Linear Models Review

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- To measure causal effects

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- Natural experiments: Assignment into "treatment" and "control" is as if randomized by nature
- Observational studies: We do not know how assignment into "treatment" and "control" was achieved

Non-Parametric Approaches

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Smokers

	Old	Young
GM	$\hat{y}_1$	$\hat{y}_2$
$\sim GM$	$\hat{y}_3$	$\hat{y}_4$

Non-smokers

	Old	Young
GM	$\hat{y}_5$	$\hat{y}_6$
$\sim GM$	$\hat{y}_7$	$\hat{y}_8$

Non-Parametric Approaches (cont.)

We need not make too many assumptions about how T affects Y after controlling for X. We could simply assume

$$p(Y|T,X) = g(T,X)$$

and use sample data to estimate $p(Y|T,X)$ at different combinations of T and X.

Smokers

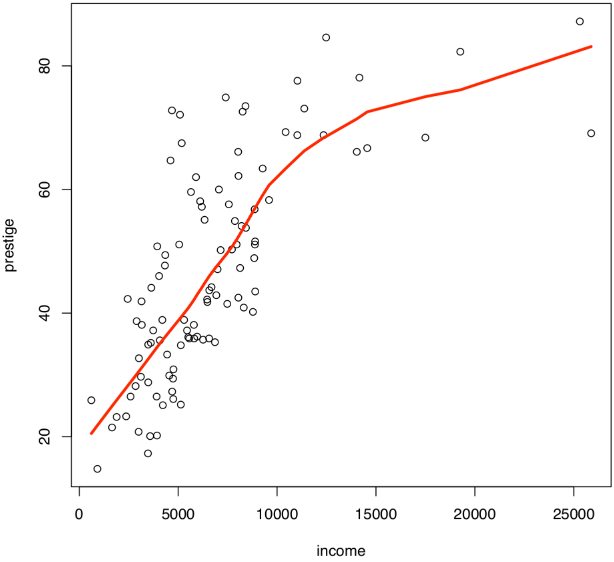
	Old	Young
GM	0.9	0.6
$\sim GM$	0.6	0.6

Non-smokers

	Old	Young
GM	0.5	0.5
$\sim GM$	0.5	0.5

Non-Parametric Approaches (cont.)

How can we learn anything about $p(Y|X)$ when X is continuous?



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Non-Parametric Approaches (cont.)

- NPR depends on less extraneous assumptions (i.e., it does not require linearity), but
- NPR is computationally expensive
- NPR cannot be easily transmitted
- NPR collapes under "curse of dimensionality"
- We need to move from "natural" NPR to parametric approaches

Substantive Statements as Probability Models

Outcome (Y)	Predictor or Cause (X)
Votes for Party A	Platforms of parties A and B
Frequency of wars	Political regimes of neighboring countries
Campaign spending ($)	Incumbent strength
Survival of democracy	Country's income level
Cancer rates	Number of phone lines

Probability Models of Data Generating Processes

The **generalized linear model** notation makes it clear that we are building a model of the probability of Y conditional on X:

$$Y_i \sim f(\theta_i)$$
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- Systematic component: $\mathbf{X}$
- Link function: $g(\cdot)$

Distribution of Random Variables

- **Random variable**: A real-valued function that is defined on a sample space

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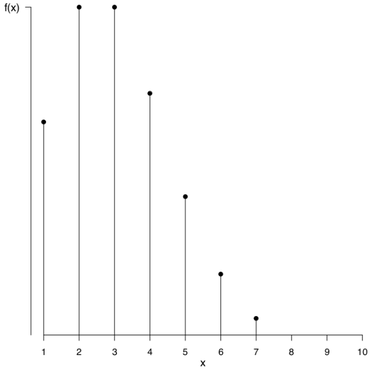
- **Random variable**: A real-valued function that is defined on a sample space
- Random variable X is characterized by a probability distribution over all possible values x that X can take

