**CURRENT STATE OF PAUL'S PROJECT**

At the moment the basic system consists of a $L' \times L$ system with periodic boundary conditions and a 50-50 configuration of $\pm 1$ Ising spins. A magnetic field of strength $H$ is applied in the region $x \in [-L'/4, L'/4]$ where the $x$ coordinates are taken to run from $-L'/2$ to $L'/2$. At zero temperature this will lead to a stripe of + spins in the middle of the system. At non-zero temperature the magnetic field will confine the + phase to this region and create two interfaces. The question is what are the statistics of these interfaces, how can they be analyzed, what is the effect of the strength of the magnetic field as well as the finite size effects? Finally what then happens when a driving force is applied in the same region as the magnetic field? All this is under local Kawasaki dynamics

The interface can be measured numerically in a number of different ways.

- **The magnetisation profile**. The system is statistically invariant under translations in the $y$ direction. We can thus define a magnetisation profile in the $x$ direction defined as

$$M_x = \frac{1}{L} \sum_y S_{x,y} \tag{1}$$

  For large systems where the correlation length $\xi$ is much smaller that the lateral system size $L'$, each of the interfaces can be fitted with a tanh profile or a more carefully calculated mean field magnetisation profile $m(x)$ from which an interface thickness $\xi_\perp$ can be estimated.
- **The bond energy along a value** $x$ As in the Abrahams paper one can compute

$$C_{x1} = \sum \frac{1}{L} \sum_y S_{x,y} S_{x,y+1} \tag{2}$$

  This is a microscopic way of measuring the presence of the interface between the two phases as a function of $x$.
- **The correlation function in the** $y$ **direction as a function of** $x$ Generalizing the Abrahams paper we compute

$$C_{xr} = \frac{1}{L} \sum_y S_{x,y} S_{x,y+r} - \frac{1}{L^2} (\sum_y S_{x,y})^2 \tag{3}$$

  An exponential fit of this function $C_{xr} \sim \exp(-r/\xi_{||}(x))$ defines an $x$ dependent correlation function $\xi(x)$ away from the interface we should find $\xi_{||}(x) \to \xi_{bulk}$, where $\xi_{bulk}$ is the bulk correlation length (with the magnetic field taken into account). Paul's simulations show that $\xi_{||}(x)$ has a Gaussian profile peaked at the point where the field switches from zero to non-zero (otherwise the $T = 0$ interface).

> I think the behavior of $C_{xr}$ should be exponential when there is an applied field. However the physics should be quite different when there is no applied field.

As well as carrying out theses basic measurements, we could consider the following. In order to get a better definition of an effective surface one can use a numerical coarse graining procedure. For a given spin replace it with a majority rule

$$S_{x,y} \to \sigma_{x,y} = \mathrm{sign} \sum_{\mathrm{x'y'} \in \Lambda_{\mathrm{xy}}} \mathrm{S_{x'y'}}, \tag{4}$$

where $\Lambda_{xy}$ denotes the point $xy$ and its nearest neighbours. This procedure can be carried out until the correlation length in the bulk becomes equal to 1 and should therefore mean the numerical measurements are detecting correlation effects due to the surface fluctuations rather than simply bulk fluctuations.

Interestingly for the strip geometry used here one can see that it should be stable (but have a global zero diffusive mode ) even when there is no applied field. At low temperatures to minimize the interfacial energy, if the periodic boundary conditions (pbcs) where not present you would expect one phase to form a drop in the middle of the other. In the stripe phase the energy relative to bulk is

$$E_{stripe} = 2L\sigma. \tag{5}$$

If you relax the pbcs, the radius of the drop formed is given by conservation of species as

$$\pi R^2 = LL'/2 \Rightarrow E_{drop} = \sigma\sqrt{2\pi LL'}. \tag{6}$$

The stripe is thus stable if $E_{stripe} < E_{drop}$

$$\frac{L}{L'} < \frac{\pi}{2} \approx 1.57. \tag{7}$$

This is related to the Rayleigh Bernard instability which is induced by large wavelength fluctuations, which are suppressed if $L$ is not large enough. I think in these systems one can take the field $H$ to very small just to prevent the stripe from diffusing (which would however happen very slowly).

On the analytical side there are some interesting questions. The basic model for the effective surface profile $h(y)$ is the energy functional (I leave as an open question how to analyze the two surface case)

$$F([h] = \frac{\sigma}{2}\int_0^L h'^2(y)dy - H\int_0^L h\theta(-h)dy, \tag{8}$$

The first term is the surface tension and the second is the term due to the magnetic field.If we imagine a magnetic field applied in the region $h < 0$ we can write the magnetic penalty of going above zero as the above due to conservation of the magnetisation to zero. The conservation of the magnetisation at zero means in the interface language that

$$\int_0^L h(y)dy = 0. \tag{9}$$

There is a very nice paper by Majumdar and Comtet on this class of problems and we could compare the Ising simulation with these results (or try and find better ways to fit data).

