Bessel's Function:

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$\frac{n(n+1)}{x} \cdot x$

Bessel's Differential equation:

The linear 2nd order differential equation of the type

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \quad \cdots \quad (1)$$

is called the Bessel equation.

The number n (which is a constant) is called the order of the Bessel's equation.

The given differential equation is named after the German mathematician and astronomer Friedrich Wilhelm Bessel who studied this equation in detail and showed (in 1824) that its solutions are expressed through a special class of functions called cylindrical functions or Bessel functions.

Bessel's equation arises when finding separable solutions to Laplace's equation and the Helmholtz equation in cylindrical or spherical coordinates. Bessel functions are therefore especially important for many problems of wave propagation and static potentials. In solving in cylindrical / spherical coordinates system, one obtains Bessel functions for the following cases:

1. Electromagnetic waves in a cylindrical wave guide
2. Heat conduction in a cylindrical object.
3. Modes of vibration of a thin circular (or annular) artificial membrane.
4. Diffusion problems on a lattice.
5. Solutions to the radial Schrödinger equation (in spherical and cylindrical coordinates) for a free particle.

8. Bessel functions also have useful properties for other problems such as signal processing.

Solution of Bessel's equation:

The Bessel's differential equation is

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \quad --- (1)$$

Let the series solution of (1) be $\rightarrow y=\sum_{m\geq 0} c_m x^{\lambda+m}$

$y = x^{\lambda}(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+----)$ where $c_0 \neq 0$

$$\Rightarrow y = c_0x^{\lambda}+c_1x^{\lambda+1}+c_2x^{\lambda+2}+c_3x^{\lambda+3}+c_4x^{\lambda+4}+---- \quad (2)$$

Then $\frac{dy}{dx} = \lambda c_0x^{\lambda-1}+(\lambda+1)c_1x^{\lambda}+(\lambda+2)c_2x^{\lambda+1}+(\lambda+3)c_3x^{\lambda+2}$
$+(\lambda+4)c_4x^{\lambda+3}+----$

and $\frac{d^2y}{dx^2} = \lambda(\lambda-1)c_0x^{\lambda-2}+(\lambda+1)\lambda c_1x^{\lambda-1}+(\lambda+2)(\lambda+1)c_2x^{\lambda}$
$+(\lambda+3)(\lambda+2)c_3x^{\lambda+1}+(\lambda+4)(\lambda+3)c_4x^{\lambda+2}+---$

Now putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$[\lambda(\lambda-1)c_0x^{\lambda}+(\lambda+1)\lambda c_1x^{\lambda+1}+(\lambda+2)(\lambda+1)c_2x^{\lambda+2}$
$+(\lambda+3)(\lambda+2)c_3x^{\lambda+3}+(\lambda+4)(\lambda+3)c_4x^{\lambda+4}+---]+[\lambda c_0x^{\lambda}$
$+(\lambda+1)c_1x^{\lambda+1}+(\lambda+2)c_2x^{\lambda+2}+(\lambda+3)c_3x^{\lambda+3}+(\lambda+4)c_4x^{\lambda+4}+--]$
$+[c_0x^{\lambda+2}+c_1x^{\lambda+3}+c_2x^{\lambda+4}+c_3x^{\lambda+5}+c_4x^{\lambda+6}+---]$
$+[-n^2c_0x^{\lambda}-n^2c_1x^{\lambda+1}-n^2c_2x^{\lambda+2}-n^2c_3x^{\lambda+3}-n^2c_4x^{\lambda+4}---]=0$

$\Rightarrow\{\lambda(\lambda-1)c_0+\lambda c_0-n^2c_0\}x^{\lambda}+\{(\lambda+1)\lambda c_1+(\lambda+1)c_1-n^2c_1\}x^{\lambda+1}$
$+\{(\lambda+2)(\lambda+1)c_2+(\lambda+2)c_2+c_0-n^2c_2\}x^{\lambda+2}$
$+\{(\lambda+3)(\lambda+2)c_3+(\lambda+3)c_3+c_1-n^2c_3\}x^{\lambda+3}$
$+\{(\lambda+4)(\lambda+3)c_4+(\lambda+4)c_4+c_2-n^2c_4\}x^{\lambda+4}+---=0$

