Homework Assignment 3

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1 Question 1

(a) The Taylor series of expansion of a function $f(x)$ around a point a is given by

$$f(x) = f(a) + f'(x)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + ...$$

For $f(x) = 0.5 + \sin(x)$, at $x = 0$, we have

$$\begin{aligned} f(0) &= 0.5 \\ f'(0) &= 1 \\ &. \\ &. \\ &. \\ f^k(0) &= \frac{(-1)^k}{(1+2k)!} \end{aligned}$$

Thus, the taylor series expansion is given by

$$\begin{aligned} f(x)_{x=0} &= 0.5 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \\ f(x)_{x=0} &= 0.5 + \sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{(1+2k)!} \end{aligned}$$

(b) The graph of the function over $[-\pi/2, \pi/2]$ is given by Figure 1. to run the code,

```
python q1.py
```

(c) Consider a quadratic polynomial $p_2(x)$ that is going to be the best uniform approximation of the function $f(x) = 0.5 + \sin(x)$. Since $f^{(3)}(x) = -\cos(x)$ does not change sign between $[-\pi/2, \pi/2]$, we can get 4 points such that the L_∞ norm of the errors are minimized and the errors at these points are alternating. Consider the points to be $\{-\pi/2, x_1, x_2, \pi/2\}$ and consider the polynomial to be fit as $p(x) = ax^2 + bx + c$. Now let $e(z)$ denote the error at point z.

$$\implies e(-\frac{\pi}{2}) = -e(x_1) = e(x_2) = -e(\frac{\pi}{2})$$

$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -0.5 - \sin(x_1) + ax_1^2 + bx_1 + c = 0.5 + \sin(x_2) - ax_2^2 - bx_2 - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$

Since the function $f(x) = 0.5 + \sin(x)$ is symmetric around the origin between $[-\pi/2, \pi/2]$, we would want the best approximating function to be symmetric between $[-\pi/2, \pi/2]$ and hence we take the points x_1 and x_2 as $-x_1 = x_2 = z$. So, now the alternating points are given by $\{-\pi/2, -z, z, \pi/2\}$. Thus the new errors are written as,

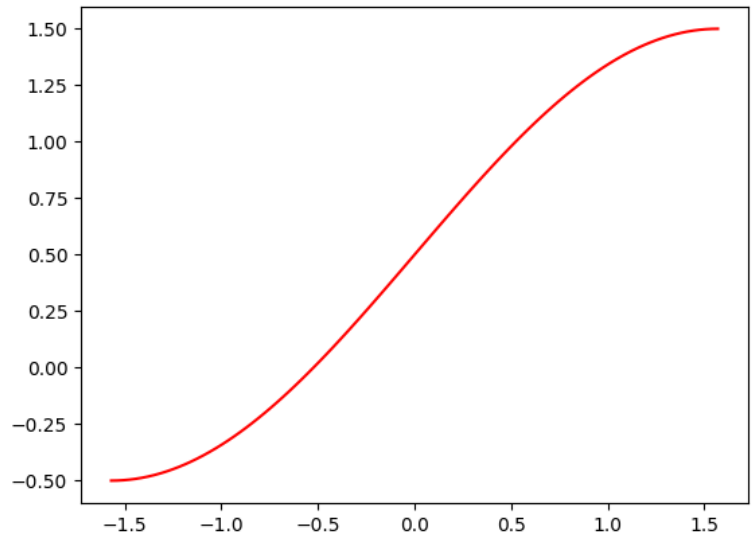

Figure 1: Function $f(x) = 0.5 + \sin(x)$

$$e(-\frac{\pi}{2}) = -e(-z) = e(z) = -e(\frac{\pi}{2})$$
$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -0.5 + \sin(z) + az^2 - bz + c = 0.5 + \sin(z) - az^2 - bz - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$

Solving the middle two equations, we get,

$$-0.5 + \sin(z) + az^2 - bz + c = 0.5 + \sin(z) - az^2 - bz - c$$
$$2az^2 + 2c = 1$$

Solving $e(-\frac{\pi}{2}) = -e(\frac{\pi}{2})$, we get,

$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$
$$2a\frac{\pi^2}{4} + 2c = 1$$

