

# Optimization and Computational Linear Algebra for Data Science

## Lecture 1: Vector spaces

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**Warning:** *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

## 1 General definitions

We present below the abstract mathematical definition of a vector space. **Please do not try to memorize it!** Simply remember that a vector space is a set whose elements are called *vectors*, that one can add vectors together and multiply them by real numbers called *scalars*.

### Definition 1.1 (*Vector space*)

A vector space (over  $\mathbb{R}$ ) consists of a set  $V$  (whose elements are called vectors) and two operations  $+$  and  $\cdot$  that verify:

1. The sum of two vectors is a vector: for all  $\vec{x}, \vec{y} \in V$  we have  $\vec{x} + \vec{y} \in V$ .

2. The vector sum is commutative and associative. For all  $\vec{x}, \vec{y}, \vec{z} \in V$  we have

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{and} \quad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

3. There exists a zero vector  $\vec{0} \in V$  that verifies  $\vec{x} + \vec{0} = \vec{x}$  for all  $\vec{x} \in V$ .

4. For all  $\vec{x} \in V$ , there exists  $\vec{y} \in V$  such that  $\vec{x} + \vec{y} = \vec{0}$ . Such  $\vec{y}$  is called the additive inverse of  $\vec{x}$  and is written  $-\vec{x}$ .

5. Scalar multiplication: for all  $\vec{x} \in V$  and all  $\alpha \in \mathbb{R}$ ,  $\alpha \cdot \vec{x} \in V$ .

6. Identity element for scalar multiplication:  $1 \cdot \vec{x} = \vec{x}$  for all  $\vec{x} \in V$ .

7. Compatibility between scalar multiplication and the usual multiplication: for all  $\alpha, \beta \in \mathbb{R}$  and all  $\vec{x} \in V$ , we have

$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$

8. Distributivity: for all  $\alpha, \beta \in \mathbb{R}$  and all  $\vec{x}, \vec{y} \in V$ ,

$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{y} \quad \text{and} \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

From now we will ignore  $\cdot$  and simply write  $\alpha\vec{x}$  instead of  $\alpha \cdot \vec{x}$ . We will also remove the arrows and write  $x$  instead of  $\vec{x}$ .

*Example 1.1.*

- The set  $V = \mathbb{R}^n$  endowed with the usual vector addition  $+$

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and the usual scalar multiplication  $\cdot$

$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

is a vector space.

- The set  $V = \mathcal{F}(\mathbb{R}, \mathbb{R}) \stackrel{\text{def}}{=} \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$  of all functions from  $\mathbb{R}$  to itself endowed with the addition  $+$  and the scalar multiplication  $\cdot$  defined by

$$\begin{aligned} f + g : \mathbb{R} &\rightarrow \mathbb{R} & \text{and} & & \alpha \cdot f : \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto f(t) + g(t) & & & t &\mapsto \alpha f(t) \end{aligned}$$

is a vector space.

- On a probability space, the set of random variables that admits a finite second moment is a vector space.

### Definition 1.2 (*Subspace*)

We say that a non-empty subset  $S$  of a vector space  $V$  is a subspace if it is stable by addition and multiplication by a scalar, that is if

- (i) for all  $x, y \in S$  we have  $x + y \in S$ ,
- (ii) for all  $x \in S$  and all  $\alpha \in \mathbb{R}$  we have  $\alpha x \in S$ .

Notice that a subspace is also a vector space!

## 2 Linear dependency

### Definition 2.1 (*Linear combination*)

Let  $V$  be a vector space and  $x_1, \dots, x_k \in V$ . We say that  $y \in V$  is a linear combination of the vectors  $x_1, \dots, x_k$  if there exists  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$  such that

$$y = \sum_{i=1}^k \alpha_i x_i.$$

Remember that a linear combination is always a *finite* sum.

**Remark 2.1.** If  $S$  is a subspace of a vector space  $V$ , any linear combination of elements of  $S$  belongs to  $S$ .

### Definition 2.2 (*Span*)

Let  $x_1, \dots, x_k$  be vectors in a vector space  $V$ . We define the linear span of  $x_1, \dots, x_k$  as the set of all linear combinations of these vectors:

$$\text{Span}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \left\{ \alpha_1 x_1 + \dots + \alpha_k x_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}.$$

One can easily verify (exercise!) that  $\text{Span}(x_1, \dots, x_k)$  is a subspace of  $V$ . One can also verify (exercise!) that

$$\text{Span}(x_1, \dots, x_k) = \bigcap_{\substack{S \text{ subspace of } V \\ x_1, \dots, x_k \in S}} S,$$

$\text{Span}(x_1, \dots, x_k)$  is therefore the **smallest** (for the inclusion  $\subset$ ) **subspace of  $V$  that contains  $x_1, \dots, x_k$** .

**Definition 2.3 (Linear dependency)**

Vectors  $x_1, \dots, x_k \in V$  are linearly dependent if there exists  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$  **that are not all zero** such that

$$\alpha_1 x_1 + \dots + \alpha_k x_k = 0.$$

They are said to be linearly independent otherwise.

