Lecture Note- 01

Topic : Fourier Series

1. Periodic Function
2. Generalized Fourier series
3. Half-range Fourier sine & Cosine Series
4. Parsevalls Identity for Fourier Series

Fourier series

Question-1 Define periodic function with examples.

Ans: A real-valued function $f(x)$ is said to be periodic with period T if $f(x+T) = f(x)$. for all x, where T is a positive constant. The smallest value of T for which $f(x+T) = f(x)$ holds is called the fundamental period or the least period or simply the period of $f(x)$. For example, the fundamental period of $f(x) = \sin x$ is $T = 2\pi$.

In real life, there are many phenomena which are periodic in Nature, such as the human heartbeat, the motion of Fan, the motion of clock, the motion of earth around the sun etc.

Exercise: What is the fundamental period of each of the following functions:

(1) $f(x) = \cos x$, (2) $f(x) = \tan x$,

(3) $f(x) = \cos 2\pi x$, (4) $f(x) = \sin \frac{4}{L} x$,

(5) $f(x) = \sin x + \sin 2x$

(6) $f(x) = \sin 2x + \cos 4x$

(7) $f(x) = \sin 3x + \cos 2x$

(8) $f(x) = A_0 + \sum_{n=1}^{\infty} \left[A_n \cos \frac{n\pi}{p} x + B_n \sin \frac{n\pi}{p} x\right]$,

where A_n & B_n depend only on n.

Note that the function $f(x) = A \sin Bx$ & $f(x) = A \cos$ has the period $= \frac{2\pi}{B}$ and Amplitude $= A$.

Also note that a constant has any positive no. as a period.

Question-② Who discovered Fourier series and what are the real-life applications of Fourier series? Also give the verbal as well as mathematical definition of Fourier series.

Ans. French physicist Joseph Fourier (1768-1830) discovered the Fourier series in 1822. In 1822, for the purpose of solving the heat equation Fourier introduced a series in terms of sines & cosines, that is known as Fourier series.

The Fourier series has many important practical applications such as in signal processing, in image processing, analysis of sound wave as well as in solving both the ordinary & partial differential equations.

Verbal Definition of Fourier series: A Fourier series is an expansion of a periodic function $f(x)$ in terms of an infinite sum of sines and cosines.

Mathematical Definition of Fourier series:

Let $f(x)$ be a function which is defined on the interval $(-L, L)$ and $f(x)$ is periodic with period $2L$, i.e. $f(x+2L) = f(x)$ for all x.

Then the Fourier series corresponding to the function $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right] \quad \text{---} (1)$$

Where $a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\, dx$, $a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\, dx$

and $b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\, dx$ are known as the Fourier coefficients of $f(x)$.

* Note that (i) physically a_0 is called the average value of the function $f(x)$ whereas a_n and b_n are called the amplitudes of cosine and sine function respectively.

(ii) The nth term of the Fourier series is $a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$ which is called the nth harmonic of $f(x)$. Also the amplitude of the nth harmonic is $A_n = \sqrt{a_n^2 + b_n^2}$.

(iii) Using Fourier series, we can express any non-sinusoidal periodic function in terms of sinusoids (sine & cosine).

Fourier series in different intervals:

(1) If the function $f(x)$ is defined on the interval $(-\pi, \pi)$ with period 2π, then the Fourier series (1) can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{---- (2)}$$

with Fourier coefficients as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\,dx.$

(2) If the function $f(x)$ is defined on the interval $(0, 2\pi)$ with period 2π, then the Fourier series (1) can be written as

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \quad \text{--- (3)}$$

with Fourier coefficients as follows:

$$a_0 = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \cos nx\,dx$$

and $b_n = \frac{1}{\pi}\int_{0}^{2\pi} f(x) \sin nx\,dx$

3 If the function $f(x)$ is defined on the interval $(0, \pi)$ with period π, then the Fourier series ① can be written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos 2nx + b_n \sin 2nx] \text{---} ④$$

with the Fourier coefficients as follows:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos 2nx\,dx$$

and $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin 2nx\,dx$.

