Optimization and Computational Linear Algebra for Data Science
Homework 1: Vector spaces

Due on September 7, 2019

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- You can work in groups but each student must write his/her own solution based on his/her own understanding of the problem. Please list on your submission the students you work with for the homework (his will note affect your grade).
- Late submissions will be graded with a penalty of 10% per day late. Weekend days do not count: from Friday to Monday count 1 day.
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact myself (`leo.miolane@gmail.com`) or to stop at the office hours.

Problem 1.1. *Let u, v be two vectors of $\mathbb{R}^2$. Show that either they are linearly dependent or that they span the whole of $\mathbb{R}^2$.*

Problem 1.2. *Are the following sets subspaces of $\mathbb{R}^3$? Justify your answer.*

(a) $E_1 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x - 2y + z = 0\}$.

(b) $E_2 = \{(x,y,z) \in \mathbb{R}^3 \,|\, x - 2y + z = 3\}$.

(c) $E_3 = \{(x,y,z) \in \mathbb{R}^3 \,|\, 5x + y^2 + z = 0\}$.

Problem 1.3. *Suppose that $v_1, \dots, v_k \in \mathbb{R}^n$ are linearly independent. Let $x \in \mathbb{R}^n$ and assume that $x \notin \mathrm{Span}(v_1, \dots, v_k)$. Show that $(v_1, \dots, v_k, x)$ are linearly independent.*

Problem 1.4. *Let S be a subspace of $\mathbb{R}^n$ and $v_1, \dots, v_k \in S$. Show (using the result of Problem 1.3) that one can find vectors $v_{k+1}, \dots, v_{k+m}$ in S such that $(v_1, \dots, v_{k+m})$ is a basis of S.*

Problem 1.5 ($\star$)**.** *Let U and V be two subspaces of $\mathbb{R}^n$. Show that if*

$$\dim(U) + \dim(V) > n,$$

then there must exist a non-zero vector in their intersection, i.e. $U \cap V \neq \{0\}$.