Saying that  $x_1, \dots, x_k$  are linearly dependent precisely means that one of the vectors  $x_1, \dots, x_k$  can be obtained as a linear combination of the others. Indeed if  $x_1, \dots, x_k$  are linearly dependent, then we can find  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$  that are not all zero (there exists  $i$  such that  $\alpha_i \neq 0$ ) such that  $\alpha_1 x_1 + \dots + \alpha_k x_k = 0$ . This leads to

$$x_i = \sum_{j \neq i} \frac{-\alpha_j}{\alpha_i} x_j,$$

i.e. the vector  $x_i$  can be expressed as a linear combinations of the vectors  $x_j$  for  $j \neq i$ . Conversely if we have for some  $i$ , and  $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

$$x_i = \alpha_1 x_1 + \dots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \dots + \alpha_k x_k = 0.$$

then  $\alpha_1 x_1 + \dots + \alpha_{i-1} x_{i-1} - x_i + \alpha_{i+1} x_{i+1} + \dots + \alpha_k x_k = 0$  which gives that  $x_1, \dots, x_k$  are linearly dependent.

### 3 Basis, dimension

**Definition 3.1 (Basis)**

A family  $(x_1, \dots, x_n)$  of vectors of  $V$  is a basis of  $V$  if

- (i)  $x_1, \dots, x_n$  are linearly independent,
- (ii)  $\text{Span}(x_1, \dots, x_n) = V$ .

**Definition 3.2 (Dimension)**

Let  $V$  be a vector space.

- If  $V$  admits a basis  $(v_1, \dots, v_n)$ , then every basis of  $V$  has also  $n$  vectors. We say that  $V$  has dimension  $n$  and write  $\dim(V) = n$ .
- Otherwise, we say that  $V$  has infinite dimension:  $\dim(V) = +\infty$ .

The dimension is therefore the minimum number of vector needed to span the vector space. In this course we are going to focus mostly on finite dimensional spaces and  $\mathbb{R}^n$  in particular.

**Proof.** We will need the following lemma:

**Lemma 3.1**

Let  $v_1, \dots, v_n \in V$  and suppose that we have vectors  $x_1, \dots, x_k \in V$  such that  $k > n$  and  $x_i \in \text{Span}(v_1, \dots, v_n)$  for all  $i \in \{1, \dots, k\}$ . Then  $x_1, \dots, x_k$  are linearly dependent.

Lemma 3.1 will be proved at the end of the notes. We proceed by contradiction and assume that there exists two basis  $(v_1, \dots, v_n)$  and  $(x_1, \dots, x_k)$  of  $V$  such that  $k \neq n$ . Without loss of generality we can assume that  $k > n$ . For  $i = 1, \dots, k$  we have

$$x_i \in V = \text{Span}(v_1, \dots, v_n),$$

because  $(v_1, \dots, v_n)$  is a basis of  $V$ . We can therefore apply Lemma 3.1 to get that  $x_1, \dots, x_{n+1}$  are linearly dependent. This contradicts the fact that  $(x_1, \dots, x_k)$  is a basis.  $\square$

*Example 3.1.* Let us define the vectors  $e_1, \dots, e_n \in \mathbb{R}^n$  by

$$\begin{aligned} e_1 &= (1, 0, 0, \dots, 0) \\ e_2 &= (0, 1, 0, \dots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \dots, 1). \end{aligned}$$

One can verify (exercise!) that the family  $(e_1, \dots, e_n)$  is a basis of  $\mathbb{R}^n$ . This basis is called the “canonical basis” of  $\mathbb{R}^n$ . We conclude that  $\mathbb{R}^n$  has dimension  $n$ .

### Proposition 3.1

Let  $V$  be a vector space that has dimension  $\dim(V) = n$ . Then

- Any family of vectors of  $V$  that are linearly independent contains at most  $n$  vectors (i.e. if  $x_1, \dots, x_k \in V$  are linearly independent, then  $k \leq n$ ).
- Any family of vectors of  $V$  that spans  $V$  contains at least  $n$  vectors (i.e. if  $x_1, \dots, x_k \in V$  are such that  $\text{Span}(v_1, \dots, v_k) = V$ , then  $k \geq n$ ).

**Exercise 3.1.** Prove Proposition 3.1 using Lemma 3.1.

### Proposition 3.2

Let  $U$  and  $V$  be two subspaces of  $\mathbb{R}^n$ . Assume that  $U \subset V$ . Then

$$\dim(U) \leq \dim(V) \leq n.$$

If moreover  $\dim(U) = \dim(V)$ , then  $U = V$ .

### Definition 3.3 (*Lines, hyperplanes*)

Let  $S$  be a subspace of  $\mathbb{R}^n$ .

- We call  $S$  a line if  $\dim(S) = 1$ .
- We call  $S$  an hyperplane if  $\dim(S) = n - 1$ .

### Proposition 3.3 (*Coordinates*)

Let  $(v_1, \dots, v_n)$  be a basis of  $V$ . Then for every  $x \in V$  there exists a unique vector  $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$  such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

We say that  $(\alpha_1, \dots, \alpha_n)$  are the coordinates of  $x$  in the basis  $(v_1, \dots, v_n)$ .

