Lecture 8.1: Singular Value Decomposition

Optimization and Computational Linear Algebra for Data Science

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PCA

- Data matrix $A \in \mathbb{R}^{n \times m}$
- "Covariance matrix" $S = A^{\mathsf{T}} A \in \mathbb{R}^{m \times m}$.
- S is symmetric positive semi-definite.
- **Spectral Theorem:** there exists an orthonormal basis $v_1, \ldots, v_m$ of $\mathbb{R}^m$ such that the v_i's are eigenvectors of S associated with the eigenvalues $\lambda_1 \geq \cdots \geq \lambda_m \geq 0$.

Singular values/vectors

For $i = 1, \ldots, m$:

- we define $\sigma_i = \sqrt{\lambda_i}$, called the i^{th} singular value of A.
- we call v_j the i^{th} right singular vector of A.

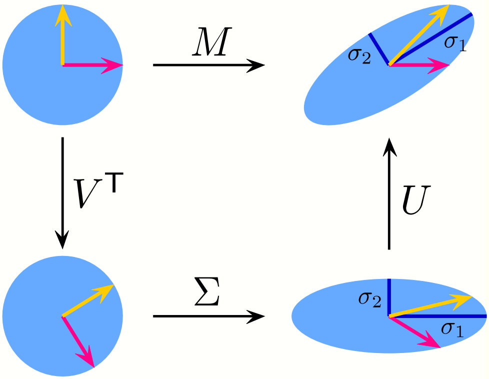
Singular Value decomposition

Theorem

Let $A \in \mathbb{R}^{n\times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{R}^{m\times m}$ and a matrix $\Sigma \in \mathbb{R}^{n\times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$, that verify

$$A = U\Sigma V^{\mathsf{T}}.$$

Geometric interpretation of $U\Sigma V^\top$



$$M = U\Sigma V^\top$$