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Lecture-02 : Analytic function

Topics Includes:

1. Analytic Function
2. Harmonic functions
3. Cauchy-Rieman equation
4. Some porblems

Analytic functions

Question Define complex function with an example.

Ans. The function defined by $w = f(z) = u(x,y) + i v(x,y)$, where $u(x,y)$ and $v(x,y)$ are real valued functions of real variables x & y, is known as complex function. Here the complex variable w is called the value of f at z or the image of z under f. Also here z is called the independent variable and w is called the dependent variable.

Note that for graphical representation of $f(z)$, it is customary to locate the points $z = (x,y)$ in the z-plane and $w = (u,v)$ in the w plane separately.

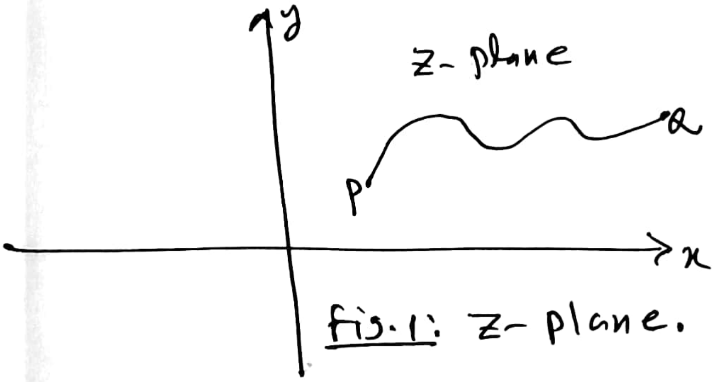

Fig.1: z-plane.

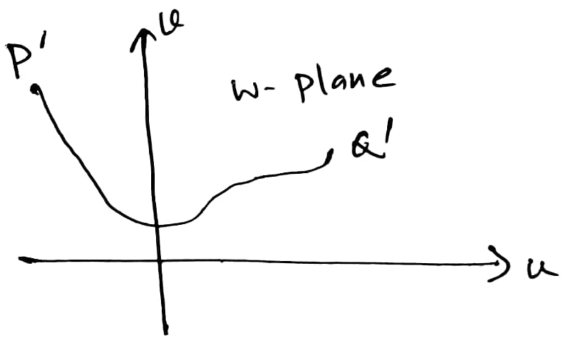

Fig.2: w-plane.

Example: The function $f(z)$ defined by $f(z) = z^2$ is an example of complex function. Now if $z = x+iy$, then $w = f(z) = u + iv = (x+iy)^2 = x^2 - y^2 + 2ixy$

So that $u = x^2 - y^2$ & $v = 2xy$. Then the image of a point $(1,2)$ in the z-plane is the point $(-3,4)$ in the w-plane.

Question: Define real and imaginary parts of a complex function with examples.

Ans: If $w = f(z) = u(x,y) + i v(x,y)$, where $z = x+iy$, then $u(x,y)$ and $v(x,y)$ are respectively called the real and imaginary parts of the complex function $w = f(z)$.

Examples: If $w = f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$ then real part of $f(z)$ is $u(x,y) = x^2 - y^2$ and imaginary part of $f(z)$ is $v(x,y) = 2xy$.

Exercise: Find the real and imaginary parts of the following complex functions:

(a) $f(z) = z\bar{z}$ **Ans.** $u(x,y) = x^2+y^2$, $v(x,y) = 0$

(b) $f(z) = z^3$, **Ans.** $u(x,y) = x^3 - 3xy^2$, $v(x,y) = 3x^2y - y^3$

(c) $f(z) = \bar{z} + 1$, **Ans.** $u(x,y) = x+1$, $v(x,y) = -y$

* **Question:** Define analytic function with examples.

Ans. **Defn:** A complex function $w = f(z)$ is said to be analytic in a region R if the derivative $f'(z)$ exists at all points z of R and is referred to as an analytic function in R or a function analytic in R. On the other hand, a function $w = f(z)$ is said to be analytic at a point z_0 if there exists a neighbourhood $|z - z_0| < \delta$ at all points of which $f'(z)$ exists.

Examples: If $f(z) = z^2$ or $f(z) = z^3$, then this two function are both analytic in the entire complex plane.

* Note that the terms Regular and Holomorphic are synonyman with Analytic.

