Homework 1 Resubmission

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Question 3

To determine the number of solutions, we have to see if b is in the column space of the A matrix. I have created an augmented matrix $B = [A|b]$.

- If the rank of B is the same as A, and both have rank equal to the span of the columns of A, then there is an unique solution of the form $\bar{x}$, where $\bar{x}$ is the SVD solution.
- If the rank of B is the same as A, but both have rank less than the span of the column space of A, then there are infinitely many solutions of the form $\bar{x} + x_N$, where $\bar{x}$ is the SVD solution.
- If the rank of B is greater than A, then there are 0 actual solutions and has a least sqaures solution - which is given by the SVD solution $\bar{x}$ such that $||A\bar{x} - b||$ is minimized.

Intuition behind forming the augmented matrix B - If the vector b belongs to the column space of A, then rank of B should be the same as A as b can be formed by a linear combination of the vectors that span the column space of A. If b does not belong to the column space, then rank of B has to be greater than rank of A by atleast 1.

(a) $A = \begin{pmatrix} 2 & 2 & 5 \\ 1 & 1 & 5 \\ 3 & 2 & 5 \end{pmatrix}$; $b = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$

Rank(A) = 3, Rank(B) = 3 and det(A) $\neq 0$. Matrix is square and invertible and system has a unique solution. Solution is $x = \begin{pmatrix} -5 & 15 & -3 \end{pmatrix}^T$

(b) $A = \begin{pmatrix} -3 & -4 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{pmatrix}$; $b = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

Rank(A) = 2, Rank(B) = 3 and det(A) = 0. Matrix is not invertible, b is not in column space of A and system has zero solutions. SVD Solution is $\bar{x} = \begin{pmatrix} -2.55 & 0.73 & 3.27 \end{pmatrix}^T + \lambda \begin{pmatrix} -0.57 & 0.57 & -0.57 \end{pmatrix}^T$.

Here $\lambda \begin{pmatrix} -0.57 & 0.57 & -0.57 \end{pmatrix}^T$ is a representation of the null space of A (space spanned by last column of V when we take SVD).

(c) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{pmatrix}$; $b = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

Rank(A) = 2, Rank(B) = 2 and det(A) = 0. Matrix is not invertible, but b is in column space of A and system has infinitely many solutions. SVD Solution is $\bar{x} = \begin{pmatrix} 1.45 & -0.55 & -1.77 \end{pmatrix}^T + \lambda \begin{pmatrix} -0.66 & -0.66 & -0.33 \end{pmatrix}^T$.

Here $\lambda \begin{pmatrix} -0.66 & -0.66 & -0.33 \end{pmatrix}^T$ is a representation of the null space of A (space spanned by last column of V when we take SVD).

For part (b) and (c), all solutions are of the form $\bar{x} + x_N$, where x_N is a column vector which belongs to the null space of A (space spanned by the last column of V (rank = 2 for both cases) when we take SVD).

Code is included, q3.m and a test code q3Test.m. xbar is the actual solution (or SVD solution for infinite or zero solution case) and result is the number of solutions. Here is a mock run of the code - on the command prompt

```
% run 1
A = [2 2 5; 1 1 5; 3 2 5];
b = [5 -5 0]';
q3Test(A,b)
```