(a) Write out $\phi_3(x)$ in the set.

(b) By construction, the set $B' = \{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is orthogonal on $[a, b]$. Demonstrate that $\phi_0(x)$, $\phi_1(x)$, and $\phi_2(x)$ are mutually orthogonal.

Discussion Problems

23. (a) Consider the set of functions $\{1, x, x^2, x^3, \ldots\}$ defined on the interval $[-1, 1]$. Apply the Gram-Schmidt process given in Problem 22 to this set and find $\phi_0(x)$, $\phi_1(x)$, $\phi_2(x)$, and $\phi_3(x)$ of the orthogonal set B'.

(b) Discuss: Do you recognize the orthogonal set?

24. Verify that the inner product (f_1, f_2) in Definition 12.1 satisfies properties (i)–(iv) given on page 653.

25. In R^3, give an example of a set of orthogonal vectors that is not complete. Give a set of orthogonal vectors that is complete.

Book: Zill & Wright

12.2 Fourier Series

Introduction We have just seen in the preceding section that if $\{\phi_0(x), \phi_1(x), \phi_2(x), \ldots\}$ is a set of real-valued functions that is orthogonal on an interval $[a, b]$ and if f is a function defined on the same interval, then we can formally expand f in an orthogonal series $c_0\phi_0(x) + c_1\phi_1(x) + c_2\phi_2(x) + \ldots$. In this section we shall expand functions in terms of a special orthogonal set of trigonometric functions.

Trigonometric Series In Problem 12 in Exercises 12.1, you were asked to show that the set of trigonometric functions

$$\left\{1, \cos\frac{\pi}{p}x, \cos\frac{2\pi}{p}x, \cos\frac{3\pi}{p}x, \ldots, \sin\frac{\pi}{p}x, \sin\frac{2\pi}{p}x, \sin\frac{3\pi}{p}x, \ldots\right\} \tag{1}$$

is orthogonal on the interval $[-p, p]$. This set will be of special importance later on in the solution of certain kinds of boundary-value problems involving linear partial differential equations. In those applications we will need to expand a function f defined on $[-p, p]$ in an orthogonal series consisting of the trigonometric functions in (1), that is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{p}x + b_n \sin\frac{n\pi}{p}x\right).$$

This is why $a_0/2$ is used instead of a_0.

The coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$, can be determined in exactly the same manner as in the general discussion of orthogonal series expansions on pages 655 and 656. Before proceeding, note that we have chosen to write the coefficient of 1 in the set (1) as $a_0/2$ rather than a_0; this is for convenience only because the formula of a_n will then reduce to a_0 for $n = 0$.

Now integrating both sides of (2) from $-p$ to p gives

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx + \sum_{n=1}^{\infty}\left(a_n \int_{-p}^{p} \cos\frac{n\pi}{p}x\,dx + b_n \int_{-p}^{p} \sin\frac{n\pi}{p}x\,dx\right). \tag{3}$$

Since $\cos(n\pi x/p)$ and $\sin(n\pi x/p)$, $n \geq 1$, are orthogonal to 1 on the interval, the right side of (3) reduces to a single term:

$$\int_{-p}^{p} f(x)\,dx = \frac{a_0}{2}\int_{-p}^{p} dx = \frac{a_0}{2}x\Big|_{-p}^{p} = pa_0.$$

Solving for a_0 yields

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx. \tag{4}$$

We multiply (2) by $\cos(m\pi x/p)$ and integrate:

$$\int_{-p}^{p} f(x)\cos\frac{m\pi}{p}x\,dx = \frac{a_0}{2}\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx$$
$$+\sum_{n=1}^{\infty}\left(a_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx + b_n\int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx\right). \tag{5}$$

By orthogonality we have

$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\,dx = 0,\quad m>0,\qquad \int_{-p}^{p}\cos\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = 0$$

and
$$\int_{-p}^{p}\cos\frac{m\pi}{p}x\cos\frac{n\pi}{p}x\,dx = \begin{cases}0, & m\neq n\\ p, & m=n.\end{cases}$$

Thus (5) reduces to
$$\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx = a_n p,$$

and so
$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx. \tag{6}$$

Finally, if we multiply (2) by $\sin(m\pi x/p)$, integrate, and make use of the results

$$\int_{-p}^{p}\sin\frac{m\pi}{p}x\,dx = 0,\quad m>0,\qquad \int_{-p}^{p}\sin\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = 0$$

and
$$\int_{-p}^{p}\sin\frac{m\pi}{p}x\sin\frac{n\pi}{p}x\,dx = \begin{cases}0, & m\neq n\\ p, & m=n,\end{cases}$$

we find that
$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{7}$$

The trigonometric series (2) with coefficients a_0, a_n, and b_n defined by (4), (6), and (7), respectively, is said to be the **Fourier series** of the function f. The coefficients obtained from (4), (6), and (7) are referred to as **Fourier coefficients** of f.

