On the Application of a Novel Model Order Reduction Algorithm for Sequentially Semi-Separable Matrices to the Identification of One-dimensional Distributed Systems

Yue Qiu*, Martin B. van Gijzen**, Jan-Willem van Wingerden* and Michel Verhaegen*

Abstract— **The sequentially semiseparable (SSS) matrix approach provides an efficient framework for solving control and identification problems of one-dimensional (1-D) spatially interconnected systems. For this approach, model order reduction algorithm is essential for obtaining a low computational complexity. In this paper, we apply a novel model reduction algorithm to identify 1-D spatially interconnected system by the extended Kalman filter (EKF). The new algorithm is based on the model reduction of linear time-varying (LTV) systems by low-rank approximation of the controllability and observability gramians. To show the efficiency and advantage of the new algorithm, the conventional algorithm is also studied. Numerical experiments illustrate that, by exploiting the structure of the EKF, both model reduction algorithms lead to linear computational complexity for identification of 1-D spatially interconnected system, where standard EKF has cubic computational complexity. Compared with the conventional model order algorithm, the new one gives a significant reduction on computational time.**

I. INTRODUCTION

Control of spatially interconnected distributed systems arises in different areas such as mobile agents [1], multi-robot formation [2], control of partial differential equations (PDEs) [3] [4], [5], distributed power control [6]. The biggest challenge for control and identification of this kind of systems is the computational cost for designing and implementing the corresponding algorithms. The size of the system state-space model of N interconnected systems, each of order n, is $nN \times nN$. Many conventional matrix algorithms require $\mathcal{O}(n^3N^3)$ floating point operations (flops). Consequently, using such algorithms make control analysis, synthesis and identification prohibitively expensive if the number of interconnected systems is too large.

During the last decades, many efforts have been dedicated to overcome this problem. In [7], optimal control of spatially invariant systems was studied. By exploiting the spatial invariance of the system, the complexity of the optimal controller design was reduced, the controller was spatially distributed and localized. In [8], a structured linear matrix inequality (LMI) approach was proposed for distributed control of spatially interconnected systems. This led to a structure for the distributed implementation of controllers for which the computational complexity is $\mathcal{O}(n^{2\alpha}N^{\alpha})$ where $2.5 < \alpha < 3.5$. Rice et. al. used the so called sequentially semi-separable (SSS) matrix approach [9], [10], for the distributed control of spatially interconnected heterogeneous systems. This structured matrix approach reduced the computational complexity to $\mathcal{O}(n^3N)$. SSS matrix operations, such as addition, multiplication and inversion, have the property that the resulting matrices have again an SSS structure. Moreover, most of the SSS operations can be done with linear, i.e. $\mathcal{O}(m)$ computational complexity, where m is the size of the matrix. This makes the control and identification of large-scale system computationally feasible.

The first extension of the SSS matrix approach for system identification was proposed by Rice and Verhaegen [11]. Their idea was to exploit the SSS matrix structure of the extended Kalman filter (EKF) for interconnected systems to reduce the computational complexity to $\mathcal{O}(n^3N)$. In succession, van Wingerden and Torres proposed an algorithm for spatially interconnected systems based on optimizing the output-error using the steepest decent method [12]. In their approach the SSS matrix structure of the global system matrix is exploited. As explained in [13], the operations for addition and multiplication for SSS matrices require a model order reduction step to achieve linear computational complexity. The model order reduction algorithm that was proposed in [13] is based on the approximation of the Hankel blocks of the SSS matrices.

In this paper, we study the application of a new model order reduction algorithm for SSS matrices that was proposed in [14] to the identification of 1D distributed systems. The idea of this new model reduction algorithm is based on the model reduction of linear time-varying (LTV) systems by low-rank approximation of the controllability and observability gramians. To compare the performance of the newly proposed model reduction algorithm for SSS matrices, the conventional model reduction algorithm in [13] is also discussed in this paper .

The outline of this paper is as follows. Section 2 gives the preliminaries on SSS matrices, and the problem formulation for identification of spatially varying interconnected systems. Section 3 describes and analyzes the model order reduction algorithms. Section 4 gives the numerical experiments. Conclusions and future work are discussed in the last section.

