

# Recitation 5

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# Concept Review: Orthogonal Matrices

- ▶ **Orthogonal** matrices have *orthonormal* columns
  - ▶ Stronger condition than having orthogonal columns
  - ▶ Bad terminology that's grandfathered in
- ▶ We will see a lot of these matrices
- ▶ Orthogonal matrices preserve angles and norms
  - ▶ This leads to a very natural *change of basis* - more later

# Questions: Orthogonal Matrices

1. Let  $Q, U \in \mathbb{R}^{n \times n}$  be an orthogonal matrix. Let  $x, y \in \mathbb{R}^n$ .
  - i. Show that  $\langle Qx, Qy \rangle = \langle x, y \rangle$ .
  - ii. Show that  $\|Qx\| = \|x\|$ .
  - iii. Show that  $QU$  is orthogonal.

# Solutions 1: Orthogonal Matrices

1. Let  $Q, U \in \mathbb{R}^{n \times n}$  be an orthogonal matrix. Let  $x, y \in \mathbb{R}^n$ .
  - ii. Show that  $\langle Qx, Qy \rangle = \langle x, y \rangle$ .

## Solution

$$\langle Qx, Qy \rangle = x^T Q^T Qy = x^T Iy = x^T y = \langle x, y \rangle$$

## Solutions 2: Orthogonal Matrices

1. Let  $Q, U \in \mathbb{R}^{n \times n}$  be an orthogonal matrix. Let  $x, y \in \mathbb{R}^n$ .
  - ii. Show that  $\|Qx\| = \|x\|$ .

### Solution

$$\|Qx\| = \langle Qx, Qx \rangle = x^T Q^T Qx = x^T Ix = x^T x = \langle x, x \rangle = \|x\|$$

## Solutions 3: Orthogonal Matrices

1. Let  $Q, U \in \mathbb{R}^{n \times n}$  be an orthogonal matrix. Let  $x, y \in \mathbb{R}^n$ .
  - iii. Show that  $QU$  is orthogonal.

### Solution

*Start by noticing that  $(QU)^T = U^T Q^T$ .*

*Now,*

$$(QU)^T(QU) = U^T Q^T QU = U^T IU = I$$

*and*

$$(QU)(QU)^T = QUU^T Q^T = QIQ^T = I$$

*Then  $QU$  is orthogonal.*

# Concept Review: Gram-Schmidt Process

- ▶ Gram-Schmidt Process turns a basis of linearly independent vectors into orthonormal vectors
- ▶ Understanding the GS process is important, but we will mainly only use its existence
  - ▶ Let  $v_1, \dots, v_n$  be a basis ...  
... and by GS process, let  $u_1, \dots, u_n$  be orthonormal with  $\text{Span}(v_1, \dots, v_n) = \text{Span}(u_1, \dots, u_n)$ :
  - ▶ Let  $u_1, \dots, u_n$  be an orthonormal basis of  $\mathbb{R}^n$
- ▶ Related to QR Factorization

# Questions: GS Process and QR Factorization

1. Let  $A \in \mathbb{R}^{n \times n}$  have linearly independent columns. Show that there is a matrix  $Q \in \mathbb{R}^{n \times n}$  and  $R \in \mathbb{R}^{n \times n}$  s.t that  $A = QR$ , where  $Q$  has orthonormal columns and  $R$  is upper triangular.  
(Hint: Recall the “linear combination of columns interpretation of matrix multiplication”).



# Solutions: GS Process and QR Factorization

1. Let  $A \in \mathbb{R}^{m \times n}$  have linearly independent columns. Show that there is a matrix  $Q \in \mathbb{R}^{m \times n}$  and  $R \in \mathbb{R}^{n \times n}$  s.t that  $A = QR$ , where  $Q$  has orthonormal columns and  $R$  is upper triangular.

## Solution

*First, let  $v_1, \dots, v_n$  be the columns of  $A$ .*

*Apply the GS process to get  $u_1, \dots, u_n$ .*

*Now, let  $Q$  have  $u_1, \dots, u_n$  as its columns.*

*Note that by the GS process, we have*

*$\text{Span}(v_1, \dots, v_i) = \text{Span}(u_1, \dots, u_i) \forall i \in \{1, \dots, n\}$ .*

*Then each column  $v_i$  is a linear combination of the columns  $u_1, \dots, u_i$ .*

*Then this exactly saying that  $A = QR$ , where  $R$  contains the coefficients that transforms  $u_1, \dots, u_i$  into  $v_1, \dots, v_i \forall i \in \{1, \dots, n\}$ !*

(!Check the next slide!)

