

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	15	16	17	18	19	20
21	22	23	24	25	26	27

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- Quizzes have to be answered on **WebAssign**.
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- Otherwise, you can scan (using dedicated app) your handwritten work. **It has to be legible!!!**
- Midterm ($\sim$ mid-October) and Final will be «take-home exams».
- Limited time: after downloading the Midterm/Final questions, you will have to upload your work within few hours.

Questions ?

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Subspaces

What are the subspaces of $\mathbb{R}^2$?

The span is always a subspace

Proposition

Let $x_1, \ldots, x_k \in V$. Then, $\mathrm{Span}(x_1, \ldots, x_k)$ is a subspace of V.

Linear dependency

A useful lemma

Lemma

Let $v_1, \ldots, v_n \in V$ and let $x_1, \ldots, x_k \in \mathrm{Span}(v_1, \ldots, v_n)$.
Then, if $k > n$, $x_1, \ldots, x_k$ are linearly dependent.

Abuse of language: Instead of saying «$x_1, \ldots, x_k$ are linearly dependent», we should have said «the family $(x_1, \ldots, x_k)$ is linearly dependent».

Basis, dimension

The dimension is well defined!

Theorem

If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.

Proof.

$\square$

Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$.

- Any family of vectors of V that spans V contains at least n vectors.

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Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If $x_1, \dots, x_n$ are linearly independent, then $(x_1, \dots, x_n)$ is a basis of V.
2. If $\mathrm{Span}(x_1, \dots, x_n) = V$, then $(x_1, \dots, x_n)$ is a basis of V.

Very useful to show that a family of vector forms a basis!

Proof.

$\square$

An inequality

Proposition

Let U and V be two subspaces of $\mathbb{R}^n$. Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

A bit of vocabulary

Definition

Let S be a subspace of $\mathbb{R}^n$.

- We call S a *line* if $\dim(S) = 1$.
- We call S an *hyperplane* if $\dim(S) = n - 1$.

Coordinates

Coordinates of a vector in a basis

Definition

If $(v_1, \ldots, v_n)$ is a basis of V, then for every $x \in V$ there exists a unique vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \ldots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \ldots, v_n)$.

Proof.

$\square$

Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of $\mathbb{R}^2$.
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y.

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Why do we care about this ?

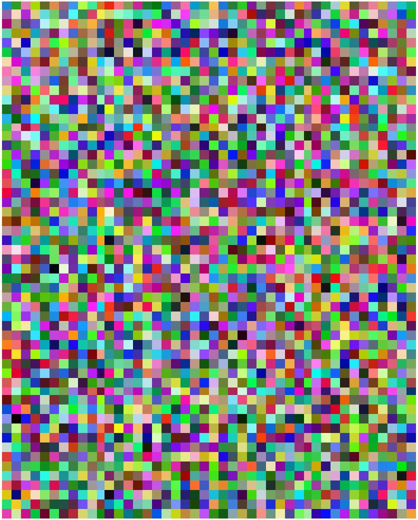
Application to image compression

- Image = Grid of pixels
- Represented as a vector $v \in \mathbb{R}^n$, for some large n.
- One need to store n numbers.

$n = 44 \times 55 = 2420$

Can we do better?

- If we were storing an arbitrary image, NO!



«Random» image

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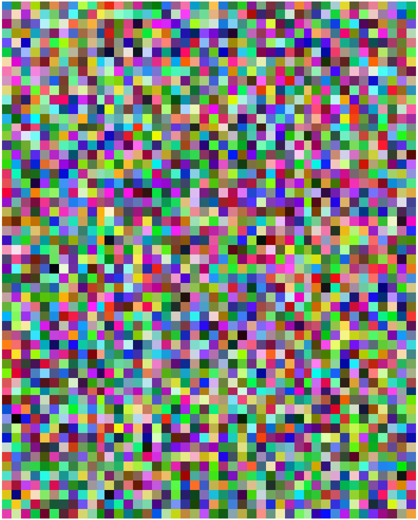
- These images have some *structure*.



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- There exists a basis $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$ in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- This means that the coordinates $(\alpha_1, \ldots, \alpha_n)$ of v in the basis $(w_1, \ldots, w_n)$ contains a lot of zeros.

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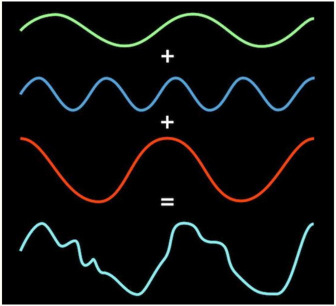
Store only the $k \ll n$ non-zero coordinates of v (in the w_i's basis') !

A toy example

Consider $n = 2$, that is images $v \in \mathbb{R}^2$ with only 2 pixels.

Examples of good bases

- **Fourier bases** (used in `.jpeg`, `.mp3`)



- `JPEG2000` uses **wavelet bases**, and achieves better performance than `JPEG`.
- In **Homework 4**, you will use wavelets to compress/denoise images.

Questions?