ME 3109: Measurement & Instrumentation

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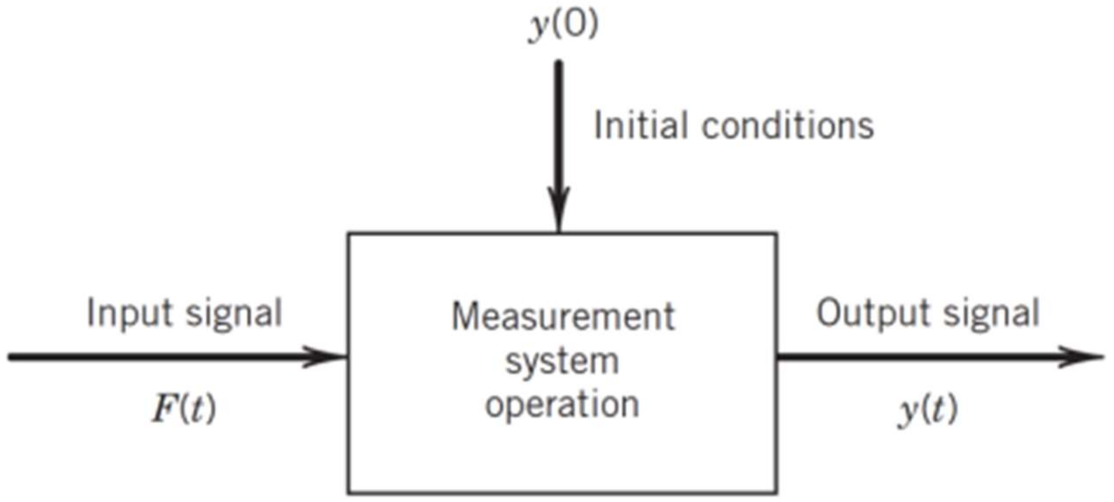
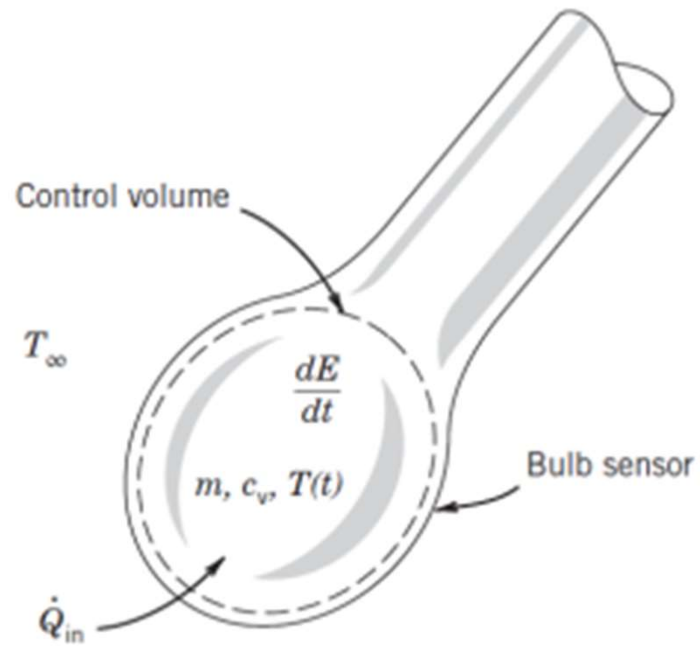
ME 3109: Measurement & Instrumentation

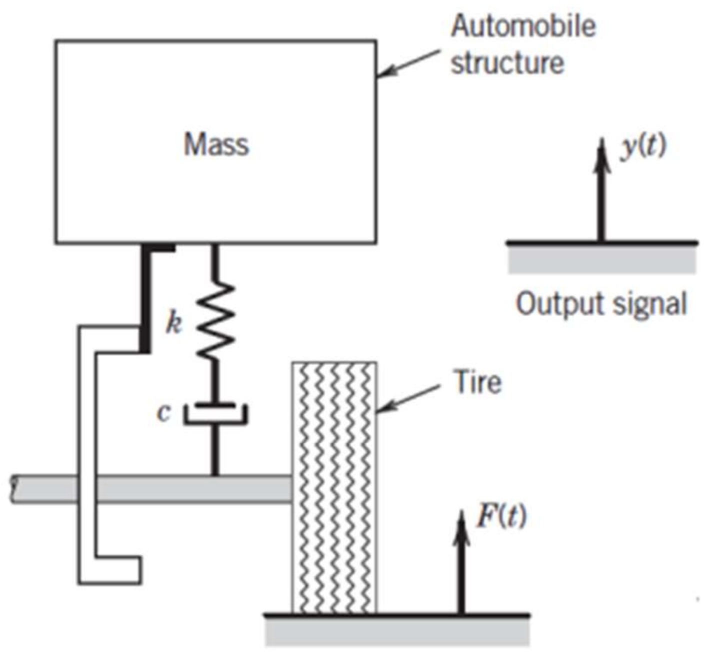
Topic_04: Measurement System Behavior[1]	
1	Dynamic Response
2	Zero Order Measurement Systems
3	First Order Measurement Systems
4	Second Order Measurement Systems
5	MATLAB and Simulink for System Dynamics

[1] Figliola's "***Theory & Design for Mechanical Measurements***", 5th Edition

1. Dynamic Response: Measurement System Operation

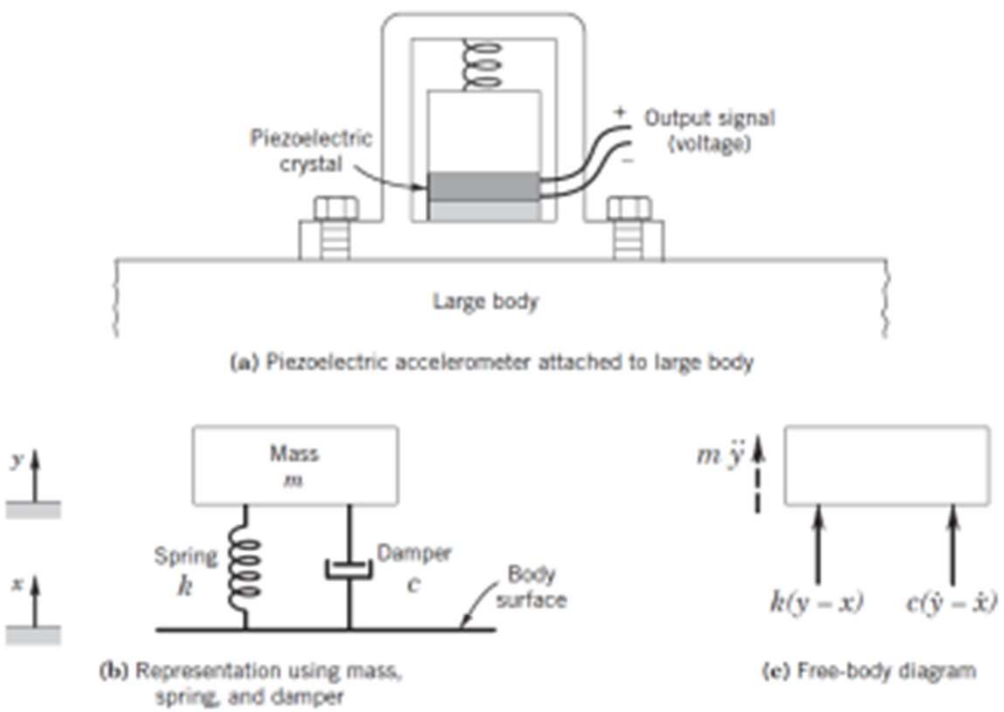




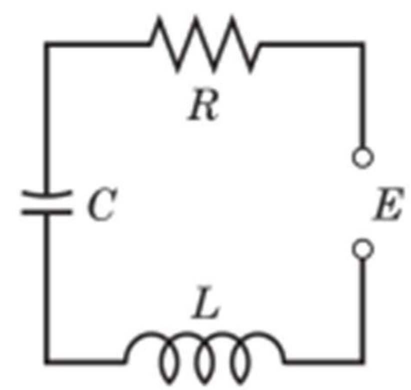


1. Dynamic Response: Lumped Parameter Model

In lumped parameter modeling, the spatially distributed physical attributes of a system are modeled as discrete elements. Governing equations of the models reduce from partial (time and space dependent) to ordinary differential equations (time dependent)

