Session 2: Linear transformations and matrices

Optimization and Computational Linear Algebra for Data Science

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 Solving linear systems



Linear maps & matrices

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Linear maps & rnatrices

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Linear map	Matrix
$L: \mathbb{R}^m o \mathbb{R}^n$	$L \in \mathbb{R}^{n \times m}$

Linear maps & rnatrices

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Rotations in \mathbb{R}^2

Let $\theta \in \mathbb{R}$. The rotation $R_{\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ of angle θ about the origin is linear.

Exercise: what is the canonical matrix of R_{θ} ?

Operations on matrices

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Addition and scalar multiplication

Sum of two matrices of the **same** dimensions:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} + \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} = \begin{pmatrix} a_{1,1} + b_{1,1} & \cdots & a_{1,m} + b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} + b_{n,1} & \cdots & a_{n,m} + b_{n,m} \end{pmatrix}$$

• Multiplication by a scalar λ :

$$\lambda \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} = \begin{pmatrix} \lambda a_{1,1} & \cdots & \lambda a_{1,m} \\ \vdots & \ddots & \vdots \\ \lambda a_{n,1} & \cdots & \lambda a_{n,m} \end{pmatrix}$$

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A new vector space!

Proposition

- $\mathbb{R}^{n \times m}$ is a vector space.
- $\operatorname{dim}(\mathbb{R}^{n \times m}) =$

Proof.

Operations on matrices

Product of two matrices

Warning:

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \times \begin{pmatrix} b_{1,1} & \cdots & b_{1,m} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,m} \end{pmatrix} \neq \begin{pmatrix} a_{1,1} \times b_{1,1} & \cdots & a_{1,m} \times b_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} \times b_{n,1} & \cdots & a_{n,m} \times b_{n,m} \end{pmatrix}$$

Operations on matrices

Matrix product

Let $L \in \mathbb{R}^{n \times m}$ and $M \in \mathbb{R}^{m \times k}$.

Definition (Matrix product)

The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$.

Matrix product

Theorem

Let $L \in \mathbb{R}^{n \times m}$ and $M \in \mathbb{R}^{m \times k}$.

The entries matrix product LM are given by

$$(LM)_{i,j} = \sum_{i=1}^{m} L_{i,\ell} M_{\ell,j}, \quad \text{for } 1 \leq i \leq n \text{ and } 1 \leq j \leq k.$$

Proof

Operations on matrices

Rotations in \mathbb{R}^2

The R_a and R_b denote respectively the matrices of the rotations of angles a and b about the origin, in \mathbb{R}^2 .

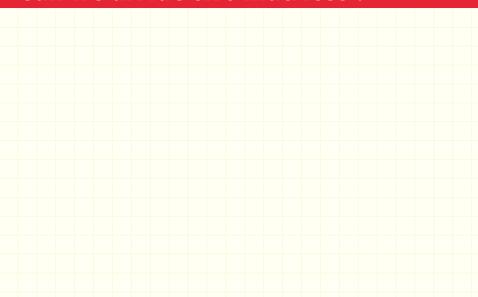
Exercise: Compute the product R_aR_b .

Matrix product properties

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Operations on matrices

Can we divide two matrices?



Operations on matrices

Invertible matrices

Definition (Matrix inverse)

A **square** matrix $M \in \mathbb{R}^{n \times n}$ is called *invertible* if there exists a matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix M^{-1} is unique and is called the *inverse* of M.

Exercise: Let $A, B \in \mathbb{R}^{n \times n}$. Show that if $AB = \mathrm{Id}_n$ then $BA = \mathrm{Id}_n$.

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Kernel and image

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Definitions

Let $L: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

Definition (Kernel)

The kernel $\mathrm{Ker}(L)$ (or nullspace) of L is defined as the set of all vectors $v \in \mathbb{R}^m$ such that L(v) = 0, i.e.

$$\operatorname{Ker}(L) \stackrel{\text{def}}{=} \{ v \in \mathbb{R}^m \, | \, L(v) = 0 \}.$$

Definition (Image)

The image $\operatorname{Im}(L)$ (or column space) of L is defined as the set of all vectors $u \in \mathbb{R}^n$ such that there exists $v \in \mathbb{R}^m$ such that L(v) = u.

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Kernel and image

Picture			

Remarks

Let $L: \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.

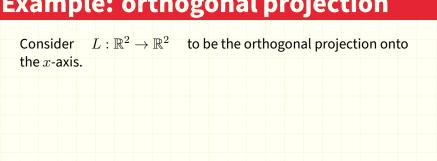
Proposition

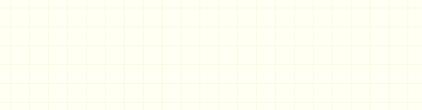
- $ightharpoonup \operatorname{Ker}(L)$ is a subspace of \mathbb{R}^m .
- $ightharpoonup \operatorname{Im}(L)$ is a subspace of \mathbb{R}^n .

Remark: ${\rm Im}(L)$ is also the Span of the columns of the matrix representation of L.

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Example: orthogonal projection







Kernel and image 20/27

Linear systems

Why do we care about this?

Assume that we given a dataset:

$$a_i = (a_{i,1}, \dots, a_{i,m}) \in \mathbb{R}^m, \quad y_i \in \mathbb{R}$$
 for $i = 1, \dots, n$.

We would like to find
$$x \in \mathbb{R}^m$$
 such that

for
$$i$$

$$=1,$$

$$\dots n$$

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$$x_1 a_{i,1} + \dots + x_m a_{i,m} = y_i$$

for all $i \in \{1, \ldots, n\}$.

Matrix notation

Why do we care about this?

Let's write $A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,m} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,m} \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{and} \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n.$

$$\stackrel{n}{\mid}\in\mathbb{F}$$

Let's find all solutions!

Why do we care about this?

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Conclusion: 3 possible cases

- 1. $y \notin \text{Im}(A)$: there is no solution to Ax = y.
- 2. $y \in \text{Im}(A)$, then there exists $x_0 \in \mathbb{R}^m$ such that $Ax_0 = y$. The set of solutions in then

$$S = \{x_0 + v \mid v \in \operatorname{Ker}(A)\}.$$

- If $Ker(A) = \{0\}$, then $S = \{x_0\}$: x_0 is the unique solution.
- If $Ker(A) \neq \{0\}$, then Ker(A) contains infinitely many vectors: there are infinitely many solutions.

 $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix} \in \mathbb{R}^{n \times m} \quad \text{and} \quad y = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} \in \mathbb{R}^n.$

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