- The interface thickness will be given by

$$\xi_\perp^2 = \langle (h - \langle h \rangle)^2 \rangle, \tag{10}$$

  note the breaking of symmetry by the field will give a non-zero value of $\langle h \rangle$.
- The presence of the interface at a given height $x$ is given by

$$p(x) = \langle \delta(x - h) \rangle. \tag{11}$$

  Without a magnetic field it is known that this has support over a region $\sqrt{L}$, i.e. the length of the stripe determines the scale of the interface fluctuations. Analytical results are known for this case (I can use the results of Comtet and Majumdar) You can also use this to compute $\xi_\perp$.
- The spin-spin correlation along the interface - in terms of the height profile we write (plus phase below and minus above) $S(x, y) = 1$ is $x < h(y)$ and $S(x, y) = -1$ if $x > h(y)$ and so we have

$$C(x, r) = \langle \text{sign}(h(y) - x)\text{sign}(h(y + r) - x) \rangle_c \tag{12}$$

  In principle these objects can be computed using path integrals/quantum mechanics techniques. I checked with Satya that they were open and he confirmed its as far as he was concerned. The business with the non-uniform magnetic field seems to open up many new questions, even before applying the non-equilibrium driving.

## 1. Single interface model

In the interface model we can identify the spin at the point $(x, y)$ as a $+$ if $x < h(y)$, i.e. the plus spins are below the interface and as $-$ if it is above the interface. This means

$$S(x, y) = \text{sign}(h(y) - x). \tag{13}$$

For consider a system with $(x, y) \in [-L'2/2, L'/2] \times [0, L]$ and apply a magnetic field

$$H(x) = -\frac{B}{2}\text{sign}(x). \tag{14}$$

This means that their is a symmetry between the $+$ and $-$ spins. Moreover I think this means that even Glauber dynamics will give rise to two phases $+$ for $x < 0$ and $-$ for $x > 0$ with an interface which fluctuates about $x = 0$. In the height model we have

$$H = \frac{\sigma}{2} \int_0^L dy h'^2(y) - \int dx dy S(x, y) H(x), \tag{15}$$

the first term is the interfacial energy and the second is the energy due to the interaction with the magnetic field

$$H_M = -\frac{B}{2}\int_0^L dy\left[\int_{-L'/2}^{0}\operatorname{sign}(h(y)-x)dx - \int_0^{L'/2}\operatorname{sign}(h(y)-x)dx\right]. \tag{16}$$

In the case where $h(y) > 0$ we have

$$\left[\int_{-L'/2}^{0}\operatorname{sign}(h(y)-x)dx - \int_0^{L'/2}\operatorname{sign}(h(y)-x)dx\right] = -L'/2 - \int_0^{h(y)} dx + \int_{h(y)}^{L'/2} dx$$

$$-2\int_0^{h(y)} dx = -2h(y) = -2|h(y)| \tag{17}$$

where as when $h(y) < 0$ we find

$$\left[\int_{-L'/2}^{0}\operatorname{sign}(h(y)-x)dx - \int_0^{L'/2}\operatorname{sign}(h(y)-x)dx\right] = 2h(y) = -2|h(y)|. \tag{18}$$

The energy due to interaction with the field is thus

$$H_M = B\int_0^L |h(y)|dy, \tag{19}$$

and so

$$H = \frac{\sigma}{2}\int_0^L h'^2(y)dy + B\int_0^L |h(y)|dy. \tag{20}$$

By construction the Hamiltonian is invariant under $h \to -h$. For an interface with $h(0) = h$ and $h(L) = h'$ the partition function is given by

$$Z(h,h',L) = \int_{h(0)=h} d[h]\delta(h(L)-h')\exp\left(-\frac{\beta\sigma}{2}\int_0^L h'^2(y)dy - \beta B\int_0^L |h(y)|dy\right). \tag{21}$$

