Homework 1

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Question 1

Code included. There are two files

- q1.m - takes in a square matrix A and then returns its LDU decomposition. It assumes that when the k^{th} diagonal element of D is zero, the k^{th} diagonal element of U is 1.
- q1Test.m - takes in a square matrix A and prints the components PA and LDU to show the validity of the code.

Here is a mock run of the code - on the command prompt

```
% run 1
A = [2 2 5; 1 1 5; 3 2 5];
q1Test(A)
```

Question 2

(a) $A_1 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \\ 1 & 0 & -1/2 \end{pmatrix}$

$$\text{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1/2 & 1 \end{pmatrix}; \text{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \text{U} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1/2 \\ 0 & 0 & 1 \end{pmatrix}; \text{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{U} = \begin{pmatrix} 0.4472 & 0.5963 & -0.6667 \\ -0.8944 & 0.2981 & -0.3333 \\ 0 & 0.7454 & 0.6667 \end{pmatrix}; \Sigma = \begin{pmatrix} 2.4495 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \text{V} = \begin{pmatrix} 0.1826 & 0.8944 & 0.4082 \\ -0.9129 & 0 & 0.4082 \\ 0.3651 & -0.4472 & 0.8165 \end{pmatrix};$$

(b) $A_2 = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 1 \end{pmatrix}$

$$\text{L} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0.25 & 0.125 & 1 \\ 1 & 0.25 & -0.375 & 0.4286 \end{pmatrix}; \text{D} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0.8750 \end{pmatrix}; \text{U} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -0.25 & 0 \\ 0 & 0 & 1 & -0.5 \\ 0 & 0 & 0 & 1 \end{pmatrix};$$

$$
\mathrm{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathrm{U} = \begin{pmatrix} -0.2273 & 0.1886 & 0.6002 & 0.3657 & -0.6472 \\ 0.0556 & 0.4309 & 0.6230 & -0.0650 & 0.6472 \\ 0.9380 & -0.0273 & 0.0653 & 0.3264 & -0.0925 \\ 0.2236 & -0.2663 & 0.3671 & -0.8170 & -0.2774 \\ -0.1238 & -0.8409 & 0.3356 & 0.2968 & 0.2774 \end{pmatrix}; \Sigma = \begin{pmatrix} 4.3870 & 0 & 0 & 0 \\ 0 & 2.5053 & 0 & 0 \\ 0 & 0 & 1.411 & 0 \\ 0 & 0 & 0 & 0.6975 \\ 0 & 0 & 0 & 0 \end{pmatrix};
$$

$$
\mathrm{V} = \begin{pmatrix} 0.0391 & -0.2473 & -0.8669 & -0.4311 \\ -0.9581 & 0.2252 & -0.0200 & -0.1760 \\ 0.2829 & 0.8324 & 0.0122 & -0.4764 \\ 0.0227 & -0.4419 & 0.4979 & -0.7458 \end{pmatrix};
$$

(c) $A_3 = \begin{pmatrix} 2 & 2 & 5 \\ 1 & 1 & 5 \\ 3 & 2 & 5 \end{pmatrix}$

$$
\mathrm{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0.6667 & 1 & 0 \\ 0.3333 & 0.5 & 1 \end{pmatrix}; \mathrm{D} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0.6667 & 0 \\ 0 & 0 & 2.5 \end{pmatrix}; \mathrm{U} = \begin{pmatrix} 1 & 0.6667 & 1.6667 \\ 0 & 1 & 2.5 \\ 0 & 0 & 1 \end{pmatrix}; \mathrm{P} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
$$

$$
\mathrm{U} = \begin{pmatrix} -0.5859 & 0.0444 & -0.8091 \\ -0.5182 & -0.7882 & 0.3319 \\ -0.6231 & 0.6138 & 0.4849 \end{pmatrix}; \Sigma = \begin{pmatrix} 9.7910 & 0 & 0 \\ 0 & 1.4162 & 0 \\ 0 & 0 & 0.3606 \end{pmatrix}; \mathrm{V} = \begin{pmatrix} -0.3635 & 0.8063 & 0.4666 \\ -0.2999 & 0.3729 & -0.8781 \\ -0.8820 & -0.4591 & 0.1062 \end{pmatrix};
$$

Code of the question is q2.m and the test file is q2Test.m. Here is a mock run of the code - on the command prompt

```
q2Test
```

Question 3

To determine the number of solutions, we have to see if b is in the column space of the A matrix. I have created an augmented matrix $B = [A|b]$.

