1. The energy equation of compressible flow for adiabatic flow is derived below:-

Assumption:
1. Steady flow
2. One dimensional flow
3. Frictionless flow
4. Equation of state is applicable.

Bernoulli's equation;

$$\frac{p}{\gamma}+\frac{v^2}{2g}+z=\text{const}$$

$$\Rightarrow \frac{p}{\rho g}+\frac{v^2}{2g}+z=\text{const}$$

$$\Rightarrow \frac{p}{\rho}+\frac{v^2}{2}+zg=\text{const}$$

$$\Rightarrow \frac{dp}{\rho}+\frac{2v\,dv}{2}+0=0$$

$$\Rightarrow \frac{dp}{\rho}+v\,dv=0 \quad \text{——} (1)$$

For adiabatic process

$$pv^k=\text{const}$$

$$\frac{p}{\rho^k}=\text{const}$$

$$\frac{p}{\rho^k}=\frac{p_1}{\rho_1^k}$$

$$\Rightarrow \frac{1}{\rho^k}=\frac{1}{p}\cdot\frac{p_1}{\rho_1^k}$$

$$\Rightarrow \frac{1}{\rho}=\left(\frac{1}{p}\right)^{1/k}\cdot\frac{p_1^{1/k}}{\rho_1}$$

$$\Rightarrow dp\left\{\left(\frac{1}{p}\right)^{1/k}\cdot\frac{p_1^{1/k}}{\rho_1}\right\}+v\,dv=0$$

$$\Rightarrow \frac{p_1^{1/k}}{\rho_1}\int_1^2\frac{dp}{p^{1/k}}+\int_1^2 v\,dv=0$$

$$\Rightarrow \frac{p_1^{1/k}}{\rho_1}\left[\frac{p^{-\frac{1}{k}+1}}{-\frac{1}{k}+1}\right]_1^2+\frac{1}{2}(v_2^2-v_1^2)=0$$

$$1-\frac{1}{k}=\frac{k-1}{k}$$

$$\Rightarrow \frac{k}{k-1}\,\frac{p_1^{1/k}}{\rho_1}\left[p_2^{\frac{k-1}{k}}-p_1^{\frac{k-1}{k}}\right]+\frac{1}{2}(v_2^2-v_1^2)=0$$

$$\Rightarrow \frac{1}{2}(v_2^2-v_1^2)=\frac{k}{k-1}\,\frac{p_1^{1/k}}{\rho_1}\left[p_1^{\frac{k-1}{k}}-p_2^{\frac{k-1}{k}}\right]$$

$$=\frac{k}{k-1}\left[\frac{1}{\rho_1}p_1^{\frac{1}{k}+\frac{k-1}{k}}-p_2^{\frac{k-1}{k}}\cdot\frac{p_1^{1/k}}{\rho_1}\right]$$

$$=\frac{k}{k-1}\left[\frac{p_1}{\rho_1}-p_2^{\frac{k-1}{k}}\cdot p_2^{1/k}\cdot\frac{1}{\rho_2}\right]$$

$$=\frac{k}{k-1}\left[\frac{p_1}{\rho_1}-\frac{p_2}{\rho_2}\right]$$

$$=\frac{k}{k-1}\,\frac{p_1}{\rho_1}\left[1-\frac{p_2}{\rho_2}\,\frac{\rho_1}{p_1}\right]$$

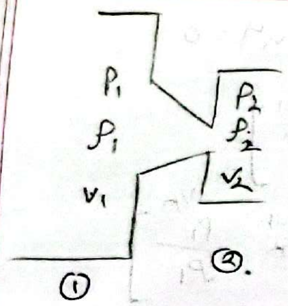
$$=\frac{k}{k-1}\,\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)\left(\frac{\rho_1}{\rho_2}\right)\right]$$

$$=\frac{k}{k-1}\,\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)\left(\frac{p_2}{p_1}\right)^{-1/k}\right]$$

$$=\frac{k}{k-1}\,\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{1-1/k}\right]$$

$$\therefore \frac{1}{2}(v_2^2-v_1^2)=\frac{k}{k-1}\,\frac{p_1}{\rho_1}\left[1-\left(\frac{p_2}{p_1}\right)^{\frac{k-1}{k}}\right]$$

(2)



Assumptions!