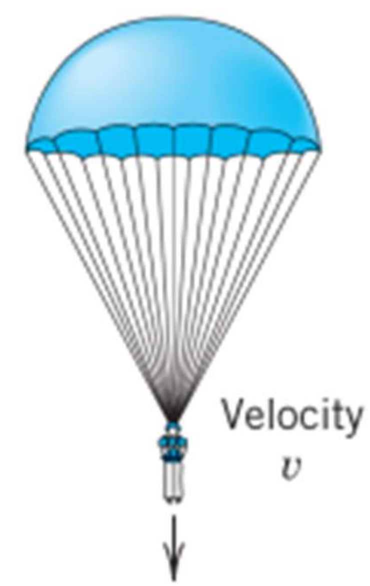


Lumped parameter model of accelerometer



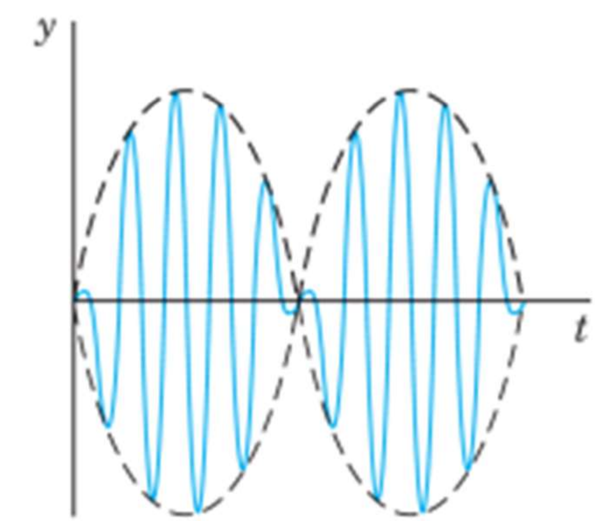
Current I in an RLC circuit

$$LI'' + RI' + \frac{1}{C}I = E'$$



Parachutist

$$mv' = mg - bv^2$$



Beats of a vibrating system

$$y'' + \omega_0^2 y = \cos \omega t, \quad \omega_0 = \omega$$

1. Dynamic Response: Mechanical System Elements

ⓐ Spring →

$$f(t) \longrightarrow \boxed{\text{Spring}} \longrightarrow x(t) \quad \Rightarrow \quad f(t) = K\,x(t)$$

$f(t)$ (tension/compression)

Spring const., K

$x(t)$ (extension/compression)

→ stores energy when displaced from equillibrium. The energy is released when it comes back to original position.

$$\boxed{E = \frac{1}{2}\frac{f^2}{K}}$$

1. Dynamic Response: Mechanical System Elements

(b) Dashpot/ Damper →

$f(t) \rightarrow$ Dash-pot $\rightarrow x(t) \Rightarrow f(t) = -cv$

$= -c\frac{dx}{dt}$

force opposing @ velocity v

Damping Coeff, c

Displacement

→ dissipates energy as heat.

Dissipated Power, $\boxed{P = cv^2}$

1. Dynamic Response: Mechanical System Elements

(c) **Mass** ⇒

$$f(t) \rightarrow \boxed{\text{mass}} \rightarrow x(t)$$

$f(t)$: force for acceleration, a

mass, m

$x(t)$: Displacement

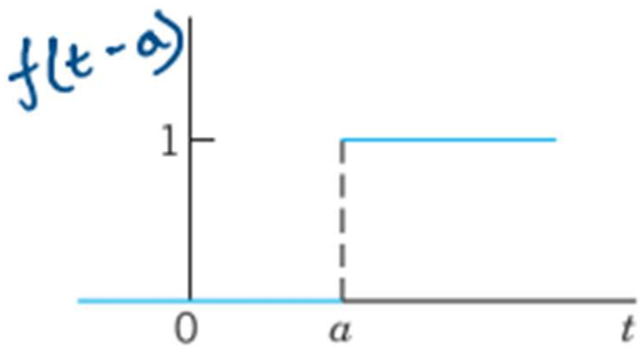
$$\Rightarrow f(t) = ma = m\frac{dv}{dt} = m\frac{d^2x}{dt^2}$$

⇒ kinetic energy is stored when a mass is moving with velocity, v

$$\boxed{E = \frac{1}{2}mv^2}$$

1. Dynamic Response: Input Functions

Unit step function

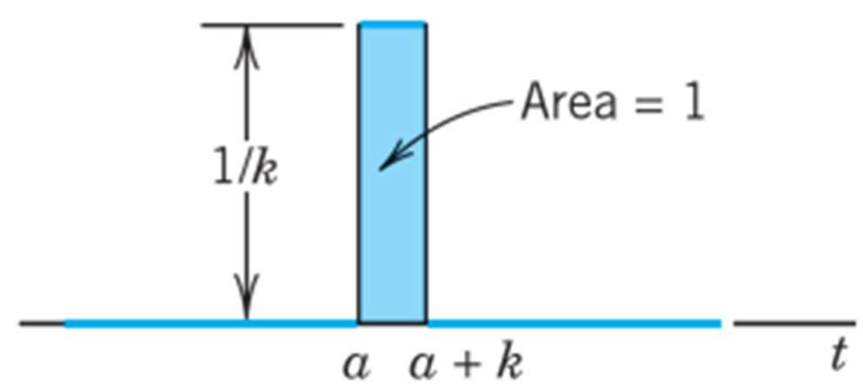


$$f(t-a) = \begin{cases} 0 & ; t \le a \\ 1 & ; t > a \end{cases}$$

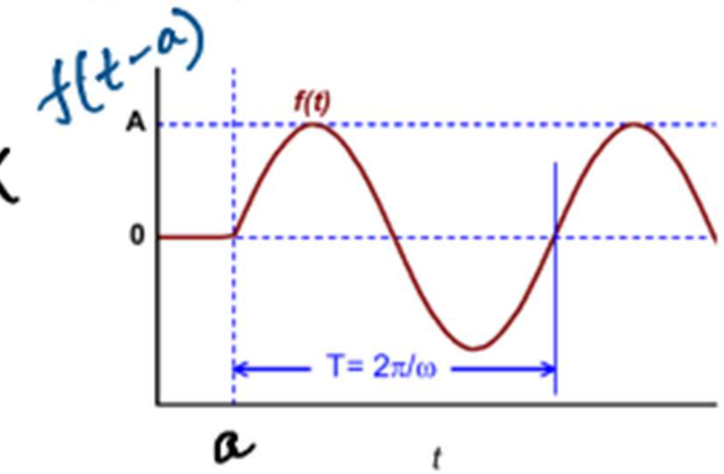
Short Impulses

$$f_K(t-a) = \begin{cases} 1/K & ; a \le t \le a+K \\ 0 & ; \text{otherwise} \end{cases}$$

$K \to 0$

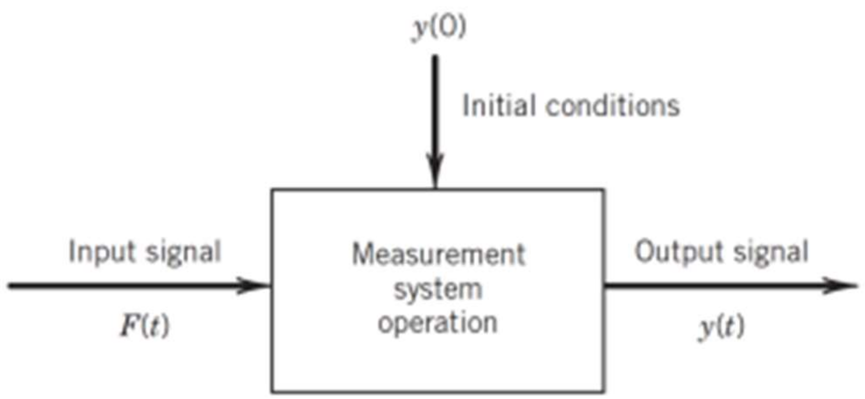


Harmonic Function



$$f(t-a) = \begin{cases} 0 & ; t \le a \\ A\sin\omega t & ; t > a \end{cases}$$

1. Dynamic Response: Modeling a General System



$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = F(t)$$

n-th order ordinary differential eqn.

Forcing function → $F(t) = b_m \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} + \cdots + b_1 \frac{dx}{dt} + b_0 x \quad m \le n$

a's and b's are physical system parameter and their values depend on the measurement system itself.

