Session 8: SVD, spectral clustering on graphs

Optimization and Computational Linear Algebra for Data Science

Contents

- 1. Singular Value Decomposition
- 2. Graphs and Graph Laplacian
- 3. Spectral clustering

Midterm next week

- Thu. Oct. 29, the questions have to be downloaded from Gradescope between 00:01 AM and 9:59 PM.
- **Duration:** 1 hour and 40 minutes to work on the problems + 20 minutes to scan and upload your work.
- Upload your work as a single PDF.
- In case the upload does not work for you, email me your work.

Singular Value Decomposition

Singular Value decomposition

Theorem

Let $A \in \mathbb{R}^{n \times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ and a matrix $\Sigma \in \mathbb{R}^{n \times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$, that verify

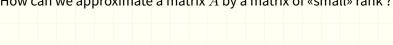
$$A = U\Sigma V^{\mathsf{T}}.$$

Singular Value Decomposition

Remarks																				

Low-rank approximation

How can we approximate a matrix A by a matrix of «small» rank?



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Singular Value Decomposition

Graphs and Graph Laplacian

Graphs

Graphs and Graph Laplacian

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Graph Laplacian

Definition

The Laplacian matrix of ${\cal G}$ is defined as

$$L = D - A.$$

Graph Laplacian

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.

For all
$$x \in \mathbb{R}^n$$
, $x^T L x = \sum_{i \sim j} (x_i - x_j)^2$.

Properties of the Laplacian

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Graphs and Graph Laplacian

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Graphs and Graph Laplacian

Algebraic connectivity

Proposition

- The multiplicity of the eigenvalue 0 of L (i.e. the number of i such that $\lambda_i = 0$) is equal to the number of connected components of G.
- In particular, G is connected if and only if $\lambda_2 > 0$.

- λ_2 is sometimes called the «algebraic connectivity» of G and measures somehow how well G is connected.
- From now, we assume that G is connected, i.e. $\lambda_2 > 0$.

Exercise: show that λ_2 increases when one adds edges to G.

Spectral clustering algorithm

Input: Graph Laplacian L, number of clusters k

- 1. Compute the first k orthonormal eigenvectors v_1,\ldots,v_k of the Laplacian matrix L.
- 2. Associate to each node i the vector $x_i = (v_2(i), \dots, v_k(i))$.
- 3. Cluster the points x_1, \ldots, x_n with (for instance) the k-means algorithm.
- 4. Deduce a clustering of the nodes of the graph.

The case of two groups

For k=2 groups:

- 1. Compute the second eigenvector v_2 of the Laplacian matrix L.
- 2. Associate to each node i the number $x_i = v_2(i)$.
- 3. Cluster the nodes in:

for some $\delta \in \mathbb{R}$.

$$S = \{i \mid v_2(i) \ge \delta\}$$
 and $S^c = \{i \mid v_2(i) < \delta\},$

Cut of a partition

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Minimal cut problem

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« Min-Cut » is NP-Hard

Spectral clustering with the Laplacian

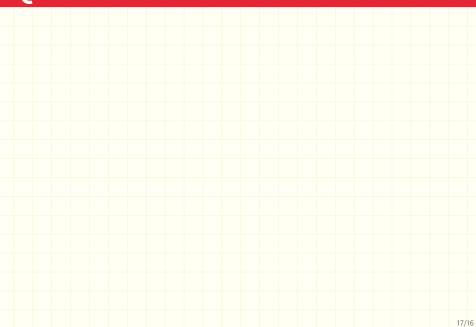
	Go	al:	minimize			$x^{T}Lx$			SU	ıbje	to	$\begin{cases} x \in \{-1, 1\}^n \\ x \perp (1, \dots, 1). \end{cases}$									

Spectral clustering as a «relaxation»

Idea: We first solve the « relaxed » problem:

minimize $v^\mathsf{T} L v$ subject to $\begin{cases} \|v\| = \sqrt{n} \\ v \perp (1, \dots, 1). \end{cases}$

Questions?



Questions?

