Exact analytic solution

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$$y' = \sin^2 x + y \cot x$$

Make a substitution:

$$y = y_1 * u$$

Solve the complementary equation:

$$y_1' - y_1 \cot x = 0$$

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = y_1 \cot x$$

$$\int \frac{\mathrm{d}y_1}{y_1} = \int \cot x \mathrm{d}x$$

$$\ln y_1 = \ln \sin x$$

$$y_1 = \sin x$$

If:

$$y = y_1 * u$$

Then:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\sin^2 x}{\sin x}$$

$$\int \mathrm{d}u = \int \sin x \mathrm{d}x$$

$$u = -\cos x + C$$

$$y = -\cos x \sin x + C \sin x$$

Solve Initial Value Problem for:

$$x_0 = 1, \quad y_0 = 1$$

$$y_0 = -\cos x_0 \sin x_0 + C \sin x_0$$

$$1 = 0 + C$$

$$C = 1$$

The solution for IVP is:

$$y = -\cos x \sin x + \sin x$$

Structure of the program

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1. Functions for computing $f(x,y)$ and $y(x)$ functions for all the numerical methods.
2. Function $find\_x(x_0, x_{max}, step)$ to compute all the x values (where x returns an array of x_i values for given x_0, x_{max} and $step$).
3. Function $exact(x_0, y_0, x_{max}, step)$ to compute all the y values for exact analytic solution, where y returns an array of y_i values respectively to x_i from previous step for given x_0, y_0, x_{max} and $step$.
4. Function $euler(x_0, y_0, x_{max}, step)$ to compute all the y values for Euler Method method, where y returns an array of y_i values respectively to x_i from the second step for given x_0, y_0, x_{max} and $step$.
5. Function $euler\_imp(x_0, y_0, x_{max}, step)$ to compute all the y values for Improved Euler method, where y returns an array of y_i values respectively to x_i from the second step for given x_0, y_0, x_{max} and $step$.
6. Function $runge\_kutta(x_0, y_0, x_{max}, step)$ to compute all the y values for Runge-Kutta method, where y returns an array of y_i values respectively to x_i from the second step for given x_0, y_0, x_{max} and $step$.
7. Function $global\_error(x_0, y_0, x_{max}, step_0, step_{max}, step\_of\_steps)$ to compute global errors for each method with respect to exact solution, where $global\_error$ returns three arrays (for each method) of $global\_error_i$ value respectively to $step_i$ value between $step_0$, $step_{max}$ with step $step\_of\_steps$.
8. Computation of all the results using mentioned functions and plotting.