$Z$ obeys the time (L) dependent Schrödinger equation

$$\frac{\partial Z}{\partial L} = \frac{1}{2\sigma\beta}\frac{\partial^2 Z}{\partial h^2} - B\beta|h|Z \tag{22}$$

with the initial conditions

$$Z(h,h',0) = \delta(h-h'). \tag{23}$$

The solution can be written in terms of the stationnary solutions to the Schrd̈inger equation

$$-\frac{1}{2\sigma\beta}\frac{\partial^2\psi_E}{\partial h^2} + \beta B|h|\psi_E = E\psi, \tag{24}$$

as

$$Z(h,h',L) = \sum_E \exp(-EL)\psi_E(h)\psi_E(h'). \tag{25}$$

In the thermodynamic limit the sum is dominated by the ground state energy $E_0$ so the free energy per unit length is given by

$$f = TE_0. \tag{26}$$

The height distribution at a given point (they are all equivalent when we use the periodic boundary conditions $h = h'$) is given by

$$p(h) = \psi^2_{E_0}(h). \tag{27}$$

The eigenfunction equation can be written as

$$-\frac{1}{2}\frac{\partial^2 \psi_\epsilon}{\partial h^2} + \lambda |h| \psi_\epsilon = \epsilon \psi_\epsilon, \tag{28}$$

where $\epsilon = E\beta\sigma$ and $\lambda = \beta^2\sigma B$. The solutions are given by the even wave functions

$$\psi_\epsilon(h) = \frac{1}{N}\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}(|h| - \frac{\epsilon}{\lambda})\right), \tag{29}$$

where the energies $\epsilon$ are determined from the condition $\psi'_\epsilon(0) = 0$. This means that $\epsilon_k = 2^{-\frac{1}{3}}\lambda^{\frac{2}{3}}\alpha_k$, where $-\alpha_k$ denote the zeros of the derivative of the Airy function (which are all negative). The ground state is given by $\alpha_0 = 1.087....$ This means that

$$E_0 = \frac{\epsilon_0}{\beta\sigma} = 2^{-\frac{1}{3}}\frac{\lambda^{\frac{2}{3}}}{\sigma}\alpha_0 = 2^{-\frac{1}{3}}\alpha_0\beta^{\frac{1}{3}}\sigma^{-\frac{1}{3}}B^{\frac{2}{3}}. \tag{30}$$

The ground state wave function is thus

$$\psi_0(h) = \frac{\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_0)\right)}{\sqrt{2\int_0^\infty dh \mathrm{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_0\right)}}, \tag{31}$$

and so

$$p(h) = \frac{\mathrm{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_0)\right)}{2\int_0^\infty dh \mathrm{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_0\right)} \tag{32}$$

This means that $z = (2\lambda)^{\frac{1}{3}}h$ is dimensionless and $(2\lambda)^{-\frac{1}{3}} = \xi_\perp$ is the interface width (the scale of $h$) so

$$\xi_\perp = \frac{1}{(2\beta^2\sigma B)^{\frac{1}{3}}}. \tag{33}$$

The pdf of the scaled variable $z$ is then just

$$p(h) = \frac{\mathrm{Ai}^2\left(|z| - \alpha_0)\right)}{2\int_0^\infty dz \mathrm{Ai}^2\left(|z| - \alpha_0\right)}. \tag{34}$$

The next exited state has odd parity and is given by

$$\psi_1(h) = \frac{\text{sign(h)Ai}\left((2\lambda)^{\frac{1}{3}}|\text{h}| - \alpha_1)\right)}{\sqrt{2\int_0^\infty dh\text{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \beta_0\right)}}, \tag{35}$$

where here $\text{Ai}(-\alpha_1) = 0$, i.e. $-\alpha_1$ is the first (negative) negative zero of the Airy function. For the higher excited states we have for the even wave functions

$$\psi_{2n}(h) = \frac{\text{Ai}\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_{2n})\right)}{\sqrt{2\int_0^\infty dh\text{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_{2n}\right)}}, \tag{36}$$

with

$$E_{2n} = 2^{-\frac{1}{3}}\alpha_{2n}\beta^{\frac{1}{3}}\sigma^{-\frac{1}{3}}B^{\frac{2}{3}}. \tag{37}$$

and for the odd wave functions

$$\psi_{2n+1}(h) = \frac{\text{sign(h)Ai}\left((2\lambda)^{\frac{1}{3}}|\text{h}| - \alpha_{\text{2n+1}})\right)}{\sqrt{2\int_0^\infty dh\text{Ai}^2\left((2\lambda)^{\frac{1}{3}}|h| - \alpha_{2n+1}\right)}}, \tag{38}$$

and with

$$E_{2n+1} = 2^{-\frac{1}{3}}\alpha_{2n+1}\beta^{\frac{1}{3}}\sigma^{-\frac{1}{3}}B^{\frac{2}{3}}. \tag{39}$$

The characteristic energy gap between levels gives an inverse correlation length parallel to the interface

$$\xi_{||} = \frac{1}{\Delta E} = 2^{\frac{1}{3}}\beta^{-\frac{1}{3}}\sigma^{\frac{1}{3}}B^{-\frac{2}{3}} \tag{40}$$

