

Optimization and Computational Linear Algebra for Data Science

Lecture 1: Vector spaces

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Warning: *This material is not meant to be lecture notes. It only gathers the main concepts and results from the lecture, without any additional explanation, motivation, examples, figures...*

1 General definitions

We present below the abstract mathematical definition of a vector space. **Please do not try to memorize it!** Simply remember that a vector space is a set whose elements are called *vectors*, that one can add vectors together and multiply them by real numbers called *scalars*.

Definition 1.1 (*Vector space*)

A vector space (over \mathbb{R}) consists of a set V (whose elements are called vectors) and two operations $+$ and \cdot that verify:

1. The sum of two vectors is a vector: for all $\vec{x}, \vec{y} \in V$ we have $\vec{x} + \vec{y} \in V$.

2. The vector sum is commutative and associative. For all $\vec{x}, \vec{y}, \vec{z} \in V$ we have

$$\vec{x} + \vec{y} = \vec{y} + \vec{x} \quad \text{and} \quad \vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}.$$

3. There exists a zero vector $\vec{0} \in V$ that verifies $\vec{x} + \vec{0} = \vec{x}$ for all $\vec{x} \in V$.

4. For all $\vec{x} \in V$, there exists $\vec{y} \in V$ such that $\vec{x} + \vec{y} = \vec{0}$. Such \vec{y} is called the additive inverse of \vec{x} and is written $-\vec{x}$.

5. Scalar multiplication: for all $\vec{x} \in V$ and all $\alpha \in \mathbb{R}$, $\alpha \cdot \vec{x} \in V$.

6. Identity element for scalar multiplication: $1 \cdot \vec{x} = \vec{x}$ for all $\vec{x} \in V$.

7. Compatibility between scalar multiplication and the usual multiplication: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x} \in V$, we have

$$\alpha \cdot (\beta \cdot \vec{x}) = (\alpha\beta) \cdot \vec{x}.$$

8. Distributivity: for all $\alpha, \beta \in \mathbb{R}$ and all $\vec{x}, \vec{y} \in V$,

$$(\alpha + \beta) \cdot \vec{x} = \alpha \cdot \vec{x} + \beta \cdot \vec{y} \quad \text{and} \quad \alpha \cdot (\vec{x} + \vec{y}) = \alpha \cdot \vec{x} + \alpha \cdot \vec{y}.$$

From now we will ignore \cdot and simply write $\alpha\vec{x}$ instead of $\alpha \cdot \vec{x}$. We will also remove the arrows and write x instead of \vec{x} .

Example 1.1.

- The set $V = \mathbb{R}^n$ endowed with the usual vector addition $+$

$$(x_1, \dots, x_n) + (y_1, \dots, y_n) = (x_1 + y_1, \dots, x_n + y_n)$$

and the usual scalar multiplication \cdot

$$\alpha \cdot (x_1, \dots, x_n) = (\alpha x_1, \dots, \alpha x_n)$$

is a vector space.

- The set $V = \mathcal{F}(\mathbb{R}, \mathbb{R}) \stackrel{\text{def}}{=} \{f \mid f : \mathbb{R} \rightarrow \mathbb{R}\}$ of all functions from \mathbb{R} to itself endowed with the addition $+$ and the scalar multiplication \cdot defined by

$$\begin{aligned} f + g : \mathbb{R} &\rightarrow \mathbb{R} & \text{and} & & \alpha \cdot f : \mathbb{R} &\rightarrow \mathbb{R} \\ t &\mapsto f(t) + g(t) & & & t &\mapsto \alpha f(t) \end{aligned}$$

is a vector space.

- The set of random variables on a given probability space is a vector space: if X and Y are two random variables and $\alpha \in \mathbb{R}$, $X + Y$ and αX are also random variables.

Definition 1.2 (*Subspace*)

We say that a non-empty subset S of a vector space V is a subspace if it is closed under addition and multiplication by a scalar, that is if

- (i) for all $x, y \in S$ we have $x + y \in S$,
- (ii) for all $x \in S$ and all $\alpha \in \mathbb{R}$ we have $\alpha x \in S$.

Notice that a subspace is also a vector space!

