Optimization and Computational Linear Algebra for Data Science
Common mistakes

Notations. Let $x \in \mathbb{R}^n$ and $A, B \in \mathbb{R}^{n \times n}$.

$$Ax = \mathrm{Im}(A), \quad \mathrm{Im}(A) + \mathrm{Ker}(A) = n, \quad \mathrm{Ker}(A) = 0.$$

The correct formulation would be $Ax \in \mathrm{Im}(A)$, $\dim(\mathrm{Im}(A)) + \dim(\mathrm{Ker}(A)) = n$, $\mathrm{Ker}(A) = \{0\}$.

Matrix multiplication #1.

$$\text{If} \quad A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix}, \quad \text{then} \quad A^2 = \begin{pmatrix} a_{1,1}^2 & \cdots & a_{1,m}^2 \\ \vdots & & \vdots \\ a_{n,1}^2 & \cdots & a_{n,m}^2 \end{pmatrix}.$$

Matrix multiplication #2. Let $A, B \in \mathbb{R}^{n \times n}$.

$$\text{If} \quad AB = 0 \quad \text{then} \quad A = 0 \quad \text{or} \quad B = 0.$$

This property (which is true when A, B are numbers) does not hold for matrices. Compute for instance

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Simplifications. Let $A \in \mathbb{R}^{k \times n}$ and $x, u \in \mathbb{R}^n$. Suppose that $Ax = Au$ and that $A \neq 0$ (A is not the all-zero matrix).

$$\text{Since} \quad Ax = Au \quad \text{one can simplify by } A\text{:} \quad \cancel{A}x = \cancel{A}u \quad \text{to get} \quad x = u.$$

This is false when $\mathrm{Ker}(A) \neq \{0\}$: take $u = 0$ and x a non-zero vector in $\mathrm{Ker}(A)$. If $\mathrm{Ker}(A) = \{0\}$, the right way to justify that $x = u$ is the following. $Ax = Au$ implies that $A(x - u) = 0$. Hence $x - u \in \mathrm{Ker}(A) = \{0\}$: $x = u$.

In the case where $k = n$ and A invertible, then one could simply multiply the equation $Ax = Au$ by A^{-1} on both sides to get that $x = u$.

Expanding the Euclidean norm. Let $u, v \in \mathbb{R}^n$

$$\|u + v\|^2 = \|u\|^2 + 2\|u\|\|v\| + \|v\|^2,$$

or

$$\|u + v\|^2 = u^2 + 2uv + v^2.$$

The first formula is not correct, while the second has no meaning (what does u^2 or uv mean when u, v are vectors?). The right formula is

$$\|u + v\|^2 = \|u\|^2 + 2\langle u, v \rangle + \|v\|^2,$$

where $\langle u, v \rangle$ is the dot product between u and v.

Invertible matrices. Let A be a $n \times m$ matrix such that $\mathrm{Ker}(A) = \{0\}$.

$$\text{Since } \mathrm{Ker}(A) = \{0\}, \quad A \text{ is invertible and } \mathrm{Im}(A) = \mathbb{R}^n.$$

This is only true in the case where $n = m$. Otherwise this makes no sense: an invertible matrix is **by definition a square matrix**. Moreover by the rank-nullity theorem $\dim(\mathrm{Im}(A)) = m - 0 = m$, which can be strictly less than n.

Matrix products. Let M be a symmetric matrix. By the spectral theorem there exists a diagonal matrix D and an orthogonal matrix P such that $M = PDP^{\mathsf{T}}$. Then we have

$$M = PDP^{\mathsf{T}} = PP^{\mathsf{T}}D,$$

and

$$M^{2019} = P^{2019}D^{2019}(P^{\mathsf{T}})^{2019}.$$

The first formula is false since for matrices A, B, $AB \neq BA$ in general. For the second, $(AB)^k = ABAB\dots AB \neq A^kB^k$ in general, since AB may be different from BA.