In the large $L$ limit the connected correlation function

$$C_f(r) = \langle f(h(0))f(h(r))\rangle_c \tag{41}$$

is given by

$$C_f(r) = \sum_{n\neq 0}\left[\int_{-\infty}^{\infty} dh\ f(h)\psi_n(h)\psi_0(h)\right]^2 \exp(-[\alpha_n - \alpha_0]\frac{r}{\xi_{||}}). \tag{42}$$

In particular we would like to study the case

$$f(h) = \text{sign}(h - x) \tag{43}$$

so $C_f(r) = C(x, r)$ (defined above) is the spin-spin correlation measured parallel to the interface $x =$ along the direction of the interface. Here we have

$$\int_{-\infty}^{\infty} dh\ f(h)\psi_n(h)\psi_0(h) = -\int_{-\infty}^{x} dh\ \psi_n(h)\psi_0(h) + \int_{x}^{\infty} dh\ \psi_n(h)\psi_0(h). \tag{44}$$

When $x > 0$, using orthogonality we have

$$\int_{-\infty}^{\infty} dh\ f(h)\psi_n(h)\psi_0(h) = 2\int_x^{\infty} dh\ \psi_n(h)\psi_0(h). \tag{45}$$

The matrix elements can be evaluated using the equation

$$-\frac{1}{2}\frac{\partial^2\psi_n}{\partial h^2} + \lambda|h|\psi_n = \epsilon_n\psi_n \tag{46}$$

Consider the integral for $n \neq 0$

$$I(n,x) = \int_x^{\infty} dh\psi_n(h)\psi_0(h) \tag{47}$$

Using the Schródinger equation in the way used to demonstrate the orthogonality of wave functions we find

$$I(n,x) = \frac{1}{2}\frac{\psi_0(x)\psi_n'(x) - \psi_n(x)\psi_0'(x)}{\epsilon_n - \epsilon_0} \tag{48}$$

and thus

$$\begin{aligned}&C(x,r) = (2\lambda)^{\frac{2}{3}}\\ &\times\sum_{n\neq 0}\frac{[\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_0\right)\mathrm{Ai}'\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_n\right) - \mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_n\right)\mathrm{Ai}'\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_0\right)]^2}{2^{-\frac{2}{3}}\lambda^{\frac{4}{3}}N_n^2N_0^2(\alpha_n - \alpha_0)^2}\exp(-[\alpha_n - \alpha_0]\frac{r}{\xi_{||}})\end{aligned} \tag{49}$$

where $N_n$ denotes the wave function normalisation

$$N_n^2 = 2\int_0^{\infty} dh\mathrm{Ai}^2\left((2\lambda)^{\frac{1}{3}}h - \alpha_n\right) = 2(2\lambda)^{-\frac{1}{3}}\int_0^{\infty} du\mathrm{Ai}^2\left(u - \alpha_n\right). \tag{50}$$

We thus find

$$\begin{aligned}&C(x,r) = \sum_{n\neq 0}\\ &\frac{[\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_0\right)\mathrm{Ai}'\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_n\right) - \mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_n\right)\mathrm{Ai}'\left((2\lambda)^{\frac{1}{3}}|x| - \alpha_0\right)]^2}{\left[\int_0^{\infty} du\mathrm{Ai}^2\left(u - \alpha_n\right)\int_0^{\infty} du\mathrm{Ai}^2\left(u - \alpha_0\right)\right](\alpha_n - \alpha_0)^2}\exp(-[\alpha_n - \alpha_0]\frac{r}{\xi_{||}}),\end{aligned} \tag{51}$$

Finally we can further simplify the above by computing the normalization terms, integration by parts gives

$$N_n' = \int_0^{\infty} du\mathrm{Ai}^2\left(u - \alpha_n\right) = [u\mathrm{Ai}^2\left(u - \alpha_n\right)]_0^{\infty} - 2\int_0^{\infty} duu\mathrm{Ai}\left(u - \alpha_n\right)\mathrm{Ai}'\left(u - \alpha_n\right), \tag{52}$$

the boundary term above is zero and using Airy's equation in the second term, i.e.

$$\mathrm{Ai}''(z) - z\mathrm{Ai}(z) = 0, \tag{53}$$

we get

(54)

$$N_n' = -2\int_0^\infty du[\mathrm{Ai}''(u-\alpha_n) + \alpha_n \mathrm{Ai}(u-\alpha_n)]\mathrm{Ai}'(u-\alpha_n) = \mathrm{Ai}'^2(-\alpha_n) + \alpha_n \mathrm{Ai}^2(-\alpha_n).$$

