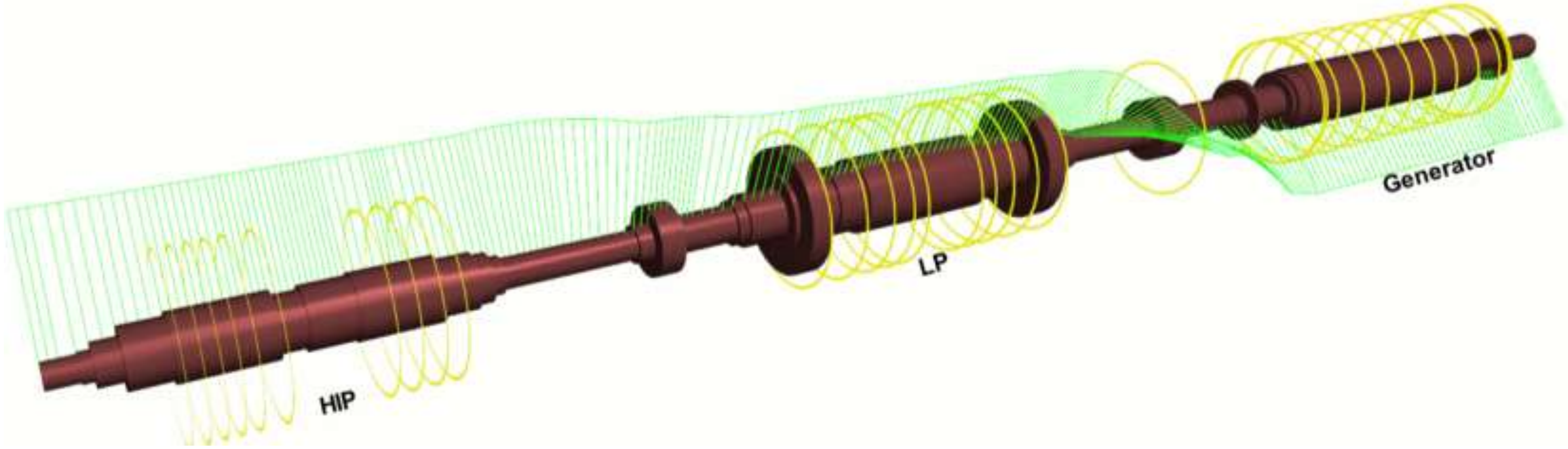
ME 3101: Mechanics of Machinery
Torsional Vibration



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Different Types of Vibration

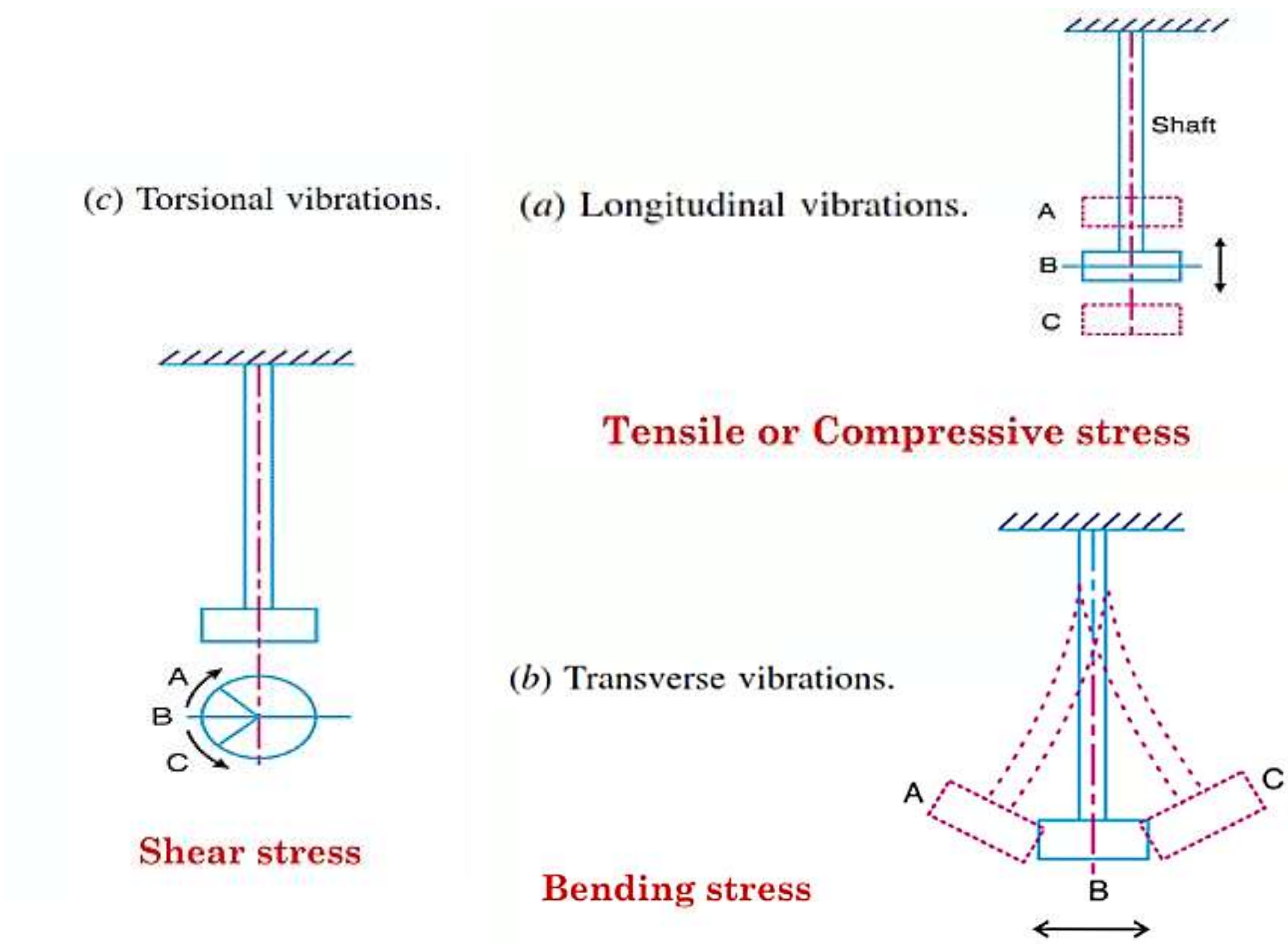
1. Free or natural vibrations

a) **Longitudinal vibration**
b) **Transverse vibration**
c) **Torsional vibration**

2. Forced vibrations

3.Damped vibrations

B *= Mean Position*
A & C *= Extreme Position*



Torsional Vibrations

- When the particles of a shaft or disc move in a circle about the axis of a shaft, then the vibrations are known as **torsional vibrations**
- In this case, the shaft is **twisted** and **untwisted** alternately and **torsional shear stresses** are induced in the shaft

$\therefore$ Restoring force $= q.\theta$ ... *(i)*

and accelerating force $= I \times \dfrac{d^2\theta}{dt^2}$ ... *(ii)*

Equating equations *(i)* and *(ii)*, the equation of motion is

$$I \times \frac{d^2\theta}{dt^2} = -q.\theta$$

or

$$I \times \frac{d^2\theta}{dt^2} + q.\theta = 0$$

$$\therefore \quad \frac{d^2\theta}{dt^2} + \frac{q}{I} \times \theta = 0 \qquad ...\ (iii)$$

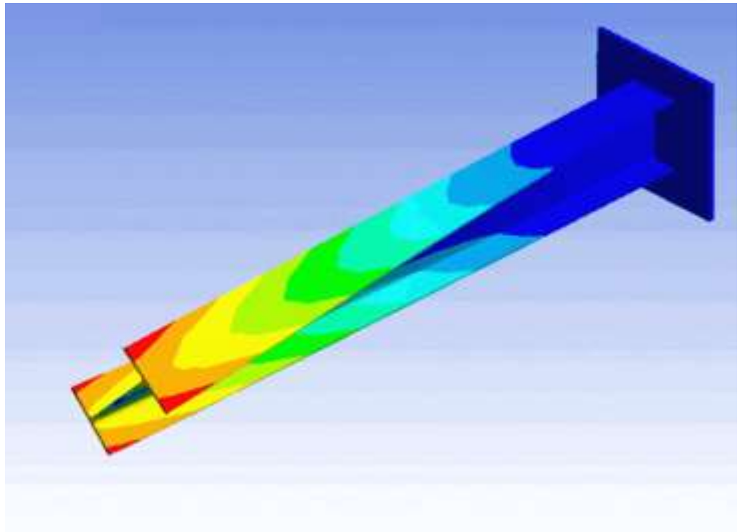
The fundamental equation of the simple harmonic motion is