Discrete Random Variables



- The probability function of X is the function f such that for every x

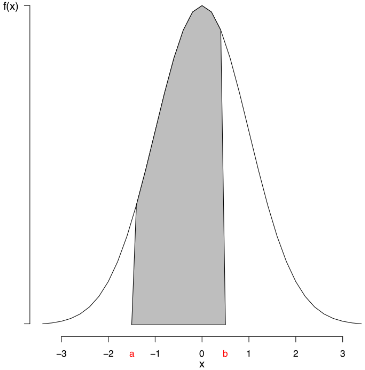
$$f(x) = \Pr(X = x)$$

1. If x is outside the sample space, then $f(x) = 0$
2. If $x_1, x_2, \ldots$ includes all values in the sample space, then

$$\Sigma_{i=1}^{\infty} f(x_i) = 1$$

3. $\Pr(X \in A) = \Sigma_{x_i \in A} f(x_i)$

Continuous Random Variables



- The probability density function $f(x)$ specifies the probability of X taking values on subsets of the sample space;
  e.g., for subset (a, b)
  1. $f(x) \geq 0, \forall x$
  2. $\int_{-\infty}^{\infty} f(x)dx = 1$
  3. $\Pr(a < x \leq b) = \int_a^b f(x)dx$

Joint Distribution of Discrete X, Y

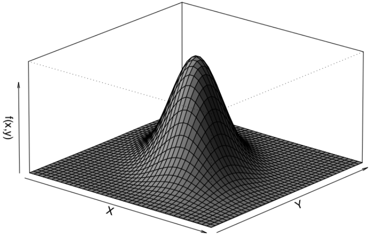
	X		
Y	1	2	3
1	0.1	0.3	0
2	0	0	0.2
3	0.1	0.1	0
4	0	0.2	0

- If X and Y are discrete random variables, their distribution is also discrete
- The joint p.f. of (X, Y) is the function f such that for every point (x, y)

$$f(x, y) = \Pr(X = x \text{ and } Y = y)$$

1. If x, y are outside the sample space, then $f(x, y) = 0$
2. $\Sigma_{(x,y)} f(x, y) = 1$

Joint Distribution of Continuous X, Y



- If X and Y are continuous random variables, their distribution is also continuous
- The joint pdf of (X, Y) is the function f such that for every region A

$$\Pr(X, Y \in A) = \int_A \int f(x, y) dx dy$$

1. $f(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) = 1$

Marginal Distribution of X

	X			
Y	1	2	3	$f_2(Y)$
1	0.1	0.3	0	0.4
2	0	0	0.2	0.2
3	0.1	0.1	0	0.2
4	0	0.2	0	0.2
$f_1(X)$	0.2	0.6	0.2	1

The distribution of X computed from the joint distribution of (X, Y) is the marginal distribution of X

- Discrete distributions:

$$f_1(x) = \Sigma_y f(x, y)$$

- Continuous distributions:

$$f_1(x) = \int_{-\infty}^{\infty} f(x, y) dy$$

Conditional Distributions

After observing $Y = y$, the probability that $X = x$ is specified by the conditional probability

$$g_1(x|y) = p(X = x|Y = y) = \frac{p(X = x \text{ and } Y = y)}{p(Y = y)} = \frac{f(x, y)}{f(y)}$$

where $g_1(x|y) \geq 0$ and $\sum_y g_1(x|y) = 1$

Conditional Distributions (cont.)

What's $g_1(X = 1|Y = 3)$?

	X			
Y	1	2	3	$f_2(Y)$
1	0.1	0.3	0	0.4
2	0	0	0.2	0.2
3	0.1	0.1	0	0.2
4	0	0.2	0	0.2
$f_1(X)$	0.2	0.6	0.2	1

