Lecture- 05

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Topics Includes:

1. Singularities
2. Poles and Residues
3. Cauchy's Residue theorem
4. Contour Integrations.

Singularity of Analytic Functions:

Question Define singularity or singular points of an analytic function with examples.

Ans. **Defn 1:** If a function $f(z)$ fails to be analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called a singular point of $f(z)$.

Defn 2: If a function $f(z)$ is analytic at all points of a bounded domain except at a finite number of points, then these exceptional points are called singular points or singularities of $f(z)$.

Examples: If $f(z) = \frac{1}{z}$, then $f(z)$ is analytic except at $z=0$. So $z=0$ is a singular point of $f(z) = \frac{1}{z}$. Similarly $f(z) = \frac{1}{z-1}$ has a singular point at $z=1$.

Question: Define poles with examples.

Ans. **Defn 1:** If there exists a positive integer n such that $\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0$ then $f(z)$ is said to have a pole of order n at $z=z_0$. If $n=1$, then z_0 is called simple pole or pole of order one, whereas if $n=2$, then z_0 is called pole of order 2 or double pole.

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Def^n-2: Suppose $f(z)$ has a Laurent's series as $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ where $a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$ for $n=0, \pm 1, \pm 2, \cdots$

Now if the principal part of the Laurent's series of $f(z)$ consists of a finite number of term such as

$$\frac{a_{-1}}{z-z_0} + \frac{a_{-2}}{(z-z_0)^2} + \cdots + \frac{a_{-n}}{(z-z_0)^n}, \text{ where } a_{-n} \neq 0$$

then $f(z)$ is said to have a pole of order n at $z = z_0$.

Note that the pole is one kind of singular point or singularity.

Examples: ① If $f(z) = \frac{1}{(z-2)^3}$, then

$$\lim_{z\to 2}\left\{(z-2)^3 . f(z)\right\} = \lim_{z\to 2}\left\{(z-2)^3 . \frac{1}{(z-2)^3}\right\} = \lim_{z\to 2} 1 = 1 \neq 0$$

Hence $f(z)$ has a pole of order 3 or a triple pole, at $z=2$.

② If $f(z) = \frac{3z-2}{(z-1)^2 (z+1)(z-4)}$, then $f(z)$ has a pole of order 2 at $z=1$ and simple poles at $z=-1$ and $z=4$.

Question: What is meant by residue of a complex function f(z) and how do you calculate or how to find the residue of a complex function at a finite pole.

Ans: The residue of a complex function at a given point is the coefficient of the term with a negative power in the Laurent series expansion of the function around that point.

Suppose the function f(z) be analytic except the point z=a and f(z) has a Laurent series as $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-a)^n$

$$\Rightarrow f(z) = a_0 + a_1(z-a) + \cdots + \frac{a_{-1}}{z-a} + \frac{a_{-2}}{(z-a)^2} + \cdots \longrightarrow (1)$$

Where $a_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}} dz, \quad n = 0, \pm 1, \pm 2, \cdots \longrightarrow (2)$

Now if $n = -1$, then from (2) we get

$$a_{-1} = \frac{1}{2\pi i} \oint_C f(z)\, dz$$ which is called the residue of f(z) at z=a (singular point or pole)

i.e. the coefficient of $\frac{1}{z-a}$ in the Laurent series.

Note that the residue plays an crucial role in calculating the complex integrals of f(z) which is singularities by using Cauchy's Residue theorem.

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Calculation of residue at a simple pole:

* To find the residue of a complex function for simple pole, we can use the following formula: $\operatorname{Res}[f(z), a] = \lim_{z \to a} \{(z-a) \cdot f(z)\}$ --- (1)

* To calculate the residue of a complex function for pole of order m at $z=a$, we can use the following formula:

$$\operatorname{Res}[f(z), a] = \lim_{z \to a} \left[\frac{1}{\lfloor m-1} \frac{d^{m-1}}{dz^{m-1}} \{(z-a)^m f(z)\} \right].$$ ---- (2)

Example-1: Find the poles and residues of the function defined by $f(z) = \frac{1}{z^2(z-1)}$.

Solution: Given that $f(z) = \frac{1}{z^2(z-1)}$.

Now the poles of $f(z)$ are given by

$z^2(z-1) = 0 \Rightarrow z = 0, 0, 1.$

Then $f(z)$ has a double pole at $z=0$ and a simple pole at $z=1$.

