Homework Assignment 2

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1 Question 1

(a) Value for $(\log_6 x)^{3/2}$ at x = 2.25 is 0.3046 (from interpolation). Code included (q1.m). Here is a demo of the code -

```
X = [1, 1.5, 2, 2.5, 3, 3.5, 4];
Y = log10(X)/log10(6);
Y = Y.^1.5;
value = q1(X,Y,2.25);
```

To see the estimated function, uncomment line 46 in the code and run the code again.

```
% display(stringsum)
```

[The display is not exactly transferrable to a matlab function. I am always printing $(x - x_i)$, even if $x_i < 0$, resulting in $(x - -x_i)$, where $x_i > 0$ and also the actual multiplication of brackets look like written format $(x - x_i)(x - x_{i+1})$. In matlab we would need $(x - x_i) * (x - x_{i+1})$].

(b) Function $f(x) = \frac{6}{1+25x^2}$

   (a) At x = 0.05 with n = 2, value of function is 5.9856

   (b) At x = 0.05 with n = 4, value of function is 5.936

   (c) At x = 0.05 with n = 40, value of function is 5.6471

Actual value of the function at x = 0.05 is 5.6470588. Code included (q1b.m, q1bTest.m). Here is a demo of the code -

```
px = q1bTest;
```

(c) Code included (q1c.m and q1cTest.m). A plot of the value E_n against the value of n is shown in Figure 1. Here is a demo of the code -

```
En = q1cTest;
```

The maximum error estimate increases with the value of n as we are using evenly spaced points in the interval [-1,1] and the actual function fit is being thrown out of shape for (-$infty$,-1) and (1,∞). In these intervals the function is being fit poorly and the value of the actual function at the edges of the intervals that is -1,1 are not being estimated properly, resulting in higher errors. The figures 2 and 3 show the estimated function between [-1,1].

Below is the graph for values of $n = 20$ and $n = 40$ for both the function and the error plots in Figures 4 and 5 respectively (the left side sub figures are the actual function values computed using the function itself).

2 Question 2

Let $P(x)$ be the linear interpolating polynomial for the function $f(x)$. Then we get that for some x, for some c_x between the minimum and maximum of x, x_0 and x_1.

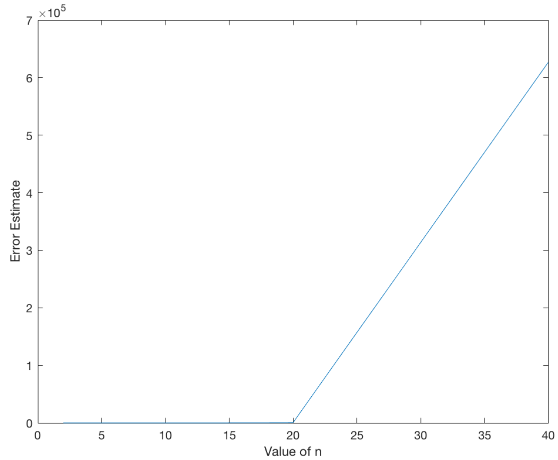

Figure 1: Error vs Value of n

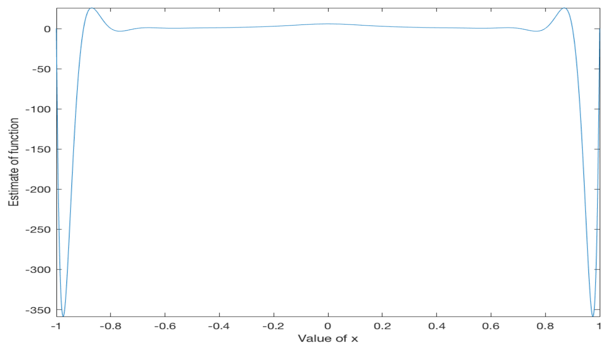

Figure 2: Function and Error Values between -1 and 1 for n=20

$$f(x) - P(x) \leq \frac{(x-x_0)(x-x_1)}{2} * f''(c_x)$$

Here x_0 and x_1 are contained in the interval $[a,\, b]$ that we are interpolating and $x_0 \leq x_1$. Bounding the error, we get,

$$|f(x) - P(x)| \leq \frac{(x-x_0)(x-x_1)}{2} * max_{x_0 \leq x \leq x_1} f''(x) \tag{1}$$

where $x \in [x_0, x_1]$. Solving for (1),

$$\begin{aligned} h &= x_0 - x_1 \\ \implies max_{x_0 \leq x \leq x_1}(x-x_0)(x-x_1) &= \frac{h^2}{4} \end{aligned}$$

Hence, the error bound is given by,

$$|f(x) - P(x)| \leq \frac{h^2}{8} * max_{x_0 \leq x \leq x_1} |f''(x)|$$

For $f(x) = \sin(x)$ between $[0, 2\pi]$, consider $0 \leq x_0 \leq x \leq x_1 \leq 2\pi$. Here $f^{''}(x) = -\sin(x)$ and $|f^{''}(x)| = \sin(x)$.

