26. Solve for $F(x)$ the integral equation

$$\int_0^\infty F(x) \sin xt \, dx = \begin{cases} 1, & 0 \le t < 1 \\ 2, & 1 \le t < 2 \\ 0, & x \ge 2. \end{cases}$$

Answer : $F(x) = \frac{2}{\pi x}(1 + \cos x - 2\cos 2x)$.

27. Using Parseval's identity for Fourier transform prove the followings :

(i) $\displaystyle\int_0^\infty \frac{dt}{(a^2+t^2)(b^2+t^2)} = \frac{\pi}{2ab(a+b)}$.

(ii) $\displaystyle\int_0^\infty \frac{\sin at}{t(a^2+t^2)} dt = \frac{\pi}{2} \cdot \frac{(1-e^{-a^2})}{a^2}$.

4.18 Applications of Fourier transforms in solving boundary value Problems.

Example 1 (a) . Find the finite Fourier sine transform and the finite Fourier cosine transform of $\frac{\partial U}{\partial x}$ where U is a function of x and t for $o < x < l$, $t > o$.

Solution : (i) By def^n of finite Fourier sine transform of $F(x)$,

$o < x < l$, we have

$$f_s[F(x)] = f_s(n) = \int_0^l F(x) \sin\frac{n\pi x}{l} dx.$$

$$\therefore \quad f_s\left(\frac{\partial U}{\partial x}\right) = \int_0^l \frac{\partial U}{\partial x} \sin\frac{n\pi x}{l} dx.$$

$$= \left[\sin\frac{n\pi x}{l} . U(x,t)\right]_0^l - \frac{n\pi}{l}\int_0^l \cos\frac{n\pi x}{l} U(x,t)\, dx$$

$$= 0 - \frac{n\pi}{l}\int_0^l U(x,t) \cos\frac{n\pi x}{l} dx$$

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$= -\frac{n\pi}{l} f_c\, U(x,t) = -\frac{n\pi}{l} f_c(U)$ (1)

Since $f_c(n) = \int_0^l F(x) \cos\frac{n\pi x}{l} dx$

$\therefore f_c[F(x)] = f_c(n).$

(ii) By defn of finite Fourier cosine transform of $F(x)$, $o < x < l$, we have

$$f_c(n) = \int_0^l F(x) \cos\frac{n\pi x}{l} dx$$

$$\therefore \quad f_c\left\{\frac{\partial U}{\partial x}\right\} = \int_0^l \frac{\partial U}{\partial x} \cos\frac{n\pi x}{l} dx$$

$$= \left[\cos\frac{n\pi x}{l} U(x,t)\right]_0^l + \frac{n\pi}{l}\int_0^l U(x,t) \sin\frac{n\pi x}{l} dx$$

$$= U(l,t) \cos n\pi - U(o,t) + \frac{n\pi}{l} f_s\{U(x,t)\}$$

$$= U(l,t) \cos n\pi - U(o,t) + \frac{n\pi}{l} f_s(U) \quad (2)$$

Example 1(b). Find the finite Fourier sine transform and the finite Fourier cosine transform of $\frac{\partial^2 U}{\partial x^2}$ where U is a function of x and t for $o < x < l$, $t > o$.

Solution : (iii) Replacing U by $\frac{\partial U}{\partial x}$ in (1)

we get $f_s\left\{\frac{\partial^2 U}{\partial x^2}\right\} = -\frac{n\pi}{l} f_c\left\{\frac{\partial U}{\partial x}\right\}$

$$= -\frac{n\pi}{l}\left[U(l,t) \cos n\pi - U(o,t) + \frac{n\pi}{l} f_s(U)\right]$$

$$= -\frac{n\pi}{l} U(l,t) \cos n\pi + \frac{n\pi}{l} U(o,t) - \frac{n^2\pi^2}{l^2} f_s(U)$$

(iv) Replacing U by $\frac{\partial U}{\partial x}$ in (2), we get

$$f_c\left\{\frac{\partial^2 U}{\partial x^2}\right\} = \frac{\partial U(l,t)}{\partial x}\cos n\pi - \frac{\partial U(o,t)}{\partial x} + \frac{n\pi}{l} f_s\left\{\frac{\partial U}{\partial x}\right\}$$

$$= \frac{\partial U(l,t)}{\partial x}\cos n\pi - \frac{\partial U(o,t)}{\partial x} - \frac{n^2\pi^2}{l^2} f_c(U).$$

Since $f_s\left\{\frac{\partial U}{\partial x}\right\} = -\frac{n\pi}{l} f_c(U)$.

