

Optimization and Computational Linear Algebra for Data Science

Homework 2: Linear transformations & matrices

Due on September 17, 2019

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- Unless otherwise stated, all answers must be mathematically justified.
 - Partial answers will be graded.
 - You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will not affect your grade).
 - Problems with a (\star) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
 - If you have any questions, feel free to contact me (1m4271@nyu.edu) or to stop at the office hours.
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Problem 2.1 (2 points). *Which of the following are linear transformations? Justify.*

(a) $T : \begin{cases} \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ (x, y) & \mapsto (x^2 + y^2, x - y) \end{cases}$

(b) $T : \begin{cases} \mathbb{R}^2 & \rightarrow \mathbb{R}^2 \\ (x, y) & \mapsto (x + y + 1, x - y) \end{cases}$

(c) $T : \begin{cases} \mathbb{R}^{n \times m} & \rightarrow \mathbb{R}^{m \times n} \\ A & \mapsto A^\top \end{cases} \quad \text{where } A^\top \text{ is transpose of } A, \text{ i.e. the } m \times n \text{ matrix defined by}$

$$(A^\top)_{i,j} = A_{j,i} \quad \text{for all } (i,j) \in \{1, \dots, m\} \times \{1, \dots, n\}.$$

(d) $T : \begin{cases} \mathbb{R}^{n \times n} & \rightarrow \mathbb{R} \\ A & \mapsto \text{Tr}(A) \end{cases} \quad \text{where } \text{Tr}(A) \text{ is the trace of the matrix } A, \text{ defined by}$

$$\text{Tr}(A) = \sum_{i=1}^n A_{i,i}.$$

Problem 2.2 (3 points). *Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation such that*

$$f(1, 2) = (1, 2, 3) \quad \text{and} \quad f(2, 2) = (1, 0, 1).$$

- (a) *Compute the matrix (canonically) associated to f .*
- (b) *Compute the set $\{x \in \mathbb{R}^2 \mid f(x) = (1, 4, 5)\}$.*
- (c) *Compute the set $\{x \in \mathbb{R}^2 \mid f(x) = (2, 4, 5)\}$.*

Problem 2.3 (2 points). Let $B \in \mathbb{R}^{3 \times 4}$ be a matrix with arbitrary entries:

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{pmatrix}.$$

Find two matrices A and C such that

$$ABC = \begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{pmatrix}$$

holds for any B defined above.

Problem 2.4 (3 points).

(a) Let A be a $n \times m$ matrix. Show that the image $\text{Im}(A)$ and the kernel $\text{Ker}(A)$ of A are subspaces of respectively \mathbb{R}^n and \mathbb{R}^m .

(b) Let

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Compute a basis of $\text{Ker}(A)$ and show that $\text{Im}(A) = \mathbb{R}^3$.

Problem 2.5 (\star). Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{k \times n}$. Prove that there exists a matrix $C \in \mathbb{R}^{m \times k}$ such that $A = CB$ if and only if $\text{Ker}(B)$ is a subspace of $\text{Ker}(A)$.