Now the lowest power of x is x^{λ} and its coefficient equated to zero gives

$$\lambda^2 c_0 - \cancel{\lambda c_0} + \cancel{\lambda c_0} - n^2 c_0 = 0$$

$$\Rightarrow c_0(\lambda^2 - n^2) = 0 \Rightarrow \lambda^2 = n^2 \Rightarrow \lambda = \pm n, \text{ since } c_0 \neq 0.$$

Equating the coefficient of $x^{\lambda+1}$ to zero, we get

$$(\lambda+1)\lambda c_1 + (\lambda+1) c_1 - n^2 c_1 = 0$$

$$(\lambda^2 + \lambda + \lambda + 1) c_1 - n^2 c_1 = 0$$

$$\Rightarrow (\lambda+1)^2 c_1 - n^2 c_1 = 0$$

$$\Rightarrow c_1\{(\lambda+1)^2 - n^2\} = 0 \Rightarrow c_1 = 0$$

Equating the coefficient of $x^{\lambda+2}$ to zero, we get

$$(\lambda+2)(\lambda+1)c_2 + (\lambda+2)c_2 + c_0 - n^2 c_2 = 0$$

$$\Rightarrow \{(\lambda+2)^2 - n^2\} c_2 = -c_0$$

$$\Rightarrow c_2 = -\frac{c_0}{(\lambda+2)^2 - n^2}$$

Equating the coefficient of $x^{\lambda+3}$ to zero, we get

$$(\lambda+3)(\lambda+2)c_3 + (\lambda+3)c_3 + c_1 - n^2 c_3 = 0$$

$$\Rightarrow \{(\lambda+3)^2 - n^2\} c_3 = -c_1$$

$$\Rightarrow c_3 = -\frac{c_1}{(\lambda+3)^2 - n^2} = 0 \quad [\text{since } c_1 = 0]$$

Equating the coefficient of $x^{\lambda+4}$, we get

$$(\lambda+4)(\lambda+3)c_4 + (\lambda+4)c_4 + c_2 - n^2 c_4 = 0$$

$$\Rightarrow \{(\lambda+4)^2 - n^2\} c_4 = -c_2$$

$$\Rightarrow c_4 = -\frac{c_2}{(\lambda+4)^2 - n^2} = \frac{c_0}{\{(\lambda+2)^2 - n^2\}\{(\lambda+4)^2 - n^2\}}$$

Similarly, $c_5 = c_7 = \cdots = 0$ and

$$c_6 = -\frac{c_0}{\{(\lambda+2)^2 - n^2\}\{(\lambda+4)^2 - n^2\}\{(\lambda+6)^2 - n^2\}}$$

[P.T.O]

Now substituting the values of $c_1, c_2, c_3, c_4, c_5, c_6$ in ② we get

$$y = c_0 x^r + 0 - \frac{c_0}{\{(r+2)^2-n^2\}} \cdot x^{r+2} + 0 + \frac{c_0}{\{(r+2)^2-n^2\}\{(r+4)^2-n^2\}} x^{r+4}$$

$$+ 0 - \frac{c_0}{\{(r+2)^2-n^2\}\{(r+4)^2-n^2\}\{(r+6)^2-n^2\}} \cdot x^{r+6} + \cdots$$

For $r = n$ we have

$$y = c_0 x^n \left[1 - \frac{1}{\{(n+2)^2-n^2\}} x^2 + \frac{1}{\{(n+2)^2-n^2\}\{(n+4)^2-n^2\}} x^4 - \frac{1}{\{(n+2)^2-n^2\}\{(n+4)^2-n^2\}\{(n+6)^2-n^2\}} x^6 + \cdots\right]$$

$$\Rightarrow y = c_0 x^n \left[1 - \frac{1}{4\lfloor 1 (n+1)} x^2 + \frac{1}{4^2 \lfloor 2 (n+1)(n+2)} x^4 - \frac{1}{4^3 \lfloor 3 (n+1)(n+2)(n+3)} x^6 + \cdots\right] \quad \text{--- ③}$$

