AHSANULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY (AUST)

ME-3105: FLUID MECHANICS

(LC-6: Rotation of Fluid Particle
and
Stream Function))

BY

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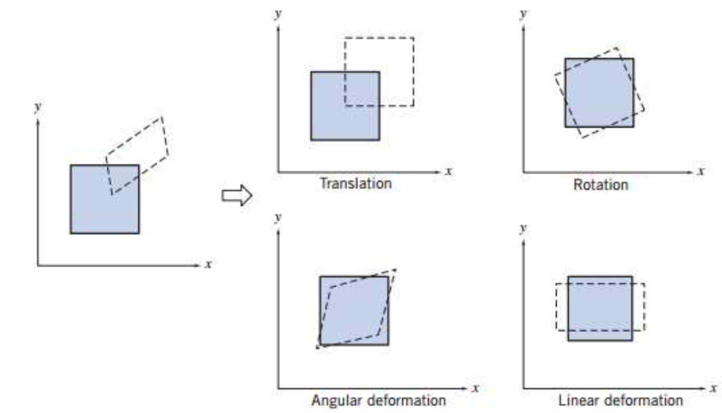
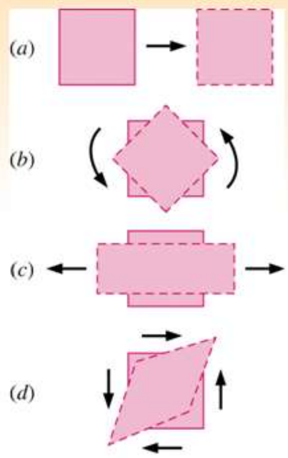
MOVEMENT OR KINEMATICS OF FLUID PERTICLE

Pictorial representation of the components of fluid motion: Velocity of the fluid particle is function of x, y, z and t or V = V (x, y, z, t);



MOVEMENT OR KINEMATICS OF FLUID PERTICLE

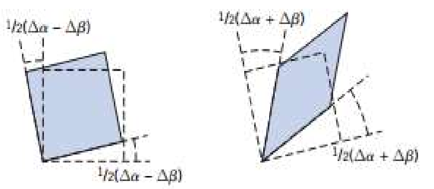


- In fluid mechanics, an element may undergo four fundamental types of motion.
  - Translation
  - Rotation
  - Linear strain
  - Shear strain
- Because fluids are in constant motion, motion and deformation are described in terms of rates
  - velocity: rate of translation
  - angular velocity: rate of rotation
  - linear strain : rate of linear strain
  - shear strain : rate of shear strain

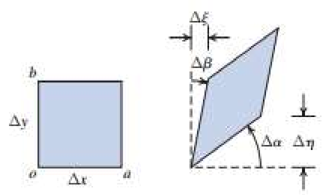
MOVEMENT OR KINEMATICS OF FLUID PERTICLE

A fluid element may move in a flow and undergo with one of the three types of motion:

- *Pure or Irrotational Translation*
- *Pure Rotation or Rotational Translation*
- *Pure Distortion or Deformation (Angular or Linear Deformation)*

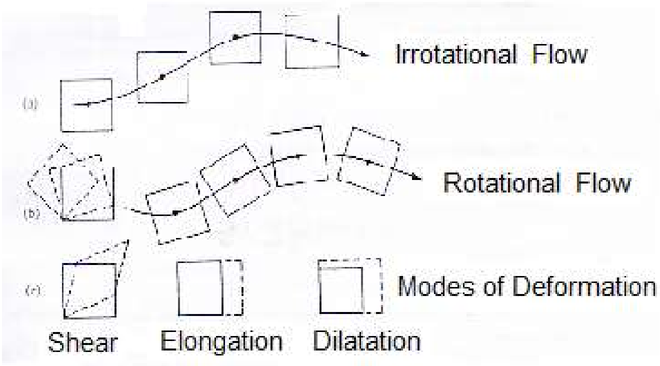


(c) Rotational component (d) Angular deformation component



(a) Original particle (b) Particle after time Δt

MOVEMENT OR KINEMATICS OF FLUID PERTICLE



LINEAR TRANSLATION AND LINEAR DEFORMATION OF FLUID ELEMENT

SEE HAND ANALYSIS FOR DERIVATION

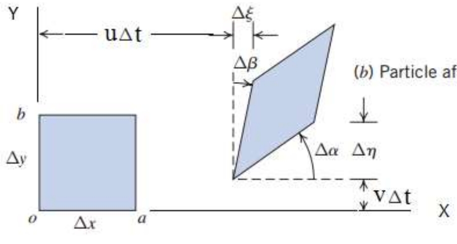
ROTATIONAL OR ANGULAR DEFORMATION RATE OF FLUID PERTICLE

A fluid particle moving in a general three-dimensional flow field may rotate about all three coordinate axes. Thus particle rotation is a vector quantity and, in general,

