Session 6 bis: Markov Chains and PageRank

Optimization and Computational Linear Algebra for Data Science

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Contents

Markov chains

An example

Stochastic matrices

Definition

A matrix $P \in \mathbb{R}^{n \times n}$ is said to be *stochastic* if:

1. $P_{i,j} \geq 0$ for all $1 \leq i, j \leq n$.
2. $\sum_{i=1}^{n} P_{i,j} = 1$, for all $1 \leq j \leq n$.

Probability vectors

The key equation

Proposition

For all $t \geq 0$

$$x^{(t+1)} = Px^{(t)} \quad \text{and consequently,} \quad x^{(t)} = P^t x^{(0)}.$$

Long-term behavior

Perron-Frobenius Theorem

Invariant measure

Definition

A vector $\mu \in \Delta_n$ is called an invariant measure for the transition matrix P if $\mu = P\mu$, i.e. if μ is an eigenvector of P associated with the eigenvalue 1.

Perron-Frobenius Theorem

Theorem

Let P be a stochastic matrix such that there exists $k \geq 1$ such that all the entries of P^k are strictly positive. Then the following holds:

1. 1 is an eigenvalue of P and there exists an eigenvector $\mu \in \Delta_n$ associated to 1.
2. The eigenvalue 1 has multiplicity 1: $\mathrm{Ker}(P - \mathrm{Id}) = \mathrm{Span}(\mu)$.
3. For all $x \in \Delta_n$, $P^t x \xrightarrow[t \to \infty]{} \mu$.

Consequence

Corollary

Let P be a stochastic matrix such that there exists $k \geq 1$ such that all the entries of P^k are strictly positive.
Then there exists a unique invariant measure μ and for all initial condition $x^{(0)} \in \Delta_n$,

$$x^{(t)} = P^t x^{(0)} \xrightarrow[t \to \infty]{} \mu.$$

Proof: Geometrical observations

Proof: contraction

We will prove the theorem in the case where $P_{i,j} > 0$ for all i, j.

Lemma

The mapping

$$\begin{array}{rccc} \varphi: & \Delta_n & \to & \Delta_n \\ & x & \mapsto & Px \end{array}$$

is a contraction mapping for the ℓ_1-norm: there exists $c \in (0,1)$ such that for all $x, y \in \Delta_n$:

