## **Session 3: The rank**

Optimization and Computational Linear Algebra for Data Science

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  Is the rank useful in practice?

## The rank

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## **Recap of the videos**

#### Definition

We define the rank of a family  $x_1, \ldots, x_k$  of vectors of  $\mathbb{R}^n$  as the dimension of its span:

$$\operatorname{rank}(x_1,\ldots,x_k) \stackrel{\text{def}}{=} \dim(\operatorname{Span}(x_1,\ldots,x_k)).$$

#### **Definition**

Let 
$$M \in \mathbb{R}^{n \times m}$$
. Let  $c_1, \dots, c_m \in \mathbb{R}^n$  be its columns. We define  $\operatorname{rank}(M) \stackrel{\mathrm{def}}{=} \operatorname{rank}(c_1, \dots, c_m) = \dim(\operatorname{Im}(M))$ .

### **Proposition**

Let  $M \in \mathbb{R}^{n \times m}$ . Let  $r_1, \dots, r_n \in \mathbb{R}^m$  be the rows of M and  $c_1, \dots, c_m \in \mathbb{R}^n$  be its columns. Then we have  $\operatorname{rank}(r_1, \dots, r_n) = \operatorname{rank}(c_1, \dots, c_m) = \operatorname{rank}(M)$ .

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### How do we compute the rank?

For  $v_1, \ldots, v_k \in \mathbb{R}^n$ , and  $\alpha \in \mathbb{R} \setminus \{0\}$ ,  $\beta \in \mathbb{R}$  we have

$$rank(v_1, ..., v_k) = rank(v_1, ..., v_{i-1}, \alpha v_i, v_{i+1}, ..., v_k)$$
  
= rank(v\_1, ..., v\_{i-1}, v\_i + \beta v\_j, v\_{i+1}, ..., v\_k)

As a consequence, the Gaussian elimination method keeps the rank of a matrix unchanged!

The rank

Let's compute the rank of  $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix}$ 

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The rank

**Example** 

## Example

The rank

Examp			

## The rank-nullity Theorem

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## **Rank-nullity Theorem**

#### Theorem

Let  $L: \mathbb{R}^m \to \mathbb{R}^n$  be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

### Proof sketch on an example

Let us solve the linear system Ax = 0.

$$\begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -1 & 5 & 2 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 4 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} (R_1) \\ (R_2) - 2(R_1) \\ (R_3) + (R_1) \end{pmatrix}$$

$$\begin{pmatrix}
1 & -1 & 0 & 1 & 0 \\
0 & 2 & 1 & -3 & 0 \\
0 & 0 & 0 & 7 & 0
\end{pmatrix}
\begin{pmatrix}
(R_1) \\
(R_2) \\
(R_3) - 2(R_2)$$

The rank-nullity Theorem

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The rank-nullity Theorem

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The rank-nullity Theorem

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The rank-nullity Theorem

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The rank-nullity Theorem

## **Invertible matrices**

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### **Invertible matrices**

#### Definition (Matrix inverse)

A **square** matrix  $M \in \mathbb{R}^{n \times n}$  is called *invertible* if there exists a matrix  $M^{-1} \in \mathbb{R}^{n \times n}$  such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix  $M^{-1}$  is unique and is called the *inverse* of M.

**Exercise**: Let  $A, B \in \mathbb{R}^{n \times n}$ . Show that if  $AB = \mathrm{Id}_n$  then  $BA = \mathrm{Id}_n$ .

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### **Invertible matrices**

#### Theorem

Let  $M \in \mathbb{R}^{n \times n}$ . The following points are equivalent:

- 1. *M* is invertible.
- 2. For all  $y \in \mathbb{R}^n$ , there exists a unique  $x \in \mathbb{R}^n$  such that Mx = y.
- 3.  $\operatorname{rank}(M) = n$ .
- 4.  $Ker(M) = \{0\}.$

Invertible matrices

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Invertible matrices

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Invertible matrices

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Invertible matrices

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## Transpose of a matrix

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## **Transpose of a matrix**

#### Definition

Let  $M \in \mathbb{R}^{n \times m}$ . We define its  $transpose\ M^\mathsf{T} \in \mathbb{R}^{m \times n}$  by

$$(M^{\mathsf{T}})_{i,j} = M_{j,i}$$

for all  $i \in \{1, ..., m\}$  and  $j \in \{1, ..., n\}$ .

#### Remark:

- We have  $(M^{\mathsf{T}})^{\mathsf{T}} = M$ .
- The mapping  $M \mapsto M^{\mathsf{T}}$  is linear.

## Properties of the transpose

#### Proposition

For all 
$$A \in \mathbb{R}^{n \times m}$$
,  $\operatorname{rank}(A) = \operatorname{rank}(A^{\mathsf{T}})$ .

## Proposition

Let 
$$A \in \mathbb{R}^{n \times m}$$
 and  $B \in \mathbb{R}^{m \times k}$ . Then 
$$(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}.$$





## **Symmetric matrices**

#### Definition

A square matrix  $A \in \mathbb{R}^{n \times n}$  is said to be symmetric if

$$\forall i, j \in \{1, \dots, n\}, A_{i,j} = A_{j,i}$$

or, equivalently if  $A = A^{\mathsf{T}}$ .

**Remark**: For all  $M \in \mathbb{R}^{n \times m}$  the matrix  $MM^{\mathsf{T}}$  is symmetric.

# Is the rank useful in practice?

## Back to the movies ratings example

Assume that you are given the matrix of movies ratings:

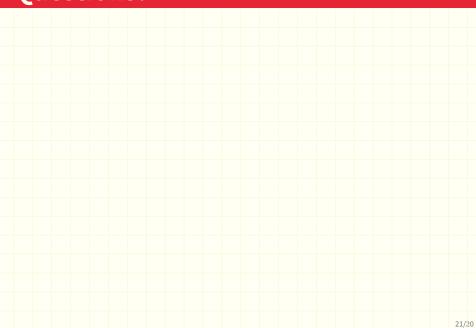
$$\begin{pmatrix} 1 & 1 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1.001 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0.0001 & 0 \\ 2.0001 & 2 & 2 & 0 & 0 \end{pmatrix}$$

**Goal:** how many different « user profiles » do we have ?

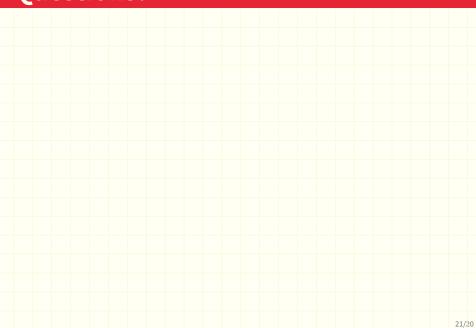
### **Conclusions**

- The rank is not «robust»!
- We need to have a way to check if a matrix is «approximately low-rank».
- Equivalentely, given m vectors, one would like to be able to see if there exists a subspace of dimension  $k \ll m$  from which the vectors are « not far ».
- The singular value decomposition (lecture 6-7) will solves our problems!

## **Questions?**



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## **Questions?**

