Optimization and Computational Linear Algebra for Data Science
Lecture 2: Linear transformations

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Warning: *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

1 General definitions

We present below the abstract mathematical definition of a vector space. **Please do not try to memorize it!** Simply remember that a vector space is a set whose elements are called *vectors*, that one can add vectors together and multiply them by real numbers called *scalars*.

Definition 1.1 (*Vector space*)

A vector space (over $\mathbb{R}$) consists of of a set V (whose elements are called vectors) and two operations $+$ and $\cdot$ that verify:

1. *The sum of two vectors is a vector: for all $\vec{x}, \vec{y} \in V$ we have $\vec{x} + \vec{y} \in V$.*
2. *The vector sum is commutative and associative. For all $\vec{x}, \vec{y}, \vec{z} \in V$ we have*
$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \qquad \text{and} \qquad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$
3. *There exists a zero vector $\vec{0} \in V$ that verifies $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$.*
4. *For all $\vec{x} \in V$, there exists $\vec{y} \in V$ such that $\vec{x} + \vec{y} = \vec{0}$. Such $\vec{y}$ is called the additive inverse of $\vec{x}$ and is written $-\vec{x}$.*
5. *Scalar multiplication: for all $\vec{x} \in V$ and all $\alpha \in \mathbb{R}$, $\alpha \cdot \vec{x} \in V$.*
6. *Identity element for scalar multiplication: $1 \cdot \vec{x} = \vec{x}$ for all $\vec{x} \in V$.*
7. *Compatibility between scalar multiplication and the usual multiplication: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$, we have*
$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$
8. *Distributivity: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,*
$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{y} \qquad \text{and} \qquad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

From now we will ignore $\cdot$ and simply write $\alpha\vec{x}$ instead of $\alpha \cdot \vec{x}$.

Example 1.1.

- *The set $V = \mathbb{R}^n$ endowed with the usual vector addition $+$*
$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$
and the usual scalar multiplication $\cdot$
$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

is a vector space.

- *The set* $V = \mathcal{F}(\mathbb{R},\mathbb{R}) \stackrel{\text{def}}{=} \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ *of all functions from* $\mathbb{R}$ *to itself endowed with the addition* $+$ *and the scalar multiplication* $\cdot$ *defined by*

$$\begin{array}{rccl} f+g: & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & f(t)+g(t) \end{array} \quad \text{and} \quad \begin{array}{rccl} \alpha \cdot f: & \mathbb{R} & \to & \mathbb{R} \\ & t & \mapsto & \alpha f(t) \end{array}$$

is a vector space.

Definition 1.2 (*Subspace*)

We say that a non-empty subset S *of a vector space* V *is a* subspace *is it is stable by addition and multiplication by a scalar, that is if*

(i) *for all* $x, y \in S$ *we have* $x + y \in S$*,*
(ii) *for all* $x \in S$ *and all* $\alpha \in \mathbb{R}$ *we have* $\alpha x \in S$*.*

Notice that a subspace is also a vector space!

2 Linear dependency

Definition 2.1 (*Span*)

Given vectors $x_1, \dots x_k \in V$ *the* span *of* $x_1, \dots, x_k$ *is the set of vectors that can be written as a linear combination of these:*

$$\mathrm{Span}(x_1, \dots, x_n) = \Big\{\alpha_1 x_1 + \cdots + \alpha_k x_k \,\Big|\, \alpha_1, \dots, \alpha_k \in \mathbb{R}\Big\}.$$

One can easily verify (exercise!) that $\mathrm{Span}(x_1, \dots, x_n)$ is a subspace of V.

