1

Lecture-01 : Complex Number

Topics Includes:

1. Complex Number & its application
2. Modulus & Arguments of complex No.
3. Geometrical Interpretation of complex No.
4. Complex Conjugate
5. Polar form of a complex Number.

Complex Number

Question: Define complex number with its examples and real-life ~~ex~~ applications.

Ans Defn: The number of the form $z = x+iy$ with $x, y \in \mathbb{R}$ is called the complex number. Here x & y are real number, i is the imaginary unit which has the property $i^2 = -1$ and z is called the complex variable.

Also the complex number can be written as $z = (x, y)$. Also if $z = x+iy$, $Re(z) = x$ and $Im(z) = y$.

Examples: (i) $z = 2+3i$, (ii) $z = -5+2i$ etc.

Real-life application: Complex numbers are used in many scientific and engineering fields, including physics, chemistry, biology, economics, electrical as well as mechanical engineering, mathematics and physics. In particular, mechanical and structural engineers use complex numbers to analyse the vibration of structures in machines, buildings and bridges, the behavior of fluid flow around aircraft, and that of wind around buildings and bridges. Also there are so many applications of complex number in a variety of fields.

Graphical or Geometrical representation of Complex Numbers:

Since a complex number $z = x+iy$ can be considered as an ordered pair of real no., so we can represent such numbers by a point in the xy plane. To each complex number there corresponds one and only one point in the xy plane and conversely, to each point in the xy plane there corresponds one and only one complex number.

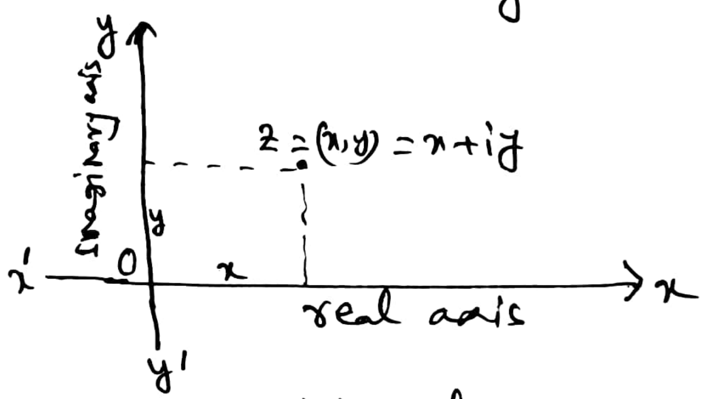

Fig.1: Graphical representation of complex No.

Question: Define modulus and amplitude of a complex number.

Ans. Modulus: The modulus or absolute value of a complex number $z = x+iy$ is the distance of the complex number z from the origin in the xy plane or Argand plane and is denoted by $\text{mod } z = |z| = \sqrt{x^2+y^2} \geq 0$ where $|z| \in \mathbb{R}$.

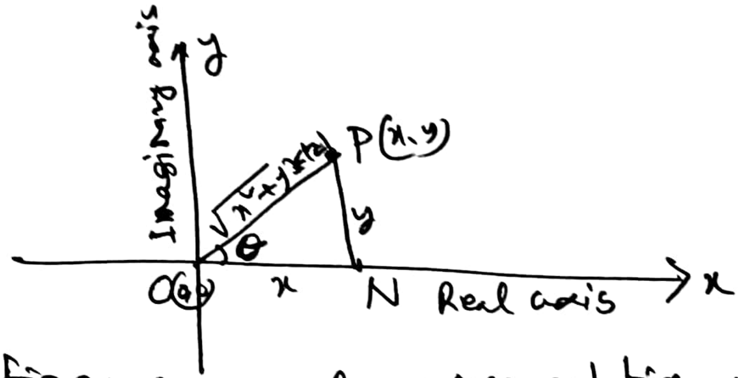

Fig.2: Graphical representation of Modulus & Argument of Complex no.

* Argument or Amplitude of a complex number:

The angle between the positive x-axis and the line joining the origin and a complex No. $z = x + iy \neq 0$ is called the argument or amplitude of z. Argument or Amplitude is denoted by $\arg z = \operatorname{amp} z = \tan^{-1}\left(\frac{y}{x}\right)$.

$\boxed{\text{Question:}}$ What is modulus and argument of $z=0$, i.e. the origin?

Ans. Here $z = 0 = 0 + i.0 = (0,0)$.

Then $|z| = \sqrt{0^2+0^2} = 0$

$\arg z = \tan^{-1}\left(\frac{0}{0}\right)$ which is undefined.

therefore $z=0$ (i.e. the origin) has modulus 0 but does not really have an angle.

Question: Define complex conjugate or conjugate complex Number.

