Optimization and Computational Linear Algebra for Data Science
Homework 2: Linear transformations & matrices

Due on September 20, 2020

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 2.1 (2 points). *Which of the following are linear transformations? Justify.*

(**a**) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x^2+y^2, x-y) \end{array} \right.$

(**b**) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (x+y+1, x-y) \end{array} \right.$

(**c**) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times m} & \to & \mathbb{R}^{m\times n} \\ A & \mapsto & A^\mathsf{T} \end{array} \right.$ *where A^T is transpose of A, i.e. the $m \times n$ matrix defined by*

$$\left(A^\mathsf{T}\right)_{i,j} = A_{j,i} \qquad \textit{for all} \quad (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\}.$$

(**d**) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R} \\ A & \mapsto & \mathrm{Tr}(A) \end{array} \right.$ *where $\mathrm{Tr}(A)$ is the trace of the matrix A, defined by*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{i,i}\,.$$

Problem 2.2 (3 points). *Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that*

$$f(1,2) = (1,2,3) \qquad \textit{and} \qquad f(2,2) = (1,0,1).$$

(**a**) *Compute the matrix (canonically) associated to f.*

(**b**) *Compute the set $\{x \in \mathbb{R}^2 \,|\, f(x) = (1,4,5)\}$.*

(**c**) *Compute the set $\{x \in \mathbb{R}^2 \,|\, f(x) = (2,4,5)\}$.*

Problem 2.3 (2 points)**.** *Let $B \in \mathbb{R}^{4\times 3}$ be a matrix with arbitrary entries:*

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \\ B_{4,1} & B_{4,2} & B_{4,3} \end{pmatrix}.$$

Find two matrices A and C such that

$$ABC = \begin{pmatrix} B_{1,2} & B_{1,1} & B_{1,3} & B_{1,2} \\ B_{2,2} + B_{3,2} & B_{2,1} + B_{3,1} & B_{2,3} + B_{3,3} & B_{2,2} + B_{3,2} \\ B_{4,2} & B_{4,1} & B_{4,3} & B_{4,2} \end{pmatrix}$$

holds for any B defined above.

Problem 2.4 (3 points)**.**

(**a**) *Let A be a $n \times m$ matrix. Show that the image* $\mathrm{Im}(A)$ *and the kernel* $\mathrm{Ker}(A)$ *of A are subspaces of respectively $\mathbb{R}^n$ and $\mathbb{R}^m$.*

(**b**) *Let*

$$A = \begin{pmatrix} 1 & 2 & 1 & 2 \\ -1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 2 \end{pmatrix}.$$

Compute a basis of $\mathrm{Ker}(A)$ *and show that* $\mathrm{Im}(A) = \mathbb{R}^3$.

Problem 2.5 ($\star$)**.** *Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{k\times n}$. Prove that there exists a matrix $C \in \mathbb{R}^{m\times k}$ such that $A = CB$ if and only if* $\mathrm{Ker}(B)$ *is a subspace of* $\mathrm{Ker}(A)$.