

# Lecture 1.2: Vector Spaces

Optimization and Computational Linear Algebra for Data Science

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# Introduction

# So far, « Vectors = arrows »

Two fundamental operations:

1. Add two vectors  $\vec{u}$  and  $\vec{v}$  to obtain another vector  $\vec{u} + \vec{v}$
2. Multiply a vector  $\vec{u}$  by a «scalar» (= a real number)  $\lambda$  to get another vector  $\lambda \cdot \vec{u}$

# Coordinate representation

- ❖ One can represent vectors using coordinates
- ❖ 2D vectors in the plane  $\vec{u} = (u_1, u_2) \in \mathbb{R}^2$
- ❖ 3D vectors in space  $\vec{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$
- ❖  $n$ -dimensional vectors  $\vec{u} = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$

- ❖  $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, \dots, u_n + v_n)$
- ❖  $\lambda \cdot \vec{u} = (\lambda u_1, \lambda u_2, \dots, \lambda u_n)$

# Vector Spaces

# Abstract definition

## Definition (simplified)

A vector space consists of a set  $V$  (whose elements are called vectors) and two operations  $+$  and  $\cdot$  such that

- ❖ The sum of two vectors is a vector: for  $\vec{x}, \vec{y} \in V$ , the sum  $\vec{x} + \vec{y}$  is a vector, i.e.  $\vec{x} + \vec{y} \in V$ .
- ❖ Multiplying a vector  $\vec{x} \in V$  by a scalar  $\lambda \in \mathbb{R}$  gives a vector  $\lambda \cdot \vec{x} \in V$ .
- ❖ The operations  $+$  and  $\cdot$  are “nice and compatible”.

# « Nice and compatible » ?

1. The vector sum is commutative and associative. For all  $\vec{x}, \vec{y}, \vec{z} \in V$ :

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{and} \quad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

2. There exists a zero vector  $\vec{0} \in V$  that verifies  $\vec{x} + \vec{0} = \vec{x}$  for all  $\vec{x} \in V$ .
3. For all  $\vec{x} \in V$ , there exists  $\vec{y} \in V$  such that  $\vec{x} + \vec{y} = \vec{0}$ . Such  $\vec{y}$  is called the additive inverse of  $\vec{x}$  and is written  $-\vec{x}$ .
4. Identity element for scalar multiplication:  $1 \cdot \vec{x} = \vec{x}$  for all  $\vec{x} \in V$ .
5. Distributivity: for all  $\alpha, \beta \in \mathbb{R}$  and all  $\vec{x}, \vec{y} \in V$ ,

$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{y} \quad \text{and} \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

6. Compatibility between scalar multiplication and the usual multiplication: for all  $\alpha, \beta \in \mathbb{R}$  and all  $\vec{x} \in V$ , we have

$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$



# Example 1: $\mathbb{R}^n$

The set  $V = \mathbb{R}^n$  endowed with the usual vector addition  $+$

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and the usual scalar multiplication  $\cdot$

$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

is a vector space.

**We will work in  $\mathbb{R}^n$  99% of the time !**

# Example 2: functions

The set  $V \stackrel{\text{def}}{=} \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$  of all functions from  $\mathbb{R}$  to itself  
endowed with the addition  $+$  and the scalar multiplication  $\cdot$   
defined by

$$\begin{array}{lll} f + g : \mathbb{R} & \rightarrow & \mathbb{R} \\ t & \mapsto & f(t) + g(t) \end{array} \quad \text{and} \quad \begin{array}{lll} \alpha \cdot f : \mathbb{R} & \rightarrow & \mathbb{R} \\ t & \mapsto & \alpha f(t) \end{array}$$

is a vector space.

**Useful in signal processing.**

# Example 3: random variables

The set of random variables on a given probability space  $\Omega$  is a vector space:

If  $X$  and  $Y$  are two random variables and  $\alpha \in \mathbb{R}$ ,  $X + Y$  and  $\alpha X$  are also random variables.

**Important to have this in mind when doing stats/probabilities!**

# Why do we need all this?

- Get geometric intuition.
- Save time: A theorem that applies to vector spaces will in particular be true for all the examples we listed before.

# Subspaces

# Definition

## Definition

We say that a non-empty subset  $S$  of a vector space  $V$  is a *subspace* if it is closed under addition and multiplication by a scalar, that is if

1. for all  $x, y \in S$  we have  $x + y \in S$ ,
2. for all  $x \in S$  and all  $\alpha \in \mathbb{R}$  we have  $\alpha x \in S$ .

**Remark:** a subspace is a also vector space.

# Examples

- ❖  $\mathbb{R}^n$  is a subspace of  $\mathbb{R}^n$ .
- ❖  $\{0\}$  is a subspace of  $\mathbb{R}^n$ .
- ❖ Any line that contains the origin is subspace of  $\mathbb{R}^2$ .