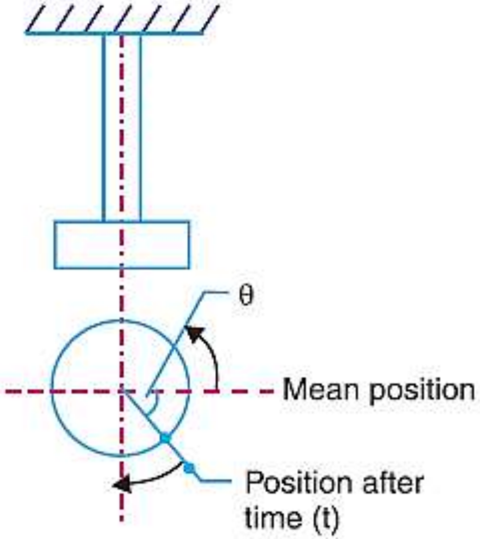
$$\frac{d^2\theta}{dt^2} + \omega^2 x = 0$$

Comparing equations *(iii)* and *(iv)*,

$\therefore$ Time period, $t_p = \dfrac{2\pi}{\omega} = 2\pi\sqrt{\dfrac{I}{q}}$

and natural frequency, $f_n = \dfrac{1}{t_p} = \dfrac{1}{2\pi}\sqrt{\dfrac{q}{I}}$





θ = Angular displacement of the shaft from mean position after time t in radians,

m = Mass of disc in kg,

I = Mass moment of inertia of disc in kg-m^2 $= m.k^2$,

k = Radius of gyration in metres,

q = Torsional stiffness of the shaft in N-m.

Torsional Vibrations

Note : The value of the torsional stiffness q may be obtained from the torsion equation,

$$\frac{T}{J} = \frac{C\theta}{l} \quad \text{or} \quad \frac{T}{\theta} = \frac{C.J}{l}$$

$$\therefore \qquad q = \frac{C.J}{l} \qquad \ldots \left(\because \frac{T}{\theta} = q \right)$$

where C = Modulus of rigidity for the shaft material,

J = Polar moment of inertia of the shaft cross-section,

$= \frac{\pi}{32} d^4$; d is the diameter of the shaft, and

l = Length of the shaft.

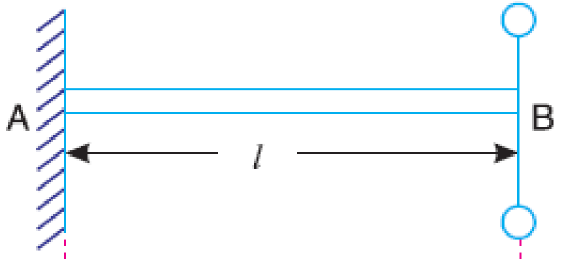
Effect of Inertia of the Constraint/Shaft on Torsional Vibrations

- When the mass moment of inertia of the constraint $\boldsymbol{I_c}$ and the mass moment of inertia of the disc $\boldsymbol{I}$ are known, then natural frequency of vibration

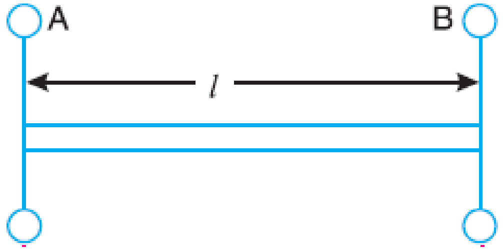
$$f_n = \frac{1}{2\pi} \sqrt{\frac{q}{I + (\frac{I_c}{3})}}$$

Free Torsional Vibrations of Rotor Systems

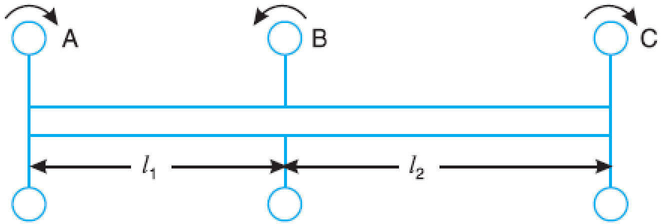
- Single rotor system



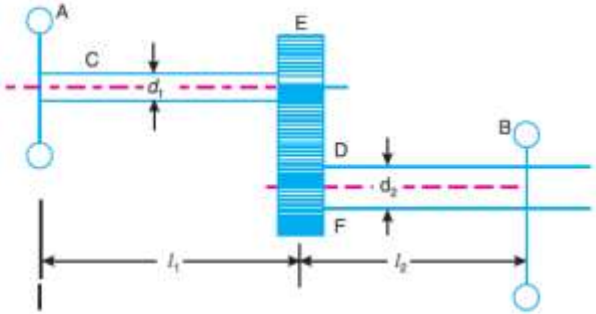
- Two rotor system



- Three rotor system



- Geared system

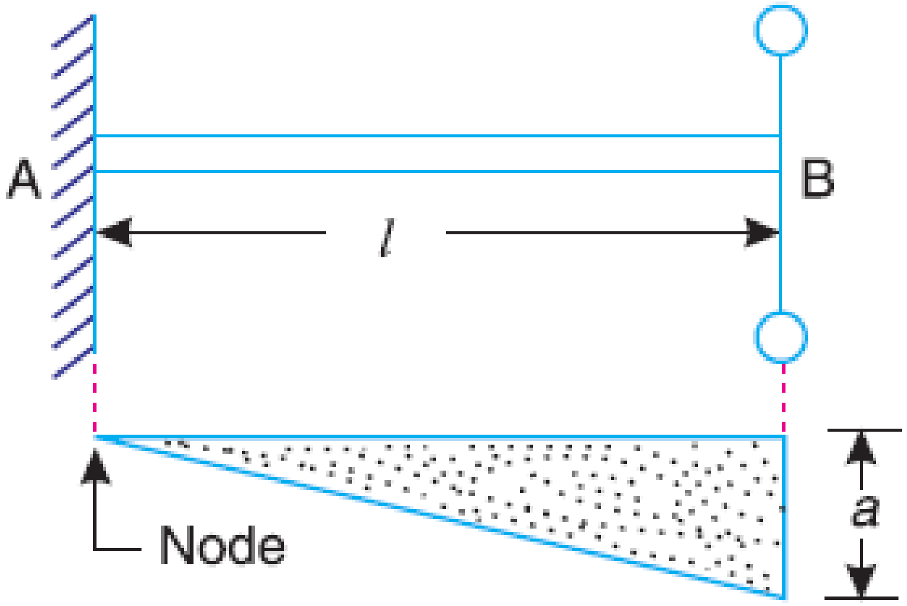


Free Torsional Vibrations of a Single Rotor Systems

Natural frequency of vibration.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{q}{I}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{C.J}{l.I}}$$



- The amplitude of vibration is **zero** at **A** and **maximum** at **B**
- It may be noted that the point or the section of the shaft whose amplitude of torsional vibration is **zero**, is known as **node**. In other words, ***at the node***, the shaft remains ***unaffected by the vibration***

Free Torsional Vibrations of a Two Rotor Systems

- In this system, the torsional vibrations occur only when the two rotors A and B move in **opposite directions** (A moves in anticlockwise, B moves in clockwise direction at the same instant and vice versa)
- Two rotors must have the same frequency.

