Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

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Subspaces

What are the subspaces of $\mathbb{R}^2$?

The span is always a subspace

Proposition

Let $x_1, \ldots, x_k \in V$. Then, $\mathrm{Span}(x_1, \ldots, x_k)$ is a subspace of V.

Linear dependency

A useful lemma

Lemma

Let $v_1, \ldots, v_n \in V$ and let $x_1, \ldots, x_k \in \mathrm{Span}(v_1, \ldots, v_n)$.
Then, if $k > n$, $x_1, \ldots, x_k$ are linearly dependent.

Abuse of language: Instead of saying «$x_1, \ldots, x_k$ are linearly dependent», we should have said «the family $(x_1, \ldots, x_k)$ is linearly dependent».

Basis, dimension

The dimension is well defined!

Theorem

If V admits a basis $(v_1, \ldots, v_n)$, then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.

Proof.

$\square$

Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$.

- Any family of vectors of V that spans V contains at least n vectors.

  i.e. if $x_1, \dots, x_k \in V$ are such that $\mathrm{Span}(x_1, \dots, x_k) = V$, then $k \geq n$.

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Proof.

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Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If $x_1, \dots, x_n$ are linearly independent, then $(x_1, \dots, x_n)$ is a basis of V.
2. If $\mathrm{Span}(x_1, \dots, x_n) = V$, then $(x_1, \dots, x_n)$ is a basis of V.

Very useful to show that a family of vector forms a basis!

Proof.

$\square$

An inequality

Proposition

Let U and V be two subspaces of $\mathbb{R}^n$. Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

A bit of vocabulary

Definition

Let S be a subspace of $\mathbb{R}^n$.

- We call S a *line* if $\dim(S) = 1$.
- We call S an *hyperplane* if $\dim(S) = n - 1$.

Coordinates

Coordinates of a vector in a basis

Definition

If $(v_1, \ldots, v_n)$ is a basis of V, then for every $x \in V$ there exists a unique vector $(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \ldots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \ldots, v_n)$.

Proof.

$\square$

Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of $\mathbb{R}^2$.
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y.

Exercise

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Why do we care about this ?

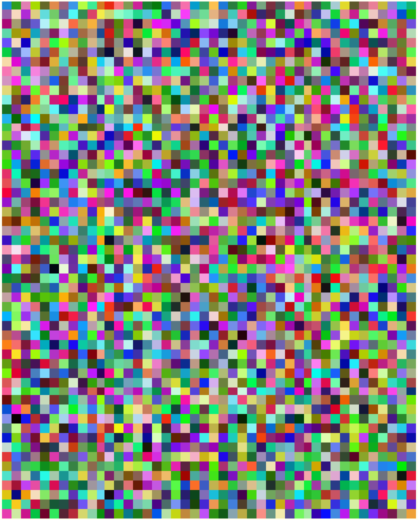
Application to image compression

- Image = Grid of pixels
- Represented as a vector $v \in \mathbb{R}^n$, for some large n.
- One need to store n numbers.

$n = 44 \times 55 = 2420$

Can we do better?

- If we were storing an arbitrary image, NO!



«Random» image

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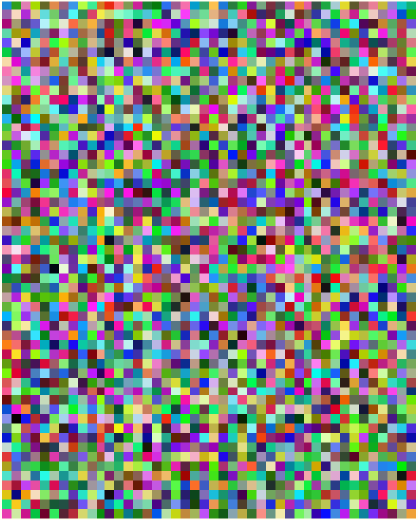
- However, we are mainly storing images coming from the « real world »
- These images have some *structure*.



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«Real» image

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- There exists a basis $(w_1, \ldots, w_n)$ of $\mathbb{R}^n$ in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- This means that the coordinates $(\alpha_1, \ldots, \alpha_n)$ of v in the basis $(w_1, \ldots, w_n)$ contains a lot of zeros.

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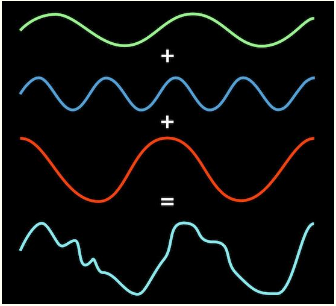
Store only the $k \ll n$ non-zero coordinates of v (in the w_i's basis') !

A toy example

Consider $n = 2$, that is images $v \in \mathbb{R}^2$ with only 2 pixels.

Examples of good bases

- **Fourier bases** (used in `.jpeg`, `.mp3`)



- `JPEG2000` uses **wavelet bases**, and achieves better performance than `JPEG`.
- In **Homework 4**, you will use wavelets to compress/denoise images.

Questions?