$= au$ (say) where $a = c_0$ for $r = n$ and

$$u = x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{1}{4^2 \lfloor 2 (n+1)(n+2)} x^4 - \frac{1}{4^3 \lfloor 3 (n+1)(n+2)(n+3)} x^6 + \cdots\right] \quad \text{--- ④}$$

Again for $r = -n$, we have

$$y = c_0 x^{-n} \left[1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \lfloor 2 (-n+1)(-n+2)} x^4 - \frac{1}{4^3 \lfloor 3 (-n+1)(-n+2)(-n+3)} x^6 + \cdots\right] \quad \text{--- ⑤}$$

$= bv$ (say) where $b = c_0$ for $r = -n$ and

$$v = x^{-n} \left[1 - \frac{1}{4(-n+1)} x^2 + \frac{1}{4^2 \lfloor 2 (-n+1)(-n+2)} x^4 - \cdots\right] \longrightarrow ⑥$$

Case-5: When n is not integral or zero.

The general solution of the Bessel's equation (1) is $y = au + bv$ where a and b are arbitrary constants.

Now if u is multiplied by the constant $\frac{1}{2^n \Gamma(n+1)} = \frac{1}{2^n . \lfloor n}$, then the product $\frac{1}{2^n \lfloor n} . u$ called the Bessel's function of 1st kind of order n and is denoted by $J_n(x)$. Thus

$$J_n(x) = \frac{1}{2^n . \lfloor n} u$$

$$\Rightarrow J_n(x) = \frac{1}{2^n . \lfloor n} x^n \left[1 - \frac{1}{4(n+1)} x^2 + \frac{x^4}{4^2 . \lfloor 2 (n+1)(n+2)} \cdots \right]$$

$$\Rightarrow J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{1}{\lfloor n} - \frac{1}{\lfloor 1 \lfloor n+1} . \left(\frac{x}{2}\right)^2 + \frac{1}{\lfloor 2 . \lfloor n+2} \left(\frac{x}{2}\right)^4 - \frac{1}{\lfloor 3 \lfloor n+3} . \left(\frac{x}{2}\right)^6 + \cdots \right]$$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty} (-1)^r . \left(\frac{x}{2}\right)^{n+2r} . \frac{1}{\lfloor r \lfloor n+r} \quad \cdots (7)$$

Similarly if v is multiplied by the constant $\frac{1}{2^{-n} . \lfloor -n}$, then the product $\frac{1}{2^{-n} . \lfloor -n} v$ is called the Bessel's function of 1st kind of order $-n$ and is denoted by $J_{-n}(x)$. Thus

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-n+2r} . \frac{1}{\lfloor r . \lfloor -n+r} \quad \cdots (8)$$

Hence the complete solution of the Bessel's equation can also be expressed in the form $y = A J_n(x) + B J_{-n}(x)$, when n is not integer.

EX-1: Write down Bessel's differential equation of order zero and solve it.

Graph of Bessel function of 1st kind:

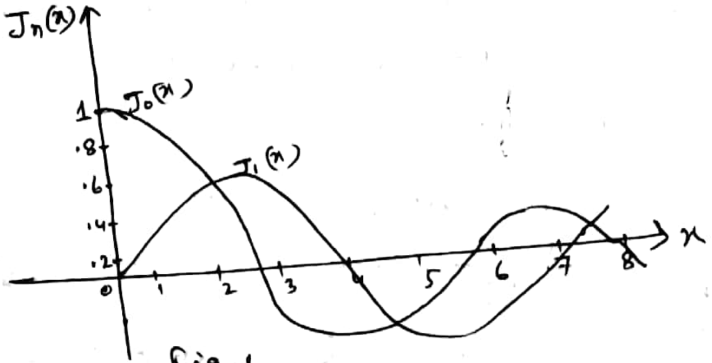

Fig-1

prove that $J_{-n}(x) = (-1)^n J_n(x)$ or equivalently Show that $J_n(x)$ and $J_{-n}(x)$ are linearly dependent for any positive or negative integer.