Natural frequency of torsional vibration for rotor A,

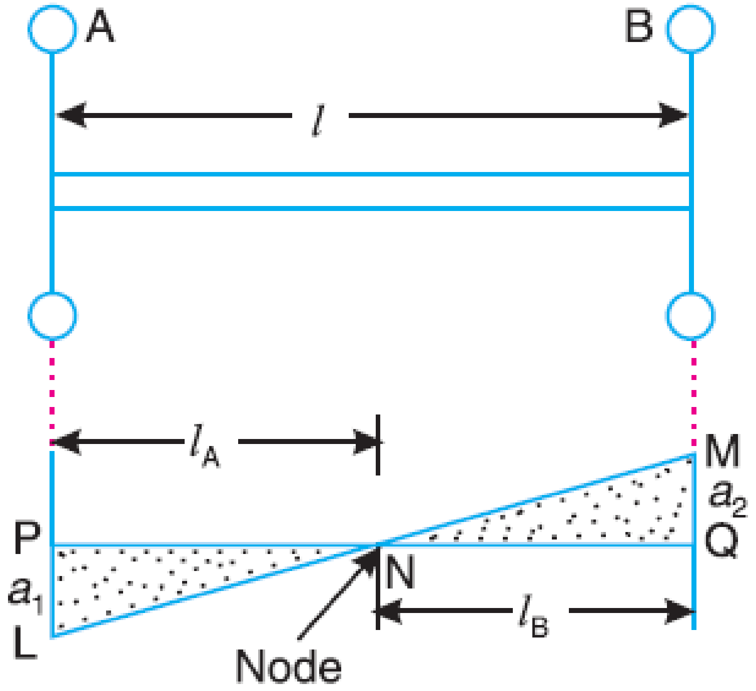
$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A I_A}}$$

Natural frequency of torsional vibration for rotor B,

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_B I_B}}$$

$$f_{nA} = f_{nB}$$

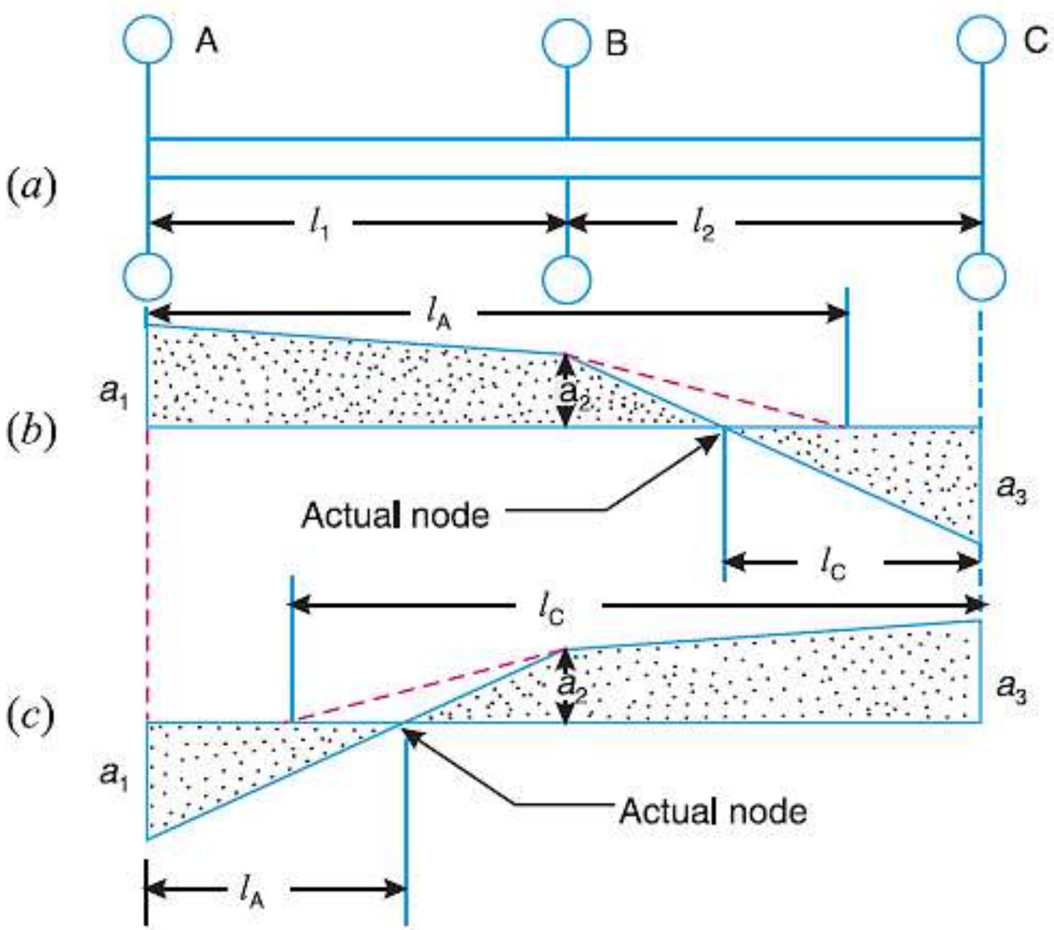
$$l_A I_A = l_B I_B$$



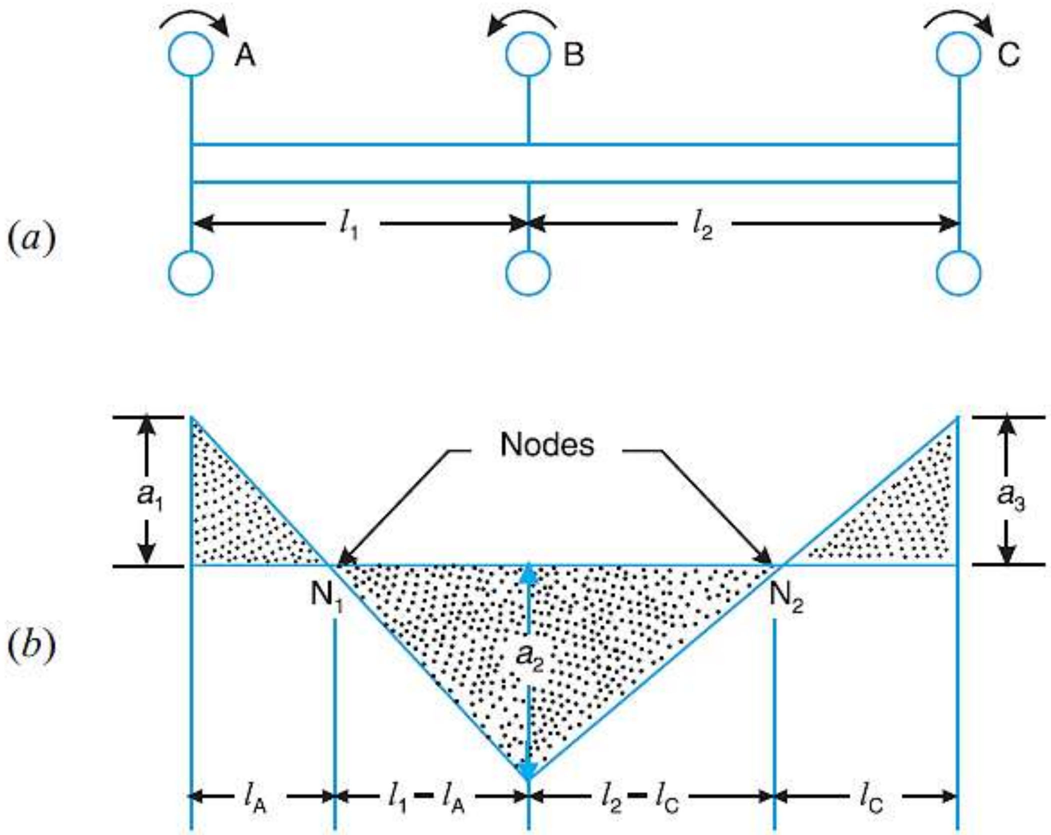
- The line **LNM** in figure is known as **elastic line** for the shaft

Free Torsional Vibrations of a Three Rotor System

(i) Single node system



(ii) Two node system



- In each case, the two rotors rotate in one direction and the third rotor rotates in opposite direction with the same frequency

Free Torsional Vibrations of a Three Rotor System

Natural frequency of torsional vibration for rotor A

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A I_A}} \ldots\ldots. \text{(i)}$$

Natural frequency of torsional vibration for rotor B

$$f_{nB} = \frac{1}{2\pi}\sqrt{\frac{C.J}{I_B}\left(\left(\frac{1}{l_1 - lA}\right) + \left(\frac{1}{l_2 - l_C}\right)\right)} \ldots\ldots \text{(ii)}$$