Example 2. Prove that the solution of the boundary value problem $\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}$

$$U(o,t) = U(2,t) = o,\ t > o$$

$$U(x, o) = x,\ o < x < 2$$

is $U(x, t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin\frac{n\pi x}{2} e^{-\frac{3}{4}n^2\pi^2 t}$. **D. U. H. 1990**

Proof : The given partial differential equation is

$$\frac{\partial U}{\partial t} = 3\frac{\partial^2 U}{\partial x^2}. \quad (1)$$

Taking the finite Fourier sine transform (with $l = 2$) of both sides of (1), we get

$$\int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2}\, dx = \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \quad (2)$$

Let $u = u(n,t) = \int_0^2 U(x,t) \sin\frac{n\pi x}{2}\, dx$

then $\frac{du}{dt} = \int_0^2 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{2}\, dx$

$$= \int_0^2 3\frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{2}\, dx \text{ using (2)}$$

(on integrating by parts)

$$= 3\left[\sin\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\right]_0^2 - \frac{3n\pi}{2}\int_0^2 \cos\frac{n\pi x}{2}\cdot\frac{\partial U}{\partial x}\,dx$$

$$= 0 - \frac{3n\pi}{2}\left[\cos\frac{n\pi x}{2}\cdot U(x,t)\right]_0^2 - \frac{3n^2\pi^2}{4}\int_0^2 \sin\frac{n\pi x}{2}\cdot U(x,t)\,dx$$

$$= 0 - \frac{3n^2\pi^2}{4}\int_0^2 U(x,t)\sin\frac{n\pi x}{2}\,dx$$

Since $U(o, t) = U(2, t) = o$

$$= -\frac{3n^2\pi^2}{4}u, \text{ Since } u = \int_0^2 U(x, t)\sin\frac{n\pi x}{2}\,dx.$$

$$\therefore \quad \frac{du}{dt} = -\frac{3n^2\pi^2}{4}u \text{ where } u = u(n, t).$$

$$\text{or,} \quad \frac{du}{u} = -\frac{3n^2\pi^2}{4}dt$$

Integrating both sides, we get

$\log u = -\dfrac{3n^2\pi^2}{4}t + \log A$, where A is an arbitrary constant.

$$\text{or,} \quad \log u = \log e^{-\frac{3n^2\pi^2 t}{4}} + \log A = \log A e^{-\frac{3n^2\pi^2}{4}t}$$

$$\therefore \quad u = u(n, t) = A e^{-\frac{3n^2\pi^2}{4}t} \qquad (3)$$

when $t = 0$, $u(n, o) = Ae^o = A$

$$\therefore \boxed{A = u(n, o)} \qquad (4)$$

$$\text{Now } u(n, t) = \int_0^2 U(x, t)\sin\frac{n\pi x}{2}\,dx$$

$$\therefore \quad u(n, o) = \int_0^2 U(x, o)\sin\frac{n\pi x}{2}\,dx$$

$$= \int_0^2 x \sin\frac{n\pi x}{2}\,dx, \text{ Since } U(x, o) = x$$

$$= \left[-\frac{2x}{n\pi}\cos\frac{n\pi x}{2}\right]_0^2 + \frac{2}{n\pi}\int_0^2 \cos\frac{n\pi x}{2}\,dx$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 + \frac{4}{n^2\pi^2}\left[\sin\frac{n\pi x}{2}\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi + 0 = -\frac{4}{n\pi}\cos n\pi.$$

Thus from (4), we have $\boxed{A = -\frac{4}{n\pi}\cos n\pi}$

putting the value of A in (3), we get

$$u(n, t) = -\frac{4}{n\pi}\cos n\pi\, e^{-\frac{3n^2\pi^2}{4}t} \qquad (5)$$

Now taking the inverse finite Fourier sine transform, we get

$$U(x,t) = \frac{2}{2}\sum_{n=1}^{\infty} -\frac{4}{n\pi}\cos n\pi.\, e^{-\frac{3}{4}n^2\pi^2 t}\sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} -\frac{4}{n\pi}(-1)^n\, e^{-\frac{3}{4}n^2\pi^2 t}\sin\frac{n\pi x}{2}$$

$$= \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{2}.\, e^{-\frac{3}{4}n^2\pi^2 t}$$

which is the required solution.

