Homework Assignment 5

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1 Question 1

Consider a plane curve $y(x)$ which is revolved around the x-axis. The constraints on the curve are that for $x_0 \leq x \leq x_1$, we have $y > 0$. When revolving such a curve around x-axis, we want a C^2 surface such that to get minimal surface area A, we want a surface passing through two circular wireframes. Thus,

$$\begin{aligned} dA &= 2\pi y ds \\ &= 2\pi y\sqrt{1+y'^2}dx \end{aligned}$$

Thus the surface area is,

$$\begin{aligned} A &= \int dA \\ &= 2\pi \int y\sqrt{1+y'^2}dx \end{aligned}$$

and the quantity that we are minimizing is $f(y) = y\sqrt{1+y'^2}$

Consider the Euler-Lagrange differential equation - since $f_x = 0$, we have

$$\begin{aligned} f - y'\frac{df}{dy'} &= C_0 \\ y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1+y'^2}} &= C_0 \\ y(1+y'^2) - yy'^2 &= C_0\sqrt{1+y'^2} \\ y &= C_0\sqrt{1+y'^2} \\ y' &= \frac{\sqrt{y^2-C_0^2}}{C_0} \\ \frac{dy}{dx} &= \frac{\sqrt{y^2-C_0^2}}{C_0} \\ dx &= \frac{C_0}{\sqrt{y^2-C_0^2}}dy \\ x &= C_0\int\frac{1}{\sqrt{y^2-C_0^2}}dy \\ x &= C_0\cosh^{-1}\left(\frac{y}{C_0}\right) + C_1 \\ y &= C_0\cosh\left(\frac{x-C_1}{C_0}\right) \end{aligned}$$

Now, we have the boundary conditions, given by $y_0 = y(x_0)$ and $y_1 = y(x_1)$, thus we get the values of C_0 and C_1 from the following implicit equations,

$$\begin{aligned} y_0 &= C_0\cosh\left(\frac{x_0-C_1}{C_0}\right) \\ y_1 &= C_0\cosh\left(\frac{x_1-C_1}{C_0}\right) \end{aligned}$$

Depending on the values of (x_0, y_0) and (x_1, y_1), we may or may not get values for C_0 and C_1. When solutions do exist, then it forms a C^2 curve. When there exist no solutions for C_0 and C_1, the physical interpretation for this is that surface breaks and forms circular disks in each ring to minimize area. However, calculus of variations cannot be used to obtain such solutions.

2 Question 2

Consider the polar co-ordinates, given by

$$\begin{aligned} x &= \rho \cos u \\ y &= \rho \sin u \\ z &= z \end{aligned}$$

Now consider the differentials, given by,

$$\begin{aligned} dx &= d\rho \cos u - \rho \sin u du \\ dy &= d\rho \sin u + \rho \cos u du \end{aligned}$$

Thus, the arc length is given by

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= (d\rho \cos u - \rho \sin u du)^2 + (d\rho \sin u + \rho \cos u du)^2 + dz^2 \\ &= d\rho^2(\cos^2 u + \sin^2 u) + \rho^2 du^2(\cos^2 u + \sin^2 u) + dz^2 \\ &= d\rho^2 + \rho^2 du^2 + dz^2 \end{aligned}$$

Now converting the polar coordinates to spherical coordinates, we get, $\rho = r \sin v$ and $z = r \cos v$. Thus, $d\rho = dr \sin v + r \cos v dv$ and $dz = dr \cos v + r \sin v dv$. Since the radius here is fixed, $r = R$, we get $dr = 0$. Thus,

$$\begin{aligned} d\rho &= R \cos v dv \\ dz &= R \sin v dv \end{aligned}$$

Substituting in the above equation, we get,

$$\begin{aligned} ds^2 &= d\rho^2 + \rho^2 du^2 + dz^2 \\ &= (R \cos v dv)^2 + (R \sin v)^2 du^2 + (R \sin v dv)^2 \\ &= R^2 dv^2(\cos^2 v + \sin^2 v) + R^2 \sin^2 v du^2 \\ &= R^2 dv^2 + R^2 \sin^2 v du^2 \end{aligned}$$

