Recitation 2

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Questions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$
2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$
3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$
4. $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$
5. $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

Solutions 1: Linear Transformations

Which of the following functions are linear? If the function is linear, what is the kernel?

1. $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a,b) = (2a, a+b)$
2. $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a,b) = (a+b, 2a+2b, 0)$
3. $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a,b) = (2a, a+b, 1)$
4. $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a,b) = \sqrt{a^2+b^2}$
5. $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x+3$

Solution

1. *Linear, Kernel is* $\{0\}$.
2. *Linear, Kernel is* $\{(c,-c) : c \in \mathbb{R}\}$.
3. *Not linear,* $f_3(0,0) = (0,0,1)$.
4. *Not linear,* $f_4(1,0) + f_4(0,1) = 2$ *and* $f_4(1,1) = \sqrt{2}$.
5. *Not linear,* $f_5(0) = 3$.

Some Etymology...

Linearity: Wikipedia

The property of a mathematical relationship (function) that can be graphically represented as a straight line.

- (!) **Huge caveat** - 'Linear' in Linear Algebra actually refers to *linear equations*.
- We will actually study many curved objects, using linear tools

Algebra: Wikipedia

The study of mathematical symbols and the rules for manipulating these symbols.

- Linear algebra is the study of *manipulating* letters/symbols which are used to represent linear transformations.
- Two types of manipulation....

Type 1: Linear Transformations as Letters

Definition (Linear Transformation)

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if

1. for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and
2. for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.

- Our linear transformation here is represented by the *letter* L.
- We saw/will see rules for manipulating linear transformations from an *algebraic* perspective, such as...
  - Associative, but not Commutative
  - When are matrices invertible?
  - How to take Derivatives? (Homework 9)
- Practice questions later

Type 2: Linear Transformations as Matrices

Theorem (Matrix Representation Theorem)

All linear transformations represent matrices;
all matrices represent linear transformations.

- Important, but boring theorem.
- Linear transformations can also be represented by matrices
$$L = \begin{bmatrix} L_{1,1} & \dots & L_{1,n} \\ \vdots & \ddots & \vdots \\ L_{m,1} & \dots & L_{m,n} \end{bmatrix}$$
- We will also examine the *mechanical* perspective of linear transformations, such as...
    - How to actually multiply?
    - Interpretation of multiplication
    - Using matrix multiplication simply for calculations. (Removing the notion of a transformation)
- (!) Think about which framework to use in your proofs!

Matrix Notation: (Reference Slide)

- A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is represented by a $m \times n$ matrix which is an element of $\mathbb{R}^{m\times n}$. (Note the order!)

$$\mathrm{T} = \begin{array}{cc} & n \\ m & \begin{pmatrix} T_{1,1} & \dots & T_{1,n} \\ \vdots & \ddots & \vdots \\ T_{m,1} & \dots & T_{m,n} \end{pmatrix} \end{array}$$

- This matrix has m rows and n columns.
- $T_{i,j}$ represents the entry in the ith row and jth column.

Definition (Matrix product)

Let $A \in \mathbb{R}^{n\times k}$ and $B \in \mathbb{R}^{k\times n}$.
AB is the $n \times m$ matrix of the $A \circ B$, with coefficients

$$(AB)_{r,c} = \sum_{i=1}^{k} A_{r,i}B_{i,c} \quad \text{for all} \quad 1 \leq r \leq n, \quad 1 \leq c \leq m.$$

(r, c) denote row,column

Matrix Multiplication Mechanics: Inner Products

- Next few slides go over "Inner Product Method" of matrix multiplication.
  - We haven't covered inner products yet
- Each entry of the resultant matrix is an *inner product* of a row of the first matrix and a column of the second matrix
- This is the *exact* definition of matrix multiplication.
- Most straightforward way to calculate a matrix product