In finding the coefficients a_0, a_n, and b_n, we assumed that f was integrable on the interval and that (2), as well as the series obtained by multiplying (2) by $\cos(m\pi x/p)$, converged in such a manner as to permit term-by-term integration. Until (2) is shown to be convergent for a given function f, the equality sign is not to be taken in a strict or literal sense. Some texts use the symbol $\sim$ in place of $=$. In view of the fact that most functions in applications are of a type that guarantees convergence of the series, we shall use the equality symbol. We summarize the results:

DEFINITION 12.5 Fourier Series

The **Fourier series** of a function f defined on the interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{p}x + b_n\sin\frac{n\pi}{p}x\right), \tag{8}$$

where
$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx \tag{9}$$

$$a_n = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi}{p}x\,dx \tag{10}$$

$$b_n = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi}{p}x\,dx. \tag{11}$$

Example 1 Expansion in a Fourier Series

Expand
$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases} \tag{12}$$
in a Fourier series.

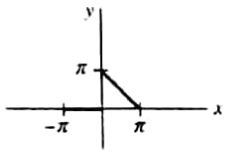


Figure 12.1 Function f in Example 1

Solution The graph of f is given in Figure 12.1. With $p = \pi$ we have from (9) and (10) that

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\,dx\right] = \frac{1}{\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{\pi} = \frac{\pi}{2}$$

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi}(\pi - x)\cos nx\,dx\right] \\ &= \frac{1}{\pi}\left[(\pi - x)\frac{\sin nx}{n}\bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\sin nx\,dx\right] \qquad [\sin n\pi = 0] \\ &= -\frac{1}{n\pi}\,\frac{\cos nx}{n}\bigg|_0^{\pi} \\ &= \frac{-\cos n\pi + 1}{n^2\pi} \qquad \leftarrow \cos n\pi = (-1)^n \\ &= \frac{1 - (-1)^n}{n^2\pi}. \end{aligned}$$

In like manner we find from (11) that

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}.$$

Therefore
$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left\{\frac{1 - (-1)^n}{n^2\pi}\cos nx + \frac{1}{n}\sin nx\right\}. \tag{13}$$ ❑

Note that a_n defined by (10) reduces to a_0 given by (9) when we set $n = 0$. But as Example 1 shows, this may not be the case *after* the integral for a_n is evaluated.

■ **Convergence of a Fourier Series** The following theorem gives sufficient conditions for convergence of a Fourier series at a point.

THEOREM 12.1 Conditions for Convergence

Let f and f' be piecewise continuous on the interval $(-p, p)$; that is, let f and f' be continuous except at a finite number of points in the interval and have only finite discontinuities at these points. Then the Fourier series of f on the interval converges to $f(x)$ at a point of continuity. At a point of discontinuity, the Fourier series converges to the average

$$\frac{f(x+) + f(x-)}{2},$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.*

*In other words, for x a point in the interval and $h > 0$,

$$f(x+) = \lim_{h\to 0} f(x + h), \quad f(x-) = \lim_{h\to 0} f(x - h).$$

proof of this theorem you are referred to the classic text by Churchill and n.*

Example 2 Convergence of a Point of Discontinuity

The function (12) in Example 1 satisfies the conditions of Theorem 12.1. Thus for every x in the interval $(-\pi, \pi)$, except at $x = 0$, the series (13) will converge to $f(x)$. At $x = 0$ the function is discontinuous, and so the series (13) will converge to

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi + 0}{2} = \frac{\pi}{2}.$$

Periodic Extension Observe that each of the functions in the basic set (1) has a different fundamental period,** namely, $2p/n$, $n \geq 1$, but since a positive integer multiple of a period is also a period we see that all of the functions have in common the period $2p$ (verify). Hence the right-hand side of (2) is $2p$-periodic; indeed, $2p$ is the fundamental period of the sum. We conclude that a Fourier series not only represents the function on the interval $(-p, p)$ but also gives the **periodic extension** of f outside this interval. We can now apply Theorem 12.1 to the periodic extension of f, or we may assume from the outset that the given function is periodic with period $T = 2p$; that is, $f(x + T) = f(x)$. When f is piecewise continuous and the right- and left-hand derivatives exist at $x = -p$ and $x = p$, respectively, then the series (8) converges to the average $[f(p-) + f(-p+)]/2$ at these endpoints and to this value extended periodically to $\pm 3p$, $\pm 5p$, $\pm 7p$, and so on. The Fourier series in (13) converges to the periodic extension of (12) on the entire x-axis. At $0, \pm 2\pi, \pm 4\pi, \ldots$, and at $\pm \pi, \pm 3\pi, \pm 5\pi, \ldots$, the series converges to the values

$$\frac{f(0+) + f(0-)}{2} = \frac{\pi}{2} \quad \text{and} \quad \frac{f(\pi+) + f(\pi-)}{2} = 0,$$

respectively. The solid dots in Figure 12.2 represent the value $\pi/2$.

We may assume that the given function f is periodic.

