POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

DEIB

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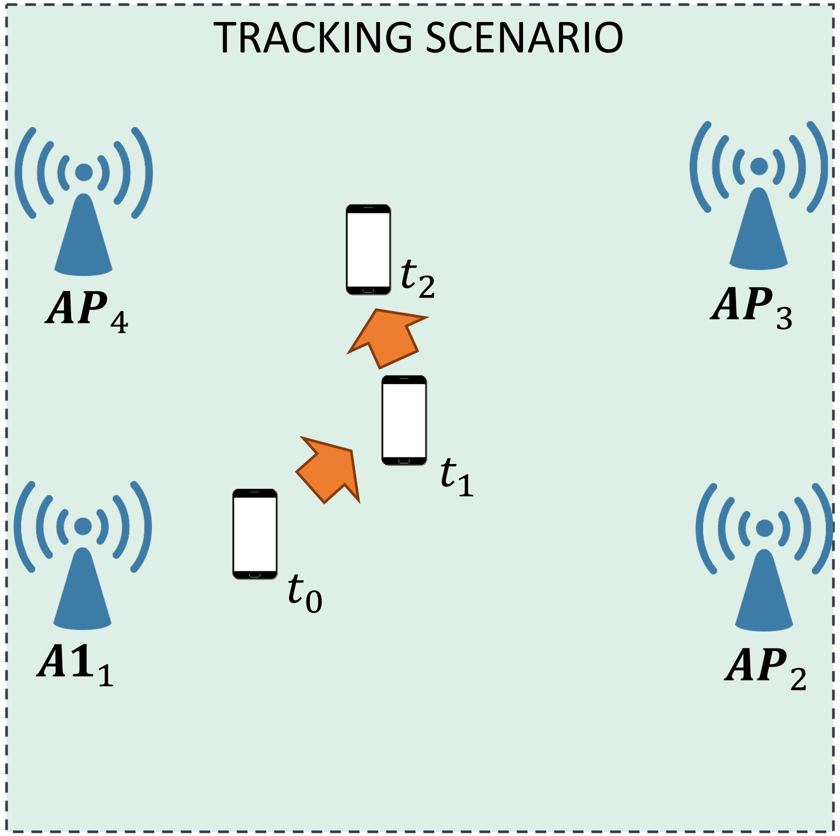
Lab 3-4: BAYESIAN TRACKING
Kalman Filter

2D tracking: problem formulation

GOAL: infer the position of a moving UE $\boldsymbol{u}_t$ from measurements $\boldsymbol{\rho}$ by Kalman Filter

$$\boldsymbol{s}_1 = [s_{1,x} \quad s_{1,y}]$$



$$\boldsymbol{\rho} = [\rho_1 \; \rho_2 \; \rho_3 \; \rho_4]$$

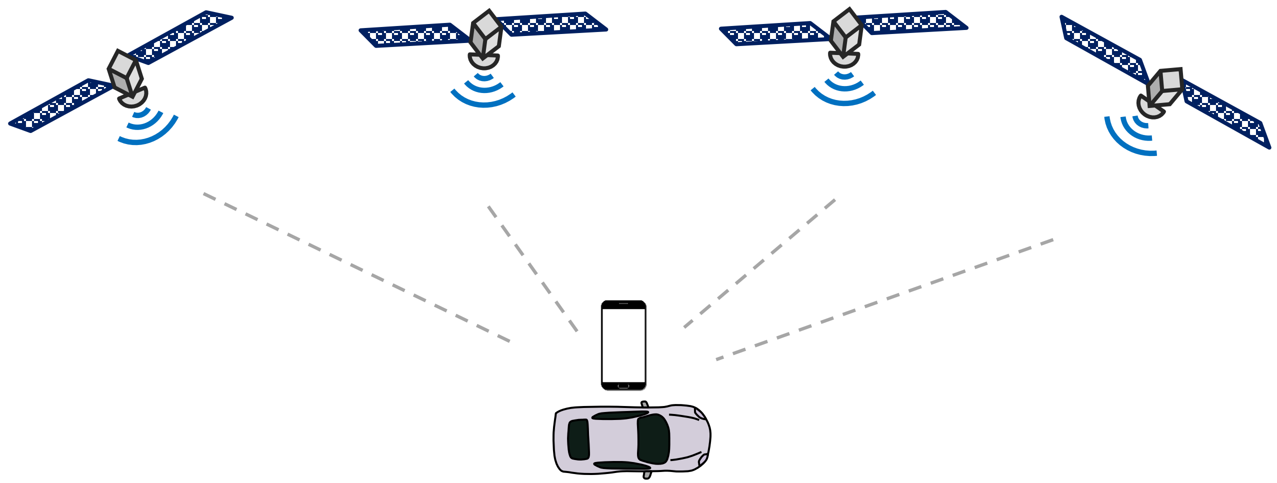
Each i–th AP has a TOA measurement at time t: $\rho_{i,t} = h_i(\boldsymbol{u}_t, \boldsymbol{s}_i) + n_i = |\boldsymbol{u}_t - \boldsymbol{s}_{i,t}\,| + n_{i,t}$

$$\rho_{i,t} = h_i(\boldsymbol{x}_t, \boldsymbol{s}_i) + n_i = |\boldsymbol{x}_t - \boldsymbol{s}_{i,t}\,| + n_{i,t}$$