Combining these two equations,

$$2a\frac{\pi^2}{4} + 2c = 1$$
$$2az^2 + 2c = 1$$

Now since $z \neq \pi/2$ or $z \neq -\pi/2$, the only possible results from this are that $a = 0$ and $c = 0.5$. Now substituting this back in the error equations, we get,

$$-1 + b\frac{\pi}{2} = \sin(z) - bz = \sin(z) - bz = -1 + b\frac{\pi}{2}$$

Since we want the error at z to be minimum, we take the differential of the error at this point and equate it to zero, given by

$$
\begin{aligned}
\frac{d}{dz}e(z) &= \frac{d}{dz}(\sin(z) - bz) = 0 \\
\implies \cos(z) - b = 0 \\
&\implies b = \cos(z) \\
&\implies z = \cos^{-1}(b)
\end{aligned}
$$

Substituting this in the error equations above, we get,

$$
\begin{aligned}
-1 + b\frac{\pi}{2} &= \sin(\cos^{-1}(b)) - b\cos^{-1}(b) \\
-1 + b\frac{\pi}{2} &= \sqrt{1-b^2} - b\cos^{-1}(b) \\
\sqrt{1-b^2} - b(\cos^{-1}(b) + \frac{\pi}{2}) + 1 = 0
\end{aligned}
$$

We have to find roots for this equation, and using Muller's method, we get $b = 0.724$. Thus the best uniform approximation for $f(x) = 0.5 + \sin(x)$ is $g(x) = 0.5 + 0.724x$.

For the $\mathcal{L}_\infty$ norm, we take max over $[-\pi/2, \pi/2]$ of the error function $e(x) = f(x) - g(x) = \sin(x) - 0.724x$,

$$
\begin{aligned}
\frac{de(x)}{dx} &= 0 \\
\cos(x) - 0.724 &= 0 \\
x &= \cos^{-1}(0.724) \\
x &= 0.761
\end{aligned}
$$

Plugging this back into the error term, we get,

$$
\begin{aligned}
\mathcal{L}_\infty &= sin(0.761) - 0.724 * 0.761 \\
&= 0.138
\end{aligned}
$$

Thus, the $\mathcal{L}_\infty$ norm is 0.138

For the $\mathcal{L}_2$ norm, we take,

$$
\begin{aligned}
\mathcal{L}_2 &= \sqrt{\int_{-\pi/2}^{\pi/2} (\sin(x) - 0.724x)^2 dx} \\
&= \sqrt{\int_{-\pi/2}^{\pi/2} \sin^2(x) + 0.524x^2 - 1.448\sin(x) dx} \\
&= \sqrt{0.174x^3 + 0.5x - 1.448\sin(x) - 0.25\sin(2x) + 1.448x\cos(x)\Big|_{-pi/2}^{pi/2}} \\
&= 0.171
\end{aligned}
$$

Thus, the $\mathcal{L}_2$ norm is 0.171.

The plotted function looks like Figure 2

(d) Taking the Legendre polynomials for the least squares approximation, we get,

$$
p(x) = \sum_{i=0}^{2} \frac{< f(x), p_i >}{< p_i, p_i >} p_i(x)
$$

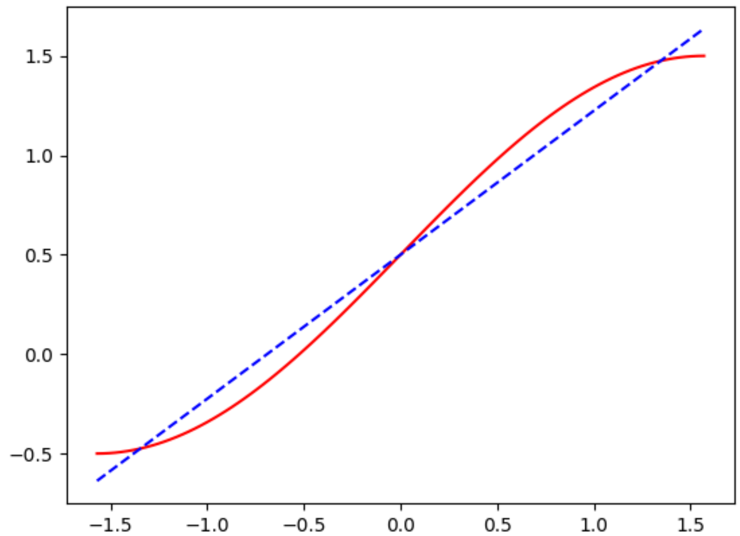

Figure 2: Plot of $f(x)$ with approximated $f(x)$

Calculating the Legendre polynomials between $[\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= [x - \frac{< x, 1 >}{< 1, 1 >}](1) \\
&= x \\
p_2(x) &= [x - \frac{< x^2, x >}{< x, x >}]x - \frac{< x, x >}{< 1, 1 >}(1) \\
&= x^2 - \frac{\pi^2}{12}
\end{aligned}$$