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{k\times m}$

Rows of first matrix "line up" with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{i=1}^{k} a_{1,i} b_{i,1} & \dots & \dots \\ \sum_{i=1}^{k} a_{2,i} b_{i,1} & \dots & \dots \\ \vdots & \dots & \dots \\ \sum_{i=1}^{k} a_{n-1,i} b_{i,1} & \dots & \dots \\ \sum_{i=1}^{k} a_{n,i} b_{i,1} & \dots & \dots \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix "line up" with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} \dots & \sum_{i=1}^{k} a_{1,i} b_{i,2} & \dots \\ \dots & \sum_{i=1}^{k} a_{2,i} b_{i,2} & \dots \\ \dots & \vdots & \dots \\ \dots & \sum_{i=1}^{k} a_{n-1,i} b_{i,2} & \dots \\ \dots & \sum_{i=1}^{k} a_{n,i} b_{i,2} & \dots \end{bmatrix}$$

Matrix Multiplication Mechanics: Inner Products

Let $A \in \mathbb{R}^{n \times k}$, $B \in \mathbb{R}^{k \times m}$

Rows of first matrix “line up” with columns of the second matrix.

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,k} \\ a_{2,1} & \dots & a_{2,k} \\ \vdots & \dots & \vdots \\ a_{n-1,1} & \dots & a_{n-1,k} \\ a_{n,1} & \dots & a_{n,k} \end{bmatrix} \begin{bmatrix} b_{1,1} & b_{1,2} & \dots & b_{1,m-1} & b_{1,m} \\ \vdots & \vdots & \dots & \vdots & \vdots \\ b_{k,1} & b_{k,2} & \dots & b_{k,m-1} & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} \dots & \dots & \sum_{i=1}^{k} a_{1,i} b_{i,m} \\ \dots & \dots & \sum_{i=1}^{k} a_{2,i} b_{i,m} \\ \dots & \dots & \vdots \\ \dots & \dots & \sum_{i=1}^{k} a_{n-1,i} b_{i,m} \\ \dots & \dots & \sum_{i=1}^{k} a_{n,i} b_{i,m} \end{bmatrix}$$

- This is the *exact* definition of matrix multiplication.
- Most straightforward way to calculate a matrix product

More M.M.M: Linear Combination of Columns

- Next few slides go over "Linear Combination of Columns" method of matrix multiplication.
- Each *column* of the result is a *linear combination of the columns* of the first matrix.
- Much more interpretable!
- (!) Keep an eye out for this
- Less straightforward way of calculating

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A.

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} | & \cdots & | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} & \cdots & \sum_{i=1}^{k} \mathbf{a_i} b_{i,m} \\ | & \cdots & | \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

Each column of the AB is a linear combination of the columns of A.

$$\begin{bmatrix} | & | & \dots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \dots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \dots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} & \dots & b_{1,m} \\ b_{2,1} & \dots & b_{2,m} \\ \vdots & \vdots & \vdots \\ b_{k-1,1} & \dots & b_{k-1,m} \\ b_{k,1} & \dots & b_{k,m} \end{bmatrix}$$

$$= \begin{bmatrix} | & \dots & | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} & \dots & \sum_{i=1}^{k} \mathbf{a_i} b_{i,m} \\ | & \dots & | \end{bmatrix}$$

More M.M.M: Linear Combination of Columns

One dimensional case (for B):

$$\begin{bmatrix} | & | & \cdots & | & | \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_{k-1}} & \mathbf{a_k} \\ | & | & \cdots & | & | \end{bmatrix} \begin{bmatrix} b_{1,1} \\ b_{2,1} \\ \vdots \\ b_{k-1,1} \\ b_{k,1} \end{bmatrix}$$

$$= \begin{bmatrix} | \\ \sum_{i=1}^{k} \mathbf{a_i} b_{i,1} \\ | \end{bmatrix}$$

- Result is in the span of columns of A!
- Much more interpretable!
- (!) Keep an eye out for this, especially if columns of A have meaning.

Questions 2: Matrix Manipulation

Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

1. Calculate AB
2. Calculate BC
3. What does A do to B?
4. What does C do to B?
5. Can you find an x s.t $Cx = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$?

