AHSANULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY (AUST)

ME-3105: FLUID MECHANICS-II (LEC-4: COMPRESSIBLE FLOW)

BY

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Compressible Fluid Flow

Introduction:

- ✓ At normal conditions, we assumed that the flow of liquid is incompressible. But the effect of compressibility of fluid must be considered if there is significant change in volume of the fluid i.e. $\Delta\rho/\rho > 0.05$.
- ✓ Compressible fluid flow problems are much more difficult than the incompressible because of thermodynamic consideration.

Classification of Compressible Flow:

- ✓ At low velocity (M < 0.30) flow is always considered incompressible .
- ✓ As fluid velocity approaches to the sonic velocity, the compressibility effect of the fluid acquire importance.
- ✓ Mach number (M) is most important parameter in compressible flow and compressible flow is categorized by the Mach number.
- ✓ $Mach\ no.\ (M) = \frac{Velocity\ of\ fluid\ or\ aircraft}{Velocity\ of\ sound} = \frac{V}{C}$

Types of Compressible Fluid Flow

✓ Based on the value of Mach number, there are six types flow defined as follows:

Types of Flow	Mach No.	Example
Subsonic incompressible flow	M <= 0.30	Fan, Blowers,Hydraulics
Subsonic compressible flow	0.3 < M < 1.0	Aircraft Turbomachines
Transsonic flow	0.9 < M < 1.0	Compressor blades
Sonic flow	M =1	Velocity of Sound
Supersonic flow	1 < M < 3	Mig 21 flight
Hypersonic flow	M > 3	Rockets, Missiles

Fundamental Equation of Compressible Fluid Flow

- Continuity equation
- Energy equation
- Momentum equation
- P, v, t relation for different process
- Equation of State
- $C_p - C_v = R$
- $C_p/C_v = k$

Compressible Fluid Flow Analysis

Taylor Series:

$$f(x) = \rho_X = \rho$$

$$f(x+h) = f(x) + f'(x)\cdot h + \frac{f''(x)}{2!}\cdot h^2 + \frac{f'''(x)}{3!}\cdot h^3 + ------$$

$$\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta^2\rho}{\delta x^2}\right)\cdot\frac{1}{2!}\cdot\left(\frac{dx}{2}\right)^2 + \left(\frac{\delta^3\rho}{\delta x^3}\right)\cdot\frac{1}{3!}\cdot\left(\frac{dx}{2}\right)^3 + --------$$

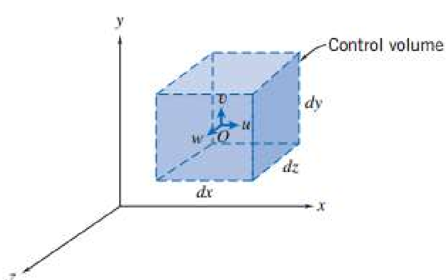
$$\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} \qquad \text{(Neglecting higher order terms)}$$

Here $h = \frac{dx}{2}$ and $f(x) = \rho_x = \rho$

Compressible Fluid Flow Analysis

Continuity Equation (Derivatin):

Consider an infinitesimal control volume with sides of length dx, dy and dz . The density of fluid at the center of the cube 'o' is 'ρ' and velocity is assumed to be $\vec{V} = u\cdot\hat{i} + v\cdot\hat{j} + w\cdot\vec{k}$

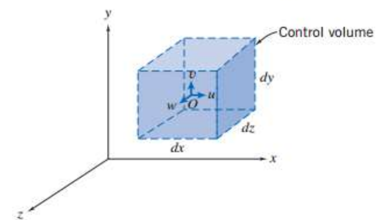


Compressible Fluid Flow Analysis

Continuity Equation (Continue):

