Recitation 1

Concept Review: Vector Spaces

Definition

A **vector space** is a set V endowed with two 'nice and compatible' operations + and \cdot that verify:

- For all $u, v \in V$, $u + v \in V$.
- For all $u \in V$ and all $\lambda \in \mathbb{R}$, $\lambda \cdot u \in V$.

Example: $V = \mathbb{R}^n$, with the usual vector addition + and scalar multiplication \cdot is a vector space.

Concept Review: Vector Spaces

In this class:

- We will always consider *real* scalars. Note that it is also possible to consider *complex* scalars.
- ightharpoonup V is (usually) \mathbb{R}^n , or (sometimes) $\mathbb{R}^{n \times m}$ (set of $n \times m$ matrices).

Concept Review: Subspaces

Definition (Subspace)

A non-empty subset S of a vector space V is called a *subspace* if it is closed under addition and scalar multiplication:

- 1. Closure under Addition: $x + y \in S$ for all $x, y \in S$.
- 2. Closure under Scalar Multiplication: $\alpha x \in S$, for all $x \in S$ and $\alpha \in \mathbb{R}$.
- A subspace is also a vector space!
- Subspaces are a recurring concept throughout this entire course.

Questions 1: Subspaces, Span

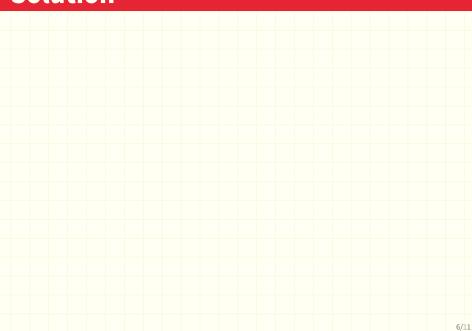
Consider the two vectors v=(1,1) and w=(-1,2). Describe the following sets geometrically. Which are subspaces of \mathbb{R}^2 ?

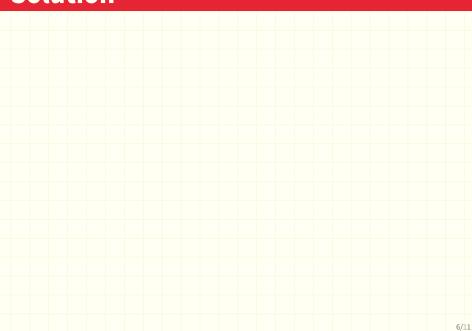
- 1. $\operatorname{Span}(v)$
- 2. $\operatorname{Span}(v, w)$
- 3. $\operatorname{Span}(v) \cup \operatorname{Span}(w)$, that is, the vectors in $\operatorname{Span}(v)$ or $\operatorname{Span}(w)$
- 4. $\operatorname{Span}(v) \cap \operatorname{Span}(w)$, that is, the vectors in both $\operatorname{Span}(v)$ and $\operatorname{Span}(w)$

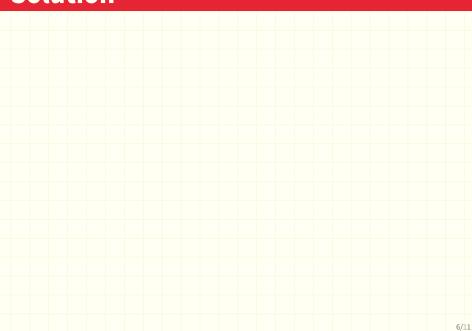
Questions 1: Subspaces, Span

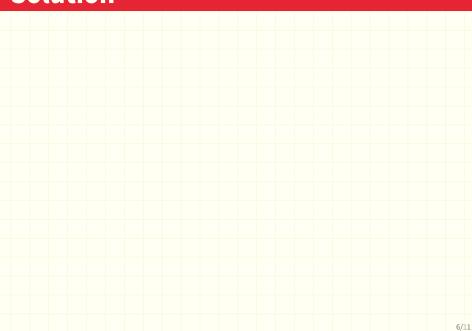
Consider the two vectors v=(1,1) and w=(-1,2). Describe the following sets geometrically. Which are subspaces of \mathbb{R}^2 ?

- 1. $\operatorname{Span}(v)$
- 2. $\operatorname{Span}(v, w)$
- 3. $\operatorname{Span}(v) \cup \operatorname{Span}(w)$, that is, the vectors in $\operatorname{Span}(v)$ or $\operatorname{Span}(w)$
- 4. $\operatorname{Span}(v) \cap \operatorname{Span}(w)$, that is, the vectors in both $\operatorname{Span}(v)$ and $\operatorname{Span}(w)$
- 5. $\{(1-t)v + tw | t \in [0,1]\}$
- 6. $\{(1-t)v + tw | t \in \mathbb{R}\}$
- 7. $\{\alpha v + \beta w | \alpha, \beta \ge 0\}$
- 8. Span(v, w, u) where u = (0, 5).
- 9. $\{(a,b) \in \mathbb{R}^2 | a^2 + b^2 \le 25\}$
- 10. $\{(a, a) \in \mathbb{R}^2 | a \in \mathbb{R} \}$









1. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$. Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\operatorname{Span}(v_1, v_2, v_3, v_4))$? (No formal proof necessary)

1. Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$. Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for $\dim(\operatorname{Span}(v_1, v_2, v_3, v_4))$? (No formal proof necessary)

2. Let $v_1,...,v_m\in\mathbb{R}^n$ be linearly dependent.

Prove that for $x\in \mathrm{Span}(v_1,\ldots,v_m)$, there exist infinitely many $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$ such that

$$x = \sum_{i=1}^{m} \alpha_i v_i.$$

2. Let $v_1,...,v_m\in\mathbb{R}^n$ be linearly dependent.

Prove that for $x\in \mathrm{Span}(v_1,\ldots,v_m)$, there exist infinitely many $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$ such that

$$x = \sum_{i=1}^{m} \alpha_i v_i.$$

3. True or False: If $B=(v_1,\ldots,v_n)$ is a basis for \mathbb{R}^n , and W is a subspace of \mathbb{R}^n , then some subset of B is a basis for W.

Questions 3: Bases, Dimension

Let V be the set of functions

$$V \stackrel{\mathrm{def}}{=} \left\{ p : \mathbb{R} \to \mathbb{R} \,\middle|\, p(x) = \sum_{k=0}^n a_k x^k, \text{ for some } a_0, \dots, a_n \in \mathbb{R}
ight\}$$

- 1. What kind of function does this set contain?
- 2. Define an addition operation $+: V \times V \to V$, and a scalar multiplication operation $\cdot: \mathbb{R} \times V \to V$, such that the triple $(V,+,\cdot)$ is a vector space.
- 3. What is the zero vector of this vector space?
- 4. Find a basis for this vector space.
- 5. What is the dimension of this vector space?







