Recitation 3

Carles Domingo

Rank Nullity Theorem

Theorem (Rank-Nullity Theorem)

Let $L : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Then

$$\operatorname{rank}(L) + \dim(\operatorname{Ker}(L)) = m.$$

- Important theorem (check that you can reproduce the proof).
- Other things to keep in mind:
  - $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$
  - For $c_1, c_2, \ldots, c_m \in \mathbb{R}^n$,

$$\text{rank}(c_1, c_2, \ldots, c_m) = \text{rank}(\begin{bmatrix} c_1 & c_2 & \ldots & c_m \end{bmatrix}) = \text{rank}\left(\begin{bmatrix} c_1 \\ c_2 \\ \ldots \\ c_m \end{bmatrix}\right)$$

Typical exercise

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 3 & -1 & -1 \\ 4 & -2 & 0 \end{bmatrix}$$

1. Find a basis of the kernel of A.
2. Find the rank of A. Did you need to perform additional computations?
3. Find a basis of the image of A. Did you need to perform additional computations?

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Questions: Rank-Nullity Theorem

Let $A \in \mathbb{R}^{l \times h}$ and $B \in \mathbb{R}^{h \times l}$, and $h > l$.
Prove or give a counterexample to the following statements.

1. $\exists A, B$ s.t AB is invertible.
2. $\exists A, B$ s.t.BA is invertible.

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Symmetric Matrices

- $A \in \mathbb{R}^{n \times n}$ symmetric if $A_{ij} = A_{ji}$ for all $i, j \in [1:n]$.
- Symmetric matrices appear often and have good properties:
  - Orthogonal Projections (Lec. 4) are symmetric.
  - Spectral Theorem (Lec. 7) “symmetric matrices have an orthonormal basis of eigenvectors".
  - PCA (Lec. 7): Covariance matrix is symmetric.
  - Convexity (Lec. 9,11): Hessian Matrix (matrix of second derivative) is symmetric

Questions: Symmetric Matrices

Let $A \in \mathbb{R}^{k \times n}$. Prove/answer the following statements.

1. Show that $\forall x \in \mathbb{R}^n$, $x^\top A^\top A x \geq 0$.
2. When is $x^\top A^\top A x = 0$?
3. Show that $\text{Ker}(A) = \text{Ker}(A^\top A)$.
4. Use this to show $\text{rank}(A) = \text{rank}(A^\top A)$.
5. Now show that $\text{rank}(A) = \text{rank}(A^\top)$.

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