# Neural Networks

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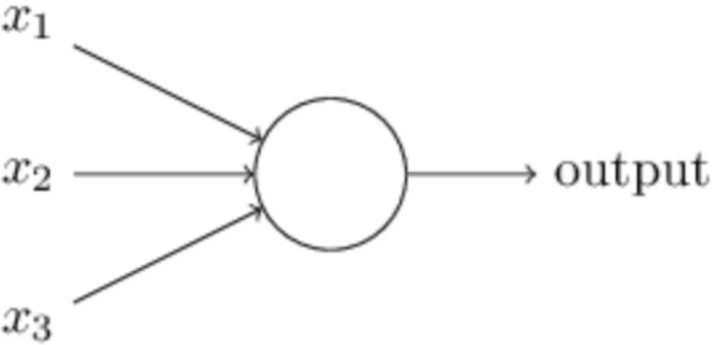
- Perceptrons no longer common — main neuron model is sigmoid neurons — but perceptrons help motivate
- Takes several binary inputs and produces a single binary output



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- Neuron's output, 0 or 1, is determined by whether weighted sum $\sum_j w_j x_j$ is less than or greater than a threshold
- Just like weights, threshold is a real number and a parameter of the neuron

$$\text{output} = \begin{cases} 0 & \text{if } \sum_j w_j x_j \leq \text{threshold} \\ 1 & \text{if } \sum_j w_j x_j > \text{threshold} \end{cases}$$

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- By varying weights and threshold, can get different models of decision-making
- A perceptron can weigh up different kinds of evidence in order to make decisions
- Complex network of perceptrons could model subtle decisions

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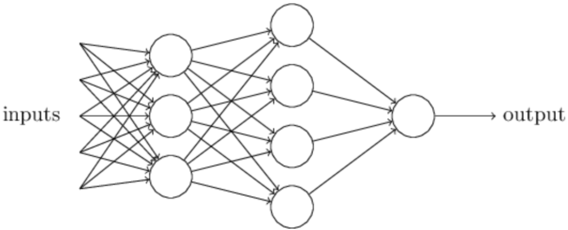


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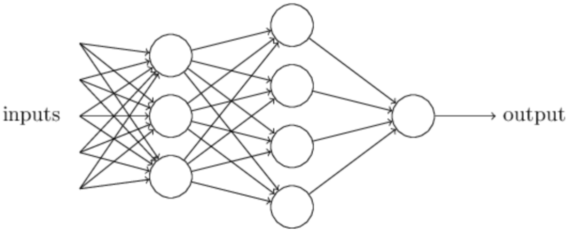


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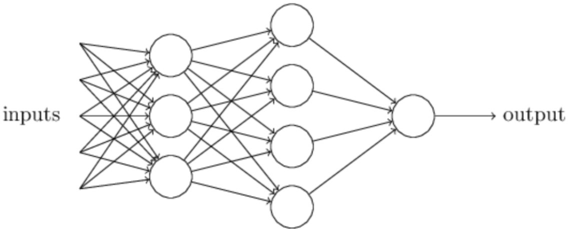


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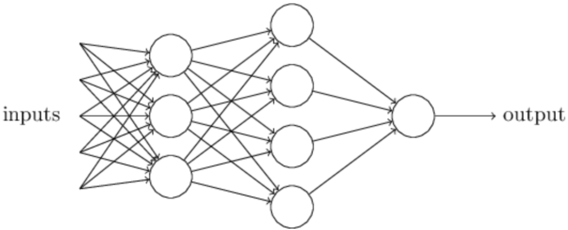


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- Many-layer network can engage in sophisticated decision-making

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- Really big bias $\rightarrow$ easy to get a 1

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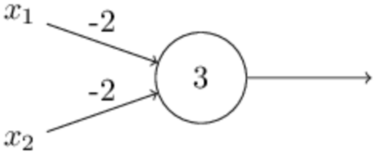

