

Recitation 8

Alex Dong

CDS, NYU

Fall 2020

Singular Value Decomposition

- ▶ $A = U\Sigma V^T$
 - ▶ $A, \Sigma \in \mathbb{R}^{n \times m}$, $U \in \mathbb{R}^{n \times n}$, $V \in \mathbb{R}^{m \times m}$
 - ▶ U, V are orthogonal
 - ▶ (U, V not necessarily the same dimension)
 - ▶ Σ is “almost diagonal”.
 - ▶ All entries off the main diagonal are 0.
- ▶ (!!) *Every* matrix has a SVD.
- ▶ SVD provides a framework for understanding the properties of A .
- ▶ Can use the SVD to create matrices (Pseudo-Inverse, Lec 10).

Questions: SVD

1. Explain what the SVD says about the action of any matrix A .
2. Recall from Recitation 3 the proof we used to show

$$\text{rank}(A) = \text{rank}(A^T A).$$

Use the SVD to show that:

$$\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A^T).$$

Solutions 1: SVD

1. Explain what the SVD says about the action of any matrix A .

Solution

Let A have SVD $A = U\Sigma V^T$.

Transforming any vector x by A is equivalent to transforming it by V^T , Σ and then U .

Applying V^T is applying a rotation/flipping.

Applying Σ scales/squashes each (standard basis) axis, and also transforms $V^T x$ to a different space.

Applying U also applies a rotation/flipping.

Solutions 2: SVD

2. Recall from Recitation 3 the proof we used to show

$$\text{rank}(A) = \text{rank}(A^T A).$$

Use the SVD to show that:

$$\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A^T).$$

Solution

Recall that if B, C are invertible, then $\text{rank}(BAC) = \text{rank}(A)$.

$$A = U\Sigma V^T$$

$$A^T A = V\Sigma^T \Sigma V^T$$

$$AA^T = U\Sigma \Sigma^T U^T$$

$$A^T = V\Sigma^T U^T$$

Based on the definition of Σ , we can easily do the matrix multiplication, and see that:

$$\text{rank}(\Sigma) = \text{rank}(\Sigma^T \Sigma) = \text{rank}(\Sigma \Sigma^T) = \text{rank}(\Sigma^T).$$

We can then deduce that:

$$\text{rank}(A) = \text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A^T).$$

Questions: More SVD

1. Give basis transformations for U , Σ , V^T .
(Your answer should look like $T(a) = b$. where you select convenient a 's that form a basis for the origin space of T . Be careful about the dimensions w/ Σ !)
2. Let $x \in \mathbb{R}^m$ s.t $x = \sum_{i=1}^m \alpha_i v_i$, where v_i are the columns of V .
and write the expressions for $V^T x$, $\Sigma V^T x$, and $U \Sigma V^T x$.
3. Which vectors span the $Im(A)$? Which vectors span the $Ker(A)$?
Which vectors span $Im(A^T)$? Which vectors span $Ker(A^T)$?

Solutions 1: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.

Let $e_{i,m}, e_{i,n}$ denote the i th standard basis vector in \mathbb{R}^m , and \mathbb{R}^n .

1. Give basis transformations for U, Σ, V^T .

(Your answer should look like $T(a) = b$. where you select convenient a 's that form a basis for the origin space of T . Be careful about the dimensions w/ Σ !)

Solution

$$V^T(v_i) = e_{i,m} \quad \text{for } i \in \{1, \dots, m\}$$

$$\Sigma(e_{i,m}) = \sigma_i e_{i,n} \quad \text{for } i \in \{1, \dots, m\} \quad (\text{Note the } m \text{ here})$$

$$U(e_{i,n}) = u_i \quad \text{for } i \in \{1, \dots, n\}$$

Solutions 2: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.

Let $e_{i,m}, e_{i,n}$ denote the i th standard basis vector in \mathbb{R}^m , and \mathbb{R}^n .

2. Let $x \in \mathbb{R}^m$ s.t $x = \sum_{i=1}^m \alpha_i v_i$, where v_i are the columns of V .

Write the expressions for $V^T x$, $\Sigma V^T x$, and $U\Sigma V^T x$.

Solution

Let $e_{i,m}$ denote the i th standard basis vector in \mathbb{R}^m .

$$x = \sum_{i=1}^m \alpha_i v_i$$

$$V^T x = \sum_{i=1}^m \alpha_i e_{i,m}$$

$$\Sigma V^T x = \sum_{i=1}^{\min(m,n)} \sigma_i \alpha_i e_{i,n}$$

$$U\Sigma V^T x = \sum_{i=1}^{\min(m,n)} \sigma_i \alpha_i u_i$$

Solutions 3: More SVD

Let $A \in \mathbb{R}^{n \times m}$, where $n > m$, have SVD $A = U\Sigma V^T$.

Let $e_{i,m}, e_{i,n}$ denote the i th standard basis vector in \mathbb{R}^m , and \mathbb{R}^n .