Now $\operatorname{Res}[f(z), 0] = \lim_{z \to 0} \frac{1}{\lfloor 2-1} \cdot \frac{d}{dz} \left\{ (z-0)^2 \cdot \frac{1}{z^2(z-1)} \right\}$

$= \lim_{z \to 0} \frac{d}{dz} \left(\frac{1}{z-1} \right) = \lim_{z \to 0} \frac{-1}{(z-1)^2} = -1$

& $\operatorname{Res}[f(z), 1] = \lim_{z \to 1} \{(z-1) \cdot f(z)\} = \lim_{z \to 1} \left\{ (z-1) \cdot \frac{1}{z^2(z-1)} \right\}$

$= \lim_{z \to 1} \frac{1}{z^2} = 1$

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Exercise: For each of the following complex functions determine the poles and the residues at the poles:

(a) $\dfrac{1+2z}{z^2-z-2}$, Ans: Poles are $z=2,-1$ and res $\frac{5}{3}$, $\frac{1}{3}$

(b) $\dfrac{z^2+2}{z-1}$ Ans, Poles is $z=1$ & res is 3

(c) $\dfrac{z^2-2z}{(z+1)^2(z+4)}$ Ans Poles are $z=-1,-1$ & $z=-4$ & res are $-5/3$, $8/3$

(d) $\dfrac{z^2-2z}{(z+1)^2(z^2+4)}$ Ans. Poles are $z=-1,-1$, $z=-2i, 2i$ Residues are $-\frac{14}{25}$, $\frac{7-i}{25}$

(e) $\dfrac{z^2}{(z-1)^3(z-2)}$

Question: State Cauchy's Residue theorem.

Statement: If $f(z)$ is single-valued and analytic inside and on a simple closed curve C except at the singularities $a, b, c, \cdots$ inside C which have residues given by $a_{-1}, b_{-1}, c_{-1}, \cdots$ then the residue theorem states that

$$\oint_C f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\cdots) \quad \text{--- (1)}$$
$$= 2\pi i \times \sum \text{Residues}$$

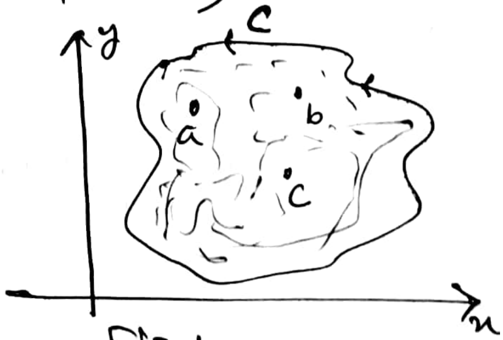

Fig.1

* Note that the residue theorem is a general form of Cauchy's theorem.

* Evaluation of Integrals using Cauchy's Residue theorem:

$\boxed{\text{Example-1:}}$ Using Cauchy's residue theorem to evaluate the following integral:

$$\oint_C \frac{5z-2}{z^2-z}\,dz, \text{ where } C: |z|=2.$$

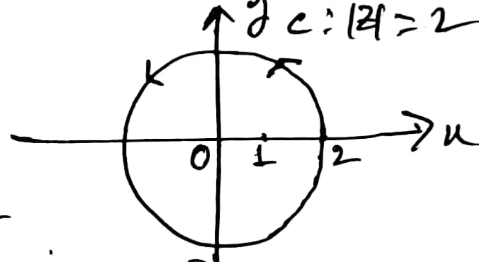

Fig-1

$\boxed{\text{Solution:}}$ Let $f(z) = \dfrac{5z-2}{z^2-z}$

Also given circle is $C: |z|=2$

The poles of $f(z)$ are given by

$$z^2-z=0 \Rightarrow z(z-1)=0 \Rightarrow z=0,1.$$

Thus $f(z)$ has two simple poles at $z=0$ & $z=1$ which lie inside C.

Now
$$\operatorname*{Res}_{z\to 0} f(z) = \lim_{z\to 0}\left\{(z-0)\cdot f(z)\right\}$$
$$= \lim_{z\to 0}\left\{z\cdot\frac{5z-2}{z(z-1)}\right\} = \lim_{z\to 0}\frac{5z-2}{z-1} = 2$$

&
$$\operatorname*{Res}_{z\to 1} f(z) = \lim_{z\to 1}\left\{(z-1)\,f(z)\right\} = \lim_{z\to 1}\left\{(z-1)\cdot\frac{5z-2}{z(z-1)}\right\}$$
$$= \lim_{z\to 1}\frac{5z-2}{z} = 3$$

Thus by Cauchy's residue theorem, we have

$$\oint_C f(z)\,dz = 2\pi i\cdot\sum \text{Residue} = 2\pi i\times(2+3) = 10\pi i$$

i.e.
$$\oint_C f(z)\,dz = \oint_C \frac{5z-2}{z^2-z}\,dz = 10\pi i$$

Ans.