$$\boldsymbol{u}_t \equiv \boldsymbol{x}_t$$

2D tracking: example

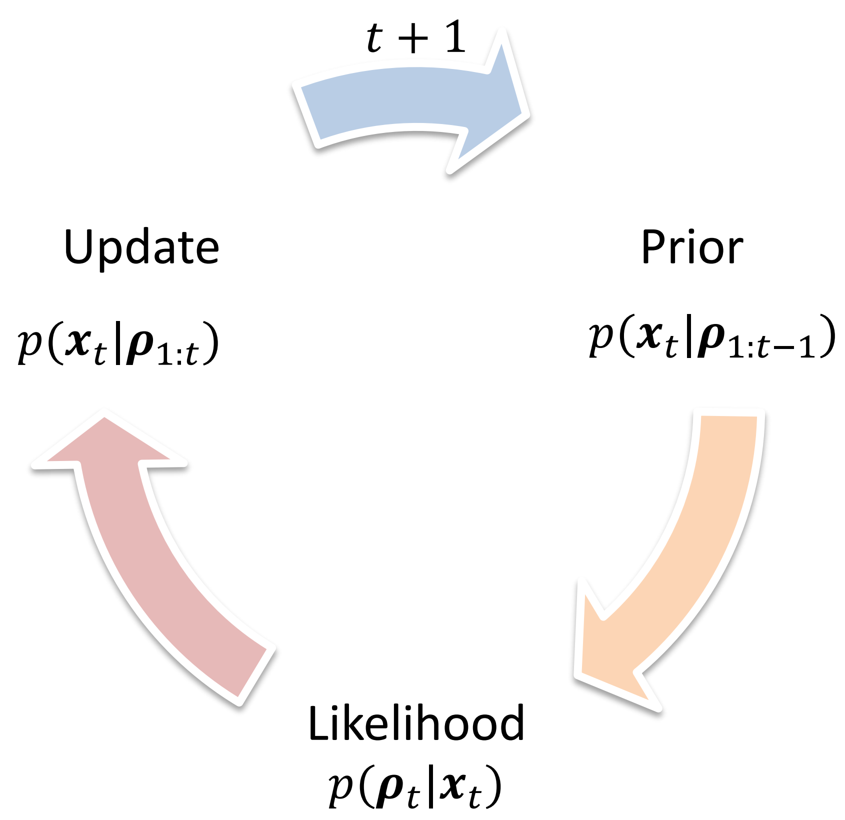
TRACKING SCENARIO

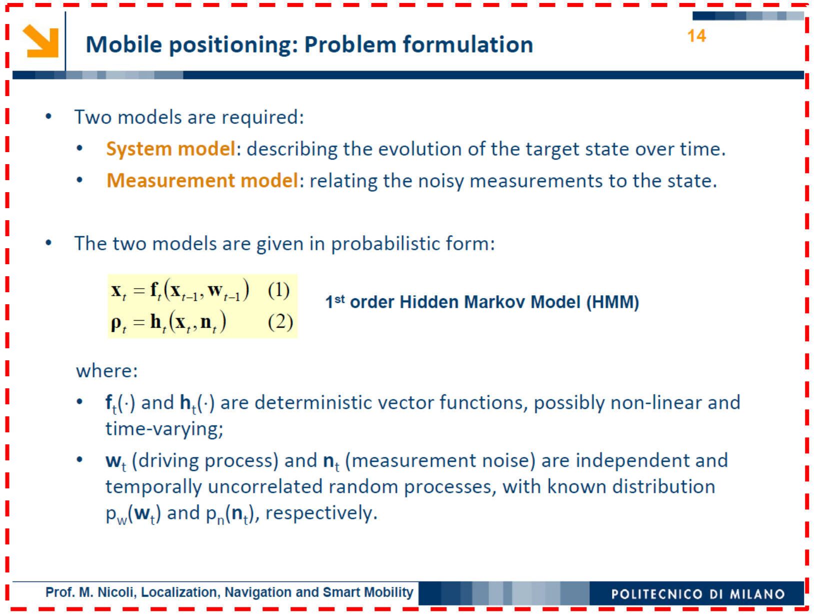


$$\boldsymbol{x}_t = \boldsymbol{u}_t = [u_{x,t} \quad u_{y,t}]$$

$$\boldsymbol{\rho}_{i,t} = \boldsymbol{x}_t + \boldsymbol{n}_i$$

Bayesian tracking principles





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Mobile positioning: Problem formulation

- Two models are required:
  - **System model**: describing the evolution of the target state over time.
  - **Measurement model**: relating the noisy measurements to the state.
- The two models are given in probabilistic form:

$$\mathbf{x}_t = \mathbf{f}_t(\mathbf{x}_{t-1}, \mathbf{w}_{t-1}) \quad (1)$$
$$\boldsymbol{\rho}_t = \mathbf{h}_t(\mathbf{x}_t, \mathbf{n}_t) \quad (2)$$

1st order Hidden Markov Model (HMM)

where:

- $\mathbf{f}_t(\cdot)$ and $\mathbf{h}_t(\cdot)$ are deterministic vector functions, possibly non-linear and time-varying;
- $\mathbf{w}_t$ (driving process) and $\mathbf{n}_t$ (measurement noise) are independent and temporally uncorrelated random processes, with known distribution $p_w(\mathbf{w}_t)$ and $p_n(\mathbf{n}_t)$, respectively.

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System (motion) model	$\boldsymbol{x}_t = \boldsymbol{f}_t(\boldsymbol{x}_{t-1}, \boldsymbol{w}_t)$
Measurement model	$\boldsymbol{\rho}_t = \boldsymbol{h}_t(\boldsymbol{x}_t, \boldsymbol{n}_t)$

$$p(\boldsymbol{x}_t|\boldsymbol{\rho}_{1:t-1}) = \int \boxed{p(\boldsymbol{x}_t|\boldsymbol{x}_{t-1})}\, p(\boldsymbol{x}_{t-1}|\boldsymbol{\rho}_{1:t-1})\,\mathrm{d}\boldsymbol{x}_{t-1}$$

$$p(\boldsymbol{x}_t|\boldsymbol{\rho}_{1:t}) \propto \boxed{p(\boldsymbol{\rho}_t|\boldsymbol{x}_t)}\, p(\boldsymbol{x}_t|\boldsymbol{\rho}_{1:t-1})$$

Bayesian tracking techniques

Kalman Filter (KF)

Optimal filter if systems and measurements are linear

Extended Kalman Filter (EKF)

Approximated solutions for non-linear Gaussian systems

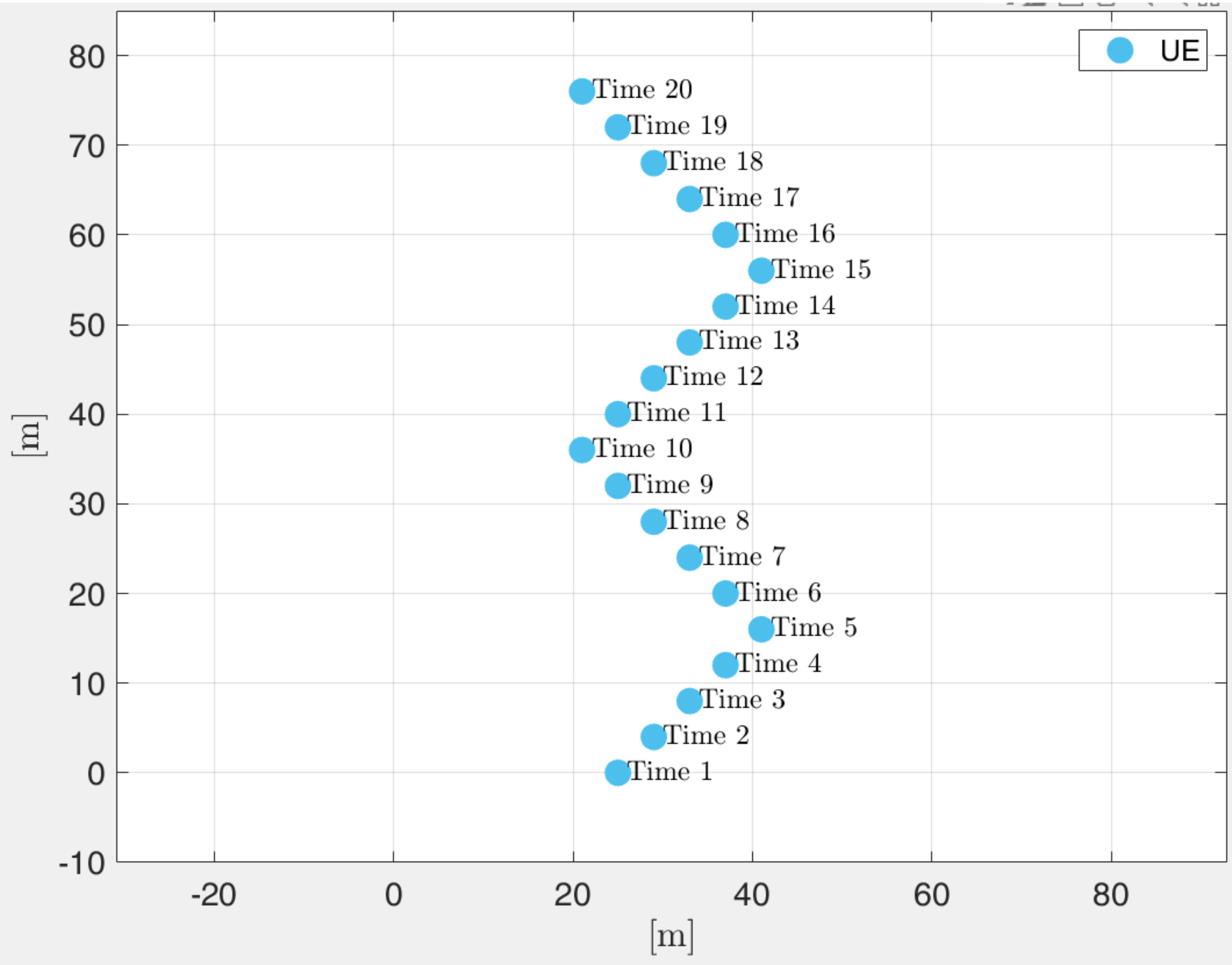
Grid Based Filter (GF)