Natural frequency of torsional vibration for rotor C

$$f_{nC} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_C I_C}} \ldots\ldots \text{(iii)}$$

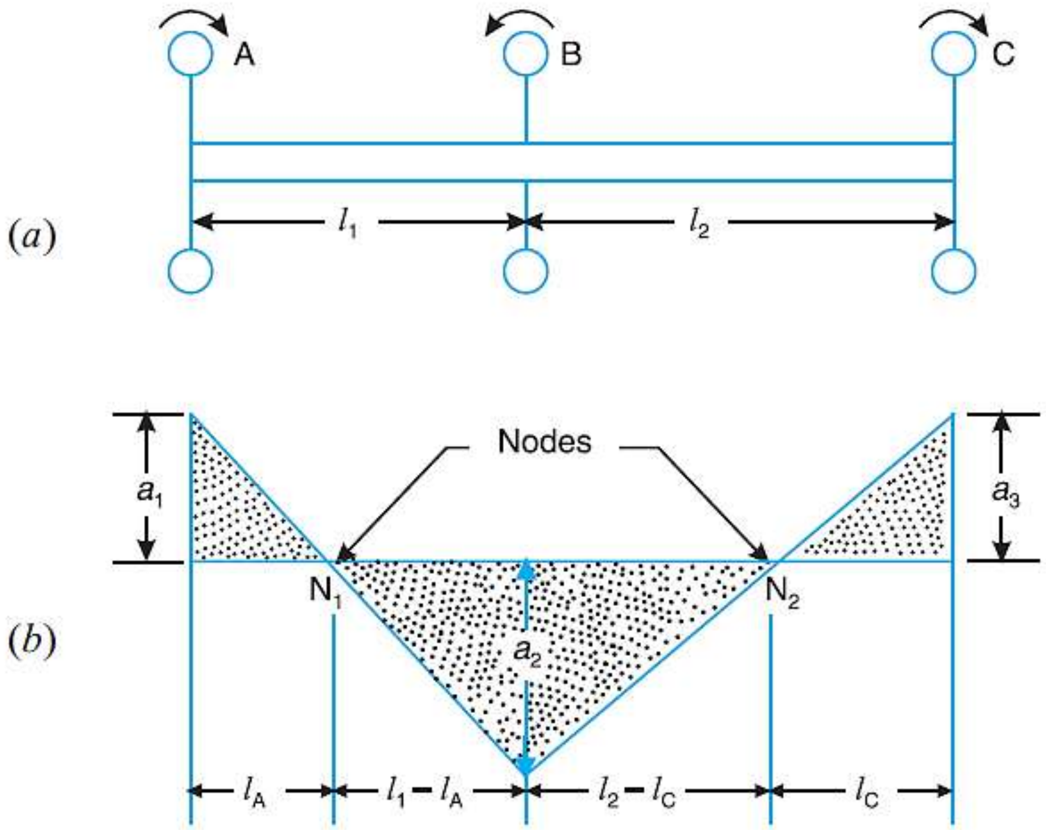
$$f_{nA} = f_{nB} = f_{nC}$$

*From **(i)** and **(iii)**,*

$$l_A I_A = l_C I_C \ldots\ldots (iv)$$

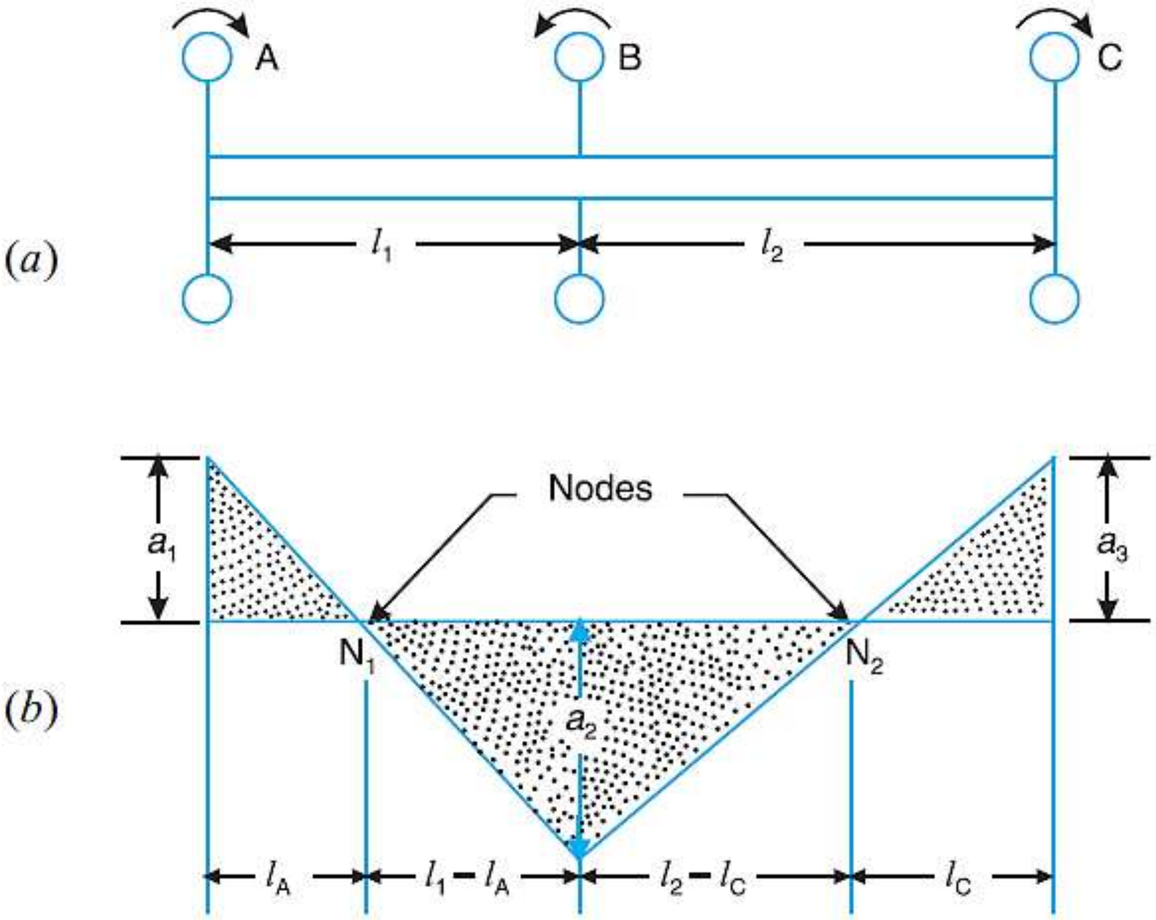
*From **(ii)** and **(iii)**,*

$$\sqrt{\frac{C.J}{I_B}\left(\left(\frac{1}{l_1 - l_A}\right) + \left(\frac{1}{l_2 - l_C}\right)\right)} = \sqrt{\frac{C.J}{l_C I_C}}$$



Free Torsional Vibrations of a Three Rotor System

On substituting the value of l_A from equation **(*iv*)** in the above expression, a quadratic equation in l_C is obtained. Therefore, there are two values of l_C and correspondingly two values of l_A. One value of l_A and the corresponding value of l_C gives the position of two nodes. The frequency obtained by substituting the value of l_A or l_C in equation **(*i*)** or **(*iii*)** is known as ***two node frequency***. But in the other pair of values, one gives the position of single node and the other is beyond the physical limits of the equation. In this case, the frequency obtained is known an ***fundamental frequency*** or ***single node frequency***.



