Session 6: Eigenvalues, eigenvectors & Markov chains

Optimization and Computational Linear Algebra for Data Science

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Introduction

Definition

Definition

Let $A \in \mathbb{R}^{n \times n}$. A **non-zero** vector $v \in \mathbb{R}^n$ is said to be an *eigenvector* of A is there exists $\lambda \in \mathbb{R}$ such that

$$Av = \lambda v.$$

The scalar λ is called the eigenvalue (of A) associated to v.

Example: diagonal matrices

Matrix with no eigenvalues/vectors

Example: orthogonal projection

Eigenspaces

Definition

If $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, the set

$$E_\lambda(A) = \{x \in \mathbb{R}^n \mid Ax = \lambda x\}$$

is called the eigenspace of A associated to λ. The dimension of $E_\lambda(A)$ is called the multiplicity of the eigenvalue λ.

Properties

Some useful facts

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ.

Fact #1

For all $\alpha \in \mathbb{R}$, $\alpha\lambda$ is an eigenvalue of the matrix αA and x is an associated eigenvector.

Some useful facts

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ.

Fact #2

For all $\alpha \in \mathbb{R}$, $\lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha \mathrm{Id}$ and x is an associated eigenvector.

Some useful facts

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ.

Fact #3

For all $k \in \mathbb{N}$, λ^k is an eigenvalue of the matrix A^k and x is an associated eigenvector.

Some useful facts

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ.

Fact #4

If A is invertible then $1/\lambda$ is an eigenvalue of the matrix inverse A^{-1} and x is an associated eigenvector.

Spectrum

Definition

The set of all eigenvalues of A is called the *spectrum* of A and denoted by $\mathrm{Sp}(A)$.

Theorem

A $n \times n$ matrix A admits at most n different eigenvalues: $\#\mathrm{Sp}(A) \leq n$.

Proof that $\#\mathrm{Sp}(A) \leq n$

Proposition

Let $v_1, \ldots, v_k$ be eigenvectors of A corresponding (respectively) to the eigenvalues $\lambda_1, \ldots, \lambda_k$.
If the λ_i are all distinct ($\lambda_i \neq \lambda_j$ for all $i \neq j$) then the vectors $v_1, \ldots, v_k$ are linearly independent.

Proof of the proposition

Proof of the proposition

Even better!

Theorem

A $n \times n$ matrix A admits at most n different eigenvalues: $\#\mathrm{Sp}(A) \leq n$.

Theorem

Let $A \in \mathbb{R}^{n \times n}$. If $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of A of multiplicities $m_1, \ldots, m_k$ respectively, then

$$m_1 + \cdots + m_k \leq n.$$

Example of application

Markov chains

An example

Stochastic matrices

Definition

A matrix $P \in \mathbb{R}^{n \times n}$ is said to be *stochastic* if:

1. $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$.
2. $\sum_{i=1}^{n} P_{i,j} = 1$, for all $1 \leq j \leq n$.

Probability vectors

The key equation

Proposition

For all $t \geq 0$

$$x^{(t+1)} = Px^{(t)} \quad \text{and consequently,} \quad x^{(t)} = P^t x^{(0)}.$$

Long-term behavior

Next week

Questions?

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