proof: We know that

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{-n+2r}}{\Gamma r+1 \cdot \Gamma -n+r+1} \quad \text{- - - - (1)}$$

since n is an integer (+,-), so for $r=0,1,2, \cdots (n-1)$; then $\Gamma -n+r+1$ is infinite and hence $\frac{1}{\Gamma -n+r+1}=0$. In this case (1) can be written as

$$J_{-n}(x) = \sum_{r=n}^{\infty} \frac{(-1)^r (x/2)^{-n+2r}}{\Gamma r+1 \cdot \Gamma -n+r+1} \quad \text{- - - - (2)}$$

Let $r-n=K \Rightarrow r = n+K$; limit $r=n \Rightarrow K=0$
$r=\infty \Rightarrow K=\infty$

Then from (2) we get

$$J_{-n}(x) = \sum_{K=0}^{\infty} \frac{(-1)^{n+K} (x/2)^{-n+2n+2K}}{\Gamma(n+K+1) \cdot \Gamma -\cancel{n}+\cancel{n}+K+1}$$

$$\Rightarrow J_{-n}(x) = (-1)^n \sum_{K=0}^{\infty} \frac{(-1)^K (x/2)^{n+2K}}{\Gamma K+1 \cdot \Gamma n+K+1} = (-1)^n J_n(x)$$

i.e. $J_{-n}(x) = (-1)^n J_n(x)$. proved.

Generating function for $J_n(x)$:

The function $e^{\{x(t-\frac{1}{t})\}/2}$ is called the generating function for $J_n(x)$.

Prove that $e^{x(t-t^{-1})/2} = \sum_{n=-\infty}^{\infty} J_n(x)\, t^n$

Proof: From the exponential series, we

now $e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} + \cdots$

$$\therefore e^{\frac{1}{2}x(t-\frac{1}{t})} = e^{\frac{1}{2}tx} \cdot e^{-\frac{x}{2t}} \quad \cdots\cdots \text{(1)}$$

$$\text{Now } e^{\frac{xt}{2}} = 1 + \frac{(xt/2)}{\lfloor 1} + \frac{(xt/2)^2}{\lfloor 2} + \frac{(xt/2)^3}{\lfloor 3} + \cdots$$

$$\Rightarrow e^{xt/2} = \sum_{r=0}^{\infty} \frac{(xt/2)^r}{\lfloor r}$$

$$\Rightarrow e^{xt/2} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r \cdot \lfloor r} \quad \cdots\cdots (2)$$

$$\text{And } e^{-(xt^{-1}/2)} = 1 - \frac{(xt^{-1}/2)}{\lfloor 1} + \frac{(xt^{-1}/2)^2}{\lfloor 2} - \frac{(xt^{-1}/2)^3}{\lfloor 3} + \cdots$$

$$\Rightarrow e^{-(xt^{-1}/2)} = \sum_{s=0}^{\infty} (-1)^s \cdot \frac{(xt^{-1}/2)^s}{\lfloor s}$$

$$\Rightarrow e^{-(xt^{-1}/2)} = \sum_{s=0}^{\infty} \frac{(-1)^s \cdot x^s t^{-s}}{2^s \cdot \lfloor s} \quad \cdots\cdots \text{(3)}$$

From (2) and (3) substituting the values of $e^{xt/2}$ and $e^{-xt^{-1}/2}$ in (1) we obtain

$$e^{\frac{1}{2}x(t-t^{-1})} = \sum_{r=0}^{\infty} \frac{x^r t^r}{2^r \cdot \lfloor r} \cdot \sum_{s=0}^{\infty} \frac{(-1)^s \cdot x^s t^{-s}}{2^s \cdot \lfloor s}$$

$$\Rightarrow e^{\frac{1}{2}x(t-t^{-1})} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \cdot x^{r+s} \cdot t^{r-s}}{2^{r+s} \cdot \lfloor r \lfloor s}$$

$$\Rightarrow e^{\frac{1}{2}x(t-t^{-1})} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^s \cdot \left(\frac{x}{2}\right)^{r+s} \cdot t^{r-s}}{\Gamma r+1 \; \Gamma s+1} \quad \text{---- (4)}$$