Definition 2.2 (*Linear dependency*)

Vectors $x_1, \dots x_k \in V$ *are* linearly dependent *is there exists* $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ ***that are not all zero*** *such that*

$$\alpha_1 x_1 + \cdots + \alpha_k x_k = 0.$$

They are said to be linearly independent *otherwise.*

Saying that $x_1, \dots, x_k$ are linearly dependent precisely means that one of the vectors $x_1, \dots, x_k$ can be obtained as a linear combination of the others. Indeed if $x_1, \dots, x_k$ are linearly dependent, then we can find $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ that are not all zero (there exists i such that $\alpha_i \neq 0$) such that $\alpha_1 x_1 + \cdots + \alpha_k x_k = 0$. This leads to

$$x_i = \sum_{j \neq i} \frac{-\alpha_j}{\alpha_i} x_j,$$

i.e. the vector x_i can be expressed as a linear combinations of the vectors x_j for $j \neq i$. Conversely if we have for some i, and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

$$x_i = \alpha_1 x_1 + \cdots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \cdots \alpha_k x_k = 0.$$

then $\alpha_1 x_1 + \cdots + \alpha_{i-1} x_{i-1} - x_i + \alpha_{i+1} x_{i+1} + \cdots \alpha_k x_k = 0$ which gives that $x_1, \dots, x_k$ are linearly dependent.

Theorem 2.1

Let $v_1, \dots, v_n \in V$ and suppose that we have vectors $x_1, \dots, x_k \in V$ such that $k > n$ and $x_i \in \operatorname{Span}(v_1, \dots, v_n)$ for all $i \in \{1, \dots, k\}$. Then $x_1, \dots, x_k$ are linearly dependent.

Theorem 2.1 will be proved in Section 3.

Definition 2.3 (*Basis*)

A family $(x_1, \dots, x_n)$ of vectors of V is a basis of V if

(i) $x_1, \dots, x_n$ *are linearly independent,*

(ii) $\operatorname{Span}(x_1, \dots, x_n) = V$.

Definition 2.4 (*Dimension*)

If the vector space V admits a base $(v_1, \dots, v_n)$, then any base of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.

Proof. We proceed by contradiction and assume that there exists two basis $(v_1, \dots, v_n)$ and $(x_1, \dots, x_k)$ of V such that $k \neq n$. Without loss of generality we can assume that $k > n$. For $i = 1, \dots, k$ we have

$$x_i \in V = \operatorname{Span}(v_1, \dots, v_n),$$

because $(v_1, \dots, v_n)$ is a basis of V. We can therefore apply Theorem 2.1 to get that $x_1, \dots, x_{n+1}$ are linearly dependent. This contradicts the fact that $(x_1, \dots, x_k)$ is a basis. $\square$

Proposition 2.1 (*Coordinates*)

Let $(v_1, \dots, v_n)$ be a basis of V. Then for every $x \in V$ there exists a unique vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n.$$

We say that $(\alpha_1, \dots, \alpha_n)$ are the coordinates of x in the basis $(v_1, \dots, v_n)$.

Proof. Existence. $(v_1, \dots, v_n)$ forms a basis of V therefore $V = \operatorname{Span}(v_1, \dots, v_n)$. We get that $x \in \operatorname{Span}(v_1, \dots, v_n)$ which gives that there exists $\alpha_1, \dots, \alpha_n$ such that $x = \alpha_1 v_1 + \cdot + \alpha_n v_n$.
Unicity. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$ such that

$$x = \alpha_1 v_1 + \cdots + \alpha_n v_n = \beta_1 v_1 + \cdots + \beta_n v_n.$$

This leads to

$$(\alpha_1 - \beta_1) v_1 + \cdots + (\alpha_n - \beta_n) v_n = 0.$$

The vectors $v_1, \dots, v_n$ are linearly independent because they forms a basis. Consequently $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \cdots \alpha_n - \beta_n = 0$, i.e. $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n)$. $\square$

3 Proof of Theorem 2.1

Notice that it suffices to prove the theorem for $k = n + 1$ because if $x_1, \dots, x_{n+1}$ are linearly dependent, so are $x_1, \dots, x_{n+1}, \dots x_k$. We will therefore show for all $n \geq 1$

$$\begin{aligned} \mathcal{H}_n : \text{« For all } v_1, \dots, v_n \in V \text{ and all } x_1, \dots x_{n+1} \in \operatorname{Span}(v_1, \dots, v_n), \\ \text{the vectors } x_1, \dots, x_{n+1} \text{ are linearly dependent. »} \end{aligned}$$

