Optimization and Computational Linear Algebra for Data Science
Homework 6: Eigenvectors and Markov chains

Due on October 18, 2020

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a $(\star)$ are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 6.1 (2 points)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that A is orthogonal if and only if its eigenvalues all have absolute value 1 (i.e. are either $+1$ or -1).*

Problem 6.2 (3 points)**.** *We say that a symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive semi-definite if for all $x \in \mathbb{R}^n$,*

$$x^\mathsf{T} M x \geq 0.$$

(a) *Let $A \in \mathbb{R}^{n\times k}$. Show that AA^T is symmetric, positive semi-definite.*

(b) *Show that a symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive semi-definite if and only if all its eigenvalues are non-negative.*

(c) *Let $M \in \mathbb{R}^{n\times n}$ be a (symmetric) positive semi-definite matrix. Let $r = \operatorname{rank}(M)$. Show that there exists $A \in \mathbb{R}^{n\times r}$ such that $M = AA^\mathsf{T}$.*

Problem 6.3 (5 points)**.** *Read section 3 of the notes of lecture 6 (available on the course's webpage). Download the Jupyter notebook `tennis_rank.ipynb` and the two files `atp.csv` and `wta.csv`. These two files contain the outcome of all the tennis games on the professional circuit of the last two decades. Follow the instructions and questions on the notebook to find out who are the best players!*

Problem 6.4 $(\star)$**.** *Let $A \in \mathbb{R}^{n\times n}$ such that for all $x \in \mathbb{R}^n$, $x^\mathsf{T} A x \geq 0$. Show that* $\operatorname{Ker}(A) = \operatorname{Ker}(A^\mathsf{T})$.