Free Torsional Vibrations of a Three Rotor System

It may be noted that

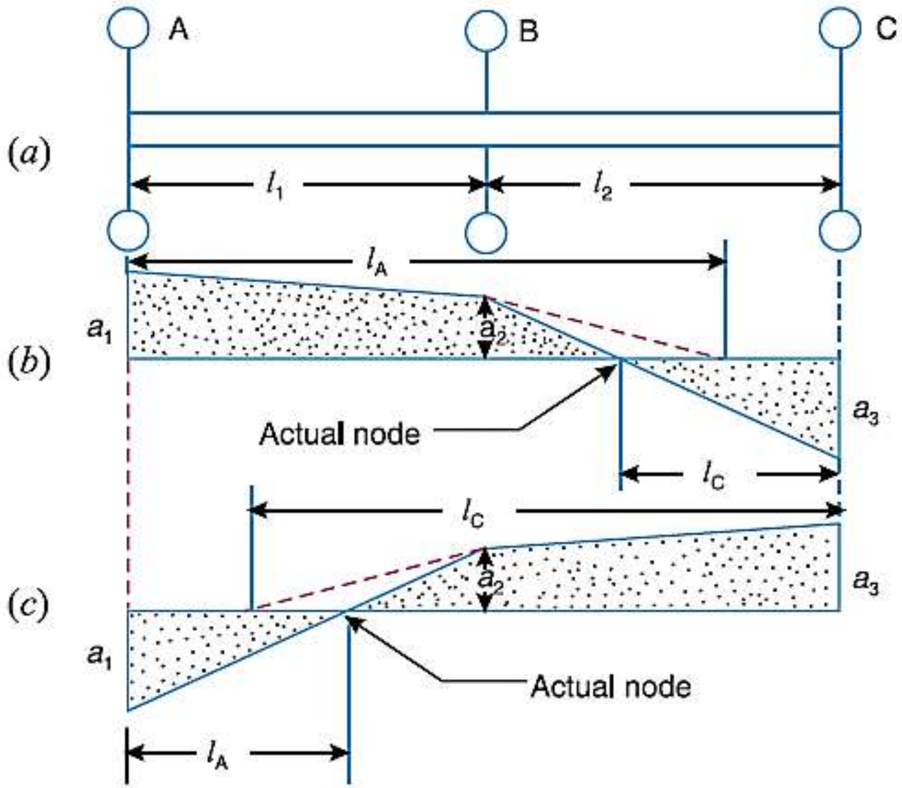
1. When the rotors A and B rotate in the same direction and the rotor C in the opposite direction, then the torsional vibrations occur with a single node, as shown in Fig.
   In this case $l_A > l_1$ *i.e.* the node lies between the rotors B and C, but it does not give the actual value of the node.
2. When the rotors B and C rotate in the same direction and the rotor A in opposite direction, then the torsional vibrations also occur with a single node as shown in Fig. In this case $l_C > l_2$ *i.e* the node lies between the rotors A and B, but it does not give the actual value of the node.
3. When the amplitude of vibration for the rotor A (a_1) is known, then the amplitude of rotor B,

$$a_2 = \frac{l_A - l_1}{l_A} a_1$$

and amplitude of rotor C,

$$a_3 = \frac{l_C}{l_c - l_2} a_2$$

As there are two values of l_A and l_C, therefore there will be two values of amplitude for one node and two node vibrations.



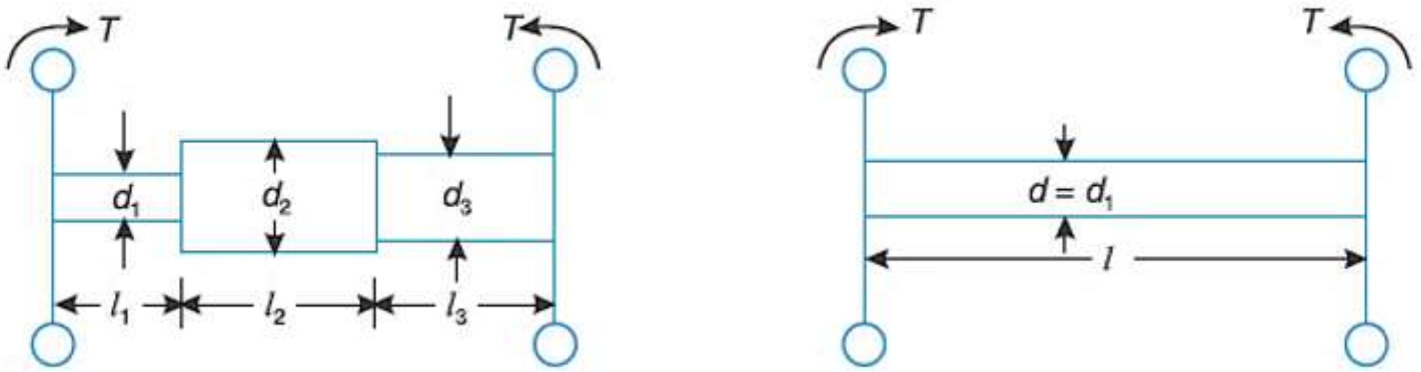
Torsionally Equivalent Shaft

Let d_1, d_2 and d_3 = Diameters for the lengths l_1, l_2 and l_3 respectively,

θ_1 θ_2 and θ_3 = Angle of twist for the lengths l_1, l_2 and l_3 respectively,

= Total angle of twist, and

J_1, J_2 and J_3 = Polar moment of inertia for the shafts of diameters d_1, d_2 and d_3 respectively.



(*a*) Shaft of varying diameters.

(*b*) Torsionally equivalent shaft.

Since the total angle of the twist of the shaft is equal to the sum of the angle of twists of the different lengths.

$$\theta = \theta_1 + \theta_2 + \theta_3$$

From the torsion Equation

$$\frac{\tau}{r} = \frac{T}{J} = \frac{G\theta}{l}$$

$$\theta = \frac{Tl}{JG}$$

Where

- τ = Shear stress (MPa)
- r = Radius of the shaft (mm)
- T = Torque (Nm)
- J = Polar moment of inertia
- G = Modulus of rigidity (MPa)
- θ = Angle of twist (rad)
- l = length of the shaft

Torsionally Equivalent Shaft

As we want to make the multiple cross-sectional shafts into a uniform diametral shaft, so we have to assume that the diameter d of the equivalent shaft should be equal to one of the diameters of the actual shaft. So we can assume d = d_1

Substitute

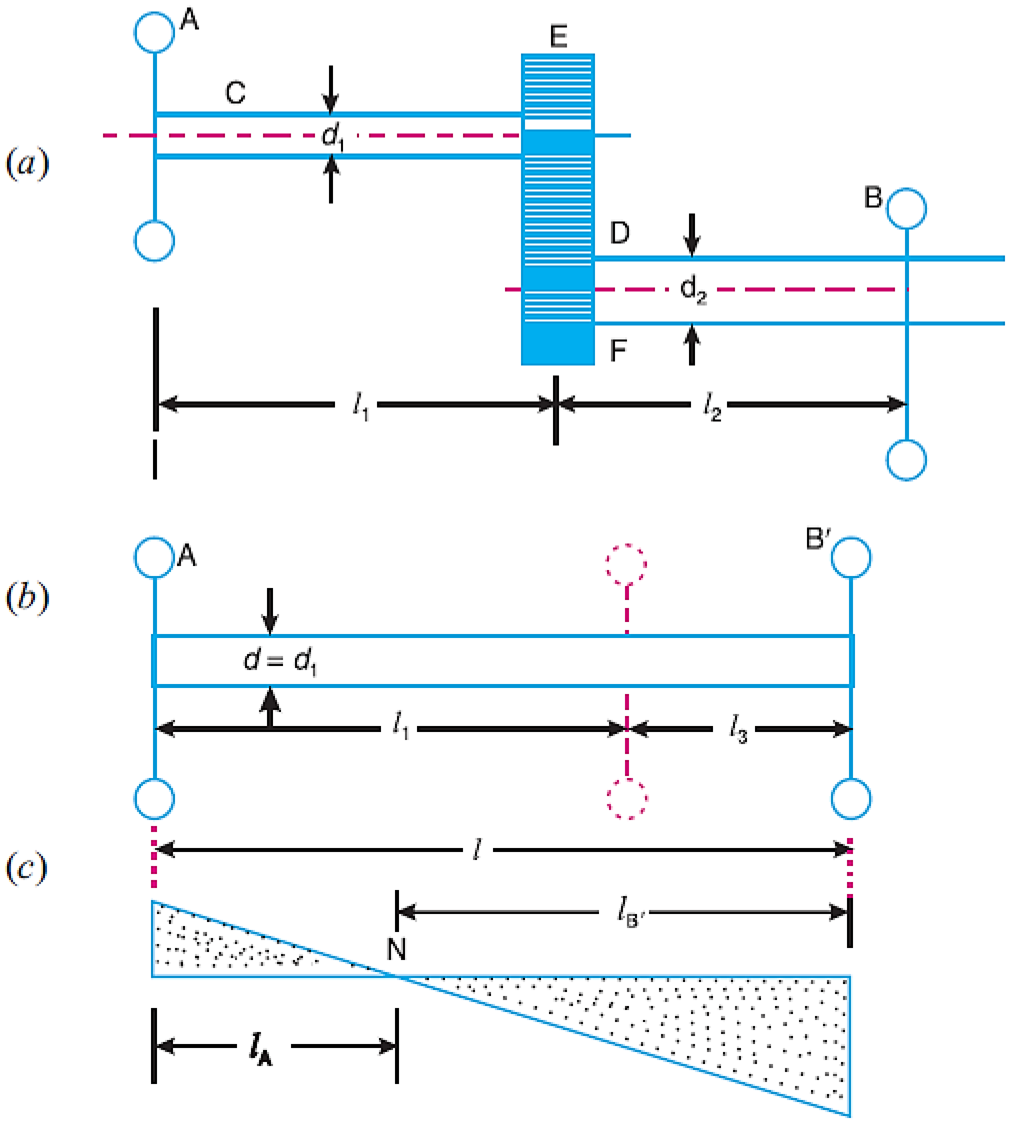
$$\frac{l}{(d_1)^4} = \frac{l_1}{(d_1)^4} + \frac{l_2}{(d_2)^4} + \frac{l_3}{(d_3)^4}$$