Exercise-1 Evaluate the following integrals using Cauchy's residue theorem:

(a) $\oint_C \frac{z^2 - z + 1}{(z-1)(z-4)(z+3)}\, dz$; $C: |z| = 5$ Ans. $2\pi i$

(b) $\oint_C \frac{1}{z^3(z+4)}\, dz$; $C: |z| = 2$, Ans. $\frac{\pi i}{32}$

(c) $\oint_C \frac{3z^3+2}{(z-1)(z^2+9)}\, dz$; $C: |z-2| = 2$, Ans. πi

(d) $\oint \frac{z\, dz}{(z^2+1)(z-3)^2}$; $C: |z| = 2$ Ans. $\frac{8\pi i}{25}$

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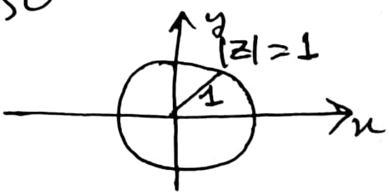
Contour Integration:

We want to evaluate the value of the definite integrals using Cauchy's residue theorem. For this, we shall choose a contour which may be a circle, semicircle etc. The process of integration along a contour is known as contour integration.

$\boxed{\text{Example-1}}$ Evaluate the following integral by using contour integration [using a unit circle as a contour]: $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.

Solution: Let $\displaystyle I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$.



We know that the equation of unit circle is $|z|=1 \Rightarrow z = e^{i\theta} \Rightarrow z = \cos\theta + i\sin\theta$

$$\Rightarrow z^{-1} = \frac{1}{z} = \cos\theta - i\sin\theta$$

$$\therefore z + \frac{1}{z} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$$

Also we have $z = e^{i\theta}$

$$\Rightarrow dz = i e^{i\theta} d\theta$$

$$\Rightarrow dz = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Then $\displaystyle I = \oint_c \frac{dz/iz}{5+4\left(\frac{z^2+1}{2z}\right)}$ where $c: |z|=1$.

$$\Rightarrow I = \frac{1}{i}\oint_c \frac{dz}{2z^2+5z+2}$$

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$\Rightarrow I = \frac{1}{i}\oint_C f(z)\,dz$, where $f(z) = \frac{1}{2z^2+5z+2}$.

$\Rightarrow \boxed{I = \frac{1}{i} \times 2\pi i \sum \text{Residue}}$ - - - - (1)

To obtain residues of $f(z)$, we 1st find the poles of $f(z)$ as follows;

And the poles of $f(z)$ are obtained by solving

$$2z^2+5z+2=0$$

$$\Rightarrow 2z^2+4z+z+2=0$$

$$\Rightarrow (2z+1)(z+2)=0 \Rightarrow z=-\frac{1}{2},\ z=-2$$

which are simple poles. But only the pole $z=-\frac{1}{2}$ lies inside the contour C.

$$\therefore \operatorname{Res}_{z\to-\frac{1}{2}} f(z) = \lim_{z\to-\frac{1}{2}} \left\{ z-\left(-\frac{1}{2}\right) . f(z) \right\}$$

$$= \lim_{z\to-\frac{1}{2}} \left\{ \frac{\cancel{2z+1}}{2} \cdot \frac{1}{\cancel{(2z+1)}(z+2)} \right\}$$

$$= \lim_{z\to-\frac{1}{2}} \left\{ \frac{1}{2(z+2)} \right\} = \frac{1}{2\left(-\frac{1}{2}+2\right)} = \frac{1}{3}$$

$$\therefore \sum \text{Residues} = \frac{1}{3}$$

Hence from (1) we get

$$I = \frac{1}{i} 2\pi i \cdot \frac{1}{3} = \frac{2\pi}{3}$$

i.e. $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}$ Ans.

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$\boxed{\text{Exercise-1}}$ Evaluate the following integrals by using contour integration:

(a) $$\int_0^{2\pi} \frac{d\theta}{5+3\sin\theta}$$

(b) $$\int_0^{2\pi} \frac{\sin 2\theta}{5-3\cos\theta}\, d\theta$$

(c) $$\int_0^{2\pi} \frac{\cos 3\theta}{5-4\cos\theta}\, d\theta$$

(d) $$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta$$

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