Optimization and Computational Linear Algebra for Data Science
Homework 6: Eigenvectors and Markov chains

Due on October 15, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 6.1 (2 points)**.** *Let $A \in \mathbb{R}^{n\times n}$ be a symmetric matrix. Show that A is orthogonal if and only if its eigenvalues all have absolute value 1.*

Problem 6.2 (3 points)**.** *We say that a symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive semi-definite if for all $x \in \mathbb{R}^n$,*

$$x^\mathsf{T} M x \geq 0.$$

(a) *Let $A \in \mathbb{R}^{n\times k}$. Show that AA^T is positive semi-definite.*

(b) *Show that a symmetric matrix $M \in \mathbb{R}^{n\times n}$ is positive semi-definite if and only if all its eigenvalues are non-negative.*

(c) *Let $M \in \mathbb{R}^{n\times n}$ be a (symmetric) positive semi-definite matrix. Let $r = \mathrm{rank}(M)$. Show that there exists $A \in \mathbb{R}^{n\times r}$ such that $M = AA^\mathsf{T}$.*

Problem 6.3 (5 points)**.** *The CSV file `nba17-18.csv` contains the scores of all the games of the NBA basketball 2017-2018 season, playoffs/finals included. The* Jupyter notebook *`nba.ipynb` contains functions to read the CSV file and construct the 30×30 (there is 30 teams) matrix `winLoss` where*

$$\texttt{winLoss}_{i,j} = \textit{number of wins of team } i \textit{ against team } j.$$

Using this information, the NBA teams are ranked in `nba.ipynb` according to their win-loss percentage.

(a) *Rank the NBA teams according the "PageRank inspired ranking" described in Section 4.2 of the lecture notes #6.*

(b) *The ranking of question (a) uses only the number of win and loss, and not the scores of the games. Propose an evolution of the PageRank method of question (a) in order to takes the scores into account, and compute the corresponding ranking.*

It is intented that you code in Python and use the provided Jupyter Notebook. Your answers to this problem have to be submitted inside the provided Jupyter notebook. Please make use of comments/markdown cells (where you can type in Latex) in order to make your code/answers readable.

Problem 6.4 ($\star$)**.** *Let* $A \in \mathbb{R}^{n\times n}$ *such that for all* $x \in \mathbb{R}^n$, $x^\mathsf{T} A x \geq 0$. *Show that* $\mathrm{Ker}(A) = \mathrm{Ker}(A^\mathsf{T})$.