Writing $du = u'$, we get, the length of the shortest curve as,

$$I = \int ds = R \int_{v_A}^{v_B} \sqrt{1 + \sin^2 v (u')^2} dv$$

Thus $F(v, u, u') = \sqrt{1 + \sin^2 v (u')^2}$. Taking the differential and setting it to zero, we get,

$$\begin{aligned} \frac{\partial F}{\partial u} &= \frac{\sin^2 v u'}{\sqrt{1 + \sin^2 v (u')^2}} \\ \implies \frac{\sin^2 v u'}{\sqrt{1 + \sin^2 v (u')^2}} &= 0 \end{aligned}$$

Taking the first integral, we get,

$$\frac{\sin^2 v u'}{\sqrt{1+\sin^2 v (u')^2}} = C_0$$
$$u' = \pm\left(\frac{C_0}{\sqrt{\sin^4 v - C_0^2 \sin^2 v}}\right)$$

Thus, integrating, we get,

$$\begin{aligned}
u &= \int u' \\
&= \int \frac{C_0}{\sqrt{\sin^4 v - C_0^2 \sin^2 v}} dv \\
&= -\sin^{-1}\left(\frac{C_0 \cot v}{\sqrt{1-C_0^2}}\right) + k \\
u - k &= -\sin^{-1}\left(\frac{C_0 \cot v}{\sqrt{1-C_0^2}}\right) \\
\sin(k-u) &= \frac{C_0 \cot v}{\sqrt{1-C_0^2}}
\end{aligned}$$

Now rearranging the equations, we get,

$$\begin{aligned}
\sin k \cos u - \cos k \sin u - \frac{C_0 \cot v}{\sqrt{1-C_0^2}} &= 0 \\
\sin k \cos u \sin v - \cos k \sin u \sin v - \frac{C_0 \cos v}{\sqrt{1-C_0^2}} &= 0 \\
R \sin k \cos u \sin v - R \cos k \sin u \sin v - R\frac{C_0 \cos v}{\sqrt{1-C_0^2}} &= 0 \\
x \sin k - y \cos k - \frac{C_0 z}{\sqrt{1-C_0^2}} &= 0
\end{aligned}$$

This is the equation of a plane passing through origin, so that the minimizing curve which lies on the sphere is obtained by the intersection of the sphere and the plane passing through the origin, which is the definition of an arc of a great-circle.

3 Question 3

Given that at the right endpoint (x_1, y_1),

$$F_{y'} - \frac{g_y F}{g_x + g_y y'} = 0$$

Here,

$$\begin{aligned}
F &= \frac{\sqrt{1+y'^2}}{\sqrt{-2g}\sqrt{y_0 - y}} \\
\implies F_{y'} &= \frac{y'}{\sqrt{-2g}\sqrt{(y_0-y)(1+y'^2)}}
\end{aligned}$$

Thus, taking the right endpoint,

$$\begin{aligned}
\frac{y'}{\sqrt{-2g}\sqrt{(y_0-y)(1+y'^2)}} - \frac{g_y}{g_x+g_y y'}\frac{\sqrt{1+y'^2}}{\sqrt{-2g}\sqrt{y_0-y}} &= 0 \\
y' - \frac{g_y}{g_x+g_y y'}(1+y'^2) &= 0 \\
y' g_x + g_y y'^2 - g_y - g_y y'^2 &= 0 \\
y' &= \frac{g_y}{g_x}
\end{aligned}$$

So at (x_1, y_1), the slope of y is in the same direction as ∇g, that is, y intersects $g(x,y)=0$ orthogonally.