Exercise 1.1. Show that every subspace S of a vector space V contains the zero vector 0 .

2 Linear dependency

Definition 2.1 (*Linear combination*)

Let V be a vector space and $x_1, \dots, x_k \in V$. We say that $y \in V$ is a linear combination of the vectors x_1, \dots, x_k if there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ such that

$$y = \sum_{i=1}^k \alpha_i x_i.$$

Remember that a linear combination is always a *finite* sum.

Remark 2.1. If S is a subspace of a vector space V , any linear combination of elements of S belongs to S .

Definition 2.2 (*Span*)

Let x_1, \dots, x_k be vectors in a vector space V . We define the linear span of x_1, \dots, x_k as the set of all linear combinations of these vectors:

$$\text{Span}(x_1, \dots, x_k) \stackrel{\text{def}}{=} \left\{ \alpha_1 x_1 + \dots + \alpha_k x_k \mid \alpha_1, \dots, \alpha_k \in \mathbb{R} \right\}.$$

One can easily verify (exercise!) that $\text{Span}(x_1, \dots, x_k)$ is a subspace of V . One can also verify (exercise!) that

$$\text{Span}(x_1, \dots, x_k) = \bigcap_{\substack{S \text{ subspace of } V \\ x_1, \dots, x_k \in S}} S,$$

$\text{Span}(x_1, \dots, x_k)$ is therefore the **smallest** (for the inclusion \subset) **subspace of S that contains x_1, \dots, x_k** .

Definition 2.3 (Linear dependency)

Vectors $x_1, \dots, x_k \in V$ are linearly dependent if there exists $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ **that are not all zero** such that

$$\alpha_1 x_1 + \dots + \alpha_k x_k = 0.$$

They are said to be linearly independent otherwise.

Saying that x_1, \dots, x_k are linearly dependent precisely means that one of the vectors x_1, \dots, x_k can be obtained as a linear combination of the others. Indeed if x_1, \dots, x_k are linearly dependent, then we can find $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ that are not all zero (there exists i such that $\alpha_i \neq 0$) such that $\alpha_1 x_1 + \dots + \alpha_k x_k = 0$. This leads to

$$x_i = \sum_{j \neq i} \frac{-\alpha_j}{\alpha_i} x_j,$$

i.e. the vector x_i can be expressed as a linear combinations of the vectors x_j for $j \neq i$. Conversely if we have for some i , and $\alpha_1, \dots, \alpha_k \in \mathbb{R}$

$$x_i = \alpha_1 x_1 + \dots + \alpha_{i-1} x_{i-1} + \alpha_{i+1} x_{i+1} + \dots + \alpha_k x_k = 0.$$

then $\alpha_1 x_1 + \dots + \alpha_{i-1} x_{i-1} - x_i + \alpha_{i+1} x_{i+1} + \dots + \alpha_k x_k = 0$ which gives that x_1, \dots, x_k are linearly dependent.

3 Basis, dimension

3.1 Definitions

Definition 3.1 (Basis)

A family (x_1, \dots, x_n) of vectors of V is a basis of V if

- (i) x_1, \dots, x_n are linearly independent,
- (ii) $\text{Span}(x_1, \dots, x_n) = V$.

Definition 3.2 (Dimension)

Let V be a vector space.

- If V admits a basis (v_1, \dots, v_n) , then every basis of V has also n vectors. We say that V has dimension n and write $\dim(V) = n$.
- Otherwise, we say that V has infinite dimension: $\dim(V) = +\infty$.

The dimension is therefore the minimum number of vector needed to span the vector space. In this course we are going to focus mostly on finite dimensional spaces and \mathbb{R}^n in particular.

Proof. We will need the following lemma:

Lemma 3.1

Let $v_1, \dots, v_n \in V$ and suppose that we have vectors $x_1, \dots, x_k \in V$ such that $k > n$ and $x_i \in \text{Span}(v_1, \dots, v_n)$ for all $i \in \{1, \dots, k\}$. Then x_1, \dots, x_k are linearly dependent.