1. Steady flow
2. One dimensional flow
3. Frictionless flow
4. Adiabatic process
5. Equation of state is applicable.

at point (2) mass flow rate $\dot{m}_2 = A_2 \rho_2 V_2$ —— (I)

if initial velocity $V_1 = 0$

energy equation for adiabatic process

$$\frac{1}{2}(V_2^2 - V_1^2) = \frac{K}{K-1}\frac{P_1}{\rho_1}\left[1-\left(\frac{P_2}{P_1}\right)^{K-1/K}\right]$$

$$\therefore V_2 = \sqrt{\frac{2K}{K-1}\frac{P_1}{\rho_1}\left[1-\left(\frac{P_2}{P_1}\right)^{K-1/K}\right]} \quad \text{—— (II)}$$

using equation (II) in equation (I) we have;

$$\dot{m}_2 = A_2 \rho_2 \sqrt{\frac{2K}{K-1}\frac{P_1}{\rho_1}\left[1-\left(\frac{P_2}{P_1}\right)^{K-1/K}\right]} \quad \text{—— (III)}$$

as

$P/\rho^K = \text{const}$

$$\therefore \frac{P_1}{\rho_1^K} = \frac{P_2}{\rho_2^K}$$

$$\Rightarrow \rho_2^K = \frac{P_2}{P_1}\rho_1^K$$

$$\therefore \rho_2 = \left(\frac{P_2}{P_1}\right)^{1/K} \cdot \rho_1 \quad \text{—— (IV)}$$

using eqn (IV) in eqn (III)

$$\dot{m}_2 = A_2\left(\frac{P_2}{P_1}\right)^{1/K} \cdot \rho_1 \sqrt{\frac{2K}{K-1}\frac{P_1}{\rho_1}\left[1-\left(\frac{P_2}{P_1}\right)^{K-1/K}\right]}$$

$$= A_2\sqrt{\frac{2K}{K-1}\cdot\frac{P_1}{\rho_1}\cdot\rho_1^2\cdot\left(\frac{P_2}{P_1}\right)^{2/K}\left[1-\left(\frac{P_2}{P_1}\right)^{K-1/K}\right]}$$

$$= A_2\sqrt{\frac{2K}{K-1}P_1\rho_1\left[\left(\frac{P_2}{P_1}\right)^{2/K}-\left(\frac{P_2}{P_1}\right)^{2/K+K-1/K}\right]}$$

$$= A_2\sqrt{\frac{2K}{K-1}P_1\rho_1\left[\left(\frac{P_2}{P_1}\right)^{2/K}-\left(\frac{P_2}{P_1}\right)^{\frac{K+1}{K}}\right]} \quad \text{—— (V)} \qquad \text{Let } \frac{P_2}{P_1} = x$$

$$= A_2\sqrt{\frac{2K}{K-1}P_1\rho_1\left[x^{2/K} - x^{K+1/K}\right]}$$

now; if $\frac{d\dot{m}_2}{dx} = 0$

$$\Rightarrow \frac{d}{dx}\left[A_2\sqrt{\frac{2K}{K-1}P_1\rho_1\left[x^{2/K}-x^{K+1/K}\right]}\right] = 0$$

$$\Rightarrow A_2\left(\frac{2K}{K-1}P_1\rho_1\right)^{1/2}\left(x^{2/K}-x^{K+1/K}\right)^{-1/2}\left[\frac{2}{K}x^{2/K-1} - \frac{K+1}{K}x^{\frac{K+1}{K}-1}\right] = 0$$

$$\Rightarrow \frac{2}{K}x^{\frac{2-K}{K}} - \frac{K+1}{K}x^{1/K} = 0$$

$$\Rightarrow \frac{2}{K}x^{2-K/K} = \frac{K+1}{K}\cdot x^{1/K}$$

$$\Rightarrow \frac{x^{2-K/K}}{x^{1/K}} = \frac{K+1}{K}\times\frac{K}{2}$$

$$\Rightarrow x^{\frac{2-K}{K}-\frac{1}{K}} = \frac{K+1}{2}$$

$$\Rightarrow x^{1-k/k} = \frac{k+1}{2}$$

$$\Rightarrow x = \left(\frac{k+1}{2}\right)^{k/1-k}$$

$$\Rightarrow x = \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}}$$

$$\therefore \frac{P_2}{P_1} = \left(\frac{2}{k+1}\right)^{k/k-1} \text{———— (6)}$$

