Optimization and Computational Linear Algebra for Data Science
Homework 5: Orthogonal matrices, eigenvalues and eigenvectors

Due on October 8, 2019

- Unless otherwise stated, all answers must be mathematically justified.

- Partial answers will be graded.

- Your submission has to be uploaded to Gradescope. Indicate Gradescope the page on which each problem is written.

- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).

- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.

- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 5.1 (2 points). *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.*
Show that if $v_1, v_2 \in \mathbb{R}^n$ are two eigenvectors of A associated to some eigenvalues $\lambda_1 \neq \lambda_2$ ($Av_1 = \lambda_1 v_1$ and $Av_2 = \lambda_2 v_2$), then $v_1 \perp v_2$.

Problem 5.2 (3 points). *Let S be a subspace of $\mathbb{R}^n$ and let P_S be the matrix of the orthogonal projection onto S. Let $M = \mathrm{Id}_n - 2P_S$.*

(**a**) *Show that the matrix M is orthogonal.*

(**b**) *Show that if $\lambda \in \mathbb{R}$ is an eigenvalue of M, then $\lambda = 1$ or $\lambda = -1$.*

Problem 5.3 (5 points). *Let $A \in \mathbb{R}^{n \times n}$. We assume that there exists a basis $(v_1, \dots, v_n)$ of $\mathbb{R}^n$ consisting of eigenvectors of A:*
$$Av_i = \lambda_i v_i$$
for all $i \in \{1, \dots, n\}$. We assume that
$$\lambda_1 > |\lambda_i| \quad \text{for all } i \in \{2, \dots, n\}.$$
We consider the following algorithm:

- *Initialize $x_0 \in \mathbb{R}^n$.*

- *Perform the updates: $x_{t+1} = \frac{Ax_t}{\|Ax_t\|}$.*

(**a**) *Show that for all $t \geq 1$,*
$$x_t = \frac{A^t x_0}{\|A^t x_0\|}.$$

(**b**) *Assume that x_0 is a unit vector ($\|x_0\| = 1$) whose direction is chosen uniformly at random (this basically means that all the possible directions for x_0 are equally likely to be chosen). Let $(\alpha_1, \dots, \alpha_n)$ be the coordinates of x_0 in the basis $(v_1, \dots, v_n)$. Explain why we can be sure that $\alpha_1 \neq 0$. You do not have to do a rigorous proof of that, just give an intuitive argument.*

(c) *Show that*

$$x_t \xrightarrow[t\to\infty]{} \frac{\alpha_1 v_1}{\|\alpha_1 v_1\|} \qquad and \qquad \|Ax_t\| \xrightarrow[t\to\infty]{} \lambda_1.$$

(d) *Code the algorithm in* `Python`: *write a function* `power_iteration` *that takes a matrix A as input and outputs x_t and $\|Ax_t\|$ for the first t such that $\|x_t - x_{t-1}\| \le 10^{-6}$. Here we assume that the matrix A given as input verifies the assumption of this problem.*

Problem 5.4 ($\star$). *Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Define the function*

$$\begin{aligned} f: \quad \mathbb{R}^n \setminus \{0\} \quad &\to \quad \mathbb{R} \\ x \quad &\mapsto \quad \frac{x^\mathsf{T} A x}{x^\mathsf{T} x}. \end{aligned}$$

Show that f has a maximum at some $x_\star \in \mathbb{R}^n \setminus \{0\}$ and that $x_\star$ verifies

$$A x_\star = \lambda x_\star, \qquad where \quad \lambda = f(x_\star).$$