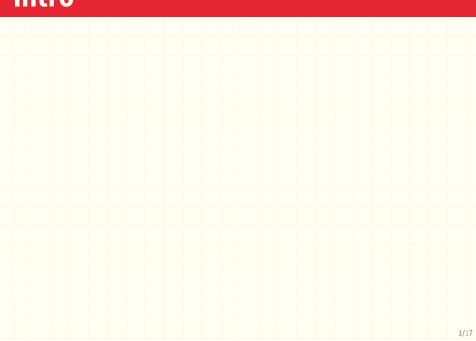
Session 6: Eigenvalues, eigenvectors & Markov chains

Optimization and Computational Linear Algebra for Data Science

Contents

- 1. Orthogonal matrices
- 2. Eigenvalues & eigenvectors
- 3. Properties
- 4. Markov chains

Intro



Definition

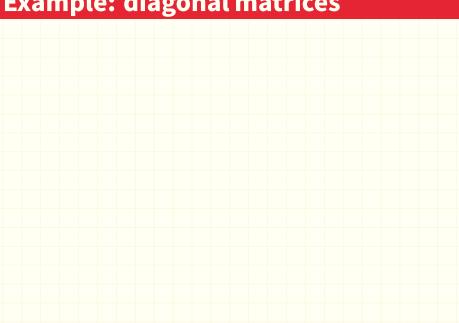
Definition

Let $A\in\mathbb{R}^{n\times n}$. A **non-zero** vector $v\in\mathbb{R}^n$ is said to be an eigenvector of A is there exists $\lambda\in\mathbb{R}$ such that

$$Av = \lambda v$$
.

The scalar λ is called the eigenvalue (of A) associated to v.

Example: diagonal matrices



Matrix with no eigenvalues/vectors



Example: orthogonal projection



Eigenspaces

Definition

If $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, the set

$$E_{\lambda}(A) = \{ x \in \mathbb{R}^n \, | \, Ax = \lambda x \}$$

is called the eigenspace of A associated to λ . The dimension of $E_{\lambda}(A)$ is called the multiplicity of the eigenvalue λ .

Properties

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Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #1

For all $\alpha\in\mathbb{R}$, $\alpha\lambda$ is an eigenvalue of the matrix αA and x is an associated eigenvector.

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #2

For all $\alpha \in \mathbb{R}$, $\lambda + \alpha$ is an eigenvalue of the matrix $A + \alpha \mathrm{Id}$ and x is an associated eigenvector.

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #3

For all $k \in \mathbb{N}$, λ^k is an eigenvalue of the matrix A^k and x is an associated eigenvector.

Let $A \in \mathbb{R}^{n \times n}$. Suppose that A has an eigenvalue $\lambda \in \mathbb{R}$ and let $x \in \mathbb{R}^n$ be an eigenvector associated to λ .

Fact #4

If A is invertible then $1/\lambda$ is an eigenvalue of the matrix inverse A^{-1} and x is an associated eigenvector.

Properties

Spectrum

Definition

The set of all eigenvalues of A is called the *spectrum* of A and denoted by $\operatorname{Sp}(A)$.

Theorem

A $n \times n$ matrix A admits at most n different eigenvalues: $\#\mathrm{Sp}(A) \leq n$.

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Proof that $\#\mathrm{Sp}(A) \leq n$

Proposition

Let v_1, \ldots, v_k be eigenvectors of A corresponding (respectively) to the eigenvalues $\lambda_1, \ldots, \lambda_k$.

If the λ_i are all distinct ($\lambda_i \neq \lambda_j$ for all $i \neq j$) then the vectors v_1, \ldots, v_k are linearly independent.

Properties 10/17

Markov chains

Markov chains 11/17

An example

Markov chains

Stochastic matrices

Definition

A matrix $P \in \mathbb{R}^{n \times n}$ is said to be *stochastic* if:

- **1.** $P_{i,j} \ge 0$ for all $1 \le i, j \le n$.
- 2. $\sum_{i=1}^{n} P_{i,j} = 1$, for all $1 \le j \le n$.

Probability vectors

Markov chains

The key equation

Proposition

For all
$$t \ge 0$$

Markov chains

$$\iota \leq 0$$

$$_{\infty}(t+)$$

$$_{\infty}(t+1)$$

$$c(t+1)$$

$$x^{(t+1)} = Px^{(t)}$$
 and consequently, $x^{(t)} = P^t x^{(0)}$.

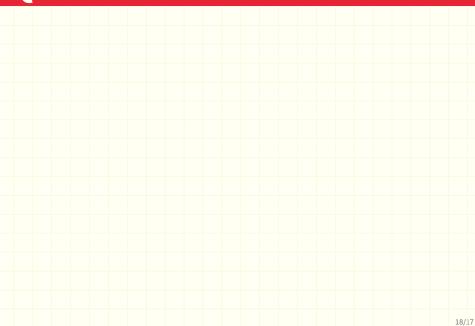
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Markov chains

Markov chains

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Questions?



Questions?