**Proof. Existence.**  $(v_1, \dots, v_n)$  forms a basis of  $V$  therefore  $V = \text{Span}(v_1, \dots, v_n)$ . We get that  $x \in \text{Span}(v_1, \dots, v_n)$  which gives that there exists  $\alpha_1, \dots, \alpha_n$  such that  $x = \alpha_1 v_1 + \dots + \alpha_n v_n$ .

**Uniqueness.** Let  $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$  such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n.$$

This leads to

$$(\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n = 0.$$

The vectors  $v_1, \dots, v_n$  are linearly independent because they form a basis. Consequently  $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots = \alpha_n - \beta_n = 0$ , i.e.  $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n)$ .  $\square$

## Proof of Lemma 3.1

Notice that it suffices to prove the theorem for  $k = n + 1$  because if  $x_1, \dots, x_{n+1}$  are linearly dependent, so are  $x_1, \dots, x_{n+1}, \dots, x_k$ . We will therefore show for all  $n \geq 1$

$$\mathcal{H}_n : \ll \text{For all } v_1, \dots, v_n \in V \text{ and all } x_1, \dots, x_{n+1} \in \text{Span}(v_1, \dots, v_n), \\ \text{the vectors } x_1, \dots, x_{n+1} \text{ are linearly dependent.} \gg$$

**Base case:**  $\mathcal{H}_1$  is true. Indeed, if  $x_1, x_2 \in \text{Span}(v_1)$ , then there exists  $\alpha_1, \alpha_2 \in \mathbb{R}$  such that  $x_1 = \alpha_1 v_1$  and  $x_2 = \alpha_2 v_1$ . If  $\alpha_1 = 0$  then  $x_1 = 0$  and  $x_1, x_2$  are therefore linearly dependent. Otherwise if  $\alpha_1 \neq 0$  then  $v_1 = \frac{1}{\alpha_1} x_1$  which then gives  $x_2 = \frac{\alpha_2}{\alpha_1} x_1$ :  $x_1, x_2$  are linearly dependent.

**Induction step:** We assume now that  $\mathcal{H}_{n-1}$  holds for some  $n \geq 2$  and we will show that  $\mathcal{H}_n$  holds. We consider therefore  $x_1, \dots, x_{n+1} \in \text{Span}(v_1, \dots, v_n)$ . We can find real numbers  $\alpha_{i,j}$  such that

$$\begin{aligned} x_1 &= \alpha_{1,1}v_1 + \cdots + \alpha_{1,n}v_n \\ x_2 &= \alpha_{2,1}v_1 + \cdots + \alpha_{2,n}v_n \\ &\vdots \\ x_{n+1} &= \alpha_{n+1,1}v_1 + \cdots + \alpha_{n+1,n}v_n. \end{aligned}$$

We have to show that  $x_1, \dots, x_{n+1}$  are linearly dependent. Let us consider the first line. If  $\alpha_{1,1} = \alpha_{1,2} = \cdots = \alpha_{1,n} = 0$ , then  $x_1 = 0$  which gives then that  $x_1, \dots, x_{n+1}$  are linearly dependent. Otherwise, there exists  $j$  such that  $\alpha_{1,j} \neq 0$ . Without loss of generality we can assume that  $\alpha_{1,1} \neq 0$ .

$$\begin{aligned} x_1 &= \alpha_{1,1}v_1 + \cdots + \alpha_{1,n}v_n \\ x_2 - \frac{\alpha_{2,1}}{\alpha_{1,1}}x_1 &= 0 + \cdots + \alpha_{2,n}v_n - \frac{\alpha_{2,1}}{\alpha_{1,1}}\alpha_{1,n}v_n \\ &\vdots \\ x_{n+1} - \frac{\alpha_{n+1,1}}{\alpha_{1,1}}x_1 &= 0 + \cdots + \alpha_{n+1,n}v_n - \frac{\alpha_{n+1,1}}{\alpha_{1,1}}\alpha_{1,n}v_n. \end{aligned}$$

If we define  $y_i \stackrel{\text{def}}{=} x_i - \frac{\alpha_{i,1}}{\alpha_{1,1}}x_1$  for  $i = 2, \dots, n+1$  we obtain have  $y_i \in \text{Span}(v_2, \dots, v_n)$ . We can now apply the induction hypothesis  $\mathcal{H}_{n-1}$  to get that  $y_2, \dots, y_{n+1}$  are linearly dependent. This means that there exists  $\beta_2, \dots, \beta_{n+1}$  that are not all zero, such that  $\beta_2 y_2 + \cdots + \beta_{n+1} y_{n+1} = 0$  which finally gives

$$\left( -\beta_2 \frac{\alpha_{2,1}}{\alpha_{1,1}} - \cdots - \beta_{n+1} \frac{\alpha_{n+1,1}}{\alpha_{1,1}} \right) x_1 + \beta_2 x_2 + \cdots + \beta_{n+1} x_{n+1} = 0.$$

Since  $\beta_2, \dots, \beta_{n+1}$  are not all zero we get that  $x_1, \dots, x_{n+1}$  are linearly dependent.  $\mathcal{H}_n$  is proved.