Example 3. Use finite Fourier transforms to solve

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ U(o, t) = o;$$

$$U(\pi,t) = o,\ U(x, o) = 2x$$

where $o < x < \pi$, $t > o$. **D. U. M.SC (F) 1986**

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2} \qquad (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t}\sin nx\,dx=\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx \qquad (2)$$

Let $u=u(n,t)=\int_0^\pi U(x,t)\sin nx\,dx$

then $\frac{du}{dt}=\int_0^\pi \frac{\partial U}{\partial t}\sin nx\,dx$

$=\int_0^\pi \frac{\partial^2 U}{\partial x^2}\sin nx\,dx$ using (2)

(on integrating by parts)

$=\left[\sin nx\frac{\partial U}{\partial x}\right]_0^\pi - n\int_0^\pi \cos nx\frac{\partial U}{\partial x}\,dx$

$=0-n\int_0^\pi \cos nx\frac{\partial U}{\partial x}\,dx$

$=-n\left[\cos nx\,U(x,t)\right]_0^\pi - n^2\int_0^\pi \sin nx\,U(x,t)\,dx$

$=0-n^2\int_0^\pi U(x,t)\sin nx\,dx$, Since $U(\pi,t)=0$

and $U(0,t)=0$.

$=-n^2u$, Since $u=\int_0^\pi U(x,t)\sin nx\,dx$

$\therefore \quad \frac{du}{dt}=-n^2u$

or, $\frac{du}{u}=-n^2dt$

Integrating both sides, we get $\log u = -n^2 t + \log A$, A being some constant of integration.

or, $\log u = \log e^{-n^2t} + \log A = \log A e^{-n^2t}$

$\therefore \quad u = Ae^{-n^2t} \quad (3)$

Now $u = u(n, t) = \int_0^\pi U(x, t) \sin nx \, dx$

$$\therefore \quad u(n, o) = \int_0^\pi U(x, o) \sin nx \, dx$$

$$= \int_0^\pi 2x \sin nx \, dx, \text{ Since } U(x, o) = 2x$$

$$= 2\left[-\frac{x\cos nx}{n}\right]_0^\pi + \frac{2}{n}\int_0^\pi \cos nx \, dx$$

$$= -\frac{2\pi}{n}\cos n\pi + o + \frac{2}{n^2}[\sin nx]_0^\pi$$

$$= -\frac{2\pi}{n}\cos n\pi \quad \boxed{\therefore u(n, o) = -\frac{2\pi}{n}\cos n\pi}$$

When $t = o$, $u(n, o) = Ae^0 = A$

$$\therefore \quad \boxed{A = -\frac{2\pi}{n}\cos n\pi}$$

Putting the value of A in (3), we get

$$u(n,t) = u = -\frac{2\pi}{n}\cos n\pi e^{-n^2t}$$

Applying the inversion formula for finite Fourier sine transform, we get

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\left(-\frac{2\pi}{n}\cos n\pi e^{-n^2t}\right)\sin nx.$$

For physical interpretation, $U(x, t)$ may be regarded as the temperature at any point x at an instant of time t in a solid bounded by the planes $x = o$ and $x = \pi$. The boundary conditions $U(o, t) = o$ and $U(\pi, t) = o$ give the zero temperature at the ends while $U(x, o) = 2x$ represents that the initial temperature is a function of x.

Example 4. Use finite Fourier transforms to solve

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2};\ U(o, t) = o\ ;\ U(4, t) = o;$$

$U(x, o) = 2x$ where $o < x < 4$, $t > o$.