Approximated solutions for non-linear non-Gaussian systems.
Uniform sampling of pdfs

Particle Filter (PF)

Approximated solutions for general non-linear non-Gaussian systems.
Non-uniform and time-varying sampling of pdfs

KF Exercise



Tracking problem:
  Moving UE with zig-zag trajectory.

Measurement availability:
  UE position

UE state:
  2D position

WE DO NOT KNOW THE UE MOBILITY,
WE JUST HAVE TO TRACK BY KF

KF: equations

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Prediction:

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{F}_t \hat{\mathbf{x}}_{t-1|t-1}$$

$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^T + \mathbf{Q}_{t-1}$$

System (motion) model $\boldsymbol{x}_t = \boldsymbol{f}_t(\boldsymbol{x}_{t-1}, \boldsymbol{w}_t)$ $\quad \boldsymbol{x}_t = \boldsymbol{F}\boldsymbol{x}_{t-1}$

$$\boldsymbol{w}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_t)$$

Update:

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + \mathbf{G}_t \underbrace{\left(\boldsymbol{\rho}_t - \mathbf{H}_t \hat{\mathbf{x}}_{t|t-1}\right)}_{\text{Innovation } \varepsilon_{t|t-1}}$$

$$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{G}_t \mathbf{H}_t \mathbf{P}_{t|t-1}$$

$$\mathbf{G}_t = \mathbf{P}_{t|t-1} \mathbf{H}_t^T \left(\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t\right)^{-1}, \quad \text{Kalman gain } K \times N$$

Measurement model $\boldsymbol{\rho}_t = \boldsymbol{h}_t(\boldsymbol{x}_t, \boldsymbol{n}_t)$ $\quad \boldsymbol{\rho}_{i,t} = \boldsymbol{x}_t + \boldsymbol{n}_i$

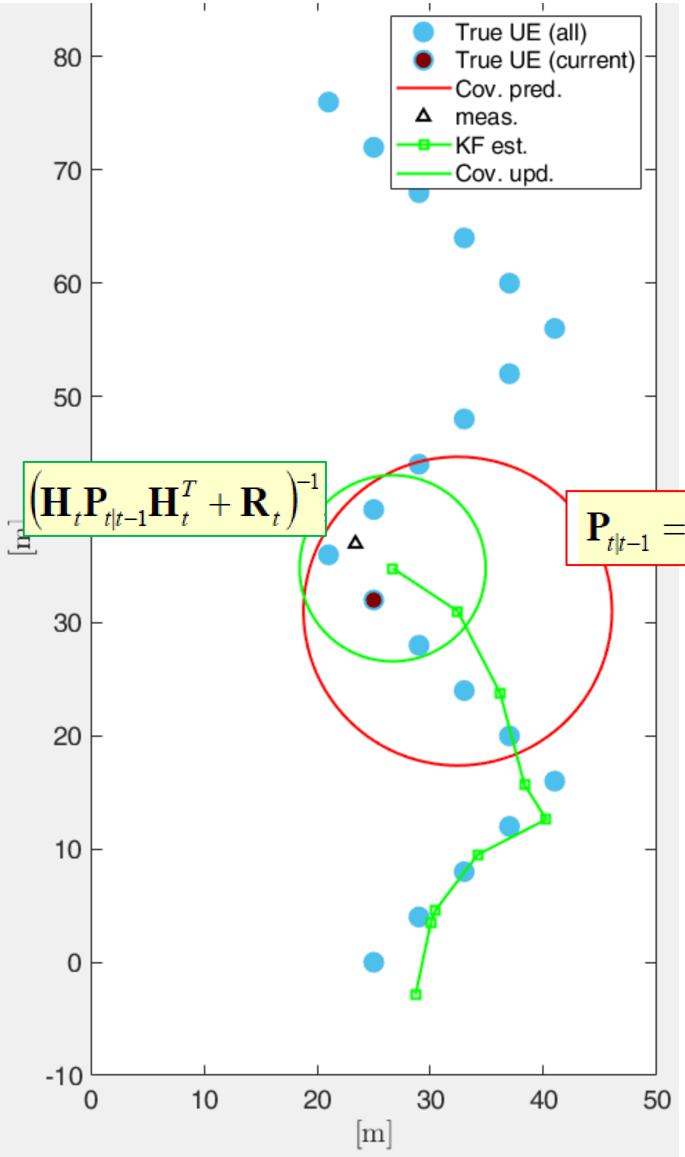
$$\boldsymbol{n}_t \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_t)$$

Initialization:

$$\hat{\mathbf{x}}_{0|0} = E[\mathbf{x}_0]$$

$$\mathbf{P}_{0|0} = \mathrm{Cov}(\mathbf{x}_0)$$

Recap



$$\left(\mathbf{H}_t \mathbf{P}_{t|t-1} \mathbf{H}_t^T + \mathbf{R}_t\right)^{-1}$$

$$\mathbf{P}_{t|t-1} = \mathbf{F}_t \mathbf{P}_{t-1|t-1} \mathbf{F}_t^T + \mathbf{Q}_{t-1}$$

Nearly constant velocity model

- The velocity is included in the unknown vector and estimated together with the position.
- A random walk model is assumed for the velocity, while the position is obtained as integration of the velocity:

$$\underbrace{\begin{bmatrix} u_{x,t} \\ u_{y,t} \\ v_{x,t} \\ v_{y,t} \end{bmatrix}}_{\mathbf{x}_t} = \underbrace{\begin{bmatrix} 1 & 0 & T & 0 \\ 0 & 1 & 0 & T \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{F}} \cdot \underbrace{\begin{bmatrix} u_{x,t-1} \\ u_{y,t-1} \\ v_{x,t-1} \\ v_{y,t-1} \end{bmatrix}}_{\mathbf{x}_{t-1}} + \underbrace{\begin{bmatrix} T^2/2 & 0 \\ 0 & T^2/2 \\ T & 0 \\ 0 & T \end{bmatrix}}_{\mathbf{L}} \cdot \underbrace{\begin{bmatrix} w_{vx,t-1} \\ w_{vy,t-1} \end{bmatrix}}_{\mathbf{w}_{a,t-1}}$$

$$\mathbf{x}_t = \begin{bmatrix} \mathbf{u}_t \\ \mathbf{v}_t \end{bmatrix} = \begin{bmatrix} u_{x,t} \\ u_{y,t} \\ v_{x,t} \\ v_{y,t} \end{bmatrix}$$

$$\mathbf{x}_t = \mathbf{F}\mathbf{x}_{t-1} + \mathbf{L}\mathbf{w}_{a,t-1}$$

$$\mathbf{w}_{a,t} \sim \mathcal{N}\left(\mathbf{0}, \sigma_a^2 \mathbf{I}_2\right) \Rightarrow \mathbf{x}_t \sim \mathcal{N}\left(\mathbf{F}\mathbf{x}_{t-1}, \sigma_a^2 \mathbf{L}\mathbf{L}^{\mathrm{T}}\right)$$

