

# Recitation 1

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# Announcements

Welcome!

- ▶ My Office hours: TBA (once I figure my schedule out)
- ▶ Rotating cohorts - if you are not scheduled to be in person, please join the online zoom call (w/ Carles or Irina)
- ▶ Recitations are for practice problems
- ▶ You will have time in recitation to solve the problems.
- ▶ Recitations will be released so you can look over/solve problems in advance.

# Why is Linear Algebra important?

Linear Algebra...

- ▶ Appears in ALL applied math, including *data science*
- ▶ Is solvable. If you can write it down, you can solve it!  
(not true for other math, e.g Diff Eq, Integrals)
- ▶ Is *fundamental* to understanding tools in machine learning

Relevant applications we will cover in the class:

- ▶ Linear Regression
- ▶ Principal Component Analysis
- ▶ Gradient Descent

First year MSDS students are *highly encouraged* to take this class.

# Concept Review: Vector Spaces

## Definition (Vector space)

$V$  is a vector space over field  $F$  when

1. Closure under Addition:  $x + y \in V$
2. Sum is commutative ( $x + y = y + x$ ) and associative  $x + (y + z) = (x + y) + z$
3. Additive Identity (in  $V$ ):  $0 \in V$  ( $x + 0 = x$ )
4. Additive Inverse: ( $\forall x, \exists -x$  s.t  $x + (-x) = 0$ )
5. Closure under Scalar Multiplication:  $\alpha x \in V$
6. Multiplicative Identity (in  $F$ ):  $\alpha x \in V$
7. Compatibility in Multiplication:  $\alpha(\beta x) = (\alpha\beta)x$
8. Distributivity:  $(\alpha + \beta)x = \alpha x + \beta \cdot \vec{y}$  and  $\alpha(x + y) = \alpha x + \alpha \cdot y$ .

Notice that “vectors” are not explicitly defined.

Optional: prove (after class) that  $\mathbb{R}^3$  with standard definitions for addition and scalar multiplication is vector space over field  $\mathbb{R}$ .

# Concept Review: Vector Spaces

- ▶ In this class,
  - ▶ Our field is always  $\mathbb{R}$ . ( $\mathbb{C}$  is also a field.)
  - ▶ Standard definitions for vector addition, and scalar multiplication.
  - ▶ But,  $V$  is (usually)  $\mathbb{R}^n$ , or (sometimes)  $\mathbb{R}^{n \times n}$ .
- ▶ Everything in linear algebra is in a vector space!
- ▶ (!) A recurring concept in data science is to “vectorize” problems
  - ▶ If you can transform/reframe your problem in linear algebra, you can (attempt) to solve it!

# Concept Review: Subspaces

## Definition (Subspace)

A subset  $S$  of a vector space  $V$  is a *subspace* if it is closed under addition and scalar multiplication.

1. Closure under Addition:  $x + y \in V$
2. Closure under Scalar Multiplication:  $\alpha x \in V$

- ▶ A subspace is also a vector space!
- ▶ Everything in linear algebra is in a vector space.
- ▶ Anything *interesting* in linear algebra is in a subspace.
- ▶ Subspaces are a recurring concept throughout this entire course.

# Questions 1: Subspaces, Span

Recall that  $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$  can be thought of as the  $xy$ -plane. Consider the two vectors  $v = (1, 1)$  and  $w = (-1, 2)$ . Describe the following sets geometrically. Which are subspaces of  $\mathbb{R}^2$ ?

1.  $\text{Span}(v)$
2.  $\text{Span}(v, w)$
3.  $\text{Span}(v) \cup \text{Span}(w)$ , that is, the vectors in  $\text{Span}(v)$  or  $\text{Span}(w)$
4.  $\text{Span}(v) \cap \text{Span}(w)$ , that is, the vectors in both  $\text{Span}(v)$  and  $\text{Span}(w)$
5.  $\{(1 - t)v + tw : t \in [0, 1]\}$
6.  $\{(1 - t)v + tw : t \in \mathbb{R}\}$
7.  $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$
8.  $\text{Span}(v, w, x)$  where  $x = (0, 5)$ .
9.  $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$
10.  $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
11.  $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
12.  $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

# Solutions 1: Subspaces, Span

Recall that  $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$  can be thought of as the  $xy$ -plane. Consider the two vectors  $v = (1, 1)$  and  $w = (-1, 2)$ . Describe the following sets geometrically. Which are subspaces of  $\mathbb{R}^2$ ?

- |  |       |
|--|-------|
| 1. $\text{Span}(v)$                                    | True  |
| 2. $\text{Span}(v, w)$                                 | True  |
| 3. $\text{Span}(v) \cup \text{Span}(w)$ ,              | False |
| 4. $\text{Span}(v) \cap \text{Span}(w)$ ,              | True  |
| 5. $\{(1 - t)v + tw : t \in [0, 1]\}$                  | False |
| 6. $\{(1 - t)v + tw : t \in \mathbb{R}\}$              | False |
| 7. $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$     | False |
| 8. $\text{Span}(v, w, x)$ where $x = (0, 5)$ .         | True  |
| 9. $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$   | False |
| 10. $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$   | True  |
| 11. $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ | False |
| 12. $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$   | False |



## Questions 2: Linear Independence, Bases,

1. Let  $v_1, v_2, v_3, v_4$  (all distinct)  $\in \mathbb{R}^3$ .

Let  $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$ .

If  $C_1$  and  $C_2$  are both linearly independent, what are the possible values for  $\dim(\text{Span}\{v_1, v_2, v_3, v_4\})$ ? (No formal proof necessary)

2. Let  $v_1, \dots, v_n \in \mathbb{R}^n$  be a basis of  $\mathbb{R}^n$ .

Prove that for  $x \in \mathbb{R}^n$ , there exists unique  $\alpha_1, \dots, \alpha_n$  such that  $x = \sum_{i=1}^n \alpha_i v_i$ .

