Homework Assignment 3

Abhay Gupta (Andrew Id: abhayg)

October 23, 2018

1 Question 1

(a) The Taylor series of expansion of a function $f(x)$ around a point a is given by

$$f(x) = f(a) + f'(x)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + ...$$

For $f(x) = 0.5 + \sin(x)$, at $x = 0$, we have

$$\begin{aligned} f(0) &= 0.5 \\ f'(0) &= 1 \\ &. \\ &. \\ &. \\ f^k(0) &= \frac{(-1)^k}{(1+2k)!} \end{aligned}$$

Thus, the taylor series expansion is given by

$$\begin{aligned} f(x)_{x=0} &= 0.5 + x - \frac{x^3}{3!} + \frac{x^5}{5!} + ... \\ f(x)_{x=0} &= 0.5 + \sum_{k=0}^{\infty} \frac{(-1)^k x^{1+2k}}{(1+2k)!} \end{aligned}$$

(b) The graph of the function over $[-\pi/2, \pi/2]$ is given by Figure 1. to run the code,

```
python q1.py
```

(c) Consider a quadratic polynomial $p_2(x)$ that is going to be the best uniform approximation of the function $f(x) = 0.5 + \sin(x)$. Since $f^{(3)}(x) = -\cos(x)$ does not change sign between $[-\pi/2, \pi/2]$, we can get 4 points such that the L_∞ norm of the errors are minimized and the errors at these points are alternating. Consider the points to be $\{-\pi/2, x_1, x_2, \pi/2\}$ and consider the polynomial to be fit as $p(x) = ax^2 + bx + c$. Now let $e(z)$ denote the error at point z.

$$\implies e(-\frac{\pi}{2}) = -e(x_1) = e(x_2) = -e(\frac{\pi}{2})$$

$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -0.5 - \sin(x_1) + ax_1^2 + bx_1 + c = 0.5 + \sin(x_2) - ax_2^2 - bx_2 - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$

Since the function $f(x) = 0.5 + \sin(x)$ is symmetric around the origin between $[-\pi/2, \pi/2]$, we would want the best approximating function to be symmetric between $[-\pi/2, \pi/2]$ and hence we take the points x_1 and x_2 as $-x_1 = x_2 = z$. So, now the alternating points are given by $\{-\pi/2, -z, z, \pi/2\}$. Thus the new errors are written as,

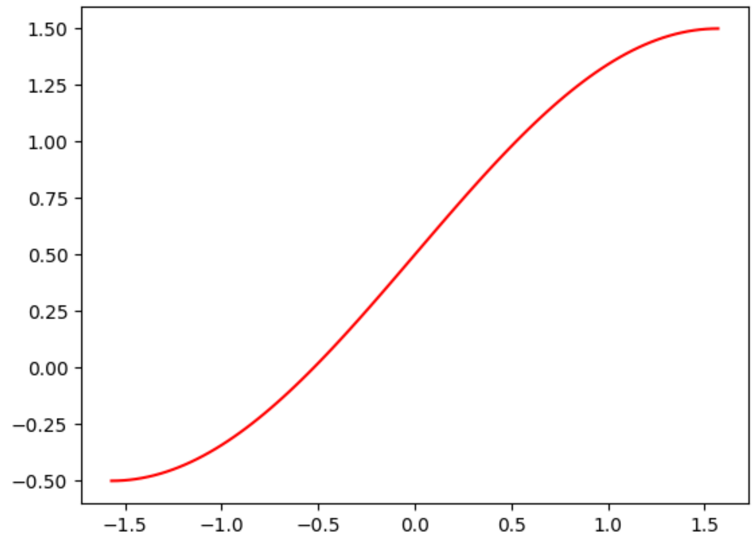

Figure 1: Function $f(x) = 0.5 + \sin(x)$

$$e(-\frac{\pi}{2}) = -e(-z) = e(z) = -e(\frac{\pi}{2})$$
$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -0.5 + \sin(z) + az^2 - bz + c = 0.5 + \sin(z) - az^2 - bz - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$

Solving the middle two equations, we get,

$$-0.5 + \sin(z) + az^2 - bz + c = 0.5 + \sin(z) - az^2 - bz - c$$
$$2az^2 + 2c = 1$$

Solving $e(-\frac{\pi}{2}) = -e(\frac{\pi}{2})$, we get,

$$-0.5 - a\frac{\pi^2}{4} + b\frac{\pi}{2} - c = -1.5 + a\frac{\pi^2}{4} + b\frac{\pi}{2} + c$$
$$2a\frac{\pi^2}{4} + 2c = 1$$

