Session 8: SVD, linear algebra & graphs

Optimization and Computational Linear Algebra for Data Science

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Contents

Midterm next week

- Thu. Oct. 29, the questions have to be downloaded from Gradescope between 00:01 AM and 9:59 PM.
- **Duration:** 1 hour and 40 minutes to work on the problems + 20 minutes to scan and upload your work.
- Upload your work **as a single PDF**.
- In case the upload does not work for you, **email me your work**.

Singular Value Decomposition

Singular Value decomposition

Theorem

Let $A \in \mathbb{R}^{n\times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ and a matrix $\Sigma \in \mathbb{R}^{n\times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$, that verify

$$A = U\Sigma V^{\mathsf{T}}.$$

Low-rank approximation

Graphs

Graph Laplacian

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Definition

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Properties of the Laplacian

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Spectal clustering with the Laplacian

Algorithm

Input: Graph Laplacian L, number of clusters k

1. Compute the first k eigenvectors $v_1, \ldots, v_k$ of the Laplacian matrix L.
2. Associate to each node i the vector $x_i = (v_2(i), \ldots, v_k(i))$.
3. Cluster the points $x_1, \ldots, x_n$ with (for instance) the k-means algorithm.
4. Deduce a clustering of the nodes of the graph.

The case of two groups

For $k = 2$ groups:

1. Compute the second eigenvector v_2 of the Laplacian matrix L.
2. Associate to each node i the number $x_i = v_2(i)$.
3. Cluster the nodes in:

$$S = \{i \,|\, v_2(i) \geq \delta\} \qquad \text{and} \qquad S^c = \{i \,|\, v_2(i) < \delta\},$$

   for some $\delta \in \mathbb{R}$.

Why does this work ?

Spectral clustering as a «relaxation»

Questions?

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