Example-① Find the Fourier series for the periodic function $f(x)$ given by

$$f(x) = \begin{cases} 0, & -\pi < x \leq 0 \\ x, & 0 < x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Solution: Since ~~the~~ given that $f(x+2\pi) = f(x)$, for all x, the given function is periodic with period 2π. Also the graph of the given function is as follows:

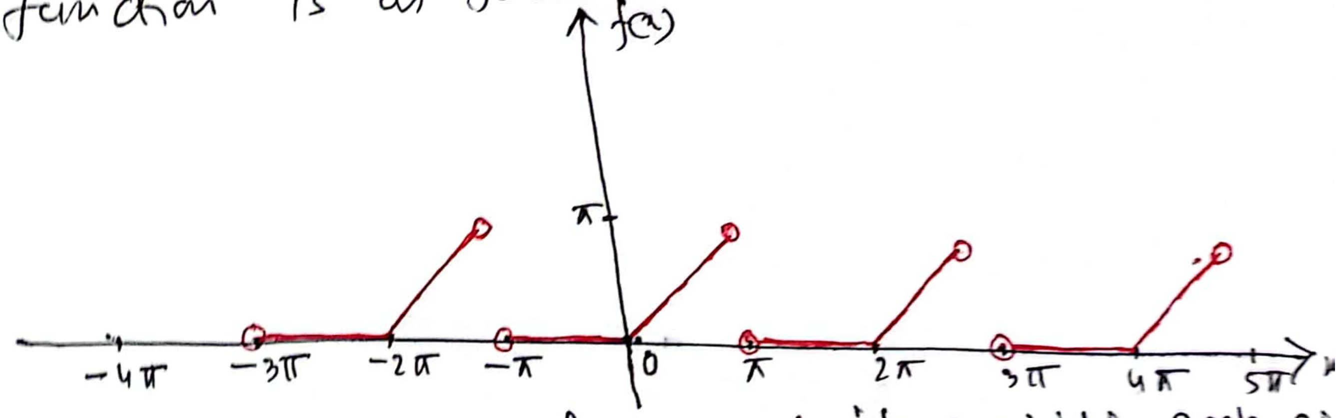

Fig.1: Graph of $f(x)$ and its periodic extensi

Now the Fourier series of $f(x)$ which is defined on the interval $(-\pi, \pi)$ with period 2π is

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \quad \text{---- (1)}$$

with Fourier coefficients are as follows:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

and $b_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$

Now $a_0 = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\,dx = \dfrac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \dfrac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$

$$\Rightarrow a_0 = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = 0 + \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi}\cdot\frac{\pi^2}{2} = \frac{\pi}{2} \Rightarrow \boxed{a_0 = \frac{\pi}{2}}$$

Again $a_n = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

$$\Rightarrow a_n = \frac{1}{\pi}\int_{-\pi}^{0} 0 \cdot \cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos nx\,dx$$

$$\Rightarrow a_n = 0 + \frac{1}{\pi}\left[x \cdot \frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_{x=0}^{\pi}$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\pi \cdot \frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$\Rightarrow a_n = \frac{1}{\pi}\left[\pi \cdot 0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] \qquad \left[\because \cos n\pi = (-1)^n,\ \sin n\pi = 0\right]$$

$$\Rightarrow \boxed{a_n = \frac{(-1)^n - 1}{n^2 \pi}}$$

Also $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin nx\,dx$$

$$\Rightarrow b_n = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot \sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cdot \sin nx\,dx$$

$$\Rightarrow b_n = 0 + \frac{1}{\pi}\left[x\cdot\left(-\frac{\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_{x=0}^{\pi}$$

$$\Rightarrow b_n = 0 + \frac{1}{\pi}\left[-\pi \cdot \frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + 0 - \frac{\sin 0}{n^2}\right]$$

$$\Rightarrow b_n = \frac{1}{\pi}\left[-\pi\frac{(-1)^n}{n} + \frac{0}{n^2} + 0 - 0\right]$$

$$\Rightarrow \boxed{b_n = -\frac{(-1)^n}{n}}$$

Now putting the values of a_0, a_n & b_n in (1) we get

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos nx - \frac{(-1)^n}{n}\sin nx\right]$$

Which is the required Fourier series for the given periodic function.

