ME 3101: Mechanics of Machinery

Longitudinal and Transverse Vibration

VIBRATION

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Terms Used in Vibratory Motion

1. *Period of vibration or time period.* It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. *Cycle.* It is the motion completed during one time period.

3. *Frequency.* It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

Types of Vibratory Motion

1. Free or natural vibrations.

When no external force acts on the body, after giving it an initial displacement, then the body is said to be under ***free or natural vibrations.*** The frequency of the free vibrations is called ***free or natural frequency.***

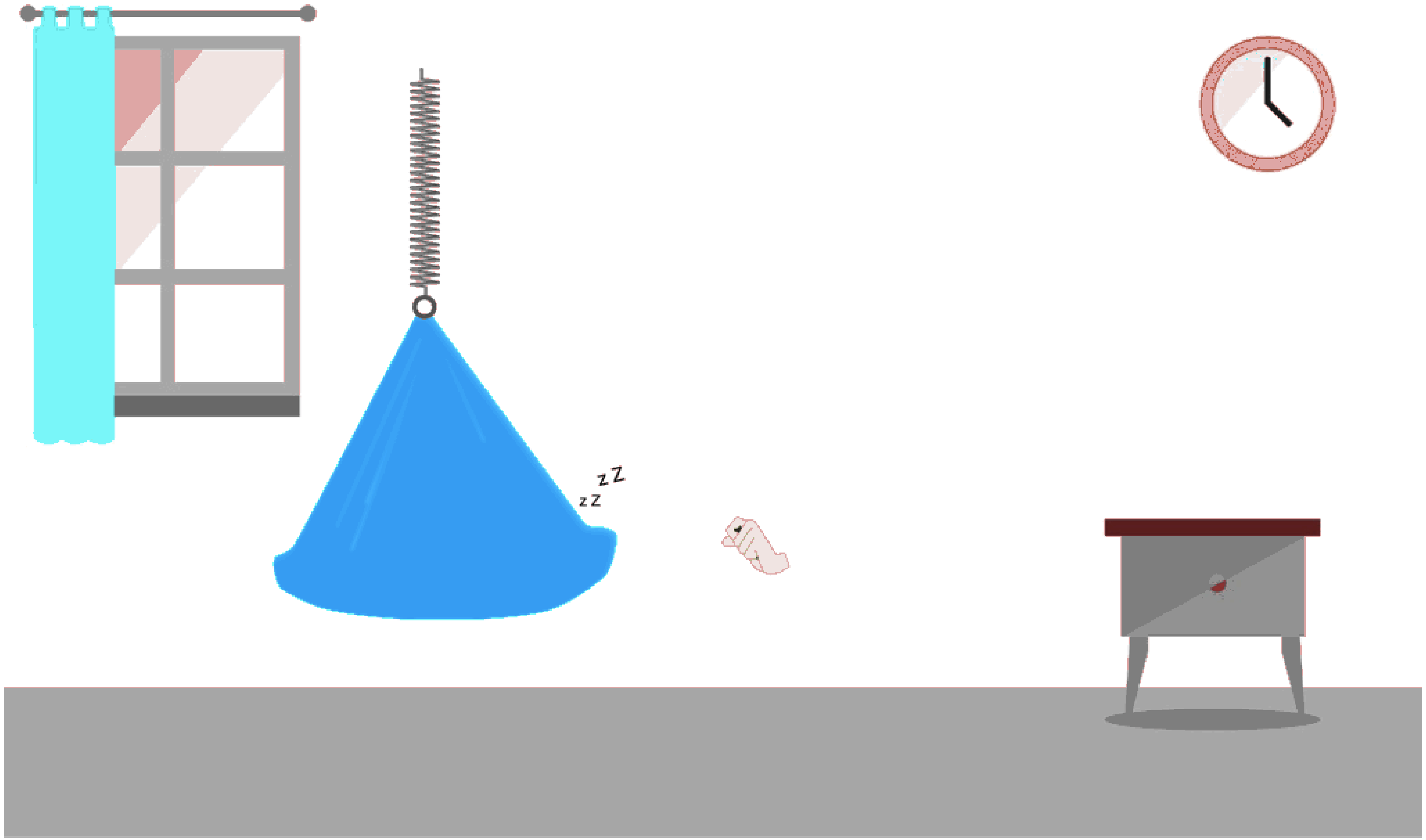
2. Forced vibrations.

