Recitation 1

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Announcements

Welcome!

- My Office hours: TBA (once I figure my schedule out)
- Rotating cohorts - if you are not scheduled to be in person, please join the online zoom call (w/ Carles or Irina)
- Recitations are for practice problems
- You will have time in recitation to solve the problems.
- Recitations will be released so you can look over/solve problems in advance.

Why is Linear Algebra important?

Linear Algebra...

- Appears in ALL applied math, including *data science*
- Is solvable. If you can write it down, you can solve it!
  (not true for other math, e.g Diff Eq, Integrals)
- Is *fundamental* to understanding tools in machine learning

Relevant applications we will cover in the class:

- Linear Regression
- Principal Component Analysis
- Gradient Descent

First year MSDS students are *highly encouraged* to take this class.

Concept Review: Vector Spaces

Definition (Vector space)

V is a vector space over field F when

1. Closure under Addition: $x + y \in V$
2. Sum is commutative $(x + y = y + x)$ and associative $x + (y + z) = (x + y) + z$
3. Additive Identity (in V): $0 \in V$ $(x + 0 = x)$
4. Additive Inverse: $(\forall x, \exists\ -x$ s.t $x + (-x) = 0$
5. Closure under Scalar Multiplication: $\alpha x \in V$
6. Multiplicative Identity (in F): $\alpha x \in V$
7. Compatibility in Multiplication: $\alpha(\beta x) = (\alpha\beta)x$
8. Distributivity: $(\alpha + \beta)x = \alpha x + \beta \cdot \vec{y}$ and $\alpha(x + y) = \alpha x + \alpha \cdot y$.

Notice that "vectors" are not explicitly defined.

Optional: prove (after class) that $\mathbb{R}^3$ with standard definitions for addition and scalar multiplication is vector space over field $\mathbb{R}$.

Concept Review: Vector Spaces

- In this class,
  - Our field is always $\mathbb{R}$. ($\mathbb{C}$ is also a field.)
  - Standard definitions for vector addition, and scalar multiplication.
  - But, V is (usually) $\mathbb{R}^n$, or (sometimes) $\mathbb{R}^{n \times n}$.
- Everything in linear algebra is in a vector space!
- (!) A recurring concept in data science is to "vectorize" problems
  - If you can transform/reframe your problem in linear algebra, you can (attempt) to solve it!

Concept Review: Subspaces

Definition (Subspace)

A subset S of a vector space V is a *subspace* if it is closed under addition and scalar multiplication.

1. Closure under Addition: $x + y \in S$
2. Closure under Scalar Multiplication: $\alpha x \in S$

- A subspace is also a vector space!
- Everything in linear algebra is in a vector space.
- Anything *interesting* in linear algebra is in a subspace.
- Subspaces are a recurring concept throughout this entire course.

Questions 1: Subspaces, Span

Recall that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ can be thought of as the xy-plane. Consider the two vectors $v = (1, 1)$ and $w = (-1, 2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1. $\mathrm{Span}(v)$
2. $\mathrm{Span}(v, w)$
3. $\mathrm{Span}(v) \cup \mathrm{Span}(w)$, that is, the vectors in $\mathrm{Span}(v)$ or $\mathrm{Span}(w)$
4. $\mathrm{Span}(v) \cap \mathrm{Span}(w)$, that is, the vectors in both $\mathrm{Span}(v)$ and $\mathrm{Span}(w)$
5. $\{(1 - t)v + tw : t \in [0, 1]\}$
6. $\{(1 - t)v + tw : t \in \mathbb{R}\}$
7. $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$
8. $\mathrm{Span}(v, w, u)$ where $u = (0, 5)$.
9. $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$
10. $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
11. $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$
12. $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$

Solutions 1: Subspaces, Span

Recall that $\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$ can be thought of as the xy-plane. Consider the two vectors $v = (1, 1)$ and $w = (-1, 2)$. Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

1. $\text{Span}(v)$ — True
2. $\text{Span}(v, w)$ — True
3. $\text{Span}(v) \cup \text{Span}(w)$, — False
4. $\text{Span}(v) \cap \text{Span}(w)$, — True
5. $\{(1-t)v + tw : t \in [0, 1]\}$ — False
6. $\{(1-t)v + tw : t \in \mathbb{R}\}$ — False
7. $\{\alpha v + \beta w : \alpha, \beta \geq 0\}$ — False
8. $\text{Span}(v, w, u)$ where $u = (0, 5)$. — True
9. $\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 25\}$ — False
10. $\{(a, a) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ — True
11. $\{(a, a^2) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ — False
12. $\{(a, 1) \in \mathbb{R}^2 : a \in \mathbb{R}\}$ — False