Ans: The complex conjugate (or simply conjugate) of a complex number $z = x+iy$ is defined as the complex number $x-iy$ and is denoted by $\bar{z} = z^* = x - iy = (x, -y)$.

Also $|\bar{z}| = \sqrt{x^2+(-y)^2} = \sqrt{x^2+y^2} = |z|$

and $\arg \bar{z} = \tan^{-1}\left(\frac{-y}{x}\right) = -\tan^{-1}\left(\frac{y}{x}\right) = -\arg z$

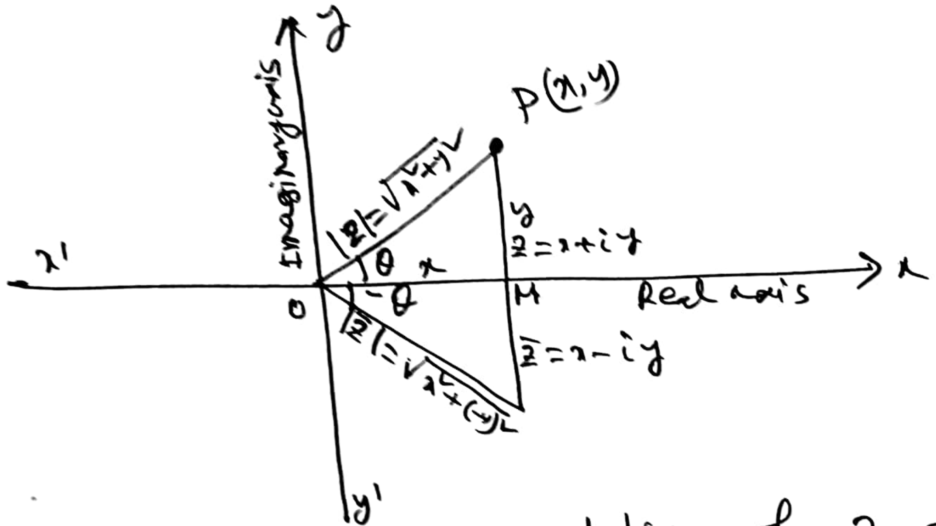

Fig. 3: Graphical representation of z and $\bar{z}$.

Note: Condition for two complex numbers z_1 & z_2 to be conjugate: (i) $|z_1| = |z_2|$ and (ii) $\arg z_1 = -\arg z_2$, i.e. $\arg z_1 + \arg z_2 = 0$.

Question: What is polar form of a complex number?

Ans: Let P be the point in the complex plane corresponding to the complex number (x, y) or $x+iy$. Then from fig.1 we see that $x = r\cos\theta$, $y = r\sin\theta$ where $r = \sqrt{x^2+y^2}$

Also by Euler's formula we know that $\cos\theta + i\sin\theta = e^{i\theta}$.

Then $z = x+iy = r(\cos\theta + i\sin\theta)$

$\Rightarrow \boxed{z = re^{i\theta}}$ is called the polar form or exponential form of a complex number, where r and θ are called modulus and argument respectively and (r, θ) is called polar coordinates.

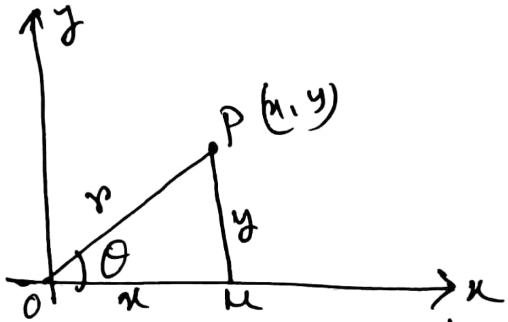

fig.1: polar form of complex number.

Examples: Find the modulus and principal arguments of the following complex numbers:

(i) $5-5i$, (ii) $\pm i$, (iii) $\left(\frac{1+i}{1-i}\right)^2$, (iv) $\frac{\sqrt{3}+i}{\sqrt{3}-i}$

Ans. (i) Let $z = 5-5i$

$\therefore |z| = |5-5i| = \sqrt{(5)^2+(-5)^2} = \sqrt{25+25} = \sqrt{50}$
$= 5\sqrt{2}$

$\therefore$ Modulus of $z = 5-5i$ is $5\sqrt{2}$.

Also the argument $z = 5-5i$ is $\theta = \tan^{-1}\left(\frac{-5}{5}\right)$

$\Rightarrow \theta = \tan^{-1}(-1) = -\tan^{-1}(\tan \pi/4) = -\pi/4$

$\therefore$ Argument of $z = 5-5i$ is $\pi/4$ Ans.

