Optimization and Computational Linear Algebra for Data Science
Lecture 2: Linear transformations

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**Warning:** *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

# 1 Linear transformations

**Definition 1.1 (*Linear transformation*)**

*A function $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if*

*(i) for all $v \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$ we have $L(\alpha v) = \alpha L(v)$ and*

*(ii) for all $v, w \in \mathbb{R}^m$ we have $L(v + w) = L(v) + L(w)$.*

Notice that $L : \mathbb{R}^m \to \mathbb{R}^n$ is linear if and only if $L(\alpha v + w) = \alpha L(v) + L(w)$ for all $v, w \in \mathbb{R}^m$ and all $\alpha \in \mathbb{R}$.

**Proposition 1.1**

*The set $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ of all linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$ is a vector space.*

**Proposition 1.2**

*If $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^k$ are two linear transformations, then the composite function $M \circ L : \mathbb{R}^m \to \mathbb{R}^k$ is also linear.*

**Theorem 1.1 (*Equality on a basis implies equality everywhere*)**

*Let $L$ and $M$ be two linear transformations from $\mathbb{R}^m$ to $\mathbb{R}^n$. Let $(v_1, \dots, v_m)$ be a basis of $\mathbb{R}^m$ and suppose that for all $i \in \{1, \dots, m\}$ we have*

$$L(v_i) = M(v_i).$$

*Then $L = M$, i.e. $L(v) = M(v)$ for all $v \in \mathbb{R}^m$.*

# 2 Matrix representation

From Theorem 1.1 we know that a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ is uniquely characterized by the image $L(v_1), \dots, L(v_m)$ of any basis $(v_1, \dots, v_m)$ of the input space.

We consider the canonical basis $(e_1, \dots, e_m)$ of $\mathbb{R}^m$ and encode $L$ by a $n \times m$ matrix (that we will write also $L$) whose columns are $L(e_1), \dots, L(e_m)$:

$$L = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_m) \\ | & | & & | \end{pmatrix} = \begin{pmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,m} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n,1} & L_{n,2} & \cdots & L_{n,m} \end{pmatrix}$$

where we write $L(e_j) = \begin{pmatrix} L_{1,j} \\ L_{2,j} \\ \vdots \\ L_{n,j} \end{pmatrix}$. The matrix $L$ is called the (canonical) matrix of the linear transformation $L$. We denote by $\mathbb{R}^{n\times m}$ the set of all $n \times m$ matrices.

*Example* 2.1 (Homothety). Let $\lambda \in \mathbb{R}$. The mapping (called "homothety of ratio $\lambda$")

$$\begin{array}{rccc} L: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & \lambda x \end{array}$$

is linear. The canonical matrix of $L$ is

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

In the case where $\lambda = 1$, $L$ is simply the identity, its matrix is called the identity matrix and denoted by

$$\mathrm{Id}_n \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

**Definition 2.1 (*Matrix product*)**

*Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{k\times n}$. The product $ML$ is the $k \times m$ matrix defined by*

$$(ML)_{i,j} = \sum_{r=1}^{n} M_{i,r} L_{r,j} \quad \text{for all} \quad 1 \le i \le k, \quad 1 \le j \le m.$$

*For $x \in \mathbb{R}^n$, we define the matrix-vector product $Mx \in \mathbb{R}^k$ by*

$$(Lx)_i = \sum_{r=1}^{n} M_{i,r} x_r, \qquad 1 \le i \le k.$$

*Notice that this corresponds – if we see the vector $x \in \mathbb{R}^n$ as a $n \times 1$ matrix – to the matrix product between $M$ and $x$.*

**Proposition 2.1**

*Let $M \in \mathbb{R}^{n\times m}$. Then for all $x \in \mathbb{R}^m$, $M(x) = Mx$.*

**Proposition 2.2 (*Matrix product means composition of linear transformations*)**

*Let $L : \mathbb{R}^m \to \mathbb{R}^n$ and $M : \mathbb{R}^n \to \mathbb{R}^k$ be two linear transformations whose matrices are also denoted by $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{k\times n}$. Then the $k \times m$ matrix $ML$ is the matrix of the linear transformation $M \circ L : \mathbb{R}^m \to \mathbb{R}^k$.*