Questions 2: Linear Independence, Bases,

1. Let v_1, v_2, v_3, v_4 (all distinct) $\in \mathbb{R}^3$.
   Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$.
   If C_1 and C_2 are both linearly independent, what are the possible values for $dim(Span\{v_1, v_2, v_3, v_4\})$? (No formal proof necessary)
2. Let $v_1, ..., v_n \in \mathbb{R}^n$ be a basis of $\mathbb{R}^n$.
   Prove that for $x \in R^n$, there exists unique $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^n \alpha_i v_i$.
3. Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
   Prove that for $x \in Span(v_1, ..., v_m)$, there exist infinitely many $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $x = \sum_{i=1}^m \alpha_i v_i$.
4. True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.

Solutions 2: Linear Independence, Bases

1. Let v_1, v_2, v_3, v_4 (all distinct) $\in \mathbb{R}^3$.
   Let $C_1 = \{v_1, v_2\}; C_2 = \{v_3, v_4\}$.
   If C_1 and C_2 are both linearly independent, what are the possible values for $dim(Span\{v_1, v_2, v_3, v_4\})$? (No formal proof necessary.)

Solution

Either $C_1 \subset Span(C_2)$ and the dimension is 2, or the dimension is 3.

Solutions 2: Linear Independence, Bases

2. Let $v_1, ..., v_n \in \mathbb{R}^n$ be a basis of $\mathbb{R}^n$.
   Prove that for $x \in R^n$, there exists unique $\alpha_1, ..., \alpha_n$ such that $x = \sum_{i=1}^n \alpha_i v_i$.

Solution

By definition of basis, $v_1, ..., v_n$ is a linearly independent set, and spans $\mathbb{R}^n$. Since $x \in \mathbb{R}^n$,

$$\exists \alpha_1, ..., \alpha_n \ s.t\ x = \sum_{i=1}^n \alpha_i v_i.$$

Let $\beta_1, ..., \beta_n$ s.t $x = \sum_{i=1}^n \beta_i v_i$. Then,

$$x - x = 0 = \sum_{i=1}^n (\alpha_i - \beta_i) v_i$$

Then by definition of linear independence,

$$\alpha_i - \beta_i = 0 \qquad \forall i \in 1, ..., n$$

So $\alpha_i = \beta_i, \forall i \in 1, ..., n$

3. Let $v_1, ..., v_m \in \mathbb{R}^n$ be linearly dependent.
Prove that for $x \in Span(v_1, ..., v_m)$, there exist infinitely many $\alpha_1, ..., \alpha_m \in \mathbb{R}$ such that $x = \sum_{i=1}^{m} \alpha_i v_i$.

Solution

By assumption, $x \in Span(v_1, \ldots, v_m)$. So

$$\exists \beta_1, ..., \beta_m \ s.t\ x = \sum_{i=1}^{m} \beta_i v_i$$

Since $v_1, \ldots, v_m$ are linearly dependent, there are $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$ such that

$$\sum_{i=1}^{m} \gamma_i v_i = 0$$

where not all $\gamma_i = 0$.
Now, let $r \in \mathbb{R}$. Then,

$$x = x + 0 = \sum_{i=1}^{m} \beta_i v_i + r \sum_{i=1}^{m} \gamma_i v_i = \sum_{i=1}^{m} (\beta_i + r\gamma_i) v_i$$

This gives infinitely many distinct α where $\alpha_i = \beta_i + r\gamma_i$ for $r \in \mathbb{R}$.

Solutions 2: Linear Independence, Bases

4. True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.

Solution

False. Consider $B = \{(1,0),(0,1)\}$ *and* $W = Span((1,1))$.

Questions 3: Bases, Dimension

Let V be the set of functions

$$V := \{p : \mathbb{R} \to \mathbb{R} \mid p(x) = \sum_{k=0}^{n} a_k x^k, \text{ where } a_0, \dots, a_n \in \mathbb{R}\}$$

1. What kind of function does this set contain?
2. Define an addition operation $+ : V \times V \to V$,
   and a scalar multiplication operation $\cdot : \mathbb{R} \times V \to V$,
   such that the triple $(V, +, \cdot)$ is a vector space.
3. Find a basis for this vector space.
4. What is the dimension of this vector space?

Solutions 3: Bases, Dimension

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Solution

1. *Polynomials evaluated at x*
2. *Standard definitions for function addition and scalar multiplication*
3. $1, x, x^2, ..., x^n$
4. $n + 1$