Exercise Find the Fourier series for the following function given by:

(1) $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Ans: $$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2 \pi} \cos nx + \frac{1}{n} \sin nx \right]$$

(2) $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Ans: $$f(x) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$$

(3) $$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 2, & 0 \le x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Ans: $$f(x) = \frac{1}{2} + \frac{3}{\pi} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$$

(4) $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Ans: $$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left[\frac{2(-1)^n}{n^2} \cos nx + \left\{ \frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3 \pi} \right\} \sin nx \right]$$

(5) $$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 \le x < \pi \end{cases} \quad \text{with } f(x+2\pi) = f(x).$$

Ans: $$f(x) = \frac{1}{\pi} + \frac{1}{2} \sin x + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)} \cos nx$$

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* Fourier series for odd and Even functions:
* Or Half-range Fourier sine and cosine series:

Even Function: A function $f(x)$ is said to be even if $f(-x) = f(x)$ for all x. Note that the graph of an even function is always symmetrical about the y-axis.

Examples: $f(x) = x^2$, $f(x) = \cos x$, $f(x) = |x|$ are examples of even function.

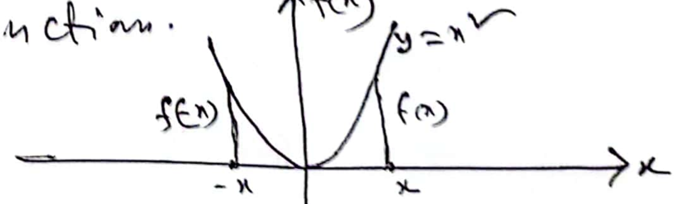

Fig.1: Graph of even function

Odd function: A function $f(x)$ is said to be odd if $f(-x) = -f(x)$ for all x. Note that the graph of an odd function is symmetrical about the origin.

Examples: $f(x) = x$, $f(x) = \sin x$, $f(x) = x^3$ are examples of odd function.

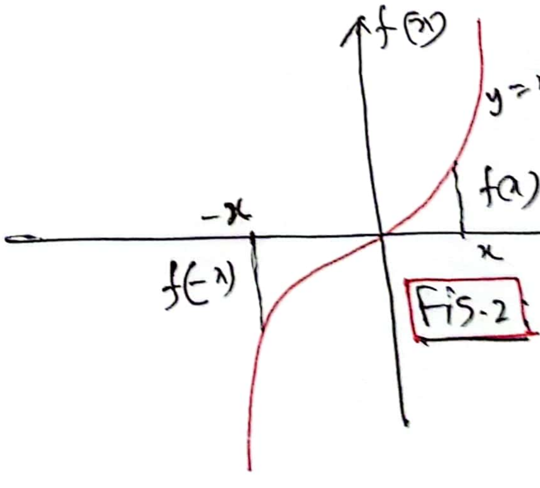

Fig.2: Graph of odd function.

Some important properties of Even/odd function:

(1) The product of two even function is even.

(2) The product of two odd function is even.

(3) The product of an even function and odd function is odd.

(4) The sum (or difference) of two even functions is even.

(5) The sum (or difference) of two odd function is odd.

(6) If $f(x)$ is even function, then $\int_{-L}^{L} f(x)\,dx = 2\int_{0}^{L} f(x)\,dx$.

(7) If $f(x)$ is odd function, then $\int_{-L}^{L} f(x)\,dx = 0$.