$$l = \frac{l_1(d_1)^4}{(d_1)^4} + \frac{l_2(d_1)^4}{(d_2)^4} + \frac{l_3(d_1)^4}{(d_3)^4}$$

$$l = l_1 + l_2\left(\frac{d_1}{d_2}\right)^4 + l_3\left(\frac{d_1}{d_3}\right)^4$$

From this expression, we can evaluate the length of the torsionally equivalent shaft.

Free Torsional Vibrations of a Geared System



1. the gear teeth are rigid and are always in contact,
2. there is no backlash in the gearing, and
3. the inertia of the shafts and gears is negligible.

Let d_1 and d_2 = Diameter of the shafts C and D,

l_1 and l_2 = Length of the shafts C and D,

I_A and I_B = Mass moment of inertia of the rotors A and B,

ω_A and ω_B = Angular speed of the rotors A and B,

$$G = \text{Gear ratio} = \frac{\text{Speed of pinion } E}{\text{Speed of wheel } F}$$

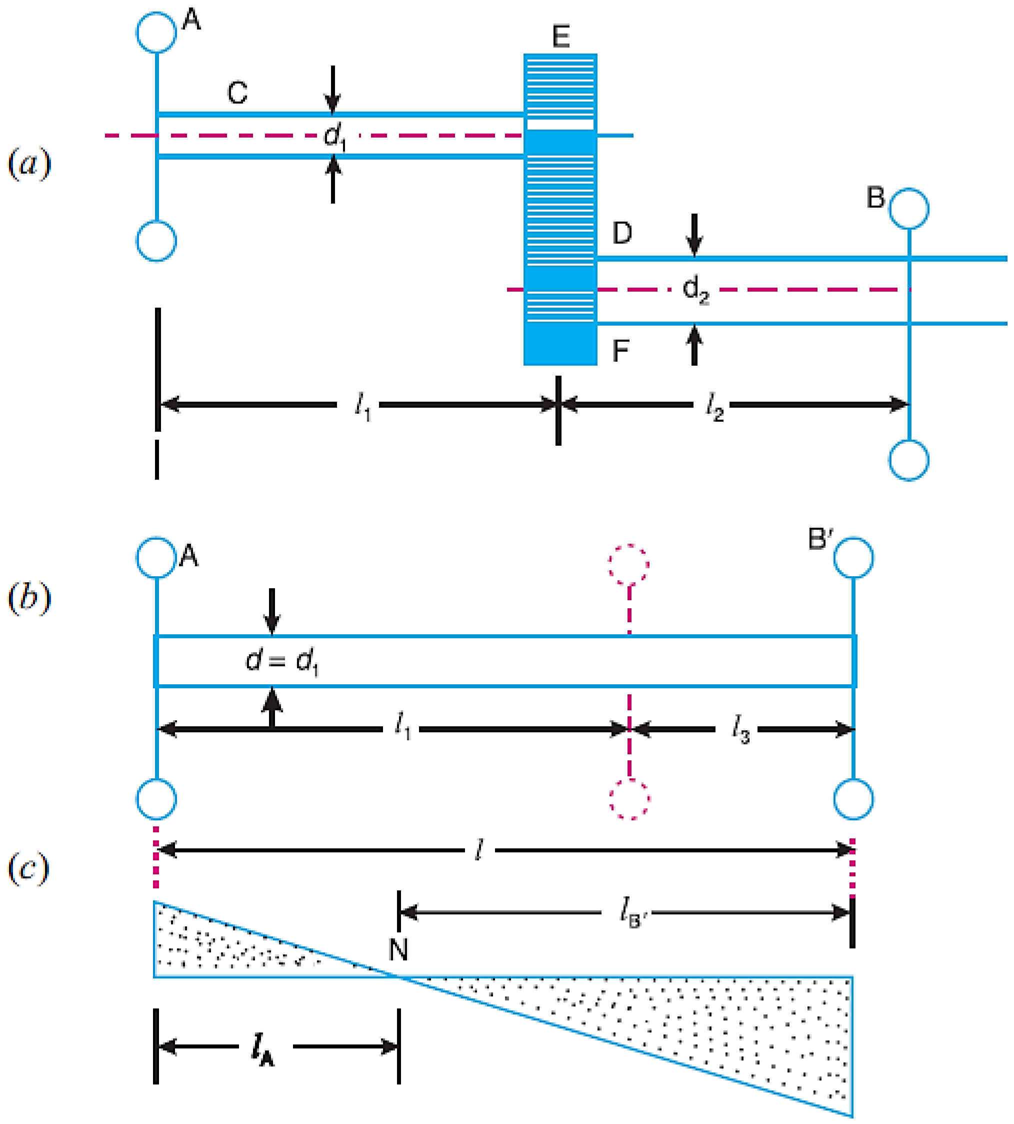
. . . (Speeds of E and F will be same as that of rotors A and B)

d = Diameter of the equivalent shaft,

l = Length of the equivalent shaft, and

I_B = Mass moment of inertia of the equivalent rotor B .

Free Torsional Vibrations of a Geared System



The following two conditions must be satisfied by an equivalent system :

1. The kinetic energy of the equivalent system must be equal to the kinetic energy of the original system.
2. The strain energy of the equivalent system must be equal to the strain energy of the original system.