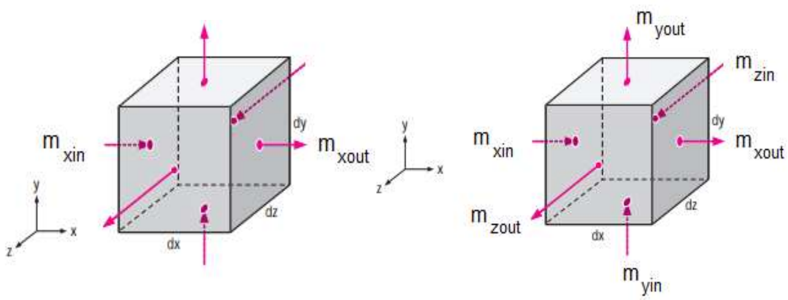
Now velocity and density of fluid at right surface (along x-axis) of the control volume $u_{x+\frac{dx}{2}} = u + \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}$ and $\rho_{x+\frac{dx}{2}} = \rho + \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}$

Again velocity and density of fluid at left surface (along x-axis) of the control volume $u_{x+\frac{dx}{2}} = u - \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}$ and $\rho_{x+\frac{dx}{2}} = \rho - \left(\frac{\delta \rho}{\delta x}\right)\cdot\frac{dx}{2}$



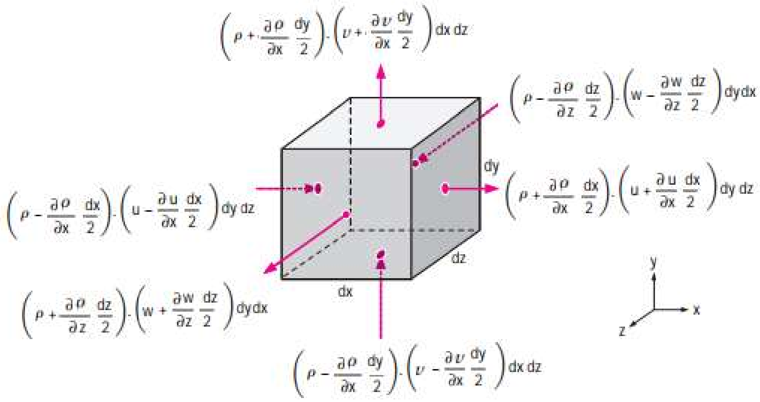
Compressible Fluid Flow Analysis

Continuity Equation (Continue):



Compressible Fluid Flow Analysis

Continuity Equation (Continue):



Compressible Fluid Flow Analysis

Continuity Equation (Continue):

$$m_{x\_in} = \left[u - \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot\left[\rho - \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho - u\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} - \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta u}{\delta x}\right)\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\left(\frac{dx}{2}\right)^2\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho - u\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} - \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$m_{x\_out} = \left[u + \left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot\left[\rho + \left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho + u\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} + \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2} + \left(\frac{\delta u}{\delta x}\right)\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\left(\frac{dx}{2}\right)^2\right]\cdot(dy\cdot dz)$$

$$= \left[u\cdot\rho + u\cdot\left(\frac{\delta\rho}{\delta x}\right)\cdot\frac{dx}{2} + \rho\cdot\left(\frac{\delta u}{\delta x}\right)\cdot\frac{dx}{2}\right]\cdot(dy\cdot dz)$$

Compressible Fluid Flow Analysis

Continuity Equation (Continue):

$$m_{x\_net} = m_{x\_in} - m_{x\_out} = -\left[u \cdot \left(\frac{\delta\rho}{\delta x}\right) + \rho \cdot \left(\frac{\delta u}{\delta x}\right)\right](dx \cdot dy \cdot dz)$$

$$= -\frac{\delta}{\delta x}(\rho\, u) \cdot (dx \cdot dy \cdot dz)$$

$$m_{y\_net} = -\frac{\delta}{\delta y}(\rho\, v) \cdot (dx \cdot dy \cdot dz)$$

$$m_{z\_net} = -\frac{\delta}{\delta z}(\rho\, w) \cdot (dx \cdot dy \cdot dz)$$

Rate of change of mass within control volume dx.dy.dz ,

$$m_{change} = \frac{\delta}{\delta t}(\rho \cdot dx \cdot dy \cdot dz) = \frac{\delta\rho}{\delta t}(dx \cdot dy \cdot dz)$$