Plugging the polynomials $p_0(x)$, $p_1(x)$ and $p_2(x)$ into the first equation, we get,

$$\begin{aligned}
< 0.5 + \sin(x), p_0(x) > &= \int_{-\pi/2}^{\pi/2} 0.5 + \sin(x) dx = \frac{\pi}{2} \\
< 0.5 + \sin(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} 0.5x + x\sin(x) dx = 2 \\
< 0.5 + \sin(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} (0.5 + \sin(x))(x^2 - \frac{\pi^2}{12}) dx = 0 \\
< p_0(x), p_0(x) > &= \int_{-\pi/2}^{\pi/2} 1 dx = \pi \\
< p_1(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} x^2 dx = \frac{\pi^3}{12} \\
< p_2(x), p_2(x) > &= \int_{-\pi/2}^{\pi/2} (x^2 - \frac{\pi^2}{12})^2 dx = \frac{\pi^5}{180}
\end{aligned}$$

Plugging the above values into the original $p(x)$ equation, we get,

$$
\begin{aligned}
p(x) &= \frac{\pi/2}{\pi} p_0(x) + \frac{2}{\frac{\pi^3}{12}} p_1(x) + 0.p_2(x) \\
p(x) &= 0.5 p_0(x) + 0.774 p_1(x) \\
p(x) &= 0.5 + 0.774x
\end{aligned}
$$

The best least square approximation by a quadratic polynomial is $p(x) = 0.5 + 0.774x$

Consider the error function $e(x) = 0.5 + \sin(x) - 0.5 - 0.774x = \sin(x) - 0.774x$. Now, $\mathcal{L}_2$ error is given by

$$
\begin{aligned}
\mathcal{L}_2 &= \sqrt{\int_{-\pi/2}^{\pi/2} (\sin(x) - 0.774x)^2 dx} \\
&= \sqrt{\int_{-\pi/2}^{\pi/2} \sin^2(x) + 0.559x^2 - 1.548 \sin(x) dx} \\
&= \sqrt{0.199x^3 + 0.5x - 1.548\sin(x) - 0.25\sin(2x) + 1.548x\cos(x) \Big|_{-pi/2}^{pi/2}} \\
&= 0.15074
\end{aligned}
$$

Thus, the $\mathcal{L}_2$ error $= 0.15074$.

For the $\mathcal{L}_\infty$ error, we take max over $[\frac{-\pi}{2}, \frac{\pi}{2}]$ of $e(x)$. For this,

$$
\begin{aligned}
\frac{de(x)}{dx} &= 0 \\
\cos(x) - 0.774 &= 0 \\
x &= \cos^{-1}(0.774) \\
x &= 0.685
\end{aligned}
$$

Plugging this back into the error term, we get,

$$
\begin{aligned}
\mathcal{L}_\infty &= sin(0.685) - 0.774 * 0.685 \\
&= 0.102
\end{aligned}
$$

Hence, the $\mathcal{L}_\infty$ error $= 0.102$

The plotted function looks like Figure 3

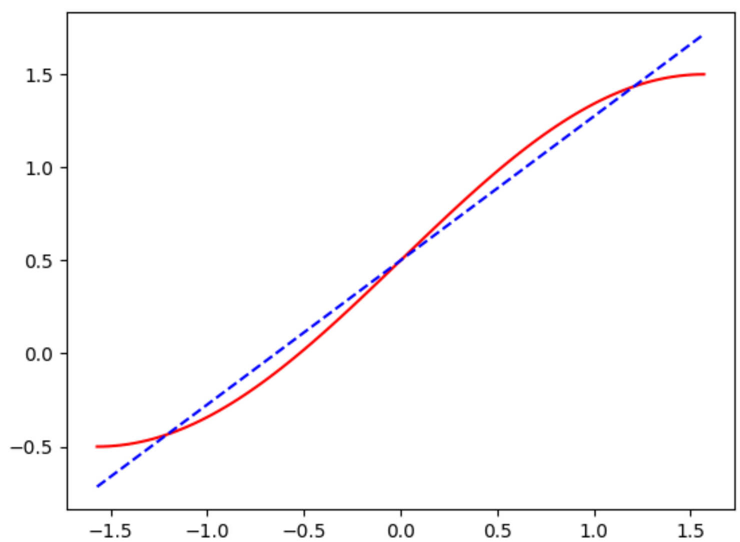

Figure 3: Plot of $f(x)$ with approximated $f(x)$

2 Question 2

Consider the points given by $x = i/100$, $i = 0, ...100$ and the corresponding $f(x_i)$ given by the txt file. Plotting the function, we get Figure 4.

