Lecture 2.2: Matrices

Optimization and Computational Linear Algebra for Data Science

Léo Miolane

Contents

Matrix associated to a linear map

The key observation

- Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation.
- Let $(e_1, \ldots, e_m)$ be the canonical basis of $\mathbb{R}^m$.

Then, for all $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$:

$$L(x) = L\Big(\sum_{i=1}^{m} x_i e_i\Big) = \sum_{i=1}^{m} x_i L(e_i).$$

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Conclusion: if you give me the vectors $L(e_1), \dots, L(e_m) \in \mathbb{R}^n$ then, I can compute $L(x)$ for any $x \in \mathbb{R}^m$.

« One needs $n \times m$ numbers to store a the linear map L on a computer »

Matrices

Definition

A $n \times m$ matrix is an array (of real numbers) with n rows and m columns. We denote by $\mathbb{R}^{n \times m}$ the set of all $n \times m$ matrices.

Canonical matrix of a linear map

We can encode a linear map $L : \mathbb{R}^m \to \mathbb{R}^n$ by a $n \times m$ matrix.

Definition

The canonical matrix of L is the $n \times m$ matrix (that we will write also L) whose columns are $L(e_1), \dots, L(e_m)$:

$$L = \begin{pmatrix} | & | & & | \\ L(e_1) & L(e_2) & \cdots & L(e_m) \\ | & | & & | \end{pmatrix} = \begin{pmatrix} L_{1,1} & L_{1,2} & \cdots & L_{1,m} \\ L_{2,1} & L_{2,2} & \cdots & L_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n,1} & L_{n,2} & \cdots & L_{n,m} \end{pmatrix}$$

where we write $L(e_j) = \begin{pmatrix} L_{1,j} \\ L_{2,j} \\ \vdots \\ L_{n,j} \end{pmatrix}$.

Example #1: identity matrix

The Identity map $\begin{array}{rccc} \mathrm{Id}: & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & x \end{array}$ is linear.

Exercise: what is the canonical matrix of Id ?

Example #2: Homothety

Let $\lambda \in \mathbb{R}$. The homothety map of ratio λ:

$$\begin{array}{rccc} H_\lambda : & \mathbb{R}^n & \to & \mathbb{R}^n \\ & x & \mapsto & \lambda x \end{array}$$

is linear.

Exercise: what is the canonical matrix of H_λ ?

Example #3: rotations in $\mathbb{R}^2$

Let $\theta \in \mathbb{R}$. The rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ of angle θ about the origin is linear.

Exercise: what is the canonical matrix of R_θ ?

Matrix product

Matrix-vector product

- We have seen: linear map $\to$ matrix
- We will see now: matrix $\to$ linear map

Definition

The linear transformation associated to a matrix $L \in \mathbb{R}^{n \times m}$ is the map

$$\begin{array}{rccc} L: & \mathbb{R}^m & \to & \mathbb{R}^n \\ & x & \mapsto & Lx \end{array}$$

where the “matrix-vector” product $Lx \in \mathbb{R}^n$ is defined by

$$(Lx)_i = \sum_{j=1}^{m} L_{i,j} x_j \qquad \text{for all } i \in \{1, \dots, n\}.$$

Visualizing the formula

$$(Lx)_i = \sum_{j=1}^{m} L_{i,j}\, x_j$$

Matrix product

Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{m\times k}$.

Definition (Matrix product)

The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$. His coefficients are given by the formula:

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell} M_{\ell,j} \quad \text{for all} \quad 1 \leq i \leq n, \quad 1 \leq j \leq k.$$

Visualizing the formula

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell}\, M_{\ell,j}$$