

# Lecture 2.1: Linear transformations

Optimization and Computational Linear Algebra for Data Science

# Contents

1. Definition of a linear transformation
2. Properties of linear transformations

# Definition

# Examples

You already know some linear transformations from high-school !

**Symmetry**

**Rotation**

# Definition

Symmetries (about a line passing through the origin) and rotations (about the origin) are mappings

$$\begin{aligned} L : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ v &\mapsto L(v), \end{aligned}$$

that are “linear”:

## Definition

A function  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear if

1. for all  $v \in \mathbb{R}^m$  and all  $\alpha \in \mathbb{R}$  we have  $L(\alpha v) = \alpha L(v)$  and
2. for all  $v, w \in \mathbb{R}^m$  we have  $L(v + w) = L(v) + L(w)$ .

# Examples of linear maps

❖  $L : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is linear  
 $(x, y) \mapsto (2x, 2y)$

❖  $M : \mathbb{R}^3 \rightarrow \mathbb{R}^2$  is linear  
 $(x, y, z) \mapsto (x + y + z, z)$

# An example of a non-linear map

The function  $F : \mathbb{R} \rightarrow \mathbb{R}$   
 $x \mapsto x^2$  is **not** linear.

# Properties



# Composition of linear maps

## Proposition

If  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $M : \mathbb{R}^n \rightarrow \mathbb{R}^k$  are both linear, then the composite function

$$\begin{array}{ccc} M \circ L : & \mathbb{R}^m & \rightarrow \mathbb{R}^k \\ & v & \mapsto M(L(v)) \end{array}$$

is also linear.

**Proof.**



# Elementary property

## Proposition

If  $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is linear, then

❖  $L(0) = 0.$

❖  $L\left(\sum_{i=1}^k \alpha_i v_i\right) = \sum_{i=1}^k \alpha_i L(v_i),$  for all  $\alpha_i \in \mathbb{R}, v_i \in \mathbb{R}^m.$

**Proof.**