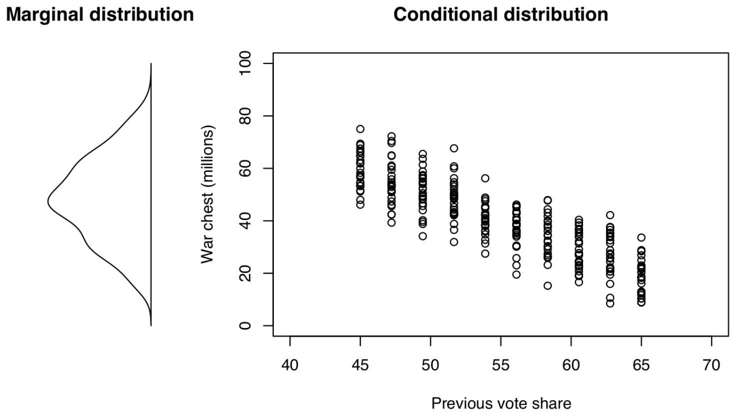
- $f_2(y = 3) = 0.2$
- $f(x = 1 \text{ and } y = 3) = 0.1$
- $f(x = 2 \text{ and } y = 3) = 0.1$
- $f(x = 3 \text{ and } y = 3) = 0$

$$\frac{f(x = 1 \text{ and } y = 3)}{f_2(y = 3)} = \frac{0.1}{0.2}$$

$$= \frac{1}{2}$$

Linear Regression as a Probability Model

War chest as a function of *support in previous election* The regression line joins $E(Y|X)$ at different values of X



Alternative Notations for OLS Regression

GLM notation

$$
\begin{aligned}
Y_i &\sim \mathcal{N}(\mu_i, \sigma^2) \\
\mu_i &= \alpha + \beta X_i
\end{aligned}
$$

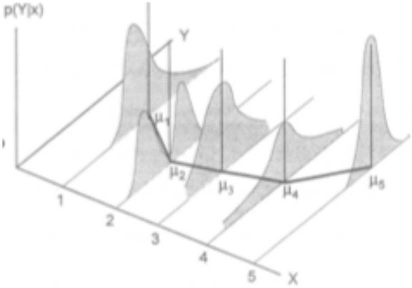
OLS notation

$$
\begin{aligned}
Y_i &= \alpha + \beta X_i + \epsilon_i \\
\epsilon_i &\sim \mathcal{N}(0, \sigma^2)
\end{aligned}
$$

In both cases, note the three main assumptions we impose:

- Normality
- Linearity
- Constant variance

Potential Pitfalls in Parametric Models



1. Failure of normality: $p(Y|x=1)$, $p(Y|x=2)$, and $p(Y|x=3)$ are not normal
2. Failure of linearity: $E[p(Y|X)] \neq \alpha + \beta X$
3. Failure of constant variance: $p(Y|x=4)$ and $p(Y|x=5)$ do not have the same spread $(\sigma_y|x=4 \neq \sigma_y|x=5)$

Mechanics of OLS Point Estimates

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- This line could be constructed in many different ways, but we make three consequential decisions:
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  - Sum of errors
  - Sum of squared errors

Multiple Regression

- In multiple regression, we fit a least squares “hyperplane” in $k+1$-dimensional space

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- $\hat{\beta}_0$ is the expected value of Y when all variables X are jointly 0
- For any slope coefficient estimate $\hat{\beta}_k$: A unit increase in $X_{i,k}$ will yield on average a $\hat{\beta}_k$ increase in Y_i, *all else constant*

Multiple Regression (cont.)

We define the multiple regression model as follows:

$$
\begin{aligned}
&Y_i = \beta_0 + \beta_1 X_{i1} + \ldots + \beta_k X_{ik} + \epsilon_i \\
&\epsilon_i \sim \mathcal{N}(0, \sigma^2)
\end{aligned}
$$

In this model, we have:

- $i = 1, \ldots, n$ observations
- k independent variables
- $k + 1$ slope and intercept parameters β_j
- one variance parameter σ^2

Statistical Theory for Linear Models

The scalar notation is cumbersome, but the model can be simplified by defining the following vectors and matrices:

- $\boldsymbol{y} = [y_1, y_2, \ldots, y_n]'$ is a vector of observations on the dependent variable
- $\boldsymbol{X} = [\mathbf{1}, \boldsymbol{x_1}, \boldsymbol{x_2}, \ldots, \boldsymbol{x_k}]$ is a matrix with a column of 1's and k columns of independent variables
- $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \ldots, \beta_k]'$
- $\boldsymbol{\epsilon} = [\epsilon_1, \epsilon_2, \ldots, \epsilon_n]'$ is a vector of random errors

Statistical Theory for Linear Models (cont.)