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial n^2} \quad (1)$$

Taking the finite Fourier sine transform (with $l = 4$) of both sides of (1), we get

$$\int_o^4 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{4}\,dx = \int_o^4 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{4}\,dx \quad (2)$$

Let $u = u(n, t) = \int_o^4 U(x, t) \sin\frac{n\pi x}{4}\,dx.$

Then $\frac{du}{dt} = \int_o^4 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{4}\,dx$

$$= \int_o^4 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{4}\,dx \text{ using (2)}$$

(on integrating by parts)

$$= \left[\sin\frac{n\pi x}{4}.\frac{\partial U}{\partial x}\right]_o^4 - \frac{n\pi}{4}\int_o^4 \cos\frac{n\pi x}{4}\frac{\partial U}{\partial x}\,dx$$

$$= 0 - \frac{n\pi}{4}\int_o^4 \cos\frac{n\pi x}{4}.\frac{\partial U}{\partial x}\,dx$$

$$= -\frac{n\pi}{4}\left[\cos\frac{n\pi x}{4}.\ U(x,t)\right]_o^4 - \frac{n^2\pi^2}{16}\int_o^4 U(x, t)\sin\frac{n\pi x}{4}\,dx$$

$$= 0 - \frac{n^2\pi^2}{16}\int_o^4 U(x,t)\sin\frac{n\pi x}{4}\,dx \text{ Since } U(o,t)=U(4,t) = o$$

$$= -\frac{n^2\pi^2}{16} u, \text{ Since } u = \int_0^4 U(x, t) \sin\frac{n\pi x}{4}\, dx$$

$$\therefore \quad \frac{du}{dt} = -\frac{n^2\pi^2}{16} u \text{ where } u = u(n, t).$$

or, $$\frac{du}{u} = -\frac{n^2\pi^2}{16} dt$$

Integrating both sides, we get

$\log \quad u = -\dfrac{n^2\pi^2 t}{16} + \log A$, A being some constant of integration.

or, $$\log u = \log e^{-\frac{n^2\pi^2 t}{16}} + \log A = \log A\, e^{-\frac{n^2\pi^2 t}{16}}$$

$$\therefore \quad u = A\, e^{-\frac{n^2\pi^2 t}{16}} \quad (3)$$

or, $$u(n, t) = A\, e^{-\frac{n^2\pi^2 t}{16}}$$

When $t = o$, $u(n, o) = A\, e^0 = A$

$$\therefore \quad \boxed{A = u(n, o)} \quad (4)$$

Now $u(n, t) = \displaystyle\int_0^4 U(x, t) \sin\frac{n\pi x}{4}\, dx$

$$\therefore \quad u(n, o) = \int_0^4 U(x, o) \sin\frac{n\pi x}{4}\, dx$$

$$= \int_0^4 2x \sin\frac{n\pi x}{4}\, dx$$

$$= \left[-2x\frac{4}{n\pi}\cos\frac{n\pi x}{4}\right]_0^4 + \frac{8}{n\pi}\int_0^4 \cos\frac{n\pi x}{4}\, dx$$

$$= -\frac{32}{n\pi}\cos n\pi + o + \frac{32}{n^2\pi^2}\left[\sin\frac{n\pi x}{4}\right]_o^4$$

$$= -\frac{32}{n\pi}\cos n\pi$$

Thus from (4), we have $\boxed{A = -\frac{32}{n\pi}\cos n\pi}$

Putting the value of A in (3), we get

$$u(n, t) = -\frac{32}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2 t}{16}} \quad (5)$$

Now applying the inversion formula for finite Fourier sine transform, we get

$$U(x, t) = \frac{2}{4}\sum_{n=1}^{\infty} -\frac{32}{n\pi}\cos n\pi\, e^{-\frac{n^2\pi^2 t}{16}} \cdot \sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty} (-1)^{n+1}\frac{e^{-\frac{n^2\pi^2 t}{16}}}{n} \cdot \sin\frac{n\pi x}{4}$$

which is the required solution.

Physical interpretation :

Physically, $U(x, t)$ represents the temperature at any point x at any time t in solid bounded by the planes $x = o$ and $x = 4$ (or a bar on the x – axis with the ends $x = o$ and $x = 4$, whose surface is insulated laterally).The condition $U(o, t) = o$ and $U(4, t) = o$ implies that the ends are kept at zero temperature while $U(x, o) = 2x$ implies that the initial temperature is a function of x.