(ii) Let $z = \pm i = 0 \pm i.1$

$\therefore |z| = \sqrt{0^2 + (\pm 1)^2} = \sqrt{0+1} = 1$

Also $\arg z = \tan^{-1}\left(\frac{\pm 1}{0}\right) = \pm \tan^{-1}\left(\tan \frac{\pi}{2}\right) = \pm \frac{\pi}{2}$

i.e. Modulus of $z = \pm i$ is 1 & $\operatorname{Arg} z = \pm \pi/2$

(iii) Let $z = \left(\frac{1+i}{1-i}\right)^2 = \frac{1+2i+i^2}{1-2i+i^2} = \frac{\cancel{1}+2i-\cancel{1}}{\cancel{1}-2i-\cancel{1}} = \frac{2i}{-2i}$

$\Rightarrow z = -1 = -1 + 0.i$

$\therefore |z| = \sqrt{(-1)^2 + 0^2} = \sqrt{1} = 1$

& $\operatorname{Arg} z = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1}(\tan \pi) = \pi$

$\therefore$ Modulus of $z = \left(\frac{1+i}{1-i}\right)^2$ is 1

& Argument of $z = \left(\frac{1+i}{1-i}\right)^2$ is π } Ans

(iv) Let $z = \frac{\sqrt{3}+i}{\sqrt{3}-i} = \frac{(\sqrt{3}+i)(\sqrt{3}+i)}{(\sqrt{3}+i)(\sqrt{3}-i)} = \frac{(\sqrt{3})^2 + 2\sqrt{3}i + i^2}{(\sqrt{3})^2 - i^2}$

$\Rightarrow z = \frac{3+2\sqrt{3}i-1}{3+1} = \frac{2+2\sqrt{3}i}{4} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore |z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = \sqrt{1} = 1$

& $\operatorname{Arg} z = \tan^{-1}\left(\frac{\sqrt{3}/2}{1/2}\right) = \tan^{-1}(\sqrt{3}) = \tan^{-1}(\tan \pi/3)$
$= \pi/3$

$\therefore$ Modulus of $z = \frac{\sqrt{3}+i}{\sqrt{3}-i}$ is 1

& Argument of $z = \frac{\sqrt{3}+i}{\sqrt{3}-i}$ is $\pi/3$ } Ans

Exercise-1 Find the modulus and principal argument of the following complex numbers:

(i) ± 1, (ii) $\frac{2-i}{2+i}$, (iii) $\frac{1+2i}{1-(1-i)^2}$.

Example-2 Describe geometrically the region determined by the following relations:

(i) $Im(z) \leq 0$, (ii) $Re(z) \geq 0$, (iii) $Re(z) > 1$

(iv) $Im(z) > 1$, (v) $|z-2| \leq |z+2|$, (vi) $1 < |z+i| \leq 2$.

Ans: (i) Given $Im(z) \leq 0$

$\Rightarrow Im(x+iy) \leq 0$ [$\because z = x+iy$]

$\Rightarrow y \leq 0$ which represents the lower portion of the x-axis including the x-axis.

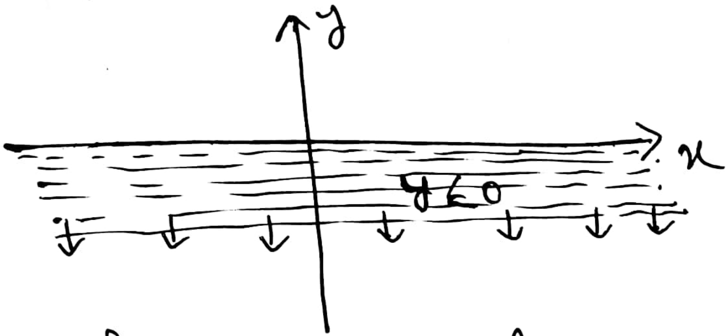

Fig.1: Graph of $Im(z) \leq 0$.

(ii) $Re(z) \geq 0 \Rightarrow Re(x+iy) \geq 0 \Rightarrow x \geq 0$ which represents the right hand sides of the y-axis including the y-axis.

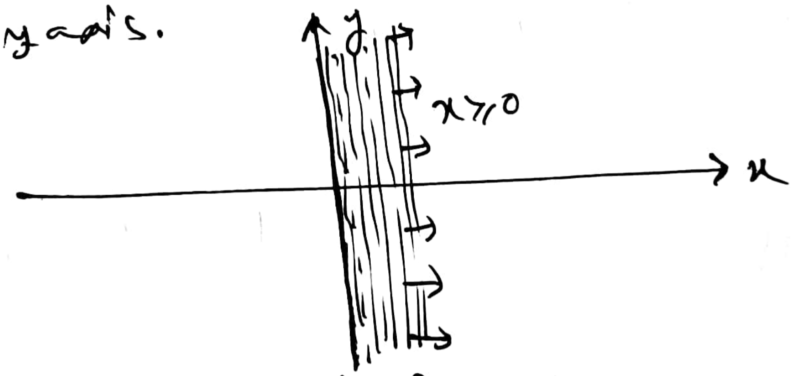

Fig.2: Graph of $Re(z) \geq 0$.