$$\vec{\omega} = \hat{i}\omega_x + \hat{j}\omega_y + \hat{k}\omega_z$$

where ω_x is the rotation about the x axis, ω_y is the rotation about the y axis, and ω_z is the rotation about the z axis. The positive sense of rotation is given by the right-hand rule.

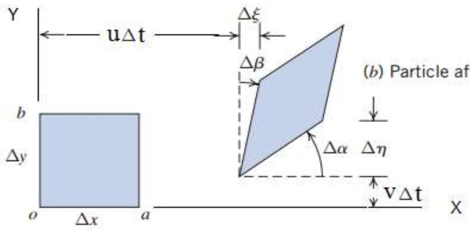
Following the right-hand rule, *counterclockwise rotation is positive*



(a) Original particle at time 't'

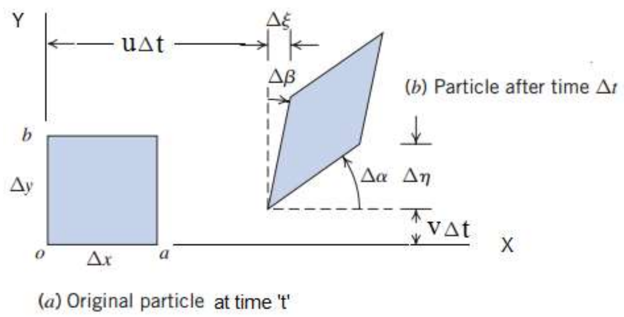
ROTATIONAL OR ANGULAR DEFORMATION RATE OF FLUID PERTICLE

Consider the xy plane view of the particle at time 't' at point O (x, y). The left and lower sides of the particle are given by the two perpendicular line segments 'oa' and 'ob' of lengths 'Δx' and 'Δy' respectively, shown in the figure (a). In general, after an interval Δt the particle will have translated to some new position, and also have rotated and deformed. A possible instantaneous orientation of the lines at time 't +Δt' is shown in the Figure b.



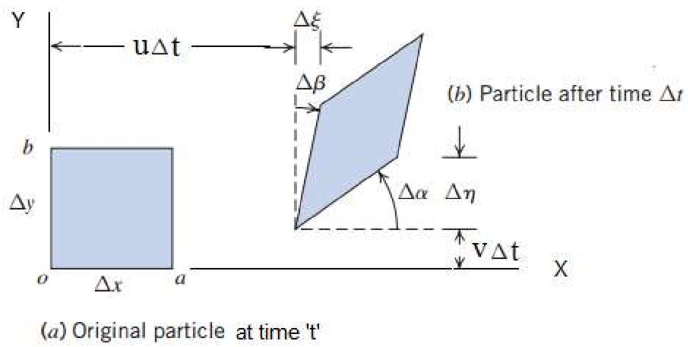
(a) Original particle at time 't'

ROTATION OF FLUID ELEMENT



SEE HAND ANALYSIS FOR DERIVATION

ANGULAR DEFORMATION OR SHEAR STRAIN OF FLUID ELEMENT



SEE HAND ANALYSIS FOR DERIVATION

ALL FORMULA/SUMMERY

$\vec{V} = u\,\vec{\imath} + v\,\vec{\jmath} + w\,\vec{k}$ and $\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$

3D Continuity equation in Cartesian coordinate for incompressible flow,

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) + \frac{\partial \rho}{\partial t} = 0 \text{ or } \vec{\nabla} \cdot \rho\vec{V} + \frac{\partial \rho}{\partial t} = 0$$

For incompressible flow, ρ = constant

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ or } \vec{\nabla} \cdot \vec{V} = DIV \cdot \vec{V}$$

3D Continuity equation in Cylindrical coordinate for incompressible flow,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho v_r) + \frac{1}{r}\frac{\partial(\rho v_\theta)}{\partial \theta} + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0$$