When the body vibrates under the influence of external force, then the body is said to be under ***forced vibrations.*** The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

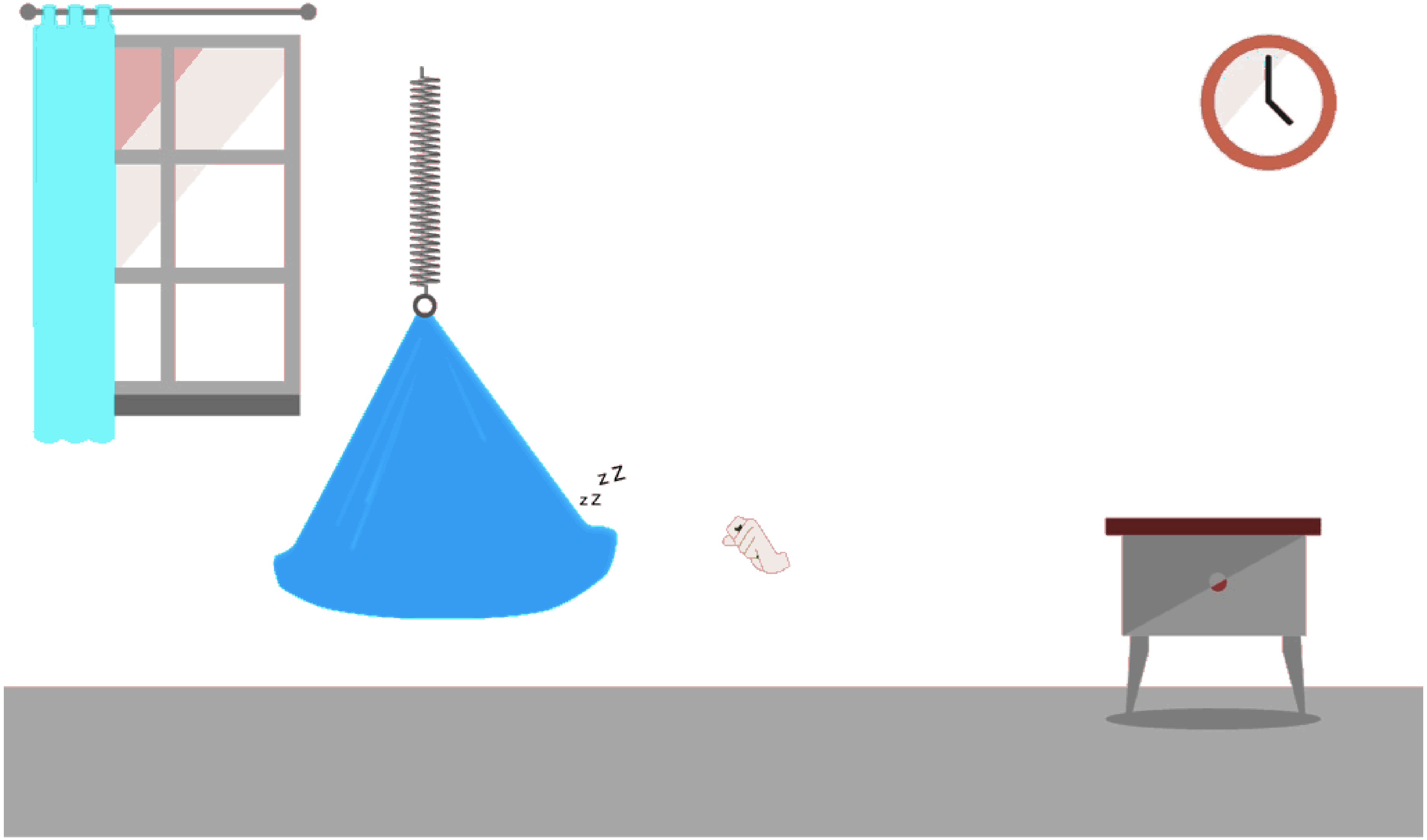
Note : When the frequency of the external force is same as that of the natural vibrations, resonance takes place.