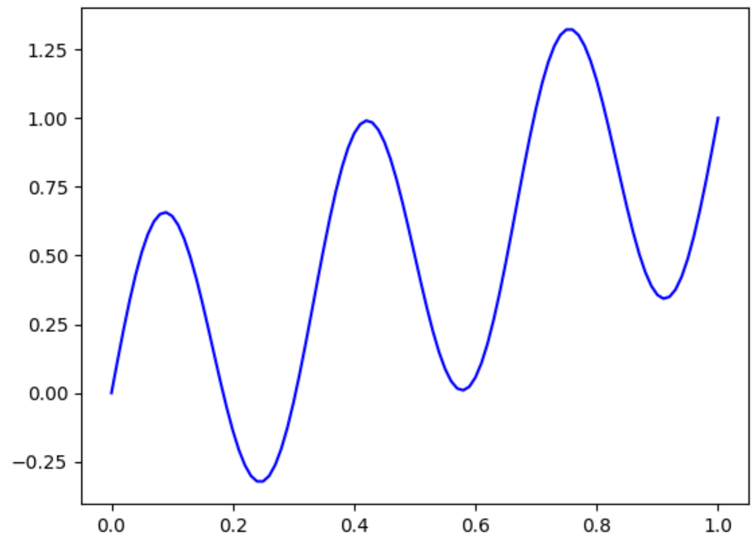

Figure 4: Actual function plotted using given values

Consider using the basis $1, x, x^2, ... \sin(\pi x), \sin(2\pi x)...$ We get the basis vector representation as $\begin{pmatrix} 1 \\ \mathrm{x} \\ \mathrm{x}^2 \\ . \\ . \\ \sin(\pi x) \\ \sin(2\pi x) \\ . \\ . \end{pmatrix}$

and the function $f(x)$ can be represented as $f(x) = c_0 + c_1 x + c_2 x^2 + ... + c_k \sin(\pi x) + c_{k+1} \sin(2\pi x) + ...$ Intuition for choosing only sin basis is that the function has periodicity 6 and the actual function is some representation of the sin function scaled with some polynomial which takes values 0 at $x = 0$ and 1 at $x = 1$.

Now taking the ϕ_i vectors (size 101x1) corresponding to the i^{th} basis vector, and we form the A matrix, where

$$A_i = \phi_i$$

and the q vector given by,

$$q_i = f_i, i = 1, 2, ...101$$

Now solve for

$$Ac = q$$

where c is the coefficient vector. Since if we keep adding more and more terms, we will keep getting more and more errors, we threshold the error to ensure we can get a sufficiently good reconstruction of the function with lesser number of terms.

The reconstructed function if given by $g_i = \phi_i c_i$ and to compute the error we take the L_2 norm of $e = f - g$. To ensure lesser number of terms, we take the following steps:

1. Check for minimum error for a given number of linear and trignometric basis.

2. Then check for the given minimum error, if the number of non-zero coefficients in the least.
3. Any function meeting these two conditions is the best approximation of the given function.

Following this, the reconstructed function is given below,

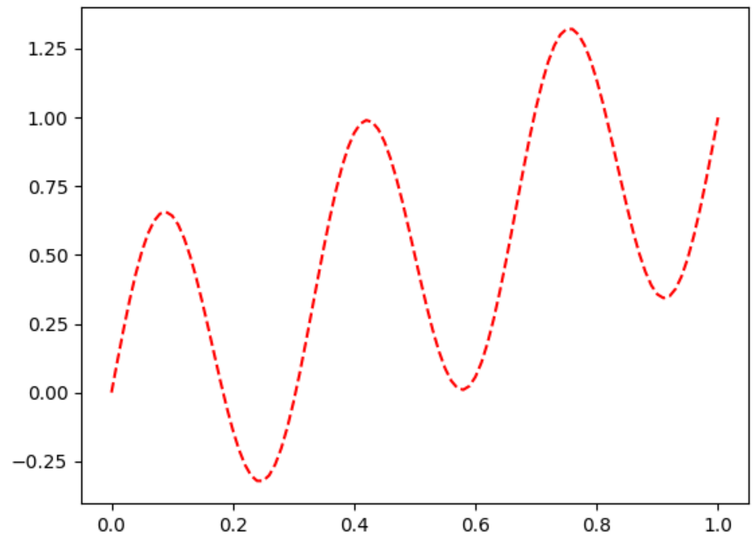

Figure 5: Fitted function f(x)