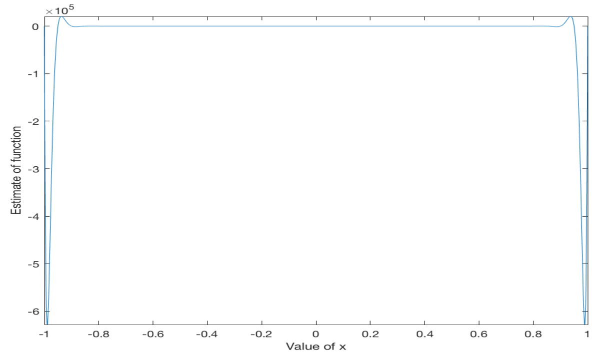

Figure 3: Function and Error Values between -1 and 1 for n=40

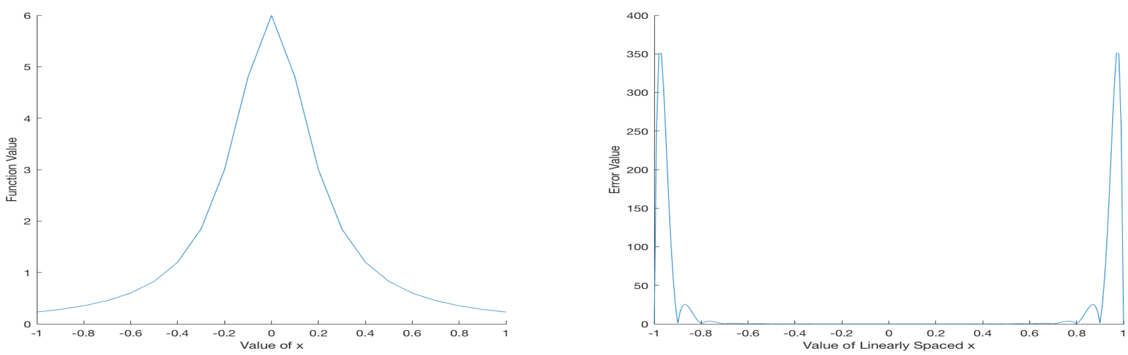

Figure 4: Function and Error Values between -1 and 1 for n=20

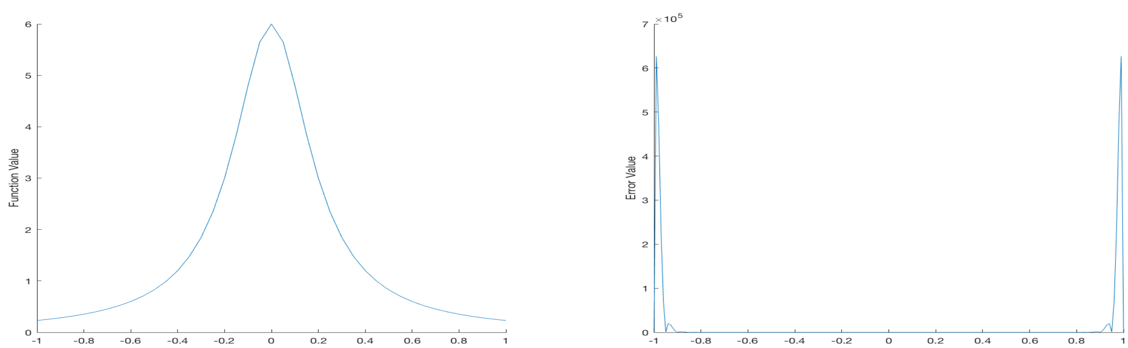

Figure 5: Function and Error Values between -1 and 1 for n=40

Considering the above defined error bound, we get,

$$|f(x) - P(x)| \leq \frac{h^2}{8} * max_{x_0 \leq x \leq x_1} \sin(x)$$

The maximum value this expression takes is when $\sin(x) = 1$

$$\implies |f(x) - P(x)| \leq \frac{h^2}{8}$$

We need ϵ accuracy such that $\frac{h^2}{8} \leq \epsilon$. For 6-digit accuracy, consider $\epsilon = 5 * 10^{-7}$ and then $\frac{h^2}{8} \leq 5 * 10^{-7}$. Solving for h, we get h = 0.002. Thus we will need **500 intervals** or **501 data points** for a 6 digit accuracy.