Zero-mean Gaussian acceleration

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EKF: equations

Linearized model:

$$\mathbf{x}_t = \hat{\mathbf{F}}_t \mathbf{x}_{t-1} + \mathbf{f}_t\left(\hat{\mathbf{x}}_{t-1|t-1}\right) - \hat{\mathbf{F}}_t \hat{\mathbf{x}}_{t-1|t-1} + \mathbf{w}_{t-1}$$

$$\boldsymbol{\rho}_t = \hat{\mathbf{H}}_t \mathbf{x}_t + \mathbf{h}_t\left(\hat{\mathbf{x}}_{t|t-1}\right) - \hat{\mathbf{H}}_t \hat{\mathbf{x}}_{t|t-1} + \mathbf{n}_t$$

We can apply the KF approach, where for the mean evaluation we use the non-linear functions, while for the covariance we use the above equations (constant terms do not change the result). We get:

Prediction:

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{f}_t\left(\hat{\mathbf{x}}_{t-1|t-1}\right)$$

$$\mathbf{P}_{t|t-1} = \hat{\mathbf{F}}_t \mathbf{P}_{t-1|t-1} \hat{\mathbf{F}}_t^T + \mathbf{Q}_{t-1}$$

Update:

$$\hat{\mathbf{x}}_{t|t} = \hat{\mathbf{x}}_{t|t-1} + \mathbf{G}_t \underbrace{\left(\boldsymbol{\rho}_t - \mathbf{h}_t\left(\hat{\mathbf{x}}_{t|t-1}\right)\right)}_{\text{Innovation } \varepsilon_{t|t-1}}$$

$$\mathbf{P}_{t|t} = \mathbf{P}_{t|t-1} - \mathbf{G}_t \hat{\mathbf{H}}_t \mathbf{P}_{t|t-1}$$

$$\mathbf{G}_t = \mathbf{P}_{t|t-1} \hat{\mathbf{H}}_t^T \left(\hat{\mathbf{H}}_t \mathbf{P}_{t|t-1} \hat{\mathbf{H}}_t^T + \mathbf{R}_t\right)^{-1}$$

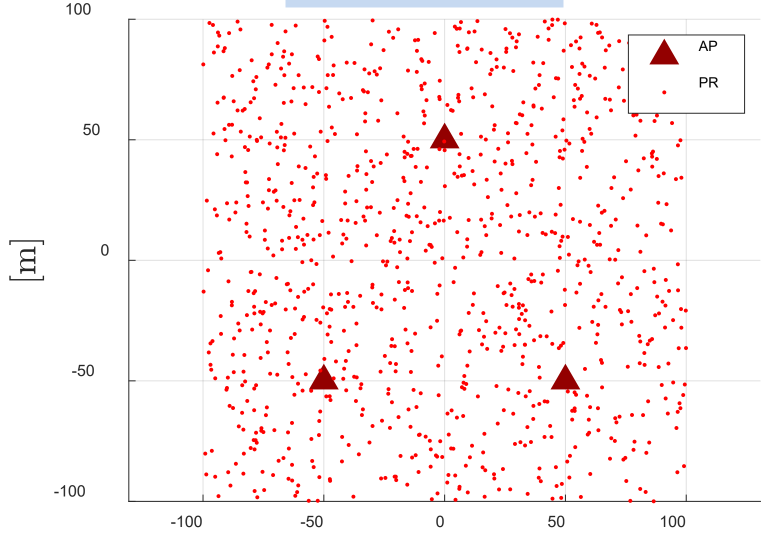
Particle Filter

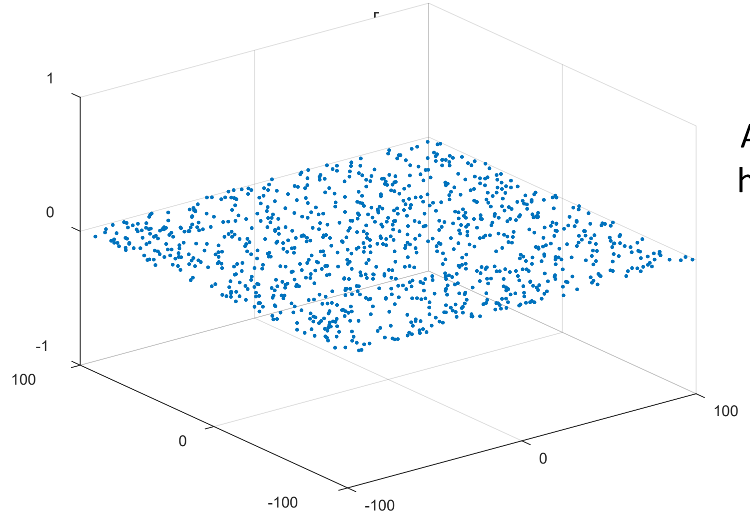
Particle filtering uses a set of particles to represent the spatial pdf.

Each particle has a likelihood weight assigned to it that represents the probability of that particle being sampled from the probability density function.

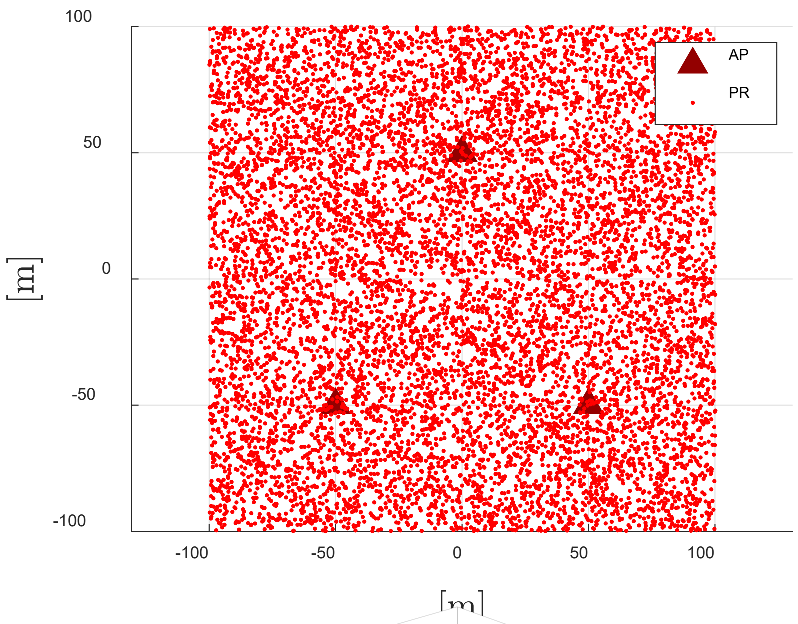
Weight collapse is a common issue encountered in these filtering algorithms -> need to resample after each observation.