Order of a system is designated by the order of the Differential Equation

1. Dynamic Response: Special Cases of the General System Model

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = F(t)$$

$n = 0 \Rightarrow a_0 y = F(t) \rightarrow$ zero-order system (potentiometer, Ideal spring, systems during static calibration)

$n = 1 \Rightarrow a_1 \dot{y} + a_0 y = F(t) \rightarrow$ first-order system (Radioactive decay, capacitor, thermometer, leaking water from a tank)

$n = 2 \Rightarrow a_2 \ddot{y} + a_1 \dot{y} + a_0 y = F(t) \Rightarrow$ second-order system (Accelerometers, RLC circuit, vibrating mass-spring-systems)

2. Zero Order System

$a_0 y = F(t)$

$\Rightarrow y = K F(t)$

$k = \frac{1}{a_0}$ = static sensitivity or gain

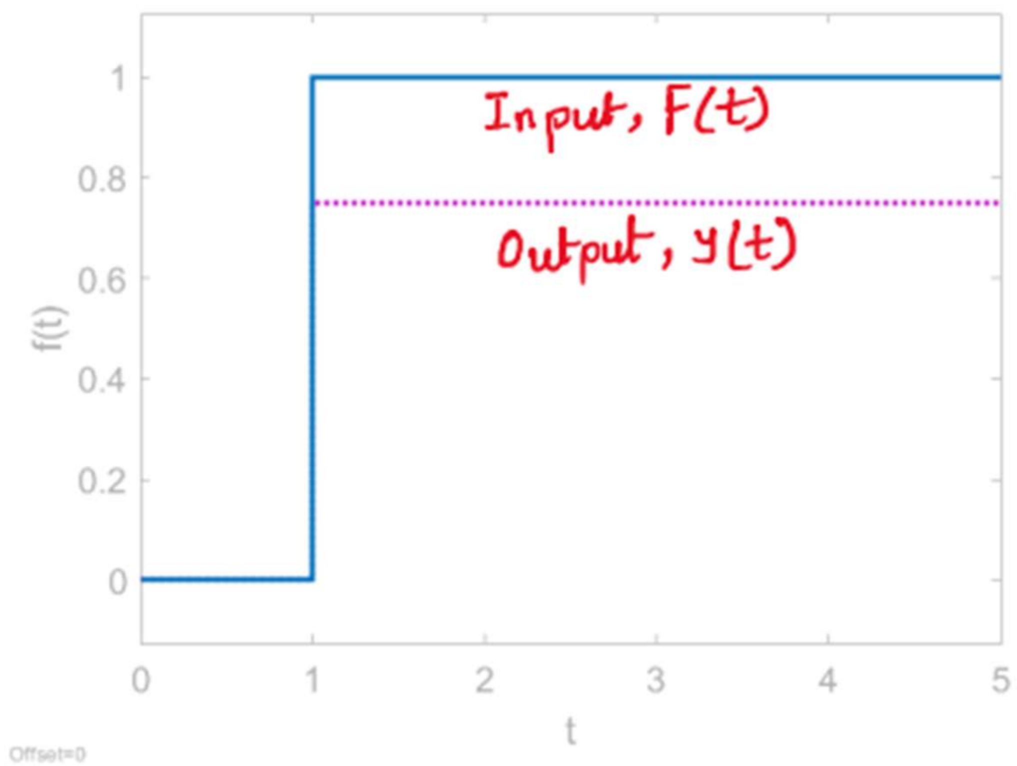
→ k is the scaling factor between input and output

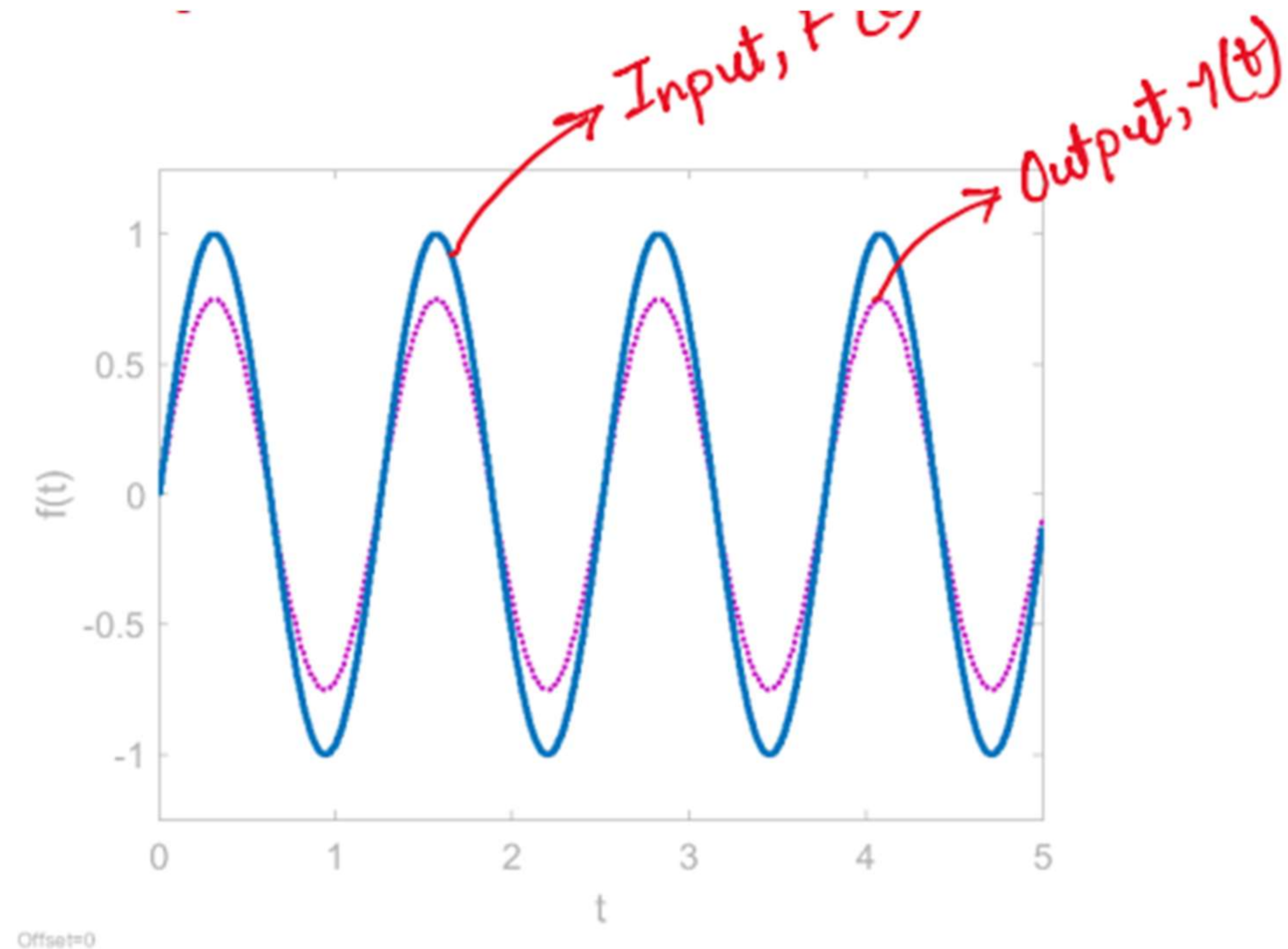
K has the unit of output devided by input

Notes:

- ✓ System output responds to input instantaneously
- ✓ Non-time dependent system response
- ✓ No equillibrium seeking force
- ✓ No distortion or time lag
- ✓ Systems during static calibration
- ✓ For dynamic input signal → zero-order model is only valid at static equilibrium

2. Zero Order System





Zero-order instrument’s response for step and harmonic inputs (for k = 0.75)

3. First Order System

$$a_1\dot{y} + a_0 y = F(t)$$

dissipation (electric/thermal resistance)

storage (electric or thermal capacitance)

$$\tau\dot{y} + y = KF(t)$$

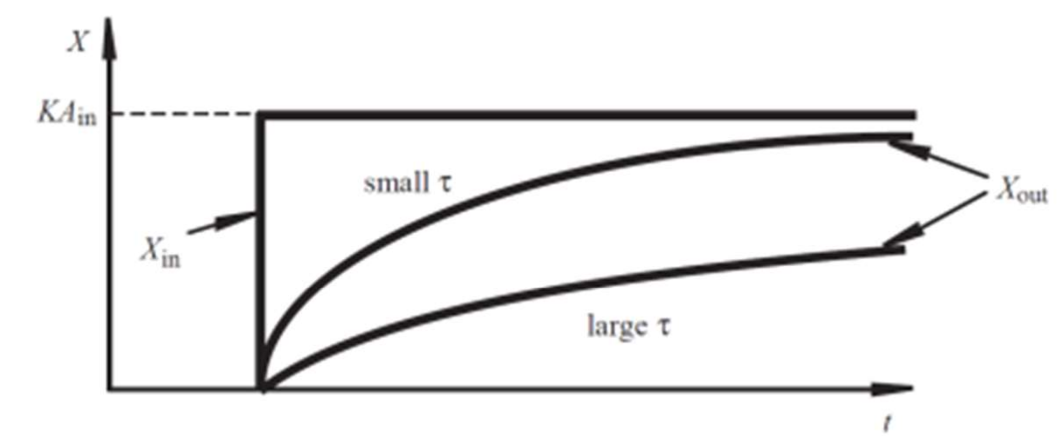
$\rightarrow \frac{a_1}{a_0}$ = Time constant; [s]

$\tau \rightarrow 0 \rightarrow$ zero-order system

Notes:

✓ system with storage and dissipative capacity but negligible inertia

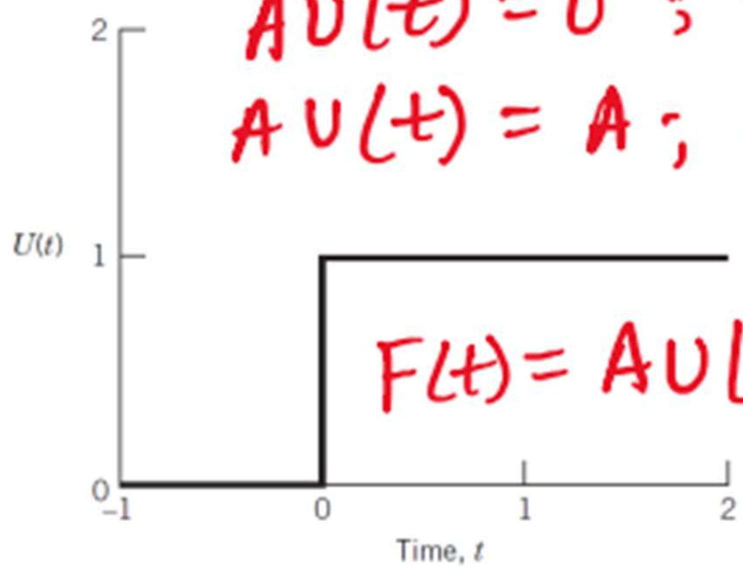
✓ Time constant provides a measure of the speed of the measurement system response



3. First Order System: Step Input Function

$AU(t) = 0 \; ; \; t \le 0^-$

$AU(t) = A \; ; \; t \ge 0^+$



$F(t) = AU(t)$

Initial Cond. $y(0) = y_0$

Solve for, $t \ge 0^+$

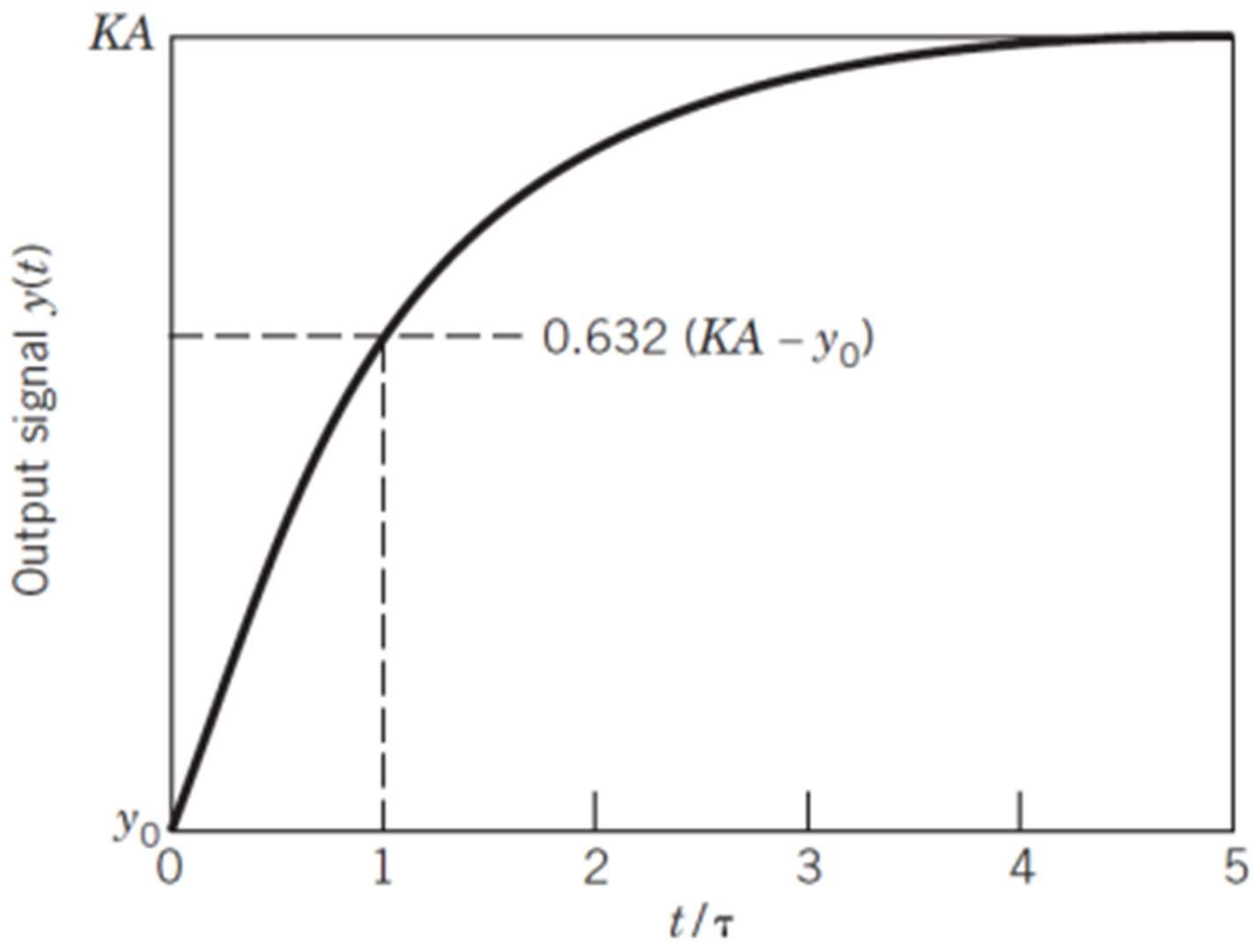
So, $\tau \frac{dy}{dt} + y = KF(t)$

$\Rightarrow \tau \frac{dy}{dt} + y = KAU(t) = KF(t)$

Solution

$$\underbrace{y(t)}_{\text{Time Response}} = \underbrace{KA}_{\text{Steady Response}} + \underbrace{(y_0 - KA)\, e^{-t/\tau}}_{\text{Transient Response}}$$

3. First Order System: Time Response



Time constant, τ: time required to complete 63.2% of the process.

Rise time, t_r: time required to achieve response from 10% to 90% of the final value

Settling time, t_s: time required for the response to reach and stay within 2% of the final value

3. First Order System: Harmonic Input Function

$F(t) = A\sin\omega t$

$\omega = 2\pi f$

ω [rad/s]

f [Hz]

and: $\tau \dfrac{dy}{dt} + y = KA\sin\omega t$

time response:

$$y(t) = \underbrace{ce^{-t/\tau}}_{\text{depends on initial condition}} + \frac{KA}{\sqrt{1+(\omega\tau)^2}} \sin\left(\omega t - \tan^{-1}(\omega\tau)\right)$$

$ce^{-t/\tau} \rightarrow$ transient response

$\dfrac{KA}{\sqrt{1+(\omega\tau)^2}} \sin\left(\omega t - \tan^{-1}(\omega\tau)\right)$