Let $P(x)$ be the quadratic interpolating polynomial for the function $f(x)$. Then we get that for some x and for some c_x between the minimum and maximum of points in x, x_0, x_1 and x_2

$$f(x) - P(x) \leq \frac{(x - x_0)(x - x_1)(x - x_2)}{6} * f'''(c_x)$$

Here x_0, x_1 and x_2 are contained in the interval $[a, b]$. Bounding the error, we get,

$$|f(x) - P(x)| \leq \frac{(x - x_0)(x - x_1)(x - x_2)}{6} * max_{x_0 \leq x \leq x_1} |f'''(x)|$$

where $x \in [x_0, x_2]$. Consider

$$x_1 = x_0 + h, \qquad x_2 = x_1 + h$$
$$max_{x_0 \leq x \leq x_2} d(x) = (x - x_0)(x - x_1)(x - x_2)$$

Solving for $d(x)$,

$$x = x_1 + th$$
$$\implies x - x_0 = (1 + t)h, \qquad x_1 = th, \qquad x_2 = (t - 1)h$$
$$\implies max_{x_0 \leq x \leq x_2} d(x) = h^3 max_{-1 \leq t \leq 1} |t - t^3|$$

Solving for t, and plugging it back in the above equation, we get,

$$max_{x_0 \leq x \leq x_2} d(x) = \frac{2h^3}{3\sqrt{3}}$$

Now, bounding the error, we get,

$$|f(x) - P(x)| \leq max_{x_0 \leq x \leq x_2} \frac{d(x)}{6} max_{x_0 \leq x \leq x_2} |f'''(x)|$$

Plugging the derived values in the equation and solving for $max_{x_0 \leq x \leq x_2} f'''(x) = -\cos(x)$, we get,

$$|f(x) - P(x)| \leq \frac{h^3}{9\sqrt{3}}$$

The max over the interval $[0, 2\pi]$ for $|-\cos(x)|$ is 1.

We need ϵ accuracy such that $\frac{h^3}{9\sqrt{3}} \leq \epsilon$. For 6-digit accuracy, consider $\epsilon = 5 * 10^{-7}$ and then $\frac{h^3}{9\sqrt{3}} \leq 5 * 10^{-7}$. Solving for h, we get h = 0.0198. Thus we will need **51 intervals** or **52 data points** for a 6 digit accuracy.

Interploation	**# Evenly Spaced Intervals**	**# Data Points for Interpolation Table**
Linear	500	501
Quadratic	51	52

Table 1: Table of Interpolation technique vs Number of intervals and data points

3 Question 3

Code included in the solution. Main implementation is q3.m and for $f(x) = x - \tan(x)$ the test function is in q3Test.m. Here is a demo run of the code -

```
q3Test
```

For $f(x) = \tan(x)$ at $x = 11$, the closest roots are $x_0 = 10.904$ and $x_1 = 14.066$. There are no other roots for the function between $[x_0, x_1]$.

4 Question 4

Consider the function $f(x)$ having an order 2 root. Let ξ represent that root, which implies that $f(\xi) = 0$ and $f^{'}(\xi) = 0$ but $f^{''}(x) \neq 0$. This means that the function $f(x)$ can be written as some function $f(x) = (x-\xi)^2 h(x)$.

Now consider Newton's method, where we get $x_{n+1} = x_n - f(x)/f^{'}(x)$. Let $g(x) = x_n - f(x)/f^{'}(x)$. We know that $x_{n+1} = g(x_n)$ and in the neighborhood of ξ, $g(x)$ is closer to ξ than x.

Using the Taylor theorem expansion around the point ξ, we get

$$\begin{aligned} g(x) &= g(\xi) + g'(\xi)(x-\xi) + \frac{1}{2}g''(\epsilon)(x-\xi)^2 \\ \implies x_{n+1} &= \xi + g'(\xi)(x_n-\xi) + \frac{1}{2}g''(\epsilon)(x_n-\xi)^2 \end{aligned}$$

Since $g(\xi) = \xi$. Now, subtracting ξ on both sides and diving by $x_n - \xi$ we get

$$\frac{x_{n+1}-\xi}{x_n-\xi} = g'(\xi) + \frac{1}{2}g''(\epsilon)(x_n-\xi)$$

When $n \to \infty$,

$$\lim_{n\to\infty} \frac{|x_{n+1}-\xi|}{|x_n-\xi|} \leq |g'(\xi)| \tag{2}$$

Now since $f(x) = (x-\xi)^2 h(x)$, we consider

$$\begin{aligned} g(x) &= x - \frac{(x-\xi)^2 h(x)}{2(x-\xi)h(x) + (x-\xi)^2 h'(x)} \\ g(x) &= x - \frac{(x-\xi)h(x)}{2h(x) + (x-\xi)h'(x)} \\ \implies g'(x) &= 1 - \frac{(h(x) + (x-\xi)h'(x))(2h(x) + (x-\xi)h'(x)) - (x-\xi)h(x)(2h'(x) + (x-\xi)h''(x) + h'(x))}{(2h(x) + (x-\xi)h'(x))^2} \\ \lim_{x\to\xi} g'(x) &= 1 - \frac{2(h(x))^2}{4(h(x))^2} \\ \implies g'(\xi) &= \frac{1}{2} \end{aligned}$$

Plugging the value of $g'(\xi)$ in equation (2), we get

$$\lim_{n\to\infty} \frac{|x_{n+1}-\xi|}{|x_n-\xi|} \le \frac{1}{2}$$

Thus, the function converges linearly and hence does not converge quadratically.