Compressible Fluid Flow Analysis

Continuity Equation (Continue):

As per principle of conservation of mass,

$$m_{x\_net} + m_{y\_net} + m_{z\_net} = m_{change}$$

$$-\frac{\delta}{\delta x}(\rho\, u) \cdot (dx \cdot dy \cdot dz) - \frac{\delta}{\delta y}(\rho\, v) \cdot (dx \cdot dy \cdot dz) - \frac{\delta}{\delta z}(\rho\, w) \cdot (dx \cdot dy \cdot dz) = \frac{\delta\rho}{\delta t}(dx \cdot dy \cdot dz)$$

$$\frac{\delta}{\delta x}(\rho\, u) + \frac{\delta}{\delta y}(\rho\, v) + \frac{\delta}{\delta z}(\rho\, w) + \frac{\delta\rho}{\delta t} = 0$$

$$\vec{V} = u\,\dot{i} + v\,\dot{j} + w\,\vec{k} \qquad \nabla = \dot{i} \cdot \frac{\delta}{\delta x} + \dot{j} \cdot \frac{\delta}{\delta y} + \vec{k} \cdot \frac{\delta}{\delta z}$$

Again,

$$\vec{V} \cdot \nabla = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta y} + \frac{\delta w}{\delta z} = \mathrm{Div}\,\vec{V} \qquad \rho\,\vec{V} = \dot{i}(u\,\rho) + \dot{j}(v\,\rho) + \vec{k}(w\,\rho)$$

$$\nabla \cdot \rho\,\vec{V} = \frac{\delta}{\delta x}(\rho u) + \frac{\delta}{\delta y}(\rho v) + \frac{\delta}{\delta z}(\rho w)$$

Continuity equation become, $\nabla \cdot \rho\,\vec{V} + \frac{\delta\rho}{\delta t} = 0$

Compressible Fluid Flow Analysis

Continuity Equation (Continue):

Continuity equation in Cartesian coordinates:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Alternative form of the continuity equation:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{V}) = \underbrace{\frac{\partial \rho}{\partial t} + \vec{V} \cdot \vec{\nabla}\rho}_{\text{Material derivative of } \rho} + \rho \vec{\nabla} \cdot \vec{V} = 0$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \vec{\nabla} \cdot \vec{V} = 0$$

Compressible Fluid Flow Analysis

Derivation of Energy Equation for Compressible Fluid Considering Both Isothermal and Adiabatic Process:

Solution:

For compressible fluid, density depends on pressure and temperature. The internal energy is also affected by the change of heat. Energy equation can be applied to any gas or system. Considering a fluid flow in a streamline from point 1 to point 2.

Assumption:

- ❑ One dimensional flow
- ❑ Steady flow
- ❑ Frictionless flow
- ❑ Equation of state is applicable.

Assume P_1, T_1 and ρ_1 are pressure, temperature and density of fluid at point 1 and P_2, T_2 and ρ_2 are pressure, temperature and density of fluid at point 2.

(SEE HAND ANALYSIS FOR DERIVATION FOR BOTH PROCESS)

Compressible Fluid Flow Analysis

Derive expression for mass flow rate of flow of compressible fluid through a converging nozzle from a large reservoir to a receiver. What is the condition for maximum flow rate for air ? Deduce also an expression for maximum flow rate.

Derivation:

Assume P_1, V_1 and ρ_1 are pressure, velocity and density of fluid at section 1 and P_2, V_2 and ρ_2 are pressure, velocity and density of fluid at section 2.

Assumptions:

- Adiabatic process
- Frictionless flow
- Steady flow
- One dimensional flow
- Equation of state is applicable.