3. Let  $v_1, \dots, v_m \in \mathbb{R}^n$  be linearly dependent.

Prove that for  $x \in \text{Span}(v_1, \dots, v_m)$ , there exist infinitely many  $\alpha_1, \dots, \alpha_m \in \mathbb{R}$  such that  $x = \sum_{i=1}^m \alpha_i v_i$ .

4. True or False: If  $B = \{v_1, \dots, v_n\}$  is a basis for  $\mathbb{R}^n$ , and  $W$  is a subspace of  $\mathbb{R}^n$ , then some subset of  $B$  is a basis for  $W$ .

## Solutions 2: Linear Independence, Bases

1. Let  $v_1, v_2, v_3, v_4$  (all distinct)  $\in \mathbb{R}^3$ .

Let  $C_1 = \{v_1, v_2\}$ ;  $C_2 = \{v_3, v_4\}$ .

If  $C_1$  and  $C_2$  are both linearly independent, what are the possible values for  $\dim(\text{Span}\{v_1, v_2, v_3, v_4\})$ ? (No formal proof necessary.)

### Solution

*Either  $C_1 \subset \text{Span}(C_2)$  and the dimension is 2, or the dimension is 3.*

## Solutions 2: Linear Independence, Bases

2. Let  $v_1, \dots, v_n \in \mathbb{R}^n$  be a basis of  $\mathbb{R}^n$ .

Prove that for  $x \in \mathbb{R}^n$ , there exists unique  $\alpha_1, \dots, \alpha_n$  such that  $x = \sum_{i=1}^n \alpha_i v_i$ .

### Solution

*By definition of basis,  $v_1, \dots, v_n$  is a linearly independent set, and spans  $\mathbb{R}^n$ . Since  $x \in \mathbb{R}^n$ ,*

$$\exists \alpha_1, \dots, \alpha_n \text{ s.t. } x = \sum_{i=1}^n \alpha_i v_i.$$

*Let  $\beta_1, \dots, \beta_n$  s.t.  $x = \sum_{i=1}^n \beta_i v_i$ . Then,*

$$x - x = 0 = \sum_{i=1}^n (\alpha_i - \beta_i) v_i$$

*Then by definition of linear independence,*

$$\alpha_i - \beta_i = 0 \quad \forall i \in 1, \dots, n$$

*So  $\alpha_i = \beta_i, \forall i \in 1, \dots, n$*

## Solutions 2: Linear Independence, Bases

3. Let  $v_1, \dots, v_m \in \mathbb{R}^n$  be linearly dependent.

Prove that for  $x \in \text{Span}(v_1, \dots, v_m)$ , there exist infinitely many  $\alpha_1, \dots, \alpha_m \in \mathbb{R}$  such that  $x = \sum_{i=1}^m \alpha_i v_i$ .

### Solution

By assumption,  $x \in \text{Span}(v_1, \dots, v_m)$ . So

$$\exists \beta_1, \dots, \beta_m \text{ s.t. } x = \sum_{i=1}^m \beta_i v_i$$

Since  $v_1, \dots, v_m$  are linearly dependent, there are  $\gamma_1, \dots, \gamma_m \in \mathbb{R}$  such that

$$\sum_{i=1}^m \gamma_i v_i = 0$$

where not all  $\gamma_i = 0$ .

Now, let  $r \in \mathbb{R}$ . Then,

$$x = x + 0 = \sum_{i=1}^m \beta_i v_i + r \sum_{i=1}^m \gamma_i v_i = \sum_{i=1}^m (\beta_i + r\gamma_i) v_i$$

This gives infinitely many distinct  $\alpha$  where  $\alpha_i = \beta_i + r\gamma_i$  for  $r \in \mathbb{R}$ .

## Solutions 2: Linear Independence, Bases

4. True or False: If  $B = \{v_1, \dots, v_n\}$  is a basis for  $\mathbb{R}^n$ , and  $W$  is a subspace of  $\mathbb{R}^n$ , then some subset of  $B$  is a basis for  $W$ .

### Solution

*False. Consider  $B = \{(1, 0), (0, 1)\}$  and  $W = \text{Span}((1, 1))$ .*

## Questions 3: Bases, Dimension

Let  $V$  be the set of functions

$$V := \{p : \mathbb{R} \rightarrow \mathbb{R} \mid p(x) = \sum_{k=0}^n a_k x^k\}$$

where  $a_0, \dots, a_n \in \mathbb{R}$ , and  $x \in \mathbb{R}$  is *constant*.

1. What kind of function does this set contain?
2. Define an addition operation  $+: V \times V \rightarrow V$ ,  
and a scalar multiplication operation  $\cdot: \mathbb{R} \times V \rightarrow V$ ,  
such that the triple  $(V, +, \cdot)$  is a vector space.
3. Find a basis for this vector space.
4. What is the dimension of this vector space?

## Solutions 3: Bases, Dimension

1. What kind of function does this set contain?
2. Define an addition operation  $+: V \times V \rightarrow V$ , and a scalar multiplication operation  $\cdot: \mathbb{R} \times V \rightarrow V$ , such that the triple  $(V, +, \cdot)$  is a vector space.
3. Find a basis for this vector space.
4. What is the dimension of this vector space?

### Solution

1. *Polynomials evaluated at  $x$*
2. *Standard definitions for function addition and scalar multiplication*
3.  $1, x, x^2, \dots, x^n$
4.  $n + 1$