II. PRELIMINARIES

A. Sequentially Semiseparable Matrices

The interconnected subsystems model considered in this paper is shown in Fig. 1. The subsystem S_i where $i =$

*Delft Center for System and Control, Delft University of Technology, 2628 CD Delft, The Netherlands, Y.Qiu@tudelft.nl

**Delft Institute of Applied Mathematics, Delft University of Technology, 2628 CD Delft, The Netherlands, M.B.vanGijzen@tudelft.nl

$1,2,\cdots N$ shown in Fig. 1 is described by a linear system model of the form:

$$S_i: \quad \begin{bmatrix} \dot{x}_i \\ f_{i+1} \\ b_{i-1} \\ y_i \end{bmatrix} = \begin{bmatrix} A_i & B_i^f & B_i^b & B_i \\ C_i^f & R_i & 0 & Q_i \\ C_i^b & 0 & W_i & V_i \\ C_i & G_i^f & G_i^b & D_i \end{bmatrix} \begin{bmatrix} x_i \\ f_i \\ b_i \\ u_i \end{bmatrix} \tag{1}$$

where $f_i \in \mathbb{R}^{n_i^l}$ and $b_i \in \mathbb{R}^{n_i^u}$ are interconnection variables that represent forward and backward information as illustrated in Fig. 1, respectively. The vectors $x_i \in \mathbb{R}^{n_i}$, $u_i \in \mathbb{R}^{p_i}$ and $y_i \in \mathbb{R}^{q_i}$ are the state, input and output of subsystem S_i. The matrices R_i and W_i terms present the information interchange between subsystems S_{i-1} and S_i, respectively. Note that each subsystem S_i can be different from each other, even they can be of arbitrarily size of states, input and output, as long as the interconnections variables f_i and b_i are of suitable size for interconnection.

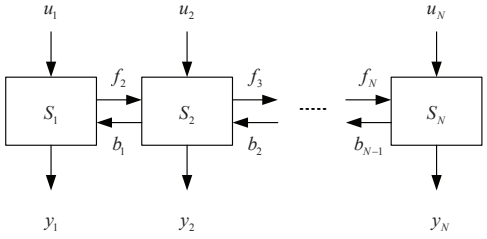

Fig. 1. 1-D spatially interconnected systems

Assume that the N subsystems in Fig. 1 are connected with zero boundary input, i.e., $f_1 = 0$ and $b_N = 0$. Then by eliminating the interconnection variables, the global model of the interconnected subsystems becomes:

$$\begin{bmatrix} \bar{\dot{x}} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{bmatrix} \begin{bmatrix} \bar{x} \\ \bar{u} \end{bmatrix} \tag{2}$$

Here $\bar{x} = \begin{bmatrix} x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}^T \in \mathbb{R}^{N_x}$ with $N_x = \sum_{i=1}^N n_i$, $\bar{u} = \begin{bmatrix} u_1^T & u_2^T & \cdots & u_N^T \end{bmatrix}^T \in \mathbb{R}^{N_u}$ with $N_u = \sum_{i=1}^N p_i$ and $\bar{y} = \begin{bmatrix} y_1^T & y_2^T & \cdots & y_N^T \end{bmatrix}^T \in \mathbb{R}^{N_y}$ with $N_y = \sum_{i=1}^N q_i$ are called lifted system states, input and output, respectively in [15]. The system matrices $\left(\bar{A}, \bar{B}, \bar{C}, \bar{D}\right)$ have a very specific structure, called "sequentially semiseparable" (SSS) [13]. Take $N = 4$ for example, the system matrices in (2) are shown in (3) - (6).