Base case: $\mathcal{H}_1$ is true. Indeed, if $x_1, x_2 \in \mathrm{Span}(v_1)$, then there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $x_1 = \alpha_1 v_1$ and $x_2 = \alpha_2 v_1$. If $\alpha_1 = 0$ then $x_1 = 0$ and x_1, x_2 are therefore linearly dependent. Otherwise if $\alpha_1 \neq 0$ then $v_1 = \frac{1}{\alpha_1} x_1$ which then gives $x_2 = \frac{\alpha_2}{\alpha_1} x_1$: x_1, x_2 are linearly dependent.

Induction step: We assume now that $\mathcal{H}_{n-1}$ holds for some $n \geq 2$ and we will show that $\mathcal{H}_n$ holds. We consider therefore $x_1, \dots, x_{n+1} \in \mathrm{Span}(v_1, \dots, v_n)$. We can find real numbers $\alpha_{i,j}$ such that

$$\begin{array}{ccccccc} x_1 & = & \alpha_{1,1} v_1 & + & \cdots & + & \alpha_{1,n} v_n \\ x_2 & = & \alpha_{2,1} v_1 & + & \cdots & + & \alpha_{2,n} v_n \\ \vdots & & & & & & \\ x_{n+1} & = & \alpha_{n+1,1} v_1 & + & \cdots & + & \alpha_{n+1,n} v_n. \end{array}$$

We have to show that $x_1, \dots, x_{n+1}$ are linearly dependent. Let us consider the first line. If $\alpha_{1,1} = \alpha_{1,2} = \cdots = \alpha_{1,n} = 0$, then $x_1 = 0$ which gives then that $x_1, \dots, x_{n+1}$ are linearly dependent. Otherwise, there exists j such that $\alpha_{1,j} \neq 0$. Without loss of generality we can assume that $\alpha_{1,1} \neq 0$.

$$\begin{array}{ccccccc} x_1 & = & \alpha_{1,1} v_1 & + & \cdots & + & \alpha_{1,n} v_n \\ x_2 - \frac{\alpha_{2,1}}{\alpha_{1,1}} x_1 & = & 0 & + & \cdots & + & \alpha_{2,n} v_n - \frac{\alpha_{2,1}}{\alpha_{1,1}} \alpha_{1,n} v_n \\ \vdots & & & & & & \\ x_{n+1} - \frac{\alpha_{n+1,1}}{\alpha_{1,1}} x_1 & = & 0 & + & \cdots & + & \alpha_{n+1,n} v_n - \frac{\alpha_{n+1,1}}{\alpha_{1,1}} \alpha_{1,n} v_n. \end{array}$$

If we define $y_i \stackrel{\mathrm{def}}{=} x_i - \frac{\alpha_{i,1}}{\alpha_{1,1}} x_1$ for $i = 2, \dots, n+1$ we obtain have $y_i \in \mathrm{Span}(v_2, \dots, v_n)$. We can now apply the induction hypothesis $\mathcal{H}_{n-1}$ to get that $y_2, \dots, y_{n+1}$ are linearly dependent. This means that there exists $\beta_2, \dots \beta_{n+1}$ that are not all zero, such that $\beta_2 y_2 + \cdots + \beta_{n+1} y_{n+1} = 0$ which finally gives

$$\Big(- \beta_2 \frac{\alpha_{2,1}}{\alpha_1, 1} - \cdots - \beta_{n+1} \frac{\alpha_{n+1,1}}{\alpha_{1,1}} \Big) x_1 + \beta_2 x_2 + \cdots + \beta_{n+1} x_{n+1} = 0.$$

Since $\beta_2, \dots, \beta_{n+1}$ are not all zero we get that $x_1, \dots, x_{n+1}$ are linearly dependent. $\mathcal{H}_n$ is proved.