ALL FORMULA/SUMMERY

Volumetric Strain,

$$\varepsilon_v = \frac{1}{V}\frac{\partial v}{\partial t} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\frac{1}{V}\frac{\partial v}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \vec{\nabla} \cdot \vec{V}$$

For incompressible flow volumetric strain is zero $\vec{\nabla} \cdot \vec{V} = 0$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

ALL FORMULA/SUMMERY

For rotational flow,

$$\vec{\omega} = \omega_x \cdot \vec{\imath} + \omega_y \cdot \vec{\jmath} + \omega_z \cdot \vec{k}$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right);\ \ \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right);\ \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\vec{\omega} = \frac{1}{2}\cdot\left(\vec{\nabla} \times \vec{V}\right) = \frac{1}{2}\left(Curl\,\vec{V}\right);\ Vorticity, \bar{\xi} = 2\vec{\omega} = \vec{\nabla} \times \vec{V}$$

Angular deformation of fluid,

$$\dot{\gamma_{xy}} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y};\ \ \dot{\gamma_{xz}} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\ ;\ \dot{\gamma_{yz}} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

VELOCITY POTENTIAL ‘ϕ (phi)‘

Velocity Potential ‘φ‘: The velocity potential is defined as a scalar function of space and time such that its negative derivative with respect to any direction gives the fluid velocity in that direction. Thus mathematically the velocity potential is defined as $\varphi = f(x, y, z, t)$ for unsteady flow and $\varphi = f(x, y, z)$ for steady flow such that

$$u = -\frac{\partial \phi}{\partial x}\ ;\ v = -\frac{\partial \phi}{\partial y}\ ;\ w = -\frac{\partial \phi}{\partial z}$$

Where u, v and w are the components of velocity in the x, y and z directions respectively. The negative signifies that φ decreases with an increases in the value of x, y and z. In the other words it indicates that the flow is always in the direction of decreasing φ .

For incompressible fluid if the flow is steady then equation of continuity is given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

VELOCITY POTENTIAL ‘ϕ (phi)

By substituting the value of u, v and w in terms of φ,

$$\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial\phi}{\partial z}\right)=0 \qquad \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

The above equation is known as Laplace Equation, express as $\nabla^2\phi = 0$ if the velocity potential satisfies the Laplace equation it represents the possible steady, incompressible, irrotational flow. Often an irrotational flow is known as **Potential flow**.

VELOCITY POTENTIAL ‘ϕ (phi)

For roatational flow, the rotation components are given by,

$$\omega_x = \frac{1}{2}\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right) \qquad \omega_z = \frac{1}{2}\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)$$

Substituting the value of u, v and w interms of ϕ,

$$\omega_x = \frac{1}{2}\left(-\frac{\delta^2\phi}{\delta y\delta z} + \frac{\delta^2\phi}{\delta z\delta y}\right) \quad \omega_y = \frac{1}{2}\left(-\frac{\delta^2\phi}{\delta z\delta x} + \frac{\delta^2\phi}{\delta x\delta z}\right) \quad \omega_z = \frac{1}{2}\left(-\frac{\delta^2\phi}{\delta x\delta y} + \frac{\delta^2\phi}{\delta y\delta x}\right)$$

However ϕ is a continuous function then,

$$\frac{\delta^2\phi}{\delta y\delta z} = \frac{\delta^2\phi}{\delta z\delta y} \qquad \frac{\delta^2\phi}{\delta y\delta x} = \frac{\delta^2\phi}{\delta x\delta z} \qquad \frac{\delta^2\phi}{\delta x\delta y} = \frac{\delta^2\phi}{\delta y\delta x}$$

so $\omega_z = 0$, $\omega_y = 0$ and $\omega_x = 0$ i.e the flow is irrotational.

STREAM FUNCTION ‘ Ψ(psi) ’

Stream Function ‘ Ψ ‘ : The stream function **Ψ** (psi) is defined as a scalar function of space and time in two dimensional flow such that its partial derivative with respect to any direction gives velocity component at right angle in that direction. Thus mathematically the velocity potential is defined as ψ = f (x, y, t) for unsteady flow and ψ = f (x, y) for steady flow such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

The Continuity equation for 2D incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

By substituting the value of u, v and w in terms of ψ,

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

So the stream function ψ satisfy the continuity equation.