Example 5. Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $o < x < 6$, $t > o$, subject to the conditions $U(o, t) = o$, $U(6, t) = o$, $U(x, o) = \begin{cases} 1, & o < x < 3 \\ 0, & 3 < x < 6 \end{cases}$

and interpret physically.

Solution : The given partial differential equation is $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (1)

Taking the finite Fourier sine transform (with $l = 6$) of both sides of (1), we get

$$\int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6}\, dx = \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6}\, dx \qquad (2)$$

Let $u = u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6}\, dx$

Then $\frac{du}{dt} = \int_0^6 \frac{\partial U}{\partial t} \sin\frac{n\pi x}{6}\, dx$

$= \int_0^6 \frac{\partial^2 U}{\partial x^2} \sin\frac{n\pi x}{6}\, dx$ using (2)

(On integrating by parts)

$$= \left[\sin\frac{n\pi x}{6} \cdot \frac{\partial U}{\partial x}\right]_0^6 - \frac{n\pi}{6}\int_0^6 \cos\frac{n\pi x}{6} \cdot \frac{\partial U}{\partial x}\, dx$$

$$= o - \frac{n\pi}{6}\left[\cos\frac{n\pi x}{6}.\ U(x, t)\right]_0^6 - \frac{n^2\pi^2}{36}\int_0^6 \sin\frac{n\pi x}{6}\ U(x, t)\, dx$$

$$= 0 - \frac{n\pi}{6}\left[\cos n\pi.\ U(6, t) - U(o, t)\right] - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6}\, dx$$

$$= 0 - \frac{n^2\pi^2}{36}\int_0^6 U(x, t) \sin\frac{n\pi x}{6}\, dx, \text{ Since } U(6,t) = U(o,t) = o$$

$$= -\frac{n^2\pi^2}{36} u, \text{ Since } u = \int_0^6 U(x,t) \sin\frac{n\pi x}{6}\, dx$$

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$\therefore \quad \frac{du}{dt} = -\frac{n^2\pi^2}{36} u$. where $u = u(n, t)$

or, $\quad \frac{du}{u} = -\frac{n^2\pi^2}{36} dt$

Integrating both sides, we get

$\log u = -\frac{n^2\pi^2}{36} t + \log A$, A being some constant of integration

or, $\log u = \log e^{-\frac{n^2\pi^2 t}{36}} + \log A = \log A e^{-\frac{n^2\pi^2 . t}{36}}$

$$\therefore \quad u = A e^{-\frac{n^2\pi^2 t}{36}} \qquad (3)$$

When $t = o$, $u(n, o) = Ae^{o} = A$

$$\therefore \quad \boxed{A = u(n, o)} \qquad (4)$$

Now $u(n, t) = \int_0^6 U(x, t) \sin\frac{n\pi x}{6} dx$

$$\therefore \quad u(n, o) = \int_0^6 U(x, o) \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 U(x, o) \sin\frac{n\pi x}{6} dx + \int_3^6 U(x, o) \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 1. \sin\frac{n\pi x}{6} dx + \int_3^6 o. \sin\frac{n\pi x}{6} dx$$

$$= \int_0^3 \sin\frac{n\pi x}{6} dx + o$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi x}{6}\right]_0^3$$

$$= -\frac{6}{n\pi}\left[\cos\frac{n\pi}{2} - 1\right]$$

$$= \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right).$$

Thus from (4), we have

$$A = \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) \qquad (5)$$

Putting the value of A in (3), we get

$$u(n, t) = \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2 t}{36}}$$

Taking the inverse Fourier sine transform we get

$$U(x, t) = \frac{2}{6}\sum_{n=1}^{\infty} \frac{6}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right) e^{\frac{-n^2\pi^2 t}{36}} . \sin\frac{n\pi x}{6}$$

$$\text{or, } \quad U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\left(1 - \cos\frac{n\pi}{2}\right) e^{-\frac{n^2\pi^2 t}{36}} . \sin\frac{n\pi x}{6}.$$

Physical interpretation

Physically U(x, t) represents the temperature at any point x at any time t in a bar with the ends x = o and x = 6 kept at zero temperature which is insulated laterally. Initially the temperature in the half bar from x = o to x = 3 is constant equal to 1 unit while the half bar from x = 3 to x = 6 is at zero temperature.