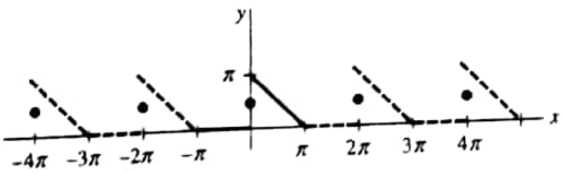


Figure 12.2 Periodic extension of the function f shown in Figure 12.1

Sequence of Partial Sums It is interesting to see how the sequence of partial sums $\{S_N(x)\}$ of a Fourier series approximates a function. For example, the first three partial sums of (13) are

$$S_1(x) = \frac{\pi}{4}, \quad S_2(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x, \quad S_3(x) = \frac{\pi}{4} + \frac{2}{\pi}\cos x + \sin x + \frac{1}{2}\sin 2x.$$

In Figure 12.3 we have used a CAS to graph the partial sums $S_5(x)$, $S_8(x)$, and $S_{15}(x)$ of (13) on the interval $(-\pi, \pi)$. Figure 12.3(d) shows the periodic extension using $S_{15}(x)$ on $(-4\pi, 4\pi)$.

*Ruel V. Churchill and James Ward Brown, *Fourier Series and Boundary Value Problems* (New York: McGraw-Hill, 2000).

**See Problem 21 in Exercises 12.1.

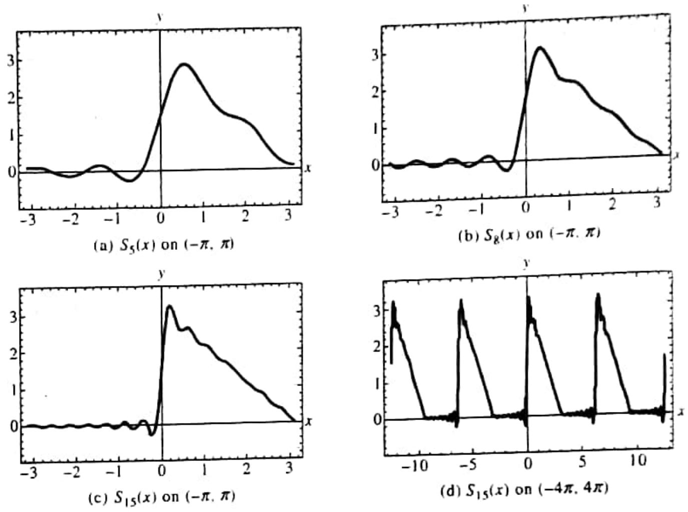


Figure 12.3 Partial sums of a Fourier series

EXERCISES 12.2

Answers to selected odd-numbered problems begin on page ANS-

In Problems 1–16, find the Fourier series of f on the given interval.

1. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

2. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases}$

3. $f(x) = \begin{cases} 1, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

4. $f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 \le x < 1 \end{cases}$

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} \pi^2, & -\pi < x < 0 \\ \pi^2 - x^2, & 0 \le x < \pi \end{cases}$

7. $f(x) = x + \pi, \quad -\pi < x < \pi$

8. $f(x) = 3 - 2x, \quad -\pi < x < \pi$

9. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases}$

10. $f(x) = \begin{cases} 0, & -\pi/2 < x < 0 \\ \cos x, & 0 \le x < \pi/2 \end{cases}$

11. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

12. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

13. $f(x) = \begin{cases} 1, & -5 < x < 0 \\ 1 + x, & 0 \le x < 5 \end{cases}$

14. $f(x) = \begin{cases} 2 + x, & -2 < x < 0 \\ 2, & 0 \le x < 2 \end{cases}$

15. $f(x) = e^x, \quad -\pi < x < \pi$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ e^x - 1, & 0 \le x < \pi \end{cases}$

17. Use the result of Problem 5 to show

$$\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

and

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. Use Problem 17 to find a series that gives the numerical value of $\pi^2/8$.

19. Use the result of Problem 7 to show

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

the result of Problem 9 to show

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \cdots.$$

The **root-mean-square value** of a function $f(x)$ defined over an interval (a, b) is given by

$$\text{RMS}(f) = \sqrt{\frac{\int_a^b f^2(x)\,dx}{b-a}}.$$

If the Fourier series expansion of f is given by (8), show that the RMS value of f over the interval $(-p, p)$ is given by

$$\text{RMS}(f) = \sqrt{\tfrac{1}{4}a_0^2 + \tfrac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)},$$

where a_0, a_n, and b_n are the Fourier coefficients in (9), (10), and (11).

12.3 Fourier Cosine and Sine Series

Review The effort expended in the evaluation of coefficients a_0, a_n, and b_n in expanding a function f in a Fourier series is reduced significantly when f is either an even or an odd function. A function f is said to be:

$$\textbf{even} \text{ if } f(-x) = f(x) \quad \text{and} \quad \textbf{odd} \text{ if } f(-x) = -f(x).$$

On a symmetric interval such as $(-p, p)$, the graph of an even function possesses symmetry with respect to the y-axis, whereas the graph of an odd function possesses symmetry with respect to the origin.

