Optimization and Computational Linear Algebra for Data Science
Homework 1: Vector spaces

Due on September 20, 2020

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 1.1 (3 points)**.** *Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer.*

(a) $E_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x - 2y + z = 0\}$.

(b) $E_2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x - 2y + z = 3\}$.

(c) $E_3 = \{(x,y,z) \in \mathbb{R}^3 \,|\, 5x + y^2 + z = 0\}$.

Problem 1.2 (2 points)**.** *Let $x_1, \dots, x_k \in \mathbb{R}^n$. Assume that $x_1 \in \mathrm{Span}(x_2, \dots, x_k)$. Show that*

$$\mathrm{Span}(x_1, \dots, x_k) = \mathrm{Span}(x_2, \dots, x_k).$$

Problem 1.3 (2 points)**.** *Suppose that $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent. Let $x \in \mathbb{R}^n$ and assume that $x \notin \mathrm{Span}(v_1, \dots, v_k)$. Show that $(v_1, \dots, v_k, x)$ are linearly independent.*

Problem 1.4 (3 points)**.** We prove in this problem Proposition 3.2 from the notes. You can use the results of Problems 1.2 and 1.3 and of course the other results from the lecture.

Let S be a subspace of $\mathbb{R}^n$ of dimension k and let $x_1, \dots, x_k \in S$.

(a) *Show that if $x_1, \dots, x_k$ are linearly independent, then $(x_1, \dots, x_k)$ is a basis of S.*

(b) *Show that if $\mathrm{Span}(x_1, \dots, x_k) = S$, then $(x_1, \dots, x_k)$ is a basis of S.*

Problem 1.5 ($\star$)**.** *Let U and V be two subspaces of $\mathbb{R}^n$. Show that if*

$$\dim(U) + \dim(V) > n,$$

then there must exist a non-zero vector in their intersection, i.e. $U \cap V \neq \{0\}$.