$$\bar{A} = \begin{bmatrix} A_1 & B_1^b C_2^b & B_1^b W_2 C_3^b & B_1^b W_2 W_3 C_4^b \\ B_2^f C_1^f & A_2 & B_2^b C_3^b & B_2 W_3 C_4^b \\ B_3^f R_2 C_1^f & B_3^f C_2^f & A_3 & B_3 C_4^b \\ B_4^f R_3 R_2 C_1^f & B_4^f R_3 C_2^f & B_4^f C_3^f & A_4 \end{bmatrix} \tag{3}$$

$$\bar{B} = \begin{bmatrix} B_1 & B_1^b V_2 & B_1^b W_2 V_3 & B_1^b W_2 W_3 V_4 \\ B_2^f Q_1 & B_2 & B_2^b V_3 & B_2^b W_3 V_4 \\ B_3^f R_2 Q_1 & B_3 Q_2 & B_3 & B_3^b V_4 \\ B_4 R_3 R_2 Q_1 & B_4^f R_3 Q_2 & B_4^f Q_3 & B_4 \end{bmatrix} \tag{4}$$

$$\bar{C} = \begin{bmatrix} C_1 & G_1^b C_2^b & G_1^b W_2 C_3^b & G_1^b W_2 W_3 C_4^b \\ G_2^f C_1^f & C_2 & G_2^b C_3^b & G_2^b W_3 C_4^b \\ G_3^f R_2 C_1^f & G_3^f C_2^f & C_3 & G_3^b C_4^b \\ G_4^f R_3 R_2 C_1^f & G_4^f R_3 C_2^f & G_4^f C_3^f & C_4 \end{bmatrix} \tag{5}$$

$$\bar{D} = \begin{bmatrix} D_1 & G_1^b V_2 & G_1^b W_2 V_3 & G_1^b W_2 W_3 V_4 \\ G_2^f Q_1 & D_2 & G_2^b V_3 & G_2^b W_3 V_4 \\ G_3^f R_2 Q_1 & G_3^f V_2 & D_3 & G_3^b V_4 \\ G_4^f R_3 R_2 Q_1 & G_4^f R_3 Q_2 & G_4^f Q_3 & D_4 \end{bmatrix} \tag{6}$$

We can also use the generators parameterized representation given in [13], [16] and [17] to denote $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{D}$, which yields:

$$\begin{aligned} \bar{A} &= SSS(B_s^f, R_s, C_s^f, A_s, B_s^b, W_s, C_s^b) \\ \bar{B} &= SSS(B_s^f, R_s, Q_s, B_s, B_s^b, W_s, V_s) \\ \bar{C} &= SSS(G_s^f, R_s, C_s^f, C_s, G_s^b, W_s, C_s^b) \\ \bar{D} &= SSS(G_s^f, R_s, Q_s, D_s, G_s^b, W_s, V_s) \end{aligned}$$

Here the arguments in the brackets of $SSS(\ )$ are called the generators of an SSS matrix. By this representation, the structure of SSS matrices can be exploited to do fast computation for matrix addition, multiplication and inversion by operations on its generators, with the cost of linear computational complexity [13] [16]. Efficient algorithms for solving more complicated problems, such as computing the QR factorization and solving Lyapunov and Ricatti equations, have been proposed in [9] [18] [19].

B. Linear Time-varying Systems Realization

The SSS matrices have a realization of the 1-D spatially distributed system that is shown by Fig. 1 in part A of this section. Note that SSS matrices can also be interpreted as linear time-varying (LTV) systems. This is studied by Dewilde et al. in [20]. Consider a mixed-causal system, which is described by the state-space model as

$$\begin{aligned} \begin{bmatrix} x_{i+1}^c \\ x_{i-1}^a \end{bmatrix} &= \begin{bmatrix} R_i & \\ & W_i \end{bmatrix} \begin{bmatrix} x_i^c \\ x_i^a \end{bmatrix} + \begin{bmatrix} C_i^b \\ C_i^f \end{bmatrix} u_i \\ y_i &= \begin{bmatrix} B_i^f & B_i^b \end{bmatrix} \begin{bmatrix} x_i^c \\ x_i^a \end{bmatrix} + A_i u_i \end{aligned} \tag{7}$$

where x^c denotes the causal system states while x^a represents the anti-causal system states. With zero initial state, the matrix $\mathscr{H}$ that describes the input-output behavior of this mix-causal system, i.e., $\overline{y} = \mathscr{H}\overline{u}$, induces an SSS matrix structure. Take, $N = 4$ for example, the matrix $\mathscr{H}$ is equal to $\overline{A}$ shown in (3).