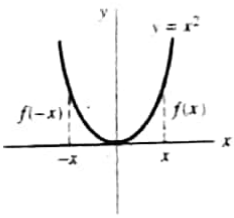


Figure 12.4 Even function

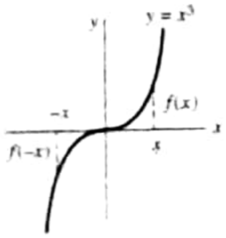


Figure 12.5 Odd function

Even and Odd Functions It is likely the origin of the words *even* and *odd* derives from the fact that the graphs of polynomial functions that consist of all even powers of x are symmetric with respect to the y-axis, whereas graphs of polynomials that consist of all odd powers of x are symmetric with respect to the origin. For example,

$$f(x) = x^{\overset{\downarrow \text{ even integer}}{2}} \text{ is even since } f(-x) = (-x)^2 = x^2 = f(x)$$

$$f(x) = x^{\overset{\downarrow \text{ odd integer}}{3}} \text{ is odd since } f(-x) = (-x)^3 = -x^3 = -f(x).$$

See Figures 12.4 and 12.5. The trigonometric cosine and sine functions are even and odd functions, respectively, since $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$. The exponential functions $f(x) = e^x$ and $f(x) = e^{-x}$ are neither even nor odd.

Properties The following theorem lists some properties of even and odd functions.

THEOREM 12.2 Properties of Even/Odd Functions

(a) The product of two even functions is even.

(b) The product of two odd functions is even.

(c) The product of an even function and an odd function is odd.

(d) The sum (difference) of two even functions is even.

(e) The sum (difference) of two odd functions is odd.

(f) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$.

(g) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Proof of (b) Let us suppose that f and g are odd functions. Then we have $f(-x) = -f(x)$ and $g(-x) = -g(x)$. If we define the product of f and g as $F(x) = f(x)g(x)$, then

$$F(-x) = f(-x)g(-x) = (-f(x))(-g(x)) = f(x)g(x) = F(x).$$

This shows that the product F of two odd functions is an even function. The proofs of the remaining properties are left as exercises. See Problem 52 in Exercises 12.3. ❑

Cosine and Sine Series If f is an even function on $(-p, p)$, then in view of the foregoing properties, the coefficients (9), (10), and (11) of Section 12.2 become

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\,dx = \frac{2}{p}\int_0^p f(x)\,dx$$

$$a_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\cos\frac{n\pi}{p}x}_{\text{even}}\,dx = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx$$

$$b_n = \frac{1}{p}\int_{-p}^{p} \underbrace{f(x)\sin\frac{n\pi}{p}x}_{\text{odd}}\,dx = 0.$$

Similarly, when f is odd on the interval $(-p, p)$,

$$a_n = 0, \qquad n = 0, 1, 2, \ldots, \qquad b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx.$$

We summarize the results in the following definition.

DEFINTION 12.6 Fourier Cosine and Sine Series

(i) The Fourier series of an even function on the interval $(-p, p)$ is the **cosine series**

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi}{p}x, \tag{1}$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx \tag{2}$$

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi}{p}x\,dx. \tag{3}$$

(ii) The Fourier series of an odd function on the interval $(-p, p)$ is the **sine series**

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi}{p}x, \tag{4}$$

where

$$b_n = \frac{2}{p}\int_0^p f(x)\sin\frac{n\pi}{p}x\,dx. \tag{5}$$

Example 1 Expansion in a Sine Series

Expand $f(x) = x$, $-2 < x < 2$, in a Fourier series.

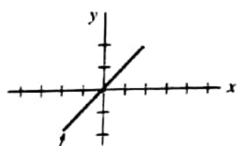


Figure 12.6 Odd function f in Example 1

Solution Inspection of Figure 12.6 shows that the given function is odd on the interval $(-2, 2)$, and so we expand f in a sine series. With the identification $2p = 4$, we have $p = 2$. Thus (5), after integration by parts, is

$$b_n = \int_0^2 x\sin\frac{n\pi}{2}x\,dx = \frac{4(-1)^{n+1}}{n\pi}.$$

Therefore
$$f(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi}{2}x. \qquad (6)$$

The function in Example 1 satisfies the conditions of Theorem 12.1. Hence the series (6) converges to the function on $(-2, 2)$ and the periodic extension (of period 4) given in Figure 12.7.