$\hookrightarrow$ steady periodic response

$$y(t) = ce^{-t/\tau} + B(\omega)\sin(\omega t + \Phi)$$

$B(\omega)$: Amplitude of the steady response

$\Phi(\omega)$; phase shift

3. First Order System: Harmonic Input Function

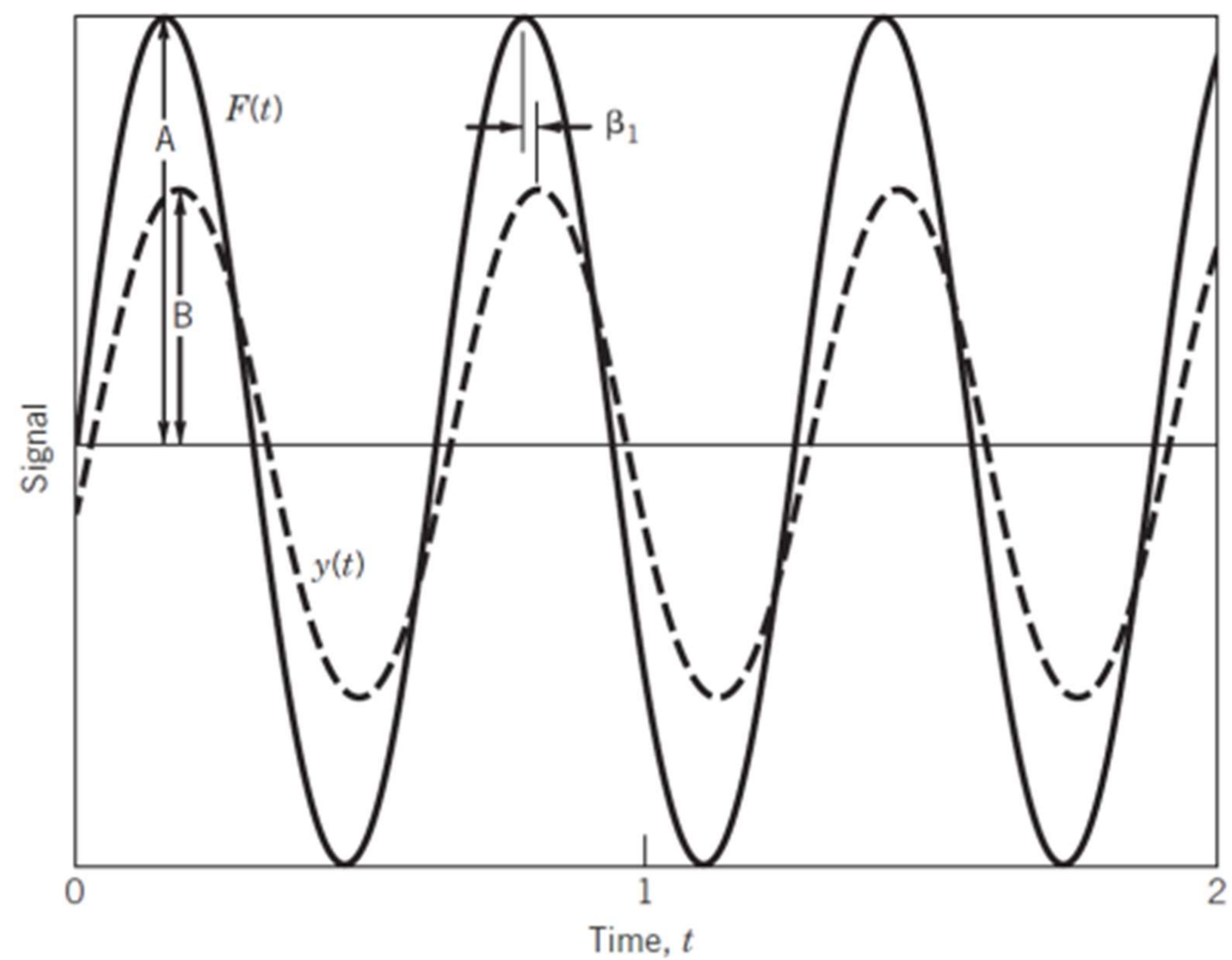
time delay, β_1

$$= \frac{\Phi}{\omega} \; [s]$$

Magnitude Ratio, $M(\omega)$

$$= \frac{B}{KA} = \frac{1}{\sqrt{1+(\omega\tau)^2}}$$

* As $\tau << T = \frac{2\pi}{\omega}$



For $\omega\tau >> 1$, response is attenuated, and time/phase is lagged from input, and for $\omega\tau << 1$, the transient response becomes very small, and response follows the input with small attenuation and time/phase lag.

3. First Order System: Frequency Response

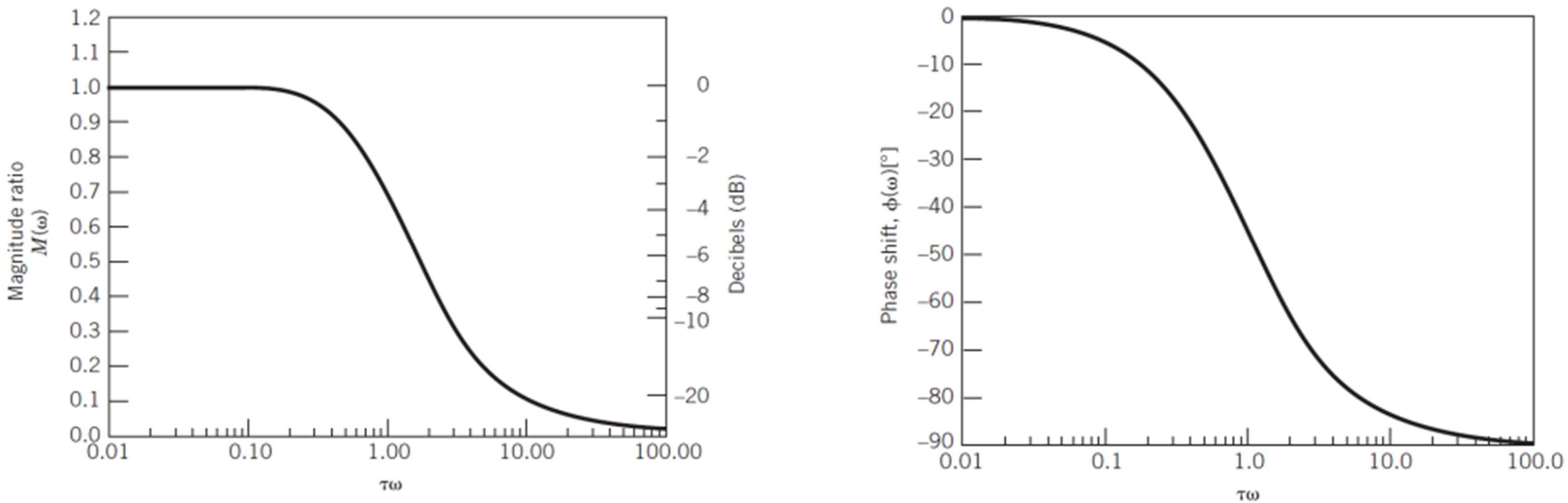
As, $t \to \infty$: steady-state solution :

$$y(t) = \frac{KA}{\sqrt{1+(\omega\tau)^2}} \sin(\omega t + \Phi)$$

The steady response of any system to which a periodic input of frequency, ω, is applied is known as the frequency response of the system.

Both τ and ω affect the frequency respone function, $M(\omega)$ and $\Phi(\omega)$ represent the frequency response of a first-order measurement system to periodic inputs

3. First Order System: Frequency Response



If we want to measure signals with high-frequency content, then we would need a system having a small τ. On the other hand, systems of large τ may be adequate to measure signals of low-frequency content

3. First Order System: Frequency Response

We normally want measurement systems to have a magnitude ratio at or near unity over the anticipated frequency band of the input signal to minimize dynamic error.

$\rightarrow \delta(\omega) = M(\omega) - 1$

Frequency Bandwidth: Frequency Band over which, $M(\omega) \geq 0.707$

or, $-3\,dB \leq M(\omega) \leq 0\,dB$

$$dB = 20 \log M(\omega)$$

The frequency response of a measurement system is found physically by a dynamic calibration

3. First Order System: Transfer Function

$$G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{F(t)\}}$$

$$F(t) \longrightarrow \boxed{\text{Linear System}} \longrightarrow y(t)$$

transfer function, $$G(s) = \frac{Y(s)}{F(s)}$$

$$F(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$

Transfer function of a linear system, G(s), is defined as the ratio of the Laplace transform (LT) of the output variable, to the LT of the input variable with all the initial conditions are assumed to be zero

Laplace operator, $s = \sigma + j\omega$. For steady-state sinusoidal input $\sigma = 0$, and system response can be evaluated by setting,

$s = j\omega$

3. First Order System: Transfer Function

Magnitude Ratio, $M(\omega) = G_a(\omega) = |G(j\omega)|$

Phase shift, $\Phi(\omega) = \angle G(j\omega)$

$$F(s) \Rightarrow \boxed{G(s)} \Rightarrow Y(s)$$

<u>then</u> $\boxed{y(t) = F(t) \times G_a \angle \Phi}$

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

3. First Order System: Transfer Function

Some Functions f(t) and their Laplace Transforms F(s)

	$f(t)$	$\mathcal{L}(f)$
1	1	$1/s$
2	t	$1/s^2$
3	t^2	$2!/s^3$
4	t^n $(n = 0, 1, \cdots)$	$\frac{n!}{s^{n+1}}$
5	t^a (a positive)	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
7	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
8	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
9	$\cosh at$	$\frac{s}{s^2-a^2}$
10	$\sinh at$	$\frac{a}{s^2-a^2}$
11	$e^{at}\cos \omega t$	$\frac{s-a}{(s-a)^2+\omega^2}$
12	$e^{at}\sin \omega t$	$\frac{\omega}{(s-a)^2+\omega^2}$

$f'(t) \rightarrow sF(s) - f(0)$

$f''(t) \rightarrow s^2F(s) - sf(0) - f'(0)$

$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t, \; \zeta < 1 \rightarrow \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

3. First Order System: Transfer Function

Transfer Function (TF) of a First Order System

$$\tau \frac{dy}{dt} + y = K\,F(t)$$

$$\Rightarrow \tau s Y(s) + Y(s) = K F(s)$$

$$\Rightarrow \frac{Y(s)}{KF(s)} = G(s) = \frac{1}{\tau s + 1}$$

$$KF(s) \Rightarrow \boxed{\frac{1}{\tau s + 1}} \Rightarrow Y(s)$$

- $s = j\omega$

$$M(\omega) = |G(j\omega)| = \left|\frac{1}{\tau j\omega + 1}\right| = \frac{1}{\sqrt{1 + (\omega\tau)^2}}$$

$$\Phi = \angle G(j\omega) = \tan^{-1}(-\omega\tau)$$

4. Second Order System

Inertia/Inductance

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = F(t)$$

$$\Rightarrow \frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = KF(t)$$

$$\omega_n = \sqrt{\frac{a_0}{a_2}} = \text{natural frequency of the system}$$

$$\zeta = \frac{a_1}{2\sqrt{a_0 a_2}} = \text{damping ratio of the system}$$

$$K = \frac{1}{a_0} = \text{static sensitivity}$$

- ✓ System posessing inertia
- ✓ Homogenous solution, $y_h(t)$ provides the transient response
- ✓ Form of the $y_h(t)$ will depend on the roots of the characteristics eqn.