Plotting the reconstructed function on top of the actual function, we get,

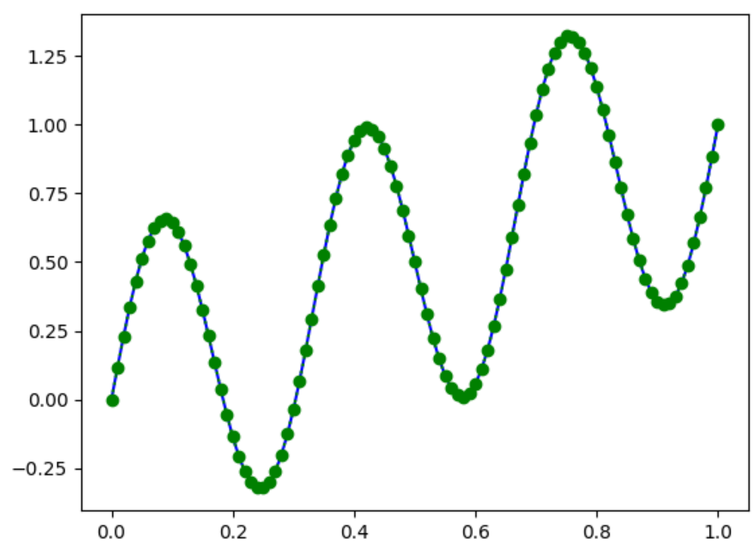

Figure 6: Dotted green points are the fitted function and blue line is the actual function

The actual fitted function is given by $g(x) = x + 0.571 \sin(6\pi x)$ with a reconstruction error of 1.07e-14. To run the code,

```
python q2.py (Assumes there is a data folder which has problem2.txt in it)
```

3 Question 3

The Chebyshev polynomials are defined as $T_k(\cos(\theta)) = \cos(k\theta)$. Now, consider the following

$$\begin{aligned}
x = \cos(\theta) &\implies \theta = \cos^{-1}(x) \\
T_0(x) &= 1 \\
T_1(x) &= cos(1 * \cos^{-1}(x)) = x \\
T_{i+1}(x) &= \cos((i+1)\theta) = \cos(i\theta)\cos(\theta) - \sin(i\theta)\sin(\theta) \\
T_{i-1}(x) &= \cos((i-1)\theta) = \cos(i\theta)\cos(\theta) + \sin(i\theta)\sin(\theta) \\
\implies T_{i+1}(x) &= 2\cos(i\theta)\cos(\theta) - T_{i-1}(x) \\
T_{i+1}(x) &= 2xT_i(x) - T_{i-1}(x)
\end{aligned}$$

Thus the polynomials have the recurrence relation given by $T_{k+1}(\theta) = 2xT_k(\theta) - T_{k-1}(x)$, with $T_0(x) = 1$ and $T_1(x) = x$.

(a) Using the relation, we get

$$\begin{aligned}
T_2(x) &= 2xT_1(x) - T_0(x) \\
&= 2x^2 - 1 \\
T_3(x) &= 2xT_3(x) - T_1(x) \\
&= 2x(2x^2 - 1) - x \\
&= 4x^3 - 3x \\
T_4(x) &= 2xT_3(x) - T_2(x) \\
&= 2x(4x^3 - 3x) - (2x^2 - 1) \\
&= 8x^4 - 8x^2 + 1 \\
T_5(x) &= 2xT_4(x) - T_3(x) \\
&= 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) \\
&= 16x^5 - 20x^3 + 5x \\
T_6(x) &= 2xT_5(x) - T_4(x) \\
&= 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) \\
&= 32x^6 - 48x^4 + 18x^2 - 1 \\
T_7(x) &= 2xT_6(x) - T_5(x) \\
&= 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x) \\
&= 64x^7 - 112x^5 + 56x^3 - 7x
\end{aligned}$$

Thus, $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$ and $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$.

