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Lecture-03 : Complex Integration

Topics includes:

1. Complex Line integral & its related problems.
2. Cauchy's theorem & its related problems.
3. Cauchy's integral formula & its related problems

Complex Integration

Question Define complex line integral.

Ans. Let a complex function $f(z)$ be continuous at all points of a rectifiable (i.e. having finite length) curve C with endpoints $z_0 = a$ and $z_n = b$.

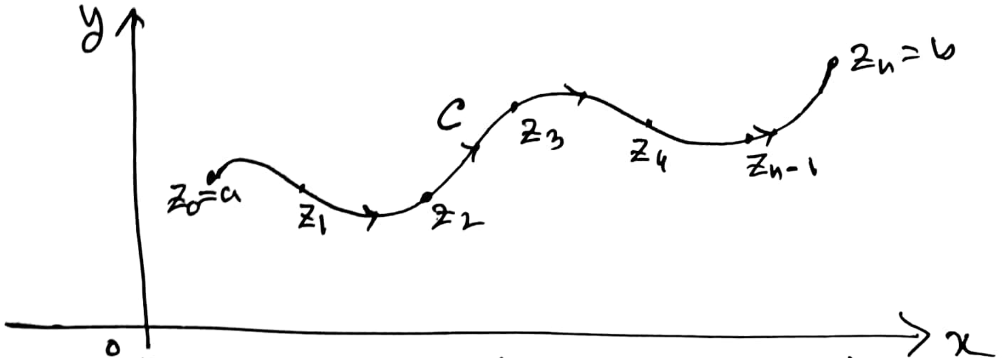


Fig. 1: Graph of complex line integral.

Then the integral $\int_a^b f(z)\,dz$ or $\int_C f(z)\,dz$ is called the complex line integral of $f(z)$ along the curve or the definite integral of $f(z)$ from a to b along the curve C.

* Note that if $f(z)$ is analytic at all points of a region R and if C is a curve lying in R, then $f(z)$ is certainly integrable along C.

Connection between Real and Complex line integrals:

If $f(z) = u + iv$, then the above complex line integral can be expressed in term of real line integrals as

$$\int_C f(z)\,dz = \int_C (u+iv)(dx + i\,dy)$$

$$\Rightarrow \int_C f(z)\,dz = \int_C \{u\,dx - v\,dy\} + i\int v\,dx + u\,dy.$$

Example-1 Using the definition of line integral to evaluate $\int_C f(z)\,dz$ where $f(z) = z^2$ and C is the line segment from $z=0$ to $z=1$ and then $z=1$ to $z=1+i$.

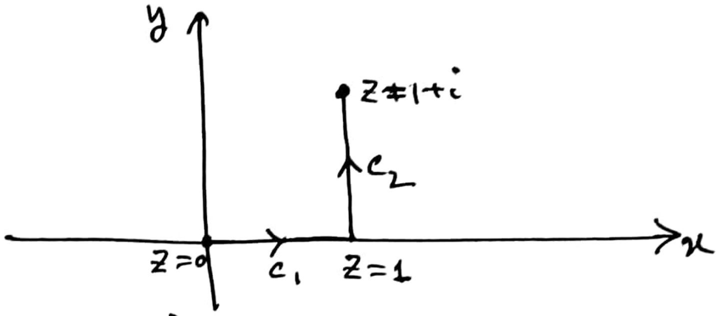


Fig. 1:

Solution:

Here $z = 0 = 0 + i.0 = (0,0)$

$z = 1 = 1 + 0i = (1,0)$

and $z = 1+i = (1,1)$

Let the curve C composed of two line segments which are c_1 & c_2. $[\because z = x+iy]$

Now on c_1: we have $z = x \Rightarrow dz = dx$ $[\because y=0]$ and limits of x are 0 to 1.

Also on c_2: we have $z = 1+iy$ $[\because x=1]$

$\Rightarrow dz = i\,dy$

Also limits of y are 0 to 1.

Thus
$$\int_C f(z)\,dz = \int_C z^2\,dz = \int_{c_1} z^2\,dz + \int_{c_2} z^2\,dz$$

$$= \int_0^1 x^2\,dx + \int_0^1 (1+iy)^2 i\,dy = \int_0^1 x^2\,dx + \int_0^1 (1+2iy-y^2)i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y + 2i.\frac{y^2}{2} - \frac{y^3}{3}\right]_0^1$$

$$= \left(\frac{1}{3} - 0\right) + i\left\{\left(1 + i - \frac{1}{3}\right) - 0\right\}$$

$$= \frac{1}{3} + i - 1 - \frac{i}{3} = -\frac{2}{3} + \frac{2}{3}i = -\frac{2}{3}(1-i)$$

Ans.

