

# Optimization and Computational Linear Algebra for Data Science

## Homework 9: Convex functions

Due on November 19, 2019

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- Unless otherwise stated, all answers must be mathematically justified.
  - Partial answers will be graded.
  - You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
  - Problems with a  $(\star)$  are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
  - If you have any questions, feel free to contact me ([lm4271@nyu.edu](mailto:lm4271@nyu.edu)) or to stop at the office hours.
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**Problem 9.1** (2 points). Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function. We assume that the minimum  $m \stackrel{\text{def}}{=} \min_{x \in \mathbb{R}^n} f(x)$  of  $f$  on  $\mathbb{R}^n$  is finite, and that the set of minimizers of  $f$

$$M \stackrel{\text{def}}{=} \{v \in \mathbb{R}^n \mid f(v) = m\}$$

is non-empty.

- (a) Show that  $M$  is a convex set.
- (b) Show that if  $f$  is strictly convex, then  $M$  has only one element.

**Problem 9.2** (2 points). Let  $M \in \mathbb{R}^{n \times n}$  be a symmetric matrix,  $b \in \mathbb{R}^n$  and  $c \in \mathbb{R}$ . For  $x \in \mathbb{R}^n$  we define

$$f(x) = x^\top Mx + \langle x, b \rangle + c.$$

$f$  is called a quadratic function.

- (a) Compute the gradient  $\nabla f(x)$  and the Hessian  $H_f(x)$  at all  $x \in \mathbb{R}^n$ . Show that  $f$  is convex if and only if  $M$  is positive semi-definite.
- (b) In this question, we assume  $M$  to be positive semi-definite. Show that  $f$  admits a minimizer if and only if  $b \in \text{Im}(A)$ .

**Problem 9.3** (3 points). We say that a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is strongly convex if there exists  $\alpha > 0$  such that the function  $x \mapsto f(x) - \frac{\alpha}{2}\|x\|^2$  is convex. In other words,  $f$  is strongly convex if there exists  $\alpha > 0$  and a convex function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$f(x) = g(x) + \frac{\alpha}{2}\|x\|^2.$$

- (a) Show that a strongly convex function is strictly convex.
- (b) Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}$  be a twice differentiable function. Show that  $g$  is strongly convex if and only if there exists  $\alpha > 0$  such that for all  $x \in \mathbb{R}^n$  the eigenvalues of  $H_\varphi(x)$  are greater or equal than  $\alpha$ .

**Problem 9.4** (3 points). Let  $A \in \mathbb{R}^{n \times m}$  and  $y \in \mathbb{R}^n$ . For  $x \in \mathbb{R}^m$  we define

$$f(x) = \|Ax - y\|^2.$$

- (a) Compute the gradient  $\nabla f(x)$  and the Hessian  $H_f(x)$  at all  $x \in \mathbb{R}^m$ . Show that  $f$  is convex.
- (b) Show that if  $\text{rank}(A) < m$ , then  $f$  is not strictly convex.
- (c) Show that if  $\text{rank}(A) = m$ , then  $f$  is strongly convex (use the definition and results of Problem 9.3).

**Problem 9.5** (★). **Notation:** For a symmetric matrix  $M \in \mathbb{R}^{n \times n}$  we denote respectively  $\lambda_{\min}(M)$  and  $\lambda_{\max}(M)$  the smallest and largest eigenvalue of  $M$ .

Let  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  a convex function that is twice differentiable. We assume that

$$\gamma \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}^n} \lambda_{\min}(H_f(x)) \quad \text{and} \quad L \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^n} \lambda_{\max}(H_f(x))$$

are both finite. Show that for all  $x, v \in \mathbb{R}^n$ :

$$f(x) + \langle \nabla f(x), h \rangle + \frac{\gamma}{2} \|h\|^2 \leq f(x+h) \leq f(x) + \langle \nabla f(x), h \rangle + \frac{L}{2} \|h\|^2.$$

