# Recitation 6

Carles Domingo

# Markov Chains

## Definition (Markov chain)

A sequence of random variables $(X_0, X_1, \dots)$ is a Markov chain with state space $E$ and "transition matrix" $P$ if for all $t \geq 0$,

$$\mathbb{P}(X_{t+1} = y \,|\, X_0 = x_0, \dots, X_t = x_t) = P(x_t, y)$$

for all $x_0, \dots, x_t$ such that $\mathbb{P}(X_0 = x_0, \dots, X_t = x_t) > 0$.

Stochastic matrix: $P_{ij} \geq 0, \sum_{i=1}^{n} P_{ij} = 1$ for all $1 \leq j \leq n$.

## Definition (Invariant measure)

A vector $\mu \in \Delta_n$ is called an invariant measure for the transition matrix $P$ if $\mu = P\mu$, i.e. if $\mu$ is an eigenvector of $P$ associated with the eigenvalue $1$.

# Perron-Frobenius theorem

### Theorem (Perron-Frobenius, stochastic case)

*Let $P$ be a stochastic matrix such that there exists $k \geq 1$ such that all the entries of $P^k$ are strictly positive. Then the following holds:*

1. *$1$ is an eigenvalue of $P$ and there exists an eigenvector $\mu \in \Delta_n$ associated to $1$.*
2. *The eigenvectors associated to $1$ are unique up to scalar multiple (i.e. $\mathrm{Ker}(P - \mathrm{Id}) = \mathrm{Span}(\mu)$).*
3. *For all $x \in \Delta_n$, $P^t x \xrightarrow[t\to\infty]{} \mu$.*

Is the condition "there exists $k \geq 1$ such that all the entries of $P^k$ are strictly positive" necessary? Let's see!

# Questions: Counterexamples

**Definition (Irreducible Markov chain)**

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

1. Show that the assumption "there exists $k \geq 1$ such that all the entries of $P^k$ are strictly positive" implies that the Markov chain is irreducible.
2. Find an example of a non-irreducible Markov chain for which several invariant measures exist.
3. But irreducibility is not enough for the Perron-Frobenius statements to hold. Show that a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   is irreducible but does not fulfill "for all $x \in \Delta_2$, $P^t x \xrightarrow[t\to\infty]{} \mu$".

# Questions: Counterexamples

Definition (Irreducible Markov chain)

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

1. Show that the assumption "there exists $k \geq 1$ such that all the entries of $P^k$ are strictly positive" implies that the Markov chain is irreducible.

# Questions: Counterexamples

### Definition (Irreducible Markov chain)

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

2. Find an example of a non-irreducible Markov chain for which several invariant measures exist.

# Questions: Counterexamples

**Definition (Irreducible Markov chain)**

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

2. Find an example of a non-irreducible Markov chain for which several invariant measures exist.

# Questions: Counterexamples

**Definition (Irreducible Markov chain)**

If for all $1 \leq i, j \leq n$, there exists $k \geq 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

3. But irreducibility is not enough for the Perron-Frobenius statements to hold. Show that a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is irreducible but does not fulfill "for all $x \in \Delta_2$, $P^t x \xrightarrow[t \to \infty]{} \mu$".

# Questions: Counterexamples

### Definition (Irreducible Markov chain)

If for all $1 \le i, j \le n$, there exists $k \ge 1$ such that $P_{ij}^k > 0$, we say that the Markov chain is irreducible.

3. But irreducibility is not enough for the Perron-Frobenius statements to hold. Show that a Markov chain with transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is irreducible but does not fulfill "for all $x \in \Delta_2$, $P^t x \xrightarrow[t \to \infty]{} \mu$".

# Questions: Counterexamples

4. Remember from the lecture that the PageRank algorithm actually computes the invariant measure of the transition matrix

$$G = \alpha P + \frac{1-\alpha}{N} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \dots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

with $\alpha \approx 0.85$. Given the previous questions, what would be the problems in taking $\alpha = 1$?

# Questions: Counterexamples

4. Remember from the lecture that the PageRank algorithm actually computes the invariant measure of the transition matrix

$$G = \alpha P + \frac{1-\alpha}{N} \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

with $\alpha \approx 0.85$. Given the previous questions, what would be the problems in taking $\alpha = 1$?

# Questions: Stochastic matrices

Remember that $P$ is a stochastic matrix when $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} P_{i,j} = 1$ for all $j$.

1. Show that $1$ is an eigenvalue of $P$.
2. Show that all eigenvalues of $P$ have absolute value less or equal than $P$.

# Questions: Stochastic matrices

Remember that $P$ is a stochastic matrix when $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} P_{i,j} = 1$ for all $j$.

1. Show that $1$ is an eigenvalue of $P$.
2. Show that all eigenvalues of $P$ have absolute value less or equal than $P$.

# Questions: Stochastic matrices

Remember that $P$ is a stochastic matrix when $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$ and $\sum_{i=1}^{n} P_{i,j} = 1$ for all $j$.

1. Show that $1$ is an eigenvalue of $P$.
2. Show that all eigenvalues of $P$ have absolute value less or equal than $P$.

# Questions: Random walks

Let us consider a variant of PageRank in which the edges are non-oriented, i.e. if page $i$ contains a link to page $j$, then page $j$ contains a link to page $i$. If we define the transition

$$P_{i,j} = \begin{cases} 1/\text{deg}(j) & \text{if link } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$

1. Show that $\pi$ defined as $\pi_j = \text{deg}(j)$ is an eigenvector of $P$ of eigenvalue 1.
2. Conclude that $x \in \Delta_n$, $P^t x \xrightarrow[t \to \infty]{} \mu$ if the Perron-Frobenius assumption holds.
3. Extra Question: Show that if each page has a link to itself and for any pair of pages $i, j$, there is a path of linked pages joining $i$ and $j$, the Perron-Frobenius assumption holds.

# Questions: Random walks

$$P_{i,j} = \begin{cases} 1/\text{deg}(j) & \text{if link } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$

1. Show that $\pi$ defined as $\pi_j = \text{deg}(j)$ is an eigenvector of $P$ of eigenvalue 1.

# Questions: Random walks

$$P_{i,j} = \begin{cases} 1/\text{deg}(j) & \text{if link } i \leftrightarrow j \\ 0 & \text{otherwise} \end{cases}$$

2. Conclude that $x \in \Delta_n$, $P^t x \xrightarrow[t\to\infty]{} \mu$ if the Perron-Frobenius assumption holds.

# Spectral theorem

**Theorem (Spectral theorem)**

*Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Then, $A$ has $n$ orthogonal eigenvectors $q_1, \dots, q_n$ and we can write $A = Q\Lambda Q^\top$, where $Q = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}$ and $\Lambda$ is diagonal.*

Remember that a matrix $A$ is diagonalizable iff it has $n$ linearly independent eigenvectors (equivalently $A = V\Lambda V^{-1}$). Thus, the spectral theorem says that symmetric matrices are diagonalizable in an orthogonal basis.

# Questions: Spectral theorem

1. Let $A, B \in \mathbb{R}^{n \times n}$ be symmetric matrices. Show that $AB = BA$ iff $A$ and $B$ diagonalize in the same basis. Does the same hold if we just assume that $A, B$ are diagonalizable?

# Questions: Spectral theorem

1. Let $A, B \in \mathbb{R}^{n\times n}$ be symmetric matrices. Show that $AB = BA$ iff $A$ and $B$ diagonalize in the same basis. Does the same hold if we just assume that $A, B$ are diagonalizable?