

Lecture 2.1: Linear transformations

Optimization and Computational Linear Algebra for Data Science

Contents

1. Definition of a linear transformation
2. Properties of linear transformations

Definition

Examples

You already know some linear transformations from high-school !

Symmetry

Rotation

Definition

Symmetries (about a line passing through the origin) and rotations (about the origin) are mappings

$$\begin{aligned} L : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ v &\mapsto L(v), \end{aligned}$$

that are “linear”:

Definition

A function $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear if

1. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$ and
2. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$.

An example

❖ $L : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is linear

$$(v_1, v_2) \mapsto (5v_1, 0, v_1 + v_2)$$

An example of a non-linear map

The function $F : \mathbb{R} \rightarrow \mathbb{R}$
 $x \mapsto x^2$ is **not** linear.

Properties

Composition of linear maps

Proposition

If $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and $M : \mathbb{R}^n \rightarrow \mathbb{R}^k$ are both linear, then the composite function

$$\begin{array}{rcl} M \circ L : & \mathbb{R}^m & \rightarrow \mathbb{R}^k \\ & v & \mapsto M(L(v)) \end{array}$$

is also linear.

Proof.



Basic properties

Proposition

If $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is linear, then

❖ $L(0) = 0$.

❖ $L\left(\sum_{i=1}^k \alpha_i v_i\right) = \sum_{i=1}^k \alpha_i L(v_i)$, for all $\alpha_i \in \mathbb{R}$, $v_i \in \mathbb{R}^m$.

Proof.