$\boxed{\text{Example-2}}$ If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,1)$ and $(2,3)$, then find the value of $\int_C (12z^2 - 4iz)\,dz$.

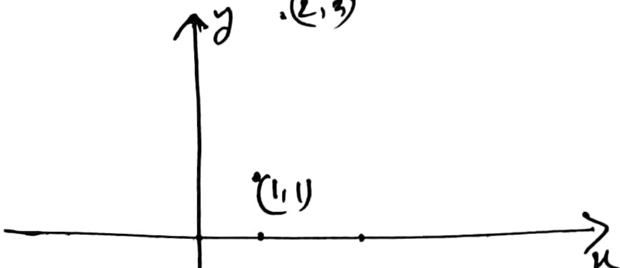


fig. 2

Solution: We have $(1,1) = 1+i$

and $(2,3) = 2+3i$

given $$\int_C (12z^2 - 4iz)\,dz = \int_{1+i}^{2+3i} (12z^2 - 4iz)\,dz$$

$$= 12\cdot\left[\frac{z^3}{3}\right]_{1+i}^{2+3i} - 4i\left[\frac{z^2}{2}\right]_{1+i}^{2+3i}$$

$$= 4\left[(2+3i)^3 - (1+i)^3\right] - 2i\left[(2+3i)^2 - (1+i)^2\right]$$

$$= (2+3i)^2\left\{4(2+3i) - 2i\right\} - (1+i)^2\left\{4(1+i) - 2i\right\}$$

$$= (4+12i-9)(8+10i) - (\cancel{1}+2i-\cancel{1})(4+2i)$$

$$= (-5+12i)(8+10i) - 2i(4+2i)$$

$$= -40+46i-120-8i+4$$

$$= -156+38i$$

i.e. $$\int_C (12z^2 - 4iz)\,dz = -156+38i$$ Ans.

Question: Define contour and closed contour with examples.

Ans: A curve which is composed of a finite number of smooth arcs is called a contour.
i.e. A contour is an arc consisting of a finite number of smooth arcs joined end to end.

If the initial point (starting point) of the first arc and the terminal point (end point) of the last arc of a contour coincide, then the contour is said to be a closed contour or a simple closed curve.

Examples: The boundaries of a circle, a semi-circle, a quadrant of a circle, a triangle, a rectangle, a square are examples of closed contours.

Fig.1: Contour not closed.

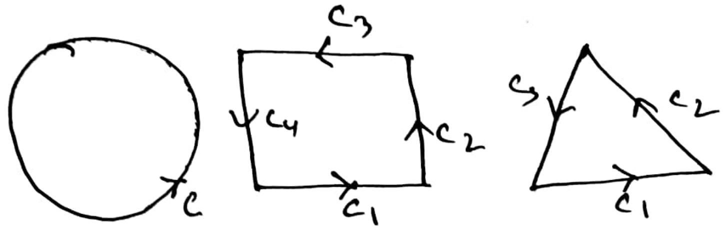


Fig.2: Closed contours.

Question: Define contour integral.

Ans: An integral of the form $\int_c f(z)\,dz$ is called a contour integral of $f(z)$ along a contour C in the z-plane. If C is a closed contour, then the integral $\oint_c f(z)\,dz$ is called a contour integral of $f(z)$ along the closed contour C in the z-plane.

- Note that although $\oint_C f(z)\,dz$ does not indicate the direction along the curve, it is conventional to take the direction positive which is anticlockwise unless indicated otherwise.

Question: State Cauchy's theorem or Cauchy's fundamental theorem or Cauchy's integral theorem.

Ans. Statement: If $f(z)$ is analytic and $f'(z)$ is continuous at all points within and on a simple closed curve C, then $\oint_C f(z)\,dz = 0$.

Example-1 Verify Cauchy's theorem for the function $z^3 - iz^2 - 5z + 2i$ if C is the circle $|z| = 1$.

Solution: Let $f(z) = z^3 - iz^2 - 5z + 2i$

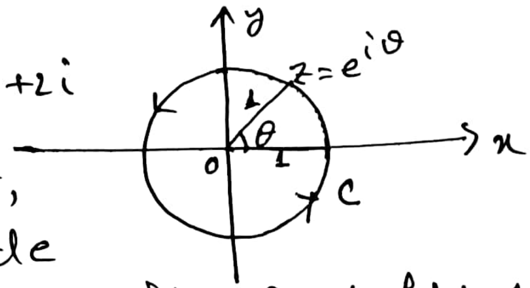


Fig.1: Graph of $|z|=1$

Then $f(z)$ is a polynomial in z, So it is analytic in the circle $|z| = 1$.