Taking the inner product with $g(x) = T_6(x)$ and $h(x) = T_7(x)$ defined as

$$\begin{aligned}
< g, h > &= \int_{-1}^{1} (1 - x^2)^{-1/2} g(x)h(x)dx \\
&= \int_{-1}^{1} (1 - x^2)^{-1/2}(32x^6 - 48x^4 + 18x^2 - 1)(64x^7 - 112x^5 + 56x^3 - 7x)dx \\
&= \int_{-1}^{1} \frac{(32x^6 - 48x^4 + 18x^2 - 1)(64x^7 - 112x^5 + 56x^3 - 7x)}{\sqrt{1 - x^2}} dx
\end{aligned}$$

For solving the integral, take $x = \sin(u)$ and $dx = \cos(u)du$ and substitute in the above equation,

$$
\begin{aligned}
< g,h > &= \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} \frac{(32\sin^6(u) - 48\sin^4(u) + 18\sin^2(u) - 1)(64\sin^7(u) - 112\sin^5(u) + 56\sin^3(u) - 7\sin(u))}{\sqrt{1-\sin^2(u)}} \cos(u)du \\
&= \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} (32\sin^6(u) - 48\sin^4(u) + 18\sin^2(u) - 1)(64\sin^7(u) - 112\sin^5(u) + 56\sin^3(u) - 7\sin(u))du \\
&= \int_{\sin^{-1}(-1)}^{\sin^{-1}(1)} (2048\sin^{13}(u) - 6656\sin^{11}(u) + 8320\sin^9(u) - 4992\sin^7(u) + 1456\sin^5(u) - 182\sin^3(u) + 7\sin(u))du
\end{aligned}
$$

All terms in the above polynomial are odd and hence their integral would be all even terms which will cancel each other out and hence the integral value would be 0. Now, since the integral value is 0, the polynomials are orthogonal to each other.

(b) Take the n^{th} order Chebyshev polynomial, and relative to the defined inner product in part (a), we get

$$
\begin{aligned}
< T_n(x), T_n(x) > &= \int_{-1}^{1} (1-x^2)^{-1/2} T_n(x) T_n(x) dx \\
&= \int_{\cos^{-1}(-1)}^{\cos^{-1}(1)} (1-\cos^2\theta)^{-1/2} T_n(\cos\theta) T_n(\cos\theta)(-\sin\theta d\theta) \\
&= -\int_{\cos^{-1}(-1)}^{\cos^{-1}(1)} \cos(n\theta)\cos(n\theta)d\theta \\
&= -\int_{\cos^{-1}(-1)}^{\cos^{-1}(1)} \cos^2(n\theta)d\theta \\
&= -\left(\frac{2n\theta + \sin(2n\theta)}{4n} + c\right)\Bigg|_{\cos^{-1}(-1)}^{\cos^{-1}(1)} \\
&= -\frac{1}{4n}\left(2n\cos^{-1}(1) - 2n\cos^{-1}(-1) + \sin(2n\cos^{-1}(1)) - \sin(2n\cos^{-1}(-1))\right)
\end{aligned}
$$

Using $\cos^{-1}(-1) = (2k+1)\pi$ and $\cos^{-1}(1) = 2k\pi$, where $k = 0, 1, ..., \infty$, we get,

$$
\begin{aligned}
&= -\frac{1}{4n}\left(2n(2k\pi) - 2n(2k+1)\pi + \sin(4nk\pi) - \sin(2n(2k+1)\pi)\right) \\
&= -\frac{1}{4n}\left(-2n\pi + 0 - 0\right) \\
&= \frac{\pi}{2}
\end{aligned}
$$

For any integer value of n and k, $\sin(4nk\pi) and \sin(4nk\pi + 2n\pi)$ are always equal to 0. Thus, the length of the polynomial is given by,

$$
\begin{aligned}
\mid T_n(x) \mid &= \sqrt{< T_n(x), T_n(x) >} \\
&= \sqrt{\frac{\pi}{2}} \\
&= 1.2533
\end{aligned}
$$

Thus, the length of $T_n(x) = 1.2533$.