Now considering that $f'''(x)$ is continuous in the neighborhood of ξ. The function is still $f(x) = (x-\xi)^2 h(x)$. Now consider the new $g(x)$,

$$\begin{aligned}
g(x) &= x - 2\frac{f(x)}{f'(x)} \\
g(x) &= x - 2\frac{(x-\xi)h(x)}{2h(x)+(x-\xi)h'(x)} \\
\implies g'(x) &= 1 - 2\frac{(h(x)+(x-\xi)h'(x))(2h(x)+(x-\xi)h'(x)) - (x-\xi)h(x)(2h'(x)+(x-\xi)h''(x)+h'(x))}{(2h(x)+(x-\xi)h'(x))^2} \\
\lim_{x\to\xi} g'(x) &= 1 - 2\frac{2(h(x))^2}{4(h(x))^2} \\
\implies g'(\xi) &= 0
\end{aligned}$$

Writing the Taylor series expansion of $g(x)$ around the point ξ, we get

$$\begin{aligned}
g(x) &= g(\xi) + g'(\xi)(x-\xi) + \frac{1}{2}g''(\xi)(x-\xi)^2 + \frac{1}{6}g'''(\epsilon)(x-\xi)^3 \\
x_{n+1} &= \xi + g'(\xi)(x_n-\xi) + \frac{1}{2}g''(\xi)(x_n-\xi)^2 + \frac{1}{6}g'''(\epsilon)(x_n-\xi)^3 \\
x_{n+1}-\xi &= 0 + \frac{1}{2}g''(\xi)(x_n-\xi)^2 + \frac{1}{6}g'''(\epsilon)(x_n-\xi)^3 \\
\frac{x_{n+1}-\xi}{(x_n-\xi)^2} &= \frac{1}{2}g''(\xi) + \frac{1}{6}g'''(\epsilon)(x_n-\xi)
\end{aligned}$$

When $n \to \infty$,

$$\lim_{n\to\infty} \frac{|x_{n+1}-\xi|}{|x_n-\xi|^2} \le \frac{1}{2}|g''(\xi)| \tag{3}$$

Since $g''(\xi) \neq 0$, this shows that the function converges quadratically when we use the iteration of the form

$$x_{n+1} = x_n - 2\frac{f(x)}{f'(x)}$$

5 Question 5

(a) Code is included as q5.m

(b) Here is a sample run of the code (q5.m)

```
f = @(x) x^3 - 4*x^2 + 6*x -4;
X0 = [0.02,0,-0.02];
[root, val] = q5(f, X0);
```

Initial Value	Root
X0 = [1,0,-1]	1-i
X0 = [-1.5,0,1.5]	2
X0 = [0.02,0,-0.02]	1+i

Table 2: Table of initial estimate values and the derived root value

The roots of the polynomial $p(x) = x^3 - 4x^2 + 6x - 4$ are 2, 1+i and 1-i. The table 2 shows some sample estimates to get the roots (for direct root finding).

We start with the original function $f(x) = x^3 - 4x^2 + 6x - 4$ and set the initial values as $X_0 = [-1.5, 0, 1.5]$, we end up getting the root at $x = 2$. Deflating the function, we get,

$$f_1(x) = \frac{f(x)}{x-2}$$
$$f_1(x) = x^2 - 2x + 2$$

and using $X_0 = [1.8, 1.9, 2.1]$ (values near the root found in first iteration), we get the root as i+i. And now deflating the function further, we get,

$$f_2(x) = \frac{f_1(x)}{x-(1+i)}$$
$$f_2(x) = x - (1-i)$$

Thus the roots, using deflating method are given by 2, i+i and i-i.

6 Question 6

(a) Consider the polynomials $p(x) = x^3 - 9x^2 + 26x - 24$ and $q(x) = x^2 + 3x - 10$. Now consider the polynomial $xp(x) = x^4 - 9x^3 + 26x^2 - 24x$ and the polynomials $xq(x) = x^3 + 3x^2 - 10x$ and $x^2q(x) = x^4 + 3x^3 - 10x^2$. Writing these solutions together we get,

$$\begin{aligned}
1.x^4 - 9x^3 + 26x^2 - 24x + 0 \\
0.x^4 + 1.x^3 - 9x^2 + 26x - 24 \\
1.x^4 + 3x^3 - 10x^2 + 0.x + 0 \\
0.x^4 + 1.x^3 + 3.x^2 - 10.x + 0 \\
0.x^4 + 0.x^3 + 1.x^2 + 3.x - 10
\end{aligned}$$