$N_P = 1000$

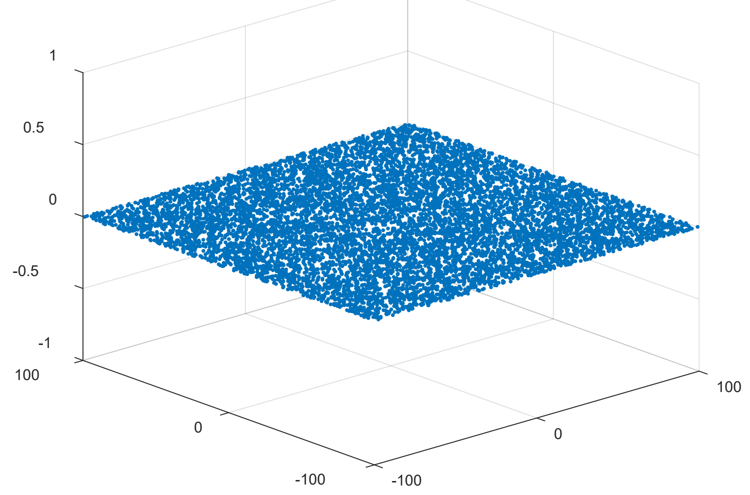




$N_P = 10000$

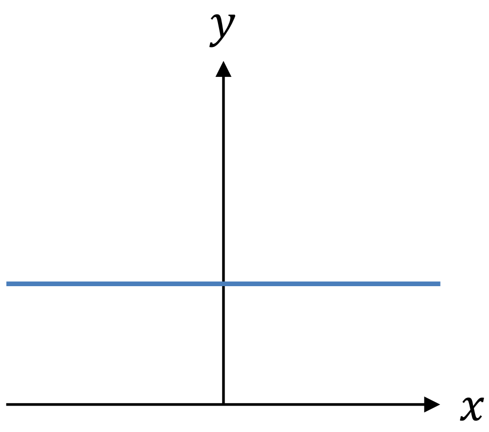


All particles have weight $^{1}/_{N_P}$

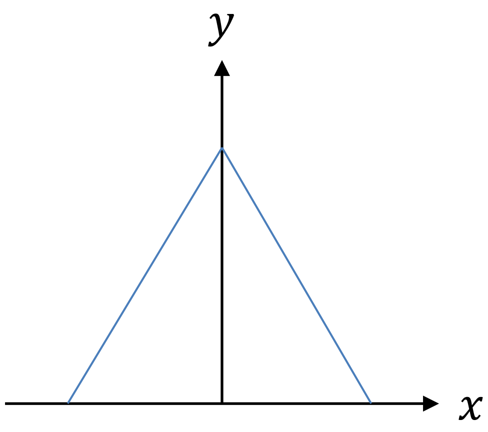


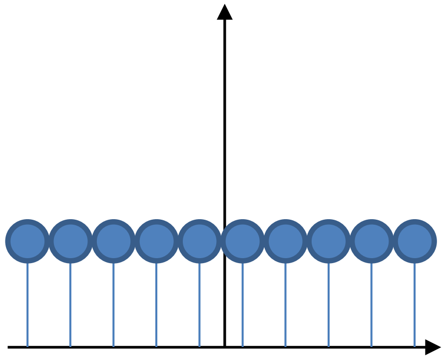
PDF sampling

Uniform PDF

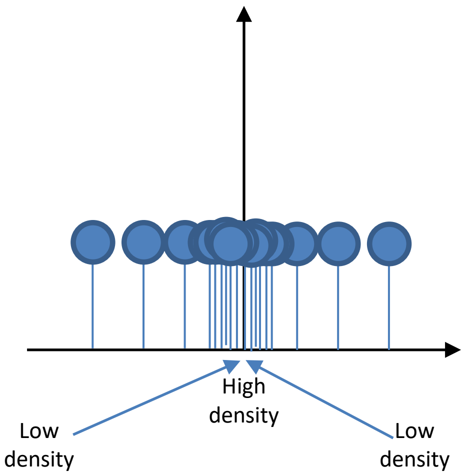


Triangular PDF



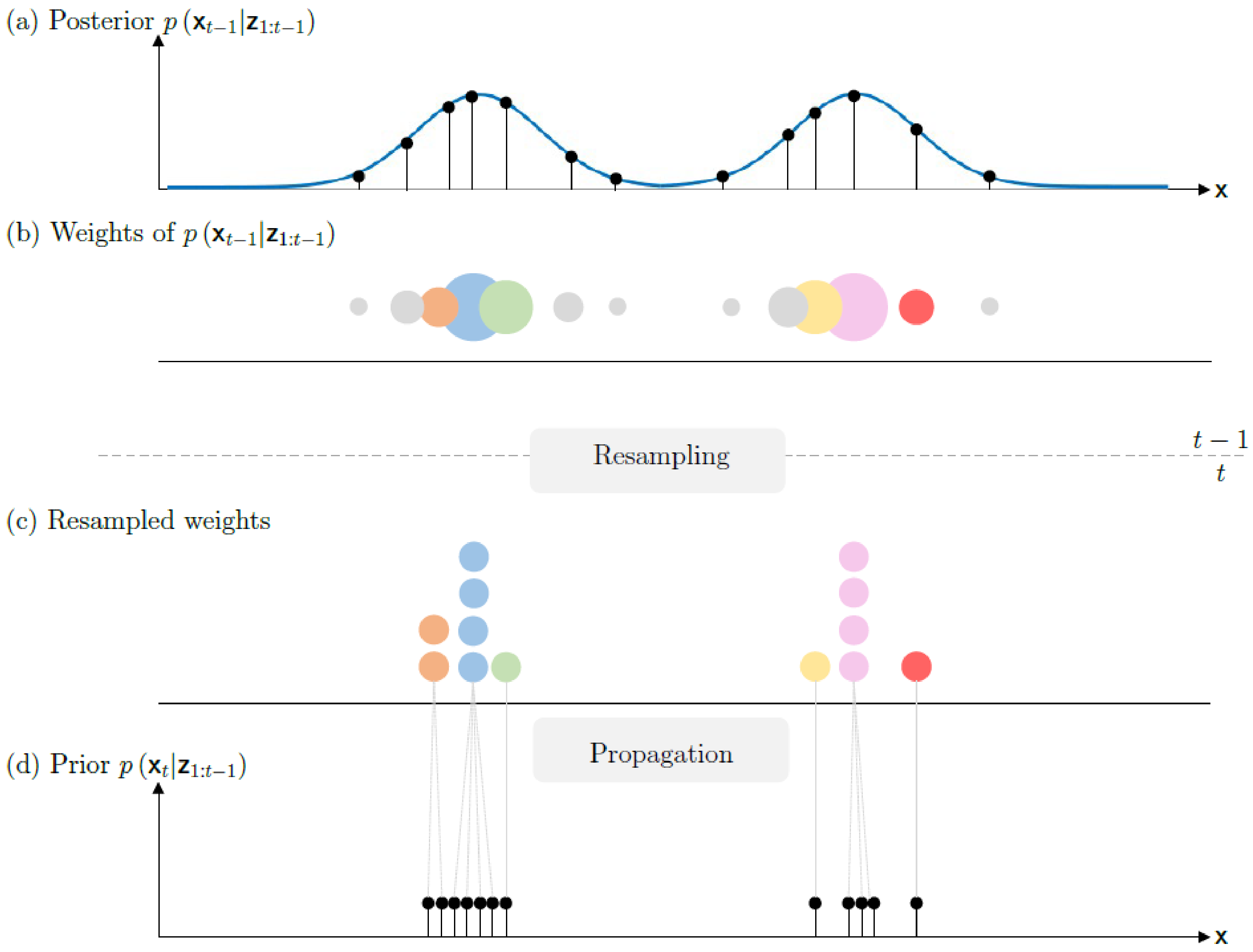


N_P particles uniformly distributed, each of them with weight $w_i = \frac{1}{N_P}$