3. Damped vibrations. When there is a reduction in amplitude over every cycle of vibration, the motion is said to be ***damped vibration.*** This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

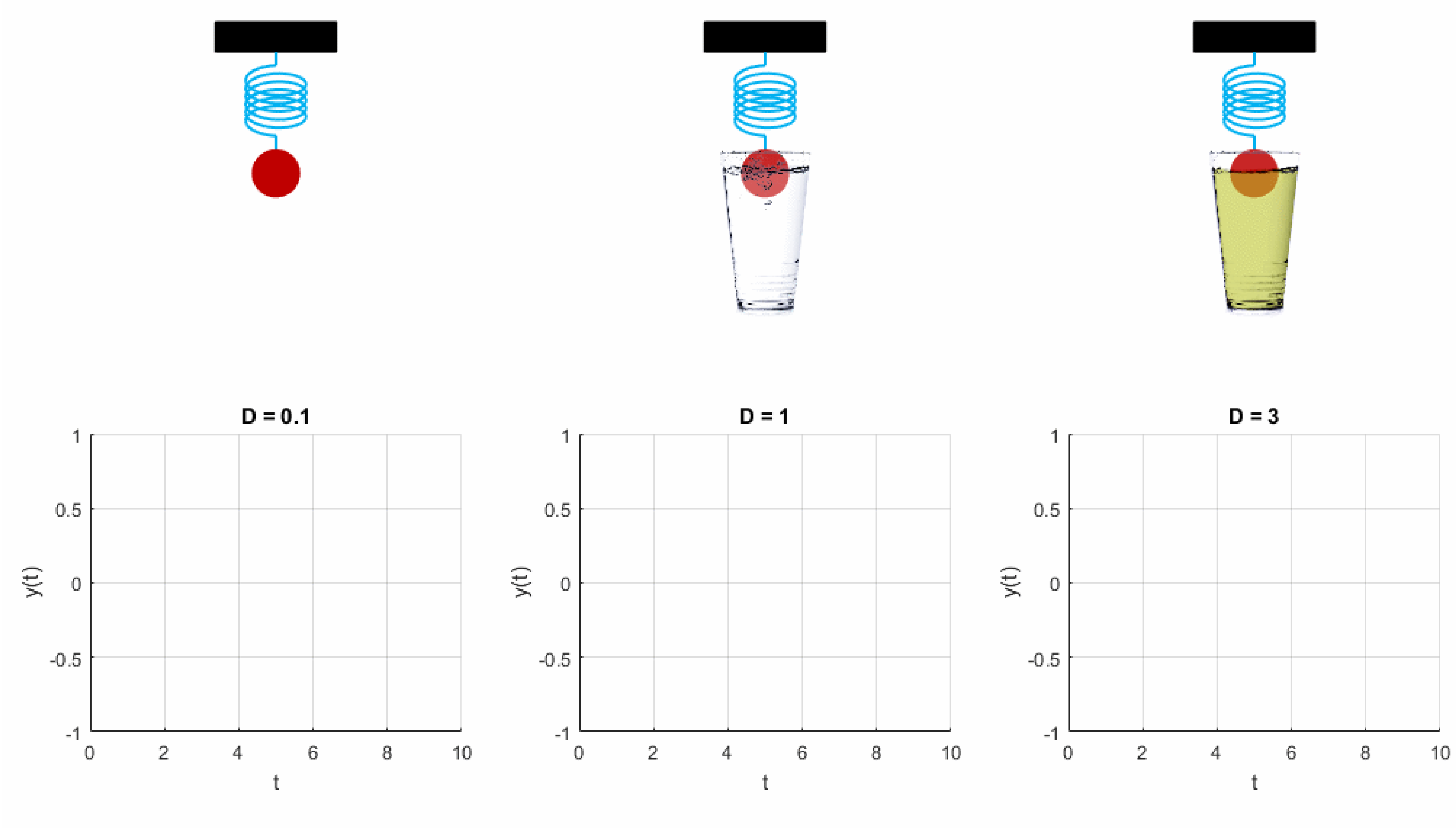
Free or Natural Vibration



Forced Vibration



Damped Vibration



Under Damped

Critically Damped

Over Damped

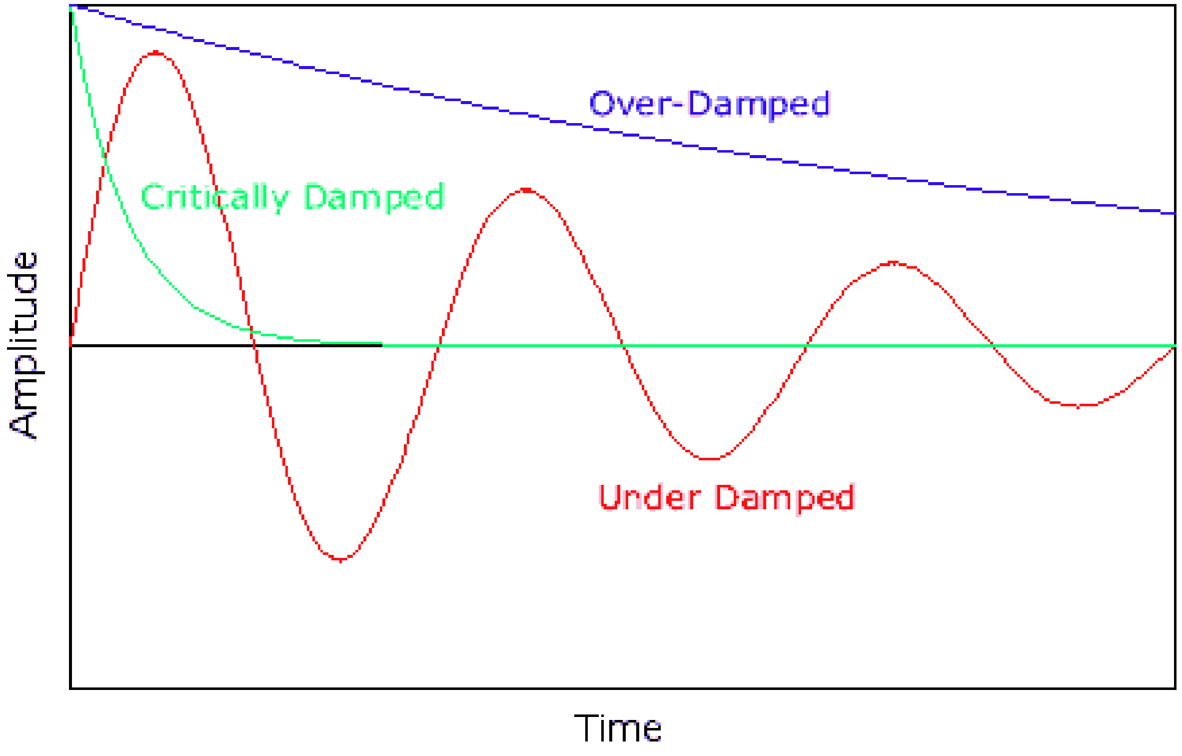
Damping Ratio/Factor (D)
dimensionless measure of damping in a system, indicating how oscillations decay over time compared to critical damping

Types of Damped Vibration

A. **Underdamped Vibration (D < 1) :** A system oscillates with gradually decreasing amplitude because the damping is not strong enough to stop the oscillations completely. **Example:** A car's suspension bouncing a few times after hitting a speed bump before settling.

B. **Critically Damped Vibration (D = 1):** A system returns to equilibrium as quickly as possible without oscillating, providing the optimal level of damping. **Example:** A door closer that shuts the door smoothly without slamming or oscillating.

C. **Overdamped Vibration (D > 1):** A system returns to equilibrium very slowly due to excessive damping, without any oscillation.
Example: A heavy door with excessive hydraulic damping closes very slowly, taking a long time to shut.



