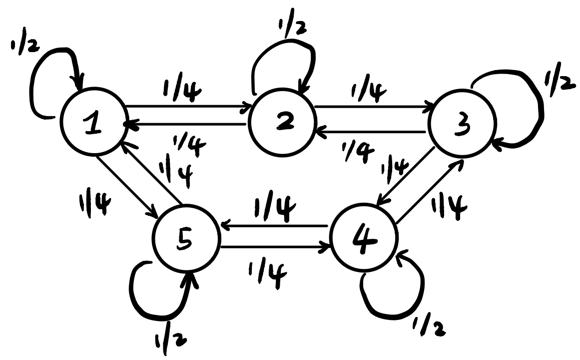
1.

(a) The state trasition diagram is as followed:



It is irreducible since we can access to every state from every state.

It is not periodic since $P_{jj} \neq 0$. For instance, if we start Markov Chain with $X_0 = 1$, $P_{0|0}^{(1)} = 1/2$. It is possible that the state keeps 1:

X	1	1	1	1	⋯
t	0	1	2	3	⋯

or

X	1	2	1	2	⋯
t	0	1	2	3	⋯

Hence this Markov Chain is not periodic.

(b) $p_{100} = p_0 P^{100}$.

According to eigenvalue decomposition. $P = VDV^{-1}$

where $D = \frac{1}{8} \operatorname{diag}(3-\sqrt{5}, 3-\sqrt{5}, 3+\sqrt{5}, 3+\sqrt{5}, 8)$

$$V = \begin{bmatrix} -1 & \frac{1}{2}(-1-\sqrt{5}) & -1 & \frac{1}{2}(-1+\sqrt{5}) & 1 \\ \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) & 1 \\ \frac{1}{2}(1-\sqrt{5}) & -1 & \frac{1}{2}(-1+\sqrt{5}) & -1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

$$V^{-1} = \frac{1}{10}\begin{bmatrix} -1+\sqrt{5} & -1+\sqrt{5} & -1-\sqrt{5} & 4 & -1-\sqrt{5} \\ -1-\sqrt{5} & -1+\sqrt{5} & -1+\sqrt{5} & -1-\sqrt{5} & 4 \\ -1-\sqrt{5} & -1-\sqrt{5} & -1+\sqrt{5} & 4 & -1+\sqrt{5} \\ -1+\sqrt{5} & -1-\sqrt{5} & -1-\sqrt{5} & -1-\sqrt{5} & 4 \\ 2 & 2 & 2 & 2 & 2 \end{bmatrix}$$

Then $P^n = (VDV^{-1})(VDV^{-1})\cdots(VDV^{-1})$
$= VD^nV^{-1}$

Since $\frac{1}{8}(3-\sqrt{5})<1$, $\frac{1}{8}(3+\sqrt{5})<1$,

$$\lim_{n\to\infty} P^n = V\cdot \mathrm{diag}(0,0,0,0,1)V^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

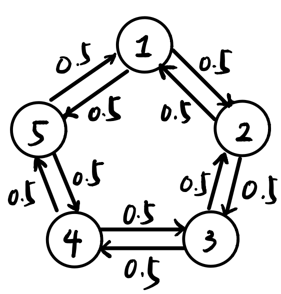
Thus $$p_\infty^T = p_0^T \lim_{t\to\infty} P^t = p_0^T\cdot\frac{1}{5}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} = [\tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}, \tfrac{1}{5}]$$

If we simulate p for 100 times, the simulated value is

$$p_{100} = [0.2000, 0.2000, 0.2000, 0.2000, 0.2000]^T$$

which is close to p_∞.

(c) Consider this situation as a Markov Chain with 5 states distributed as $p_0 = [25\ \ 20\ \ 35\ \ 24\ \ 46]^T$.



Consider that talking to neighbors as a trasition between 2 states.

Then, based on the observation of (a) & (c), we can estimate the average age:

(1) Randomly start from one person.

(2) Let person talk to one of his/her neighbors to tell him/her the current

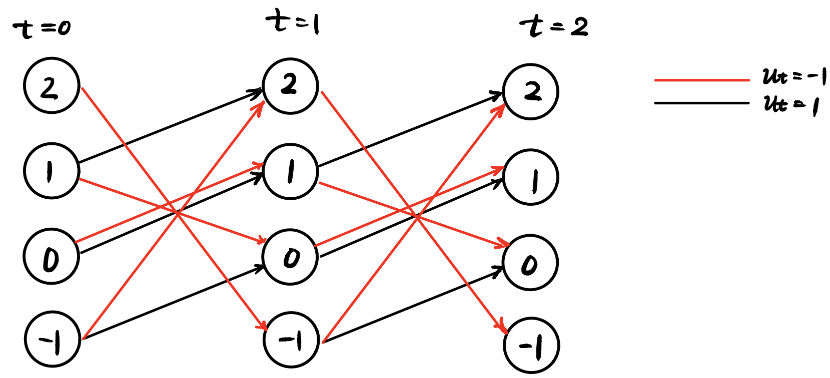
iterations and all the ages he/she knows.

(3) Repeat (2) until $\frac{\text{all the ages}}{\#\text{ iterations}} < \epsilon$

By doing this, we're actually simulating the process of (a) and (b), just changing the probability to actual ages.

2. State: $x_t \in \{-1, 0, 1, 2\}$, control input: $u_t \in \{-1, 1\}$

Motion model: $x_{t+1} = x_t u_t + u_t^2 = x_t u_t + 1$



(a) Cost: stage cost $l_t(x_t, u_t) = x_t u_t$, terminal cost $q(x) = x^2$.

