Session 3: The rank

Optimization and Computational Linear Algebra for Data Science

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Contents

The rank

Recap of the videos

Definition

We define the rank of a family $x_1, \dots, x_k$ of vectors of $\mathbb{R}^n$ as the dimension of its span:

$$\operatorname{rank}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \dim(\operatorname{Span}(x_1, \dots, x_k)).$$

Definition

Let $M \in \mathbb{R}^{n \times m}$. Let $c_1, \dots, c_m \in \mathbb{R}^n$ be its columns. We define

$$\operatorname{rank}(M) \stackrel{\text{def}}{=} \operatorname{rank}(c_1, \dots, c_m) = \dim(\operatorname{Im}(M)).$$

Proposition

Let $M \in \mathbb{R}^{n \times m}$. Let $r_1, \dots, r_n \in \mathbb{R}^m$ be the rows of M and $c_1, \dots, c_m \in \mathbb{R}^n$ be its columns. Then we have

$$\operatorname{rank}(r_1, \dots, r_n) = \operatorname{rank}(c_1, \dots, c_m) = \operatorname{rank}(M).$$

How do we compute the rank ?

For $v_1, \ldots, v_k \in \mathbb{R}^n$, and $\alpha \in \mathbb{R} \setminus \{0\}$, $\beta \in \mathbb{R}$ we have

$$\operatorname{rank}(v_1, \ldots, v_k) = \begin{cases} \operatorname{rank}(v_1, \ldots, v_{i-1},\ \alpha v_i\ , v_{i+1}, \ldots, v_k) \\ \\ \operatorname{rank}(v_1, \ldots, v_{i-1},\ v_i + \beta v_j\ , v_{i+1}, \ldots, v_k) \end{cases}$$

As a consequence, the Gaussian elimination method keeps the rank of a matrix unchanged!

Example

Let's compute the rank of $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 2 & 0 & 1 & -1 \\ -1 & 5 & 2 & 0 \end{pmatrix}$

Example

The rank-nullity Theorem

Rank-nullity Theorem

Theorem

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

Intuition

Let us solve the linear system $Ax = 0$.

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & -1 & 0 \\ -1 & 5 & 2 & 0 & 0 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 4 & 2 & 1 & 0 \end{array}\right) \begin{array}{l} (R_1) \\ (R_2) - 2(R_1) \\ (R_3) + (R_1) \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ 0 & 2 & 1 & -3 & 0 \\ 0 & 0 & 0 & 7 & 0 \end{array}\right) \begin{array}{l} (R_1) \\ (R_2) \\ (R_3) - 2(R_2) \end{array}$$

Proof of the rank-nullity Theorem

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Invertible matrices

Invertible matrices

Definition (Matrix inverse)

A **square** matrix $M \in \mathbb{R}^{n\times n}$ is called *invertible* if there exists a matrix $M^{-1} \in \mathbb{R}^{n\times n}$ such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix M^{-1} is unique and is called the *inverse* of M.

Exercise: Let $A, B \in \mathbb{R}^{n\times n}$. Show that if $AB = \mathrm{Id}_n$ then $BA = \mathrm{Id}_n$.

Invertible matrices

Theorem

Let $M \in \mathbb{R}^{n \times n}$. The following points are equivalent:

1. M is invertible.
2. $\operatorname{rank}(M) = n$.
3. $\operatorname{Ker}(M) = \{0\}$.
4. For all $y \in \mathbb{R}^n$, there exists a unique $x \in \mathbb{R}^n$ such that $Mx = y$.

Proof

Proof

Proof

Proof

Transpose of a matrix

Transpose of a matrix

Definition

Let $M \in \mathbb{R}^{n \times m}$. We define its *transpose* $M^{\mathsf{T}} \in \mathbb{R}^{m \times n}$ by

$$(M^{\mathsf{T}})_{i,j} = M_{j,i}$$

for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

Remark:

- We have $(M^{\mathsf{T}})^{\mathsf{T}} = M$.
- The mapping $M \mapsto M^{\mathsf{T}}$ is linear.

Properties of the transpose

Proposition

For all $A \in \mathbb{R}^{n \times m}$, $\quad \operatorname{rank}(A) = \operatorname{rank}(A^\mathsf{T})$.

Proposition

Let $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times k}$. Then

$$(AB)^\mathsf{T} = B^\mathsf{T} A^\mathsf{T}.$$

Proof.

$\square$

Symmetric matrices

Definition

A square matrix $A \in \mathbb{R}^{n\times n}$ is said to be *symmetric* if

$$\forall i, j \in \{1, \dots, n\},\ A_{i,j} = A_{j,i}$$

or, equivalently if $A = A^{\mathsf{T}}$.

Remark: For all $M \in \mathbb{R}^{n\times m}$ the matrix MM^{T} is symmetric.

Is the rank useful in practice?

Back to the movies ratings example

Assume that you are given the matrix of movies ratings:

$$\begin{pmatrix} 1 & 1 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0 & 0 \\ 1 & 1.001 & 5 & 5 & 5 \\ 2 & 2 & 2 & 0.0001 & 0 \\ 2.0001 & 2 & 2 & 0 & 0 \end{pmatrix}$$

Goal: how many different « user profiles » do we have ?

Conclusion

- The rank is not «robust» !
- We need to have a way to check if a matrix has «approximately a small rank».
- Equivalentely, given m vectors, one would like to be able to see if there exists a subspace of dimension $k \ll m$ from which the vectors are « close ».
- The singular value decomposition (lecture 6-7) will solves our problems !

Questions?

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