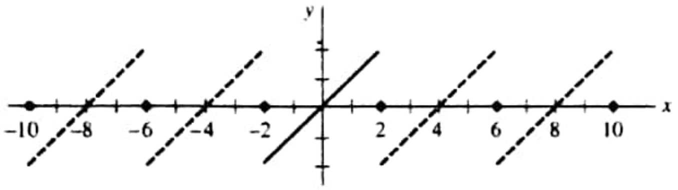


Figure 12.7 Periodic extension of the function f shown in Figure 12.6

Example 2 Expansion in a Sine Series

The function $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$ shown in Figure 12.8 is odd on the interval $(-\pi, \pi)$. With $p = \pi$ we have from (5)

$$b_n = \frac{2}{\pi}\int_0^{\pi}(1)\sin nx\,dx = \frac{2}{\pi}\frac{1-(-1)^n}{n},$$

and so
$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx. \qquad (7)$$

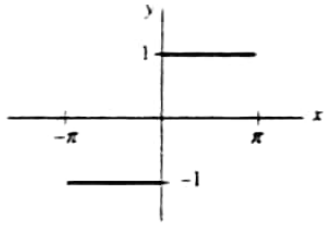


Figure 12.8 Odd function f in Example 2

Gibbs Phenomenon With the aid of a CAS we have plotted in Figure 12.9 the graphs $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_{15}(x)$ of the partial sums of nonzero terms of (7). As seen in Figure 12.9(d) the graph of $S_{15}(x)$ has pronounced spikes near the discontinuities at $x = 0$, $x = \pi$, $x = -\pi$, and so on. This "overshooting" by the partial sums S_N from the functional values near a point of discontinuity does not smooth out but remains fairly constant, even when the value N is taken to be large. This behavior of a Fourier series near a point at which f is discontinuous is known as the **Gibbs phenomenon**.

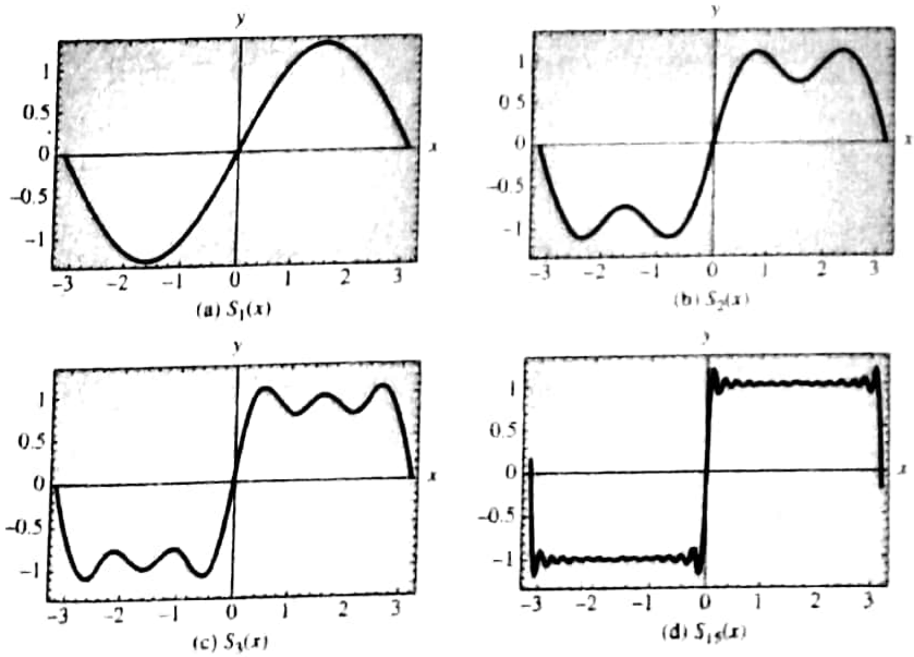


Figure 12.9 Partial sums of sine series (7) on $(-\pi, \pi)$

The periodic extension of f in Example 2 onto the entire x-axis is a meander function (see page 226).

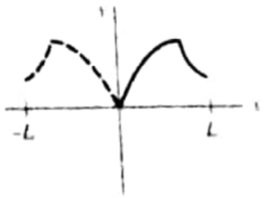


Figure 12.10 Even reflection

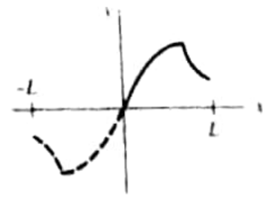
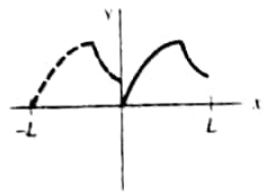


Figure 12.11 Odd reflection



$f(x) = f(x + L)$

Figure 12.12 Identity reflection

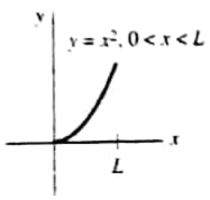


Figure 12.13 Function f in Example 3

■ **Half-Range Expansions** Throughout the preceding discussion it was understood that a function f was defined on an interval with the origin as midpoint, that is, $-p < x < p$. However, in many instances we are interested in representing a function that is defined only for $0 < x < L$ by a trigonometric series. This can be done in many different ways by supplying an arbitrary *definition* of the function on the interval $-L < x < 0$. For brevity we consider the three most important cases. If $y = f(x)$ is defined on the interval $0 < x < L$,

(*i*) reflect the graph of the function about the y-axis onto $-L < x < 0$; the function is now even on $-L < x < L$ (see Figure 12.10); or

(*ii*) reflect the graph of the function through the origin onto $-L < x < 0$; the function is now odd on $-L < x < L$ (see Figure 12.11); or

(*iii*) define f on $-L < x < 0$ by $f(x) = f(x + L)$ (see Figure 12.12).