Now on the circle $C: |z| = 1$, we put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$

Then $dz = ie^{i\theta}\,d\theta$.

$$\therefore \oint_C f(z)\,dz = \oint_C (z^3 - iz^2 - 5z + 2i)\,dz$$

$$= \int_0^{2\pi} (e^{i3\theta} - ie^{i2\theta} - 5e^{i\theta} + 2i)\, ie^{i\theta}\,d\theta$$

$$= i\int_0^{2\pi} e^{i4\theta}\,d\theta + \int_0^{2\pi} e^{i3\theta}\,d\theta - 5i\int_0^{2\pi} e^{i2\theta}\,d\theta - 2\int_0^{2\pi} e^{i\theta}\,d\theta \quad \text{---} (1)$$

$$\Rightarrow \oint_C f(z)\,dz = 0+0-0-0=0.$$

$$\left[\because \int_0^{2\pi} e^{ik\theta}\,d\theta = \left[\frac{e^{ik\theta}}{ik}\right]_{\theta=0}^{2\pi} = \frac{1}{ik}\left[e^{i2k\pi}-e^{0}\right] = \frac{1}{ik}[1-1]=0 \text{ for } k=1,2,3,4\right]$$

Hence Cauchy's theorem is verified.

$\boxed{\text{Exercise:}}$ (a) verify Cauchy's theorem for the function $z^3-iz^2-5z+2i$ if C is the circle $|z-1|=2$.

(b) verify Cauchy's theorem for the function $5\sin 2z$ if C is the square with vertices $1\pm i$, $-1\pm i$.

Consequence of Cauchy's theorem:

(i) If $z=a$ is a point outside the closed curve C, then $\oint_C \frac{dz}{z-a}=0$.

(ii) If $z=a$ is a point inside the closed curve C, then $\oint_C \frac{dz}{z-a} = 2\pi i$

$\boxed{\text{Example-2}}$ Using Cauchy's integral theorem to evaluate $\oint_C \frac{3z^2+7z-1}{z-2}\,dz$ where C is the curve $|z|=1$.

Solution: Given $|z|=1 \Rightarrow x^2+y^2=1$ which is the circle with center $(0,0)$ and radius 1.

Also $z-2=0 \Rightarrow z=2$ which lies outside the circle $|z|=1$.

Hence by using Cauchy's integral theorem

$$\oint_C \frac{3z^2+7z-1}{z-2}\,dz = 0 \quad \underline{\text{Ans.}}$$

$\boxed{\text{Exercise-1}}$ Using Cauchy's integral theorem to evaluate the following integrals;

(a) $\oint_C \frac{2z+5}{(z-1)(z-2)}\,dz$ where is the circle $|z|=\frac{1}{2}$ $\left(\Rightarrow x^2+y^2=\frac{1}{4}\right)$ $(0,0)$ $r=\frac{1}{2}$

(=0 Ans)

(b) $\oint_C \frac{4z^2-6z+1}{z-4}\,dz$ where C is the circle $|z-1|=2$ $\left(\Rightarrow (x-1)^2+y^2=4\right)$ $(1,0)$ $r=2$

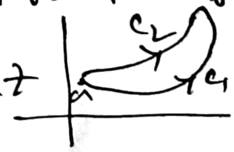
(=0 Ans.)

(c) $\oint_C \frac{e^z}{z-2}\,dz$ where C is the circle $|z|=1$.

(=0 Ans.)

* Important Consequence of Cauchy's theorem: If $f(z)$ is analytic in a simple-connected region R, then $\int_a^b f(z)dz$ is independent of the path in R joining any two points a & b in R.

i.e. $\int_a^b f(z)dz = \int_{C_1} f(z)dz = \int_{C_2} f(z)dz$



Question: State Cauchy's integral formula.

Statement: If $f(z)$ is analytic inside and on a simple closed curve C and 'a' is any point inside C, then $f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a}\,dz$ - - (i)

Where C is traversed in the positive sense.