Combining these two equations,

$$2a\frac{\pi^2}{4} + 2c = 1$$
$$2az^2 + 2c = 1$$

Now since $z \neq \pi/2$ or $z \neq -\pi/2$, the only possible results from this are that $a = 0$ and $c = 0.5$. Now substituting this back in the error equations, we get,

$$-1 + b\frac{\pi}{2} = \sin(z) - bz = \sin(z) - bz = -1 + b\frac{\pi}{2}$$

Since we want the error at z to be minimum, we take the differential of the error at this point and equate it to zero, given by

$$
\begin{aligned}
\frac{d}{dz}e(z) &= \frac{d}{dz}(\sin(z) - bz) = 0\\
\implies \cos(z) - b = 0\\
&\implies b = \cos(z)\\
&\implies z = \cos^{-1}(b)
\end{aligned}
$$

Substituting this in the error equations above, we get,

$$
\begin{aligned}
-1 + b\frac{\pi}{2} &= \sin(\cos^{-1}(b)) - b\cos^{-1}(b)\\
-1 + b\frac{\pi}{2} &= \sqrt{1-b^2} - b\cos^{-1}(b)\\
\sqrt{1-b^2} - b(\cos^{-1}(b) + \frac{\pi}{2}) + 1 = 0
\end{aligned}
$$

We have to find roots for this equation, and using Muller's method, we get $b = 0.724$. Thus the best uniform approximation for $f(x) = 0.5 + \sin(x)$ is $g(x) = 0.5 + 0.724x$.

For the $\mathcal{L}_\infty$ norm, we take max over $[-\pi/2, \pi/2]$ of the error function $e(x) = f(x) - g(x) = \sin(x) - 0.724x$,

$$
\begin{aligned}
\frac{de(x)}{dx} &= 0\\
\cos(x) - 0.724 &= 0\\
x &= \cos^{-1}(0.724)\\
x &= 0.761
\end{aligned}
$$

Plugging this back into the error term, we get,

$$
\begin{aligned}
\mathcal{L}_\infty &= sin(0.761) - 0.724 * 0.761\\
&= 0.138
\end{aligned}
$$

Thus, the $\mathcal{L}_\infty$ norm is 0.138

For the $\mathcal{L}_2$ norm, we take,

$$
\begin{aligned}
\mathcal{L}_2 &= \sqrt{\int_{-\pi/2}^{\pi/2} (\sin(x) - 0.724x)^2 dx}\\
&= \sqrt{\int_{-\pi/2}^{\pi/2} \sin^2(x) + 0.524x^2 - 1.448\sin(x)dx}\\
&= \sqrt{0.174x^3 + 0.5x - 1.448\sin(x) - 0.25\sin(2x) + 1.448x\cos(x)\Big|_{-pi/2}^{pi/2}}\\
&= 0.171
\end{aligned}
$$

Thus, the $\mathcal{L}_2$ norm is 0.171.

The plotted function looks like Figure 2

(d) Taking the Legendre polynomials for the least squares approximation, we get,

$$
p(x) = \sum_{i=0}^{2} \frac{< f(x), p_i >}{< p_i, p_i >} p_i(x)
$$

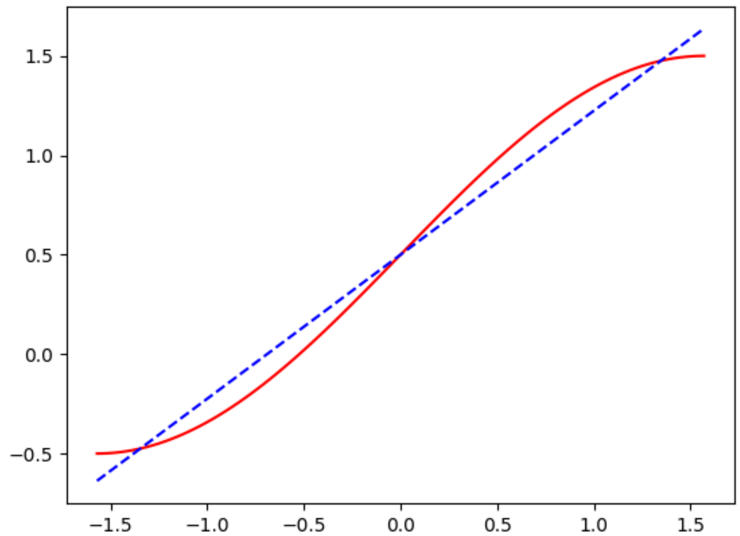

Figure 2: Plot of $f(x)$ with approximated $f(x)$

Calculating the Legendre polynomials between $[\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned}
p_0(x) &= 1 \\
p_1(x) &= [x - \frac{< x, 1 >}{< 1, 1 >}](1) \\
&= x \\
p_2(x) &= [x - \frac{< x^2, x >}{< x, x >}]x - \frac{< x, x >}{< 1, 1 >}(1) \\
&= x^2 - \frac{\pi^2}{12}
\end{aligned}$$