4 Question 4

The equation of a normal to the plane $n = [A, B, C]$ and a point $p = (x_0, y_0, z_0)$ on the plane is given by $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, which can be written as $Ax + By + Cz + D = 0$, where $D = -Ax_0 - By_0 - Cz_0$. The distance of any point $P = (x_1, y_1, z_1)$ to the plane is given by

$$
\begin{aligned}
v &= (x_1 - x_0, y_1 - y_0, z_1 - z_0) \\
n' &= n/||n|| \\
d &= |v.n'|
\end{aligned}
$$

(a) The plane equation is given by $0.095X + 0.994Y + 0.0484Z + 0.492 = 0$. The average distance to the plane is 0.00273 and the plot of the plane is shown in Figure 7

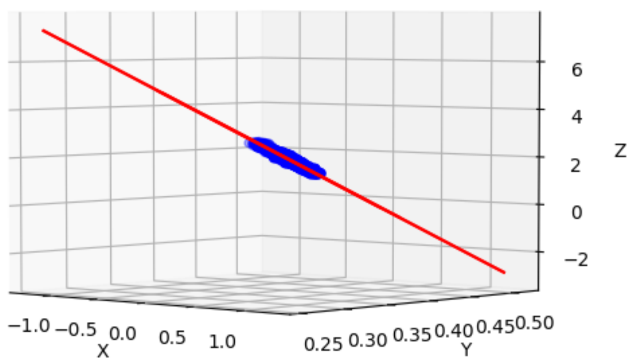

Figure 7: Plane fitting through clear-table

To run the code for this system

```
python q4.py '../data/clear-table.txt'
```

(b) The plane equation is given by $0.100X + 0.990Y + +0.0947Z + 0.565 = 0$. The average distance to the plane is 0.0219 and the plot of the plane is shown in Figure 8.

The plane does not fit well to the data points as there are many spurious points (outlier points that cause the normal of the surface to shift direction). To run the code for this system

```
python q4.py '../data/cluttered-table.txt'
```

(c) The equation for the dominant plane is given by $0.101X + 0.9937Y + 0.0491Z + 0.4938 = 0$. The average distance to the dominant plane is 0.0151 and the plot of the plane is shown in Figure 9

The dominant plane is found by using RANSAC. We compute the best inliers in the system by computing the set of points whose absolute distance to the plane meets a certain threshold and we declare them as inliers. We iterate this for 2000 iterations with a threshold of 0.003 to account for a fit of data atleast as good as part (a), where average distance to the plane was 0.002. Despite finding the dominant plane, the average distance to the system does not decrease by much as there are points that are located far away from the calculated plane and thus the average distance from the plane is still high. To run the code for this system

```
python q4c.py
```

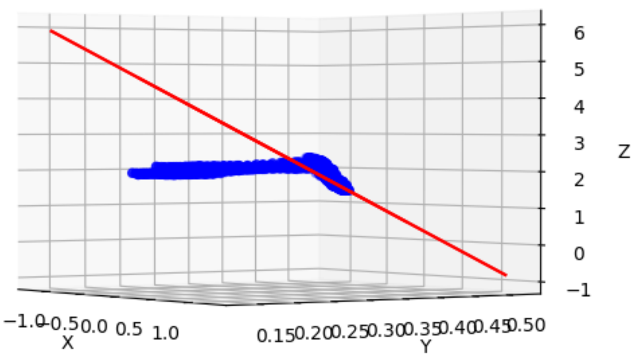

Figure 8: Plane fitting through cluttered-table

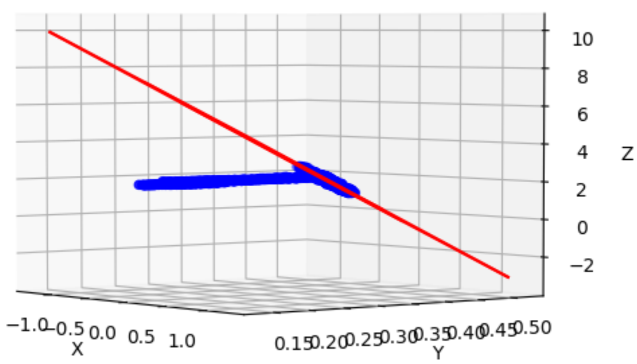

Figure 9: Dominant plane fitting through cluttered-table

(d) We extend the RANSAC code here from part(c). For that, we run RANSAC for the first time and then remove the inlier points from the set of total points and repeat the process 3 times. We set 500 iterations and a threshold of 0.01 for the data distance metric. The planes are visualized in Figure 10.

