# Lecture 4.2: Inner product

Optimization and Computational Linear Algebra for Data Science

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# The Euclidean dot product

## Definition

We define the Euclidean dot product of two vectors $x$ and $y$ of $\mathbb{R}^n$ as:

$$x \cdot y \;=\; \sum_{i=1}^{n} x_i \, y_i \;=\; x_1 y_1 + \cdots + x_n y_n .$$

# Inner product

Let $V$ be a vector space.

**Definition**

An inner product on $V$ is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{R}$ that verifies the following points:

1. *Symmetry*: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.
2. *Linearity*: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.
3. *Positive definiteness*: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

# Other example

If $V$ is the set of all random variables (on a probability space $\Omega$) that have a finite second moment, then

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \mathbb{E}[XY]$$

is an inner product on $V$.

# Norm induced by an inner product

**Proposition**

If $\langle \cdot, \cdot \rangle$ is an inner product on $V$ then

$$\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$$

is a norm on $V$. We say that the norm $\| \cdot \|$ is induced by the inner product $\langle \cdot, \cdot \rangle$.

# Example

Consider again the set $V$ of all random variables (on a probability space $\Omega$) that have a finite second moment, with the inner product:

$$\langle X, Y \rangle \stackrel{\text{def}}{=} \mathbb{E}[XY].$$

# Cauchy Schwarz inequality

## Theorem (Cauchy-Schwarz inequality)

*Let* $\|\cdot\|$ *be the norm induced by the inner product* $\langle\cdot,\cdot\rangle$ *on the vector space* $V$. *Then for all* $x, y \in V$*:*

$$|\langle x, y\rangle| \leq \|x\|\,\|y\|. \tag{1}$$

*Moreover, there is equality in* (1) *if and only if* $x$ *and* $y$ *are linearly dependent, i.e.* $x = \alpha y$ *or* $y = \alpha x$ *for some* $\alpha \in \mathbb{R}$.

# Examples