# Session 8: SVD, spectral clustering on graphs

Optimization and Computational Linear Algebra for Data Science

#### **Contents**

- 1. Singular Value Decomposition
- 2. Graphs and Graph Laplacian
- 3. Spectral clustering

#### Midterm next week

- Thu. Oct. 29, the questions have to be downloaded from Gradescope between 00:01 AM and 9:59 PM.
- **Duration:** 1 hour and 40 minutes to work on the problems + 20 minutes to scan and upload your work.
- Upload your work as a single PDF.
- In case the upload does not work for you, email me your work.

## Singular Value Decomposition

#### **Singular Value decomposition**

#### Theorem

Let  $A \in \mathbb{R}^{n \times m}$ . Then there exists two orthogonal matrices  $U \in \mathbb{R}^{n \times n}$  and  $V \in \mathbb{R}^{m \times m}$  and a matrix  $\Sigma \in \mathbb{R}^{n \times m}$  such that  $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$  and  $\Sigma_{i,j} = 0$  for  $i \neq j$ , that verify

$$A = U\Sigma V^{\mathsf{T}}.$$

Singular Value Decomposition

Remarks																			

## Low-rank approximation

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Singular Value Decomposition

## **Graphs and Graph Laplacian**

#### **Graphs**

Graphs and Graph Laplacian

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#### **Graph Laplacian**

#### Definition

The Laplacian matrix of  ${\cal G}$  is defined as

$$L = D - A.$$

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For all 
$$x \in \mathbb{R}^n$$
,  $x^T L x = \sum_{i \sim j} (x_i - x_j)^2$ .

### **Properties of the Laplacian**

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Graphs and Graph Laplacian

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Graphs and Graph Laplacian

#### **Algebraic connectivity**

#### Proposition

- The multiplicity of the eigenvalue 0 of L (i.e. the number of i such that  $\lambda_i = 0$ ) is equal to the number of connected components of G.
- In particular, G is connected if and only if  $\lambda_2 > 0$ .

- $\lambda_2$  is sometimes called the «algebraic connectivity» of G and measures somehow how well G is connected.
- From now, we assume that G is connected, i.e.  $\lambda_2 > 0$ .

**Exercise**: show that  $\lambda_2$  increases when one adds edges to G.

#### Spectral clustering algorithm

**Input:** Graph Laplacian L, number of clusters k

- 1. Compute the first k orthonormal eigenvectors  $v_1,\ldots,v_k$  of the Laplacian matrix L.
- 2. Associate to each node i the vector  $x_i = (v_2(i), \dots, v_k(i))$ .
- 3. Cluster the points  $x_1, \ldots, x_n$  with (for instance) the k-means algorithm.
- 4. Deduce a clustering of the nodes of the graph.

#### The case of two groups

#### For k=2 groups:

- 1. Compute the second eigenvector  $v_2$  of the Laplacian matrix L.
- 2. Associate to each node i the number  $x_i = v_2(i)$ .
- 3. Cluster the nodes in:

for some  $\delta \in \mathbb{R}$ .

$$S = \{i \mid v_2(i) \ge \delta\}$$
 and  $S^c = \{i \mid v_2(i) < \delta\},$ 

#### **Cut of a partition**

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## Minimal cut problem

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#### « Min-Cut » is NP-Hard

Spectral clustering with the Laplacian

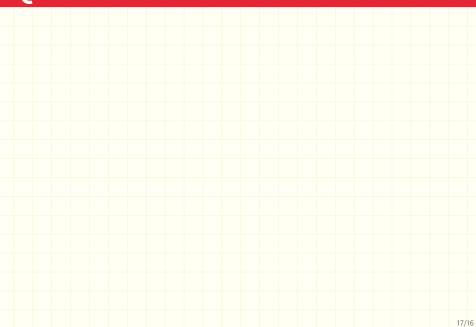
	Goal:			minimize			$x^{T}Lx$			SU	ıbje	ect 1	to	$\begin{cases} x \in \{-1, 1\}^n \\ x \perp (1, \dots, 1). \end{cases}$								

#### Spectral clustering as a «relaxation»

Idea: We first solve the « relaxed » problem:

minimize  $v^\mathsf{T} L v$  subject to  $\begin{cases} \|v\| = \sqrt{n} \\ v \perp (1, \dots, 1). \end{cases}$ 

## **Questions?**



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