VELOCITY POTENTIAL ‘ϕ (phi) ‘ AND STREAM FUNCTION ‘ Ψ(psi) ’

Stream Function ‘ Ψ ‘ :

From definition of stream line,

$$\frac{dx}{u} = \frac{dy}{v} \qquad u \cdot dy - v \cdot dx = 0 \qquad \frac{\delta \psi}{\delta y} \cdot dy + \frac{\delta \psi}{\delta x} \cdot dx = 0$$

Again for steady flow, stream function ψ = ψ(x, y)

$$d\psi = \frac{\delta \psi}{\delta y} \cdot dy + \frac{\delta \psi}{\delta x} \cdot dx \qquad d\psi = 0 \qquad \psi = \text{Constant}$$

So ψ is a constant along a streamline. If we know the function ψ(x,y) we can plot constant line of ψ to provide the family of streamline that are helpful in visualizing flow pattern.

VELOCITY POTENTIAL ‘ϕ (phi) ‘ AND STREAM FUNCTION ‘ Ψ(psi) ’

Stream Function ‘ Ψ ‘ :

Difference in ψ between streamlines is equal to volume flow rate between streamlines. It can also be defined as the flux or flow rate between two streamlines. The unit of ψ is m^3/s (discharge per unit thickness of flow). Existence of ψ means a possible case of fluid flow.

EQUIPOTENTIAL LINES AND STREAM LINES ARE PERPENDICULAR

For two dimensional flow,

$$u = \frac{\delta\psi}{\delta y} \;;\; v = -\frac{\delta\psi}{\delta x}$$ (for stream line) and $\psi = \psi(x,y)$

$$u = -\frac{\delta\phi}{\delta x} \;;\; v = -\frac{\delta\phi}{\delta y}$$ (for velocity potential) and $\phi = \phi(x,y)$

$$d\psi = \frac{\delta\psi}{\delta x}\cdot dx + \frac{\delta\psi}{\delta y}\cdot dy = -v\cdot dx + u\cdot dy$$ and

$$d\phi = \frac{\delta\phi}{\delta x}\cdot dx + \frac{\delta\phi}{\delta y}\cdot dy = -u\cdot dx - v\cdot dy$$

The line of constant ψ are streamline i.e $d\psi = 0$ $\left.\frac{dy}{dx}\right|_{\psi_{constant}} = \frac{v}{u}$

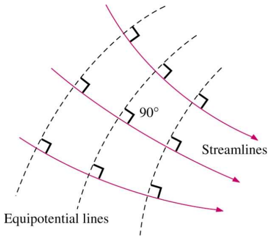
Along a line of constant ϕ, $d\phi = 0$ $\left.\frac{dy}{dx}\right|_{\phi_{constant}} = -\left(\frac{u}{v}\right)$

EQUIPOTENTIAL LINES AND STREAM LINES ARE PERPENDICULAR

$$\left.\frac{dy}{dx}\right|_{\psi_{constant}} = \frac{v}{u} \qquad \left.\frac{dy}{dx}\right|_{\phi_{constant}} = -\left(\frac{u}{v}\right)$$

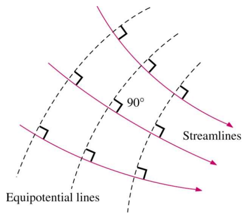
The product of slope of ψ = constant line and velocity potential line ϕ = constant line is negative 1 i.e. both lines are perpendicular where they intersect. The lines of constant ϕ are called equipotential lines.

Flow Net: Flow net is consists of a family of streamlines and equipotential lines that represent the graphical flow pattern of the flow for visualization.



DIFFERENCE BETWEEN 'ϕ (phi) ' AND ' Ψ(psi) '

- The stream function is defined by continuity; the Laplace equations for ψ results from irrotationality. The velocity potential is defined by irrotationality and the Laplace equations for ϕ results from continuity.
- Curves of constant values of ψ define streamlines of the flow. Curves of constant values of ϕ define equipotential lines of the flow.
- **The stream line with constant ψ and velocity potential line with constant φ are perpendicular at the point they intersect.**



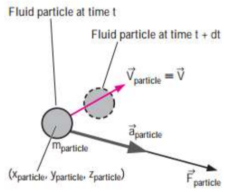
ACCELERATION OF FLUID PERTICLE (Note)

