

Session 1: Vector spaces

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

1. Recap of the videos
2. More about the dimension
3. Coordinates
4. Why do we care about all these things ?
Application to data science: image compression

Logistics

The teaching team

❖ **Lecturer:** Léo Miolane – lm4271nyu.edu
leomiolane.github.io/linalg-for-ds.html

❖ **Sections leaders:**

Alex



In person

Irina



Remote

Carles



Remote

Course components

Three main components:

1. Videos

2-3 short videos to watch **before** each lecture

2. Lectures

Deepens the concepts introduced in the videos

3. Recitations

Practice!

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Grades:

1. Weekly quizzes (5%)

2. Weekly homeworks (40%)

3. Exams: Midterm (20%) + Final (35%)

Weekly timeline

Mon	Tue	Wed	Thu	Fri	Sat	Sun
14	15	16	17	18	19	20
21	22	23	24	25	26	27

Weekly Quizzes and Homeworks

- ❖ Quizzes have to be answered on **Gradescope**, after viewing the videos, but before the associated lecture.
- ❖ Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.

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- ❖ Homeworks questions are available on the **course's webpage** and have to be submitted on **Gradescope**.
- ❖ I encourage you to type your homeworks using LaTeX.
Some instructions and template available on the course's webpage.
- ❖ Otherwise, you can scan (using dedicated app) your handwritten work. **It has to be legible!!!**

Gradescope

DS-GA 1014

Fall 2020

Entry Code: **M2ND83**

DESCRIPTION

Edit your course description on the [Course Settings](#) page.

THINGS TO DO



Review and publish grades for [Quiz 1](#) now that you're all done grading.

◆ ACTIVE ASSIGNMENTS	RELEASED	DUE (EDT) ▼	◆ SUBMISSIONS	% GRADED ◆	PUBLISHED	REGRADES
Homework 1	<div><div></div></div> SEP 02	SEP 20 AT 11:00PM	0	<div><div></div></div> 0%	<input type="radio"/>	ON ⋮
Quiz 2	<div><div></div></div> SEP 03	SEP 10 AT 2:00PM	0	<div><div></div></div> 0%	<input type="radio"/>	ON ⋮
Quiz 1	<div><div></div></div> AUG 23	SEP 10 AT 2:00PM	4	<div><div></div></div> 100%	<input type="radio"/>	ON ⋮

Midterm and Final

- ❖ **Midterm** (~ mid-October) and **Final** will be «take-home exams».
- ❖ Limited time: after downloading the Midterm/Final questions, you will have to upload your work within few hours.

Check out the syllabus on the course webpage!

Office hours + feedback

- ❖ I will have 2 office hours slots (+appointments):
 - ❖ One during New York 'standard hours'.
 - ❖ One early morning or late evening for students with a big time difference.
- ❖ Please fill the Google form with you preferences.
- ❖ Feedback, remarks about the lectures / videos / recitations / homeworks ... :
 - ❖ email me!
 - ❖ link for anonymous feedback on the course's website.

Questions on logistics ?

Vector spaces and subspaces

Quick recap of video 1.2

A **vector space** is a set V endowed with two 'nice and compatible' operations $+$ and \cdot that verify:

- ❖ For all $u, v \in V$, $u + v \in V$.
- ❖ For all $u \in V$ and all $\lambda \in \mathbb{R}$, $\lambda \cdot u \in V$.

Example: $V = \mathbb{R}^n$, with the usual vector addition $+$ and scalar multiplication \cdot is a vector space.

Quick recap of video 1.2

A non-empty subset S of a vector space V is called a **subspace** if it is closed under addition and multiplication by a scalar.

Example: For all $v \in \mathbb{R}^n$,

$$\text{Span}(v) = \{\lambda v \mid \lambda \in \mathbb{R}\}$$

is a subspace of \mathbb{R}^n .

Remarks, questions ?

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Review of Span and linear dependency

Span

The *linear span* of vectors x_1, \dots, x_k as the set of all linear combinations of these vectors.

Linear dependency

- Vectors x_1, \dots, x_k are *linearly dependent* if one of them can be expressed as a linear combination of the others.
- They are said to be *linearly independent* otherwise.

Abuse of language: Instead of saying « x_1, \dots, x_k are linearly dependent», we should say «the family (x_1, \dots, x_k) is linearly dependent».

Basis

A family (x_1, \dots, x_n) of vectors of V is a basis of V if

1. x_1, \dots, x_n are linearly independent,
2. $\text{Span}(x_1, \dots, x_n) = V$.

The dimension

A useful lemma

Lemma

Let $v_1, \dots, v_n \in V$ and let $x_1, \dots, x_k \in \text{Span}(v_1, \dots, v_n)$.
Then, if $k > n$, x_1, \dots, x_k are linearly dependent.

Definition of the dimension

Definition

We say that a vector space V has dimension n if it admits a basis (v_1, \dots, v_n) with n vectors.

The dimension is well defined!

Theorem

If V admits a basis (v_1, \dots, v_n) , then every basis of V has also n vectors.

Proof.



Properties of the dimension

Proposition

Let V be a vector space that has dimension $\dim(V) = n$. Then

1. Any family of vectors of V that spans V contains at least n vectors.
2. Any family of vectors of V that are linearly independent contains at most n vectors.