N_P particles non uniformly distributed, each of them with weight $w_i = \frac{1}{N_P}$

PF prediction



PF steps

When new measurements $\boldsymbol{\rho}$ are available, we need to update the particles accordingly.

1. Evaluate the likelihood of the measurement for all the particles

$$p(\boldsymbol{\rho}|\boldsymbol{x}_i) \quad \forall i \quad \longrightarrow \quad \text{Product of likelihoods of each AP}$$

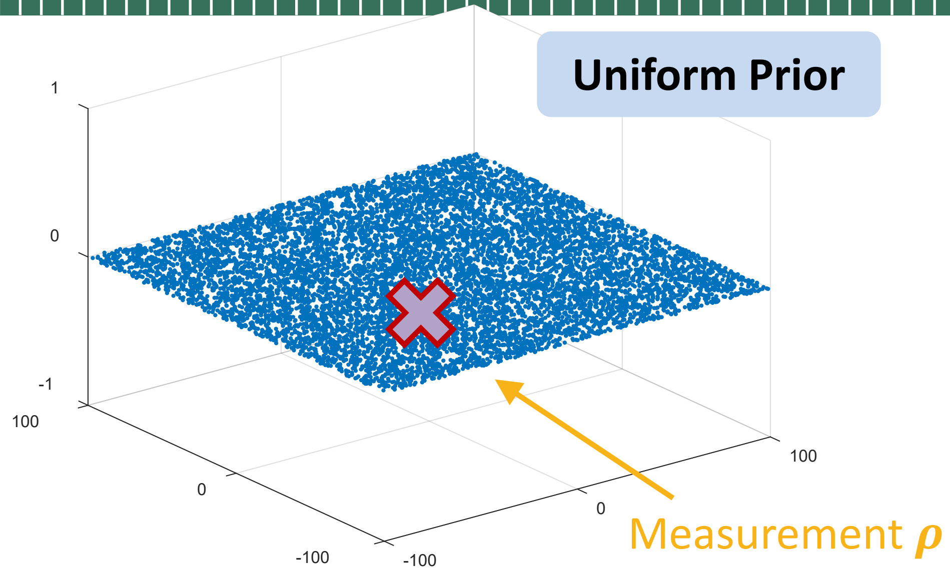
2. The value of the likelihood is the new weight w_i of the particle

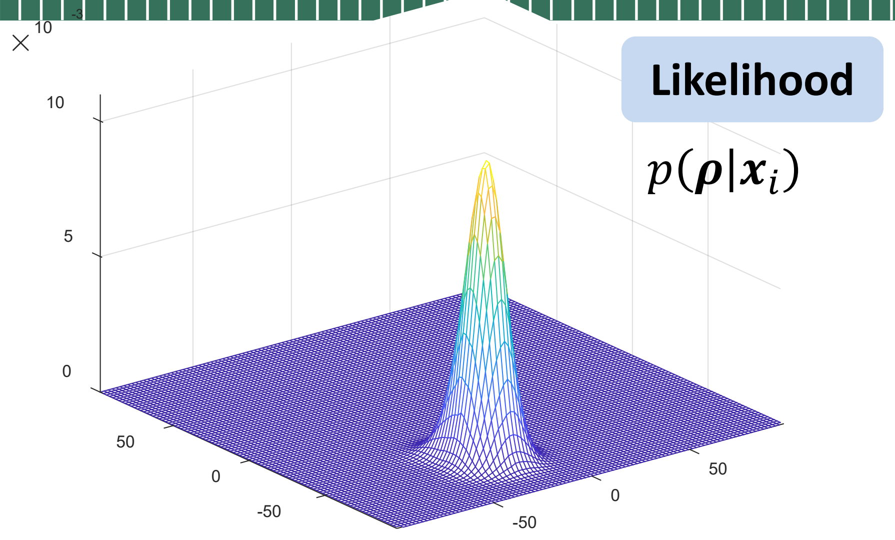
$$w_i = p(\boldsymbol{\rho}|\boldsymbol{x}_i) \quad \forall i$$

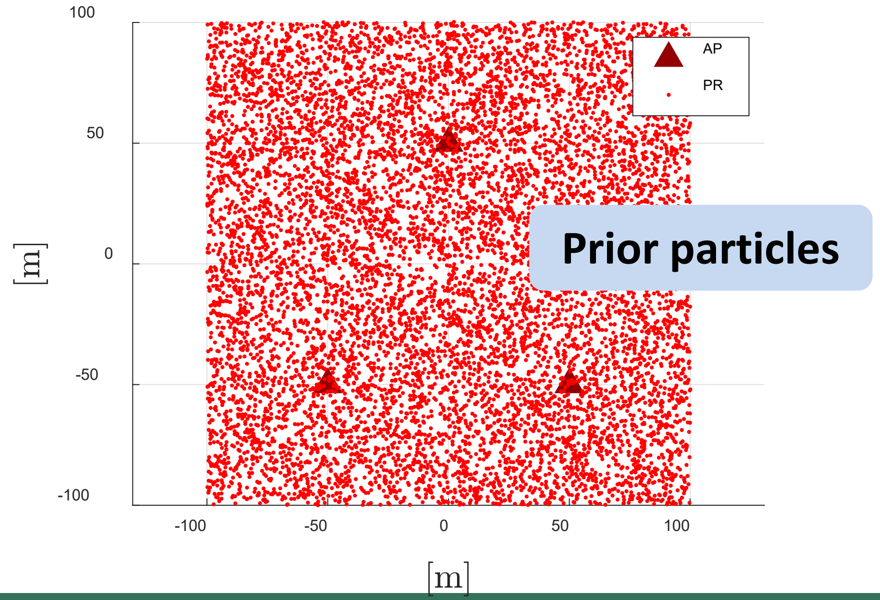
3. Now, $\sum_{i=1}^{N_P} w_i \neq 1$ so we have to normalize the weights.

