Homework Assignment 5

Abhay Gupta (Andrew Id: abhayg)

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1 Question 2

Consider the spherical coordinates given by

$$
\begin{aligned}
x &= R\sin v\cos u \\
y &= R\sin v\sin u \\
z &= R\cos v
\end{aligned}
$$

Now consider the differentials given by dx, dy, dz, we get,

$$
\begin{aligned}
dx &= R\cos v\cos u dv - R\sin v\sin u du \\
dy &= R\cos v\sin u dv + R\sin v\cos u du \\
dz &= -R\sin v dv
\end{aligned}
$$

Now computing the value of ds, we get,

$$
\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 \\
&= (R\cos v\cos u dv - R\sin v\sin u du)^2 + (R\cos v\sin u dv + R\sin v\cos u du)^2 + (-R\sin v dv)^2 + (-R\sin v dv)^2 \\
&= R^2(dv)^2 + R^2\sin^2 v(du)^2 \\
&= R^2(dv)^2(1 + \sin^2 v(\frac{du}{dv})^2)
\end{aligned}
$$

Writing $du/dv = u'$, we get, the length of the shortest curve as,

$$
\begin{aligned}
ds &= R\sqrt{1 + \sin^2 v(u')^2} dv \\
I &= \int ds = R\int_{v_A}^{v_B} \sqrt{1 + \sin^2 v(u')^2} dv
\end{aligned}
$$

Thus $F(v, u, u') = \sqrt{1 + \sin^2 v(u')^2}$. Taking the differential and setting it to zero, we get,

$$
\begin{aligned}
\frac{\partial F}{\partial u} &= \frac{\sin^2 v u'}{\sqrt{1 + \sin^2 v(u')^2}} \\
\implies \frac{\sin^2 v u'}{\sqrt{1 + \sin^2 v(u')^2}} &= 0
\end{aligned}
$$

Taking the first integral, we get,

$$
\begin{aligned}
\frac{\sin^2 v u'}{\sqrt{1 + \sin^2 v(u')^2}} &= C_0 \\
u' &= \frac{C_0}{\sqrt{\sin^4 v - C_0^2\sin^2 v}}
\end{aligned}
$$

Thus, integrating, we get,

$$
\begin{aligned}
u &= \int u' \\
&= \int \frac{C_0}{\sqrt{\sin^4 v - C_0^2 \sin^2 v}} dv \\
&= -\sin^{-1}\left(\frac{C_0 \cot v}{\sqrt{1-C_0^2}}\right) + k \\
u - k &= -\sin^{-1}\left(\frac{C_0 \cot v}{\sqrt{1-C_0^2}}\right) \\
\sin(k-u) &= \frac{C_0 \cot v}{\sqrt{1-C_0^2}}
\end{aligned}
$$

Now rearranging the equations, we get,

$$
\begin{aligned}
\sin k \cos u - \cos k \sin u - \frac{C_0 \cot v}{\sqrt{1-C_0^2}} &= 0 \\
\sin k \cos u \sin v - \cos k \sin u \sin v - \frac{C_0 \cos v}{\sqrt{1-C_0^2}} &= 0 \\
R \sin k \cos u \sin v - R \cos k \sin u \sin v - R\frac{C_0 \cos v}{\sqrt{1-C_0^2}} &= 0 \\
x \sin k - y \cos k - \frac{C_0 z}{\sqrt{1-C_0^2}} &= 0
\end{aligned}
$$

A great circle is formed with a sphere of radius R, if a plane which is at a distance less than R from the center of the sphere intersects the circle. The above equation derived is that of a plane passing through origin, so that the minimizing curve which lies on the sphere is obtained by the intersection of the sphere and the plane passing through the origin, which is the definition of an arc of a great-circle.