Lemma 3.1 will be proved at the end of the notes. We proceed by contradiction and assume that there exists two basis (v_1, \dots, v_n) and (x_1, \dots, x_k) of V such that $k \neq n$. Without loss of generality we can assume that $k > n$. For $i = 1, \dots, k$ we have

$$x_i \in V = \text{Span}(v_1, \dots, v_n),$$

because (v_1, \dots, v_n) is a basis of V . We can therefore apply Lemma 3.1 to get that x_1, \dots, x_{n+1} are linearly dependent. This contradicts the fact that (x_1, \dots, x_k) is a basis. \square

Example 3.1. Let us define the vectors $e_1, \dots, e_n \in \mathbb{R}^n$ by

$$e_1 = (1, 0, 0, \dots, 0)$$

$$e_2 = (0, 1, 0, \dots, 0)$$

$$\vdots$$

$$e_n = (0, 0, 0, \dots, 1).$$

One can verify (exercise!) that the family (e_1, \dots, e_n) is a basis of \mathbb{R}^n . This basis is called the “canonical basis” of \mathbb{R}^n . We conclude that \mathbb{R}^n has dimension n .

Definition 3.3 (*Lines, hyperplanes*)

Let S be a subspace of \mathbb{R}^n .

- We call S a line if $\dim(S) = 1$.
- We call S an hyperplane if $\dim(S) = n - 1$.

3.2 Some useful properties

Proposition 3.1

Let V be a vector space that has dimension $\dim(V) = n$. Then

- Any family of vectors of V that are linearly independent contains at most n vectors (i.e. if $x_1, \dots, x_k \in V$ are linearly independent, then $k \leq n$).
- Any family of vectors of V that spans V contains at least n vectors (i.e. if $x_1, \dots, x_k \in V$ are such that $\text{Span}(x_1, \dots, x_k) = V$, then $k \geq n$).

Exercise 3.1. Prove Proposition 3.1 using Lemma 3.1.

The next proposition is very useful for proving that a family (x_1, \dots, x_n) is a basis of a vector space V of dimension n :

Proposition 3.2

Let V be a vector space of dimension n and let $x_1, \dots, x_n \in V$.

1. If x_1, \dots, x_n are linearly independent, then (x_1, \dots, x_n) is a basis of V .
2. If $\text{Span}(x_1, \dots, x_n) = V$, then (x_1, \dots, x_n) is a basis of V .

In words, if the family we consider has as many vectors as the dimension of V , then we only have to check one of the two points of Definition 3.1!

Proposition 3.3

Let U and V be two subspaces of \mathbb{R}^n . Assume that $U \subset V$. Then

$$\dim(U) \leq \dim(V) \leq n.$$

If **moreover** $\dim(U) = \dim(V)$, then $U = V$.

Exercise 3.2. Prove Proposition 3.3 using Propositions 3.1 and 3.2.

Proposition 3.4 (Coordinates)

Let (v_1, \dots, v_n) be a basis of V . Then for every $x \in V$ there exists a unique vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n.$$

We say that $(\alpha_1, \dots, \alpha_n)$ are the coordinates of x in the basis (v_1, \dots, v_n) .

Proof. Existence. (v_1, \dots, v_n) forms a basis of V therefore $V = \text{Span}(v_1, \dots, v_n)$. We get that $x \in \text{Span}(v_1, \dots, v_n)$ which gives that there exists $\alpha_1, \dots, \alpha_n$ such that $x = \alpha_1 v_1 + \dots + \alpha_n v_n$.

Uniqueness. Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{R}$ such that

$$x = \alpha_1 v_1 + \dots + \alpha_n v_n = \beta_1 v_1 + \dots + \beta_n v_n.$$

This leads to

$$(\alpha_1 - \beta_1)v_1 + \dots + (\alpha_n - \beta_n)v_n = 0.$$

The vectors v_1, \dots, v_n are linearly independent because they form a basis. Consequently $\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \dots = \alpha_n - \beta_n = 0$, i.e. $(\alpha_1, \dots, \alpha_n) = (\beta_1, \dots, \beta_n)$. \square