4. Second Order System: Homogeneous Solution

$$\Rightarrow \frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = 0 \xrightarrow[\text{Eqn.}]{\text{Characteristics}} \frac{1}{\omega_n^2}\lambda^2 + \frac{2\zeta}{\omega_n}\lambda + 1 = 0$$

roots, $\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

$0 \leq \zeta < 1$ (underdamped system solution)

$$y_h(t) = Ce^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \Theta)$$ → Oscillatory

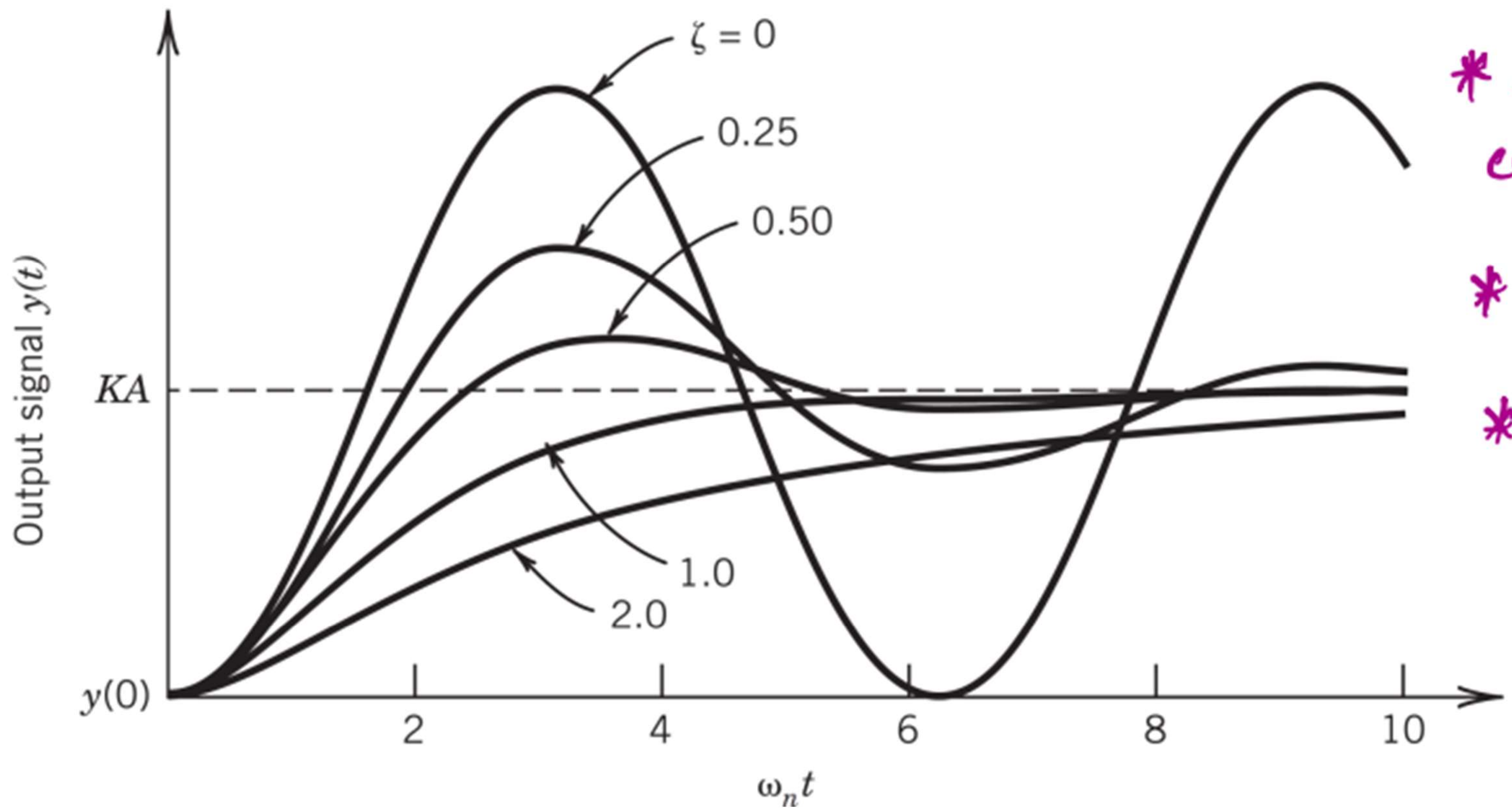
$\zeta = 1$ (critically damped system solution)

$$y_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$$

$\zeta > 1$ (overdamped system solution)

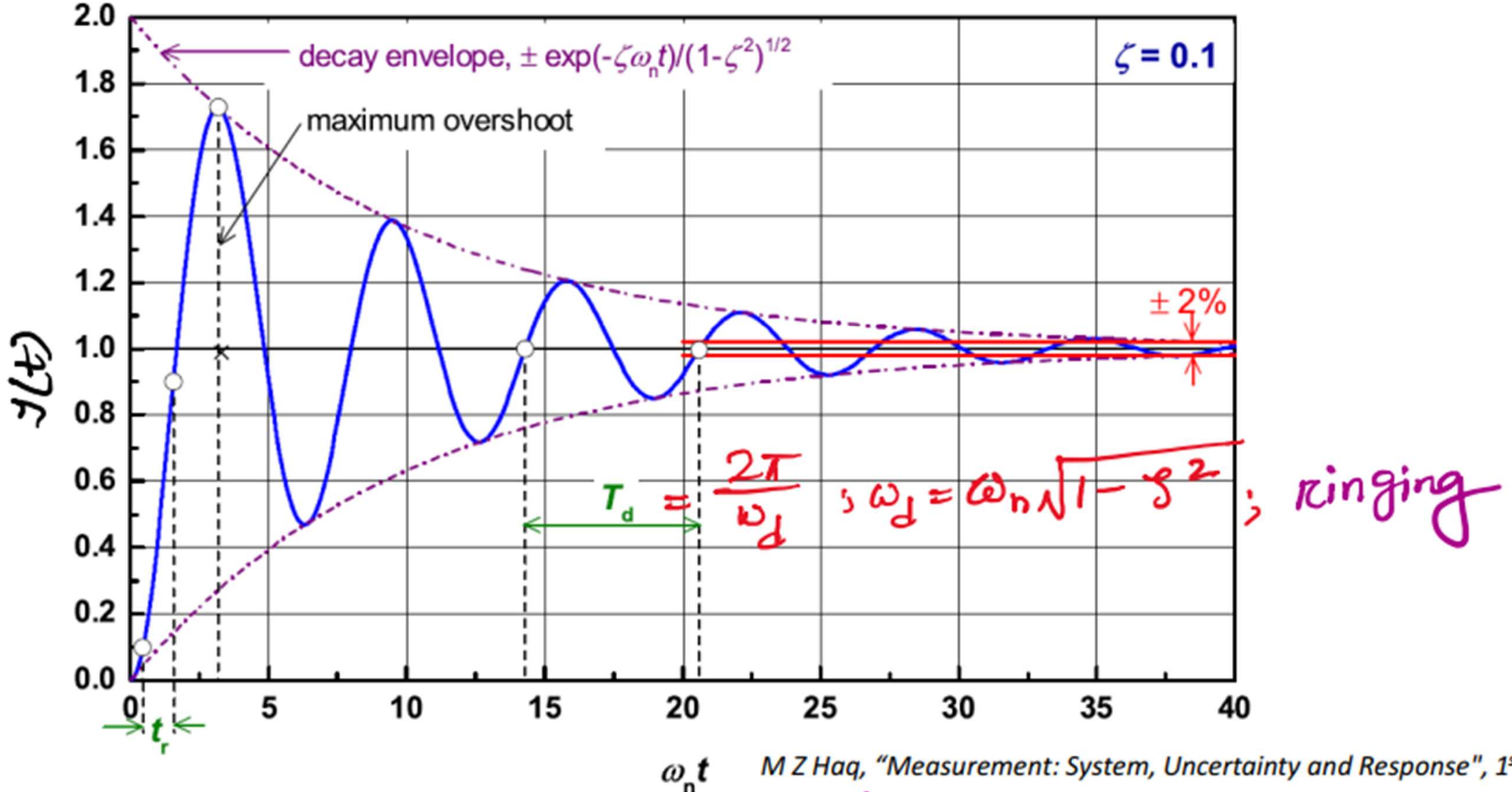
$$y_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t}$$ → Non- Oscillatory