For the even wave functions we have

$$N_n' = \alpha_n \mathrm{Ai}^2(-\alpha_n), \tag{55}$$

while for the odd wave functions

$$N_n' = \mathrm{Ai}'^2(-\alpha_n). \tag{56}$$

This then gives

$$C(x,r) = \sum_{n\neq 0} \frac{[\mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_{2n}\right) - \mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_{2n}\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)]^2}{\left[\alpha_{2n}\mathrm{Ai}^2(-\alpha_{2n})\alpha_0\mathrm{Ai}^2(-\alpha_0)\right](\alpha_{2n}-\alpha_0)^2} \exp(-[\alpha_{2n}-\alpha_0]\frac{r}{\xi_{||}})$$

$$+ \sum_n \frac{[\mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_{2n+1}\right) - \mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_{2n+1}\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)]^2}{\left[\mathrm{Ai}'^2(-\alpha_{2n+1})\alpha_0\mathrm{Ai}^2(-\alpha_0)\right](\alpha_{2n+1}-\alpha_0)^2} \exp(-[\alpha_{2n+1}-\alpha_0]\frac{r}{\xi_{||}})$$

Perhaps this can be simplified further ? However it means that the correlation function plotted in $(x,r)$ space has a universal form if plotted in terms of the variables $u = x/\xi_\perp$ and $\rho = r/\xi_{||}$.

In the limit of large $r$ the term coming from $n=1$, the next lowest eigenstate dominates and we find

(57)

$$C(x,r) \approx \frac{[\mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_1\right) - \mathrm{Ai}\left(\frac{|x|}{\xi_\perp} - \alpha_1\right)\mathrm{Ai}'\left(\frac{|x|}{\xi_\perp} - \alpha_0\right)]^2}{\left[\mathrm{Ai}'^2(-\alpha_1)\alpha_0\mathrm{Ai}^2(-\alpha_0)\right](\alpha_1-\alpha_0)^2} \exp(-[\alpha_1-\alpha_0]\frac{r}{\xi_{||}}).$$

This is still quite complicated but if we restrict attention to $x=0$ we find

$$C(0,r) \approx \frac{1}{(\alpha_1-\alpha_0)^2\alpha_0} \exp(-[\alpha_1-\alpha_0]\frac{r}{\xi_{||}}). \tag{58}$$

However at $x=0$ we can actually do better using the boundary conditions that $\mathrm{Ai}(-\alpha_{2n+1}) = 0$ and $\mathrm{Ai}'(-\alpha_{2n}) = 0$ to give

$$C(0,r) = \sum_{n=0}^{\infty} \frac{1}{(\alpha_{2n+1}-\alpha_0)^2\alpha_0} \exp(-[\alpha_{2n+1}-\alpha_0]\frac{r}{\xi_{||}}). \tag{59}$$

We know that $C(0,0) = 1$ and so (unless there is a mistake in the calculations) we have the remarkable identity

$$\sum_{n=0}^{\infty} \frac{1}{(\alpha_{2n+1}-\alpha_0)^2\alpha_0} = 1. \tag{60}$$

You can prove the above in the following way, define

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{(z-\alpha_{2n+1})^2}, \tag{61}$$

where $-\alpha_{2n+1}$ are the zeros of the Airy function. We now write

$$f(z)=\int_C\frac{d\zeta}{2\pi i}\frac{1}{(z+\zeta)^2}\frac{\mathrm{Ai}'(\zeta)}{\mathrm{Ai}(\zeta)}, \tag{62}$$

we now deform the integration contour to a circle $c$ about $\zeta=-z$ to find

$$f(z)=-\int_c\frac{d\zeta}{2\pi i}\frac{1}{(z+\zeta)^2}\frac{\mathrm{Ai}'(\zeta)}{\mathrm{Ai}(\zeta)}, \tag{63}$$

as the contour goes in the opposite direction and we have taken $C$ to be anti-clockwise. Evaluating the residue at $\zeta=-z$ we find

$$f(z)=-\left[\frac{\mathrm{Ai}''(-z)}{\mathrm{Ai}(-z)}-\frac{\mathrm{Ai}'^2(-z)}{\mathrm{Ai}^2(-z)}\right] \tag{64}$$

Now use Airy's equation $\mathrm{Ai}''(z)-z\mathrm{Ai}(z)=0$ to find

$$f(z)=z+\frac{\mathrm{Ai}'^2(-z)}{\mathrm{Ai}^2(-z)}. \tag{65}$$

Now set $z=\alpha_0$ and use $\mathrm{Ai}'(-\alpha)=0$ to find

$$f(\alpha_0)=\alpha_0 \tag{66}$$

hence proving the result.

The above calculation has some strange consequences, we see that for all $x$ the correlation function for large $r$ decays as

$$A(\frac{x}{\xi_\perp})\exp(-[\alpha_1-\alpha_0]\frac{r}{\xi_{||}}), \tag{67}$$

where $A(x)$ is the $x$ dependent amplitude, so the correlation length at large $r$ is independent of the position $x$, now going into the bulk this predicts that $\xi_{||}=\xi_b$ where $\xi_b$ is the bulk correlation length (but in the presence of the magnetic field).