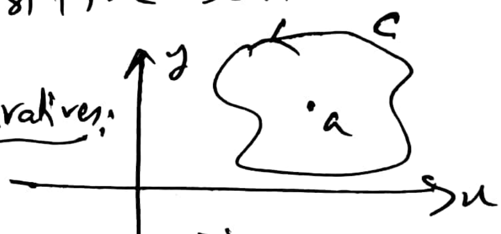


Fig. 1:

* Cauchy's integral formula for 1st & 2nd derivatives:

$$f'(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^2} \quad \text{- - - (ii)}$$

$$f''(a) = \frac{\underline{|2}}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^3} \quad \text{- - - (iii)}$$

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$$f^n(a) = \frac{\underline{|n}}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-a)^{n+1}} \quad \text{- - - (iv)}$$

for $n = 0, 1, 2, 3 \cdots$

* The results (i), (ii), (iii) - - (iv) are quite remarkable because they show that if a function $f(z)$ is known on the simple closed curve C, then the value of the function and all its derivatives can be found at all points inside C.

$\boxed{\text{Example-1}}$ Using Cauchy's integral formula to evaluate $\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$, where c is the circle $|z| = 3$.

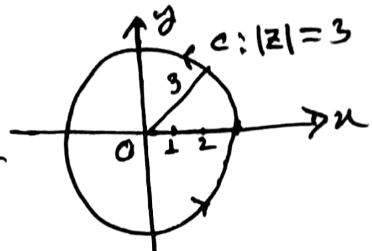


Fig. 1:

Solution: Let $f(z) = \sin \pi z^2 + \cos \pi z^2$

Now $\frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

$$\therefore \oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = \oint_C \frac{f(z)}{z-2} dz - \oint_C \frac{f(z)\,dz}{z-1} \quad \text{--- (1)}$$

Here $f(z)$ is analytic within and on c.

Also the points $z = 1$ & $z = 2$ both lie inside c.

Hence by Cauchy's integral formula we get from (1)

$$\oint_C \frac{f(z)\,dz}{z-1} = 2\pi i \cdot f(1) = 2\pi i \cdot (\sin \pi + \cos \pi) = -2\pi i \quad \text{--- (2)}$$

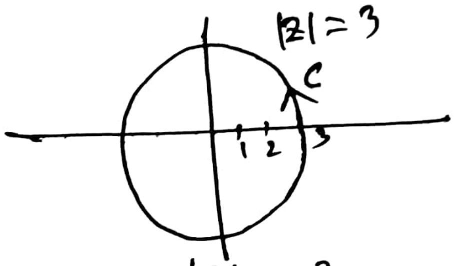
$$\& \oint_C \frac{f(z)\,dz}{z-2} = 2\pi i \cdot f(2) = 2\pi i\,(\sin 4\pi + \cos 4\pi) = 2\pi i \quad \text{--- (3)}$$

Thus using (2) & (3) into (1) we get

$$\oint_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = 2\pi i - (-2\pi i) = 4\pi i$$

Ans.

Example-2 Using Cauchy's integral formula to evaluate $\oint_C \frac{z^2+2}{z-2}\,dz$ where C is the closed curve $|z|=3$.



Solution: Let $f(z) = z^2+2$

which is analytic within & on $C: |z|=3$.

Also the point $z=2$ lies inside C.

Again since $f(2) = 2^2+2 = 6$

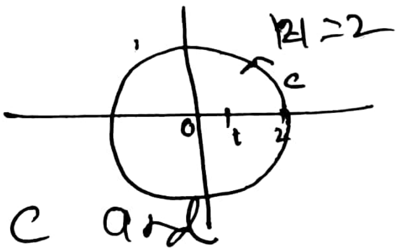
then by using Cauchy's integral formula we get

$$f(2) = \frac{1}{2\pi i}\oint_C \frac{z^2+z}{z-2}\,dz$$

$$\Rightarrow \oint_C \frac{z^2+2}{z-2}\,dz = 2\pi i \cdot f(2) = 2\pi i \cdot 6 = 12\pi i$$

Ans.

Example 3 Show that $\oint_C \frac{dz}{z+1} = 2\pi i$ where $C: |z|=2$.



Soln: Here $|z|=2$, Let $f(z)=1$

Then $f(-1) = 1$.

Also the point $z=-1$ lies inside C and

$f(z)$ is analytic within & on C.

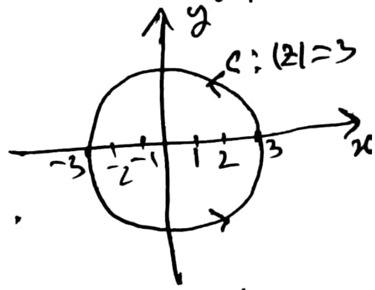
Hence by Cauchy's integral formula

$$f(-1) = \frac{1}{2\pi i}\oint_C \frac{dz}{z-(-1)}\,dz$$

$$\Rightarrow \oint_C \frac{dz}{z+1} = 2\pi i \cdot 1 = 2\pi i$$

[Proved.]