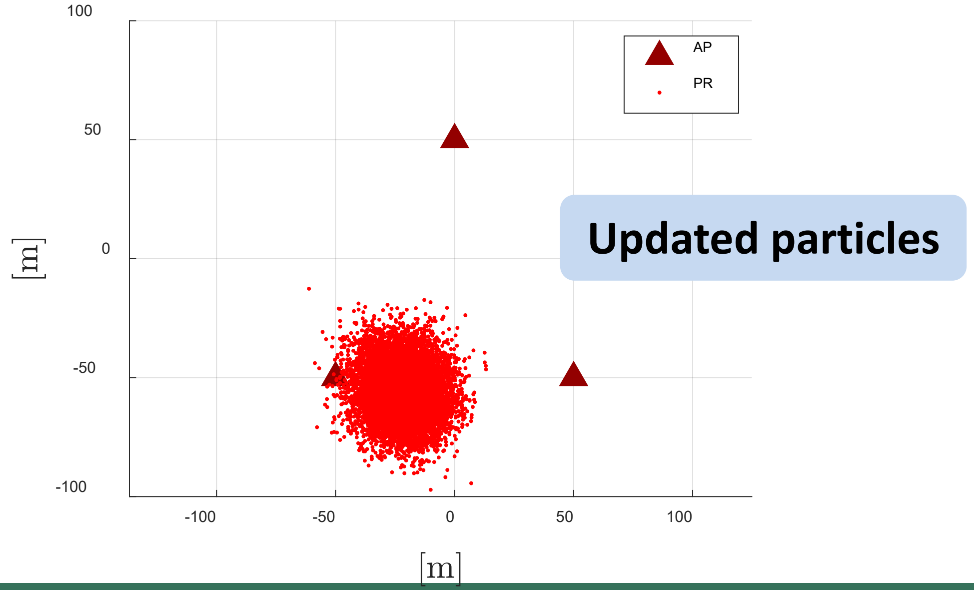
4. Resample according to the updated weights (particles will become concentrated around the position with higher weights).

PF example


Uniform Prior
1
0
-1
100
0
-100
100
0
-100
Measurement $\boldsymbol{\rho}$


Likelihood
$p(\boldsymbol{\rho}|\boldsymbol{x}_i)$
$\times 10^{-3}$
10
5
0
50
0
-50
50
0
-50


Prior particles
AP
PR
100
50
0
-50
-100
-100
-50
0
50
100
[m]
[m]

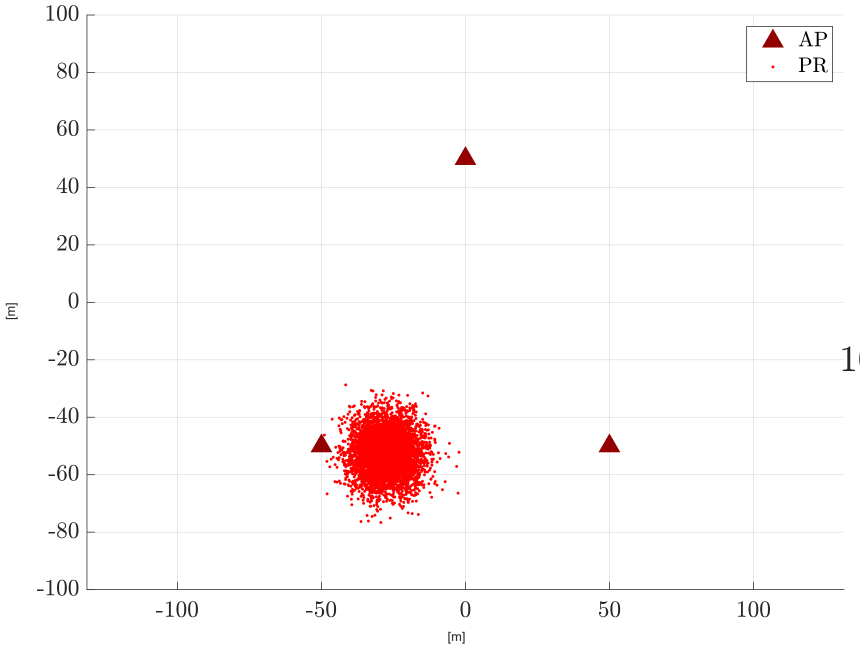

Updated particles
AP
PR
100
50
0
-50
-100
-100
-50
0
50
100
[m]
[m]