# More M.M.M: Linear Combination of Columns

(From Recitation 2)

Each column of the  $AB$  is a linear combination of the columns of  $A$ .

$$\begin{bmatrix} \begin{array}{|c|c|} \hline \mathbf{a}_1 & \mathbf{a}_2 \\ \hline \end{array} & \dots & \begin{array}{|c|c|} \hline \mathbf{a}_{k-1} & \mathbf{a}_k \\ \hline \end{array} \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$
$$= \begin{bmatrix} \sum_{i=1}^k \mathbf{a}_i b_{i,1} & \dots & \sum_{i=1}^k \mathbf{a}_i b_{i,m} \end{bmatrix}$$

# A Note About Determinants

- ▶ Eigenvalues of a matrix can be determined by using determinants
- ▶ **Not covered in this course!**
  - ▶ “too long to define, a bit complex, and slightly useless in data science...” - Léo
- ▶ Determinants lead to a lot of cool things
  - ▶  $\text{Trace}(A)$  = sum of eigenvalues of  $A$  (with multiplicity)
  - ▶ (&) A matrix satisfies it's own *characteristic polynomial* - Cayley Hamilton Theorem
  - ▶ (&) Matrix polynomial rabbit hole runs deep (Jordan Normal Form)
- ▶ Interesting from a pure math perspective

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<sup>0</sup>(&) denotes extra material not covered in this course

- ▶ *eigen*values and *eigen*vectors
- ▶ What does *eigen* mean anyway?
- ▶ German word for...
  1. own
  2. innate
  3. peculiar
  4. **intrinsic**
- ▶ A square matrix ‘owns’ certain vectors... or there are certain vectors that are intrinsic to a matrix.

# Importance of Eigenvalues and Eigenvectors

!!! *SERIOUSLY IMPORTANT* !!!

- ▶ Eigen-val/vec will show up *continuously* throughout this course
- ▶ Connections to...
  - ▶ Projections and Orthogonal Projections (Lec 4)
  - ▶ Markov Chains (Lec 6)
  - ▶ Spectral Theorem (HW 6, Lec 7)
  - ▶ SVD (Lec 7)
  - ▶ Spectral Clustering (!!??) (Lec 8)
  - ▶ Positive definite and positive semi-definite matrices (Lec 10,11)
- ▶ Many other applications not covered in this course

# $Av = \lambda v$ . So what's the big deal?

- ▶ Square matrices get their own name - *operators*.
  - ▶ Can construct powers of operators ( $A^k$ )
  - ▶ Operators can be invertible
  - ▶ Operators can be symmetric
- ▶ Sometimes a matrix  $A$  'prefers' certain directions
- ▶ (!!!) These directions act as *anchors* for understanding the action of a matrix.
- ▶ We will see how to exploit these directions in order to simplify our understanding of matrices. (Diagonalizability, Lec 7)

# Questions: Eigen

Let  $A \in \mathbb{R}^{n \times n}$  have eigenvalue  $\lambda$  associated to eigenvector  $v$ . Show that:

1.  $\forall \alpha \in \mathbb{R}$ ,  $\lambda + \alpha$  is an eigenvalue of  $A + \alpha I$  w/ eigenvector  $v$ .
2.  $\forall k \in \mathbb{N}$ ,  $\lambda^k$  is an eigenvalue of  $A^k$  w/ eigenvector  $v$ .
3. Let  $A \in \mathbb{R}^{n \times n}$  have eigenvalue-vector pairs  $\lambda_1, \dots, \lambda_n$  and  $v_1, \dots, v_n$ .

Also, assume that  $\lambda_1 > \dots > \lambda_n$ .

Prove that  $v_1, \dots, v_n$  are linearly independent.

Hint: First assume transform all  $\lambda_i$  to be positive.

# Solutions 1: Eigen

Let  $A \in \mathbb{R}^{n \times n}$  have eigenvalue  $\lambda$  associated to eigenvector  $v$ . Show that:

1.  $\forall \alpha \in \mathbb{R}$ ,  $\lambda + \alpha$  is an eigenvalue of  $A + \alpha I$  w/ eigenvector  $v$ .