Exercise- Determine whether the following functions are even or odd or neither:-

(1) $f(x) = \sin 3x$, (2) $f(x) = x\cos x$, (3) $f(x) = x^2 + x$,

(4) $f(x) = x^3 - 4x$, (5) $f(x) = e^{|x|}$, (6) $f(x) = e^{x} - e^{-x}$,

(7) $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \leq x < 1 \end{cases}$. (8) $f(x) = \begin{cases} x+5, & -2 < x < 0 \\ -x+5, & 0 \leq x < 2 \end{cases}$

(9) $f(x) = x^3,\ 0 \leq x \leq 2$, (10) $f(x) = |x^5|$.

* Fourier series for odd Function:

Question: Show that an odd function have only sine terms in its Fourier expansion.

Or If $f(x)$ is an odd function over the interval $(-L, L)$ with period $2L$, then prove that

(a) $a_0 = 0$, (b) $a_n = 0$, (c) $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

and (d) $f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$.

Proof: Since the function $f(x)$ is defined on the interval $(-L, L)$ with period $2L$, So the Fourier series corresponding to $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \quad \text{--- (1)}$$

with the Fourier coefficients given below;

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

and $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Now since the given function is odd, so $f(-x) = -f(x)$.

$$\therefore a_0 = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)}_{\text{odd}}\,dx = 0 \quad \text{[by using the property]}$$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{\text{odd}}\,dx = 0$$

$$b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$$

Then, putting this values of a_0, a_n & b_n in (1) we get

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}} \quad \text{-----} \quad (2)$$

where $b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$

Which is called half-range fourier sine series as it contains only sine terms in its fourier expansion. Since the interval $(0, L)$ is half of the interval $(-L, L)$, thus accounting for the name half-range.

$\boxed{\text{Aliter question}}$ Show that an odd function can have no cosine terms in its fourier expansion.

Fourier series for Even Function:

Question If $f(x)$ is an even function which is defined on the interval $(-L, L)$ with period $2L$, then prove that (a) $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$,

(b) $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$, (c) $b_n = 0$

and (d) $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$.

Or Show that an even function have only cosine terms in its Fourier expansion.

proof: Since $f(x)$ is defined on the interval $(-L, L)$ with period $2L$, so the Fourier series corresponding to $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \quad \text{--- (1)}$$

with the Fourier coefficients given below:

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

and $$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Again since the given function is even, so $f(-x) = f(x)$ for all x.

Now using the property of even function, we get $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$

$$a_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\cos\frac{n\pi x}{L}}_{\text{even}}\,dx = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

and $b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{f(x)\sin\frac{n\pi x}{L}}_{\text{odd}}\,dx = 0$

putting this values of a_0, a_n & b_n in (1)

we get $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ ---(2)

Where $a_0 = \frac{2}{L}\int_{0}^{L} f(x)\,dx$ & $a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$

is known as half range fourier cosine series as it contains only cosine terms in its Fourier expansion.

Aliter question: Show that an even function can have no sine terms in its fourier expansion.

Example-① Expand $f(x) = x$, $0 < x < 2$ in a half-range ⓐ Fourier Sine Series ⓑ Fourier cosine series.

Solution (a) Half-range Fourier sine series:

First we see that the function $f(x) = x$, which is defined on the interval $(0, 2)$, is neither even nor odd. Then in order to find the half-range Fourier sine series, we extend the definition of the given function to that of the odd function of period 4, shown in Fig. 1(a) below. This is called odd extension of $f(x)$. Then the period is $2L = 4 \Rightarrow L = 2$. With this extension, the graph of the given function $f(x) = x$ within the interval $(-2, 2)$ is as follows:

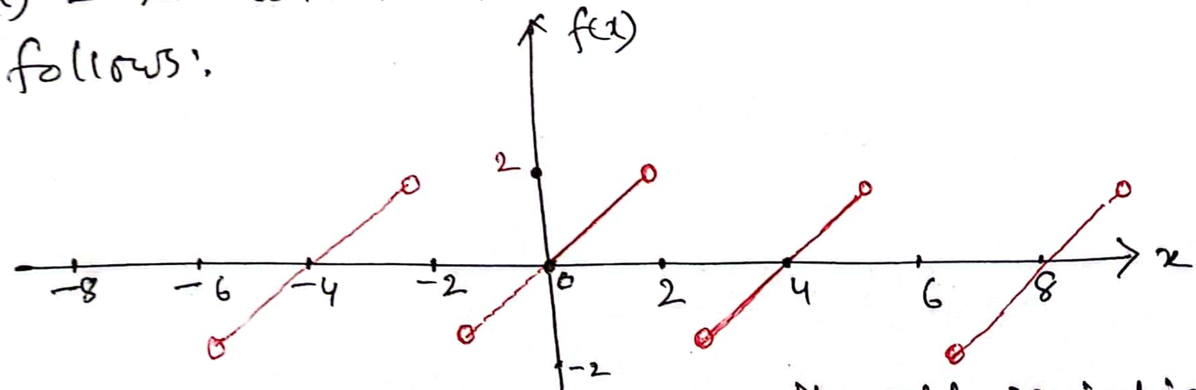

Fig. 1(a): Graph of $f(x)$ and its odd periodic extension.

Now for odd function/extension we know that $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L} dx$

and the Fourier sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{--- ①} \quad \text{with } L = 2.$$

$$\therefore \; b_n = \frac{2}{2}\int_0^2 x.\,\sin\frac{n\pi x}{2}\,dx$$

$$\Rightarrow b_n = \left[x.\left(\frac{-\cos\frac{n\pi x}{2}}{n\pi/2}\right) - 1.\left\{\frac{-\sin\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)^2}\right\}\right]_{x=0}^{2}$$

$$\Rightarrow b_n = \left[\frac{-2x}{n\pi}\cos\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right]_{x=0}^{2}$$

$$\Rightarrow b_n = \left[\frac{-2.2}{n\pi}\cos n\pi + \frac{4}{n^2\pi^2}\sin n\pi + 0 - 0\right]$$

$$\Rightarrow \boxed{b_n = \frac{-4(-1)^n}{n\pi}} \qquad \left[\because \sin n\pi = 0 \;\&\; \cos n\pi = (-1)^n\right]$$

Now putting the values of b_n in (1) we get

$$f(x) = \sum_{n=1}^{\infty}\frac{-4.(-1)^n}{n\pi}\sin\frac{n\pi x}{2}$$ which is the required half-range fourier sine series as its contains only the sine terms.

(b) Half-range fourier cosine series:

In order to find the half range fourier cosine series, we extend the definition of $f(x)$ to that of the even function of period 4 as shown in Fig. 1 (b). This is called even extension of $f(x)$. So period $2L = 4$ $\Rightarrow L = 2$. With this extension, the graph of $f(x)$ is given in the next page(17).

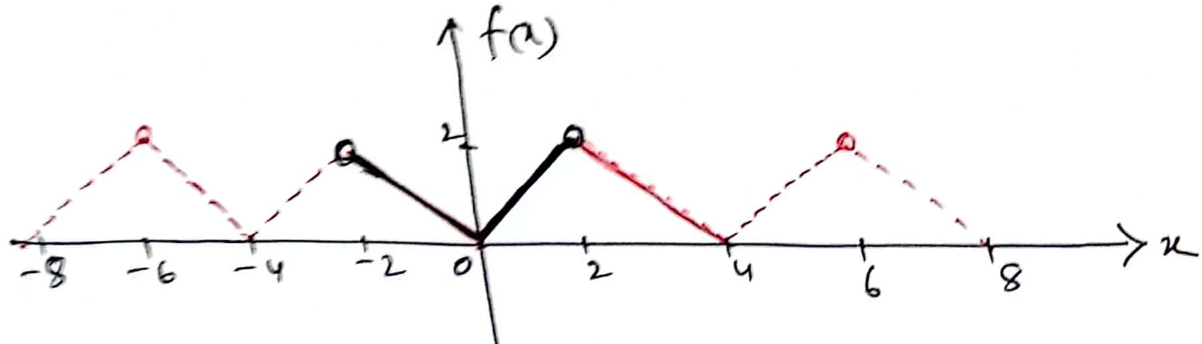

Fig. 1(b): Graph of f(x) & its even periodic extension.