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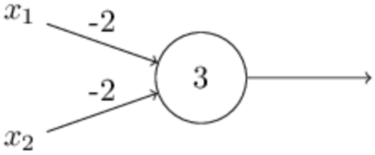


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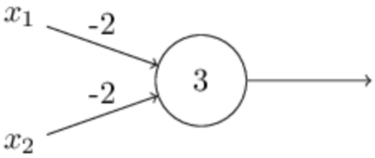


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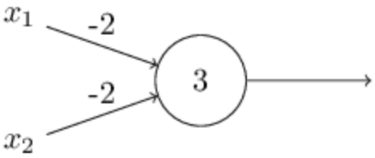


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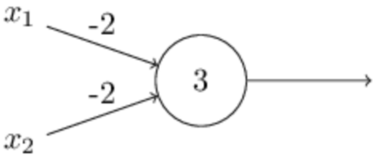


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- We can build any computation up from NAND gates

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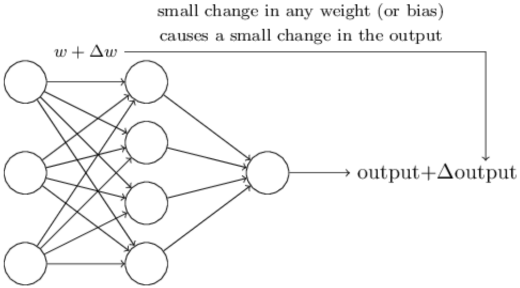
- To see how learning might work, suppose we make a small change in some weight (or bias) in the network
- What we'd like is for this small change in weight to cause only a small corresponding change in the output from the network



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- That flip may then cause the behaviour of the rest of the network to completely change in some very complicated way

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- The output is no longer 0 or 1, it is $\sigma(w \cdot x + b)$, where $\sigma$ is the sigmoid function (also called logistic function and logistic neurons)

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- When very negative, close to 0, when very positive, close to 1

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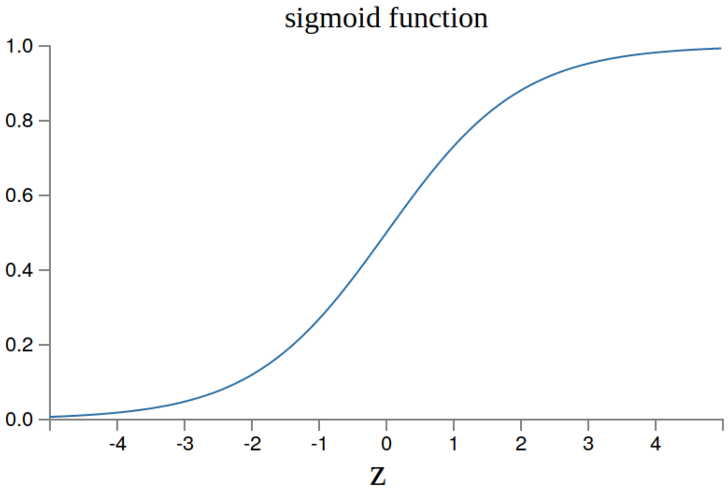

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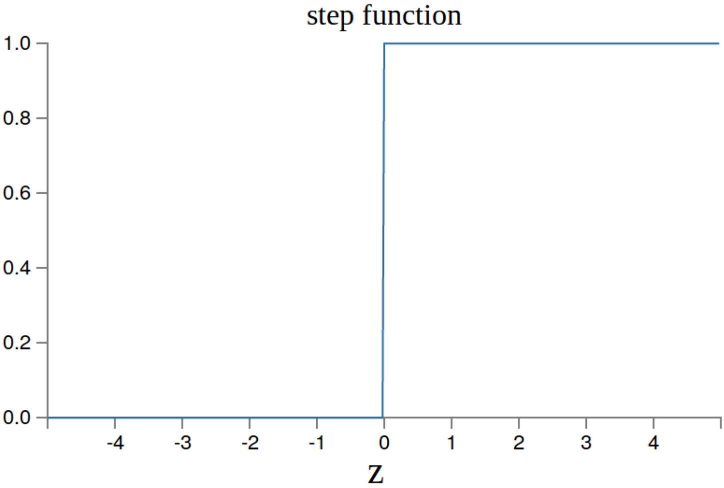