Proof.



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Properties of the dimension

Proposition

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If x_1, \dots, x_n are linearly independent, then (x_1, \dots, x_n) is a basis of V .
2. If $\text{Span}(x_1, \dots, x_n) = V$, then (x_1, \dots, x_n) is a basis of V .

Very useful to show that a family of vector forms a basis:

Example: $x_1 = (12, 37)$ and $x_2 = (-9, 17)$ form a basis of \mathbb{R}^2 .

An inequality

Proposition

Let U and V be two subspaces of \mathbb{R}^n . Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

Proof

A bit of vocabulary

Definition

Let S be a subspace of \mathbb{R}^n .

- We call S a *line* if $\dim(S) = 1$.
- We call S a *hyperplane* if $\dim(S) = n - 1$.

Coordinates

Coordinates of a vector in a basis

Definition & Theorem

If (v_1, \dots, v_n) is a basis of V , then for every $x \in V$ there exists a unique vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

We say that $(\alpha_1, \dots, \alpha_n)$ are the coordinates of x in the basis (v_1, \dots, v_n) .

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Coordinates of a vector in a basis

Exercise

1. Show that the vectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ form a basis of \mathbb{R}^2 .
2. Express the coordinates of $u = (x, y)$ in the basis (v_1, v_2) in terms of x and y .

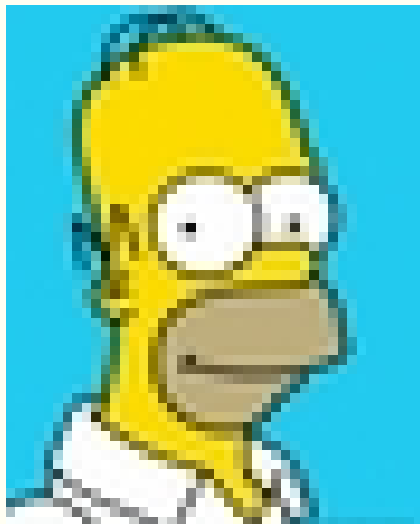
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Why do we care about this ?

Application to image compression

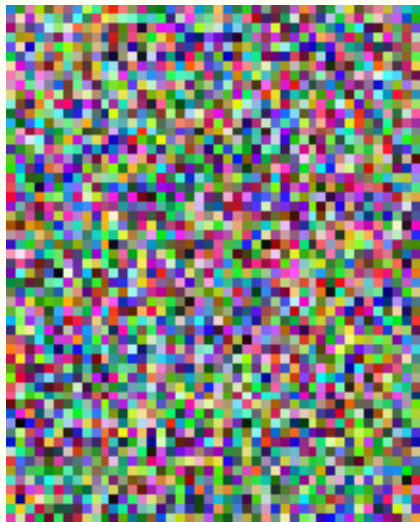
- Image = Grid of pixels
- Represented as a vector $v \in \mathbb{R}^n$, for some large n .
- One needs to store n numbers.



$$n = 44 \times 55 = 2420$$

Can we do better?

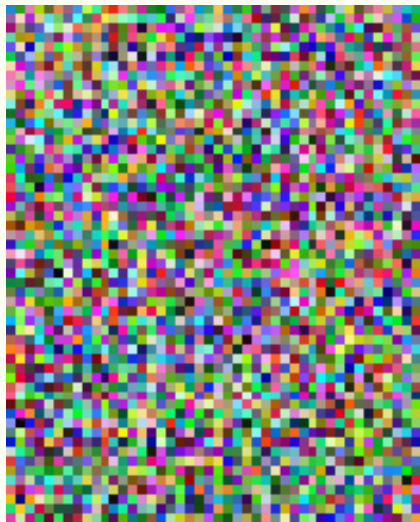
- ❖ If we want to store an arbitrary image, NO!



«Random» image

Can we do better?

- ❖ If we want to store an arbitrary image, NO!
- ❖ However, we are mainly storing images coming from the « real world »
- ❖ These images have some *structure*.



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«Real» image

What do we mean by « structure » ?

Neighboring pixels are very likely to have similar colors.

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Neighboring pixels are very likely to have similar colors.

- ❖ There exists a basis (w_1, \dots, w_n) of \mathbb{R}^n in which «real» images $v \in \mathbb{R}^n$ are (approximately) **sparse**.
- ❖ This means that the coordinates $(\alpha_1, \dots, \alpha_n)$ of v in the basis (w_1, \dots, w_n) contains a lot of zeros.

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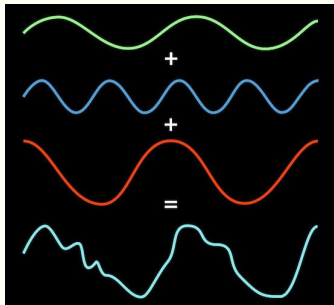
Store only the $k \ll n$ non-zero coordinates of v (in the w_i 's basis') !

A toy example

Consider $n = 2$, that is images $v \in \mathbb{R}^2$ with only 2 pixels.

Examples of good bases

- Fourier bases (used in .jpeg, .mp3)



- JPEG2000 uses **wavelet bases**, and achieves better performance than JPEG.
- In **Homework 4**, you will use wavelets to compress/denoise images.
- The course **DS-GA 1013** deepens these concepts!

Questions?

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