Now to obtain the Coefficient of t^n, putting $r-s=n \Rightarrow r=n+s$ in (4), then

$$\text{Coefficient of } t^n = \sum_{s=0}^{\infty} \frac{(-1)^s \cdot (x/2)^{n+2s}}{\Gamma s+1 \cdot \Gamma n+s+1} = J_n(x) \text{ - (5)}$$

Again to obtain the coefficient of t^{-n}, we put $r-s=-n$ or $r=-n+s$ in (4), then

$$\text{Coefficient of } t^{-n} = \sum_{s=0}^{\infty} \frac{(-1)^s \; (x/2)^{-n+2s}}{\Gamma s+1 \cdot \Gamma -n+s+1} = J_{-n}(x) \quad \text{--- (6)}$$

Also to obtain the coefficient of t^0, we put $r-s=0$ or $r=s$ in (4), then

$$\text{Coefficient of } t^0 = \sum_{s=0}^{\infty} \frac{(-1)^s \; (x/2)^{0+2s}}{\Gamma s+1 \cdot \Gamma 0+s+1} = J_0(x) \quad \text{-- (7)}$$

Thus all the integral power of t, both positive and negative occur in (5), (6) & (7)

$$\therefore e^{\frac{1}{2}x(t-t^{-1})} = \sum_{n=1}^{\infty} J_{-n}(x)\, t^{-n} + J_0(x)\, t^0 + \sum_{n=1}^{\infty} J_n(x)\, t^n$$

$$= t^{-1} J_{-1}(x) + t^{-2} J_{-2}(x) + \cdots + J_0(x) + t\, J_1(x) + t^2 J_2(x) + \cdots = \sum_{n=-\infty}^{\infty} t^n J_n(x).$$

Orthogonal properties of Bessel's function:

If α and β be the roots of $J_n(x)=0$, then prove that

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \begin{cases} 0, & \text{when } \alpha \neq \beta \\ \frac{1}{2} J_{n+1}^2(\alpha), & \text{if } \alpha = \beta. \end{cases}$$

Proof: Bessel's differential equation is

$$t^2 \frac{d^2y}{dt^2} + t\frac{dy}{dt} + (t^2 - n^2)y = 0$$

$$\Rightarrow \frac{d^2y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + \left(1 - \frac{n^2}{t^2}\right)y = 0 \quad \text{--- (1)}$$

Since $J_n(t)$ is a solution of (1), so we ~~get~~

let $t = \alpha x$ and $y = u$

$$\therefore 1 = \alpha \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{1}{\alpha} \quad \text{--- (2)}$$

Now $\frac{dy}{dt} = \frac{du}{dt} = \frac{du}{dx}\cdot\frac{dx}{dt} = \frac{1}{\alpha}\cdot\frac{du}{dx}$

Similarly $\frac{d^2y}{dt^2} = \frac{1}{\alpha^2}\frac{d^2u}{dx^2}$

$\therefore$ (1) implies that

$$\frac{1}{\alpha^2}\frac{d^2u}{dx^2} + \frac{1}{\alpha x}\cdot\frac{1}{\alpha}\frac{du}{dx} \left(1 - \frac{n^2}{\alpha^2 x^2}\right)u = 0$$

Multiplying both sides by $\alpha^2 x^2$, we get

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2x^2 - n^2)u = 0$$

$$\Rightarrow x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \quad \text{--- (3)}$$

Similarly we put $t = \beta x$, $y = v$ in (1) we get

$$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \quad \text{--- (4)}$$

Now $(3) \times \frac{v}{x} - (4) \times \frac{u}{x}$ implies that

$$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)xuv = 0$$

$$\Rightarrow \frac{d}{dx}\{x(u'v - uv')\} + (\alpha^2 - \beta^2)xuv = 0 \quad \cdots (5)$$

Since $u = J_n(\alpha x)$, $v = J_n(\beta x)$ so

$$u' = \alpha J_n'(\alpha x), \quad v' = \beta J_n'(\beta x)$$

$\therefore (5) \Rightarrow$

$$\frac{d}{dx}\left[x\{\alpha J_n'(\alpha x) J_n(\beta x) - J_n(\alpha x)\beta J_n'(\beta x)\}\right] + (\alpha^2 - \beta^2)xJ_n(\alpha x)J_n(\beta x) = 0$$