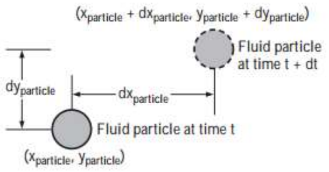
Newton's second law of motion applied for fluid particle,

$$\vec{F} = \left(M_{particle}\right)\left(\overrightarrow{a_{particle}}\right) \qquad \overrightarrow{a_{particle}} = \frac{d\left(\overrightarrow{V_{particle}}\right)}{dt}$$

Fluid particle velocity vector V is function of position of fluid particle position vector ($X_{particle}$, $Y_{particle}$, $Z_{particle}$ and time 't').

$$\overrightarrow{a_{particle}} = \frac{d\overrightarrow{V_{particle}}}{dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}\left(x_{particle}, y_{particle}, z_{particle}, t\right)}{dt}$$





ACCELERATION OF FLUID PERTICLE (Note)

Derivative of a function which value depends on four or multiple variables

$f = f(x, y, z, t)$ (value of f depends on four variable)

$$df = \frac{\delta f}{\delta x}\cdot dx + \frac{\delta f}{\delta y}\cdot dy + \frac{\delta f}{\delta z}\cdot dz + \frac{\delta f}{\delta t}\cdot dt$$

$$\frac{df}{dt} = \frac{\delta f}{\delta x}\cdot\frac{dx}{dt} + \frac{\delta f}{\delta y}\cdot\frac{dy}{dt} + \frac{\delta f}{\delta z}\cdot\frac{dz}{dt} + \frac{\delta f}{\delta t}\cdot\frac{dt}{dt}$$

$$= \frac{\delta f}{\delta x}\cdot\frac{dx}{dt} + \frac{\delta f}{\delta y}\cdot\frac{dy}{dt} + \frac{\delta f}{\delta z}\cdot\frac{dz}{dt} + \frac{\delta f}{\delta t}$$

$$\overrightarrow{a_{particle}} = \frac{\delta\vec{V}}{\delta t}\cdot\frac{dt}{dt} + \frac{\delta\vec{V}}{\delta x_{particle}}\cdot\frac{dx_{particle}}{dt} + \frac{\delta\vec{V}}{\delta y_{particle}}\cdot\frac{dy_{particle}}{dt} + ..$$

$$\frac{\delta V}{\delta z_{particle}}\cdot\frac{dz_{particle}}{dt}$$

$$u = \frac{dx_{particle}}{dt} \qquad v = \frac{dy_{particle}}{dt} \qquad w = \frac{dz_{particle}}{dt}$$

ACCELERATION OF FLUID PERTICLE (Note)

$$\overrightarrow{a_{particle}} = \frac{\delta \vec{V}}{\delta t}\cdot\frac{dt}{dt} + \frac{\delta \vec{V}}{\delta x_{particle}}\cdot\frac{dx_{particle}}{dt} + \frac{\delta \vec{V}}{\delta y_{particle}}\cdot\frac{dy_{particle}}{dt} + ..$$

$$\frac{\delta V}{\delta z_{particle}}\cdot\frac{dz_{particle}}{dt}$$

$$u = \frac{dx_{particle}}{dt} \qquad v = \frac{dy_{particle}}{dt} \qquad w = \frac{dz_{particle}}{dt}$$

$$\overrightarrow{a_{particle}}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + u\frac{\delta \vec{V}}{\delta x} + v\frac{\delta \vec{V}}{\delta y} + w\frac{\delta \vec{V}}{\delta z}$$

$$\vec{\nabla} = \vec{i}\cdot\frac{\delta}{\delta x} + \vec{j}\cdot\frac{\delta}{\delta y} + \vec{k}\cdot\frac{\delta}{\delta z} = \left(\frac{\delta}{\delta x},\frac{\delta}{\delta y},\frac{\delta}{\delta z}\right) \qquad \vec{V} = \vec{i}\cdot u + \vec{j}\cdot v + \vec{k}\cdot w$$

