Session 8: SVD, linear algebra & graphs

Optimization and Computational Linear Algebra for Data Science

Contents

- 1. Singular Value Decomposition
- 2. Graphs
- 3. Graph Laplacian
- 4. Spectral clustering

Midterm next week

- Thu. Oct. 29, the questions have to be downloaded from Gradescope between 00:01 AM and 9:59 PM.
- **Duration:** 1 hour and 40 minutes to work on the problems + 20 minutes to scan and upload your work.
- Upload your work as a single PDF.
- In case the upload does not work for you, email me your work.

Singular Value Decomposition

Singular Value decomposition

Theorem

Let $A \in \mathbb{R}^{n \times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ and a matrix $\Sigma \in \mathbb{R}^{n \times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$, that verify

$$A = U\Sigma V^{\mathsf{T}}.$$

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Singular Value Decomposition

Low-rank approximation

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Singular Value Decomposition

Graphs

Graphs

Graphs, degree

Graphs

Graph Laplacian

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Graph Laplacian

Definition

The Laplacian matrix of ${\cal G}$ is defined as

$$L = D - A.$$

Graph Laplacian

Definition

The Laplacian matrix of G is defined as

$$L = D - A.$$

For all
$$x \in \mathbb{R}^n$$
, $x^T L x = \sum_{i \sim j} (x_i - x_j)^2$.

Properties of the Lanlacian

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	Foi	rall	x	$\in \mathbb{R}$	n,		x	$^{T}L^{\mathfrak{J}}$	x =	$\sum_{i\sim i}$	•	_i –	x_j	$)^{2}.$				

Graph Laplacian

Properties of the Lanlacian

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Graph Laplacian

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Graph Laplacian

Algebraic connectivity

Proposition

The graph G is connected if and only if $\lambda_2>0$. More generally, the multiplicity of the eigenvalue 0 of L (i.e. the number of i such that $\lambda_i=0$) is equal to the number of connected components of G.

 λ_2 is sometimes called the «algebraic connectivity» of G and measure (heuristically) how well G is connected.

Exercise: show that λ_2 increases when one adds edges to G.

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Spectal clustering with the Laplacian

Algorithm

Input: Graph Laplacian L, number of clusters k

- 1. Compute the first k orthonormal eigenvectors v_1, \ldots, v_k of the Laplacian matrix L.
- 2. Associate to each node i the vector $x_i = (v_2(i), \dots, v_k(i))$.
- 3. Cluster the points x_1, \ldots, x_n with (for instance) the k-means algorithm.
- 4. Deduce a clustering of the nodes of the graph.

The case of two groups

For k=2 groups:

- 1. Compute the second eigenvector v_2 of the Laplacian matrix L.
- 2. Associate to each node i the number $x_i = v_2(i)$.
- 3. Cluster the nodes in:

$$S = \{i \,|\, v_2(i) \ge \delta\}$$
 and $S^c = \{i \,|\, v_2(i) < \delta\},$

for some $\delta \in \mathbb{R}$.

Cut of a partition

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Spectal clustering with the Laplacian

Minimal cut problem

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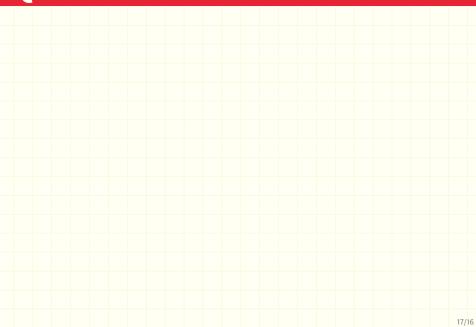
Spectal clustering with the Laplacian

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Spectal clustering with the Laplacian

Questions?



Questions?