Solutions 2: Matrix Manipulation

Solution

1. $AB = \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$
2. $BC = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 4 & 1 \end{bmatrix}$
3. *Five times first row, switch second and third row*
4. *First column becomes twice the second column plus one times third column, second column stays the same, switch 3rd and fourth columns.*
5. *No,* $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *is not in the span of the columns of* C

Linear Transformations and Subspaces

- Linear transformations are *fundamentally connected* to subspaces.
- We will spend a lot of time on investigating the *action* of a linear transformation *from* subspaces, and *to* subspaces
- Key questions in linear algebra:
  - How do linear transformations cut up vector spaces? (Kernel, Image)
  - What are "nice" combinations of 1-dimensional subspaces? (Lec 4, 5)
  - For a given linear transformation, are there certain, *special* subspaces? (Lec 6,7)

Questions 3: Invertibility

Definition (Matrix inverse)

A matrix $M \in \mathbb{R}^{n \times n}$ is called *invertible* if there exists a matrix $M^{-1} \in \mathbb{R}^{n \times n}$ such that

$$MM^{-1} = M^{-1}M = \mathrm{Id}_n.$$

Such matrix M^{-1} is unique and is called the *inverse* of M.

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.
Let S and U be invertible.

1. Prove that $Ker(S) = \{0\}$.

Now, prove or give a counter example to the following statements:

2. $Ker(T) = Ker(TU)$
3. $Ker(ST) = Ker(T)$

Solutions 3: Invertibility

Let $S \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{n \times k}$ and $U \in \mathbb{R}^{k \times k}$.
Let S and U be invertible.

1. Prove that $Ker(S) = \{0\}$.

Solution

We prove by contradiction.
Suppose that $Ker(S) \neq 0$. *Then* $\exists x \neq 0$ *s.t* $Sx = 0$.
Now, consider $S^{-1}Sx$.

$(S^{-1}S)x = Ix = x,$
and $S^{-1}(Sx) = 0.$

We have reached a contradiction, so $Ker(S) = 0$

Solution

2. $Ker(T) = Ker(TU)$. ***False***

   $Consider\ T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} U = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

   $Ker(T) = \{\begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R}\}.$

   $\mathrm{Ker}(TU) = \{\begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R}\}$

3. $Ker(ST) = Ker(T)$. ***True***

   We'll show that $Ker(ST) \subset Ker(T)$.
   Let $x \in Ker(ST)$.
   So, $STx = 0$.
   Since S *is invertible, then* $Ker(S) = 0$.
   Therefore, $Tx = 0$, *and* $x \in Ker(T)$.
   $Ker(T) \subset Ker(ST)$ *is straightforward.*

Question 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$. Show that:
   $Ker(T) \cap Im(T) = \{0\} \iff$ If $T^2 v = 0$, then $Tv = 0$

   (Second part means that $\forall v \in \mathbb{R}^n$ s.t $T^2 v = 0$, we have $Tv = 0$.)

Solution 4: Kernel and Image

1. Let $T \in \mathbb{R}^{n \times n}$, and let $v \in R^n$. Show that:
   $Ker(T) \cap Im(T) = \{0\} \iff$ If $T^2 v = 0$, then $Tv = 0$

Solution

$(\implies)$
Assume that $Ker(T) \cap Im(T) = \{0\}$.
Assume that $T^2 v = 0$. *We will show that* $Tv = 0$
Since $T^2 v = 0$, *then* $T(Tv) = 0$, *so* $Tv \in Ker(T)$.
Now, by definition, $Tv \in Im(T)$, *so* $Tv \in Ker(T) \cap Im(T)$, *and* $Tv = 0$
$(\impliedby)$
Assume that $T^2 v = 0 \implies Tv = 0$
Let $y \in Ker(T)$, *and* $y \in \mathrm{Im}(T)$. *We show that* $y = 0$.
Since $y \in Ker(T)$, *then* $Ty = 0$.
Since $y \in Im(T)$, *then* $\exists x$ *s.t* $Tx = y$.
Then $0 = Ty = T(Tx) = T^2 x$. *Since* $T^2 x = 0$, *then* $Tx = 0$. *So* $y = Tx = 0$.