Lecture 4.1: Inner product

Optimization and Computational Linear Algebra for Data Science

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The Euclidean dot product

Definition

We define the Euclidean dot product of two vectors x and y of $\mathbb{R}^n$ as:

$$x \cdot y = \sum_{i=1}^{n} x_i \, y_i.$$

Inner product

Let V be a vector space.

Definition

An inner product on V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$ that verifies the following points:

1. *Symmetry*: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.
2. *Linearity*: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.
3. *Positive definiteness*: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Other example

If V is the set of all random variables (on a probability space Ω) that have a finite second moment, then

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \mathbb{E}[XY]$$

is an inner product on V.

Norm induced by an inner product

Proposition

If $\langle \cdot, \cdot \rangle$ is an inner product on V then

$$\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$$

is a norm on V. We say that the norm $\| \cdot \|$ is induced by the inner product $\langle \cdot, \cdot \rangle$.

Cauchy Schwarz inequality

Theorem (Cauchy-Schwarz inequality)

Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x, y \in V$:

$$|\langle x, y\rangle| \leq \|x\| \, \|y\|. \tag{1}$$

Moreover, there is equality in (1) *if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.*

Examples

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