In order to satisfy the condition (1) for a given load,

K.E. of section l_1 + K.E. of section l_3

= K.E. of section l_1 + K. E. of section l_2

K.E. of section l_3 = K.E. of section l_2

In order to satisfy the condition (2) for a given shaft diameter,

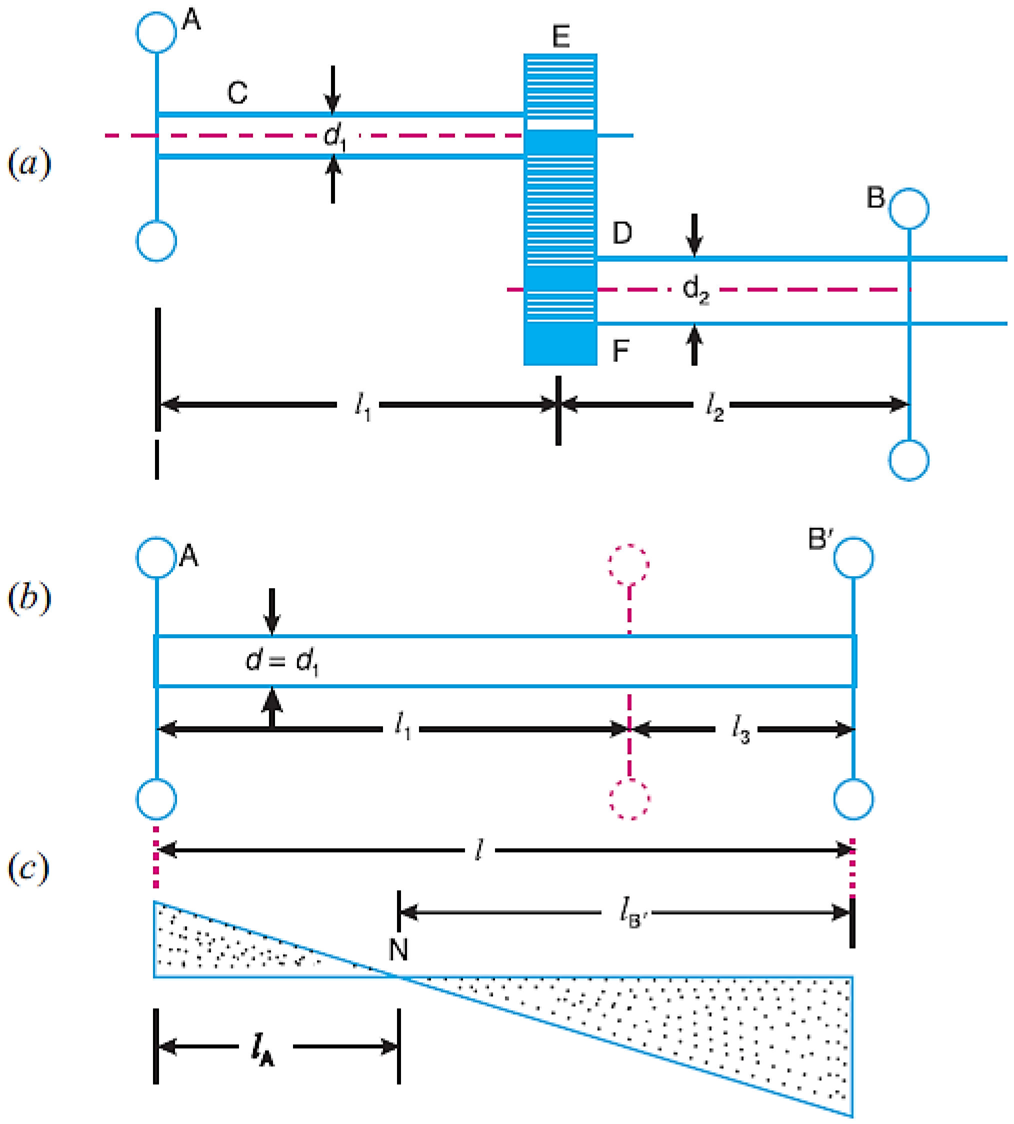
Strain energy of l_1 and l_2 = Strain energy of l_1 and l_2

Strain energy of l_3 = Strain energy of l_2

Length of the equivalent shaft

$$l = l_1 + l_3 = l_1 + G^2.l_2 \left(\frac{d_1}{d_2}\right)^4$$

Free Torsional Vibrations of a Geared System



Natural frequency of the torsional vibration of rotor A

$$f_{nA} = \frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}}$$

Natural frequency of the torsional vibration of rotor B

$$f_{nB} = \frac{1}{2}\sqrt{\frac{C.J}{l_{B'}.I_{B'}}}$$

We know that $f_{nA} = f_{nB'}$

$$\frac{1}{2}\sqrt{\frac{C.J}{l_A.I_A}} = \frac{1}{2}\sqrt{\frac{C.J}{l_{B'}.I_{B'}}}$$

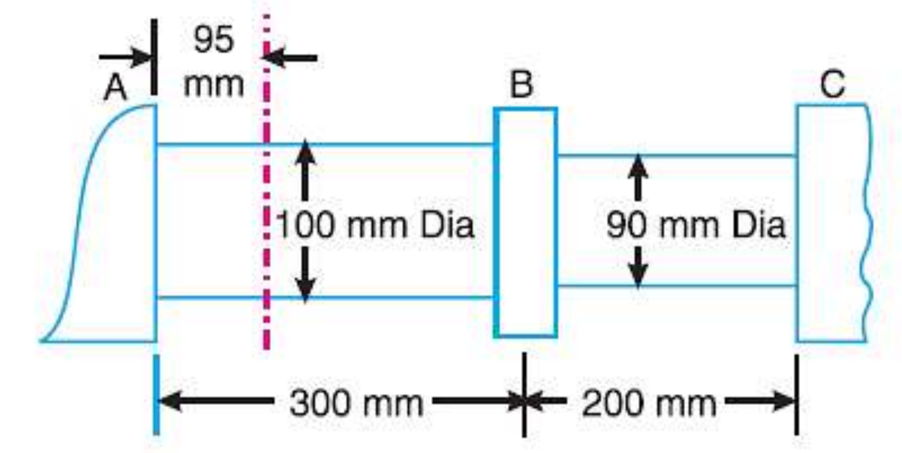
or $$l_A.I_A = l_{B'}.I_{B'}$$

Also $$l_A + l_{B'} = l$$

PROBLEM

- A motor generator set, as shown in figure, consists of two armatures A and C connected with flywheel between them at B. The modulus of rigidity of the connecting shaft is 84 GN/m^2. The system can vibrate torsionally with one node at 95 mm from A, the flywheel being at antinode. Find:
- 1. the position of another node
- 2. the natural frequency of vibration
- 3. the radius of gyration of the armature C
- The other data are given below:

Particulars	*A*	*B*	*C*
Radius of gyration, mm	300	375	–
Mass, kg	400	500	300



Ans:
a. 0.13m
b. 78.1 Hz
c. 0.233 m

Solution

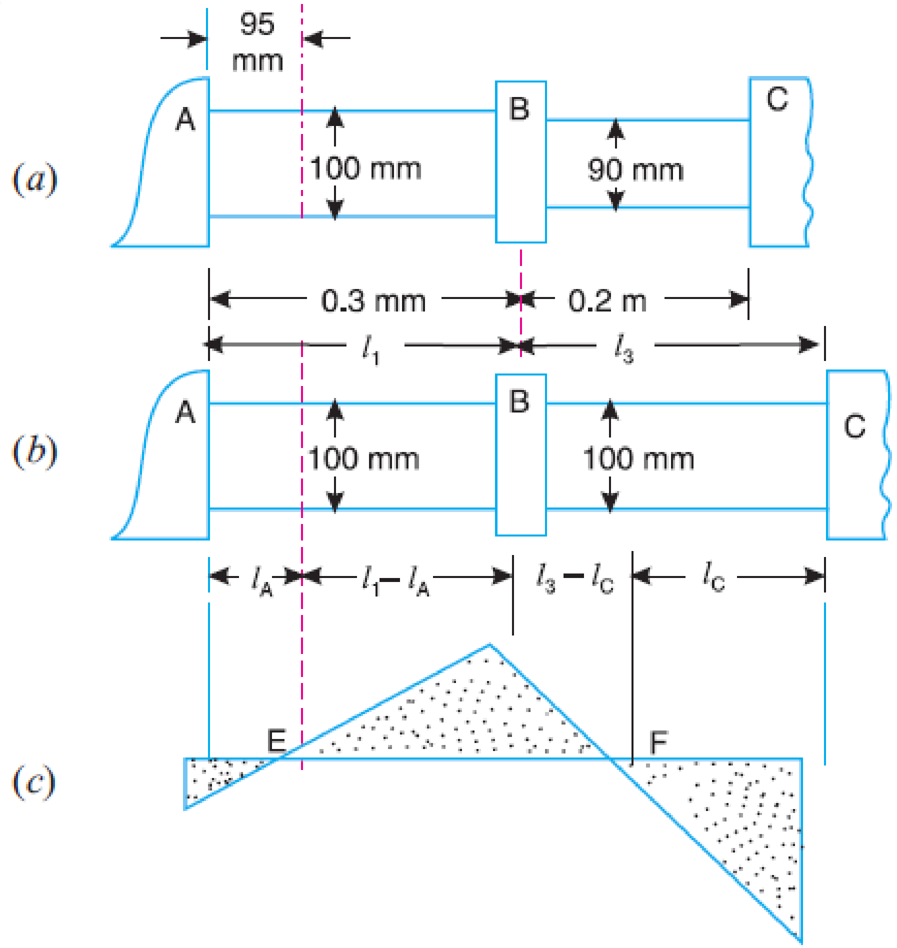
Solution. Given : $C = 84\ GN/m^2 = 84 \times 10^9\ N/m^2$; $d_1 = 100$ mm $= 0.1$ m ; $d_2 = 90$ mm $= 0.09$ m ; $l_1 = 300$ mm $= 0.3$ m ; $l_2 = 200$ mm $= 0.2$ m ; $l_A = 95$ mm $= 0.095$ m ; $m_A = 400$ kg; $k_A = 300$ mm $= 0.3$ m ; $m_B = 500$ kg ; $k_B = 375$ mm $= 0.375$ m ; $m_C = 300$ kg