Example-4 Using Cauchy's integral formula to evaluate $\oint_C \frac{e^{2z}}{(z+1)^4} dz$ where C is the circle $|z|=3$.



Solution: Let $f(z) = e^{2z}$ which is analytic inside and on the circle $|z|=3$.

Also the point $z=-1$ lies inside the circle $|z|=3$.

Hence by applying Cauchy's integral formula for 3rd derivatives, we get

$$f'''(-1) = \frac{\lfloor 3}{2\pi i} \oint_C \frac{e^{2z}}{\{z-(-1)\}^4} dz$$

$\because f(z) = e^{2z}$

so $f'''(-1) = 8e^{-2}$

$$\Rightarrow \oint_C \frac{e^{2z}}{(z+1)^4} dz = f'''(-1) \times \frac{2\pi i}{\lfloor 3}$$

$$= 8e^{-2} \cdot \frac{\cancel{2}\pi i}{\cancel{6}\,3} = \frac{8\pi i e^{-2}}{3}$$

i.e. $$\oint_C \frac{e^{2z}}{(z+1)^4} dz = \frac{8\pi i e^{-2}}{3}$$ Ans.

Example-5 Evaluate $\frac{1}{2\pi i} \oint_C \frac{z^2}{z^2+4} dz$, where C is the square with vertices at ± 2, $\pm 2+4i$.

Sol^n: Here C is the square with vertices at $\pm 2, \pm 2+4i$. i.e. $(2,0)$, $(-2,0)$, $(2,4)$ & $(-2,4)$ described in the positive sense.

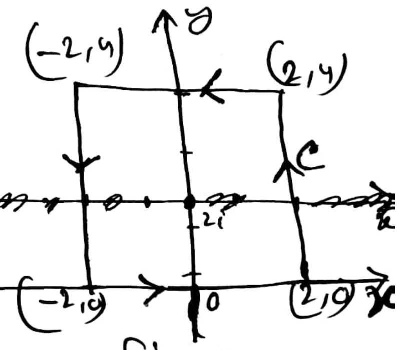


fig.1:

Now $$\frac{1}{z^2+4} = \frac{1}{(z-2i)(z+2i)} = \frac{1}{4i}\left[\frac{1}{z-2i} - \frac{1}{z+2i}\right]$$

$$\therefore \oint_C \frac{z^2 dz}{z^2+4} = \frac{1}{4i}\left[\oint_C \frac{z^2 dz}{z-2i} - \oint_C \frac{z^2 dz}{z+2i}\right] \quad \text{---} \quad (1)$$

Here $f(z)=z^2$ is analytic within and on C.
Also the point $z=2i$ lies inside C and $z=-2i$ does not lie inside C. So by Cauchy's integral formula and Cauchy's integral theorem, we get respectively

$$*\oint_C \frac{z^2\,dz}{z-2i} = 2\pi i\cdot f(2i) = 2\pi i\cdot(-4) = -8\pi i \quad [\because f(z)=z^2] \quad \text{- - - (2)}$$

$$\text{and } *\oint_C \frac{z^2\,dz}{z+2i} = 0 \quad \text{- - - (3)}$$

Hence using (2) & (3) into (1) we get

$$\frac{1}{2\pi i}\int \frac{z^2\,dz}{z^2+4} = \frac{1}{\cancel{2\pi i}}\cdot\frac{1}{4i}[-\cancel{8\pi} i - 0] = i \quad \text{Ans.}$$

$\boxed{\text{Exercise-1}}$ Using Cauchy's integral formula/theorem to evaluate the followings:

(a) $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$, where $C: |z| = \frac{1}{2}$ Ans. 0

(b) $\oint_C \frac{z\,dz}{(9-z^2)(z+i)}$, Where $C: |z|=2$ Ans. $\frac{\pi}{5}$.

(c) $\oint_C \frac{\cos 2\pi z\,dz}{(2z-1)(z-3)}$, where $C: |z|=1$ Ans. $\frac{2\pi i}{5}$

(d) $\oint_C \frac{dz}{z(z-2)^4}$, where $C: |z|=1$ Ans. $\frac{\pi i}{8}$

(e) $\oint_C \frac{dz}{z(z-2)^2}$ where $C: |z|=1$