Optimization and Computational Linear Algebra for Data Science
Homework 3: Rank

Due on September 24, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will note affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`lm4271@nyu.edu`) or to stop at the office hours.

Problem 3.1 (2 points)**.** *Let $A \in \mathbb{R}^{n\times n}$.*

(a) *Show that if $A = \alpha \mathrm{Id}_n$ for some $\alpha \in \mathbb{R}$, then for all $B \in \mathbb{R}^{n\times n}$ we have $AB = BA$.*

(b) *Conversely, show that if for all $B \in \mathbb{R}^{n\times n}$ we have $AB = BA$, then there exists $\alpha \in \mathbb{R}$ such that $A = \alpha \mathrm{Id}_n$.*

Problem 3.2 (2 points)**.** *Let $M \in \mathbb{R}^{n\times m}$ and $r = \mathrm{rank}(M)$. Show that there exists $A \in \mathbb{R}^{n\times r}$ and $B \in \mathbb{R}^{r\times m}$ such that $M = AB$.*

Problem 3.3 (3 points)**.** *Let $A \in \mathbb{R}^{n\times m}$.*

(a) *Let $M \in \mathbb{R}^{m\times m}$ be an invertible matrix. Show that*

$$\mathrm{rank}(AM) = \mathrm{rank}(A).$$

(b) *Let $M \in \mathbb{R}^{n\times n}$ be an invertible matrix. Show that*

$$\mathrm{rank}(MA) = \mathrm{rank}(A).$$

Problem 3.4 (3 points)**.** *Let $A \in \mathbb{R}^{n\times n}$ be an "upper triangular matrix", i.e. a matrix of the form*

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & a_{n-1,n} \\ 0 & \cdots & \cdots & 0 & a_{n,n} \end{pmatrix}.$$

Show that A is invertible if and only if *its diagonal coefficients $a_{1,1}, a_{2,2}, \dots, a_{n,n}$ are all non-zero.*

Problem 3.5 ($\star$)**.** *The trace* $\mathrm{Tr}(M)$ *of a* $k \times k$ *matrix* M *is defined as the sum of its diagonal coefficients, i.e.*

$$\mathrm{Tr}(M) = \sum_{i=1}^{k} M_{i,i}.$$

Let $A \in \mathbb{R}^{n\times m}$ *and* $B \in \mathbb{R}^{m\times n}$*. Show that*

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$