Chain Rule

Given $f(z) = \log_e(1+z)$ where $z = x^T x$, $x \in \mathbb{R}^d$

Sol^n:

If $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Then, $x^T = [x_1 \; x_2 \; \cdots \; x_d]$

$x^T x = [x_1^2 + x_2^2 + \cdots \; x_d^2]$

Applying chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{\partial}{\partial z}(\log(1+z)) \cdot \frac{\partial}{\partial x}(x^T \cdot x)$$

$$= \frac{1}{1+z} \cdot \frac{\partial}{\partial z}(z) \cdot \frac{\partial}{\partial x}(x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \cdots \; 2x_d)$$

$$= \frac{1}{1+Z} \cdot 2(x_1 + x_2 + \dots x_d)$$

$$= \frac{2}{1+Z} \sum_{i=1}^{d} x_i$$

(Ans)

\# $f(z) = e^{-z/2}$, where $z = g(z)$, $g(y) = y^T S^{-1} y$,

$y = h(x)$, $h(x) = x - \mu$

Sol^n

Applying chain rule

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

Now, $\frac{\partial f}{\partial x} = \frac{\partial}{\partial z}\left(e^{-z/2}\right)$

$$= -\frac{e^{-z/2}}{2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(y^T S^{-1} y\right)$$

$$= \lim_{h \to 0} \frac{(y^T + h)\, S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{(y^T S^{-1} + h S^{-1})(y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{h(y^T S^{-1} + S^{-1} y + h S^{-1})}{h}$$

$$= \lim_{h \to 0} \left(y^T S^{-1} + S^{-1} y + h S^{-1}\right)$$

$$= y^T S^{-1} + S^{-1} y$$

$$\frac{\partial y}{\partial x} = \frac{\partial (x - \mu)}{\partial x} = 1$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= -\frac{e^{-z/2}}{2} \left(y^T S^{-1} + S^{-1} y \right) \cdot 1$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{1} \left(y^T + y \right)$$

(Ans)