Recitation 5

Carles Domingo

Fall 2020

Orthogonal matrices

Definition (Orthogonal matrix)

Given a dot product $\langle \cdot, \cdot \rangle$, an orthogonal matrix is a real square matrix whose columns are *orthonormal* vectors.

Equivalently, Q is an orthonormal matrix if its inverse is Q^T.

Questions: Orthogonal matrices

1. Show the equivalence of the previous page: that Q is an orthogonal matrix iff $Q^{-1} = Q^\top$.
2. Show that $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all x, y with the appropriate dimensions and Q orthogonal.
3. Show that $\|Qx\| = \|x\|$ for all x and Q orthogonal.

Questions: Orthogonal matrices

1. Show the equivalence of the previous page: that Q is an orthogonal matrix iff $Q^{-1} = Q^\top$.
2. Show that $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all x, y with the appropriate dimensions and Q orthogonal.
3. Show that $\|Qx\| = \|x\|$ for all x and Q orthogonal.

Questions: Orthogonal matrices

1. Show the equivalence of the previous page: that Q is an orthogonal matrix iff $Q^{-1} = Q^\top$.
2. Show that $\langle Qx, Qy \rangle = \langle x, y \rangle$ for all x, y with the appropriate dimensions and Q orthogonal.
3. Show that $\|Qx\| = \|x\|$ for all x and Q orthogonal.

Eigenvalues and eigenvectors

Definition

Let $A \in \mathbb{R}^{n \times n}$. A **non-zero** vector $v \in \mathbb{R}^n$ is said to be an *eigenvector* of A is there exists $\lambda \in \mathbb{R}$ such that

$$Av = \lambda v.$$

The scalar λ is called the *eigenvalue* (of A) associated to v. The set

$$E_\lambda(A) = \{x \in \mathbb{R}^n \,|\, Ax = \lambda x\} = \mathrm{Ker}(A - \lambda \mathrm{Id})$$

is called the *eigenspace* of A associated to λ. The dimension of $E_\lambda(A)$ is called the multiplicity of the eigenvalue λ.

Eigenvalues and eigenvectors

Recall:

- If a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_k$ with eigenvectors $v_1, \ldots, v_k$ resp., then $v_1, \ldots, v_k$ are linearly independent. $\implies$ A has at most n different eigenvalues.
- More strongly, if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_k$

$$\sum_{i=1}^{k} \mathrm{dim}(E_{\lambda_i}(A)) \leq n$$

Note: To compute eigenvalues and eigenvectors using determinants and characteristic polynomials, see Léo's video. Recommended but optional and not covered in this recitation.

Questions: Eigendecomposition

1. Let $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$. Show that a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $v_1, \ldots, v_n$ iff $A = V \text{diag}((\lambda_i)_{i=1}^n) V^{-1}$. In this case, we say that A is a *diagonalizable* matrix.
2. Show that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_n$, A is diagonalizable.
3. Write the expression of a matrix in $\mathbb{R}^{2 \times 2}$ for which $[2, -1]$ is an eigenvector of eigenvalue 2 and $[1, 3]$ is an eigenvector of eigenvalue -1.

Questions: Eigendecomposition

1. Let $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$. Show that a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $v_1, \ldots, v_n$ iff $A = V \operatorname{diag}((\lambda_i)_{i=1}^n) V^{-1}$. In this case, A is a *diagonalizable* matrix.

Questions: Eigendecomposition

1. Let $V = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{n \times n}$. Show that a matrix $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ with linearly independent eigenvectors $v_1, \ldots, v_n$ iff $A = V \operatorname{diag}((\lambda_i)_{i=1}^n) V^{-1}$. In this case, A is a *diagonalizable* matrix.

Questions: Eigendecomposition

2. Show that if $A \in \mathbb{R}^{n \times n}$ has eigenvalues $\lambda_1 < \cdots < \lambda_n$, A is diagonalizable.

Questions: Eigendecomposition

3. Write the expression of a matrix in $\mathbb{R}^{2\times 2}$ for which $[2, -1]$ is an eigenvector of eigenvalue 2 and $[1, 3]$ is an eigenvector of eigenvalue -1.

Questions: Eigendecomposition

3. Write the expression of a matrix in $\mathbb{R}^{2\times 2}$ for which $[2, -1]$ is an eigenvector of eigenvalue 2 and $[1, 3]$ is an eigenvector of eigenvalue -1.

Questions: Eigenvalues & trace

1. Show that the trace is invariant by change of basis, i.e. if $X \in \mathbb{R}^{n \times n}$ is invertible and $A \in \mathbb{R}^{n \times n}$, $\text{tr}(A) = \text{tr}(XAX^{-1})$. (Hint: $\text{tr}(BC) = \text{tr}(CB)$).
2. Show that if $A \in \mathbb{R}^{n \times n}$ is diagonalizable and has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Questions: Eigenvalues & trace

1. Show that the trace is invariant by change of basis, i.e. if $X \in \mathbb{R}^{n\times n}$ is invertible and $A \in \mathbb{R}^{n\times n}$, $\mathrm{tr}(A) = \mathrm{tr}(XAX^{-1})$. (Hint: $\mathrm{tr}(BC) = \mathrm{tr}(CB)$).

Questions: Eigenvalues & trace

1. Show that the trace is invariant by change of basis, i.e. if $X \in \mathbb{R}^{n\times n}$ is invertible and $A \in \mathbb{R}^{n\times n}$, $\mathrm{tr}(A) = \mathrm{tr}(XAX^{-1})$. (Hint: $\mathrm{tr}(BC) = \mathrm{tr}(CB)$).

Questions: Eigenvalues & trace

2. Show that if $A \in \mathbb{R}^{n \times n}$ is diagonalizable and has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\text{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Questions: Eigenvalues & trace

2. Show that if $A \in \mathbb{R}^{n \times n}$ is diagonalizable and has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $\mathrm{tr}(A) = \sum_{i=1}^{n} \lambda_i$.

Questions: No (real) eigenvalues

Some matrices do not admit (real) eigenvalues and eigenvectors.

1. Show that if $\theta \in [0, 2\pi)$,

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

   does not admit real eigenvalues and eigenvectors in general.

2. Find the matrices, and the real eigenvalues and eigenvectors for the values of θ for which they exist.

Questions: No (real) eigenvalues

1. Show that if $\theta \in [0, 2\pi)$,

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

does not admit real eigenvalues and eigenvectors in general.

Questions: No (real) eigenvalues

1. Show that if $\theta \in [0, 2\pi)$,

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

does not admit real eigenvalues and eigenvectors in general.

Questions: No (real) eigenvalues

2. Find the matrices, and the real eigenvalues and eigenvectors for the values of θ for which they exist.

Questions: No (real) eigenvalues

2. Find the matrices, and the real eigenvalues and eigenvectors for the values of θ for which they exist.

Extra Q: Gram-Schmidt and QR

Let $A \in \mathbb{R}^{n \times n}$ have linearly independent columns. Show that there is a matrix $Q \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{n \times n}$ s.t that $A = QR$, where Q has orthonormal columns and R is upper triangular.

(Hint: Recall the “linear combination of columns interpretation of matrix multiplication").

Extra Q: Gram-Schmidt and QR

Let $A \in \mathbb{R}^{n \times n}$ have linearly independent columns. Show that there is a matrix $Q \in \mathbb{R}^{n \times m}$ and $R \in \mathbb{R}^{n \times n}$ s.t that $A = QR$, where Q has orthonormal columns and R is upper triangular.

(Hint: Recall the “linear combination of columns interpretation of matrix multiplication").