Homework Assignment 2 Resubmission

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1 Question 2

Let $P(x)$ be the linear interpolating polynomial for the function $f(x)$. Then we get that for some x, for some c_x between the minimum and maximum of x, x_0 and x_1.

$$f(x) - P(x) \leq \frac{(x-x_0)(x-x_1)}{2} * f''(c_x)$$

Here x_0 and x_1 are contained in the interval $[a, b]$ that we are interpolating and $x_0 \leq x_1$. Bounding the error, we get,

$$|f(x) - P(x)| \leq \frac{(x-x_0)(x-x_1)}{2} * max_{x_0 \leq x \leq x_1} f''(x) \tag{1}$$

where $x \in [x_0, x_1]$. Solving for (1),

$$h = x_0 - x_1$$
$$\implies max_{x_0 \leq x \leq x_1}(x-x_0)(x-x_1) = \frac{h^2}{4}$$

Hence, the error bound is given by,

$$|f(x) - P(x)| \leq \frac{h^2}{8} * max_{x_0 \leq x \leq x_1}|f''(x)|$$

For $f(x) = \sin(x)$ between $[0, 2\pi]$, consider $0 \leq x_0 \leq x \leq x_1 \leq 2\pi$. Here $f^{''}(x) = -\sin(x)$ and $|f^{''}(x)| = \sin(x)$.

Considering the above defined error bound, we get,

$$|f(x) - P(x)| \leq \frac{h^2}{8} * max_{x_0 \leq x \leq x_1} \sin(x)$$

The maximum value this expression takes is when $\sin(x) = 1$

$$\implies |f(x) - P(x)| \leq \frac{h^2}{8}$$

We need ϵ accuracy such that $\frac{h^2}{8} \leq \epsilon$. For 6-digit accuracy, consider $\epsilon = 5 * 10^{-7}$ and then $\frac{h^2}{8} \leq 5 * 10^{-7}$. Solving for h, we get h = 0.002. Now since the interval is $[0, 2\pi]$, we take the number of intervals as ceil($2\pi/0.002$), getting **3142** intervals or **3143** data points.

Let $P(x)$ be the quadratic interpolating polynomial for the function $f(x)$. Then we get that for some x and for some c_x between the minimum and maximum of points in x, x_0, x_1 and x_2

$$f(x) - P(x) \leq \frac{(x-x_0)(x-x_1)(x-x_2)}{6} * f'''(c_x)$$

Here x_0, x_1 and x_2 are contained in the interval $[a,\, b]$. Bounding the error, we get,

$$|f(x) - P(x)| \leq \frac{(x-x_0)(x-x_1)(x-x_2)}{6} * max_{x_0 \leq x \leq x_1} |f'''(x)|$$

where $x \in [x_0, x_2]$. Consider

$$x_1 = x_0 + h, \qquad x_2 = x_1 + h$$
$$max_{x_0 \leq x \leq x_2} d(x) = (x-x_0)(x-x_1)(x-x_2)$$

Solving for $d(x)$,

$$x = x_1 + th$$
$$\implies x - x_0 = (1+t)h, \qquad x_1 = th, \qquad x_2 = (t-1)h$$
$$\implies max_{x_0 \leq x \leq x_2} d(x) = h^3 max_{-1 \leq t \leq 1} |t - t^3|$$

Solving for t, and plugging it back in the above equation, we get,

$$max_{x_0 \leq x \leq x_2} d(x) = \frac{2h^3}{3\sqrt{3}}$$

Now, bounding the error, we get,

$$|f(x) - P(x)| \leq max_{x_0 \leq x \leq x_2} \frac{d(x)}{6} max_{x_0 \leq x \leq x_2} |f'''(x)|$$

Plugging the derived values in the equation and solving for $max_{x_0 \leq x \leq x_2} f'''(x) = -\cos(x)$, we get,

$$|f(x) - P(x)| \leq \frac{h^3}{9\sqrt{3}}$$

The max over the interval $[0, 2\pi]$ for $|-\cos(x)|$ is 1.

We need ϵ accuracy such that $\frac{h^3}{9\sqrt{3}} \leq \epsilon$. For 6-digit accuracy, consider $\epsilon = 5 * 10^{-7}$ and then $\frac{h^3}{9\sqrt{3}} \leq 5 * 10^{-7}$. Solving for h, we get h = 0.0198. Now since the interval is $[0, 2\pi]$, we take the number of intervals as ceil($2\pi/0.0198$), getting **318** intervals or **319** data points.

Interploation	**# Evenly Spaced Intervals**	**# Data Points for Interpolation Table**
Linear	3142	3143
Quadratic	318	319

Table 1: Table of Interpolation technique vs Number of intervals and data points