Other geometries. Now imagine an interface at a height $h$ above an impenetrable wall. We are interested in the case where the average height $\langle h\rangle=\overline{h}$ is fixed. Here the partition function obeys the Schrödinger equation

$$\frac{\partial Z}{\partial L}=\frac{1}{2\sigma\beta}\frac{\partial^2 Z}{\partial h^2}-B\beta hZ, \tag{68}$$

where $B$ now plays the role of a Lagrange multiplier to fix the average height. The boundary condition on the wave functions is now $\psi_\epsilon(0)=0$. Again we find

$$\psi_\epsilon(h)=\frac{1}{N}\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}(h-\frac{\epsilon}{\lambda})\right) \tag{69}$$

However we now have

$$2^{\frac{1}{3}}\lambda^{-\frac{2}{3}}\epsilon_n = \alpha_n \tag{70}$$

where $-\alpha_n$ are now the zeros of the Airy function $Ai(-\alpha_n) = 0$. The ground state energy is thus

$$\epsilon_0 = 2^{-\frac{1}{3}}\lambda^{\frac{2}{3}}\alpha_0. \tag{71}$$

The probability density of $z = (2\lambda)^{\frac{1}{3}}h$ is given by

$$p(z) = \frac{\mathrm{Ai}^2\left(z-\alpha_0)\right)}{\int_0^\infty dz \mathrm{Ai}^2\left(z-\alpha_0\right)}, \tag{72}$$

from this we find

$$\overline{h} = (2\lambda)^{-\frac{1}{3}}\frac{\int_0^\infty z\mathrm{Ai}^2\left(z-\alpha_0)\right)}{\int_0^\infty dz \mathrm{Ai}^2\left(z-\alpha_0\right)} = z_0(2\lambda)^{-\frac{1}{3}}, \tag{73}$$

where

$$z_0 = \frac{\int_0^\infty dz\ z\mathrm{Ai}^2\left(z-\alpha_0)\right)}{\int_0^\infty dz \mathrm{Ai}^2\left(z-\alpha_0\right)}. \tag{74}$$

We thus find

$$\lambda = \frac{1}{2}\frac{z_0^3}{\overline{h}_0^3} \tag{75}$$

The free energy per unit length of the interface is given by

$$f = \frac{\epsilon_0}{\beta^2\sigma} = \frac{2^{-\frac{1}{3}}\lambda^{\frac{2}{3}}\alpha_0}{\beta^2\sigma} \tag{76}$$

and so as a function of the average height

$$f = \frac{\alpha_0 z_0^2}{2\beta^2\sigma\overline{h}^2} \tag{77}$$

An alternative derivation, is

$$\begin{aligned} \overline{h} &= \frac{\partial}{\partial B}f = \frac{\partial\lambda}{\partial B}\frac{\partial}{\partial\lambda}f \\ &= \frac{2^{\frac{2}{3}}}{3}\alpha_0\lambda^{-\frac{1}{3}}, \end{aligned} \tag{78}$$

which yields

$$f = \frac{2\alpha_0^3}{9\beta^2\sigma\overline{h}^2}. \tag{79}$$

These two different calculations therefore give

$$z_0 = \frac{2}{3}\alpha_0, \tag{80}$$

which can be verified numerically. Now we consider the case where there is an upper hard surface at $h = L$, when the average height $\overline{h}$ is fixed. Here the wave functions take the form

$$\psi_\epsilon(h) = a\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}(h - \frac{\epsilon}{\lambda})\right) + b\mathrm{Bi}\left((2\lambda)^{\frac{1}{3}}(h - \frac{\epsilon}{\lambda})\right), \tag{81}$$

the boundary conditions $\psi_\epsilon(0) = \psi_\epsilon(L) = 0$ give the eigenvalue equation for $\epsilon$

$$\mathrm{Ai}\left(-(2\lambda)^{\frac{1}{3}}\frac{\epsilon}{\lambda})\right)\mathrm{Bi}\left((2\lambda)^{\frac{1}{3}}(L - \frac{\epsilon}{\lambda})\right) - \mathrm{Bi}\left(-(2\lambda)^{\frac{1}{3}}\frac{\epsilon}{\lambda})\right)\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}(L - \frac{\epsilon}{\lambda})\right) = 0 \tag{82}$$

We now write

$$\epsilon = \lambda L u \tag{83}$$

to obtain

$$\mathrm{Ai}\left(-(2\lambda)^{\frac{1}{3}}Lu)\right)\mathrm{Bi}\left((2\lambda)^{\frac{1}{3}}L(1-u)\right) - \mathrm{Bi}\left(-(2\lambda)^{\frac{1}{3}}Lu\right)\mathrm{Ai}\left((2\lambda)^{\frac{1}{3}}L(1-u)\right) = 0 \tag{84}$$