Example 6 : Solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $x > o$, $t > o$

Subject to the conditions U(o, t) = o,

$$U(x, o) = \begin{cases} 1 & o < x < 1 \\ o & x \geq 1 \end{cases}$$

and U (x, t) is bounded.

Solution : Given partial differential equation is $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ (1)

Taking the Fourier sine transform of both sides of (1), we get

$$\int_0^\infty \frac{\partial U}{\partial t} \sin nx\, dx = \int_0^\infty \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \qquad (2)$$

Let $u = u(n, t) = \int_0^\infty U(x,t) \sin nx\, dx$

then $\frac{du}{dt} = \int_0^\infty \frac{\partial U(x,t)}{\partial t} \sin nx\, dx$

$= \int_0^\infty \frac{\partial^2 U}{\partial x^2} \sin nx\, dx$ by (2)

(on integrating by parts)

$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\infty - n \int_0^\infty \cos nx \frac{\partial U}{\partial x} dx$

$= 0 - n \int_0^\infty \cos nx \frac{\partial U}{\partial x} dx$ Since $\frac{\partial U}{\partial x} \to 0$ as $x \to \infty$

$= -n\,[\cos nx\, U(x, t)]_0^\infty - n^2 \int_0^\infty \sin nx\, U(x, t)\, dx.$

$= -n\,[0 - U(0, t)] - n^2 u$ Since $U \to 0$ as $x \to \infty$

$= n\,U(0, t) - n^2 u.$

$\therefore \quad \frac{du}{dt} = n\,U(0, t) - n^2 u \qquad (3)$

From the given condition, we have $U(0, t) = 0$

$\therefore$ from (2), we have $\frac{du}{dt} = -n^2 u$

or, $\frac{du}{u} = -n^2\, dt$

Integrating both sides, we have

$\log u = -n^2 t + \log A$, A being some constant of integration.

or, $\log u = \log e^{-n^2 t} + \log A = \log A e^{-n^2 t}$

$\therefore \quad u = Ae^{-n^2 t} \qquad (4)$

Now $u(n, t) = \int_0^\infty U(x, t) \sin nx\, dx$

$\therefore \quad u(n, o) = \int_0^\infty U(x, o) \sin nx\, dx$

$= \int_0^1 U(x, o) \sin nx\, dx + \int_1^\infty U(x, o) \sin nx\, dx$

$= \int_0^1 1. \sin nx\, dx + o$ Since $U(x, o) = \begin{cases} 1, & o < x < 1 \\ o, & x \geq 1 \end{cases}$

$= \int_0^1 \sin nx\, dx = -\frac{1}{n} [\cos nx]_0^1$

$= -\frac{1}{n} (\cos n - \cos o)$

$= \frac{1}{n} (1 - \cos n)$

Therefore initially, when $t = o$, $u(n, t) = u(n,o) = \dfrac{1 - \cos n}{n}$

Thus from (4), we get

$$\frac{1 - \cos n}{n} = Ae^o = A \quad \therefore \boxed{A = \frac{1 - \cos n}{n}}$$

putting the value of A in (4), we get

$$u = u(n, t) = \frac{1 - \cos n}{n} e^{-n^2 t}$$

Note : Inverse Fourier sine transform of $f_s(n)$ is defined as

$$F(x) = \frac{2}{\pi} \int_0^\infty f_s(n) \sin nx\, dn$$

Now applying the inversion formula for Fourier sine transform, we have

$$U(x, t) = \frac{2}{\pi} \int_0^\infty u(n, t) \sin nx\, dn$$

$$= \frac{2}{\pi} \int_0^\infty \frac{1 - \cos n}{n} e^{-n^2 t} \sin nx\, dn$$

which gives the required solution, physically interpreted as the temperature at any point x at any time t in a solid $x > 0$.

Example 7. Solve the boundary value problem $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$,

$U(0, t) = 1$, $U(\pi, t) = 3$, $U(x, 0) = 2$, where $0 < x < \pi$, $t > 0$.

Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \qquad (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t} \sin nx \, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx \qquad (2)$$

Let $u = u(n, t) = \int_0^\pi U(x, t) \sin nx \, dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx \, dx$

$$= \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx \, dx \quad \text{using (2)}$$

(on integrating by parts)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - n \int_0^\pi \cos nx \frac{\partial U}{\partial x} dx$$

$$= 0 - n\,[\cos nx\, U(x, t)]_0^\pi - n^2 \int_0^\pi \sin nx\, U(x, t)\, dx$$

$$= -n\,[\cos n\pi\, U(\pi, t) - U(0, t)] - n^2 \int_0^\pi U(x, t) \sin nx dx$$

$$= -n\,[3 \cos n\pi - 1] - n^2 u$$

$\therefore \quad \frac{du}{dt} = n\,(1 - 3 \cos n\pi) - n^2 u.$

or, $\frac{du}{dt} + n^2 u = n(1 - 3 \cos n\pi) \qquad (3)$

which is a linear differential equation of first order.

$$\text{I.F} = e^{\int n^2 dt} = e^{n^2 t}.$$

Therefore, solution of (3) is

$$ue^{n^2t} = n\,(1 - 3\cos n\pi)\int e^{n^2t}\,dt$$

$$= \frac{n(1 - 3\cos n\pi)}{n^2}\,e^{n^2t} + A$$

$$= \frac{(1 - 3\cos n\pi)}{n}\,e^{n^2t} + A.$$

or, $\quad u = u(n, t) = \dfrac{1 - 3\cos n\pi}{n} + Ae^{-n^2t} \quad (4)$

When $t = o$, $u(n, o) = \dfrac{1 - 3\cos n\pi}{n} + A \quad (5)$

$$u = u(n, t) = \int_0^{\pi} U(x, t)\sin nx\,dx$$

$$\therefore \quad u(n, o) = \int_0^{\pi} U(x, o)\sin nx\,dx$$

$$= \int_0^{\pi} 2\sin nx\,dx$$

$$= \frac{-2}{n}\Big[\cos nx\Big]_0^{\pi}$$

$$= -\frac{2}{n}(\cos n\pi - 1) = \frac{2}{n}(1 - \cos n\pi)$$

Thus from (5), we get

$$\frac{2}{n}(1 - \cos n\pi) = \frac{1 - 3\cos n\pi}{n} + A$$

$$\therefore \quad A = \frac{1}{n}(2 - 2\cos n\pi - 1 + 3\cos n\pi)$$

or, $\quad A = \dfrac{1}{n}(1 + \cos n\pi)$

putting the value of A in (4), we get

$$u = (n, t) = \frac{1 - 3\cos n\pi}{n} + \frac{1}{n}(1 + \cos n\pi)\,e^{-n^2t}$$

Taking inverse finite Fourier sine transform we have

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1-3\cos n\pi}{n} \sin nx$$

$$+\frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{n} (1 + \cos n\pi)\, e^{-n^2 t} \sin nx.$$

Example 8. Solve the boundary value problem

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2},\ U(o, t) = 1,\ U(\pi, t) = 3$$

$$U(x, o) = 1, \text{ where } o < x < \pi,\ t > o.$$

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Solution : The given partial differential equation is

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2} \quad (1)$$

Taking the finite Fourier sine transform of both sides of (1), we get

$$\int_0^\pi \frac{\partial U}{\partial t} \sin nx\, dx = \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \quad (2)$$

Let $u = u(n, t) = \int_0^\pi U(x, t) \sin nx\, dx$

then $\frac{du}{dt} = \int_0^\pi \frac{\partial U}{\partial t} \sin nx\, dx$

$$= \int_0^\pi \frac{\partial^2 U}{\partial x^2} \sin nx\, dx \quad \text{using (2)}$$

(on integrating by parts)

$$= \left[\sin nx \frac{\partial U}{\partial x}\right]_0^\pi - n\int_0^\pi \cos nx \frac{\partial U}{\partial x}\, dx$$

$$= o - n\,[\cos nx\, U(x, t)]_0^\pi - n^2 \int_0^\pi \sin nx\, U(x, t)\, dx$$

Method

$$= -n\,[\cos n\pi\, U(\pi, t) - U(o, t)] - n^2 \int_o^\pi U(x, t) \sin nx\, dx$$

$$= -n\,(3 \cos n\pi - 1) - n^2 u$$

$$= n\,(1 - 3 \cos n\pi) - n^2 u.$$

or, $$\frac{du}{dt} = n\,(1 - 3\cos n\pi) - n^2 u$$

or, $$\frac{du}{dt} + n^2 u = n(1 - 3\cos n\pi) \qquad (3)$$

which is a linear differential equation of first order.