Integrating w.r.to x from 0 to 1, we get

$$\left[x\{\alpha J_n'(\alpha x) J_n(\beta x) - \beta J_n(\alpha x) J_n'(\beta x)\}\right]_{x=0}^{1} + (\alpha^2 - \beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

$$\Rightarrow \alpha J_n'(\alpha) J_n(\beta) - \beta J_n(\alpha) J_n'(\beta) + (\alpha^2 - \beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0 \quad \cdots (6)$$

Since α, β are different roots of $J_n(x) = 0$,

so $J_n(\alpha) = 0$, $J_n(\beta) = 0 \quad \cdots (7)$

From (6) and (7) we get

$$0 + (\alpha^2 - \beta^2)\int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0$$

$$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\,dx = 0, \quad \text{since } \alpha \neq \beta.$$

i.e. $\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$, if $\alpha \neq \beta$. $\longrightarrow$ (8)

<u>Case-II:</u> Let the roots of $J_n(x)$ be equal

i.e. $\alpha = \beta$.

Now multiplying (3) by $2u'$, we then get

$$x^2 u' u'' + 2x u'^2 + 2(\alpha^2 x^2 - n^2) u u' = 0$$

$$\Rightarrow \frac{d}{dx}(x^2 u'^2) + \frac{d}{dx}(-n^2 u^2) + \frac{d}{dx}(\alpha^2 x^2 u^2) = 2\alpha^2 x u^2$$

$$\Rightarrow \frac{d}{dx}\left(x^2 u'^2 - n^2 u^2 + \alpha^2 x^2 u^2\right) = 2\alpha^2 x u^2$$

$$\Rightarrow 2\alpha^2 \int_0^1 x u^2\, dx = \int_0^1 \frac{d}{dx}\left(x^2 u'^2 - n^2 u^2 + \alpha^2 x^2 u^2\right) dx$$

$$\Rightarrow 2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \left[x^2 u'^2 - n^2 u^2 + \alpha^2 x^2 u^2\right]_{x=0}^{1}$$

$$\Rightarrow 2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \left[x^2 \alpha^2 J_n'^2(\alpha x) - n^2 J_n^2(\alpha x) + \alpha^2 x^2 J_n^2(\alpha x)\right]_{x=0}^{1}$$

$$\Rightarrow 2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \left[\alpha^2 J_n'^2(\alpha) - n^2 J_n^2(\alpha) + \alpha^2 J_n^2(\alpha) - 0 + n^2 J_n^2(0) - 0\right]$$

$$\Rightarrow 2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \alpha^2 J_n'^2(\alpha) \quad \cdots (9)$$

since $J_n(\alpha) = 0$ &
$J_n(0) = 0$

[P.T.O]

From the recurrence relation we know that $\frac{d}{dx}\left(J_n(x)\right) = \frac{n}{x} J_n(x) - J_{n+1}(x)$

Replacing x by αx, then we get

$$\frac{1}{\alpha}\frac{d}{dx}\left(J_n(\alpha x)\right) = \frac{n}{\alpha x} J_n(\alpha x) - J_{n+1}(\alpha x)$$

$$\Rightarrow J_n'(\alpha x) = \frac{n}{\alpha x} J_n(\alpha x) - J_{n+1}(\alpha x)$$

$$\Rightarrow \left. J_n'(\alpha x)\right]_{x=1} = \left[\frac{n}{\alpha x} J_n(\alpha x) - J_{n+1}(\alpha x)\right]_{x=1}$$

$$\Rightarrow J_n'(\alpha) = \frac{n}{\alpha} J_n(\alpha) - J_{n+1}(\alpha)$$

$$\Rightarrow J_n'(\alpha) = 0 - J_{n+1}(\alpha) \quad ; \quad \text{since } J_n(\alpha) = 0$$

$$\Rightarrow J_n'^2(\alpha) = J_{n+1}^2(\alpha) \quad \cdots\cdots (10)$$

$\therefore$ From (9) & (10) we get

$$2\alpha^2 \int_0^1 x J_n^2(\alpha x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha) \quad \cdots\cdots (11)$$

Thus combining (8) & (11) we get

$$\int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{1}{2} J_{n+1}^2(\alpha)\, \delta_{\alpha\beta}$$

where $\delta_{\alpha\beta} = \begin{cases} 0, & \text{if } \alpha \neq \beta \\ 1, & \text{if } \alpha = \beta. \end{cases}$

proved.

