# Session 4: Norms and inner-products

Optimization and Computational Linear Algebra for Data Science

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# **Orthogonality**

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### **Definition**

#### Definition

- We say that vectors x and y are orthogonal if  $\langle x, y \rangle = 0$ . We write then  $x \perp y$ .
- We say that a vector x is orthogonal to a set of vectors A if x is orthogonal to all the vectors in A. We write then  $x \perp A$ .

**Exercise:** If x is orthogonal to  $v_1, \ldots, v_k$  then x is orthogonal to any linear combination of these vectors i.e.  $x \perp \operatorname{Span}(v_1, \ldots, v_k)$ .

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### Pythagorean Theorem

Theorem (Pythagorean theorem)

Let  $\|\cdot\|$  be the norm induced by  $\langle\cdot,\cdot\rangle$ . For all  $x,y\in V$  we have

$$x \perp y \iff ||x + y||^2 = ||x||^2 + ||y||^2.$$







App	licatio	n to rai	ndom var	iables

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Orthogonality

### **Orthogonal & orthonormal families**

#### Definition

We say that a family of vectors  $(v_1, \ldots, v_k)$  is:

- orthogonal if the vectors  $v_1, \ldots, v_n$  are pairwise orthogonal, i.e.  $\langle v_i, v_i \rangle = 0$  for all  $i \neq j$ .
- orthonormal if it is orthogonal and if all the  $v_i$  have unit norm:  $||v_1|| = \cdots = ||v_k|| = 1$ .

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### Coordinates in an orthonormal basis

#### **Proposition**

A vector space of finite dimension admits an orthonormal basis.

#### Proposition

Assume that  $\dim(V) = n$  and let  $(v_1, \ldots, v_n)$  be an **orthonormal** basis of V. Then the coordinates of a vector  $x \in V$  in the basis  $(v_1, \ldots, v_n)$  are  $(\langle v_1, x \rangle, \ldots, \langle v_n, x \rangle)$ :

$$x = \langle v_1, x \rangle v_1 + \dots + \langle v_n, x \rangle v_n.$$

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Orthogonality

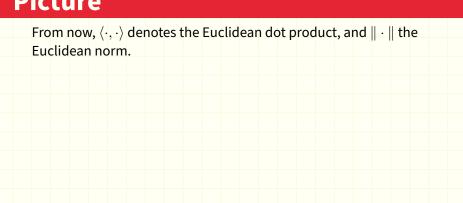
Orthogonality

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## **Orthogonal projection**

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### **Picture**



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Orthogonal projection

### Orthogonal projection and distance

#### Definition

Let S be a subspace of  $\mathbb{R}^n$ . The *orthogonal projection* of a vector x onto S is defined as the vector  $P_S(x)$  in S that minimizes the distance to x:

$$P_S(x) \stackrel{\text{def}}{=} \underset{y \in S}{\arg \min} \|x - y\|.$$

The distance of x to the subspace S is then defined as

$$d(x, S) \stackrel{\text{def}}{=} \min_{y \in S} ||x - y|| = ||x - P_S(x)||.$$

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### **Computing orthogonal projections**

#### **Proposition**

Let S be a subspace of  $\mathbb{R}^n$  and let  $(v_1, \dots, v_k)$  be an **orthonormal** basis of S. Then for all  $x \in \mathbb{R}^n$ ,

$$P_S(x) = \langle v_1, x \rangle v_1 + \dots + \langle v_k, x \rangle v_k.$$

### Proof

Orthogonal projection

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### Consequence

Orthogonal projection

### Consequence

#### Corollary

For all  $x \in \mathbb{R}^n$ ,

- $x P_S(x)$  is orthogonal to S.
- $||P_S(x)|| \le ||x||.$

# Proof of Cauchy-Schwarz inequality

### **Cauchy-Schwarz inequality**

#### Theorem

Let  $\|\cdot\|$  be the norm induced by the inner product  $\langle\cdot,\cdot\rangle$  on the vector space V. Then for all  $x,y\in V$ :

$$|\langle x, y \rangle| \le ||x|| \, ||y||. \tag{1}$$

Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e.  $x = \alpha y$  or  $y = \alpha x$  for some  $\alpha \in \mathbb{R}$ .

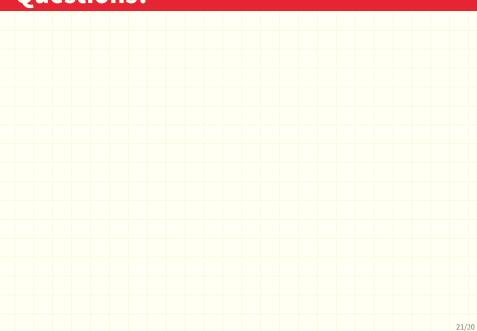
Proof of Cauchy-Schwarz inequality

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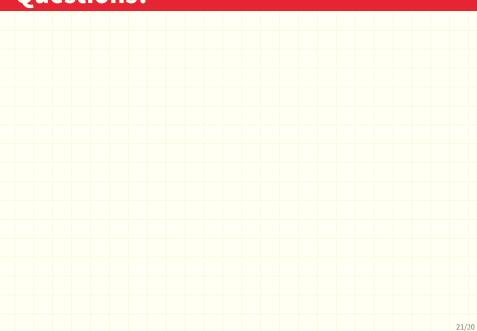
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### **Questions?**



### **Questions?**



### **Orthogonal matrices**

#### Definition

A matrix  $A \in \mathbb{R}^{n \times n}$  is called an *orthogonal matrix* if its columns are an orthonormal family.

### **A proposition**

#### Proposition

Let  $A \in \mathbb{R}^{n \times n}$ . The following points are equivalent:

- 1. A is orthogonal.
- $2. A^{\mathsf{T}} A = \mathrm{Id}_n.$
- $3. AA^{\mathsf{T}} = \mathrm{Id}_n$

### **Orthogonal matrices & norm**

#### **Proposition**

Let  $A\in\mathbb{R}^{n\times n}$  be an orthogonal matrix. Then A preserves the dot product in the sense that for all  $x,y\in\mathbb{R}^n$ ,

$$\langle Ax, Ay \rangle = \langle x, y \rangle.$$

In particular if we take x=y we see that A preserves the Euclidean norm:  $\|Ax\|=\|x\|$ .