Note that the coefficients of the series (1) and (4) utilize only the definition of the function on $0 < x < p$ (that is, half of the interval $-p < x < p$). Hence in practice there is no actual need to make the reflections described in (*i*) and (*ii*). If f is defined on $0 < x < L$, we simply identify the half-period as the length of the interval $p = L$. The coefficient formulas (2), (3), and (5) and the corresponding series yield either an even or an odd periodic extension of period $2L$ of the original function. The cosine and sine series obtained in this manner are known as **half-range expansions.** Lastly, in case (*iii*) we are defining the functional values on the interval $-L < x < 0$ to be the same as the values on $0 < x < L$. As in the previous two cases, there is no real need to do this. It can be shown that the set of functions in (1) of Section 12.2 is orthogonal on $a \le x \le a + 2p$ for any real number a. Choosing $a = -p$, we obtain the limits of integration in (9), (10), and (11) of that section. But for $a = 0$ the limits of integration are from $x = 0$ to $x = 2p$. Thus if f is defined over the interval $0 < x < L$, we identify $2p = L$ or $p = L/2$. The resulting Fourier series will give the periodic extension of f with period L. In this manner the values to which the series converges will be the same on $-L < x < 0$ as on $0 < x < L$.

Example 3 Expansion in Three Series

Expand $f(x) = x^2$, $0 < x < L$, (a) in a cosine series, (b) in a sine series, (c) in a Fourier series.

Solution The graph of the function is given in Figure 12.13.
(a) We have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2 \cos\frac{n\pi}{L}x\,dx = \frac{4L^2(-1)^n}{n^2\pi^2},$$

where integration by parts was used twice in the evaluation of a_n.

Thus
$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi}{L}x. \tag{8}$$

(b) In this case we must again integrate by parts twice:

$$b_n = \frac{2}{L}\int_0^L x^2 \sin\frac{n\pi}{L}x\,dx = \frac{2L^2(-1)^{n+1}}{n\pi} + \frac{4L^2}{n^3\pi^3}[(-1)^n - 1].$$

Hence
$$f(x) = \frac{2L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{(-1)^{n+1}}{n} + \frac{2}{n^3\pi^2}[(-1)^n - 1]\right\}\sin\frac{n\pi}{L}x. \tag{9}$$

(c) With $p = L/2$, $1/p = 2/L$, and $n\pi/p = 2n\pi/L$, we have

$$a_0 = \frac{2}{L}\int_0^L x^2\,dx = \frac{2}{3}L^2, \quad a_n = \frac{2}{L}\int_0^L x^2\cos\frac{2n\pi}{L}x\,dx = \frac{L^2}{n^2\pi^2}$$

$$b_n = \frac{2}{L}\int_0^L x^2 \sin \frac{2n\pi}{L} x \, dx = -\frac{L^2}{n\pi}.$$

Therefore
$$f(x) = \frac{L^2}{3} + \frac{L^2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n^2\pi}\cos\frac{2n\pi}{L}x - \frac{1}{n}\sin\frac{2n\pi}{L}x\right\}. \tag{10}$$

The series (8), (9), (10) converge to the $2L$-periodic even extension of f, the $2L$-periodic odd extension of f, and the L-periodic extension of f, respectively. The graphs of these periodic extensions are shown in Figure 12.14. ❑

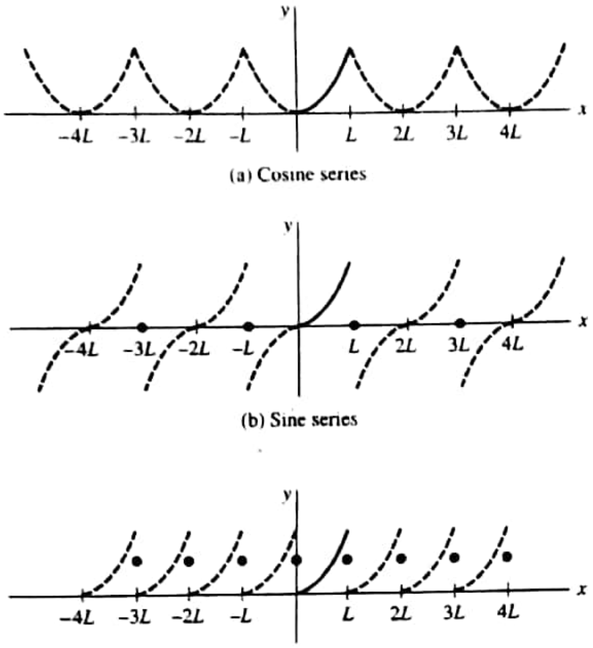


Figure 12.14 Different periodic extensions of the function f

■ **Periodic Driving Force** Fourier series are sometimes useful in determining a particular solution of a differential equation describing a physical system in which the input or driving force $f(t)$ is periodic. In the next example we find a particular solution of the differential equation