Recurrence relation for $J_n(x)$:

Prove that (i) $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$.

(ii) $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

(iii) $2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(iv) $2n J_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

(v) $\frac{d}{dx}\left(x^{n} J_n(x)\right) = x^n J_{n-1}(x)$.

(vi) $\frac{d}{dx}\left[x^{-n} J_n'(x)\right] = -x^{-n} J_{n+1}(x)$. X

Proof (i): We know $J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r}$ --- (i)

Differentiating (i) w.r.to x, we get

$$J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \cdot \Gamma(n+r+1)} (n+2r) \cdot \left(\frac{x}{2}\right)^{n+2r-1} \cdot \frac{1}{2}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \cdot \frac{n}{2} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$+ \sum_{r=0}^{\infty} \frac{(-1)^r \cdot r}{\lfloor r \, \Gamma(n+r+1)} \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{\lfloor r \, \Gamma(n+r+1)} \cdot \frac{n}{\cancel{2}} \cdot \frac{\cancel{2}}{x} \cdot \left(\frac{x}{2}\right)^{n+2r}$$

$$+ \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r-1 \cdot \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \frac{n}{x} J_n(x) + \sum_{r=1}^{\infty} \frac{(-1)^r}{\lfloor r-1 \cdot \Gamma(n+r+1)} \cdot \left(\frac{x}{2}\right)^{n+2r-1}$$

$$= \frac{n}{x} J_n(x) + \sum_{s+1=1}^{\infty} \frac{(-1)^{s+1}}{\lfloor s \cdot \Gamma(n+s+1+1)} \cdot \left(\frac{x}{2}\right)^{n+2s+1}$$

Put $r = s+1$
$\Rightarrow r-1 = s$

$$= \frac{n}{x} J_n(x) + \sum_{s=0}^{\infty} \frac{(-1)^s \cdot (-1)^1}{\lfloor s \, \Gamma((n+1)+s+1)} \cdot \left(\frac{x}{2}\right)^{(n+1)+2s}$$

$$\Rightarrow J_n'(x) = \frac{n}{x} J_n(x) - J_{n+1}(x)$$

i.e. $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$ proved.

prove that (i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$.

(ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$.

(iii) $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x}{x} - \cos x \right]$

(iv) $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{\cos x}{x} + \sin x \right]$

Proof of (i): We know that

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \cdot \left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} - \cdots \right] \quad (1)$$

$$\therefore J_{1/2}(x) = \frac{x^{\frac{1}{2}}}{2^{\frac{1}{2}} \Gamma(\frac{1}{2}+1)} \cdot \left[1 - \frac{x^2}{2 \cdot 2 \cdot (\frac{1}{2}+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (\frac{1}{2}+1)(\frac{1}{2}+2)} - \cdots \right]$$

$$= \frac{\sqrt{x}}{\sqrt{2} \cdot \frac{1}{2} \Gamma(\frac{1}{2})} \cdot \left[1 - \frac{x^2}{2 \cdot \cancel{2} \cdot \frac{3}{\cancel{2}}} + \frac{x^4}{2 \cdot 4 \cdot 4 \cdot \frac{3}{2} \cdot \frac{5}{2}} - \cdots \right]$$

$$= \frac{2\sqrt{x}}{\sqrt{2}\sqrt{\pi}} \cdot \left[1 - \frac{x^2}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} - \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \cdot x \left[\frac{x}{\lfloor 1} - \frac{x^2}{\lfloor 3} + \frac{x^4}{\lfloor 5} - \cdots \right]$$

$$\therefore J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$$ proved.

Ex. 1: Solve the Bessel's differential equation $x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$ in terms of Neumann function, when n is an integer.

Integral representation of $J_n(x)$:

Ex. 2 prove that $J_n(x) = \frac{1}{\pi} \int_0^{\pi} \cos(n\phi - x\sin\phi)\, d\phi$

for all integral n.