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- In words, $\Delta$output is a linear function of the changes to weights and biases
- This linearity makes it easy to choose small changes in the weights and biases to achieve any desired small change in the output

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- Be explicit about convention used

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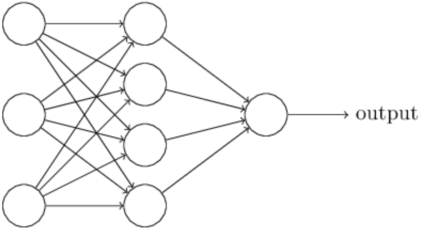


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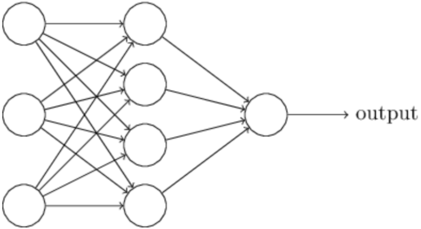


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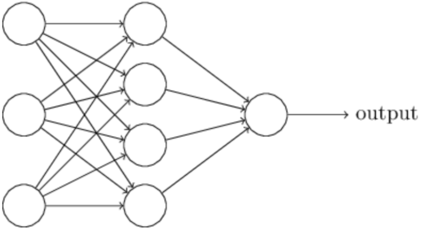


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- Middle layer is hidden layer, neither inputs nor outputs

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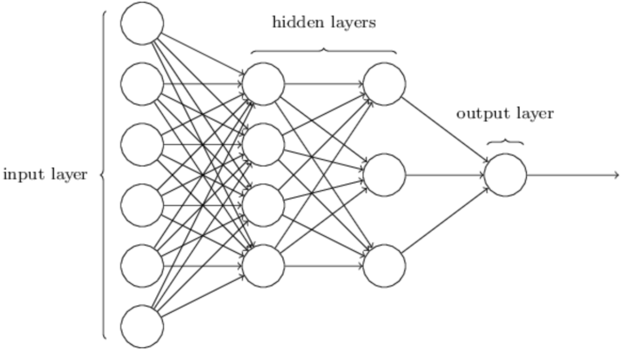


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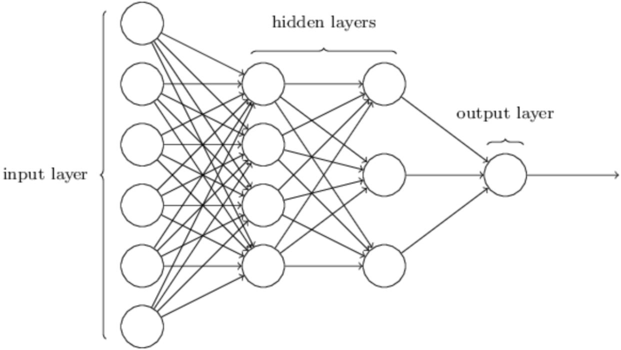


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- Confusingly, sometimes multiple layer networks are called multilayer perceptrons or MLPs

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- If the image is a 64 by 64 greyscale image, then we'd have $4,096 = 64 \times 64$ input neurons, with the intensities scaled appropriately between 0 and 1
- The output layer will contain just a single neuron, with output values of less than 0.5 indicating input image is not a 9, and values greater than 0.5 indicating input image is a 9

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- Instead, neural networks researchers have developed many design heuristics for the hidden layers, which help people get the behaviour they want out of their nets
- For example, such heuristics can be used to help determine how to trade off the number of hidden layers against the time required to train the network.