Writing out the above set of equations in matrix form, we get,

$$Q\bar{x} = \begin{pmatrix} 1 & -9 & 26 & -24 & 0 \\ 0 & 1 & -9 & 26 & -24 \\ 1 & 3 & -10 & 0 & 0 \\ 0 & 1 & 3 & -10 & 0 \\ 0 & 0 & 1 & 3 & -10 \end{pmatrix} \begin{pmatrix} x^4 \\ x^3 \\ x^2 \\ x \\ 1 \end{pmatrix}$$

To solve for the common root, we need to solve the equations of the form $Q\bar{x} = 0$, which for some x, will only have a solution iff $det(Q)$ is 0.

Calculating the determinant of Q, we get $det(Q) = 0$. Since the determinant is zero, we can say that the two polynomials $p(x)$ and $q(x)$ share a common root.

(b) Applying the ratio method for finding the root, we get $x = -det(Q1)/det(Q2)$, where

$$Q1 = \begin{pmatrix} -9 & 26 & -24 & 0 \\ 1 & -9 & 26 & -24 \\ 3 & -10 & 0 & 0 \\ 1 & 3 & -10 & 0 \end{pmatrix} \text{ and } Q2 = \begin{pmatrix} 1 & 26 & -24 & 0 \\ 0 & -9 & 26 & -24 \\ 1 & -10 & 0 & 0 \\ 0 & 3 & -10 & 0 \end{pmatrix}$$

To apply the ratio method, we need a matrix of the form $(n-1)$x$(n-1)$. Here, $n = 5$. Since removing a column from Q, gives a 4x5 matrix, we remove the last row to ensure get a matrix of of the form 4x4. Hence, $Q1$ has the first column and the last row of the matrix removed. $Q2$ has the second column and the last row of the matrix removed.

Now, $det(Q1) = 13824$ and $det(Q2) = -6912$, we get $x = -(13824)/(-6912) = 2$.

The common root is $\boldsymbol{x}$**=2**.

7 Question 7

(a) Plotting the zero contours of the function $p(x)$ and $q(x)$, we get Figure 6. Code is included for this - q7a.m. To run this code

```
q7a
```

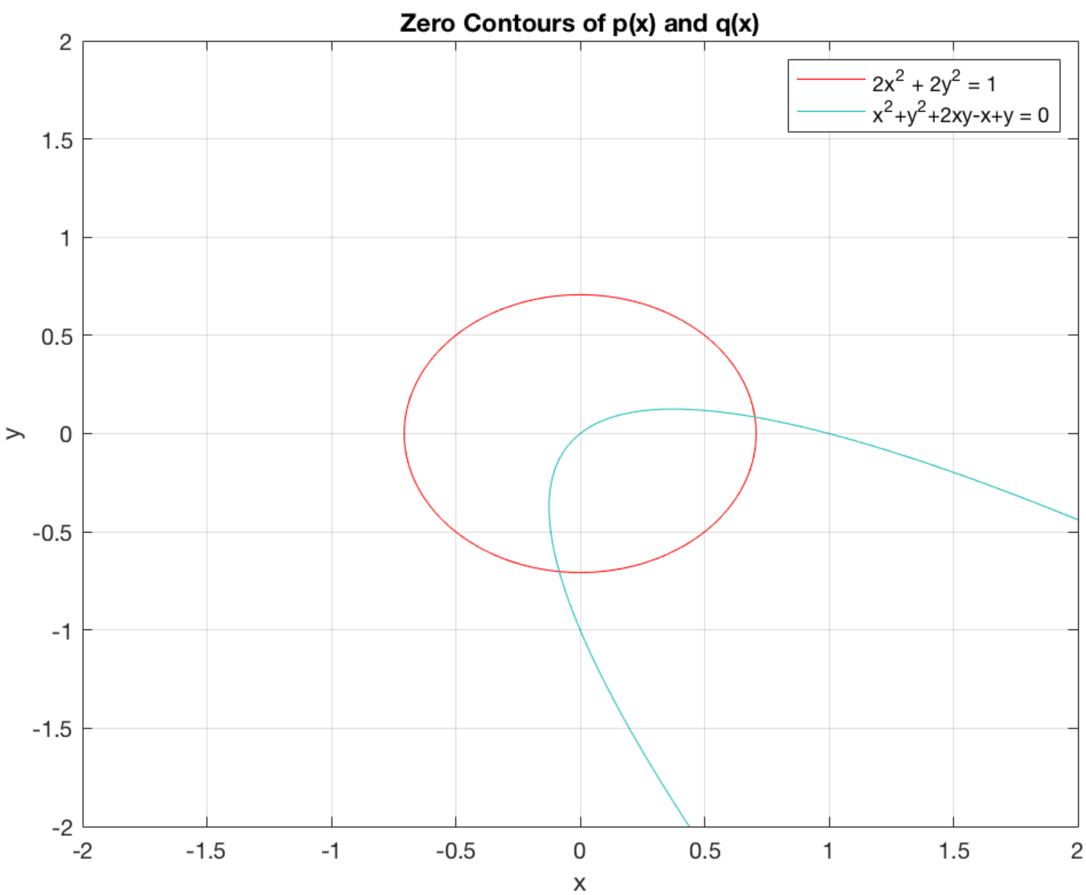

Figure 6: Zero Contours of the two functions

(b) For the elimination of y, we can consider the polynomials re-written in this form,