- If the rank of B is the same as A, and both have rank equal to the span of the columns of A, then there is an unique solution of the form $\bar{x}$, where $\bar{x}$ is the SVD solution.
- If the rank of B is the same as A, but both have rank less than the span of the column space of A, then there are infinitely many solutions of the form $\bar{x} + x_N$, where $\bar{x}$ is the SVD solution.
- If the rank of B is greater than A, then there are 0 actual solutions and has a least sqaures solution - which is given by the SVD solution $\bar{x}$ such that $||A\bar{x} - b||$ is minimized.

Intuition behind forming the augmented matrix B - If the vector b belongs to the column space of A, then rank of B should be the same as A as b can be formed by a linear combination of the vectors that span the column space of A. If b does not belong to the column space, then rank of B has to be greater than rank of A by atleast 1.

(a) $A = \begin{pmatrix} 2 & 2 & 5 \\ 1 & 1 & 5 \\ 3 & 2 & 5 \end{pmatrix}$; $b = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}$

Rank(A) = 3, Rank(B) = 3 and det(A) $\neq 0$. Matrix is square and invertible and system has a unique solution. Solution is $x = \begin{pmatrix} -5 & 15 & -3 \end{pmatrix}^T$

(b) $A = \begin{pmatrix} -3 & -4 & -1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{pmatrix}$; $b = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

Rank(A) = 2, Rank(B) = 3 and det(A) = 0. Matrix is not invertible, b is not in column space of A and system has zero solutions. SVD Solution is $\bar{x} = 1.0{*}e^{16}\begin{pmatrix} 2.8957 & -2.8957 & 2.8957 \end{pmatrix}^T$

(c) $A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & -2 \end{pmatrix}$; $b = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}$

Rank(A) = 2, Rank(B) = 2 and det(A) = 0. Matrix is not invertible, but b is in column space of A and system has infinitely many solutions. SVD Solution is $\bar{x} = \begin{pmatrix} 4.0061 & 2.0061 & -0.4970 \end{pmatrix}^T$. All solutions are of the form $\bar{x} + x_N$, where x_N is a column vector which belongs to the null space of A.

Code is included, q3.m and a test code q3Test.m. xbar is the actual solution (or SVD solution for infinite or zero solution case) and result is the number of solutions. Here is a mock run of the code - on the command prompt

```
% run 1
A = [2 2 5; 1 1 5; 3 2 5];
b = [5 -5 0]';
q3Test(A,b)
```

Question 4

$A = I$ - uu^T is a symmetric matrix with rank n-1.

(a) Given any vector v that belongs to $\mathcal{R}^n$, we get $Av = (I - uu^T)v = v - (uu^T)v = v - u(u \cdot v)$.

Consider the vector product $(Av)u$, which is given by $Avu = (v - u(u \cdot v))u = (vu - vu) = 0$, since $u^T u = 1$, [u is a unit vector]. Thus, for any vector v, the matrix A, projects the vector v perpendicular to u, on the hyperplane containing u and v.

(b) The eigenvalues of A are 1 (with multiplicity n-1) and 0 (with multiplicity 1). To calculate the eigen values, we take the characteristic polynomial given by $|A - \lambda I| = |I - uu^T - \lambda I| = |(1-\lambda)I - uu^T|$.

Using the matrix determinant lemma, $det(A + uv^T) = det(A)(1 + v^T A^{-1} u)$, we get, $|(1-\lambda)I - uu^T| = det(1-\lambda)I$ (1-$u^T((1-\lambda)I)^{-1})u)$ = $det(1-\lambda)I$ (1-$u^T(1/(1-\lambda)I)u)$ = $det(1-\lambda)I$ (1-1/$(1-\lambda))$ = $det(1-\lambda)I - 1/(1-\lambda)det(1-\lambda)I = (1-\lambda)^n - (1-\lambda)^{n-1} = (1-\lambda)^{n-1} * (-\lambda)$.

