Recitation 9

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Adjacency matrix and Laplacian

Definition (Adjacency matrix)

We define the adjacency matrix $A \in \mathbb{R}^{n \times n}$ of a graph G with n nodes as

$$A_{ij} = \begin{cases} 1 & \text{if } i \sim j \\ 0 & \text{otherwise} \end{cases}$$

Definition (Laplacian of a matrix)

The Laplacian matrix of G is defined as

$$L = D - A$$

where D is the degree matrix $D = \text{diag}(\text{deg}(1), \ldots, \text{deg}(n))$.

Remember that A is symmetric and positive semidefinite, and 1 is an eigenvalue of A with eigenvector $\mathbb{1}$.

Laplacian

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Normalized Laplacian

A bipartite graph is a graph such that the set of nodes can be split into two subsets such that all the edges of the graph are between nodes in different subsets.

2. Show that the largest eigenvalue of $L_{norm} = D^{-1/2} L D^{-1/2}$ is less or equal than 2.

3. Show that the largest eigenvalue of L_{norm} is equal to 2 if and only if the graph has some bipartite connected component.

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Adjacency matrix

Show that the largest eigenvalue λ_n of the adjacency matrix A is larger or equal than the average of the degrees of the nodes and smaller or equal than the maximum degree.

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Complete graphs

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Spectral clustering

Reminder of the spectral clustering algorithm:
Input: Graph Laplacian G, number of clusters k.

1. Compute the first k eigenvectors of the graph Laplacian.
2. Associate to each node i the vector $x_i = (v_2(i), \ldots, v_k(i))$.
3. Cluster the points $x_1, \ldots, x_n$ with k-means, for example.

Extra SVD Question

Midterm 2019 Q6: Let $M \in \mathbb{R}^{n \times m}$. Let $n \geq m$, and M have full rank. Let M have SVD $M = U \Sigma V^T$.

1. Show that $M^T M$ is invertible.
2. Which vectors span the $Im(M)$? Write the matrix of orthogonal projection onto $Im(M)$ and give a basis transformation for that matrix.
3. Let $w \in \mathbb{R}^n$, and u be the orthogonal projection of w onto $Im(M)$. Show that $M^T u = M^T w$.
4. Show that $M(M^T M)^{-1} M^T$ is the matrix of an orthogonal projection onto $Im(M)$.

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