* Also Note that there exists both real analytic functions and complex analytic functions. Functions of each types are infinitely differentiable, but complex analytic functions exhibit properties that do not generally hold for real analytic function. Also the analytic functions have convergent power series.

** Typical examples of analytic function are: (i) All polynomial function, (ii) The exponential function, (iii) The trigonometric function, logarithmic function, the power functions, (iv) Most special functions such as hypergeometric functions, Bessel's functions and gamma function.

Also the typical examples of function that are not analytic are: (i) The absolute value of function when defined on the set of real No. or complex No. is not everywhere analytic since it is not differentiable at 0. (ii) Also the piecewise defined functions, etc.

* Some real-life applications of Analytic functions:

There are various applications of analytic function in mathematics, physics and engineering fields such as in Electromagnetic Field Analysis, in signal processing, in Fluid Dynamics, in Financial Modeling, in Computer Science, in control system etc. Thus one can conclude that analytical function serves as powerful mathematical tools with applications ranging from theoretical mathematics to practical engineering and computational sciences. particularly, in Fluid Dynamics, the velocity field of a fluid, pressure distribution and other properties can be represented by analytic functions. The behavior of fluids in pipes, channels, or around obstacles can be modeled using these functions.

Question: Define entire function with an example.

Ans: A function $f(z)$ is said to be entire function if it is analytic in the whole complex plane.

Example: Since $f(z) = z^2$ is analytic everywhere in the finite z-plane and so it is an entire function.

Question: Define harmonic function with examples.

Ans. Any real-valued function $u(x,y)$ of two variables x & y having continuous partial derivatives of 1st and 2nd order in a region R and also satisfying Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, is called a harmonic function.

Examples: If $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$, then

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 6x \Rightarrow \frac{\partial^2 u}{\partial x^2} = +6x + 6$$

& $$\frac{\partial u}{\partial y} = -6xy - 6y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x - 6$$

So that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cancel{6x} + \cancel{6} - \cancel{6x} - \cancel{6} = 0$ which shows that u satisfies the Laplace's equation. Hence u is a harmonic function.

Question: Define harmonic conjugate function.

Ans. If $f(z) = u + iv$ is an analytic function, then v (imaginary part) is called harmonic conjugate of u (real part).

* **Theorem:** If a function $f(z) = u(x,y) + iv(x,y)$ is analytic in a domain D, then its component functions u and v are both harmonic in D.

✲ An Important Theorem on Analytic Functions:

The necessary and sufficient condition for a complex function $w = f(z) = u + iv$ to be analytic in a region R is that the four partial derivatives u_x, u_y, v_x, v_y should exist in R and also satisfy the Cauchy-Riemann equation: $u_x = v_y$ & $u_y = -v_x$.

Polar form of a Cauchy-Riemann equation:

If $w = f(z) = u + iv$ is an analytic function then the polar form of the Cauchy-Riemann equations are: $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ & $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$.

Question: Prove that the function $f(z) = z^2 + 5iz + 3 - i$ satisfies the Cauchy-Riemann equations.

Proof: Here $f(z) = u + iv = z^2 + 5iz + 3 - i$

$$\Rightarrow u + iv = (x+iy)^2 + 5i(x+iy) + 3 - i \quad [\because z = x+iy]$$

$$\Rightarrow u + iv = x^2 - y^2 + 2ixy + 5ix - 5y + 3 - i$$

$$\Rightarrow u + iv = (x^2 - y^2 - 5y + 3) + i(2xy + 5x - 1) \quad \text{--- (1)}$$

Now equating real & imaginary parts from both sides we get,

$$u = x^2 - y^2 - 5y + 3 \quad \& \quad v = 2xy + 5x - 1$$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x, \frac{\partial u}{\partial y} = -2y - 5 \quad \& \quad \frac{\partial v}{\partial x} = 2y + 5, \frac{\partial v}{\partial y} = 2x$$

Then $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow \boxed{u_x = v_y}$

Also $\frac{\partial u}{\partial y} = -2y - 5 = -\frac{\partial v}{\partial x}$

$\Rightarrow \boxed{u_y = -v_x}$

Hence satisfies the C-R equation.

Example-2 Is the function $2x-3y+i(3x+2y)$ analytic? Justify your answer.

Solution: Let $f(z) = u+iv = 2x-3y+i(3x+2y)$.