4. Second Order System: Step Input Function



* Ringing frequency
* 2^{nd}-Order time-constant, $\tau = \frac{1}{\zeta \omega_n}$
* Rise time, t_r
* settling time, t_s

4. Second Order System: Step Input Function



M Z Haq, "Measurement: System, Uncertainty and Response", 1st Edition

Under-damped System

4. Second Order System: Harmonic Input Function

response of a 2^{nd}-order system,

$$y(t) = y_h + \frac{KA\sin[\omega t + \Phi(\omega)]}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2\right\}^{1/2}}$$

slide 6!

phase shift,

$$\Phi(\omega) = \tan^{-1}\left(-\frac{2\zeta\omega/\omega_n}{1-(\omega/\omega_n)^2}\right)$$

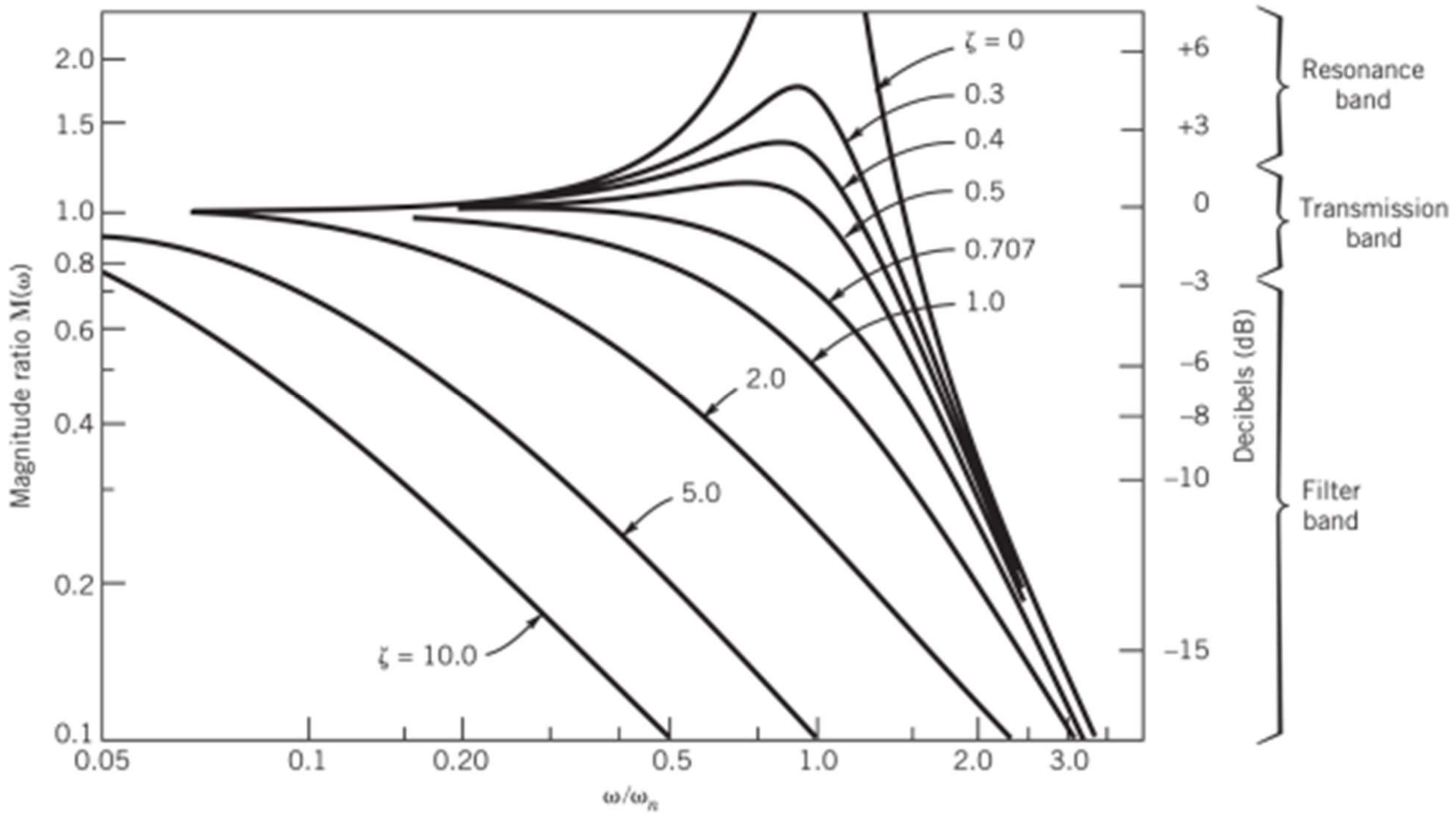
steady state response,

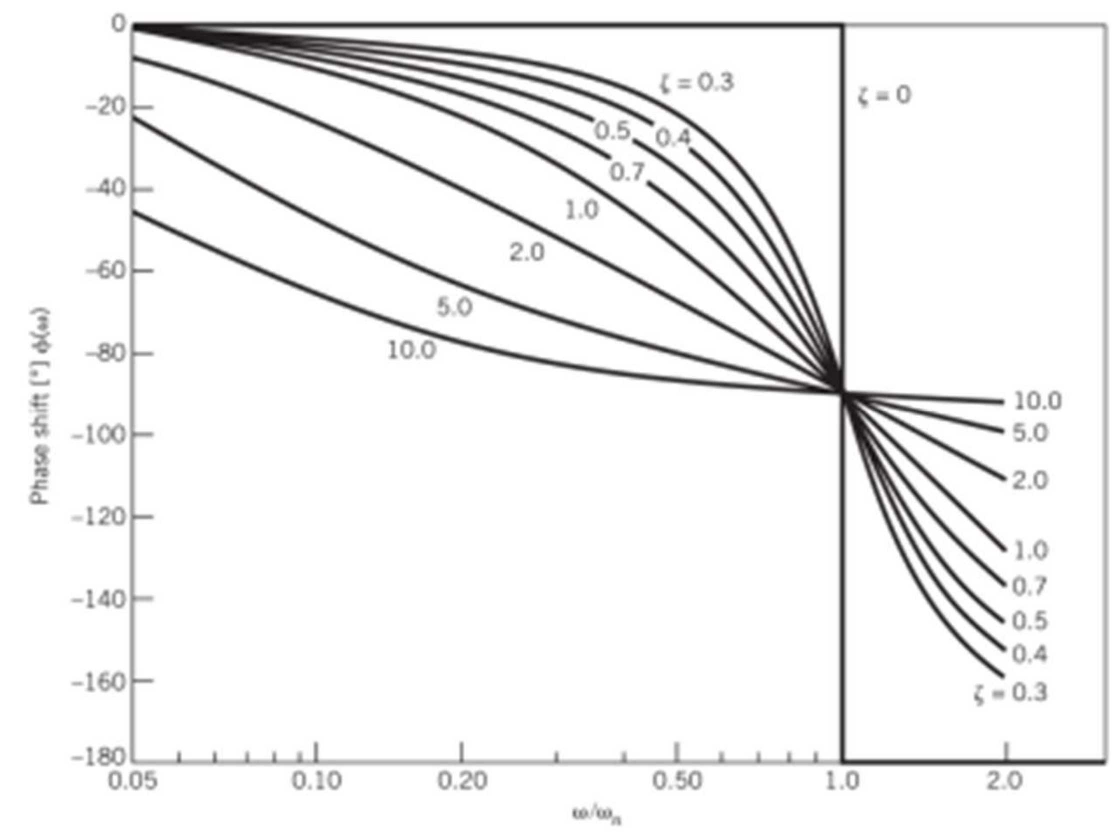
$$y_{\text{steady}}(t) = y(t\to\infty) = B(\omega)\sin[\omega t + \Phi(\omega)]$$

Magnitude ratio,

$$M(\omega) = \frac{B(\omega)}{KA} = \frac{1}{\left\{\left[1-(\omega/\omega_n)^2\right]^2 + \left[2\zeta\omega/\omega_n\right]^2\right\}^{1/2}}$$

4. Second Order System: Harmonic Input Function





As a rule of thumb, the choice of = 0.707 is optimal since it results in the best combination of amplitude linearity and phase linearity over the widest range of frequencies.