Again for even extension / even function, we know that $b_n = 0$ and the corresponding Fourier cosine series is given below:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \quad \text{------} \quad (2)$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$ & $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

with $L = 2$.

Now $a_0 = \frac{2}{2}\int_0^2 x\,dx = \left[\frac{x^2}{2}\right]_{x=0}^{2} = \frac{4}{2} = 2$

& $a_n = \frac{2}{2}\int_0^2 x.\cos\frac{n\pi x}{2}\,dx$

$$\Rightarrow a_n = \left[x.\frac{2}{n\pi}\sin\frac{n\pi x}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right]_{x=0}^{2}$$

$$\Rightarrow a_n = \left[\frac{4}{n\pi}\sin n\pi + \frac{4}{n^2\pi^2}\cos n\pi - 0 - \frac{4}{n^2\pi^2}\right]$$

$$\Rightarrow a_n = \left[\frac{4}{n\pi}\cdot 0 + \frac{4}{n^2\pi^2}(-1)^n - \frac{4}{n^2\pi^2}\right]$$

$$\Rightarrow \boxed{a_n = -\frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]}$$

Now putting the values of a_0 & a_n in (2) we get

$$f(x) = \frac{2}{2} + \sum_{n=1}^{\infty} \left(\frac{-4}{n^2\pi^2}\right)\left[1-(-1)^n\right] \cos\frac{n\pi x}{2}$$

$$\Rightarrow f(x) = 1 - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \left[\frac{1-(-1)^n}{n^2}\right] \cdot \cos\frac{n\pi x}{2}$$

Which is the required half-range Fourier cosine series.

Exercise-(1) Expand $f(x) = x^2$, $0 < x < 2$ in a half-range (a) Fourier sine series and (b) halfrange Fourier cosine series.

Exercise-(2) Expand $f(x) = x$, $-2 < x < 2$ in half-range Fourier sine series.

Exercise-(3) Expand $f(x) = \sin x$, $0 < x < \pi$ in a half-range Fourier cosine series.

Exercise-(4) Expand $f(x) = \cos x$, $0 < x < \pi$ in a half-range Fourier sine series.

☆ **Exercise-(5)** Find the half-range Fourier sine and cosine series of the function $f(x)$ defined by $f(x) = x$, $0 \le x \le \pi$.

Parseval's identity for Fourier series:

Question: If the Fourier series corresponding to the periodic function $f(x)$ converses uniformly in the interval $(-L, L)$, then prove that —

$$\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}[a_n^2 + b_n^2],$$ where

a_0, a_n and b_n denote the Fourier coefficients.

Proof: We know that if the function $f(x)$ is defined on the interval $(-L, L)$ with period $2L$, then the Fourier series corresponding to $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right] \quad \text{--- (1)}$$

where the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$\left.\text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx \right\} \quad \text{--- (2)}$$

Now multiplying (1) by $f(x)$ and integrating term by term from $-L$ to L, we obtain

$$\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0}{2}\int_{-L}^{L} f(x)\,dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx\right\}$$

$$\Rightarrow \int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2}L + L\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad [\text{using (2)}]$$

$$\Rightarrow \boxed{\frac{1}{L}\int_{-L}^{L}\{f(x)\}^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)} \quad \cdots (3)$$

Which gives the relation between the average value of the square of the function $f(x)$ and the Fourier coefficients a_0, a_n & b_n.

Note that the above relation (3) is known as parseval's identity according to the name of French Mathematician Marc Antoine parseval (1755—1836).

Applications of parseval's Identity:

In electronics, parseval's identity is used to find the average power of an electrical circuit.

Notice that a_0^2 is the power in the dc components, while $(a_n^2 + b_n^2)$ is the ac power in the nth harmonics. Thus parseval's identity states that the average power (P) in a periodic signal is the sum of the average power in its dc components and the average power in its harmonics.