Then $u = 2x-3y$ and $v = 3x+2y$

$$\Rightarrow \frac{\partial u}{\partial x} = 2, \frac{\partial u}{\partial y} = -3 \text{ and } \frac{\partial v}{\partial x} = 3, \frac{\partial v}{\partial y} = 2$$

i.e. $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\Rightarrow u_x = v_y$ $\Rightarrow u_y = -v_x$

Then the Cauchy-Riemann equation are satisfied for the given function. Also the four partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ & $\frac{\partial v}{\partial y}$ are constants, so u & v are both continuous. Hence the function $2x-3y+i(3x+2y)$ is analytic.

Example-3 Show that the function ~~from~~ $u = x^2-y^2-2xy-2x+3y$ is harmonic and find the harmonic conjugate v. Also find $f(z) = u+iv$ so that $f(z)$ is analytic.

Solution: Given that $u = x^2-y^2-2xy-2x+3y$

$$\therefore \frac{\partial u}{\partial x} = 2x-2y-2 \text{ and } \frac{\partial u}{\partial y} = -2y-2x+3$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 2 \text{ and } \frac{\partial^2 u}{\partial y^2} = -2$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2-2 = 0$$

$\therefore$ u satisfies Laplace's equation. Hence u is a harmonic function.

2nd part: (Direct method) We have by Cauchy-Riemann equations

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 2x-2y-2 \quad \text{--- (1)} \quad \text{and} \quad -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} = -2y-2x+3. \quad \text{--- (2)}$$

Now $\frac{\partial v}{\partial y} = 2x-2y-2$

$\Rightarrow \partial v = (2x-2y-2)\partial y$

$\Rightarrow \int dv = \int (2x-2y-2)\,dy$ [Integrating w.r.to y keeping x as constant]

$\Rightarrow v = 2xy - 2\cdot\frac{y^2}{2} - 2y + F(x)$, where $F(x)$ is a function x.

$\Rightarrow v = 2xy - y^2 - 2y + F(x)$ ---- (3)

Now differentiating (3) w.r.to x partially we get

$$\frac{\partial v}{\partial x} = 2y + F'(x)$$

$\Rightarrow \cancel{2y} + 2x - 3 = \cancel{2y} + F'(x)$ [using (2)]

$\Rightarrow F'(x) = 2x-3$

$\Rightarrow \int F'(x)\,dx = \int (2x-3)\,dx$ [Integrating]

$\Rightarrow F(x) = 2\cdot\frac{x^2}{2} - 3x + c$

$\Rightarrow F(x) = x^2 - 3x + c$ ---- (4)

So that eqn (3) becomes

$v = 2xy - y^2 - 2y + x^2 - 3x + c$

$\Rightarrow v = x^2 - y^2 + 2xy - 3x - 2y + c$

3rd part: (Construction of analytic function)

$f(z) = u + iv = x^2 - y^2 - 2xy - 2x + 3y + i(x^2 - y^2 + 2xy - 3x - 2y + c)$

$\Rightarrow f(z) = x^2 + i^2y^2 + 2ixy + i(x^2 + i^2y^2 + 2ixy) - 2(x+iy) - 3i(x+iy) + ic$

$\Rightarrow f(z) = (x+iy)^2 + i(x+iy)^2 - 2(x+iy) - 3i(x+iy) + ic$

$\Rightarrow f(z) = (1+i)z^2 - 2z - 3iz + ic$ Ans.

$\boxed{\text{Exercise-1}}$ Determine which of the following functions u are harmonic. For each harmonic function find the conjugate harmonic function v and express $u+iv$ as an analytic function of z.

(i) $u = 2xy + 3xy^2 - 2y^3 \longrightarrow$ Not harmonic function

(ii) $u = xe^x \cos y - ye^x \sin y \longrightarrow$ Harmonic function

(iii) $u = e^{-x}(x \sin y - y \cos y) \rightarrow$ Harmonic function.

(iv) $u = 2x - x^3 + 3xy^2 \longrightarrow$ Harmonic function.

(v) $u = e^x \cos y \longrightarrow$ Harmonic function.

(vi) $u = (x-1)^3 - 3xy^2 + 3y^2 \rightarrow$ Harmonic function

(vii) $u = -x^3 + 3xy^2 + 2y + 1 \rightarrow$ Harmonic function