$$m\frac{d^2x}{dt^2} + kx = f(t) \tag{11}$$

by first representing f by a half-range sine expansion and then assuming a particular solution of the form

$$x_p(t) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{p} t. \tag{12}$$

Example 4 Particular Solution of a DE

An undamped spring-mass system, in which the mass $m = \frac{1}{16}$ slug and the spring constant $k = 4$ lb/ft, is driven by the 2-periodic external force $f(t)$ shown in Figure 12.15. Although the force $f(t)$ acts on the system for $t > 0$, note that if we extend the graph of the function in a 2-periodic manner to the negative t-axis, we obtain an odd function. In practical terms this means that we need only find the half-range sine expansion of $f(t) = \pi t$, $0 < t < 1$. With $p = 1$ it follows from (5) and integration by parts that

$$b_n = 2\int_0^1 \pi t \sin n\pi t \, dt = \frac{2(-1)^{n+1}}{n}.$$

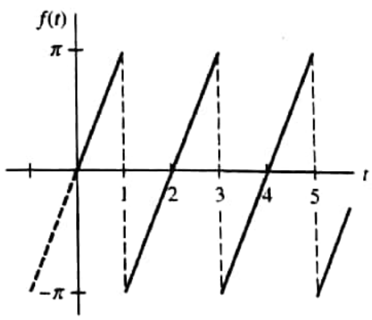


Figure 12.15 Periodic forcing function f in Example 4

From (11) the differential equation of motion is seen to be

$$\frac{1}{16}\frac{d^2x}{dt^2} + 4x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin n\pi t. \tag{13}$$

To find a particular solution $x_p(t)$ of (13), we substitute (12) into the equation and equate coefficients of $\sin n\pi t$. This yields

$$\left(-\frac{1}{16}n^2\pi^2 + 4\right)B_n = \frac{2(-1)^{n+1}}{n} \quad \text{or} \quad B_n = \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)}.$$

Thus

$$x_p(t) = \sum_{n=1}^{\infty} \frac{32(-1)^{n+1}}{n(64 - n^2\pi^2)} \sin n\pi t. \tag{14}$$ ❑

Observe in the solution (14) that there is no integer $n \geq 1$ for which the denominator $64 - n^2\pi^2$ of B_n is zero. In general, if there *is* a value of n, say N, for which $N\pi/p = \omega$, where $\omega = \sqrt{k/m}$, then the system described by (11) is in a state of pure resonance. In other words, we have pure resonance if the Fourier series expansion of the driving force $f(t)$ contains a term $\sin(N\pi/L)t$ (or $\cos(N\pi/L)t$) that has the same frequency as the free vibrations.

Of course, if the $2p$-periodic extension of the driving force f onto the negative t-axis yields an even function, then we expand f in a cosine series.

EXERCISES 12.3

Answers to selected odd-numbered problems begin on page ANS-31

In Problems 1–10, determine whether the function is even, odd, or neither.

1. $f(x) = \sin 3x$
2. $f(x) = x\cos x$
3. $f(x) = x^2 + x$
4. $f(x) = x^3 - 4x$
5. $f(x) = e^{|x|}$
6. $f(x) = e^x - e^{-x}$
7. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$
8. $f(x) = \begin{cases} x + 5, & -2 < x < 0 \\ -x + 5, & 0 \leq x < 2 \end{cases}$
9. $f(x) = x^3, 0 \leq x \leq 2$
10. $f(x) = |x^5|$

In Problems 11–24, expand the given function in an appropriate cosine or sine series.

11. $f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \leq x < \pi \end{cases}$
12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ 0, & -1 < x < 1 \\ 1, & 1 < x < 2 \end{cases}$
13. $f(x) = |x|, -\pi < x < \pi$
14. $f(x) = x, -\pi < x < \pi$
15. $f(x) = x^2, -1 < x < 1$
16. $f(x) = x|x|, -1 < x < 1$
17. $f(x) = \pi^2 - x^2, -\pi < x < \pi$
18. $f(x) = x^3, -\pi < x < \pi$
19. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \leq x < \pi \end{cases}$
20. $f(x) = \begin{cases} x + 1, & -1 < x < 0 \\ x - 1, & 0 \leq x < 1 \end{cases}$
21. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$
22. $f(x) = \begin{cases} -\pi, & -2\pi < x < -\pi \\ x, & -\pi \leq x < \pi \\ \pi, & \pi \leq x < 2\pi \end{cases}$
23. $f(x) = |\sin x|, -\pi < x < \pi$
24. $f(x) = \cos x, -\pi/2 < x < \pi/2$

In Problems 25–34, find the half-range cosine and sine expansions of the given function.

