

Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

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1. Subspaces
2. Linear dependency
3. Properties of the dimension
4. Coordinates
5. Why do we care about all these things ?

Application to data science: image compression

Logistics

The teaching team

❖ **Lecturer:** Léo Miolane – *lm4271nyu.edu*
`leomiolane.github.io/linalg-for-ds.html`

❖ **Course assistant:** Chen Li

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❖ **Course assistant:** Chen Li

❖ **Sections leaders:**

Alex



Irina



Carles



Course components

Three main components:

1. Videos

2-3 short videos to watch **before** each lecture

2. Lectures

Deepens the concepts introduced in the videos

3. Recitations

Practice!

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Practice!

Grades:

1. Weekly quizzes (5%)

2. Weekly homeworks (40%)

3. Exams: Midterm (20%) + Final (35%)

Weekly timeline

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	15	16	17	18	19	20
21	22	23	24	25	26	27

Grading

- ❖ Quizzes have to be answered on **WebAssign**.
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- ❖ Midterm (~ mid-October) and Final will be «take-home exams».
- ❖ Limited time: after downloading the Midterm/Final questions, you will have to upload your work within few hours.

Questions ?

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Subspaces

What are the subspaces of \mathbb{R}^2 ?

The span is always a subspace

Proposition

Let $x_1, \dots, x_k \in V$. Then, $\text{Span}(x_1, \dots, x_k)$ is a subspace of V .

Linear dependency

A useful lemma

Lemma

Let $v_1, \dots, v_n \in V$ and let $x_1, \dots, x_k \in \text{Span}(v_1, \dots, v_n)$.
Then, if $k > n$, x_1, \dots, x_k are linearly dependent.

Abuse of language: Instead of saying « x_1, \dots, x_k are linearly dependent», we should have said «the family (x_1, \dots, x_k) is linearly dependent».

Basis, dimension

The dimension is well defined!

Theorem

If V admits a basis (v_1, \dots, v_n) , then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.

Proof.



Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors.

i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$.

- Any family of vectors of V that spans V contains at least n vectors.

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Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If x_1, \dots, x_n are linearly independent, then (x_1, \dots, x_n) is a basis of V .
2. If $\text{Span}(x_1, \dots, x_n) = V$, then (x_1, \dots, x_n) is a basis of V .

Very useful to show that a family of vector forms a basis!

Proof.



An inequality

Proposition

Let U and V be two subspaces of \mathbb{R}^n . Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

A bit of vocabulary

Definition

Let S be a subspace of \mathbb{R}^n .

- ✚ We call S a *line* if $\dim(S) = 1$.
- ✚ We call S a *hyperplane* if $\dim(S) = n - 1$.

Coordinates

Coordinates of a vector in a basis

Definition

If (v_1, \dots, v_n) is a basis of V , then for every $x \in V$ there exists a unique vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

We say that $(\alpha_1, \dots, \alpha_n)$ are the coordinates of x in the basis (v_1, \dots, v_n) .

Proof.



Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of \mathbb{R}^2 .
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y .

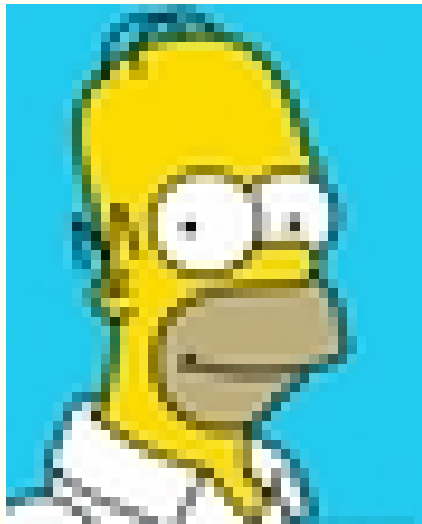
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Why do we care about this ?

Application to image compression

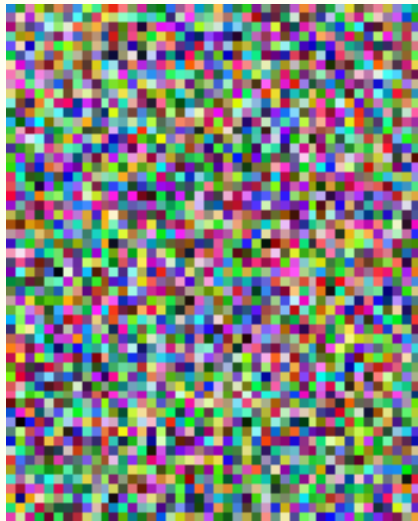
- Image = Grid of pixels
- Represented as a vector $v \in \mathbb{R}^n$, for some large n .
- One need to store n numbers.



$$n = 44 \times 55 = 2420$$

Can we do better?

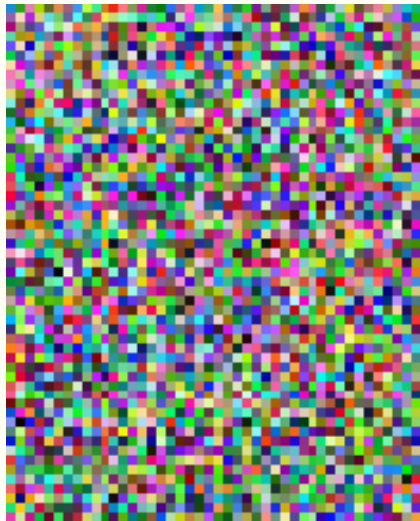
- ✚ If we were storing an arbitrary image, NO!



«Random» image

Can we do better?

- ❖ If we were storing an arbitrary image, NO!
- ❖ However, we are mainly storing images coming from the « real world »
- ❖ These images have some *structure*.



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What do we mean by « structure » ?

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- ❖ There exists a basis (w_1, \dots, w_n) of \mathbb{R}^n in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- ❖ This means that the coordinates $(\alpha_1, \dots, \alpha_n)$ of v in the basis (w_1, \dots, w_n) contains a lot of zeros.

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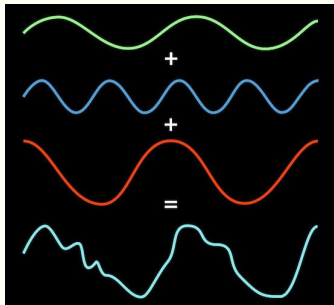
Store only the $k \ll n$ non-zero coordinates of v (in the w_i 's basis') !

A toy example

Consider $n = 2$, that is images $v \in \mathbb{R}^2$ with only 2 pixels.

Examples of good bases

- ❖ **Fourier bases** (used in .jpeg, .mp3)



- ❖ JPEG2000 uses **wavelet bases**, and achieves better performance than JPEG.
- ❖ In **Homework 4**, you will use wavelets to compress/denoise images.

Questions?