Proof of Lemma 3.1

Notice that it suffices to prove the theorem for $k = n + 1$ because if x_1, \dots, x_{n+1} are linearly dependent, so are $x_1, \dots, x_{n+1}, \dots, x_k$. We will therefore show for all $n \geq 1$

$$\mathcal{H}_n : \ll \text{For all } v_1, \dots, v_n \in V \text{ and all } x_1, \dots, x_{n+1} \in \text{Span}(v_1, \dots, v_n), \\ \text{the vectors } x_1, \dots, x_{n+1} \text{ are linearly dependent.} \gg$$

Base case: \mathcal{H}_1 is true. Indeed, if $x_1, x_2 \in \text{Span}(v_1)$, then there exists $\alpha_1, \alpha_2 \in \mathbb{R}$ such that $x_1 = \alpha_1 v_1$ and $x_2 = \alpha_2 v_1$. If $\alpha_1 = 0$ then $x_1 = 0$ and x_1, x_2 are therefore linearly dependent. Otherwise if $\alpha_1 \neq 0$ then $v_1 = \frac{1}{\alpha_1} x_1$ which then gives $x_2 = \frac{\alpha_2}{\alpha_1} x_1$: x_1, x_2 are linearly dependent.

Induction step: We assume now that \mathcal{H}_{n-1} holds for some $n \geq 2$ and we will show that \mathcal{H}_n holds. We consider therefore $x_1, \dots, x_{n+1} \in \text{Span}(v_1, \dots, v_n)$. We can find real numbers $\alpha_{i,j}$ such that

$$\begin{aligned} x_1 &= \alpha_{1,1}v_1 + \dots + \alpha_{1,n}v_n \\ x_2 &= \alpha_{2,1}v_1 + \dots + \alpha_{2,n}v_n \\ &\vdots \\ x_{n+1} &= \alpha_{n+1,1}v_1 + \dots + \alpha_{n+1,n}v_n. \end{aligned}$$

We have to show that x_1, \dots, x_{n+1} are linearly dependent. Let us consider the first line. If $\alpha_{1,1} = \alpha_{1,2} = \dots = \alpha_{1,n} = 0$, then $x_1 = 0$ which gives then that x_1, \dots, x_{n+1} are linearly dependent. Otherwise, there exists j such that $\alpha_{1,j} \neq 0$. Without loss of generality we can

assume that $\alpha_{1,1} \neq 0$.

$$\begin{array}{rclclcl}
x_1 & = & \alpha_{1,1}v_1 & + & \cdots & + & \alpha_{1,n}v_n \\
x_2 - \frac{\alpha_{2,1}}{\alpha_{1,1}}x_1 & = & 0 & + & \cdots & + & \alpha_{2,n}v_n - \frac{\alpha_{2,1}}{\alpha_{1,1}}\alpha_{1,n}v_n \\
\vdots & & & & & & \\
x_{n+1} - \frac{\alpha_{n+1,1}}{\alpha_{1,1}}x_1 & = & 0 & + & \cdots & + & \alpha_{n+1,n}v_n - \frac{\alpha_{n+1,1}}{\alpha_{1,1}}\alpha_{1,n}v_n.
\end{array}$$

If we define $y_i \stackrel{\text{def}}{=} x_i - \frac{\alpha_{i,1}}{\alpha_{1,1}}x_1$ for $i = 2, \dots, n+1$ we obtain have $y_i \in \text{Span}(v_2, \dots, v_n)$. We can now apply the induction hypothesis \mathcal{H}_{n-1} to get that y_2, \dots, y_{n+1} are linearly dependent. This means that there exists $\beta_2, \dots, \beta_{n+1}$ that are not all zero, such that $\beta_2 y_2 + \cdots + \beta_{n+1} y_{n+1} = 0$ which finally gives

$$\left(-\beta_2 \frac{\alpha_{2,1}}{\alpha_{1,1}} - \cdots - \beta_{n+1} \frac{\alpha_{n+1,1}}{\alpha_{1,1}} \right) x_1 + \beta_2 x_2 + \cdots + \beta_{n+1} x_{n+1} = 0.$$

Since $\beta_2, \dots, \beta_{n+1}$ are not all zero we get that x_1, \dots, x_{n+1} are linearly dependent. \mathcal{H}_n is proved.