$$p(x) = 2y^2 + 0.y + (2x^2 - 1)$$
$$q(x) = y^2 + y(2x + 1) + (x^2 - x)$$

Now for forming the resultant matrix, consider the equations

$$\begin{aligned} yp(x) &= 2y^3 + 0.y^2 + (2x^2 - 1)y + 0 \\ p(x) &= 0.y^3 + 2y^2 + 0.y + (2x^2 - 1) \\ yq(x) &= y^3 + y^2(2x + 1) + y(x^2 - x) + 0 \\ q(x) &= 0.y^3 + y^2 + y(2x + 1) + (x^2 - x) \end{aligned}$$

Thus the resultant matrix for eliminating y is given by $Q = \begin{pmatrix} 2 & 0 & (2x^2-1) & 0 \\ 0 & 2 & 0 & (2x^2-1) \\ 1 & (2x+1) & (x^2-x) & 0 \\ 0 & 1 & (2x+1) & (x^2-x) \end{pmatrix}$ and we have to solve for $\bar{y} = \begin{pmatrix} y^3 \\ y^2 \\ y \\ 1 \end{pmatrix}$, by using $Q\bar{y} = 0 \implies det(Q) = 0$.

Solving for the determinant we get, $det(Q) = 0 \implies 16x^4 + 16x^3 - 12x - 1 = 0$, which has the real roots $x = -0.084$ and $x = 0.702$.

Solving for y, by putting it in the equation $p(x) = 0$, we get, the following roots

$$x = -0.084, y = -0.702$$
$$x = 0.702, y = 0.084$$

(c) Marking the roots in the plot from part (a), we get Fig 7. Code is included for this - q7c.m. To run the code,

```
q7c
```

8 Question 8

(a) Consider three 2D points $A = (x^i, y^i)$, $B = (x^j, y^j)$, $C = (x^k, y^k)$. Any point P which can be represented as a linear combination of these 3 points is given by $P = uA + vB + wC$. To ensure that the point lies within the triangle formed by A, B, C, we have to ensure that, $u, v, w > 0$, as taking negative values are equal to reflecting the point in space and we may end up selecting a point outside it and taking zero values may end up with a point on an edge formed by the three vertices of the triangle.

To put this in terms of matrix form, we get,

$$P = uA + vB + wC$$
$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = u \begin{pmatrix} x^i \\ y^i \end{pmatrix} + v \begin{pmatrix} x^j \\ y^j \end{pmatrix} + w \begin{pmatrix} x^k \\ y^k \end{pmatrix}$$
$$\begin{pmatrix} p_x \\ p_y \end{pmatrix} = \begin{pmatrix} x^i & x^j & x^k \\ y^i & y^j & y^k \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

To also ensure that we do not end up scaling the vector representing these points in space by some really high arbitray values of u, v, w, we impose another constraint that $u, v, w < 1$ (any value equal to 1 would mean we need negative values on the other two coefficients - but that may result in a point outside the triangle). Combining the two conditions on u, v, w and homogenizing the solution to get a better representation in the form of $Ax = b$, we get,