Explanations of methods

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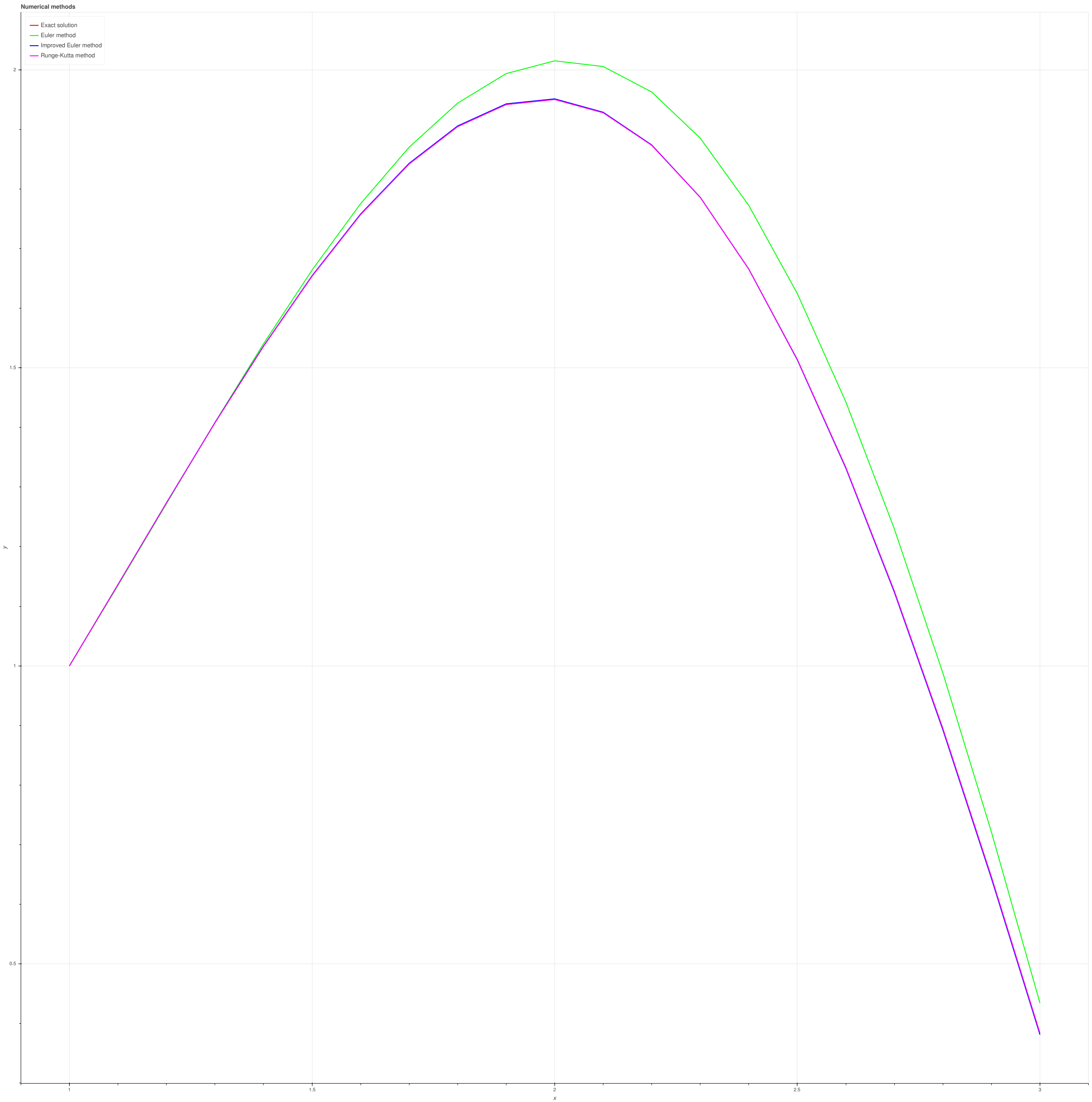
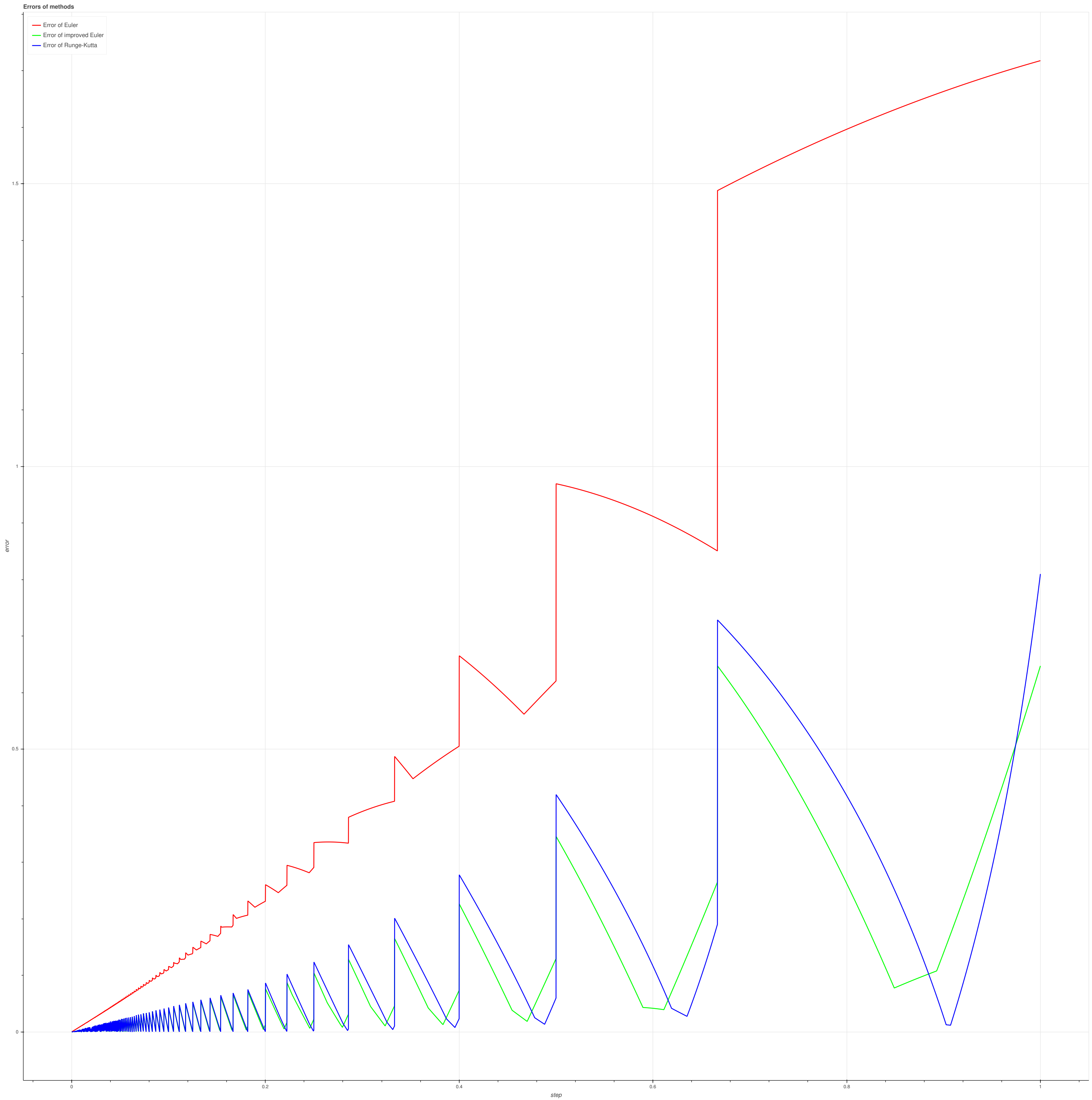
Euler method: The main idea of Euler method is to plot a function, which makes an approximation to the exact solution. How would we do that? We use slope of tangent lines to the original curve at some points x, so that all these pieces (lines from x_i, y_i to x_{i+1}, y_{i+1}) will be parallel to the original plot at point x_i, y_i. To count parallel pieces we consider our plot as some function of original function's derivative.

Improved Euler method: The idea is just the same as for Euler method, but we need our pieces from x_i, y_i to x_{i+1}, y_{i+1} to be parallel not to the original plot in x_i, y_i, but in $\frac{x_i+x_{i+1}}{2}$ (so in the center of our original piece).

Runge-Kutta method: The idea of this method is not trivial at all, but we consider some approximation of 4 different point on this piece from x_i, y_i to x_{i+1}, y_{i+1} using some coefficients, instead of a half of our exact plot's piece to make a slope of it's tangent line.


Numerical methods
Exact solution
Euler method
Improved Euler method
Runge-Kutta method
2
1.5
1
0.5
y
1
1.5
2
2.5
3
x


Errors of methods
Error of Euler
Error of improved Euler
Error of Runge-Kutta
error
step
0
0.5
1
1.5
0
0.2
0.4
0.6
0.8
1

Link to GutHub

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Just a link to my GitHub (click).