From this we have that the smallest solution for $u$ is such that

$$u_0 = u_0((2\lambda)^{\frac{1}{3}}L) \tag{85}$$

The free energy is given by

$$f = \frac{\lambda L u_0((2\lambda)^{\frac{1}{3}}L)}{\beta^2\sigma} \tag{86}$$

1.1. **Numerically simulating the interface problem.** The discrete Hamiltonian for the interface problem on a lattice with lattice size $a$ is

$$H = \frac{\sigma}{2}\sum_i \frac{(h_i - h_{i+1})^2}{a} - B\sum_i |h_i| a \tag{87}$$

In the limit $a \to 0$ we recover the continuum model. A stochastic dynamics leading to the correct equilibrium distribution is

$$\dot{h}_i = -\frac{\partial H}{\partial h_i} + \eta_i, \tag{88}$$

where $\eta_i(t)\eta_j(t) = 2T\delta_{ij}\delta(t - t')$. The temporally discrete version of this is

$$h_i(t + \Delta t) = h_i(t) - \frac{\partial H}{\partial h_i}\Delta t + \sqrt{2T\Delta t}\sigma_i, \tag{89}$$

where the $\sigma_i$ are independent zero mean Gaussian random variables of variance 1. The discrete time equation is thus, where we have set $a = 1$,

$$h_i(t + \Delta t) = h_i(t) + (\sigma[h_{i+1} - 2h_i + h_{i-1}] - B\mathrm{sign}(|h_i|))\Delta t + \sqrt{2T\Delta t}\sigma_i. \tag{90}$$

The idea would be to simulate this equation and evaluate the height distribution etc by making equilibrium measurements.

If the magnetisation is fixed then the equation of motion is

$$\dot{h}_i = -\frac{\partial H}{\partial h_i} - \mu + \eta_i, \tag{91}$$

where $\mu$ is a dynamical Lagrange multiplier enforcing the constraint $\sum_i h_i = 0$. Summing the above over $i$ shows

$$\mu = \frac{1}{L}\sum_i -\frac{\partial H}{\partial h_i} + \eta_i = \frac{1}{L}\sum_i B\mathrm{sign}(|h_i|)) + \eta_i \tag{92}$$

In the presence of a drift or flow $v(y)$in the direction $x$ the model A dynamics becomes

$$\frac{\partial h(x,t)}{\partial t} + v(h(x,t))\frac{\partial h(x,t)}{\partial x} = -\frac{\delta H}{\delta h(x,t)} + \eta(x,t). \tag{93}$$

This gives the discrete equation
(94)

$$h_i(t+\Delta t) = h_i(t) + (\sigma[h_{i+1} - 2h_i + h_{i-1}] - B\mathrm{sign}(|h_i|) - \frac{v(h_i)}{2}[h_{i+1} - h_{i-1}])\Delta t + \sqrt{2T\Delta t}\sigma_i.$$

Here we take $v(h) = \alpha\mathrm{sign}(h)$.

You can check the Airy scaling for the basic case and verify what happens when you add the flow field. You can also do this for a capillary wave model where

$$H = \frac{\sigma}{2}\sum_i \frac{(h_i - h_{i+1})^2}{a} - \lambda\sum_i h_i^2 a, \tag{95}$$

take $a = 1$

## 2. Finite size effects for the solid on solid model with no external field

Consider the vector denoted by $[a]$ which has components

$$[a]_i = a^i, \tag{96}$$

where $i$ is an index ranging from 0 to $H$. There are thus $H + 1$ sites in this SOS model. The action of the SOS transfer matrix on this vector is given by

$$[T\ [a]]_i = \sum_{j=0}^{H} \exp(-\beta|i-j|)a^j \tag{97}$$

and we find

$$
\begin{aligned}
[T\ [a]]_i &= \exp(-\beta i)\sum_{j=0}^{i}\exp(\beta j)a^j + \exp(\beta i)\sum_{j=i+1}^{H}\exp(-\beta j)a^j \\
&= \exp(-\beta i)\sum_{j=0}^{i}\exp(\beta j)a^j + \exp(\beta i)\sum_{k=0}^{H-i-1}\exp(-\beta[i+1+k])a^{i+1+k} \\
&= \exp(-\beta i)\frac{1-\exp(\beta(i+1))a^{i+1}}{1-\exp(\beta)a} + \exp(-\beta)a^{i+1}\frac{1-\exp(-\beta(H-i))a^{H-i}}{1-\exp(-\beta)a} \\
&= \left[\frac{\exp(-\beta)a}{1-\exp(-\beta)a} - \frac{\exp(\beta)a}{1-\exp(\beta)a}\right]a^i + \frac{\exp(-\beta i)}{1-\exp(\beta)a} - \frac{\exp(-\beta(H+1))a^{H+1}\exp(\beta i)}{1-\exp(-\beta)a} \qquad (98)\\
&= \left[\frac{ra}{1-ra} - \frac{\frac{a}{r}}{1-\frac{a}{r}}\right]a^i + \frac{r^i}{1-\frac{a}{r}} - \frac{r^{H+1}a^{H+1}\frac{1}{r^i}}{1-ra} \qquad (99)
\end{aligned}
$$