## Solution

*Let  $\alpha \in \mathbb{R}$ , and  $v$  be an eigenvector of  $A$ .*

*Consider the matrix  $A + \alpha I$ .*

$$\begin{aligned}(A + \alpha I)v &= Av + \alpha Iv \\ &= \lambda v + \alpha v \\ &= (\lambda + \alpha)v\end{aligned}$$

*So  $\lambda + \alpha$  is an eigenvalue of  $A + \alpha I$  with eigenvector  $v$ .*



## Solutions 2: Eigen

Let  $A \in \mathbb{R}^{n \times n}$  have eigenvalue  $\lambda$  associated to eigenvector  $v$ . Show that:

2.  $\forall k \in \mathbb{N}$ ,  $\lambda^k$  is an eigenvalue of  $A^k$  w/ eigenvector  $v$ .

### Solution

*Let  $k \in \mathbb{N}$ , and  $v$  be an eigenvector of  $A$ .*

*Consider the matrix  $A^k$ .*

$$A^k v = A \dots A v \quad k \text{ times}$$

$$A^k v = A \dots A(\lambda v) \quad k-1 \text{ times}$$

$$A^k v = \lambda^k v$$

*So  $\lambda^k$  is an eigenvalue of  $A^k$  with eigenvector  $v$ .*

## Solutions 3: Eigen

3. Let  $A \in \mathbb{R}^{n \times n}$  have eigenvalue-vector pairs  $\lambda_1, \dots, \lambda_n$  and  $v_1, \dots, v_n$ . Also, assume that  $\lambda_1 > \dots > \lambda_n$ . Prove that  $v_1, \dots, v_n$  are linearly independent.

### Solution

Let  $B = A + |\lambda_n|I$ . (This is so all eigenvalues of  $B$  are  $\geq 0$ .)

Let  $\gamma_i = \lambda_i + |\lambda_n|$  (Problem 1, eigenvecs of  $B$  are also eigenvecs of  $A$ ).

Let  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$  s.t.  $\sum_{i=1}^n \alpha_i v_i = 0$ . We will show that all  $\alpha_i = 0$ .

Consider  $0 = B^k(\sum_{i=1}^n \alpha_i v_i)$ .

$$0 = B^k(\sum_{i=1}^n \alpha_i v_i)$$

$$0 = \sum_{i=1}^n B^k \alpha_i v_i$$

$$0 = \sum_{i=1}^n \gamma_i^k \alpha_i v_i$$

$$0 = \gamma_1^k \sum_{i=1}^n \left(\frac{\gamma_i}{\gamma_1}\right)^k \alpha_i v_i$$

$$0 = \lim_{k \rightarrow \infty} \gamma_1^k \sum_{i=1}^n \left(\frac{\gamma_i}{\gamma_1}\right)^k \alpha_i v_i$$

$$0 = \alpha_1 v_1 \quad \text{since } \frac{\gamma_i}{\gamma_1} < 1 \text{ for } i \neq 1$$

Then  $0 = (\sum_{i=2}^n \alpha_i v_i)$ . Repeat the previous logic to find that each  $\alpha_i v_i = 0$ .

Then all  $\alpha_i = 0$ . So  $v_1, \dots, v_n$  are linearly independent.

## Questions 2: Properties of Orthogonal Matrices

Let  $Q \in \mathbb{R}^{n \times n}$  be orthogonal.

1. Does  $Q$  necessarily have eigenvalues and eigenvectors?

Assume that  $Q$  has eigenvalues  $\lambda_1, \dots, \lambda_k$ .

2. Describe the eigenvalues of  $Q$ .

## Solutions 2: Properties of Orthogonal Matrices

Let  $Q \in \mathbb{R}^{n \times n}$  be orthogonal.

1. Does  $Q$  necessarily have eigenvalues and eigenvectors?

### Solution

No, consider the matrix  $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  (90 deg CCW rotation in  $\mathbb{R}^2$ ).

Assume that  $Q$  has eigenvalues  $\lambda_1, \dots, \lambda_k$ .

2. Describe the eigenvalues of  $Q$ .

### Solution

Since  $Q$  is orthogonal then  $\forall x \in \mathbb{R}^n$

$$\|Qx\| = \langle Qx, Qx \rangle$$

$$\|Qx\| = x^T Q^T Qx$$

$$\|Qx\| = x^T Ix$$

$$\|Qx\| = \|x\|$$

Now, if  $x$  is an eigenvector of  $Q$  with eigenvalue  $\lambda$ , then we have

$$\|x\| = \|Qx\| = \|\lambda x\| = |\lambda| \|x\|. \text{ So } \lambda = \pm 1.$$