# **Recitation 1**

# **Concept Review: Vector Spaces**

### Definition

A **vector space** is a set V endowed with two 'nice and compatible' operations + and  $\cdot$  that verify:

- For all  $u, v \in V$ ,  $u + v \in V$ .
- For all  $u \in V$  and all  $\lambda \in \mathbb{R}$ ,  $\lambda \cdot u \in V$ .

**Example**:  $V = \mathbb{R}^n$ , with the usual vector addition + and scalar multiplication  $\cdot$  is a vector space.

## **Concept Review: Vector Spaces**

### In this class:

- We will always consider real scalars. Note that it is also possible to consider complex scalars.
- ightharpoonup V is (usually)  $\mathbb{R}^n$ , or (sometimes)  $\mathbb{R}^{n \times m}$  (set of  $n \times m$  matrices).

# **Concept Review: Subspaces**

### **Definition** (Subspace)

A subset S of a vector space V is a *subspace* if it is closed under addition and scalar multiplication:

- 1. Closure under Addition:  $x + y \in S$  for all  $x, y \in S$ .
- 2. Closure under Scalar Multiplication:  $\alpha x \in S$ , for all  $x \in S$  and  $\alpha \in \mathbb{R}$ .
- A subspace is also a vector space!
- Subspaces are a recurring concept throughout this entire course.

### **Questions 1: Subspaces, Span**

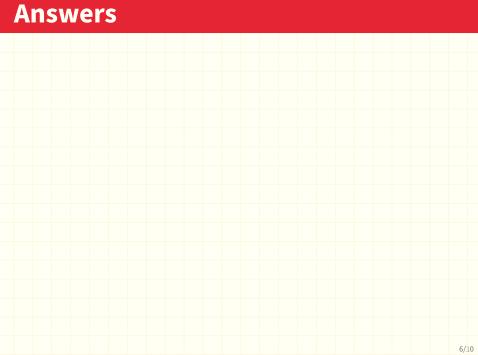
Consider the two vectors v=(1,1) and w=(-1,2). Describe the following sets geometrically. Which are subspaces of  $\mathbb{R}^2$ ?

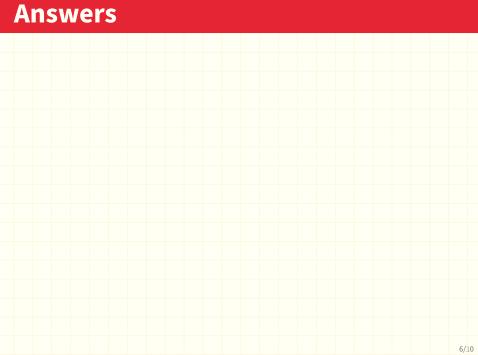
- 1.  $\operatorname{Span}(v)$
- 2.  $\operatorname{Span}(v, w)$
- 3.  $\operatorname{Span}(v) \cup \operatorname{Span}(w)$ , that is, the vectors in  $\operatorname{Span}(v)$  or  $\operatorname{Span}(w)$
- 4.  $\operatorname{Span}(v) \cap \operatorname{Span}(w)$ , that is, the vectors in both  $\operatorname{Span}(v)$  and  $\operatorname{Span}(w)$

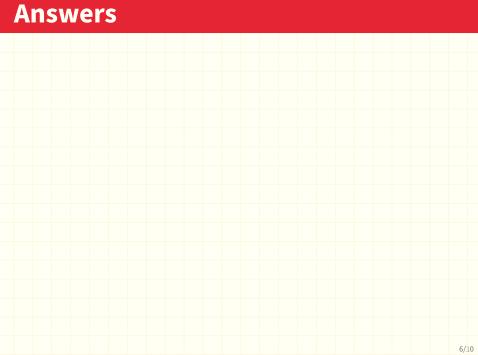
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- 5.  $\{(1-t)v + tw | t \in [0,1]\}$
- 6.  $\{(1-t)v + tw | t \in \mathbb{R}\}$
- 7.  $\{\alpha v + \beta w | \alpha, \beta \ge 0\}$
- 8. Span(v, w, u) where u = (0, 5).
- 9.  $\{(a,b) \in \mathbb{R}^2 | a^2 + b^2 \le 25\}$
- 10.  $\{(a,a) \in \mathbb{R}^2 | a \in \mathbb{R}\}$
- 11.  $\{(a, a^2) \in \mathbb{R}^2 | a \in \mathbb{R} \}$
- 12.  $\{(a,1) \in \mathbb{R}^2 | a \in \mathbb{R} \}$







1. Let  $v_1,v_2,v_3,v_4\in\mathbb{R}^3$ . Let  $C_1=\{v_1,v_2\};C_2=\{v_3,v_4\}$ . If  $C_1$  and  $C_2$  are both linearly independent, what are the possible values for  $\dim(\mathrm{Span}(v_1,v_2,v_3,v_4))$ ? (No formal proof necessary)

2. Let  $v_1,...,v_m\in\mathbb{R}^n$  be linearly dependent.

Prove that for  $x\in \mathrm{Span}(v_1,\ldots,v_m)$ , there exist infinitely many  $\alpha_1,\ldots,\alpha_m\in\mathbb{R}$  such that

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3. True or False: If  $B=(v_1,\ldots,v_n)$  is a basis for  $\mathbb{R}^n$ , and W is a subspace of  $\mathbb{R}^n$ , then some subset of B is a basis for W.

# **Questions 3: Bases, Dimension**

### Let V be the set of functions

$$V \stackrel{\mathrm{def}}{=} \left\{ p : \mathbb{R} \to \mathbb{R} \,\middle|\, p(x) = \sum_{k=0}^n a_k x^k, \text{ for some } a_0, \dots, a_n \in \mathbb{R} 
ight\}$$

- 1. What kind of function does this set contain?
- 2. Define an addition operation  $+: V \times V \to V$ , and a scalar multiplication operation  $\cdot: \mathbb{R} \times V \to V$ , such that the triple  $(V,+,\cdot)$  is a vector space.
- 3. What is the zero vector of this vector space?
- 4. Find a basis for this vector space.
- 5. What is the dimension of this vector space?