III. COMPUTATIONAL METHODS

To keep this paper self-contained, we review some definitions and concepts, see also [14]. For computational complexity, the notation $\mathscr{O}$ was used as in the beginning of this paper.

A. Model Reduction Algorithm for SSS Matrices

As shown in [17] [13], the addition, multiplication of two SSS matrices will yield an SSS matrix with a bigger semiseparable order. This induces the growth of the computational complexity. For the semiseparable, we refer to [17].

The aim of model reduction of an SSS matrix $\overline{A} = \mathscr{SSS}(B_s^f,\ R_s,\ C_s^f,\ A_s,\ B_s^b,\ W_s,\ C_s^b)$ is to find matrices $\hat{B}_s^f,\ \hat{R}_s,\ \hat{C}_s^f,\ \hat{B}_s^b,\ \hat{W}_s,\ \hat{C}_s^b$ of much smaller size compared to the generators $B_s^f,\ R_s,\ C_s^f,\ B_s^b,\ W_s,\ C_s^b$ such that $\hat{A} = \mathscr{SSS}(\hat{B}_s^f,\ \hat{R}_s,\ \hat{C}_s^f,\ A_s,\ \hat{B}_s^b,\hat{W}_s,\hat{C}_s^b)$ is with semiseparable order smaller than or equal to the minimal semiseparable order of $\overline{A}$ such that $\hat{A}$ is equivalent with A up to a tolerance ε, i.e., $\|\overline{A}-\hat{A}\| < \varepsilon$.

Since the strictly lower-triangular part and the strictly upper-triangular part have similar structures, here only the strictly lower-triangular part is analyzed. To reduce the semiseparable order of the strictly lower-triangular part of the SSS matrices $\overline{A}$ in (3), the order of B_i^f, C_i^f and R_i should be reduced. This corresponds to the model reduction of the corresponding linear causal time-varying system described in (7).

Model reduction for LTV systems has been studied in [21] [22]. In [22], the linear matrix inequality (LMI) was introduced to solve the Lyapunov inequalities for controllability and observability gramians. In [21], the low-rank Smith method was presented to approximate the square-root of the controllability and observability gramians for LTV systems. Consider the linear causal time-varying system described by the following state-space model

$$\begin{cases} x_{k+1} = A_k x_k + B_k u_k \\ y_k = C_k x_k \end{cases}$$

over time interval $[k_o, k_f]$ where the matrices and vectors are of compatible sizes for matrix-vector operations. The controllability gramian $\mathscr{G}_c(k)$ and observability gramian $\mathscr{G}_o(k)$ can be obtained from the following Stein recurrence formulas:

$$\mathscr{G}_c(k+1) = A_k \mathscr{G}_c(k) A_k^T + B_k B_k^T \quad (8)$$

$$\mathscr{G}_o(k) = A_k^T \mathscr{G}_o(k+1) A_k + C_k^T C_k \quad (9)$$

with initial conditions $\mathscr{G}_c(k_o) = 0$ and $\mathscr{G}_o(k_f+1) = 0$.

Note that the controllability gramian $\mathscr{G}_c(k)$ and observability gramian $\mathscr{G}_o(k)$ are positive semi-definite matrices, thus their eigenvalues are non-negative. The eigenvalues often have an early cut-off as pointed in [23] [24], which suggests to approximate both gramians at each step by a low-rank factorization. Since the controllability and observability gramians have similar recursions, we will only focus on the controllability gramian.

The key point of the low-rank approximation is to substitute the Cholesky factorization of the controllability gramian

$$\mathscr{G}_c(k) = L_k L_k^T$$

where $\mathscr{G}_c(k),\ L_k \in \mathbb{R}^{N\times N}$ in each iteration by its approximate Cholesky factorization,

$$\mathscr{G}_c(k) \approx \tilde{L}_k \tilde{L}_k^T$$

with rank n_k factor $\tilde{L}_k \in \mathbb{R}^{N\times n_k}$ because of the low numerical rank of $\mathscr{G}_c(k)$ where $N > n_k$ at each step. Typically, n_k is kept constant, i.e., $n_k = n$ at each step. This factorization is always feasible because of the positive semi-definiteness of $\mathscr{G}_c(k)$.