25. $f(x) = \begin{cases} 1, & 0 < x < \frac{1}{2} \\ 0, & \frac{1}{2} \leq x < 1 \end{cases}$
26. $f(x) = \begin{cases} 0, & 0 < x < \frac{1}{2} \\ 1, & \frac{1}{2} \leq x < 1 \end{cases}$
27. $f(x) = \cos x, 0 < x < \pi/2$
28. $f(x) = \sin x, 0 < x < \pi$
29. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \leq x < \pi \end{cases}$
30. $f(x) = \begin{cases} 0, & 0 < x < \pi \\ x - \pi, & \pi \leq x < 2\pi \end{cases}$

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 1, & 1 \le x < 2 \end{cases}$$

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \end{cases}$$

33. $f(x) = x^2 + x,\ 0 < x < 1$

34. $f(x) = x(2 - x),\ 0 < x < 2$

In Problems 35–38, expand the given function in a Fourier series.

35. $f(x) = x^2, 0 < x < 2\pi$ 36. $f(x) = x, 0 < x < \pi$

37. $f(x) = x + 1, 0 < x < 1$ 38. $f(x) = 2 - x, 0 < x < 2$

In Problems 39 and 40, proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = 1$, $k = 10$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is odd.

39. $f(t) = \begin{cases} 5, & 0 < t < \pi \\ -5, & \pi < t < 2\pi \end{cases}; \quad f(t + 2\pi) = f(t)$

40. $f(t) = 1 - t,\ 0 < t < 2; \quad f(t + 2) = f(t)$

In Problems 41 and 42, proceed as in Example 4 to find a particular solution $x_p(t)$ of equation (11) when $m = \frac{1}{4}$, $k = 12$, and the driving force $f(t)$ is as given. Assume that when $f(t)$ is extended to the negative t-axis in a periodic manner, the resulting function is even.

41. $f(t) = 2\pi t - t^2, 0 < t < 2\pi; \quad f(t + 2\pi) = f(t)$

42. $f(t) = \begin{cases} t, & 0 < t < \frac{1}{2} \\ 1 - t, & \frac{1}{2} < t < 1 \end{cases}; \quad f(t + 1) = f(t)$

43. (a) Solve the differential equation in Problem 39, $x'' + 10x = f(t)$, subject to the initial conditions $x(0) = 0, x'(0) = 0$.

(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

44. (a) Solve the differential equation in Problem 41, $\frac{1}{4}x'' + 12x = f(t)$, subject to the initial conditions $x(0) = 1, x'(0) = 0$.

(b) Use a CAS to plot the graph of the solution $x(t)$ in part (a).

45. Suppose a uniform beam of length L is simply supported at $x = 0$ and at $x = L$. If the load per unit length is given by $w(x) = w_0 x/L,\ 0 < x < L$, then the differential equation for the deflection $y(x)$ is

$$EI\frac{d^4y}{dx^4} = \frac{w_0 x}{L},$$

where E, I, and w_0 are constants. (See (4) in Section 3.9.)

(a) Expand $w(x)$ in a half-range sine series.

(b) Use the method of Example 4 to find a particular solution $y(x)$ of the differential equation.

46. Proceed as in Problem 45 to find a particular solution $y(x)$ when the load per unit length is as given in Figure 12.16.

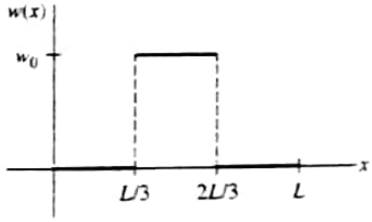


Figure 12.16 Graph for Problem 46

Computer Lab Assignments

In Problems 47 and 48, use a CAS to graph the partial sums $\{S_N(x)\}$ of the given trigonometric series. Experiment with different values of N and graphs on different intervals of the x-axis. Use your graphs to conjecture a closed-form expression for a function f defined for $0 < x < L$ that is represented by the series.

47. $$f(x) = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos nx + \frac{1 - 2(-1)^n}{n}\sin nx\right]$$

48. $$f(x) = -\frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi}{2}x$$

Discussion Problems

49. Is your answer in Problem 47 or in Problem 48 unique? Give a function f defined on a symmetric interval about the origin $-a < x < a$ that has the same trigonometric series as in Problem 47, as in Problem 48.

50. Discuss why the Fourier cosine series expansion of $f(x) = e^x$, $0 < x < \pi$ converges to e^{-x} on the interval $-\pi < x < 0$.

51. Suppose $f(x) = e^x$, $0 < x < \pi$ is expanded in a cosine series, and then $f(x) = e^x$, $0 < x < \pi$ is expanded in a sine series. If the two series are added and then divided by 2 (that is, the average of the two series) we get a series with cosines and sines that also represents $f(x) = e^x$ on the interval $0 < x < \pi$. Is this a full Fourier series of f? [*Hint*: What does the averaging of the cosine and sine series represent on the interval $-\pi < x < 0$?]

52. Prove properties (a), (c), (d), (f), and (g) in Theorem 12.2.