For maximum mass flow rate using eq^n (5)

$$\dot{m}_2 = A_2\sqrt{\frac{2k}{k-1} P_1\rho_1 \left[\left(\frac{P_2}{P_1}\right)^{2/k} - \left(\frac{P_2}{P_1}\right)^{k+1/k}\right]}$$

$$\dot{m}_2 = A_2\sqrt{\frac{2k}{k-1} P_1\rho_1 \left[\left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\cdot\frac{2}{k}} - \left(\frac{2}{k+1}\right)^{\frac{k}{k-1}\cdot\frac{k+1}{k}}\right]}$$

$$\dot{m}_2 = A_2\sqrt{\frac{2k}{k-1}P_1\rho_1}\sqrt{\left(\frac{2}{k+1}\right)^{2/k-1} - \left(\frac{2}{k+1}\right)^{k+1/k-1}}$$

$$= A_2\sqrt{\frac{2k}{k-1}P_1\rho_1}\cdot\cancel{\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}}\sqrt{\left(\frac{2}{k+1}\right)^{\frac{2}{k-1}}\left[1-\left(\frac{2}{k+1}\right)^{\frac{k+1}{k-1}-2/k-1}\right]}$$

$$= A_2\sqrt{\frac{2k}{k-1}P_1\rho_1}\cdot\left(\frac{2}{k+1}\right)^{1/k-1}\sqrt{\left[1-\left(\frac{2}{k+1}\right)^1\right]}$$

$$= A_2\sqrt{\frac{2k}{k-1}P_1\rho_1}\left(\frac{2}{k+1}\right)^{1/k-1}\cdot\left[\frac{k+1-2}{k+1}\right]^{1/2}$$

$$\dot{m}_2 = A_2\left(\frac{2}{k+1}\right)^{1/k-1}\sqrt{\frac{2k}{k-1}P_1\rho_1}\sqrt{\frac{k-1}{k+1}}$$

$$= A_2\sqrt{2kP_1\rho_1}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\cdot\cancel{\frac{1}{1}}\cdot\sqrt{\frac{k-1}{k+1}}\cdot\frac{1}{\sqrt{k-1}}$$

$$= A_2\sqrt{kP_1\rho_1}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}}\cdot\left(\frac{2}{k+1}\right)^{1/2}$$

$$= A_2\sqrt{kP_1\rho_1}\left(\frac{2}{k+1}\right)^{\frac{1}{k-1}+\frac{1}{2}}$$

$$\frac{k-1+2}{2(k-1)}$$

$$= A_2\sqrt{kP_1\rho_1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}$$

$$\therefore \dot{m}_{2(max)} = A_2\sqrt{kP_1\rho_1}\left(\frac{2}{k+1}\right)^{\frac{k+1}{2(k-1)}}$$

⑦ The equation for relationship between area, velocity and Mach no. for compressible flow is derived below:

Assumption:-

① Steady flow

② One dimensional flow

③ Frictionless flow

④ adiabatic process;

Mach no. $M = \frac{\text{velocity of fluid}}{\text{velocity of sound}}$

$M = \frac{V}{C}$

from Bernoulli's equation

$\frac{dp}{\rho} + v\,dv = 0$

$\Rightarrow \frac{1}{\rho} Kp \frac{d\rho}{\rho} + v\,dv = 0$

$\Rightarrow Kp \frac{d\rho}{\rho^2} + v\,dv = 0$

$\Rightarrow c^2 \rho \frac{d\rho}{\rho^2} + v\,dv = 0$

$\Rightarrow c^2 \frac{d\rho}{\rho} + v\,dv = 0$

$\Rightarrow \frac{c^2}{V^2} \frac{d\rho}{\rho} + \frac{dv}{v} = 0$

$\Rightarrow \frac{1}{M^2} \frac{d\rho}{\rho} + \frac{dv}{v} = 0$

$\therefore d\rho/\rho = -M^2 \frac{dv}{v}$

as adiabatic process

$\frac{P}{\rho^K} = \text{const}$

Bulk modulus of elastic

$K = -\frac{dp}{dv/v} = \frac{dp}{d\rho/\rho} = E$

$E = Kp$

$\frac{dp}{d\rho/\rho} = Kp$

$\therefore dp = Kp \frac{d\rho}{\rho}$

$c = \sqrt{\frac{Kp}{\rho}}$

$c^2 = \frac{Kp}{\rho}$

$\boxed{c^2 \rho = Kp}$

as steady flow mass flow rate will be same.

