

Optimization and Computational Linear Algebra for Data Science

Lecture 4: Norm and inner product

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Warning: *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

1 Norm

Definition 1.1 (*Norm*)

Let V be a vector space. A norm $\|\cdot\|$ on V is a function from V to $\mathbb{R}_{\geq 0}$ that verifies the following points:

- (i) Homogeneity: $\|\alpha v\| = |\alpha| \times \|v\|$ for all $\alpha \in \mathbb{R}$ and $v \in V$.
- (ii) Triangular inequality: $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$.
- (iii) Positive definiteness: if $\|v\| = 0$ for some $v \in V$, then $v = 0$.

Example 1.1. One can consider various norms over \mathbb{R}^n :

- The Euclidean norm $\|x\|_2 \stackrel{\text{def}}{=} \sqrt{\sum_{i=1}^n x_i^2}$.
- The ℓ_1 norm $\|x\|_1 \stackrel{\text{def}}{=} \sum_{i=1}^n |x_i|$.
- More generally, given $p \geq 1$, the ℓ_p -norm $\|x\|_p \stackrel{\text{def}}{=} (\sum_{i=1}^n |x_i|^p)^{1/p}$.
- The infinity-norm $\|x\|_\infty \stackrel{\text{def}}{=} \max(|x_1|, \dots, |x_n|)$.

2 Inner product

Definition 2.1 (*Inner product*)

Let V be a vector space. An inner product on V is a function $\langle \cdot, \cdot \rangle$ from $V \times V$ to \mathbb{R} that verifies the following points:

- (i) Symmetry: $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$.
- (ii) Linearity: $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \alpha v, w \rangle = \alpha \langle v, w \rangle$ for all $u, v, w \in V$ and $\alpha \in \mathbb{R}$.
- (iii) Positive definiteness: $\langle v, v \rangle \geq 0$ with equality if and only if $v = 0$.

Example 2.1.

- For $V = \mathbb{R}^n$, the Euclidean inner product $\langle x, y \rangle = \sum_{i=1}^n x_i y_i = x^\top y$ is an inner product.

- If V is the set of all continuous functions on $[0, 1]$, then $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$ is an inner product.

Proposition 2.1 (Norm induced by an inner product)

If $\langle \cdot, \cdot \rangle$ is an inner product on V then $\|v\| \stackrel{\text{def}}{=} \sqrt{\langle v, v \rangle}$ is a norm on V . We say that the norm $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$.

Remark 2.1. The Euclidean norm $\|\cdot\|_2$ on $V = \mathbb{R}^n$ is induced by the Euclidean inner product $x \cdot y = \sum_{i=1}^n x_i y_i$. Indeed, for $x \in \mathbb{R}^n$,

$$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2} = \sqrt{x \cdot x}.$$

Exercise 2.1. Let $\langle \cdot, \cdot \rangle$ be an inner product on \mathbb{R}^n , and let $\|\cdot\|$ be the induced norm by $\langle \cdot, \cdot \rangle$.

(a) Show that for all $x, y \in \mathbb{R}^n$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

(b) Deduce from the previous question that the ℓ_1 norm $\|\cdot\|_1$ and the infinity norm $\|\cdot\|_\infty$ are **not** induced by an inner product.

Theorem 2.1 (Cauchy-Schwarz inequality)

Let $\|\cdot\|$ be the norm induced by the inner product $\langle \cdot, \cdot \rangle$ on the vector space V . Then for all $x, y \in V$:

$$|\langle x, y \rangle| \leq \|x\| \|y\|. \quad (1)$$

Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

Proof. If $x = 0$ or $y = 0$ the result is obvious, we assume therefore to be in the case where $x \neq 0$ and $y \neq 0$. For $t \in \mathbb{R}$ we define the function $f(t) = \|tx - y\|^2$. Since the norm $\|\cdot\|$ is induced by the inner product $\langle \cdot, \cdot \rangle$ we have

$$f(t) = \langle tx - y, tx - y \rangle = t^2\|x\|^2 - 2t\langle x, y \rangle + \|y\|^2.$$

f is therefore a quadratic function of t . Notice that f is non-negative because $f(t) = \|tx - y\|^2 \geq 0$. This gives that its discriminant Δ is non-positive:

$$\Delta = (2\langle x, y \rangle)^2 - 4\|x\|^2\|y\|^2 \leq 0,$$

which proves (1). We have equality in (1) if and only if $\Delta = 0$ that is if and only if f admits a zero α , which is equivalent to $\alpha x - y = 0$, i.e. $y = \alpha x$. \square

3 Orthogonality

In this section we consider an inner product $\langle \cdot, \cdot \rangle$ (that induces a norm $\|\cdot\|$) on a vector space V . For simplicity one may think of $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to be the usual Euclidean dot product and norm on $V = \mathbb{R}^n$.

Definition 3.1 (*Orthogonality*)

- We say that vectors x and y are orthogonal if $\langle x, y \rangle = 0$. We write then $x \perp y$.
- We say that a vector x is orthogonal to a set of vectors $A \subset V$ if x is orthogonal to all the vectors in A , i.e. $\forall y \in A, \langle x, y \rangle = 0$. We write then $x \perp A$.
- More generality we say that $A \subset V$ and $B \subset V$ are orthogonal if $\langle x, y \rangle = 0$ for all $x \in A$ and all $y \in B$. As before, we write $A \perp B$.

