Recitation 1

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Concept Review: Vector Spaces

Definition

A **vector space** is a set V endowed with two 'nice and compatible' operations $+$ and $\cdot$ that verify:

- For all $u, v \in V$, $u + v \in V$.
- For all $u \in V$ and all $\lambda \in \mathbb{R}$, $\lambda \cdot u \in V$.

Example: $V = \mathbb{R}^n$, with the usual vector addition $+$ and scalar multiplication $\cdot$ is a vector space.

Concept Review: Vector Spaces

In this class:

- We will always consider *real* scalars. Note that it is also possible to consider *complex* scalars.

- V is (usually) $\mathbb{R}^n$, or (sometimes) $\mathbb{R}^{n \times m}$ (set of $n \times m$ matrices).

Concept Review: Subspaces

Definition (Subspace)

A non-empty subset S of a vector space V is called a *subspace* if it is closed under addition and scalar multiplication:

1. Closure under Addition: $x + y \in S$ for all $x, y \in S$.
2. Closure under Scalar Multiplication: $\alpha x \in S$, for all $x \in S$ and $\alpha \in \mathbb{R}$.

- A subspace is also a vector space!
- Subspaces are a recurring concept throughout this entire course.

Questions 1: Subspaces, Span

Consider the two vectors $v = (1, 1)$ and $w = (-1, 2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1. $\text{Span}(v)$
2. $\text{Span}(v, w)$
3. $\text{Span}(v) \cup \text{Span}(w)$, that is, the vectors in $\text{Span}(v)$ or $\text{Span}(w)$
4. $\text{Span}(v) \cap \text{Span}(w)$, that is, the vectors in both $\text{Span}(v)$ and $\text{Span}(w)$

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4. $\mathrm{Span}(v) \cap \mathrm{Span}(w)$, that is, the vectors in both $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$
5. $\{(1-t)v + tw | t \in [0, 1]\}$
6. $\{(1-t)v + tw | t \in \mathbb{R}\}$
7. $\{\alpha v + \beta w | \alpha, \beta \geq 0\}$
8. $\mathrm{Span}(v, w, u)$ where $u = (0, 5)$.
9. $\{(a, b) \in \mathbb{R}^2 | a^2 + b^2 \leq 25\}$
10. $\{(a, a) \in \mathbb{R}^2 | a \in \mathbb{R}\}$

Solution

Solution

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Questions 2: Linear Independence

1. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$. Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$.
   If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\mathrm{Span}(v_1, v_2, v_3, v_4))$? (No formal proof necessary)

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Questions 2: Linear Independence

2. Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
   Prove that for $x \in \operatorname{Span}(v_1, \ldots, v_m)$, there exist infinitely many $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$ such that
$$x = \sum_{i=1}^{m} \alpha_i v_i.$$

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Questions 2: Linear Independence

3. True or False: If $B = (v_1, \ldots, v_n)$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.

Questions 3: Bases, Dimension

Let V be the set of functions

$$V \stackrel{\text{def}}{=} \left\{ p : \mathbb{R} \to \mathbb{R} \,\middle|\, p(x) = \sum_{k=0}^{n} a_k x^k, \text{ for some } a_0, \ldots, a_n \in \mathbb{R} \right\}$$

1. What kind of function does this set contain?
2. Define an addition operation $+ : V \times V \to V$, and a scalar multiplication operation $\cdot : \mathbb{R} \times V \to V$, such that the triple $(V, +, \cdot)$ is a vector space.
3. What is the zero vector of this vector space?
4. Find a basis for this vector space.
5. What is the dimension of this vector space?

Solution

Solution

Solution

Solution