In [25], approximate balanced truncation algorithm for model reduction of large-scale linear time-invariant system was introduced. In this paper, we extend that algorithm to linear time-varying system to get a reduced order linear time-varying system, which corresponds to an SSS matrix with reduced order of its generators in the strictly lower-triangular part.

Based on the approximate balanced truncation, the reduced LTV system is given by

$$\begin{cases} \hat{x}_{k+1} = \Pi_l(k+1) A_k \Pi_r(k) \hat{x}_k + \Pi_l(k+1) B_k u_k \\ y_k = C_k \Pi_r(k) \hat{x}_k. \end{cases}$$

where the left and right projectors $\Pi_l(k) \in \mathbb{R}^{n\times N}$ and $\Pi_r(k) \in \mathbb{R}^{N\times n}$ that satisfy $\Pi_l(k)\Pi_r(k) = I_n$. Here N is the size of system states before truncation and n is the size of the truncated states, I_n is the identity matrix with size $n \times n$. For the sake of space limitation, we refer to [14] for the details of this model reduction algorithm.

B. Conventional Model Reduction Algorithm

The conventional model reduction algorithm for SSS matrices in [13] [26] is based on approximation of the Hankel blocks of SSS matrices, where the Hankel blocks of an SSS matrix A are defined by definition 1.

Definition 1: [13] Hankel blocks denote the off-diagonal blocks that extend from the diagonal to the northeast corner (for the upper case) or to the southwest corner (for the lower case).

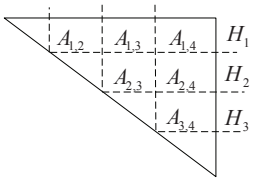

Fig. 2. Hankel blocks of an SSS matrix A

Take a 4×4 SSS matrix for example, the Hankel blocks for the upper-triangular part is shown in Fig. 2. The model reduction algorithm in [13] [26] is a *Hankel norm* optimal order reduction. That is, given an SSS matrix A, we can get an approximate matrix $\hat{A}$ with lower SSS order that achieves

$$\inf\ \|A-\hat{A}\|_H \quad (10)$$

where $\|A\|_H = \max_i \|H_i(A)\|_2$ and $H_i(A)$ are the Hankel blocks defined by definition 1. Due to the page limitation, the conventional model reduction algorithm will not be described further in this paper. For the details of the description of the conventional model reduction algorithm, we refer to [13] and [17].

Remark 1: The upper bound for model reduction of LTV systems based on balanced truncation is illustrated in [22]

by the induced L_2-norm while the model reduction for SSS matrices in [13] returns the optimal *Hankel norm* approximation. Since the *Hankel norm* is always smaller than or equal to the induced L_2-norm [22] for the same system and the approximate balanced truncation gives a system close to the optimal induced L_2-norm approximation, the upper bound for the approximated balanced truncation is bigger than the Hankel blocks approximation. This will be illustrated by a numerical experiment in section IV.

C. Flops Count

As analyzed in [14], the flops count for the new model reduction algorithm $\mathscr{F}_N$ is

$$\mathscr{F}_N = \mathscr{O}(M^2N), \tag{11}$$

and the flops count for the conventional model reduction algorithm $\mathscr{F}_c$ is

$$\mathscr{F}_c = \mathscr{O}(M^3N), \tag{12}$$

where M is the semiseparable order before model reduction and N is the number of blocks of SSS matrices.

From the flops count (12) and (11), we can see that both algorithms have linear computational complexity and the new model reduction algorithm is computationally cheaper than the conventional model reduction algorithm, which will be illustrated by the numerical experiments.

IV. NUMERICAL EXPERIMENTS

In this part, we use the the numerical example shown in [11] to illustrate the efficiency of the new model reduction algorithm in the application of identification of 1-D spatially interconnected system by extended Kalman filter with SSS structure.