Theorem 3.1 (*Pythagorean theorem*)

Let $x, y \in V$. Then

$$x \perp y \iff \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

Definition 3.2 (*Orthogonal and orthonormal families of vectors*)

Let v_1, \dots, v_k be vectors of V . We say that the family of vectors (v_1, \dots, v_k) is

- orthogonal if the vectors v_1, \dots, v_n are pairwise orthogonal, i.e. $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- orthonormal if it is orthogonal and if all the v_i have unit norm: $\|v_1\| = \dots = \|v_k\| = 1$.

Orthonormal basis are particularly convenient for computing coordinates of vectors:

Proposition 3.1

Assume that $\dim(V) = n$ and let (v_1, \dots, v_n) be an **orthonormal** basis of V . Then the coordinates of a vector $x \in V$ in the basis (v_1, \dots, v_n) are $(\langle v_1, x \rangle, \dots, \langle v_n, x \rangle)$:

$$x = \langle v_1, x \rangle v_1 + \dots + \langle v_n, x \rangle v_n.$$

Moreover

$$\|x\| = \sqrt{\langle v_1, x \rangle^2 + \dots + \langle v_n, x \rangle^2}.$$

4 Orthogonal projection and distance to a subspace

We assume in this section that $V = \mathbb{R}^n$ and that $\langle \cdot, \cdot \rangle, \|\cdot\|$ are respectively the Euclidean dot product and Euclidean norm.

Definition 4.1 (*Orthogonal projection and distance to a subspace*)

Let S be a subspace of \mathbb{R}^n . The orthogonal projection of a vector x onto S is defined as the vector $P_S(x)$ in S that minimizes the distance to x :

$$P_S(x) \stackrel{\text{def}}{=} \arg \min_{y \in S} \|x - y\|.$$

The distance of x to the subspace S is then defined as

$$d(x, S) \stackrel{\text{def}}{=} \min_{y \in S} \|x - y\| = \|x - P_S(x)\|.$$

Proposition 4.1

Let S be a subspace of \mathbb{R}^n and let (v_1, \dots, v_k) be an **orthonormal basis** of S . Then for all $x \in \mathbb{R}^n$,

$$P_S(x) = \langle v_1, x \rangle v_1 + \dots + \langle v_k, x \rangle v_k.$$

In other words, if we let

$$V = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_k \\ | & | & \cdots & | \end{pmatrix} \in \mathbb{R}^{n \times k},$$

then P_S is a linear transformation whose matrix is VV^\top :

$$\forall x \in \mathbb{R}^n, \quad P_S(x) = VV^\top x.$$

Proof. Let us add vectors v_{k+1}, \dots, v_n to the basis (v_1, \dots, v_k) to obtain an orthonormal basis of \mathbb{R}^n . (This is made possible by the Gram-Schmidt orthonormalization principle that we will see in the next lecture.) Let $\alpha_1 = \langle x, v_1 \rangle, \dots, \alpha_n = \langle x, v_n \rangle$ be the coordinates of x in the basis (v_1, \dots, v_n) . Let $y \in S$, and let β_1, \dots, β_k be its coordinates in the basis (v_1, \dots, v_k) . By Proposition 3.1:

$$\|x - y\|^2 = \sum_{i=1}^k (\alpha_i - \beta_i)^2 + \sum_{i=k+1}^n \alpha_i^2.$$

Minimizing this quantity over $y \in S$ is equivalent to minimizing it over the coordinates β_1, \dots, β_k of y . The minimum is uniquely achieved for $\beta_i = \alpha_i$ for all i , hence

$$P_S(x) \stackrel{\text{def}}{=} \arg \min_{y \in S} \|x - y\| = \alpha_1 v_1 + \dots + \alpha_k v_k = \langle v_1, x \rangle v_1 + \dots + \langle v_k, x \rangle v_k.$$

The second part of the proposition is a rewriting of this last equation, obtained by noticing that

$$V^\top x = \begin{pmatrix} -v_1 - \\ \vdots \\ -v_k - \end{pmatrix} x = \begin{pmatrix} \langle v_1, x \rangle \\ \vdots \\ \langle v_k, x \rangle \end{pmatrix}.$$

□

Corollary 4.1

For all $x \in \mathbb{R}^n$,

- $x - P_S(x)$ is orthogonal to S .
- $\|P_S(x)\| \leq \|x\|$.

Definition 4.2 (Orthogonal complement)

Let S be a subspace of \mathbb{R}^n . The orthogonal complement of S is defined by

$$S^\perp \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \mid x \perp S\} = \{x \in \mathbb{R}^n \mid \forall y \in S, \langle x, y \rangle = 0\}.$$

Proposition 4.2

Let S be a subspace of \mathbb{R}^n . Then S^\perp is also a subspace of \mathbb{R}^n with dimension

$$\dim(S^\perp) = n - \dim(S).$$