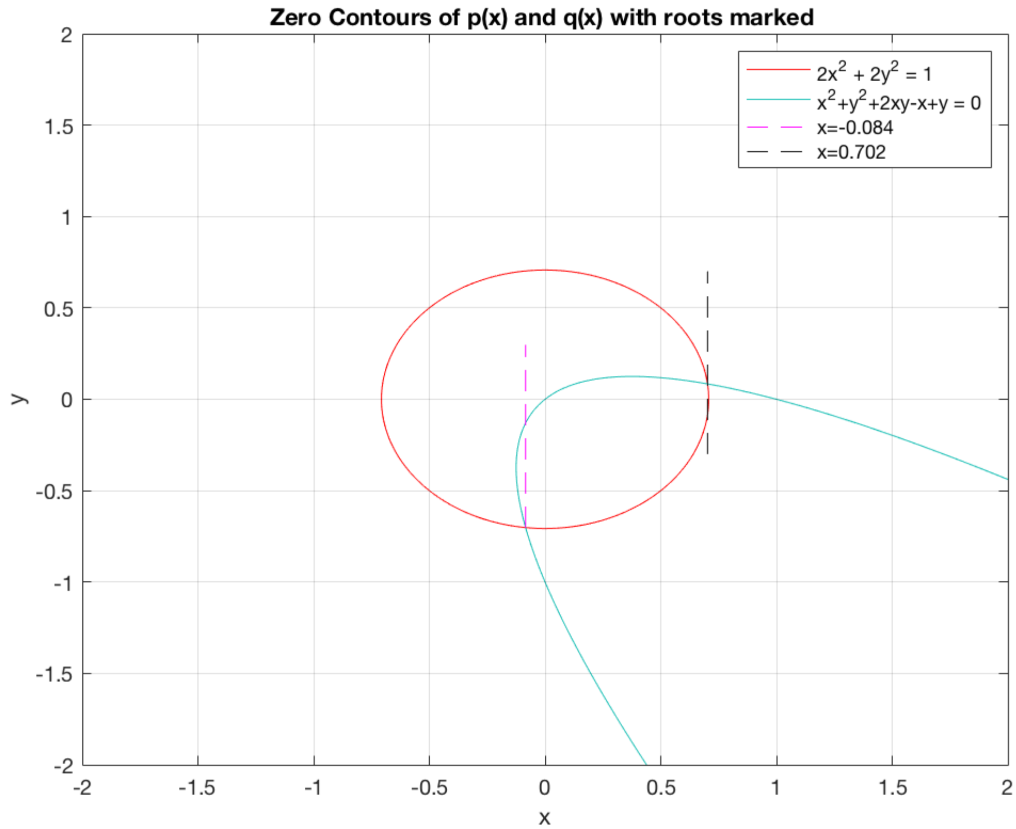

Figure 7: Functions with roots marked

$$\begin{pmatrix} p_x \\ p_y \\ 1 \end{pmatrix} = \begin{pmatrix} x^i & x^j & x^k \\ y^i & y^j & y^k \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{4}$$

which enforces the condition that $u + v + w = 1$. So the final set of conditions for the values of the u, v, w are $0 < u, v, w < 1$ and $u + v + w = 1$.

(b) The code is included as q8.m. There is test code included as q8Test.m. Here is how to run the code -

```
p0 = [0.8 1.8];
t = 41.5;
[alpha, point_t] = q8Test(p0,t);
```

The q8Test function returns the alpha values $\alpha_i, \alpha_j, \alpha_k$ that we have computed and the point on the interpolated path at the given time t.

The algorithm follows the followin design decisions:

(a) **Decision 1**: We first look at the p_0 point given and then decide the path we want to take based on the values of x, y in the point. If $x > y$, we want to select paths that have values $x_0 > y_0$ and vice versa. In the event that $x_0 = y_0$ for any point, we check the next point (x_1, y_1) in the path and make a decision according to the rule stated above.

(b) **Decision 2**: We then use the equation (4) above to get the the values of $\alpha_i, \alpha_j, \alpha_k$. We ensure that $\alpha_i, \alpha_j, \alpha_k > 0$ by selecting three random paths following the design decision 1, until all three values are positive, to ensure that the point lies in the triangle.

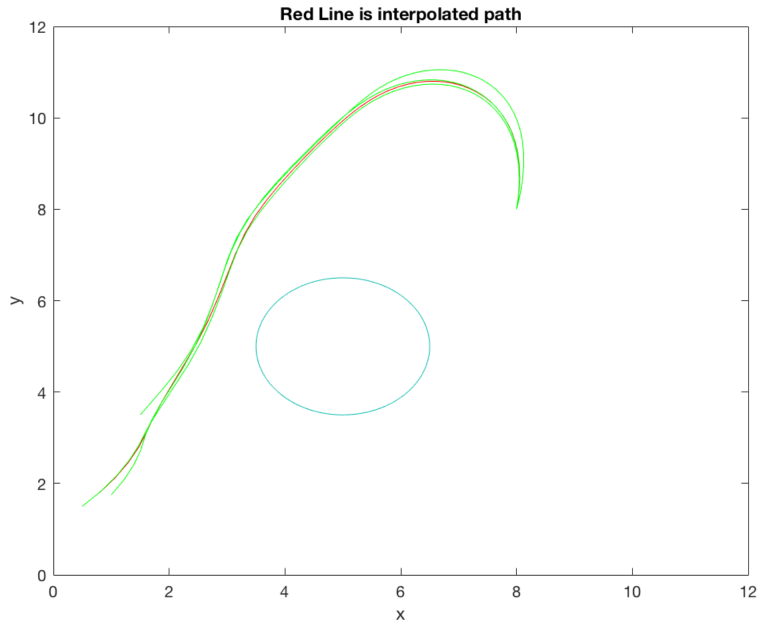

Figure 8: Plot when $p0 = [0.8\ 1.8]$

(c) **Decision 3**: We then calculate the points $p(t)$ using the equation for the path given. To interpolate the function, we use a piecewise linear strategy for each set of interpolated points and calculate the slope and intercepts based on a $y = mx + c$ strategy. Here the values of m, c are

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$
$$c = \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1}$$

The time scale is $\boldsymbol{t=1}$.