The heat conduction equation [11] is expressed by the following partial differential equation(PDE),

$$\dot{T}(x,t) = k(x)\frac{\partial^2}{\partial x^2}(T(x,t)) - \frac{1}{2}T(x,t) \tag{13}$$

with boundary condition

$$T = 0 \text{ on boundary } x = 0 \text{ and } 1 \tag{14}$$

By doing spatial discretization, time discretization and adding the input and output to the discretized system, the PDE (13) system is transformed to a linear system that is described by

$$\begin{cases} T_{k+1} = A(\theta)T_k + B(\theta)u_k \\ y_k = C(\theta)T_k \end{cases}$$

where system matrices $A(\theta)$, $B(\theta)$ and $C(\theta)$ all have the SSS matrix structure, and θ is some known vector of parameters that is related to the heat conduction coefficients $k(x)$ and the input/output weights. By adding the process noise sequence $\{\omega_k\}$ and output noise sequence $\{v_k\}$, we make it a 'real' system described by

$$\begin{cases} T_{k+1} = A(\theta)T_k + B(\theta)u_k + \omega_k \\ y_k = C(\theta)T_k + v_k \end{cases} \tag{15}$$

where $E(\omega_k\omega_j^T) = Q\delta_{kj}$, $E(v_kv_j^T) = R\delta_{jk}$, $E(\omega_kv_j^T) = S\delta_{kj}$, $E(x_0) = 0$, $E(x_0x_0^T) = \Pi_x$ and $E(\theta\theta^T) = \Pi_\theta$.

The EKF for identifying the system (15) is described by

$$\begin{cases} \widehat{T}_{k+1} = A(\widehat{\theta}_k)\widehat{T}_k + B(\widehat{\theta}_k)u_k + K_k\left(y_k - C(\widehat{\theta}_k)\widehat{T}_k\right) \\ \widehat{\theta}_{k+1} = \widehat{\theta}_k + L_k\left(y_k - C(\widehat{\theta}_k)\widehat{T}_k\right) \end{cases} \tag{16}$$

where the Kalman gains K_k and L_k are computed by

$$\begin{bmatrix} K_k \\ L_k \end{bmatrix} = \left(\begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix}\Xi_k\begin{bmatrix} C_k & N_k \end{bmatrix}^T + \begin{bmatrix} S \\ 0 \end{bmatrix}\right)P_k^{-1}. \tag{17}$$

Here $A_k = A(\widehat{\theta}_k)$, $M_k = \frac{\partial}{\partial\theta}\left(A(\theta)\widehat{T}_k + B(\theta)u_k\right)|_{\theta=\widehat{\theta}_k}$, $C_k = C(\widehat{\theta}_k)$, $N_k = \frac{\partial}{\partial\theta}\left(C(\theta)\widehat{T}_k\right)|_{\theta=\widehat{\theta}_k}$ and T_k is computed by

$$P_k = \begin{bmatrix} C_k & N_k \end{bmatrix}\Xi_k\begin{bmatrix} C_k & N_k \end{bmatrix}^T + R\ . \tag{18}$$

The covariance matrix Ξ_k is computed by solving the following recursive Riccati equation,

$$\Xi_{k+1} = \begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix}\Xi_k\begin{bmatrix} A_k & M_k \\ 0 & I \end{bmatrix}^T - \begin{bmatrix} K_k \\ L_k \end{bmatrix}T_k\begin{bmatrix} K_k \\ L_k \end{bmatrix}^T + \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \tag{19}$$

with the initial Ξ_0 as $\Xi_0 = \begin{bmatrix} \Pi_x & 0 \\ 0 & \Pi_\theta \end{bmatrix}$.

Note that for computing equation (17) - (19) with matrices of SSS matrices blocks, the permutation operations need to be performed [9].

For the parameterizations of $A(\theta)$, $B(\theta)$, $C(\theta)$ and the computation of partial derivatives $\frac{\partial}{\partial\theta}(A(\theta)T_k)$ etc, we refer to [11] and [12], details will not be discussed in this paper.