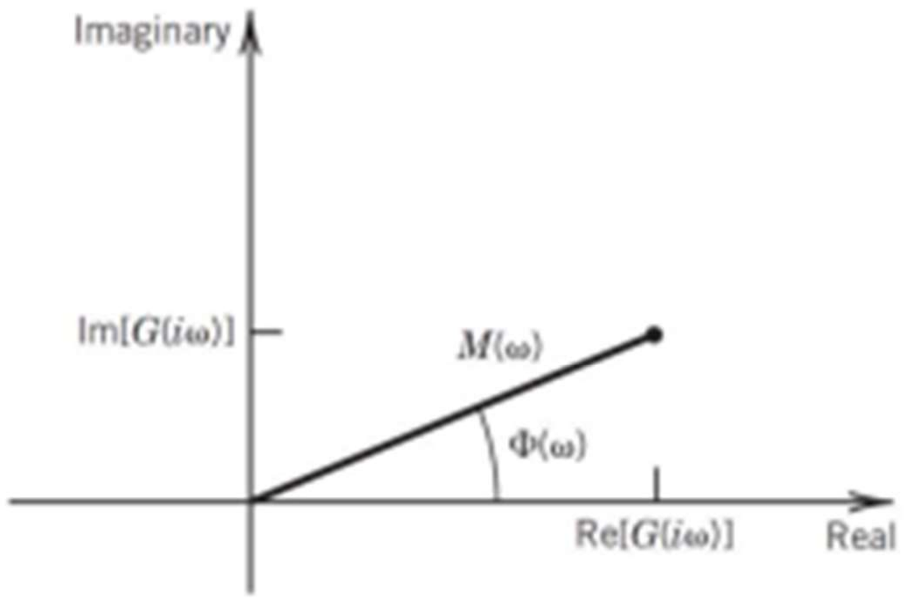
4. Second Order System: Transfer Function

$$\frac{1}{\omega_n^2}\ddot{y} + \frac{2\zeta}{\omega_n}\dot{y} + y = KF(t) \quad : \quad y(0) = 0 \text{ and } \dot{y}(0) = 0$$

$$\Rightarrow \frac{1}{\omega_n^2}\left[s^2 Y(s) - s\,y(0) - \dot{y}(0)\right] + \frac{2\zeta}{\omega_n}\left[s\,Y(s) - y(0)\right] + Y(s) = kF(s)$$

TF, $G(s) = \frac{Y(s)}{KF(s)} = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$

$$G(s = i\omega) = \frac{1}{\frac{(i\omega)^2}{\omega_n^2} + \frac{2\zeta}{\omega_n}(i\omega) + 1} = M(\omega)\, e^{i\Phi(\omega)}$$



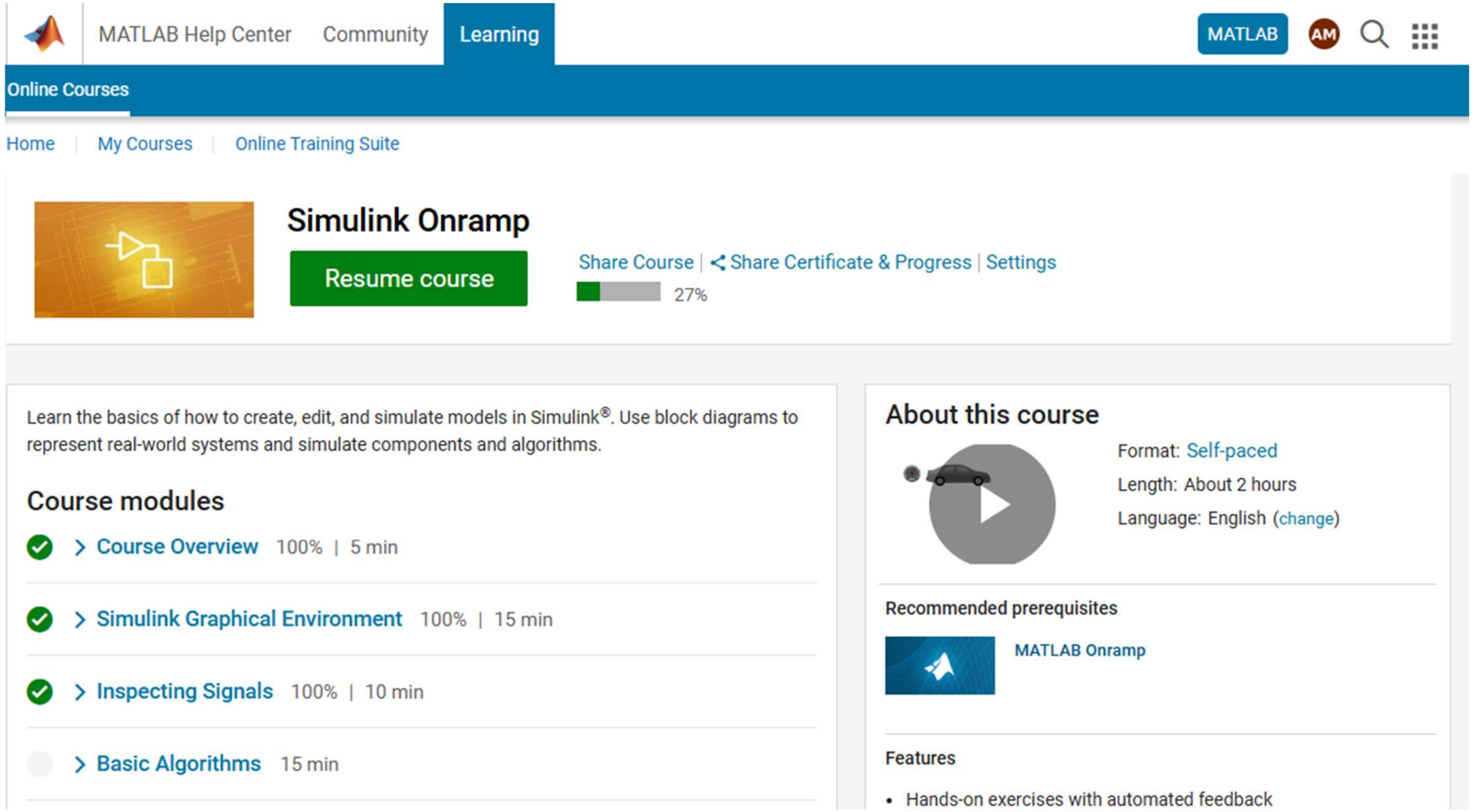
4. Second Order System: Modeling Analogies

Generic quantity	**Mechanical translation**	**Mechanical rotation**	**Electrical**	**Hydraulic**
Effort (E)	Force (F)	Torque (T)	Voltage (V)	Pressure (P)
Flow (F)	Speed (v)	Angular speed (ω)	Current (i)	Volumetric flow rate (Q)
Displacement (q)	Displacement (x)	Angular displacement (θ)	Charge (q)	Volume ($\cancel{V}$)
Momentum (p)	Linear momentum ($p = mv$)	Angular momentum ($h = J\omega$)	Flux linkage ($I = N\Phi = Li$)	Momentum/area ($\Gamma = IQ$)
Resistor (R)	Damper (b)	Rotary damper (B)	Resistor (R)	Resistor (R)
Capacitor (C)	Spring ($1/k$)	Torsion spring ($1/k$)	Capacitor (C)	Tank (C)
Inertia (I)	Mass (m)	Moment of inertia (J)	Inductor (L)	Inertance (I)
Inertia energy storage (special case)	$F = \dot{p}$ ($F = ma$)	$T = \dot{h}$ ($T = J\alpha$)	$V = \dot{\lambda}$ ($V = L\,di/dt$)	$P = \dot{\Gamma}$ ($P = I\,dQ/dt$)
Capacitor energy storage	$F = kx$	($T = k\theta$)	$V = (1/C)q$	$P = (1/C)\cancel{V}$
Dissipative	$F = bv$	$T = B\omega$	$V = Ri$	$P = RQ$

5. MATLAB and Simulink for System Dynamics

- Go through the "**Simulink Onramp**" self-paced Online course from MathWorks. (Website Link)

- Complete the course and submit the Assignment as per instructions.



Problems: Homework

1. Consider modeling a temperature sensor as a sphere having a thermal conductivity of 91 W/m-K, a density of 8900 kg/m^3, and a specific heat of 444 J/kg-K. The sensor is in an environment where the heat transfer coefficient is 100 W/m^2-K. Determine the maximum allowable diameter of the sensor if the 90% response time to a step-change in the fluid temperature, T_∞, must be: a. 10 s; b. 1 s and c. 0.01 s.

2. A single-loop RLC electrical circuit can be modeled as a second-order system in terms of current. Show that the differential equation for such a circuit subjected to a forcing function potential E(t) is given by

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = E(t)$$

   Determine the natural frequency and damping ratio for this system. For a forcing potential, E(t) = 1 + 0.5 sin 2000t V, determine the system steady response when L = 2 H, C = 1 μF, and R = 10 kΩ. Plot the steady output signal and input signal versus time. $I(0) = \dot{I}(0) = 0$.

3. A temperature-measuring device with a time constant of 0.15 s outputs a voltage that is linearly proportional to temperature. The device is used to measure an input signal of the form T(t) = 115 + 12 sin 2t °C. Plot the input signal and the predicted output signal with time, assuming first-order behavior and a static sensitivity of 5 mV/°C.

Reading

Theory and Design for Mechanical Measurements, 7th Edition, *Richard S. Figliola & Donald E. Beasley*

- ***Chapter 3***
  - ✓ *pp. 64-91*
  - ✓ *Example problem: 3.1, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11*

Thank you