(iii) Given $Re(z) > 1 \Rightarrow Re(x+iy) > 1$

$\Rightarrow x > 1$ which represents the region of the right hand sides of the line $x=1$, excluding the line itself.

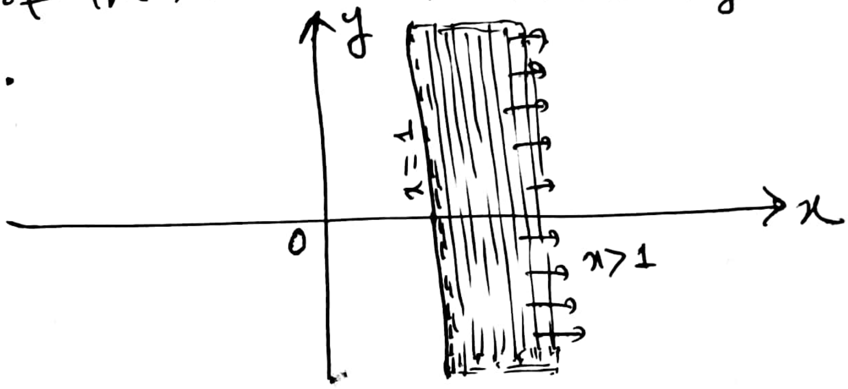

Fig. 3: Graph of $x > 1$.

Given that

(v) $|z-2| \leq |z+2|$

$\Rightarrow |x+iy-2| \leq |x+iy+2|$ $[\because z = x+iy]$

$\Rightarrow |(x-2)+iy| \leq |(x+2)+iy|$

$\Rightarrow \sqrt{(x-2)^2+y^2} \leq \sqrt{(x+2)^2+y^2}$

$\Rightarrow (x-2)^2+y^2 \leq (x+2)^2+y^2$

$\Rightarrow (x-2)^2 \leq (x+2)^2$

$\Rightarrow x^2-4x+4 \leq x^2+4x+4$

$\Rightarrow 8x \geq 0 \Rightarrow x \geq 0$ which represents the region of the right hand side of y-axis including y-axis.

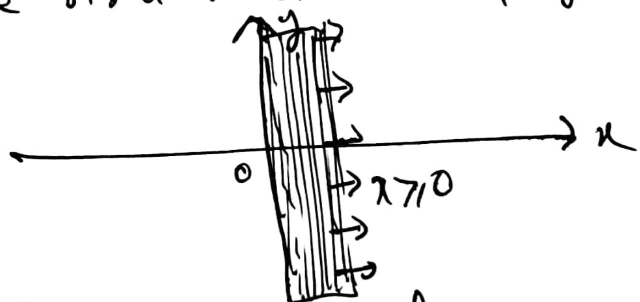

Fig. 5: Graph of $x \geq 0$.

(vi) Given $1 < |z+i| \le 2$

$\Rightarrow 1 < |(x+iy+i)| \le 2$ $\quad [\because z = x+iy]$

$\Rightarrow 1 < |\{x+i(y+1)\}| \le 2$

$\Rightarrow 1 \le \sqrt{x^2+(y+1)^2} \le 2$

$\Rightarrow 1^2 \le x^2+(y+1)^2 \le 2^2$ which represents the annular region between the concentric circles $x^2+(y+1)^2=1^2$ and $x^2+(y+1)^2=2^2$ including the outer circle and excluding the inner circle.

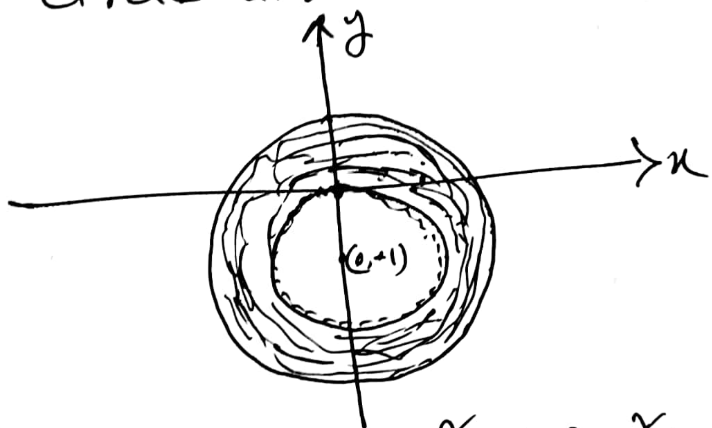

Fig.6: Graph of $1^2 < x^2+(y+1)^2 \le 2$.