# Session 4: Norms and inner-products

Optimization and Computational Linear Algebra for Data Science

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# Orthogonality

# Definition

## Definition (Orthogonality)

- We say that vectors $x$ and $y$ are *orthogonal* if $\langle x, y \rangle = 0$. We write then $x \perp y$.
- We say that a vector $x$ is orthogonal to a set of vectors $A \subset V$ if $x$ is orthogonal to all the vectors in $A$, i.e. $\forall y \in A,\ \langle x, y \rangle = 0$. We write then $x \perp A$.
- More generality we say that $A \subset V$ and $B \subset V$ are orthogonal if $\langle x, y \rangle = 0$ for all $x \in A$ and all $y \in B$. As before, we write $A \perp B$.

If $x$ is orthogonal to $v_1, \ldots, v_k$ then $x$ is orthogonal to any linear combination of these vectors i.e. $x \perp \mathrm{Span}(v_1, \ldots, v_k)$.

# Pythagorean Theorem

### Theorem (Pythagorean theorem)

*Let* $x, y \in V$. *Then*

$$x \perp y \iff \|x + y\|^2 = \|x\|^2 + \|y\|^2.$$

**Proof.**

$\square$

# Orthogonal & orthonormal families

### Definition

Let $v_1, \ldots, v_k$ be vectors of $V$. We say that the family of vectors $(v_1, \ldots, v_k)$ is

- *orthogonal* if the vectors $v_1, \ldots, v_n$ are pairwise orthogonal, i.e. $\langle v_i, v_j \rangle = 0$ for all $i \neq j$.
- *orthonormal* if it is orthogonal and if all the $v_i$ have unit norm: $\|v_1\| = \cdots = \|v_k\| = 1$.

# A toy example

Orthonormal basis are particularly convenient for computing coordinates of vectors:

### Proposition

Assume that $\dim(V) = n$ and let $(v_1, \dots, v_n)$ be an **orthonormal** basis of $V$. Then the coordinates of a vector $x \in V$ in the basis $(v_1, \dots, v_n)$ are $(\langle v_1, x\rangle, \dots, \langle v_n, x\rangle)$:

$$x = \langle v_1, x\rangle v_1 + \cdots + \langle v_n, x\rangle v_n.$$

Moreover, for all $y \in V$, we have $\langle x, y\rangle = \langle v_1, x\rangle\langle v_1, y\rangle + \cdots + \langle v_n, x\rangle\langle v_n, y\rangle$. Taking $y = x$ leads to

$$\|x\| = \sqrt{\langle v_1, x\rangle^2 + \cdots + \langle v_n, x\rangle^2}.$$

# Questions?