The linear model can then be represented succinctly as:

$$\begin{aligned}
\boldsymbol{y} &= \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
E(\boldsymbol{\epsilon}) &= [0, 0, \ldots, 0]' = \boldsymbol{0} \\
\text{var}(\boldsymbol{\epsilon}) = E(\boldsymbol{\epsilon}\boldsymbol{\epsilon'}) &= \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots & 0 \\ 0 & \sigma^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma^2 \end{bmatrix} = \sigma^2 \boldsymbol{I}_n
\end{aligned}$$

Statistical Theory for Linear Models (cont.)

$$\boldsymbol{Y} = \boldsymbol{X}\beta + \epsilon$$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n\times 1} = \underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1k} \\ 1 & x_{21} & x_{22} & \cdots & x_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & x_{n2} & \cdots & x_{nk} \end{bmatrix}}_{n\times(k+1)} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_k \end{bmatrix}}_{(k+1)\times 1} + \underbrace{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}}_{n\times 1}$$

For observation 2:

$$y_2 = \beta_0 \cdot 1 + \beta_1 X_{2,1} + \beta_2 X_{2,2} + \ldots + \beta_k X_{2,k} + \epsilon_2$$

Derivation of OLS Estimators

- Define the vector of residuals as

$$\boldsymbol{e} = \boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}$$

- The sum of squared errors is defined as

$$\boldsymbol{e}'\boldsymbol{e} = (\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)$$

- Compute $\frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{e}'\boldsymbol{e})$

$$\begin{aligned}
\frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{e}'\boldsymbol{e}) &= \frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}) \\
&= \frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{Y}' - \boldsymbol{b}'\boldsymbol{X}')(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b}) \\
&= \frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{Y}'\boldsymbol{Y} - \boldsymbol{Y}'\boldsymbol{X}\boldsymbol{b} - \boldsymbol{b}'\boldsymbol{X}'\boldsymbol{Y}' + \boldsymbol{b}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}) \\
&= \frac{\partial}{\partial \boldsymbol{b}}(\boldsymbol{Y}'\boldsymbol{Y} - 2\boldsymbol{b}'\boldsymbol{X}'\boldsymbol{Y} + \boldsymbol{b}'\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}) \\
&= -2\boldsymbol{X}'\boldsymbol{Y} + 2\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b}
\end{aligned}$$

Derivation of OLS Estimators (cont.)

- Set the first derivative of $\boldsymbol{e'e}$ with respect to $\boldsymbol{b}$ equal to 0

$$\begin{aligned} -2\boldsymbol{X'Y} + 2\boldsymbol{X'Xb} &= 0 \\ 2\boldsymbol{X'Xb} &= 2\boldsymbol{X'Y} \\ \boldsymbol{X'Xb} &= \boldsymbol{X'Y} \end{aligned}$$

- Compute the inverse $(\boldsymbol{X'X})^{-1}$ of $\boldsymbol{X'X}$ and use it to pre-multiply both sides of the previous equation:

$$\begin{aligned} (\boldsymbol{X'X})^{-1}\boldsymbol{X'Xb} &= (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y} \\ \boldsymbol{b} &= (\boldsymbol{X'X})^{-1}\boldsymbol{X'Y} \end{aligned}$$

- $\boldsymbol{b}$ is uniquely defined as long as $\boldsymbol{X'X}$ is a full-rank matrix

Derivation of OLS Estimators (cont.)