For sake of simplicity, we just identify the system matrix A and assume system matrices B and C are diagonal matrices, i.e., $B = diag(b_s)$ and $C = diag(c_s)$. We sample k_s, b_s and c_s from the uniform distribution for each $s \in \{\ 1\quad 2\quad \cdots\quad N\ \}$, i.e., $k_s \in 1 + \frac{2}{5}u[-1,\ 1]$, $b_s \in 3 + \frac{9}{10}u[-1,\ 1]$ and $c_s \in 3 + \frac{9}{10}u[-1,\ 1]$. Since we are mainly interested in the computational complexity, not the convergence of the EKF, we sample the noise sequence $\{\omega_k\}$ and $\{v_k\}$ from the normal distribution $\mathscr{N}(0, 10^{-6})$. The initial guess of Ξ_k was set to be an identity matrix, i.e., $\Xi_0 = I$ and the initial state T_0 to be the zero vector, $\hat{T}_0$ and k_s to be all ones. Since k_s, b_s and c_s were generated randomly, we repeated the numerical experiment 5 times for each problem size. Here we have run this scenario for EKF with SSS matrix structure using approximate balanced truncation and using Hankel blocks approximation. For comparison, the unstructured EKF for the same initial conditions, k_s, b_s and c_s is also implemented to compare the computational complexity of the 2 model reduction algorithms. The stopping criterion was set by $\frac{\|\hat{\theta}_k - \theta^*\|_2}{\|\theta^*\|_2} \le 5 \times 10^{-4}$ where $\hat{\theta}_k$ is the estimated parameters vector at step k and θ^* is the true parameters vector.

The average total computational time for SSS EKF by approximate balanced truncation, Hankel blocks approximation and the unstructured EKF is shown in table I - III and the average total computational time for SSS EKF by approximate balanced truncation and Hankel blocks approximation is shown in Fig. 3. The asymptotic convergence of SSS EKF by approximate balanced truncation is shown in Fig. 4 by computing $\frac{\|\hat{\theta}_k-\theta^*\|_2}{\|\theta^*\|_2}$ when $N=100$. The estimated parameters in different EKF steps and true parameters are illustrated in Fig. 5 for $N=100$ by approximate balanced truncation for SSS model order reduction.

Set the maximum semiseparable order to be 4, the matrix Ξ_k with reduced semiseparable order was denoted by $\hat{\Xi}_k$. The truncation error of the matrix Ξ_k at EKF step $k=10$, denoted as

$$e_k = \frac{\|\Xi_k - \hat{\Xi}_k\|_2}{\|\Xi_k\|_2} \tag{20}$$

for different numbers of subsystems k, where $k\in\{50,\ 100,\ 200,\ 400,\ 600,\ 800,\ 1000\}$ is shown in table IV.

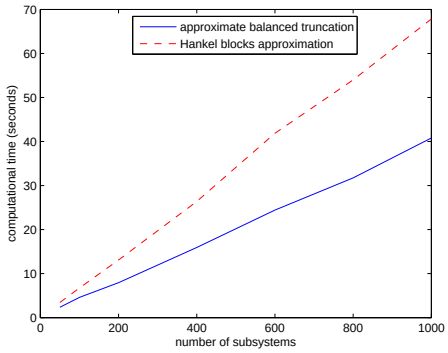

Fig. 3. Computation time of EKF by approximate balanced truncation and Hankel blocks approximation

Remark 2: As stated in equation (11) and (12), the new model reduction algorithm need less flops than the conventional method and both algorithm have linear computational complexity. This is verified by table I - II and Fig. 3.

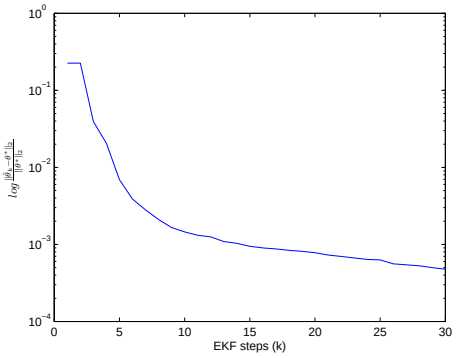

Fig. 4. Asymptotic convergence of estimation by EKF

Remark 3: As shown in table I - III, the average iteration steps for SSS EKF by approximate balanced truncation and Hankel blocks approximation for the same number of subsystems are almost the same while it is smaller than the iteration steps of unstructured EKF. This is because that the operations for inversion of SSS matrices in computing the Kalman gain in (17) preserve the matrix structure such that a certain relation between parameters is kept while inversion of unstructured matrices does not.