(SEE HAND NOTE ANALYSIS FOR DERIVATION)

Compressible Fluid Flow Analysis

Derive expression for the area velocity relationship for one dimensional compressible flow. Show the effects of variation of area on subsonic, sonic, and supersonic flows.

Derivation:

Assume P, V and ρ are pressure, velocity and density of the fluid. M is the Mach number and C is the velocity of sound.

Assumptions:

- One dimensional flow
- Steady flow
- Frictionless flow
- Adiabatic conditions

(SEE HAND NOTE ANALYSIS FOR DERIVATION)

Body Moving at a Sonic, Subsonic, and Supersonic Speed

Subsonic Velocity:

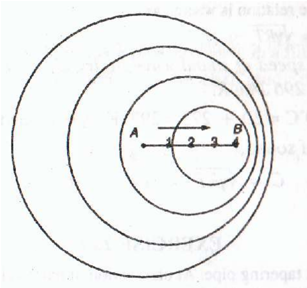


Fig. 23·3. Subsonic velocity.

The velocity ‘v’ of projectile is always less than velocity of wave ‘c’. The nose of the projectile is always behind the wave front of the pressure wave. The projectile is always penetrating in an area of disturbed fluid. The disturbed fluid has some effect on the fluid resistance to the motion of the projectile.

Body Moving at a Sonic, Subsonic, and Supersonic Speed

Sonic Velocity:

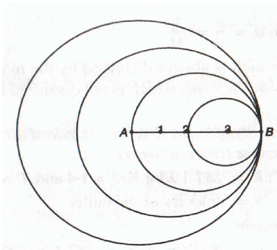


Fig. 23·4. Sonic velocity.

The velocity ‘v’ of the projectile is equal to the velocity of wave ‘c’. The nose of the projectile and the wave front always move through a common point. The nose of the projectile pushes wave with intense pressure, which moves along with it and is known as shock wave. It causes a great resistance to the motion of the projectile.

Body Moving at a Sonic, Subsonic, and Supersonic Speed

Supersonic Velocity:

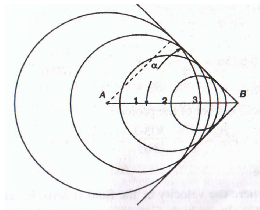


Fig. 23·5. Supersonic velocity.

The velocity 'v' of the projectile greater than the velocity of wave 'c'. The pressure wave always behind the nose of projectile. Projectile moves ahead of pressure wave. The entire system of spherical pressure waves forms a conical figure with vertex at point B. The half of angle α of the cone vertex is called Mach angle. From geometry of the figure,

$$\text{SIN}\alpha = c/v = 1/M$$

Compressible Fluid Flow Analysis

Problem-1:

A certain gas is flowing through a duct in which change in cross-section occurs and at a particular cross-section the velocity of gas is 415 m/s , absolute pressure 82 KN/m^2 and temperature 38^0 C respectively. Assuming isentropic conditions, Calculate the velocity and Mach number where the pressure is 132 KN/m^2 absolute. (R = 0.297 KJ/Kg ^{0}K and K = 1.4).

SEE HAND ANALYSIS

Compressible Fluid Flow Analysis

Problem-2:

Air is flowing through a duct with a velocity of 350 m/s at a certain section of the duct, the temperature and pressure are 40^0 C and 80 KPa respectively. Considering isentropic flow. Find the velocity and temperature at another section where the absolute pressure is 120 KPa. Also find the Mach number at the two sections. Assume R = 287 J/kg ^{0}K and k =1.4 .

SEE HAND ANALYSIS

Compressible Fluid Flow Analysis

Problem-3:

Air at 690 KPa absolute and 38^0C in a large tank enters a converging-diverging nozzle. The exit flow from the nozzle discharges into atmosphere. The nozzle exit diameter is 50 mm. Calculate the max. mass flow rate and the throat diameter.

SEE HAND ANALYSIS