(d) **Decision 4**: Finally, to get the path value at any time t, we use the implicit representation for x, given by $x_t = x_{\lfloor t \rfloor} + \beta(x_{\lceil t \rceil} - x_{\lfloor t \rfloor})$, where $\beta = t - \lfloor t \rfloor$ (since we are doing piecewise linear strategy) and then calculate the value of y given as $y = mx_t + c$, where m, c are the slope and intercept between the two points $(x_{\lceil t \rceil}, y_{\lceil t \rceil})$ and $(x_{\lfloor t \rfloor}, x_{\lfloor t \rfloor})$. Here, $\lceil t \rceil$ refers to the ceil of the value of t and $\lfloor t \rfloor$ refers to the floor of the value of t.

(c) The plots are plotted below in Figures 8, 9 and 10.

(d) The algorithm generalizes when we use any number of obstacles (assuming given paths to interpolate are not touching any obstacle). This is because we always ensure that the selected alpha values $\alpha_i, \alpha_j, \alpha_k$ are positive, so the points selected always lies in the triangle considering three points in the selected paths and the piecewise linear strategy ensures that the paths constructed are also not touching any obstacle.

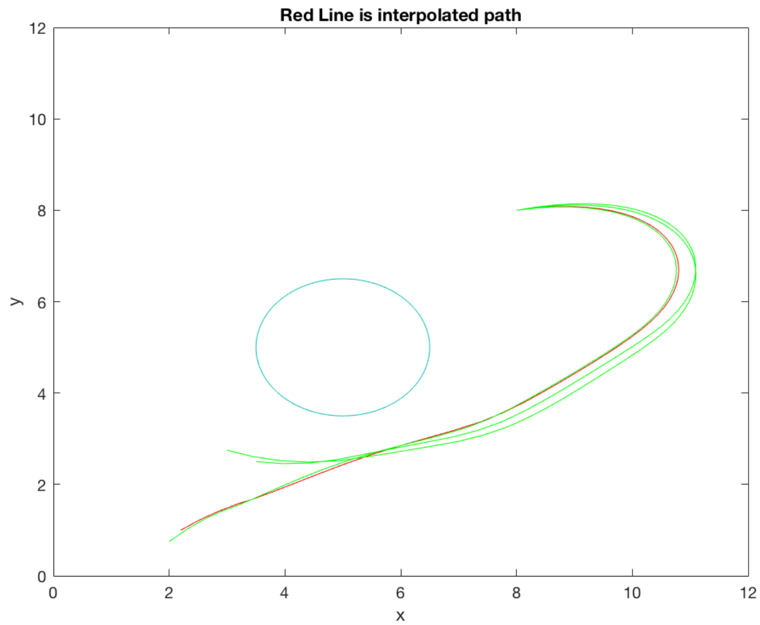

Figure 9: Plot when $p0 = [2.2\ 1]$

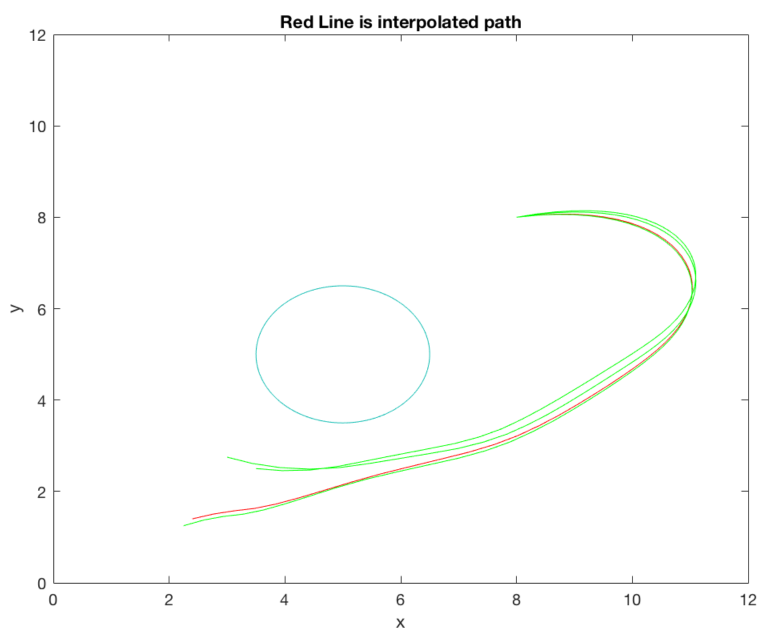

Figure 10: Plot when $p0 = [2.7\ 1.4]$