We know that mass moment of inertia of armature A,

$$I_A = m_A (k_A)^2 = 400\ (0.3)^2 = 36 \text{ kg-m}^2$$

and mass moment of inertia of flywheel B,

$$I_B = m_B (k_B)^2 = 500\ (0.375)^2 = 70.3 \text{ kg-m}^2$$



1. *Position of another node*

First of all, replace the original system, as shown in Fig., by an equivalent system as shown in Fig. (b). It is assumed that the diameter of the equivalent shaft is $d_1 = 100$ mm $= 0.1$m because the node lies in this portion. We know that the length of the equivalent shaft,

$$l = l_1 + l_2 \left(\frac{d_1}{d_2}\right)^4$$

Solution

$$l = l_1 + l_2 \left(\frac{d_1}{d_2}\right)^4 = 0.3 + 0.2\left(\frac{0.1}{0.09}\right)^4 = 0.605 \text{ m}$$

The first node lies at E at a distance 95 mm from rotor A *i.e.* l_A = 95 mm = 0.095 m, as shown in Fig.

Let l_C = Distance of node F in an equivalent system from rotor C, and

l_3 = Distance between flywheel B and armature C in an equivalent system = $l - l_1$ = 0.605 – 0.3 = 0.305 m

$$\sqrt{\frac{C.J}{I_B}\left(\left(\frac{1}{l_1 - l_A}\right) + \left(\frac{1}{l_2 - l_C}\right)\right)} = \sqrt{\frac{C.J}{l_A I_A}}$$

$$or, \quad \frac{1}{I_B}\left(\left(\frac{1}{l_1 - l_A}\right) + \left(\frac{1}{l_2 - l_C}\right)\right) = \frac{1}{l_A I_A}$$

$$or, \quad \frac{1}{70.3}\left(\left(\frac{1}{0.3 - 0.095}\right) + \left(\frac{1}{0.305 - l_C}\right)\right) = \frac{1}{0.095 \times 36}$$

$$l_c = 0.21\ m$$

Corresponding value of $\boldsymbol{l_c}$ in an original system from rotor C

$$0.21\left(\frac{d_2}{d_1}\right)^4 = 0.21\left(\frac{0.09}{0.1}\right)^4 = 0.13\ m\ (\boldsymbol{Ans})$$

Solution

2. Natural Frequency of Vibration

We know that polar moment of inertia of the equivalent shaft,

$$J = \frac{\pi}{32}(d_1)^4 = \frac{\pi}{32}(0.1)^4 = 9.82 \times 10^{-6} m^4$$

Natural frequency of vibrations,

$$f_{nA} = \frac{1}{2\pi}\sqrt{\frac{C.J}{l_A I_A}} = \frac{1}{2\pi}\sqrt{\frac{84\times10^9\times9.82\times10^{-6}}{0.095\times36}} = 78.1\ Hz\ (Ans)$$

3. Radius of gyrataion of armature C

Let k_c= Radius of gyration of armature C in metres, and

I_c = Mass moment of inertia of armature C = mc $(k_c)^2$ in kgm^2

We know that

$$l_A I_A = l_C I_C = l_C m_C (k_c)^2$$

$$or, \quad 0.095 \times 36 = 0.21 \times 300 \times (k_c)^2$$

$$k_c = 0.233\ (Ans)$$

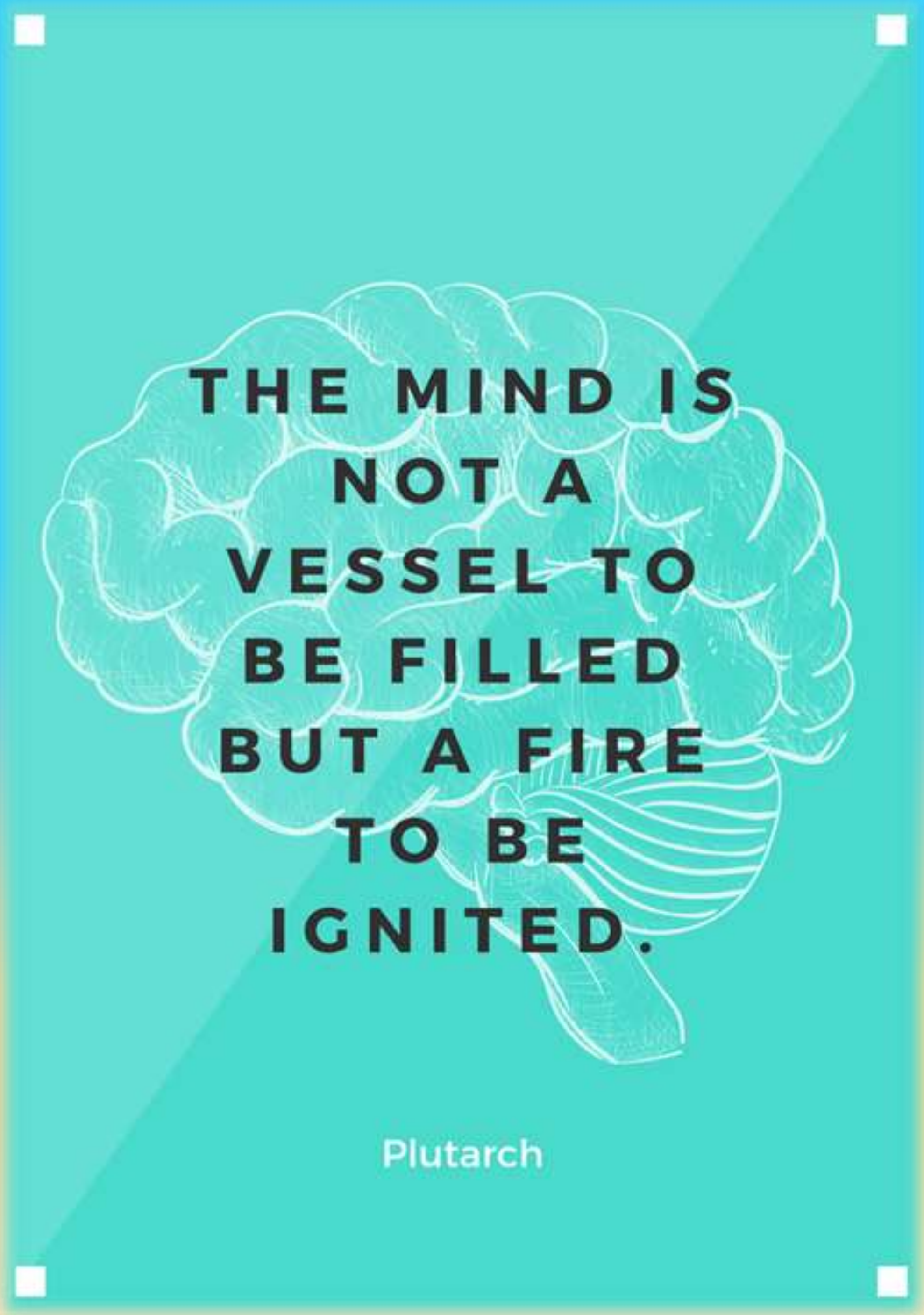
Solve by Yourself

Book: Mechanics of Machines: Advanced Theory and Examples
Chapter 16

Example: 4
Exercise: 11,12,21,22,31

Book: Theory of Machines by Khurmi
Chapter 24

Example: 24.3,24.5,24.6,24,10
Exercise: 5,8


THE MIND IS NOT A VESSEL TO BE FILLED BUT A FIRE TO BE IGNITED.
Plutarch