Second order condition: The matrix of second derivatives of $\boldsymbol{e'e}$ (Hessian matrix) should be positive definite, hence a global minimum:

$$\frac{\partial^2(\mathbf{e'e})}{\partial\mathbf{b}\partial\mathbf{b'}} = \begin{bmatrix} \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0^2}} & \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0}\partial\mathbf{b_1}} & \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0}\partial\mathbf{b_2}} & \cdots & \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0}\partial\mathbf{b_k}} \\ \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0}\partial\mathbf{b_1}} & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_0}\partial\mathbf{b_n}} & \cdots & \cdots & \cdots & \frac{\partial^2\mathbf{e'e}}{\partial\mathbf{b_k^2}} \end{bmatrix} = 2\mathbf{X'X}$$

Derivation of the Moments of $\boldsymbol{b}$

We can also find the variance of $\boldsymbol{b}$:

$$\begin{aligned}
var(\boldsymbol{b}) &= var(\boldsymbol{AY}) \\
&= \boldsymbol{A} var(\boldsymbol{Y})\boldsymbol{A}' \\
&= [(\boldsymbol{X'X})^{-1}\boldsymbol{X'}] var(\boldsymbol{Y})[(\boldsymbol{X'X})^{-1}\boldsymbol{X'}]' \\
&= [(\boldsymbol{X'X})^{-1}\boldsymbol{X'}]\sigma^2\boldsymbol{I}[(\boldsymbol{X'X})^{-1}\boldsymbol{X'}]' \\
&= \sigma^2[(\boldsymbol{X'X})^{-1}\boldsymbol{X'}][(\boldsymbol{X'X})^{-1}\boldsymbol{X'}]' \\
&= \sigma^2[(\boldsymbol{X'X})^{-1}\boldsymbol{X'}][\boldsymbol{X}(\boldsymbol{X'X})^{-1}] \\
&= \sigma^2(\boldsymbol{X'X})^{-1}
\end{aligned}$$

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- Non-stochastic X: X is fixed in repeated sampling

Basic Assumptions of OLS ("conditional" approach)

- No perfect multicollinearity
- Variability in X: $var(X) > 0$ but finite
- Linearity: $Y_i = \alpha + \beta X_i + \epsilon_i$
- Conditional zero mean: $E(\epsilon_i|\boldsymbol{X}) = 0$
- Conditional homoskedasticity: $var(\epsilon_i|\boldsymbol{X}) = \sigma^2$
- Conditional non-autocorrelation: $cov(\epsilon_i, \epsilon_j|\boldsymbol{X}) = 0, \forall i \neq j$

Maximum-Likelihood Estimation

- We use data to make inferences about a set θ of parameters (ex., $\theta = (\beta_0, \beta_1, \ldots, \beta_k)$)
- We observe

$$\boldsymbol{Y} = (y_1, y_2, \ldots, y_n)'$$
$$\boldsymbol{X} = (\boldsymbol{x_1, x_2, \ldots, x_k})$$

  and assume
  - that each draw y_i is drawn from the same distribution with parameter θ and
  - that the draws $i = (1, \ldots, n)$ are independent
- In short: $y_i \overset{iid}{\sim} f(\theta, \boldsymbol{X}_i)$

Maximum-Likelihood Estimation (cont.)

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- Which parameters $\theta = (\mu, \sigma^2)$ are most likely to have generated the values $\boldsymbol{y} = (3, 5, 7)$ if $Y \sim \mathcal{N}(\mu, \sigma^2)$?

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- Which parameters $\theta = (\mu, \sigma^2)$ are most likely to have generated the values $\boldsymbol{y} = (3, 5, 7, 6, 1, 2, 4, 5, 5, 9, 8)$ if $Y \sim \mathcal{N}(\mu, \sigma^2)$?

Maximum-Likelihood Estimation (cont.)

- We can rewrite the linear regression model as a probability model:

$$\boldsymbol{Y} \sim \mathcal{N}_n(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2 \boldsymbol{I})$$

- Which means that, for any observation:

$$y_i \sim \mathcal{N}(\boldsymbol{x_i'}\boldsymbol{\beta}, \sigma^2)$$

$$p(y_i|\boldsymbol{x}_i, \boldsymbol{\beta}, \sigma^2) = \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{1}{2\sigma^2}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right]$$

MLE for the Linear Regression Model

Because of our assumption of independence, we can write the joint pdf of $\boldsymbol{Y}$ as

$$
\begin{aligned}
p(\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\beta}, \sigma^2) &= p(y_1) \times p(y_2) \times \cdots \times p(y_n) = \prod_{i=1}^{n} p(y_i) \\
&= \prod_{i=1}^{n} \frac{1}{(2\pi\sigma^2)^{1/2}} exp\left[-\frac{1}{2\sigma^2}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right] \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \boldsymbol{x}_i'\boldsymbol{\beta})^2\right] \\
&= \frac{1}{(2\pi\sigma^2)^{n/2}} exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})\right]
\end{aligned}
$$

MLE for the Linear Regression Model (cont.)

