Lecture 4.1: Norms

Optimization and Computational Linear Algebra for Data Science

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Introduction: the Euclidean norm

Definition

We define the Euclidean norm of $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as:

$$\|x\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}.$$

General norms

Let V be a vector space.

Definition

A norm $\|\cdot\|$ on V is a function from V to $\mathbb{R}_{\geq 0}$ that verifies the following points:

1. *Homogeneity*: $\|\alpha v\| = |\alpha| \times \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
2. *Triangular inequality*: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.
3. *Positive definiteness*: if $\|v\| = 0$ for some $v \in V$, then $v = 0$.

Other examples

- The ℓ_1 norm

$$\|x\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^{n} |x_i| = |x_1| + \cdots + |x_n|.$$

Other examples

- The ℓ_p norm, for $p \geq 1$

$$\|x\|_p \stackrel{\text{def}}{=} \Big(|x_1|^p + \cdots + |x_n|^p\Big)^{1/p}.$$

- The infinity-norm

$$\|x\|_\infty \stackrel{\text{def}}{=} \max(|x_1|, \ldots, |x_n|).$$

Balls

For each of the norms $\| \cdot \|_2$, $\| \cdot \|_1$ and $\| \cdot \|_\infty$, draw the «ball»:

$$B = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}.$$