Session 4: Norms and inner-products

Optimization and Computational Linear Algebra for Data Science

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Orthogonality

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Definition

Definition

- We say that vectors x and y are orthogonal if $\langle x, y \rangle = 0$. We write then $x \perp y$.
- We say that a vector x is orthogonal to a set of vectors A if x is orthogonal to all the vectors in A. We write then $x \perp A$.

Exercise: If x is orthogonal to v_1, \ldots, v_k then x is orthogonal to any linear combination of these vectors i.e. $x \perp \operatorname{Span}(v_1, \ldots, v_k)$.

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Pythagorean Theorem

Theorem (Pythagorean theorem)

Let $x, y \in V$. Then

Proof.

Orthogonality

 $x \perp y \iff ||x + y||^2 = ||x||^2 + ||y||^2.$

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App	licatio	n to rai	ndom var	iables

Orthogonality

Orthogonal & orthonormal families

Definition

Let v_1, \ldots, v_k be vectors of V. We say that the family of vectors (v_1, \ldots, v_k) is

- orthogonal if the vectors v_1, \ldots, v_n are pairwise orthogonal, i.e. $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- orthonormal if it is orthogonal and if all the v_i have unit norm: $||v_1|| = \cdots = ||v_k|| = 1$.

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Coordinates in an orthonormal basis

Proposition

Assume that $\dim(V)=n$ and let (v_1,\ldots,v_n) be an **orthonormal** basis of V. Then the coordinates of a vector $x\in V$ in the basis (v_1,\ldots,v_n) are $(\langle v_1,x\rangle,\ldots,\langle v_n,x\rangle)$:

$$x = \langle v_1, x \rangle v_1 + \dots + \langle v_n, x \rangle v_n.$$

Orthogonal projection

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Orthogonal projection

Picture		

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Orthogonal projection and distance

Definition

Let S be a subspace of \mathbb{R}^n . The *orthogonal projection* of a vector x onto S is defined as the vector $P_S(x)$ in S that minimizes the distance to x:

$$P_S(x) \stackrel{\text{def}}{=} \underset{y \in S}{\arg \min} \|x - y\|.$$

The distance of x to the subspace S is then defined as

$$d(x, S) \stackrel{\text{def}}{=} \min_{y \in S} ||x - y|| = ||x - P_S(x)||.$$

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Important proposition

Proposition

Let S be a subspace of \mathbb{R}^n and let (v_1, \dots, v_k) be an **orthonormal** basis of S. Then for all $x \in \mathbb{R}^n$,

$$P_S(x) = \langle v_1, x \rangle v_1 + \dots + \langle v_k, x \rangle v_k.$$

Consequence

Corollary

For all $x \in \mathbb{R}^n$,

- $x P_S(x)$ is orthogonal to S.
- $||P_S(x)|| \le ||x||.$

Orthogonal matrices

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Orthogonal matrices

Definition

A matrix $A \in \mathbb{R}^{n \times n}$ is called an *orthogonal matrix* if its columns are an orthonormal family (and therefore a basis of \mathbb{R}^n because it is a linearly independent family of size $n = \dim(\mathbb{R}^n)$).

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A proposition

Proposition

Let $A \in \mathbb{R}^{n \times n}$. The following points are equivalent:

- 1. *A* is orthogonal.
- **2.**A^T A = Id_n.
- $3. AA^{\mathsf{T}} = \mathrm{Id}_n$

Orthogonal matrices & norm

Proposition

Let $A\in\mathbb{R}^{n\times n}$ be an orthogonal matrix. Then A preserves the dot product in the sense that for all $x,y\in\mathbb{R}^n$,

$$\langle Ax, Ay \rangle = \langle x, y \rangle.$$

In particular if we take x=y we see that A preserves the Euclidean norm: $\|Ax\|=\|x\|$.

Cauchy-Schwarz inequality

Theorem

Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x,y\in V$:

$$|\langle x, y \rangle| \le ||x|| \, ||y||. \tag{1}$$

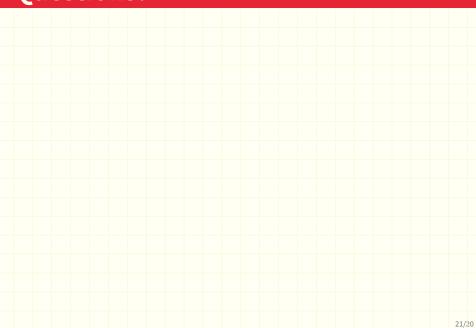
Moreover, there is equality in (1) if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.

Pro	of										

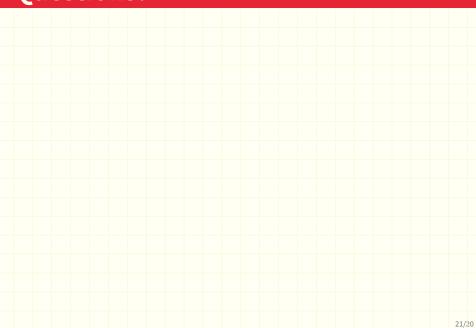
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Questions?



Questions?



Questions?