When the parameters $\boldsymbol{\beta}, \sigma^2$ are expressed as a function of the data Y, X, the joint pdf is called the likelihood function:

$$\mathcal{L}(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) = \frac{1}{(2\pi\sigma^2)^{n/2}} exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]$$

- Note that $\mathcal{L}(\cdot)$ returns a real number for every combination of $\boldsymbol{Y}, \boldsymbol{X}, \beta, \sigma^2$
- Maximum likelihood estimates of $\boldsymbol{\beta}$ are simply the values of $\boldsymbol{b}$, $\hat{\sigma}^2$ that maximize $\mathcal{L}(\cdot)$ given $\boldsymbol{Y}, \boldsymbol{X}$

MLE for the Linear Regression Model (cont.)

- In the linear model, $\mathcal{L}(\cdot)$ is a concave function (thus $\boldsymbol{b}$ and $\hat{\sigma}^2$ exist and are unique) but has a very difficult form
- Fortunately, we can appeal to the invariance property of maximum likelihood estimators to transform $\mathcal{L}(\cdot)$ into something manageable, like the log (ℓ):

$$
\begin{aligned}
\ell(\boldsymbol{\beta}, \sigma^2 | \boldsymbol{Y}, \boldsymbol{X}) &= ln\mathcal{L}(\cdot) \\
&= ln\left(\frac{1}{(2\pi\sigma^2)^{n/2}} exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)\right]\right) \\
&= -\frac{n}{2} ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(\boldsymbol{Y} - \boldsymbol{X}\beta)'(\boldsymbol{Y} - \boldsymbol{X}\beta)
\end{aligned}
$$

MLE for the Linear Regression Model (cont.)

- We can now solve the (relatively) simpler problem of choosing $\boldsymbol{\beta}$ and σ^2 to maximize ℓ:

$$\frac{\partial \ell}{\partial \boldsymbol{\beta}} = \frac{1}{2\sigma^2}(2\boldsymbol{X}'\boldsymbol{Y} - 2\boldsymbol{X}'\boldsymbol{X}\boldsymbol{\beta} = 0$$
$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{\beta}) = 0$$

- The ML estimates are:

$$\boldsymbol{b}_{ML} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$
$$\hat{\sigma}^2_{ML} = \frac{(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})'(\boldsymbol{Y} - \boldsymbol{X}\boldsymbol{b})}{n} = \frac{\boldsymbol{e}'\boldsymbol{e}}{n}$$

Dichotomous Dependent Variables: LPM

The linear probability model is based on the assumption of linearity to model dichotomous dependent variables

$$Y_i = \underbrace{\alpha + \beta x_i}_{E(Y|x_i)=\pi_i} + \varepsilon_i$$

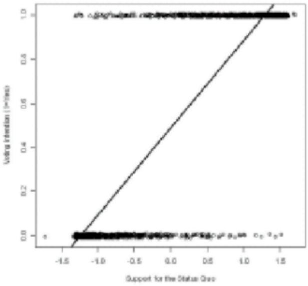
$$\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$$

LPM is BLAT, but:

1. *Errors are not normally distributed*
2. *Heteroskedasticity*

$$\text{var}(\varepsilon_i) = \pi_i(1 - \pi_i)$$

3. *Linearity assumption*



Generalized Linear Models

So far, we have discussed the identity link, but many possible options for $g(\cdot)$

- Binary (dichotomous)
  - Logit, Probit, or c-log-log
- Unordered categorical (polytomous)
  - Multinomial logit
- Ordered categorical
  - Ordered logit or Probit
- Counts
  - Poisson
- Rare counts
  - Negative binomial
- Zero-inflated models for rare non–zero outcomes (e.g. war)