$\dot{m} = A V \rho = \text{const}$

$\Rightarrow A v \rho = \text{const}$

$\Rightarrow \log A + \log v + \log \rho = \log c$

$\Rightarrow \frac{dA}{A} + \frac{dv}{v} + \frac{d\rho}{\rho} = 0$

$\Rightarrow \frac{dA}{A} + \frac{dv}{v} - M^2 \frac{dv}{v} = 0$

$\Rightarrow \frac{dA}{A} + \frac{dv}{v}(1 - M^2) = 0$

$\Rightarrow \frac{dA}{A} = -\frac{dv}{v}(1 - M^2)$

$\Rightarrow \frac{dA}{A} = \frac{dv}{v}(M^2 - 1)$

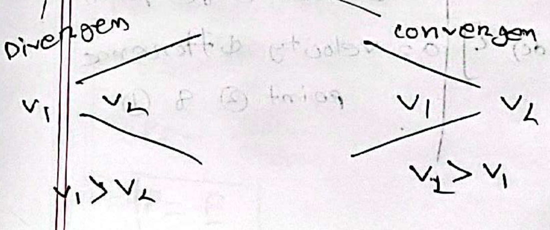
$\Rightarrow \frac{dA}{dv} = \frac{A}{v}(M^2 - 1)$

(i) If Subsonic flow $M < 1$

$\therefore \frac{dA}{dv} = (-)ve$

$dA\,(+)$ $dv = -$

$dA\,(-)$ $dv = +$

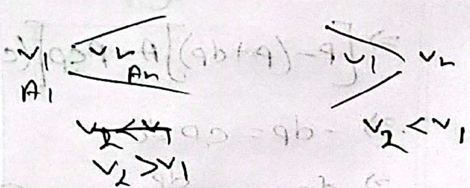


(ii) If sonic flow $M > 1$

$\therefore \frac{dA}{dv} = (+)ve$

$dA\,(+)$ $dv\,(+)$

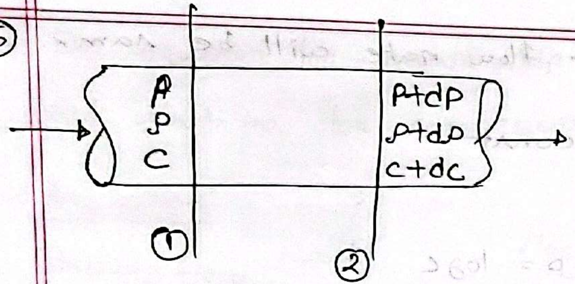
$dA\,(-)$ $dv\,(-)$



(iii) supersonic $M > 1$

$\therefore \frac{dA}{dv} = 0$

(5)



Assumptions :-

1. Steady flow
2. One dimensional flow
3. Frictionless flow
4. Adiabatic process.

as mass flow rate same so we can write

$\dot{m}_1 = \dot{m}_2$

$\Rightarrow A\rho c = A(\rho + d\rho)(c + dc)$

$\Rightarrow \quad = A(\rho c + \rho\, dc + c\, d\rho + \cancel{d\rho\, dc})$ very small

$\Rightarrow A\rho c = A\rho c + A\rho\, dc + Ac\, d\rho$

$\Rightarrow -\rho\, dc = c\, d\rho$

$\Rightarrow dc = -c \dfrac{d\rho}{\rho}$ —— (1)

we know $F = PA$

and from newton's 2nd law of gravity

$F = \dot{m} a$

$\Rightarrow PA = \dot{m} a$

$\Rightarrow [P - (P + dP)]A = Ac\rho[(c + dc) - c]$

P = Pressure difference between (1) & (2) point

a = velocity difference point (2) & (1)