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- That'd be hard to make sense of, and so we don't allow such loops

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- Loops don't cause problems in such a model, since a neuron's output only affects its input at some later time, not instantaneously

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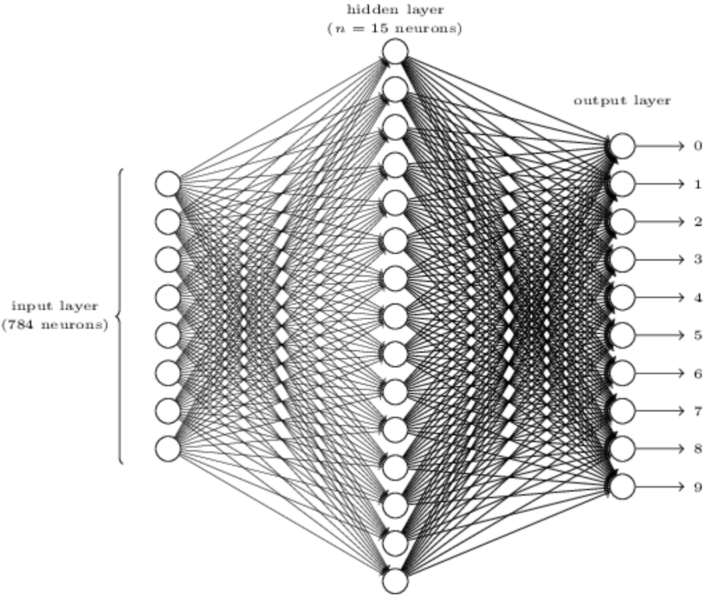
- So instead of worrying about segmentation we'll concentrate on developing a neural network which can solve the more interesting and difficult problem, namely, recognizing individual handwritten digits
- To recognize individual digits we will use a three-layer neural network

# Three-Layer Neural Network



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- The input pixels are greyscale, with a value of 0.0 representing white, a value of 1.0 representing black, and in between values representing gradually darkening shades of grey

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- If that neuron is, say, neuron number 6, then our network will guess that the input digit was a 6

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- That's why we focus first on minimizing the quadratic cost, and only after that will we examine the classification accuracy.

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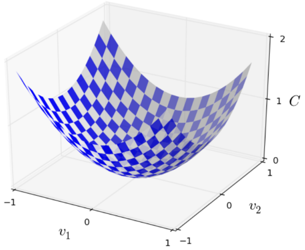

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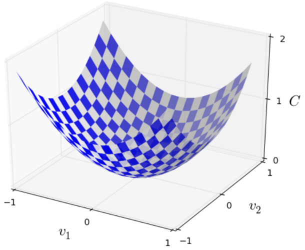


- We'd like to find where $C$ achieves its global minimum

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- Using calculus to analytically solve the minimum will not work with several (often billions of) variables
- Randomly choose a starting point for an (imaginary) ball, and then simulate the motion of the ball as it rolled down to the bottom of the valley
- We could do this simulation simply by computing derivatives (and perhaps some second derivatives) of $C$ — those derivatives would tell us everything we need to know about the local shape of the valley, and therefore how our ball should roll

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- We're going to find a way of choosing $\Delta v_1$ and $\Delta v_2$ so as to make $\Delta C$ negative; i.e., we'll choose them so the ball is rolling down into the valley

# Mathematical Notation

$$\nabla C \equiv \left(\frac{\partial C}{\partial v_1}, \frac{\partial C}{\partial v_2}\right)^T$$

$$\Delta C \approx \nabla C \cdot \Delta v$$

$$\Delta v = -\eta \nabla C, \eta > 0$$

$$\Delta C \approx -\eta \nabla C \cdot \nabla C = -\eta ||\nabla C||^2$$

$$v \rightarrow v' = v - \eta \nabla C$$

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- We'll see later how this works

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- Averaging over small sample produces a good estimate of the true gradient and speeds up learning
- Of course, the estimate won't be perfect — there will be statistical fluctuations — but it doesn't need to be perfect: all we really care about is moving in a general direction that will help decrease $C$, and that means we don't need an exact computation of the gradient