When $t = T = 2$, $V_2 = q(x_2) = \begin{cases} 1, & x_2 = \pm 1 \\ 0, & x_2 = 0 \\ 4, & x_2 = 2 \end{cases}$

For all $x_t \in \{-1, 0, 1, 2\}$ and $t = 1, 0$:

$$V_t^*(x_t) = \min_{u_t \in \{-1,1\}} \{ l_t(x_t, u_t = -1) + V_{t+1}^{u_t=-1}(x_{t+1}), l_t(x_t, u_t = 1) + V_{t+1}^{u_t=1}(x_{t+1}) \}$$

$$= \min_{u_t \in \{-1,1\}} \left\{ -x_t + V_{t+1}^{u_t=-1}(x_{t+1}), x_t + V_{t+1}^{u_t=1}(x_{t+1}) \right\}.$$

For $t = 1$,

if $x_1 = 2$, $V_1^*(2) = \min_{u_t \in \{-1,1\}} \{ -2 + V_2(-1) \}$

$= -2 + 1 = -1$

$\pi_1^*(2) = -1$

if $x_1 = 1$, $V_1^*(1) = \min_{u_t \in \{-1,1\}} \{ -1 + V_2(0), 1 + V_2(2) \}$

$= \min \{ -1, 5 \} = -1$

$\pi_1^*(1) = -1$

if $x_1 = 0$, $V_1^*(0) = \min_{u_t \in \{-1,1\}} \{ V_2(1), V_2(1) \} = 1$

$\pi_1^*(0) = -1$ or 1

if $x_1 = -1$, $V_1^*(-1) = \min_{u_t \in \{-1,1\}} \{1 + V_2(2), -1 + V_2(0)\}$

$= \min\{5, -1\} = -1$

$\pi_1^*(-1) = 1$

For $\underline{t=0}$,

if $x_0 = 2$, $V_0^*(2) = \min_{u_t \in \{-1,1\}} \{-2 + V_1(-1)\}$

$= -3$

$\pi_0^*(2) = -1$

if $x_0 = 1$, $V_0^*(1) = \min_{u_t \in \{-1,1\}} \{-1 + V_1(0), 1 + V_1(2)\}$

$= \min_{u_t \in \{-1,1\}} \{-1 + 1, 1 - 1\} = 0$

$\pi_0^*(1) = -1$

if $x_0 = 0$, $V_0^*(0) = \min_{u_t \in \{-1,1\}} \{V_1(1), V_1(1)\}$

$= V_1(1) = -1$

$\pi_0^*(0) = -1$ or 1

if $x_0 = -1$, $V_0^*(-1) = \min_{u_t \in \{-1,1\}} \{1 + V_1(2), -1 + V_1(0)\}$

$= \min\{1 + (-1), -1 + 1\} = 0$

$\pi_0^*(-1) = -1$ or 1

(b) When $x_0 = 2$, $x_1 = -1$ with $u_0 = -1$

cost $\lambda_0(x_0 = 2, u_0 = -1) = \ell_0(x_0, u_0) = -2$.

$x_1 = -1$, based on (a),

$\pi_1^*(-1) = 1$, i.e. $u_1 = 1$. then

cost $\lambda_1(x_1 = -1, u_1 = 1) = \lambda_0 + \ell_1(x_1, u_1)$

$= (-2) + (-1) \times 1$

$= -3$

and $x_2 = 0$, with terminal cost $q(0) = 0$.

Hence, the optimal cost is -3,

control sequences are $[u_0, u_1] = [-1, 1]$

state trajectory is $[x_0, x_1, x_2] = [2, -1, 0]$.

3. (a) Consider to use Mathematical Induction to prove the optimal value function $V_t^*(x)$ is the minimum of 2^{T-t} p.d. quadratic functions.

① $t = T$, $2^{T-t} = 2^0 = 1$

$$V_T^*(x_T) = q(x_T) = \frac{1}{2}x_T^T x_T = \min\{\frac{1}{2}x_T^T x_T\}$$

is the minimum of $2^{T-t} = 1$ p.d. quadratic function.

② When $t = k$. Suppose that $V_k^*(x_k)$ is the minimum of 2^{T-k} p.d. quadratic functions.

Then $t = k-1$.

$$V_{k-1}^*(x_{k-1}) = \frac{1}{2}x_{k-1}^T x_{k-1} + \min\{V_k(Ax_{k-1}), V_k(Bx_{k-1})\}$$

Because $V_k(Ax_{k-1})$ and $V_k(Bx_{k-1})$ are the minimum of 2^{T-k} p.d. quadratic functions (according to assumption).

i.e. $V_k = \min_i\{x_{k-1}^T C_i x_{k-1}\}$, $C_i \in \mathbb{R}^{n\times n}$

Then $V_{k-1}^*(x_{k-1}) = \min_i\{x_{k-1}^T(\frac{1}{2}I + C_i)x_{k-1}\}$. with $\frac{1}{2}I + C_i > 0$.

\# quadratic functions $= 2^{T-k} \cdot 2 = 2^{T-(k-1)}$

Thus, $t = k-1$. $V_{k-1}^*(x_{k-1})$ is the minimum of $2^{T-(k-1)}$ p.d. quadratic functions.

Based on ①②. $V_t^*(x)$ is the minimum of 2^{T-t} p.d. quadratic functions.

(b) See below.