Recitation 4

Carles Domingo

Norms

Definition (Norm)

A norm $\|\cdot\|$ on V is a function $\|\cdot\| : V \to \mathbb{R}_{\geq 0}$ that verifies

1. *Positive definiteness*: if $\|v\| = 0 \implies v = 0$.
2. *Homogeneity*: $\|\alpha v\| = |\alpha| \times \|v\|$
3. *Triangle inequality*: $\|u + v\| \leq \|u\| + \|v\|$

A norm defines a distance $d(x, y) = \|x - y\|$ on the vector space V.

Definition (Distance)

A distance d on a set S (not necessarily a vector space) is a function $d : S \times S \to \mathbb{R}$.

1. *Positive definiteness*: if $d(x, y) \geq 0$ for all $x, y \in S$ and $d(x, y) = 0$ iff $x = y$.
2. *Symmetry*: $d(x, y) = d(y, x)$ for all $x, y \in S$.
3. *Triangle inequality*: $d(x, y) \leq d(x, z) + d(z, y)$

Inner Products

Definition (Inner product)

Let V be a vector space. An inner product on V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$ that verifies the following points:

1. *Symmetry*: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.
2. *Linearity*: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.
3. *Positive definiteness*: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

- Important: An inner product defines a norm $\|u\| = \sqrt{\langle u, u \rangle}$.
- Most used inner product: the Euclidean inner product: $\langle u, v \rangle = u^\top v$. The corresponding norm is $\|u\| = \sqrt{u^\top u} = \sqrt{\sum_{i=1}^n u_i^2}$.
- Given an inner product, we can define the angle between two vectors: $cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$.

Questions: Norms & Inner Products

1. Which of the following functions are inner products for $x, y \in \mathbb{R}^3$?
   i. $f(x, y) = x_1 y_2 + x_2 y_3 + x_3 y_1$
   ii. $f(x, y) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_1^2 y_1^2$
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Questions: Norms & Inner Products

2. For $A \in \mathbb{R}^{m \times n}$ and $x \in \mathbb{R}^n$, prove that

$$\|Ax\| \leq \|x\| \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} A_{i,j}^2}$$

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Questions: Orthogonality

Recall from the lecture:

1. Two vectors u, v are orthogonal if $\langle u, v \rangle = 0$.
2. If U is a subspace of of a vector space V with inner product $\langle , \rangle$, the orthogonal projection $P_U : V \to U$ is defined as $x \mapsto \arg\min_{u \in U} \|x - u\|$.

Exercises:

1. Let $v_1, ..., v_k$ be a list of orthogonal vectors. Show that $v_1, ..., v_k$ are linearly independent.
2. Let U be the subspace of $\mathbb{R}^n$ with orthonormal basis $u_1, ..., u_k$.
   i. Prove that the orthogonal projection of $v \in \mathbb{R}^n$ onto U can be expressed as $P_U(v) = \sum_{i=0}^{k} \langle v, u_i \rangle u_i$. (Use the fact that the orthonormal basis for a subspace of $\mathbb{R}$ can be extended to obtain an orthonormal basis for $\mathbb{R}$).
   ii. Prove that $\|P_U(v)\| \leq \|v\|$.
   iii. Prove that $v - P_U(v)$ is orthogonal to $P_U(v)$.

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Questions: Orthogonal Complement

1. Let S, U be subspaces of a vector space V.
   Prove the following statement: $S \subset U \implies S^{\perp} \supset U^{\perp}$

2. Let $A \in \mathbb{R}^{n \times m}$. Assume the Euclidean inner product. Prove that $Im(A^T) = Ker(A)^{\perp}$.
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Questions: Idempotence

Definition (Idempotence)

An matrix P is idempotent when $P^2 = P$.

1. Show that $X(X^TX)^{-1}X^T$ is idempotent.
2. Show that all orthogonal projections are idempotent.
3. Give an example of an idempotent matrix that is not an orthogonal projection.
   (Hint: Show that your matrix does not minimize the distance to subspace it projects onto.)

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