Efficient Preconditioners for PDE-Constrained Optimization Problems with Multilevel Sequentially Semiseparable Structure

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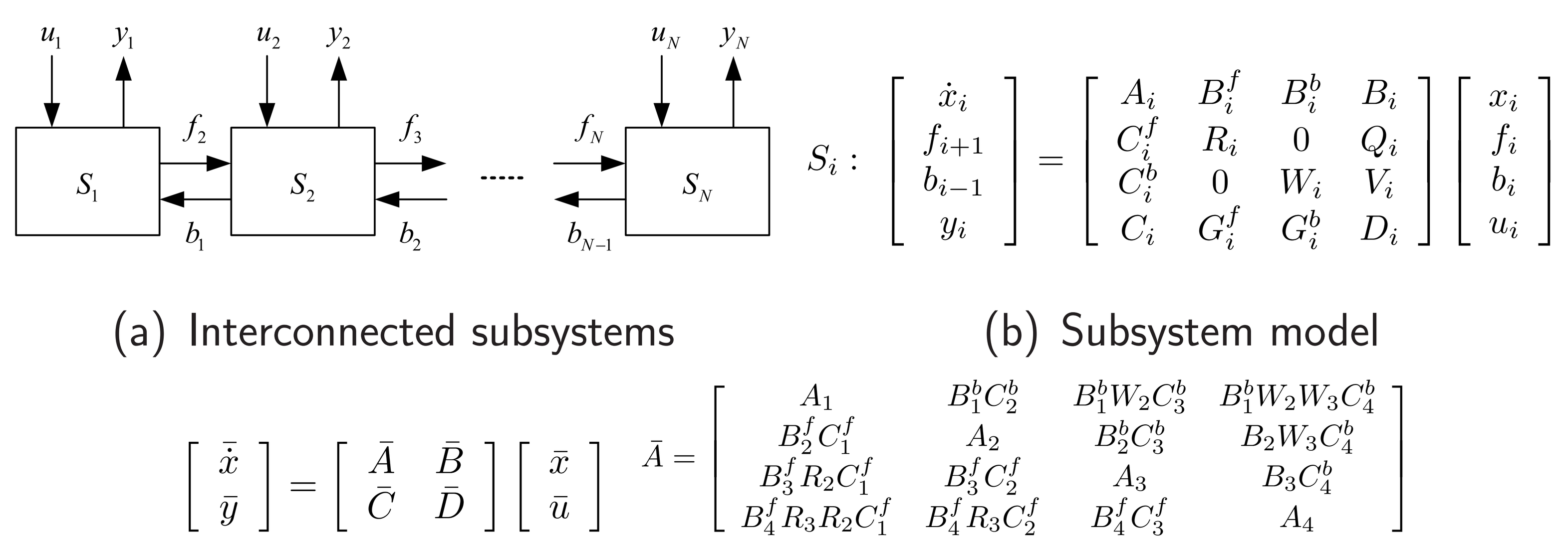
Introduction

PDE-constrained optimization problems arise in applications such as optimal flow control, diffuse optical tomography, linear (nonlinear) model predictive control. The solution of this problem is obtained by solving a large-scale linear saddle point type system. Our research focus on the multilevel sequentially semiseparable structure of the sub-matrices of the saddle-point system. By exploiting the structure, we get a class of efficient preconditioners of linear complexity.

Multilevel Sequentially Semiseparable Matrices

From a input-output system point of view, a discretized 1D PDE can be considered as a string of interconnected systems [1] that is shown in figure 1. The interconnected system induce a special structure of the system matrix, which is called sequentially semiseparable (SSS)[2].



(a) Interconnected subsystems (b) Subsystem model

(c) Global system model (d) SSS matrix when $N = 4$

Figure 1: 1D interconnected model and global SSS matrices model

SSS matrices have low rank off-diagonal blocks which enables many operations such as matrix-matrix product, matrix-vector product and matrix inversion to be performed in linear complexity [2]. If we lump the subsystems from the discretized 2D PDEs into one dimension, we get a system that has a multilevel (2-level) sequentially semiseparable (MSSS) structure.

MSSS Preconditioning Technique

Consider the problem of the optimal control of the convection-diffusion equation on $\Omega = [0,\ 1] \times [0,\ 1]$.

$$\min_{u,\ f} \frac{1}{2}\|u - \hat{u}\|^2 + \beta\|f\|^2$$

$$s.t. \ -\epsilon\nabla^2 u + \overrightarrow{\omega}.\nabla u = f \text{ in } \Omega$$

$$u = u_D \text{ on } \partial\Omega_D,\ \frac{\partial u}{\partial n} = u_N \text{ on } \partial\Omega_N$$

This yields the linear saddle-point system

$$\begin{bmatrix} 2\beta M & 0 & -M \\ 0 & M & K^T \\ -M & K & 0 \end{bmatrix} \begin{bmatrix} \mathbf{f} \\ \mathbf{u} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \\ \mathbf{d} \end{bmatrix}.$$

The preconditioner is chosen as the block-diagonal preconditioner $P = \begin{bmatrix} 2\beta M & & \\ & M & \\ & & -S \end{bmatrix}$ where the Schur complement $S = -\left(\frac{1}{2\beta}M + KM^{-1}K^T\right)$ is approximated by MSSS matrix computations. To keep the low-rank of the off-diagonal blocks to do fast computation, the model reduction algorithm described in [3] is used. Numerical experiment result shown in figure 2 illustrates the efficiency of our method. Both the construction of the preconditioner and iterative solution method are of linear complexity.

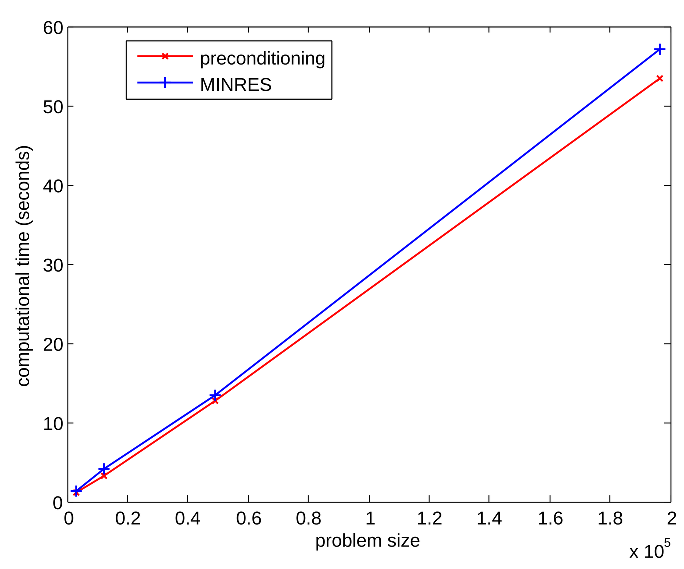


Figure 2: Computational results for optimal control of the convection-diffusion equation

References

[1] J.K. Rice. *Efficient Algorithms for Distributed Control: a Structured Matrix Approach*. PhD thesis, Delft University of Technology, 2010.

[2] S. Chandrasekaran, P. Dewilde, M. Gu, T. Pals, X. Sun, A.J. van der Veen, and D. White. Some fast algorithms for sequentially semiseparable representations. *SIAM Journal on Matrix Analysis and Applications*, 27(2):341–364, 2005.

[3] Y. Qiu, M.B. van Gijzen, J.W. van Wingerden, and M. Verhaegen. Efficient preconditioners for PDE-constrained optimization problems with a multi-level sequentially semiseparable matrix structure. Technical Report 13-04, Delft Institution of Applied Mathematics, Delft University of Technology, 2013.