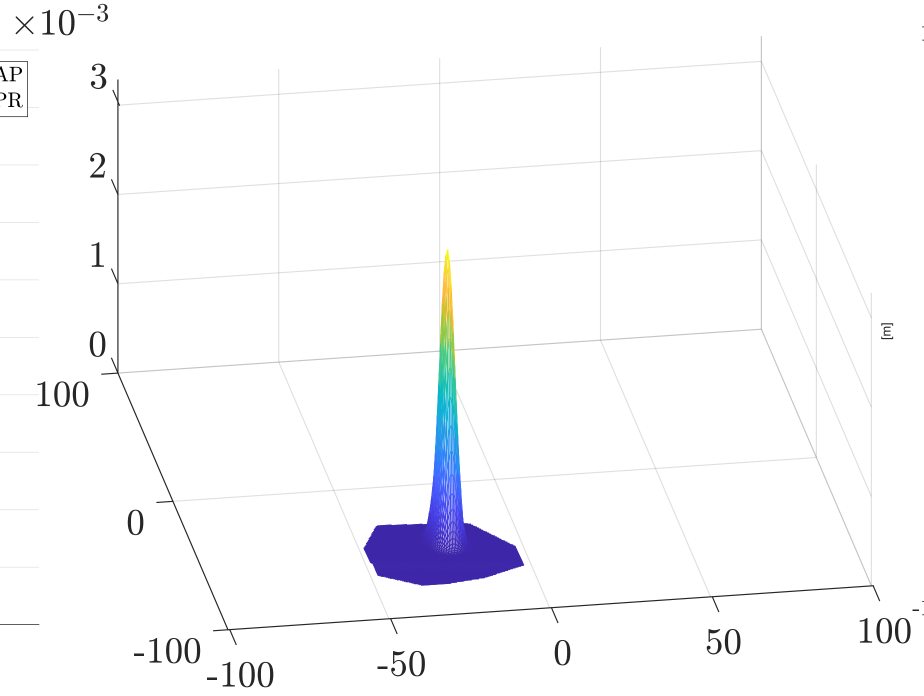
PF example

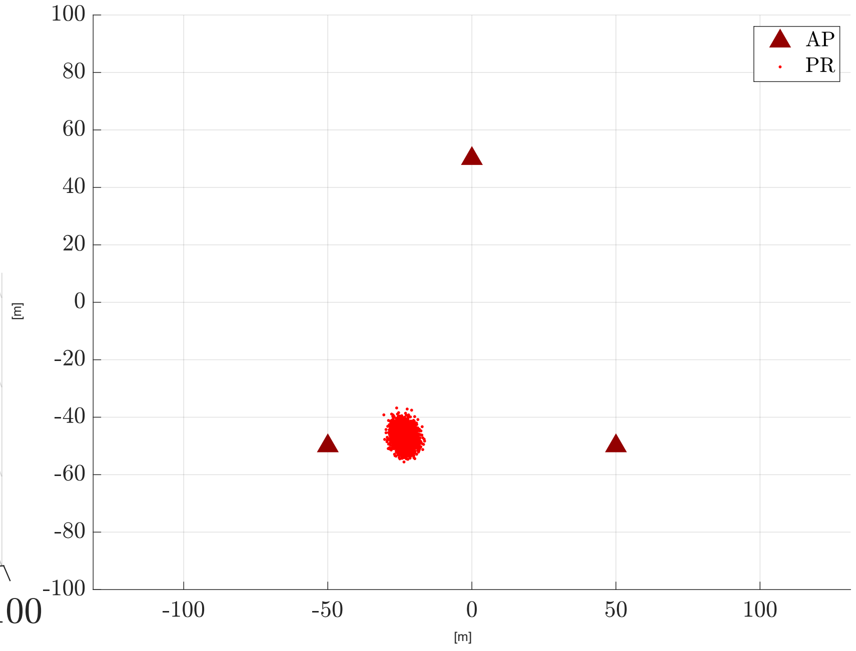
Prior

Likelihood

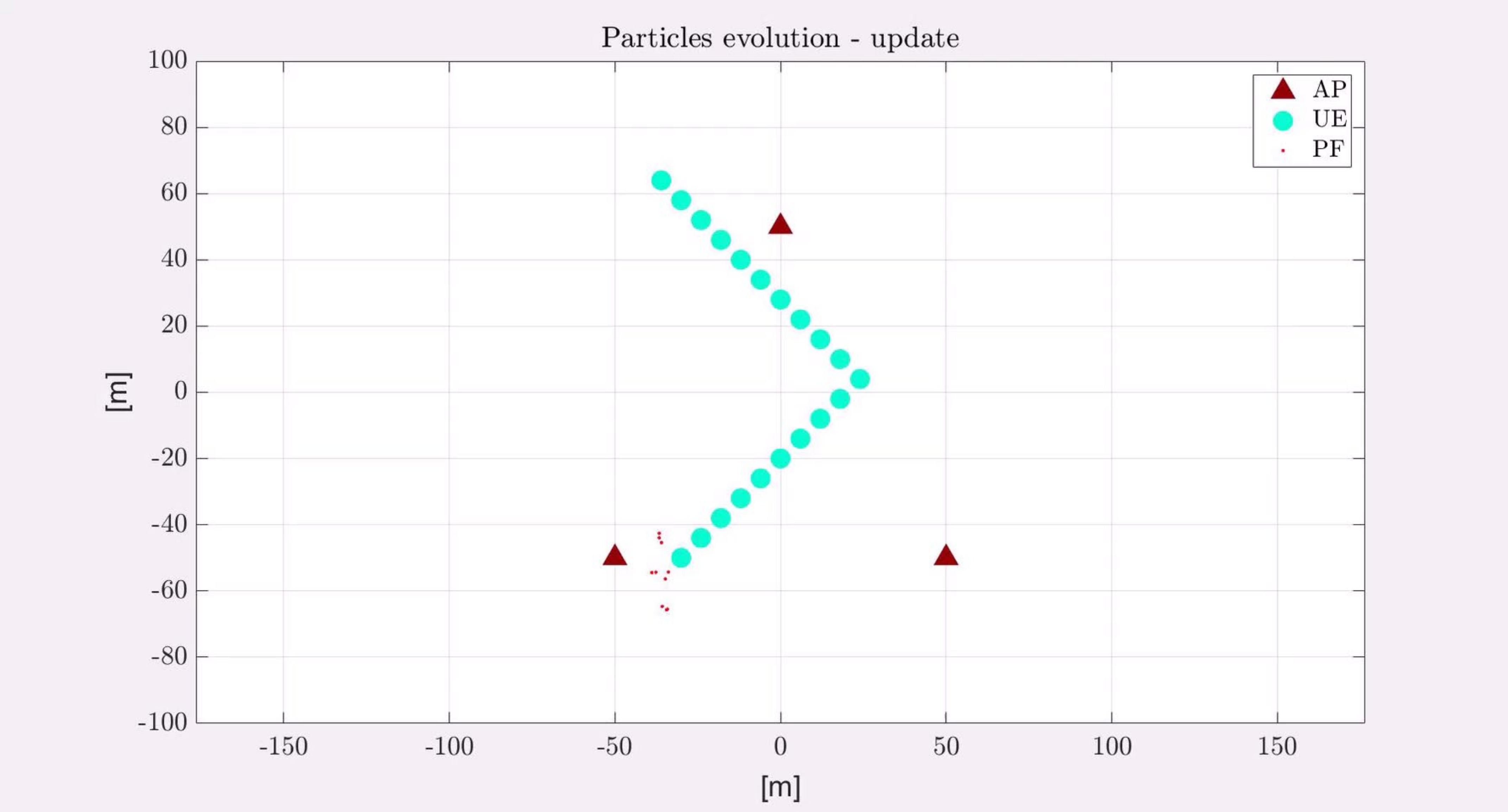
Posterior





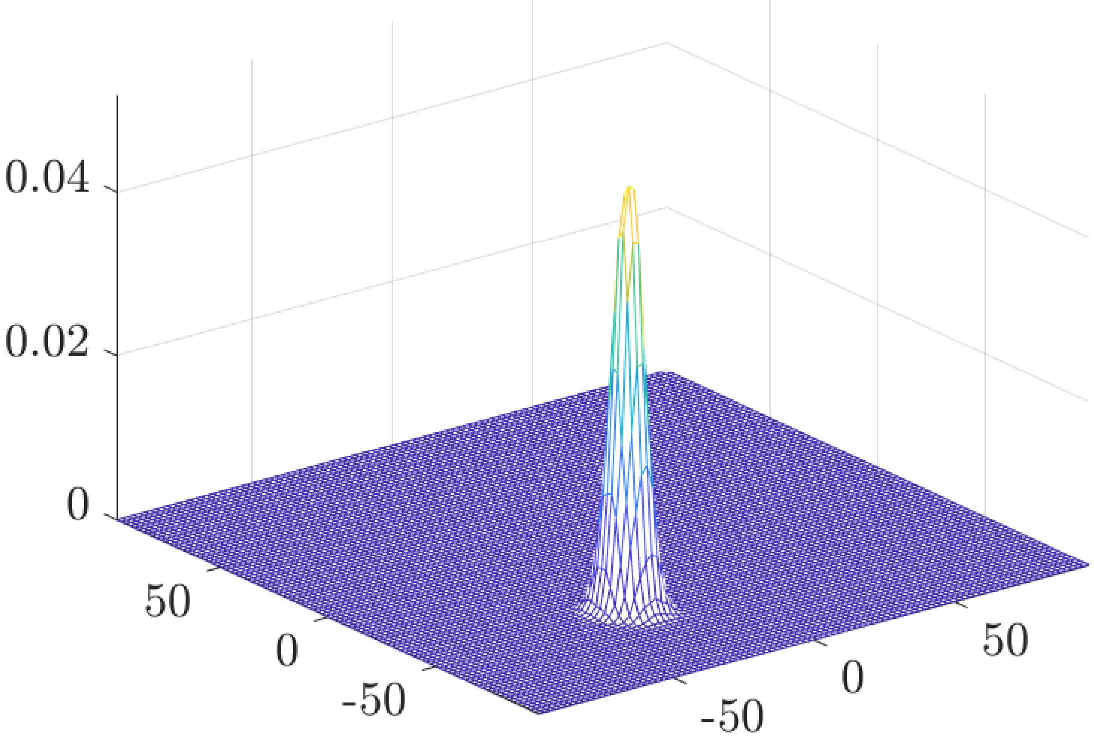


Tracking by PF - video



PF: when?

Unimodal (Gaussian) pdf



Bimodal pdf