3. Which vectors span the $Im(A)$? Which vectors span the $Ker(A)$?

Which vectors span $Im(A^T)$? Which vectors span $Ker(A^T)$?

Solution

Let $\sigma_1, \dots, \sigma_k > 0$, and $\sigma_{k+1}, \dots, \sigma_{\min(m,n)} = 0$.

Let u_1, \dots, u_n be the columns of U and v_1, \dots, v_m be the columns of V .

u_1, \dots, u_k span the $Im(A)$.

v_{k+1}, \dots, v_m span $Ker(A)$.

v_1, \dots, v_k span $Im(A^T)$.

u_{k+1}, \dots, u_n span $Ker(A^T)$.

Note the u 's and v 's!

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

1. Show that $M^T M$ is invertible.
2. Which vectors span the $Im(M)$? Write the matrix of orthogonal projection onto $Im(M)$ and give a basis transformation for that matrix.
3. Let $w \in \mathbb{R}^n$, and u be the orthogonal projection of w onto $Im(M)$. Show that $M^T u = M^T w$.
4. Show that $M(M^T M)^{-1} M^T$ is the matrix of an orthogonal projection onto $Im(M)$.

Solutions 1: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

Solution

1. Show that $M^T M$ is invertible.

“Questions: SVD” Question 2 earlier in recitation.

M is full rank in this case.

2. Which vectors span the $\text{Im}(M)$? Write the matrix of orthogonal projection onto $\text{Im}(M)$ and give a basis transformation for that matrix.

“Questions: More SVD” Question 3 earlier in recitation.

u_1, \dots, u_m span $\text{Im}(M)$. $P_{\text{Im}(M)} = U_r U_r^T$ where:

$$U_r = \begin{bmatrix} | & & | \\ u_1 & \dots & u_m \\ | & & | \end{bmatrix}$$

$$U_r U_r^T(u_i) = u_i \text{ for } i \in \{1, \dots, m\}.$$

$$U_r U_r^T(u_i) = 0 \text{ for } i \in \{m+1, \dots, n\}.$$

Solutions 2: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

3. Let $w \in \mathbb{R}^n$, and u be the orthogonal projection of w onto $\text{Im}(M)$. Show that $M^T u = M^T w$.

Solution

Let $w = \sum_{i=1}^n \alpha_i u_i$.

$u = P_{\text{Im}(M)}(w)$, so

$$u = U_r U_r^T w, \text{ and}$$

$$u = \sum_{i=1}^m \alpha_i u_i. \text{ (note change in summation).}$$

Now,

$$M^T u = \sum_{i=1}^m \sigma_i \alpha_i v_i, \text{ and}$$

$$M^T w = \sum_{i=1}^{\min(m,n)} \sigma_i \alpha_i v_i \quad (\text{From More SVD Q2})$$

Since $n > m$, $M^T u = M^T w$.

4. Show that $M(M^T M)^{-1} M^T$ is the matrix of an orthogonal projection onto $\text{Im}(M)$.

Solutions 2: SVD, Midterm, Regression

Midterm Q6 using SVD Let $M \in \mathbb{R}^{n \times m}$. Let $n > m$, and M have full rank. Let M have SVD $M = U\Sigma V^T$.

4. Show that $M(M^T M)^{-1}M^T$ is the matrix of an orthogonal projection onto $\text{Im}(M)$.

Solution

First, note that $(M^T M)^{-1} = (V\Sigma^T \Sigma V^T)^{-1}$, and that $\Sigma^T \Sigma$ has rank m and is invertible, where the diagonal entries of $(\Sigma^T \Sigma)^{-1}$ are reciprocals of the entries in $(\Sigma^T \Sigma)$.

Now,

$$M(M^T M)^{-1}M^T = (U\Sigma V^T)(V(\Sigma^T \Sigma)^{-1}V^T)(V\Sigma^T U^T)$$

$$M(M^T M)^{-1}M^T = U\Sigma(\Sigma^T \Sigma)^{-1}\Sigma^T U^T$$

Doing the matrix multiplication of $\Sigma(\Sigma^T \Sigma)^{-1}\Sigma^T$ gives

$$\Sigma(\Sigma^T \Sigma)^{-1}\Sigma^T = I_{m,n},$$

where $I_{m,n}$ is the $n \times n$ matrix where the first m diagonal entries are 1.

Finally, its easy to show that $UI_{m,n}U^T = U_r U_r^T$.

Concluding Remarks

- ▶ Throughout the course, we've emphasized viewing problems from different points of view
- ▶ Matrix multiplication
 - ▶ Inner product interpretation
 - ▶ Linear combination of columns interpretation
- ▶ Linear transformations
 - ▶ as matrices (matrix mechanics)
 - ▶ as letters (transformations on basis vectors)
- ▶ This recitation really emphasizes fluid switching between all of these frameworks
- ▶ All the things we've covered are deeply connected to each other.