The equations of the planes are given by

```
-0.1692X+0.9849Y+0.03689Z+0.2086 = 0 [red plane]
0.1706X-0.9846Y-0.03791Z-1.8096 = 0 [green plane]
-0.9646X-0.1709Y+0.2010Z+-0.2158 = 0 [black plane]
-0.9631X-0.1733Y+0.2061Z-1.8103 = 0 [yellow plane]
```

Since the hallway points are very clean, we do not have to force the RANSAC to run for more than 500 iterations,

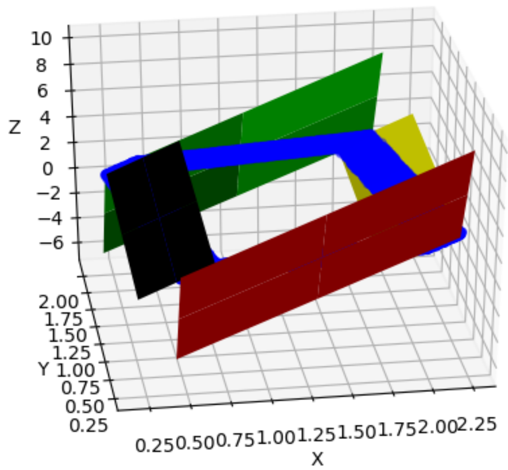

Figure 10: Planes fitting Clean Hallway

as it converges very quickly to the set of inliers for a given plane. To run the code for this system

```
python q4d.py '../data/clean_hallway.txt' 500 500 500 500 0.01 0.01 0.01 0.01
```

where each of the numbers represent the number of iterations to run the corresponding RANSAC system for.

(e) For the smoothness of a surface, we consider the root mean squared distance of the set of points that the surface defines. Here, the distance of the function is d (defined above) and the RMS distance for n points is given by

$$RMSDist = \sqrt{\frac{\sum_{i=1}^{n} d^2}{n}}$$

For the 4 planes that we form, the plane with the least root mean squared distance has the best smoothness. This is because when the average distance of the points from the plane they define is the least, the surface has lesser rises and falls and is better centered around the points, making it possible for the cat to traverse it more easily.

We extend the RANSAC code here from part(c). For that, we run RANSAC for the first time and then remove the inlier points from the set of total points and repeat the process 3 times. We set different number of iterations to force the RANSAC algorithm to converge more number of inliers, with a threshold value of 0.01 for the first two runs and 0.1 for the next two runs of the RANSAC. The planes are visualized in Figure 10.

The equations of the planes and the corresponding smoothness is given by

```
[red plane] Eq: -0.0116X-0.995Y+0.0981Z-0.591 = 0, Smoothness: 0.00412
[green plane]  Eq: 0.0171X+0.996Y-0.088Z-0.523 = 0, Smoothness: 0.0044
[black plane] Eq: -0.995X-0.0347Y-0.0984Z+0.9438 = 0, Smoothness: 0.05485
[yellow plane] Eq: 0.998X+0.00573Y-0.060Z+0.792 = 0, Smoothness: 0.05238
```

Thus the cat should climb along the red plane and the smoothness of the plane is 0.00412.

To run the code for this system

```
python q4d.py '../data/cluttered_hallway.txt' 500 1000 1500 2500 0.01 0.01 0.1 0.1
```

where each of the numbers represent the number of iterations to run the corresponding RANSAC system for.

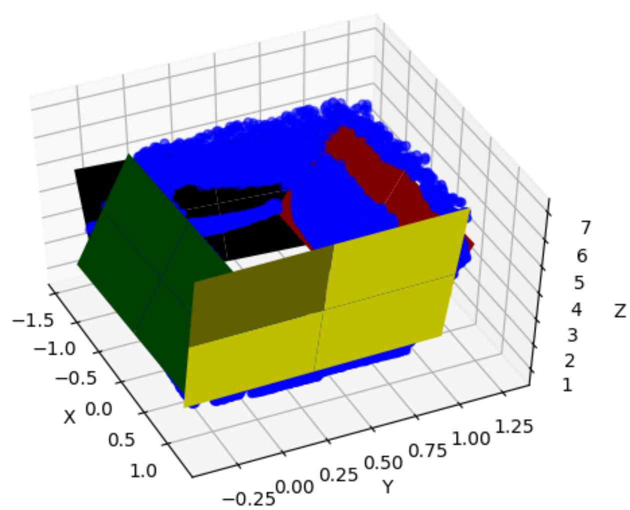

Figure 11: Planes fitting Cluttered Hallway