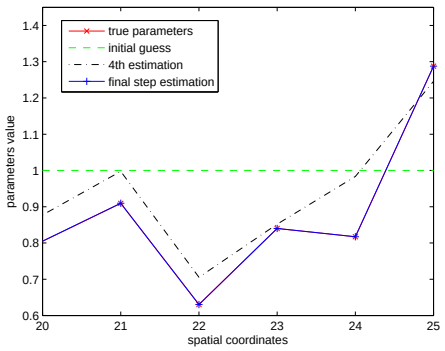

Fig. 5. Estimated parameters and true parameters

Remark 4: Fig. 5 shows that with a homogeneous initial guess, the SSS EKF estimation recovered the true parameters quickly. This is shown by the results for the subsystem with spatial coordinate from 20 to 25.

Remark 5: Table IV shows the truncation error by the 2 model reduction algorithms. Both algorithms lead to a satisfied accuracy. As pointed in remark 1, the upper bound of the approximate balanced truncation is bigger than the Hankel blocks approximation, because of that we expect the approximate balanced truncation is less accurate as the Hankel blocks approximation, which is confirmed by table IV.

V. CONCLUSIONS AND FUTURE WORKS

In this paper, a novel model reduction algorithm for sequentially semiseparable (SSS) matrices has been described. It is an approximate balanced truncation model reduction algorithm for linear time-varying systems. The approximate balanced truncation makes use of the low numerical rank property of the controllability and observability gramians. With this new model reduction algorithm for SSS matrices, the extended Kalman filter had been performed to identify the 1-D spatially interconnected systems with linear computational complexity. To compare the performance the newly proposed model reduction algorithm, convention model reduction algorithm was also studied. The new model reduction algorithm yields satisfied accuracy and is computationally cheaper than the conventional one. Numerical experiment had illustrated the effectiveness and efficiency of the new model reduction algorithm.

The next step of this research is to extend the model reduction algorithm to multi-level sequentially semiseparable (MSSS) matrices, where all generators are MSSS matrices of a lower-level. The goal is to make control and identification of 2-D or even 3-D interconnected systems could be performed with linear computational complexity within the framework of MSSS matrices computations.

TABLE I

NUMERICAL EXPERIMENT RESULTS OF SSS EKF BY APPROXIMATE BALANCED TRUNCATION

Number of subsystems	50	100	200	400	600	800	1000
number of iterations	31.4	32.8	29	29.4	30.02	29.4	30.2
computation time (seconds)	2.36	4.59	7.93	15.94	24.43	31.74	40.81
time per iteration (seconds)	0.075	0.140	0.273	0.542	0.814	1.080	1.351

TABLE II

NUMERICAL EXPERIMENT RESULTS OF SSS EKF BY HANKEL BLOCKS APPROXIMATION

Number of subsystems	50	100	200	400	600	800	1000
number of iterations	31.6	31.8	31	31.2	33.2	32	32.2
computation time (seconds)	3.43	6.74	13.10	26.38	41.88	54.00	67.82
time per iteration (seconds)	0.109	0.212	0.423	0.846	1.261	1.688	2.106

TABLE III

NUMERICAL EXPERIMENT RESULTS OF UNSTRUCTURED EKF

Number of subsystems	50	100	200	250	300	400
number of iterations	45.0	45.8	45.0	46.0	49.0	50.2
computation time (seconds)	0.062	0.392	14.048	46.614	112.65	279.142
time per iteration (seconds)	0.0014	0.0086	0.312	1.013	2.299	5.561

TABLE IV

TRUNCATION ERROR OF Ξ_k BY THESE TWO ALGORITHMS

Matrix size	100	200	400	800	1200	1600	2000
Approx. Balanced Truncation	5.24×10^{-4}	5.13×10^{-4}	2.27×10^{-3}	5.07×10^{-3}	6.82×10^{-3}	9.52×10^{-4}	3.51×10^{-3}
Hankel Blocks Approx.	7.11×10^{-6}	1.06×10^{-5}	4.16×10^{-6}	4.39×10^{-6}	1.28×10^{-5}	5.52×10^{-6}	4.55×10^{-6}

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