Solving for the characteristic polynomial, $(1-\lambda)^{n-1} * (-\lambda) = 0$, we get, $\lambda = 0, 1$.

(c) Take any scalar multiple of the vector u, we get $A(\alpha u) = \alpha(Au) = \alpha(I - uu^T)u = \alpha(u - uu^T u) = \alpha(u - u(u^T u)) = \alpha(u - u) = 0$. Thus the nullspace is given by the span of the vector u. In terms of SVD, take $A = U\Sigma V^T$, we then get the nullspace of A as the span of the last column of V (rank of matrix is $n-1$).

(d) $A^2 = A * A = (I - uu^T)(I - uu^T) = (I - 2uu^T + uu^T uu^T)$. Since u is a unit length vector, $u^T u = 1$ and hence $A^2 = (I - 2uu^T + u.1.u^T) = (I - 2uu^T + uu^T) = (I - uu^T) = A$. Hence, $A^2 = A$.

Question 5

Let $P = p_1, p_2, ...p_n$ denote the set of points before the rigid body transform and let $Q = q_1, q_2, ...q_n$ denote the set of points after the transform. To find Q from P, we perform a rotation R followed by a translation t we have to find $(R, t) = \text{argmin}_{R,t} \sum_{i=1}^{n} \|(Rp_i + t) - q_i\|^2$.

To find t, we can find the derivative of the function $RGD(t) = \sum_{i=1}^{n} \|(Rp_i + t) - q_i\|^2$ with respect to t and take roots. Taking $\partial RGD/\partial t = 0$, we get, $2t(n) + 2R\sum_{i=1}^{n}(p_i) - 2\sum_{i=1}^{n} q_i = 0$ and solving, we get $t = \bar{q} - R\bar{p}$, where $\bar{p} = \sum_{i=1}^{n}(p_i)/n$ and $\bar{q} = \sum_{i=1}^{n}(q_i)/n$.

Plugging the solution into the original RGD function, we get $(R) = \text{argmin}_R \sum_{i=1}^{n} \|R(p_i - \bar{p}) - (q_i - \bar{q})\|^2$. Let $a_i = (p_i - \bar{p})$ and $b_i = (q_i - \bar{q})$, we get $(R) = \text{argmin}_R \sum_{i=1}^{n} \|Ra_i - b_i\|^2$. Solving for $\|Ra_i - b_i\|^2$, we get $\|Ra_i - b_i\|^2 = a_i^T a_i - 2b_i^T Ra_i + b_i^T b_i$ and plugging into the equation above, we get, $(R) = \text{argmin}_R \sum_{i=1}^{n}(a_i^T a_i - 2b_i^T Ra_i + b_i^T b_i) = \text{argmin}_R \sum_{i=1}^{n}(-2b_i^T Ra_i) = \text{argmax}_R \sum_{i=1}^{n}(b_i^T Ra_i)$, which is maximizing the $tr(B^T RA)$. To maximize the $tr(B^T RA)$, we can maximize the $tr(RAB^T)$ [using property $tr(AB) = tr(BA)$].

Taking the SVD of AB^T, we get, $AB^T = U\Sigma V^T$, and equating it back in RAB^T, we get $tr(RAB^T) = tr(RU\Sigma V^T) = tr(\Sigma V^T RU)$. Since R, V, U are orthogonal matrices, $V^T RU$ is an orthogonal matrix and hence its columns x_i are such that $x_i^T x_i = 1$ (x_i's are orthonormal) and any component of the matrix $c_{ij}[C = V^T RU]$ is always less than 1 (x_i's are orthonormal vectors, which are of unit length $\implies c_{ij} \leq 1$).