I. F $= e^{\int n^2 dt} = e^{n^2 t}$.

Therefore, solution of (3) is

$$u e^{n^2 t} = n(1 - 3\cos n\pi) \int e^{n^2 t}\, dt$$

$$= \frac{n(1 - 3\cos n\pi)}{n^2}\, e^{n^2 t} + A$$

or, $$u = u(n, t) = \frac{1}{n}(1 - 3\cos n\pi) + A e^{-n^2 t} \qquad (4)$$

When $t = o$, $u(n, o) = \int_o^\pi U(x, o) \sin nx\, dx$

$$= \int_o^\pi 1.\, \sin nx\, dx$$

$$= -\frac{1}{n}\,[\cos nx]_o^\pi$$

$$= -\frac{1}{n}\,(\cos n\pi - 1)$$

$$= \frac{1}{n}\,(1 - \cos n\pi).$$

Again, when $t = o$, from (4), we get

$$u(n, o) = \frac{1}{n}\,(1 - 3\cos n\pi) + A$$

$$\therefore \quad \frac{1}{n}(1 - \cos n\pi) = \frac{1}{n}(1 - 3\cos n\pi) + A.$$

or, $A = \frac{1}{n}(1 - \cos n\pi - 1 + 3\cos n\pi) = \frac{2\cos n\pi}{n}$

putting the value of A in (4), we get

$$u = u(n, t) = \frac{1}{n}(1 - 3\cos n\pi) + \frac{2\cos n\pi}{n} e^{-n^2 t}$$

Taking inverse finite Fourier sine transform, we get

$$U(x, t) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1 - 3\cos n\pi)}{n} \sin nx$$

$$+ \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{2\cos n\pi}{n} e^{-n^2 t} \sin nx.$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(1 - 3\cos n\pi)}{n} \sin nx + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} e^{-n^2 t} \sin nx \quad (5)$$

The finite Fourier sine transform of $F(x)$, $o < x < l$, is defined as

$$f_s(n) = \int_0^l F(x) \sin \frac{n\pi x}{l} dx$$

Here $l = \pi$

$$\therefore \quad f_s(n) = \int_0^\pi F(x) \sin nx\, dx$$

$$f_s\{F(x)\} = \int_0^\pi F(x) \sin nx\, dx.$$

$\boxed{F(x) = 1}$ $\quad f_s\{1\} = \int_0^\pi 1 \sin x dx$

$$= -\frac{1}{n}[\cos nx]_0^\pi$$

$$= -\frac{1}{n}(\cos n\pi - 1) = \frac{1}{\pi}(1 - \cos n\pi).$$

$\therefore$ Taking inverse finite Fourier sine transform, we get

$$1 = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 - \cos n\pi}{n} \sin nx.$$

$\boxed{F(x) = x}$ $\quad f_s(x) = \int_0^{\pi} x \sin nx \, dx$

$$= \left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos nx}{n}\,dx$$

$$= -\frac{\pi\cos n\pi}{n} + 0 + \frac{1}{n^2}\left[\sin nx\right]_0^{\pi}$$

$$= -\frac{\pi\cos n\pi}{n}$$

Taking inverse finite Fourier sine transform we get

$$x = \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{\pi\cos n\pi}{n}\sin nx$$

Therefore, $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx + \frac{2}{\pi}\sum_{n=1}^{\infty} -\frac{2\cos n\pi}{n}\sin nx$$

$$= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx + \frac{2}{\pi}\cdot\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{-\pi\cos n\pi}{n}\sin nx$$

$$= 1 + \frac{2}{\pi}\cdot x$$

Thus $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-3\cos n\pi}{n}\sin nx = 1 + \frac{2x}{\pi}$

Hence from (5), we have

$$U(x, t) = 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n}e^{-n^2t}\sin nx$$

$$= 1 + \frac{2x}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2t}\sin nx.$$