# Lecture 7.1: Consequences of the spectral theorem

Optimization and Computational Linear Algebra for Data Science

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# The Spectral theorem

### Theorem

Let $A \in \mathbb{R}^{n \times n}$ be a **symmetric** matrix. Then there is a orthonormal basis of $\mathbb{R}^n$ composed of eigenvectors of $A$.

That means that if $A$ is symmetric, then there exists an orthonormal basis $(v_1, \ldots, v_n)$ of $\mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$Av_i = \lambda_i v_i \qquad \text{for all} \;\; i \in \{1, \ldots, n\}.$$

### Theorem (Matrix formulation)

Let $A \in \mathbb{R}^{n \times n}$ be a **symmetric** matrix. Then there exists an orthogonal matrix $P$ and a diagonal matrix $D$ of sizes $n \times n$ such that

$$A = PDP^{\mathsf{T}}.$$

# Consequences

If

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^\mathsf{T}$$

for some orthogonal matrix $P$ then:

**Consequence #1**: $\lambda_1, \ldots, \lambda_n$ are the only eigenvalues of $A$, and the number of time that an eigenvalue appear on the diagonal equals its multiplicity.

# Proof sketch on an example

Consider $n = 3$ and

$$A = P \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} P^{\mathsf{T}} \qquad \text{where} \qquad P = \begin{pmatrix} | & | & | \\ v_1 & v_2 & v_3 \\ | & | & | \end{pmatrix}$$

is an orthogonal matrix.

# Consequences

If

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^{\mathsf{T}}$$

for some orthogonal matrix $P$ then:

**Consequence #2**: The rank of $A$ equals to the number of non-zero $\lambda_i$'s on the diagonal:

$$\operatorname{rank}(A) = \#\{i \mid \lambda_i \neq 0\}.$$

# Proof

# Consequences

If

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^{\mathsf{T}}$$

for some orthogonal matrix $P$ then:

**Consequence #3**: $A$ is invertible if and only if $\lambda_i \neq 0$ for all $i$. In such case

$$A^{-1} = P \begin{pmatrix} 1/\lambda_1 & 0 & \cdots & 0 \\ 0 & 1/\lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/\lambda_n \end{pmatrix} P^{\mathsf{T}}$$

# Proof

# Consequences

If

$$A = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^{\mathsf{T}}$$

for some orthogonal matrix $P$ then:

**Consequence #4**: $\mathrm{Tr}(A) = \lambda_1 + \cdots + \lambda_n$.