where we have introduced

$$r = \exp(-\beta). \tag{100}$$

We now define

$$\lambda(a) = \frac{ra}{1-ra} - \frac{\frac{a}{r}}{1-\frac{a}{r}} \tag{101}$$

and notice that

$$\lambda(a) = \lambda(a^{-1}) \tag{102}$$

so we can write

$$[T\ [a]]_i = \lambda(a)a^i + \frac{r^i}{1-\frac{a}{r}} - \frac{r^{H+1}a^{H+1}\frac{1}{r^i}}{1-ra}. \tag{103}$$

Now consider the action of the transfer matrix on the vector $[a^{-1}]$, we find

$$\left[T\ [a^{-1}]\right]_i = \lambda(a)a^{-i} + \frac{r^i}{1-\frac{1}{ra}} - \frac{r^{H+1}a^{-(H+1)}\frac{1}{r^i}}{1-\frac{r}{a}}. \tag{104}$$

We now look for an eigenvector of the form

$$\mathbf{v} = [a] + c[a^{-1}]. \tag{105}$$

The action of $T$ on $\mathbf{v}$ is this

(106)

$$\left[T\ ([a]+c[a^{-1}]]\right]_i = \lambda(a)[a^i + ca^{-i}] + r^i\left(\frac{1}{1-\frac{a}{r}} + \frac{c}{1-\frac{1}{ra}}\right) - \frac{r^{H+1}}{r^i}\left(\frac{a^{H+1}}{1-ra} + c\frac{a^{-(H+1)}}{1-\frac{r}{a}}\right).$$

and we see that $\mathbf{v}$ is an eigenvector, with eigenvalue $\lambda(a)$, if

$$\frac{1}{1-\frac{a}{r}}+\frac{c}{1-\frac{1}{ra}} = 0 \tag{107}$$

$$\frac{a^{H+1}}{1-ra}+c\frac{a^{-(H+1)}}{1-\frac{r}{a}} = 0. \tag{108}$$

The above equations imply that

$$c=-\frac{ra-1}{a(r-a)} \tag{109}$$

and.

$$c^2=a^{2H}, \tag{110}$$

Therefore we find

$$v_i=a^i\pm a^{H-i}, \tag{111}$$

however we expect the ground state eigenvector (corresponding to the largest eigenvalue) to be symmetric about the middle of the system and so

$$v_i=v_{H-i}, \tag{112}$$

which implies that we should have $c=a^H$. This then gives

$$a^{H+1}=\frac{1-ra}{r-a}. \tag{113}$$

As a check on the above derivation we can consider the case $H=1$ so we have two sites. Here we see that the transfer matrix is given explicitly by

$$T=\begin{pmatrix} 1 & r \\ r & 1 \end{pmatrix} \tag{114}$$

and the largest eigenvector is easily seen to be given by

$$\lambda_0=1+r \tag{115}$$

Now for $H=1$ we find that

$$a^2=\frac{1-ra}{r-a} \tag{116}$$

has three solutions

$$a_1 = -1 \tag{117}$$

$$a_2 = \frac{1}{2}\left(-\sqrt{r^2+2r-3}+r+1\right) \tag{118}$$

$$a_3 = \frac{1}{2}\left(\sqrt{r^2+2r-3}+r+1\right) \tag{119}$$

We see that $a_2=1/a_3$, and $|a_2|=|a_3|=1$, also

$$\lambda(-1)=\frac{1-r}{1+r} \tag{120}$$

while

$$\lambda(a_2) = \lambda(a_3) = 1 + r \tag{121}$$

corresponds to the maximal eigenvalue. The equation determining $a$ can also be written as

$$a^H = -\frac{r - \frac{1}{a}}{r - a}, \tag{122}$$

we see that if we write $a = \exp(i\theta)$ then

$$\exp(iH\theta) = \exp(2i\psi - i\pi) \tag{123}$$

where $\tan(\psi) = \sin(\theta)/(r - \cos(\theta))$ and so

$$\tan(H\theta) = -2\frac{\frac{\sin(\theta)}{(r-\cos(\theta))}}{(1 - \frac{\sin^2(\theta)}{(r-\cos(\theta))^2})} = \tag{124}$$