4 Question 4

(a) Consider the manipulator given in the question. For mass m_1, we have

$$\begin{aligned}
v_1 &= l_1\dot{\theta_1} \\
T_1 &= \frac{1}{2}m_1 l_1^2 \dot{\theta_1}^2
\end{aligned}$$

For the two masses m_2, consider the upper mass as m_{21} and the lower mass as m_{22}.

$$\begin{aligned}
x_{21} &= l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\
y_{21} &= l_1\sin\theta_1 + l_2\sin(\theta_1+\theta_2) \\
\dot{x_{21}} &= -l_1\sin\theta_1\dot{\theta_1} - l_2\sin(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \\
\dot{y_{21}} &= l_1\cos\theta_1\dot{\theta_1} + l_2\cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \\
v_{21}^2 &= \dot{x_{21}}^2 + \dot{y_{21}}^2 \\
&= l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2) \\
T_{21} &= \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2))
\end{aligned}$$

$$\begin{aligned}
x_{22} &= l_1\cos\theta_1 - l_2\cos(\theta_1+\theta_2) \\
y_{22} &= l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) \\
\dot{x_{22}} &= -l_1\sin\theta_1\dot{\theta_1} + l_2\sin(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \\
\dot{y_{22}} &= l_1\cos\theta_1\dot{\theta_1} - l_2\cos(\theta_1+\theta_2)(\dot{\theta_1}+\dot{\theta_2}) \\
v_{21}^2 &= \dot{x_{21}}^2 + \dot{y_{21}}^2 \\
&= l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 - 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2) \\
T_{22} &= \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 - 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2))
\end{aligned}$$

For the entire system, since gravity is zero, the potential energy (V) is 0. Thus, the Lagrangian is given by,

$$\begin{aligned}
L &= T - V \\
&= T_1 + T_{21} + T_{22} \\
&= \frac{1}{2}m_1l_1^2\dot{\theta_1}^2 + \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 + 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2)) + \frac{1}{2}m_2(l_1^2\dot{\theta_1}^2 + l_2^2(\dot{\theta_1}+\dot{\theta_2})^2 - 2l_1l_2\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})cos(\theta_2)) \\
&= \frac{1}{2}m_1l_1^2\dot{\theta_1}^2 + m_2l_1^2\dot{\theta_1}^2 + m_2l_2^2(\dot{\theta_1}+\dot{\theta_2})^2
\end{aligned}$$

To get the torques, we take

$$\tau_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

For θ_1, we have,

$$\begin{aligned}
\frac{\partial L}{\partial \theta_1} &= 0 \\
\frac{\partial L}{\partial \dot{\theta}_1} &= m_1 l_1^2 \dot{\theta}_1 + 2m_2 l_1^2 \dot{\theta}_1 + 2m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= m_1 l_1^2 \ddot{\theta}_1 + 2m_2 l_1^2 \ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_2 \\
\tau_1 &= m_1 l_1^2 \ddot{\theta}_1 + 2m_2 l_1^2 \ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_1 + 2m_2 l_2^2 \ddot{\theta}_2
\end{aligned}$$

For θ_2, we have,

$$\begin{aligned}
\frac{\partial L}{\partial \theta_2} &= 0 \\
\frac{\partial L}{\partial \dot{\theta}_2} &= 2m_2 l_2^2 (\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= 2m_2 l_2^2 \ddot{\theta}_2 + 2m_2 l_2^2 \ddot{\theta}_1 \\
\tau_2 &= 2m_2 l_2^2 \ddot{\theta}_2 + 2m_2 l_2^2 \ddot{\theta}_1
\end{aligned}$$

Thus, writing in matrix form, we get

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + 2m_2 l_1^2 + 2m_2 l_2^2 & 2m_2 l_2^2 \\ 2m_2 l_2^2 & 2m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

(b) When $\ddot{\theta}_2 = 0$, we get,

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} m_1 l_1^2 + 2m_2 l_1^2 + 2m_2 l_2^2 & 2m_2 l_2^2 \\ 2m_2 l_2^2 & 2m_2 l_2^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ 0 \end{bmatrix}$$

For τ_1, considering $\ddot{\theta}_1$ terms, we get,

$$\begin{aligned}
\frac{\partial \tau_1}{\partial \ddot{\theta}_1} &= m_1 l_1^2 + 2m_2 l_1^2 + 2m_2 l_2^2 \\
&= m_1 l_1^2 + 2m_2 (l_1^2 + l_2^2)
\end{aligned}$$

This is just the inertial term for the $\ddot{\theta}_1$ component.