

Lecture 2.3: Matrix product

Optimization and Computational Linear Algebra for Data Science

Matrix-vector product

Consider a linear map $L : \mathbb{R}^m \rightarrow \mathbb{R}^n$ and its associated matrix $\tilde{L} \in \mathbb{R}^{n \times m}$.

Question: Can we use the matrix \tilde{L} to compute the image $L(x)$ of a vector $x \in \mathbb{R}^m$?

Proposition

For all $x \in \mathbb{R}^m$ coordinates of $L(x) \in \mathbb{R}^n$ are given by the formula:

$$(L(x))_i = \sum_{j=1}^m \tilde{L}_{i,j} x_j \quad \text{for all } i \in \{1, \dots, n\}.$$

Visualizing the formula

$$(L(x))_i = \sum_{j=1}^m \tilde{L}_{i,j} x_j$$

Matrix-vector product

- ❖ We have seen: linear map \rightarrow matrix
- ❖ We will see now: matrix \rightarrow linear map

Definition

The linear map associated to a matrix $\tilde{L} \in \mathbb{R}^{n \times m}$ is the map

$$\begin{aligned} L : \mathbb{R}^m &\rightarrow \mathbb{R}^n \\ x &\mapsto \tilde{L}x \end{aligned}$$

where the “matrix-vector” product $\tilde{L}x \in \mathbb{R}^n$ is defined by

$$(\tilde{L}x)_i = \sum_{j=1}^m \tilde{L}_{i,j} x_j \quad \text{for all } i \in \{1, \dots, n\}.$$

Remark: for all $x \in \mathbb{R}^m$, $L(x) = \tilde{L}x$.

Matrix product

Let $L \in \mathbb{R}^{n \times m}$ and $M \in \mathbb{R}^{m \times k}$.

Definition - Proposition

- ❖ The matrix product LM is the $n \times k$ matrix of the linear map $L \circ M$.
- ❖ Its coefficients are given by the formula:

$$(LM)_{i,j} = \sum_{\ell=1}^m L_{i,\ell} M_{\ell,j} \quad \text{for all} \quad 1 \leq i \leq n, \quad 1 \leq j \leq k.$$

Visualizing the formula

$$(LM)_{i,j} = \sum_{\ell=1}^m L_{i,\ell} M_{\ell,j}$$