Optimization and Computational Linear Algebra for Data Science
Lecture 9: Convex functions

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Warning: *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

1 Convex sets

Definition 1.1 (*Convex set*)

A set $C \subset \mathbb{R}^n$ is convex *if for all $x, y \in C$ and all $\alpha \in [0,1]$,*

$$\alpha x + (1-\alpha) y \in C.$$

Definition 1.2 (*Convex combination*)

We say that $y \in \mathbb{R}^n$ is a convex combination of $x_1, \dots, x_k \in \mathbb{R}^n$ if there exists $\alpha_1, \dots, \alpha_k \geq 0$ such that

$$y = \sum_{i=1}^{k} \alpha_i x_i \qquad \text{and} \qquad \sum_{i=1}^{k} \alpha_i = 1.$$

Proposition 1.1

If C is convex then all convex combination of elements of C remains in C.

2 Convex functions

Definition 2.1

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0,1]$,

$$f(\alpha x + (1-\alpha) y) \leq \alpha f(x) + (1-\alpha) f(y). \tag{1}$$

Notice that a linear function is also a convex function since it verifies (1) with equality.

Exercise 2.1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ a convex function and $a \in \mathbb{R}$. Show that the set*

$$\{x \in \mathbb{R}^n \mid f(x) \leq a\}$$

is convex.

3 Convex function and differential

3.1 1-D case

Proposition 3.1

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then f is convex if and only if f' is non-decreasing.

Proposition 3.2

Let $f : \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function. Then f is convex if and only if $\forall x \in \mathbb{R},\ f''(x) \geq 0$.

3.2 General case

Proposition 3.3

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. Then f is convex if and only

$$\forall x, y \in \mathbb{R}^n,\ \langle x - y, \nabla f(x) - \nabla f(y) \rangle \geq 0.$$

Proposition 3.4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice-differentiable function. We denote by H_f the Hessian matrix of f. Then f is convex if and only if for all $x \in \mathbb{R}^n$, $H_f(x)$ is positive semi-definite.

Further reading

See [2] for a very nice introduction to spectral clustering and [1] for lecture notes on spectral graph theory.

References

[1] Daniel Spielman. Spectral graph theory. *Lecture Notes, Yale University*, http://www.cs.yale.edu/homes/spielman/561/2012/, 2012.

[2] Ulrike Von Luxburg. A tutorial on spectral clustering. *Statistics and computing*, 17(4):395–416, 2007.