$\Rightarrow -dP = c\rho\, dc$

$\Rightarrow dc = -\dfrac{dP}{c\rho}$ —— (11)

from equation (1) & (11)

$-c\dfrac{d\rho}{\rho} = -\dfrac{dP}{c\rho}$

$\Rightarrow c^2 = \dfrac{dP}{d\rho}$

$\Rightarrow c = \sqrt{\dfrac{dP}{d\rho}}$

Bulk modulus of elasticity

$K = -\dfrac{dP}{dV/V} = \dfrac{dP}{d\rho/\rho} = E$

$\Rightarrow \dfrac{dP}{d\rho} = \dfrac{E}{\rho}$

$\therefore \boxed{c = \sqrt{\dfrac{E}{\rho}}}$ or $\boxed{c = \sqrt{\dfrac{K}{\rho}}}$.

(6) For isentropic process; also for adiabatic process.

$PV^K = \text{const}$

$V^K dP + P \cdot K V^{K-1} dV = 0$

$\Rightarrow V^K\left[dP + \dfrac{PK}{V} dV\right] = 0$

$\Rightarrow dP = -KP \dfrac{dV}{V}$

$\Rightarrow KP = -\dfrac{dP}{dV/V}$

$\Rightarrow \boxed{KP = E}$

$\therefore$ velocity of sound

$c = \sqrt{\dfrac{E}{\rho}} = \sqrt{\dfrac{KP}{\rho}}$

$= \sqrt{\dfrac{KP}{P/RT}}$

$\therefore \boxed{c = \sqrt{KRT}}$

extra

For isothermal process

$PV = \text{const}$

$P\,dV + dP \cdot V = 0$

$\Rightarrow P = -\dfrac{dP\,V}{dV}$

$\Rightarrow P = -\dfrac{dP}{dV/V}$

$\Rightarrow \boxed{P = E}$

$\therefore c = \sqrt{\dfrac{E}{\rho}}$.

if $P = \rho RT$

$\rho = P/RT$

$\therefore \boxed{c = \sqrt{\dfrac{P}{\rho}}}$

$\therefore c = \sqrt{RT}$

(9) Prove that maximum velocity at the throat of the convergence divergence nozzle is equal to the velocity of sound.

From adiabatic energy equation:

$$\frac{1}{2}(v_2^2 - v_1^2) = \frac{K}{K-1}\left[\frac{P_1}{\rho_1} - \frac{P_2}{\rho_2}\right] \qquad P_1 = \rho_1 R T_1$$

$$= \frac{K}{K-1}\left[\frac{P_1}{P_1/RT_1} - \frac{P_2}{P_2/RT_2}\right]$$

$$= \frac{KR}{K-1}\left[T_1 - T_2\right]$$

$$= \frac{KR}{K-1} T_2\left[\frac{T_1}{T_2} - 1\right]$$

$$= \frac{KR}{K-1} T_2\left[\left(\frac{P_2}{P_1}\right)^{1-K/K} - 1\right]$$

$$= \frac{KR}{K-1} T_2\left[\left(\frac{2}{K+1}\right)^{\frac{K}{K-1}\cdot\frac{1-K}{K}} - 1\right] \qquad \frac{P_2}{P_1} = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}$$

$$= \frac{KR}{K-1} T_2\left[\left(\frac{2}{K+1}\right)^{\frac{K}{K-1}\cdot\frac{-(K-1)}{K}} - 1\right]$$

$$= \frac{KR}{K-1} T_2\left[\left(\frac{2}{K+1}\right)^{-1} - 1\right]$$

$$= \frac{KR}{K-1} T_2\left[\frac{K+1}{2} - 1\right]$$

$$= \frac{KR}{K-1} T_2 \cdot \frac{K-1}{2}$$

$$\Rightarrow \frac{1}{2}(v_2^2 - v_1^2) = \frac{KRT_2}{2} \qquad \text{if } v_1 = 0$$

$$\therefore \frac{1}{2}v_2^2 = \frac{KRT_2}{2}$$

$$\therefore v_2 = \sqrt{KRT_2}$$

$$\therefore \boxed{v_2 = c}$$ proved.