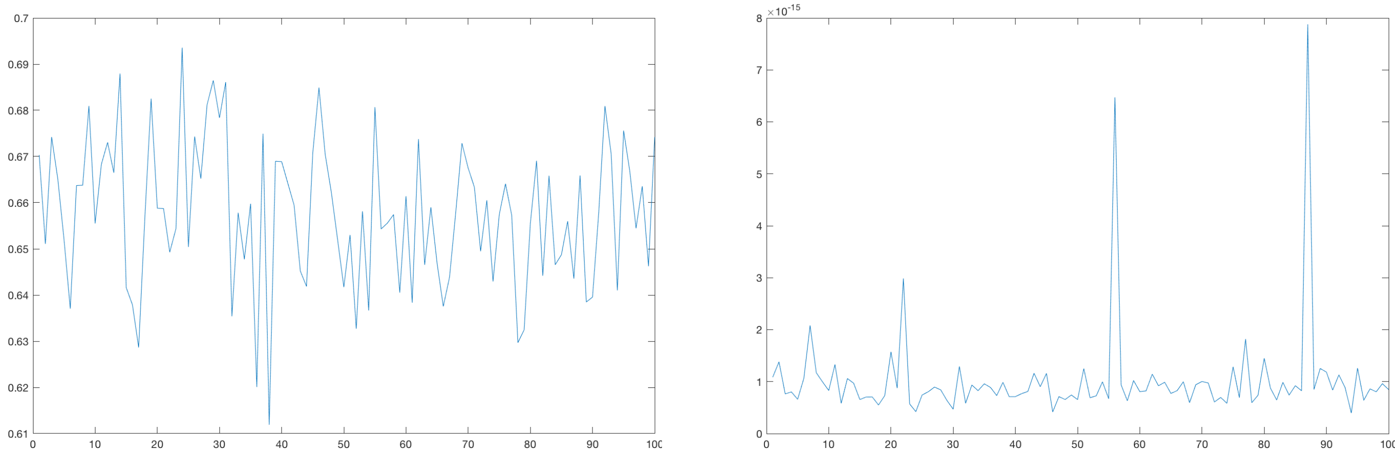
Now, we have $tr(\Sigma C) = \begin{pmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} = \sum_{i=1}^{3} \sigma_i c_{ii} \leq \sum_{i=1}^{3} \sigma_i$

To maximize the product $\Sigma C = \Sigma V^T RU$ (where $V^T RU$ is orthogonal), C has to be the identity matrix $I \implies V^T RU = I \implies R = VU^T$. However, with possibility $det(UV^T)$ can be -1, and in this case the matrix will perfectly reflect the set of points p to q. In this case, to avoid reflection, and account for only rotations, we consider the sum $\sum_{i=1}^{3} \sigma_i c_{ii}$ and the extrema case of $c_{ii} = -1(or)1$. With $c_{33} = 1$ we are obtaining a reflection, so we take the case $c_{33} = -1$ to ensure that there is no reflection going on. In terms of U and V, we multiply the 3rd column of V with -1 in the case that $det(UV^T)$ = -1.

Thus the rotation matrix is given by $R = UV^T$ translation vector t is given by $t = \bar{q} - UV^T\bar{p}$.

Code is included as q5.m (actual solver using above explanations), q5CalcError.m (calculates root mean square error (least square error)) and q5Test.m (with 2 cases).

For case 1, there are random sets of points P, Q and we calculate the R and t components. For the case 1, plotting the rmse for number of points from 1 to 100, we get Figure 1a. For case 2, Q is an exact rigid body transform of P, we calculate the R and t components. For the case 2, plotting the rmse for number of points from 1 to 100, we get Figure 1b.



(a) Error when P and Q are random points in the space

(b) Error when Q is an exact rigid body transform of P

Figure 1: Errors in two different scenarios of Rigid Body Transformation

An intuition as to why taking random points gives an error - the function for generating data assumes a normal distribution of data Q - thus inducing some noise like component in the system (which is always observed in real data) and a rmse with lower errors guarantees that the calculated R and t are the best approximations to the actual rotation and translation components R_{actual} and t_{actual}. An intuition as to why not taking random points gives an error - the function for getting $Q = R * P + t$ never induces any noise in the system and hence we get rmse $\longrightarrow 0$ always and the calculated R and t are equal to the actual rotation and translation components R_{actual} and t_{actual}. Here is a mock run of the code - on the command prompt,

```
q5Test
```