Plugging the polynomials $p_0(x)$, $p_1(x)$ and $p_2(x)$ into the first equation, we get,

$$\begin{aligned}
< 0.5 + \sin(x), p_0(x) > &= \int_{-\pi/2}^{\pi/2} 0.5 + \sin(x) dx = \frac{\pi}{2} \\
< 0.5 + \sin(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} 0.5x + x \sin(x) dx = 2 \\
< 0.5 + \sin(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} (0.5 + \sin(x))(x^2 - \frac{\pi^2}{12}) dx = 0 \\
< p_0(x), p_0(x) > &= \int_{-\pi/2}^{\pi/2} 1 dx = \pi \\
< p_1(x), p_1(x) > &= \int_{-\pi/2}^{\pi/2} x^2 dx = \frac{\pi^3}{12} \\
< p_2(x), p_2(x) > &= \int_{-\pi/2}^{\pi/2} (x^2 - \frac{\pi^2}{12})^2 dx = \frac{\pi^5}{180}
\end{aligned}$$

Plugging the above values into the original $p(x)$ equation, we get,

$$\begin{aligned}
p(x) &= \frac{\pi/2}{\pi} p_0(x) + \frac{2}{\frac{\pi^3}{12}} p_1(x) + 0.p_2(x) \\
p(x) &= 0.5 p_0(x) + 0.774 p_1(x) \\
p(x) &= 0.5 + 0.774x
\end{aligned}$$

The best least square approximation by a quadratic polynomial is $p(x) = 0.5 + 0.774x$

Consider the error function $e(x) = 0.5 + \sin(x) - 0.5 - 0.774x = \sin(x) - 0.774x$. Now, $\mathcal{L}_2$ error is given by

$$\begin{aligned}
\mathcal{L}_2 &= \sqrt{\int_{-\pi/2}^{\pi/2} (\sin(x) - 0.774x)^2 dx} \\
&= \sqrt{\int_{-\pi/2}^{\pi/2} \sin^2(x) + 0.559x^2 - 1.548\sin(x) dx} \\
&= \sqrt{0.199x^3 + 0.5x - 1.548\sin(x) - 0.25\sin(2x) + 1.548x\cos(x) \Big|_{-pi/2}^{pi/2}} \\
&= 0.15074
\end{aligned}$$

Thus, the $\mathcal{L}_2$ error $= 0.15074$.

For the $\mathcal{L}_\infty$ error, we take max over $[\frac{-\pi}{2}, \frac{\pi}{2}]$ of $e(x)$. For this,

$$\begin{aligned}
\frac{de(x)}{dx} &= 0 \\
\cos(x) - 0.774 &= 0 \\
x &= \cos^{-1}(0.774) \\
x &= 0.685
\end{aligned}$$

Plugging this back into the error term, we get,

$$\begin{aligned}
\mathcal{L}_{\infty estimate} &= \sin(0.685) - 0.774 * 0.685 \\
&= 0.102
\end{aligned}$$

Now calculating the values at the endpoints $\{\frac{-\pi}{2}, \frac{\pi}{2}\}$ we get,

$$\begin{aligned}
\mathcal{L}_{\infty \pi/2} &= |\sin(\pi/2) - 0.774 * \pi/2| = 0.2158 \\
\mathcal{L}_{\infty -\pi/2} &= |\sin(-\pi/2) + 0.774 * \pi/2| = 0.2158
\end{aligned}$$

Hence, the $\mathcal{L}_\infty$ error $= 0.2158$

The plotted function looks like Figure 3

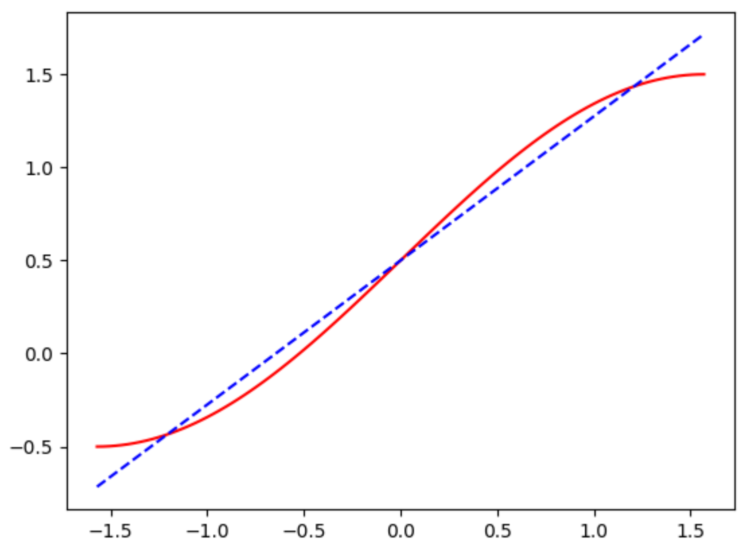

Figure 3: Plot of $f(x)$ with approximated $f(x)$