ACCELERATION OF FLUID PERTICLE (Note)

$$\overrightarrow{a_{particle}}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + u\frac{\delta \vec{V}}{\delta x} + v\frac{\delta \vec{V}}{\delta y} + w\frac{\delta \vec{V}}{\delta z}$$

$$\vec{\nabla} = \vec{i}\cdot\frac{\delta}{\delta x} + \vec{j}\cdot\frac{\delta}{\delta y} + \vec{k}\cdot\frac{\delta}{\delta z} = \left(\frac{\delta}{\delta x},\frac{\delta}{\delta y},\frac{\delta}{\delta z}\right) \qquad \vec{V} = \vec{i}\cdot u + \vec{j}\cdot v + \vec{k}\cdot w$$

$$\vec{a}\cdot(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}$$

$$a_x = \frac{\delta u}{\delta t} + u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y} + w\frac{\delta u}{\delta z}$$

$$a_y = \frac{\delta v}{\delta t} + u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta y} + w\frac{\delta v}{\delta z}$$

$$a_z = \frac{\delta w}{\delta t} + u\frac{\delta w}{\delta x} + v\frac{\delta w}{\delta y} + w\frac{\delta w}{\delta z}$$

MATERIAL DERIVATIVE

The total derivative operator $\frac{d}{dt}$ is given a special name, the material derivative $\frac{D}{Dt}$. It is formed by a fluid particle as it move through the flow field. Other names for the material derivative are total, particle, Lagrangian, Eulerian and substantial derivative.

Material Derivative, $\frac{D}{Dt} = \frac{d}{dt} = \frac{\delta}{\delta t} + \left(\vec{V} \cdot \vec{\nabla}\right)$

The material derivative $\frac{D}{Dt}$ is composed of a local or unsteady part, $\frac{\delta}{\delta t}$ and a convective or advective part, $\left(\vec{V} \cdot \vec{\nabla}\right)$.

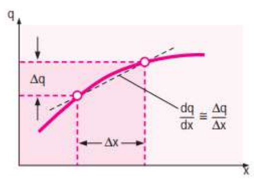
$$\underbrace{\frac{D}{Dt}}_{\text{Material derivative}} = \underbrace{\frac{\partial}{\partial t}}_{\text{Local}} + \underbrace{\left(\vec{V} \cdot \vec{\nabla}\right)}_{\text{Advective}}$$

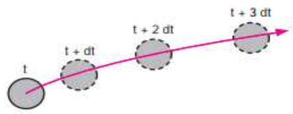
MATERIAL ACCELERATION AND PRESSURE DERIVATIVE (Note)

Material acceleration or velocity derivative,

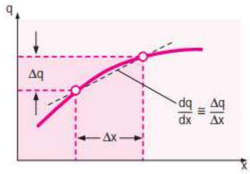
$$\vec{a}(x, y, z, t) = \frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + \left(\vec{V} \cdot \vec{\nabla}\right) \cdot \vec{V}$$

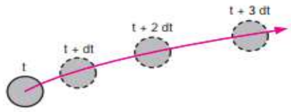
Material derivative of pressure, $\frac{DP}{Dt} = \frac{dP}{dt} = \frac{\delta P}{\delta t} + \left(\vec{V} \cdot \vec{\nabla}\right) \cdot P$





MATERIAL ACCELERATION AND PRESSURE DERIVATIVE (Note)





A first-order finite difference approximation for derivative dq/dx is simply the change in dependent variable (q) divided by the change in independent variable (x).

The material derivative D/Dt is defined by following a fluid particle as it moves throughout the flow field. In this illustration, the fluid particle is accelerating to the right as it moves up and to the right.

All Formula

Continuity Equation, $$\frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = 0$$

For rotational flow,

$$\omega_x = \frac{1}{2}\cdot\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right) \quad \omega_y = \frac{1}{2}\cdot\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right) \quad \omega_z = \frac{1}{2}\cdot\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)$$

$$\vec{\omega} = \frac{1}{2}\cdot(\vec{\nabla}\times\vec{V}) = \frac{1}{2}(Curl\,\vec{V});\ Vorticity, \vec{\xi} = 2\vec{\omega} = \vec{\nabla}\times\vec{V}$$

Angular deformation of fluid,

$$\dot{\gamma}_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y};\ \dot{\gamma}_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z};\ \dot{\gamma}_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

All Formula

For potential function,

$$u = -\frac{\delta\phi}{\delta x} \qquad v = -\frac{\delta\phi}{\delta y} \qquad w = -\frac{\delta\phi}{\delta z}$$

Stream function,

$$u = \frac{\delta\psi}{\delta y} \qquad v = -\frac{\delta\psi}{\delta x}$$

Laplace equation, $$\frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} + \frac{\delta^2\phi}{\delta z^2} = 0$$