**Proposition 2.3**

*Let $A \in \mathbb{R}^{p\times q}$, $B \in \mathbb{R}^{q\times r}$ and $C \in \mathbb{R}^{r\times s}$. Then*

$$(AB)C = A(BC).$$

**Proposition 2.4 (*Matrix inverse*)**

*Let $M \in \mathbb{R}^{n\times n}$. Assume that there exists a matrix $M^{-1} \in \mathbb{R}^{n\times n}$ such that*

$$MM^{-1} = \mathrm{Id}_n \quad \text{or, such that} \quad M^{-1}M = \mathrm{Id}_n.$$

*Then $MM^{-1} = M^{-1}M = \mathrm{Id}_n$ and $M^{-1}$ is the unique matrix that verifies this property. We say that $M$ is* invertible *and the matrix $M^{-1}$ is called the* inverse *of $M$.*

**Remark 2.1.** *$M \in \mathbb{R}^{n\times n}$ is invertible if and only if the linear transformation associated to $M$ is a bijection. In that case, $M^{-1}$ is the matrix associated to the inverse transformation.*

# 3 Kernel and image

**Definition 3.1 (*Kernel*)**

*The kernel* $\mathrm{Ker}(L)$ *(or nullspace) of a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ is defined as the set of all vectors $v \in \mathbb{R}^m$ such that $L(v) = 0$, i.e.*

$$\mathrm{Ker}(L) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^m \,|\, L(v) = 0\}.$$

**Definition 3.2 (*Image*)**

*The image* $\mathrm{Im}(L)$ *(or column space) of a linear transformation $L : \mathbb{R}^m \to \mathbb{R}^n$ is defined as the set of all vectors $u \in \mathbb{R}^n$ such that there exists $v \in \mathbb{R}^m$ such that $L(v) = u$.* $\mathrm{Im}(L)$ *is also the Span of the columns of the matrix representation of $L$.*

**Proposition 3.1**

$\mathrm{Ker}(L)$ *and* $\mathrm{Im}(L)$ *are subspaces of respectively $\mathbb{R}^m$ and $\mathbb{R}^n$. We have*

$$L \text{ injective} \iff \mathrm{Ker}(L) = \{0\}$$

*and*

$$L \text{ surjective} \iff \mathrm{Im}(L) = \mathbb{R}^m.$$

**Application: Solutions of a linear system.** We are interested into solving the system of equations in $x = (x_1, \dots, x_m) \in \mathbb{R}^m$

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \dots & + & a_{1,m}x_m & = & y_1 \\ \quad\vdots \\ a_{n,1}x_1 & + & a_{n,2}x_2 & + & \dots & + & a_{n,m}x_m & = & y_n \end{cases} \tag{1}$$

where $a_{i,j} \in \mathbb{R}$ and $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. If we define the matrix $A \in \mathbb{R}^{n\times m}$ by $A_{i,j} = a_{i,j}$ the system (1) can be rewritten as

$$Ax = y.$$

Solving (1) precisely mean « finding the inverse image of $y$ by $A$ ». From the definition of $\mathrm{Im}(A)$ we get that **the equation $Ax = y$ admits (at least) a solution $x_0$ if and only if** $y \in \mathrm{Im}(A)$.

We suppose now to be in that case. We would now like to know if there are other solutions. Let $x$ be another solution to $Ax = y$. By subtraction we get

$$A(x - x_0) = y - y = 0.$$

This means that $(x - x_0) \in \mathrm{Ker}(A)$: any solution of $Ax = y$ can therefore be written as $x = x_0 + v$ with $v \in \mathrm{Ker}(A)$. Conversely, one can verify easily that any vector of this form is a solution. We conclude that if the equation $Ax = y$ admits a solution $x_0$, then the set of **all** solutions is

$$x_0 + \mathrm{Ker}(A) \stackrel{\text{def}}{=} \{x_0 + v \,|\, v \in \mathrm{Ker}(A)\}.$$

In particular, $x_0$ **is the unique solution if and only if** $\mathrm{Ker}(A) = \{0\}$.