$$\|Px - Py\|_1 \leq c\|x - y\|_1.$$

Geometric picture

End of the proof

End of the proof

End of the proof

PageRank

Ordering the Web

The random surfer

The random surfer

PageRank Algorithm

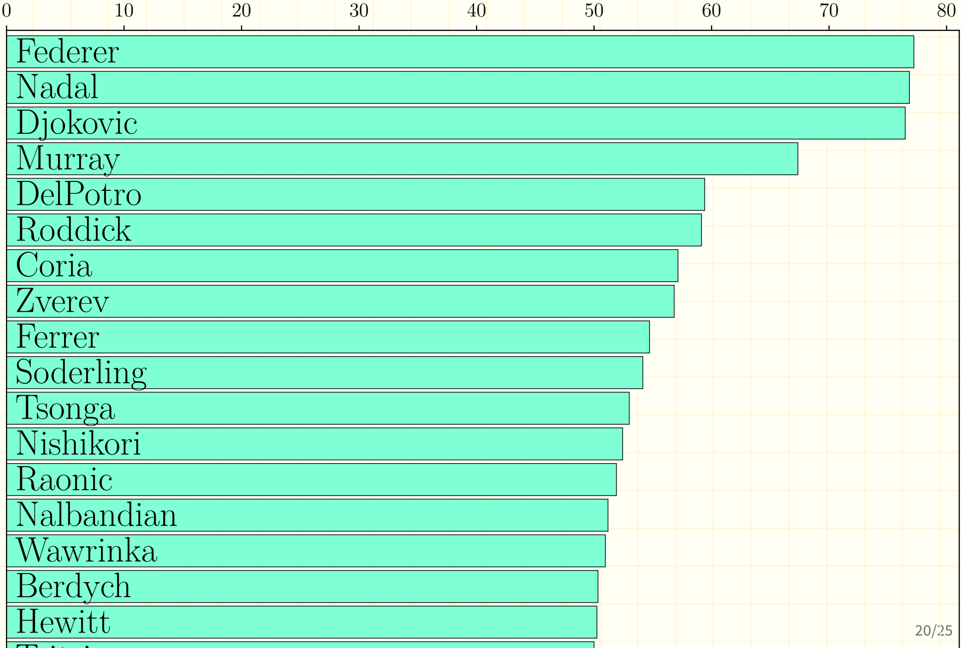
Application: ranking Tennis players

Goal: ranking the following players

Federer, Nadal, Djokovic, Murray, Del Potro, Roddick, Coria, Zverev, Ferrer, Soderling, Tsonga, Nishikori, Raonic, Nalbandian, Wawrinka, Berdych, Hewitt, Tsitsipas, Monfils, Gonzalez, Thiem, Ljubicic, Davydenko, Cilic, Pouille, Safin, Isner, Dimitrov, Medvedev, Ferrero, Goffin, Bautista Agut, Sock, Gasquet, Simon, Blake, Monaco, Coric, Stepanek, Khachanov, Almagro, Robredo, Verdasco, Anderson, Youzhny, Baghdatis, Dolgopolov, Kohlschreiber, Fognini, Melzer, Paire, Querrey, Tomic, Basilashvili.

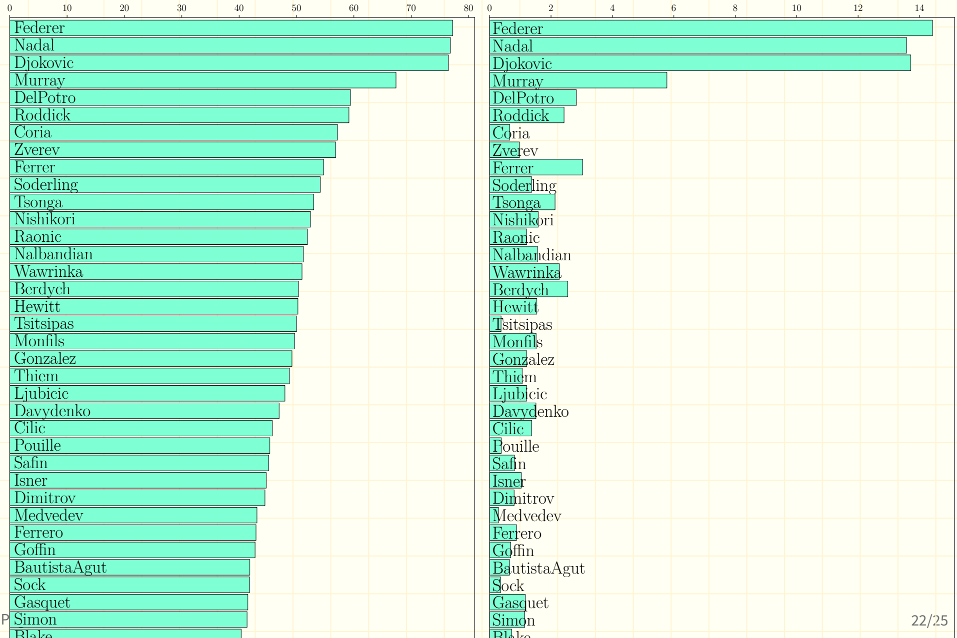
Data: Head-to Head records (number of times that player x has defeated player y)

Ranking by % of victories



The random spectator

Naive ranking vs PageRank



The Spectral Theorem

The spectral theorem

Theorem

Let $A \in \mathbb{R}^{n \times n}$ be a **symmetric** matrix. Then there is a orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of A.

Matrix formulation

Theorem

Let $A \in \mathbb{R}^{n\times n}$ be a **symmetric** matrix. Then there exists an orthogonal matrix P and a diagonal matrix D of sizes $n \times n$ such that

$$A = PDP^{\mathsf{T}}.$$

Questions?

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