Types of Free Vibrations

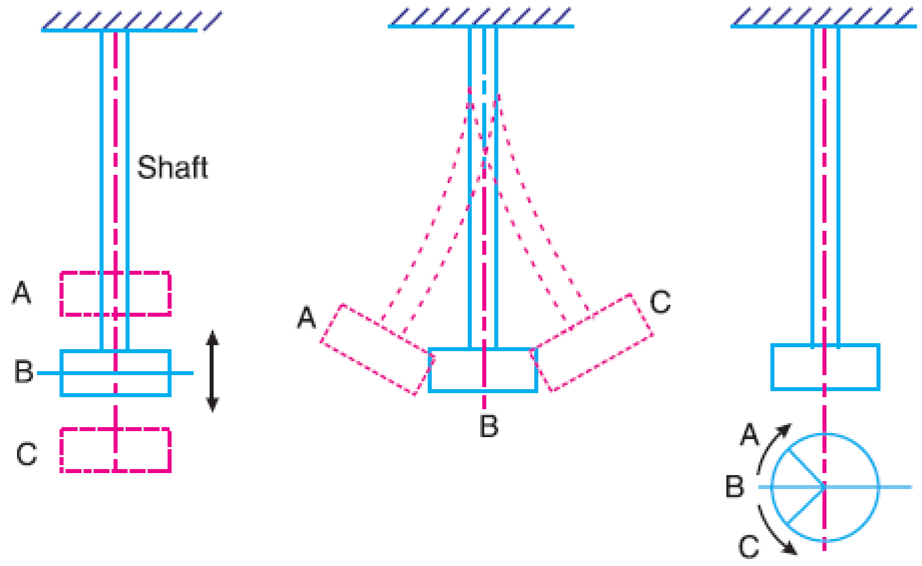
Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig. This system may execute one of the three above mentioned types of vibrations.

1. *Longitudinal vibrations*. When the particles of the shaft or disc moves parallel to the axis of the shaft.

In this case, the shaft is **elongated** and **shortened** alternately and thus the **tensile** and **compressive** stresses are induced **alternately** in the shaft.

2. *Transverse vibrations*. When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft

In this case, the shaft is **straight** and **bent alternately** and **bending stresses** are induced in the shaft.



B = Mean position ; A and C = Extreme positions.

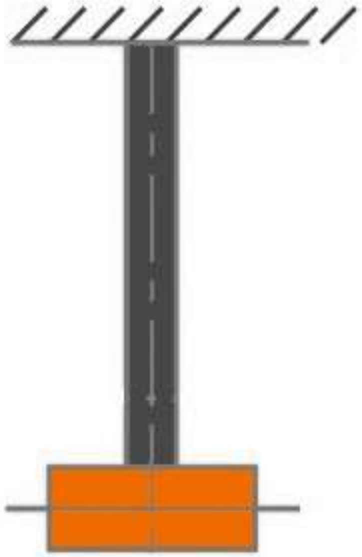
(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.

3. *Torsional vibrations*. When the particles of the shaft or disc move in a circle about the axis of the shaft

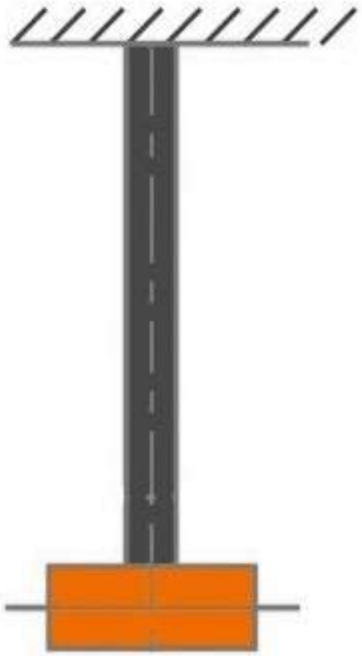
In this case, the shaft is **twisted** and **untwisted alternately** and the **torsional shear** stresses are induced in the shaft.

If the limit of proportionality (*i.e.* stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, *simple harmonic*.

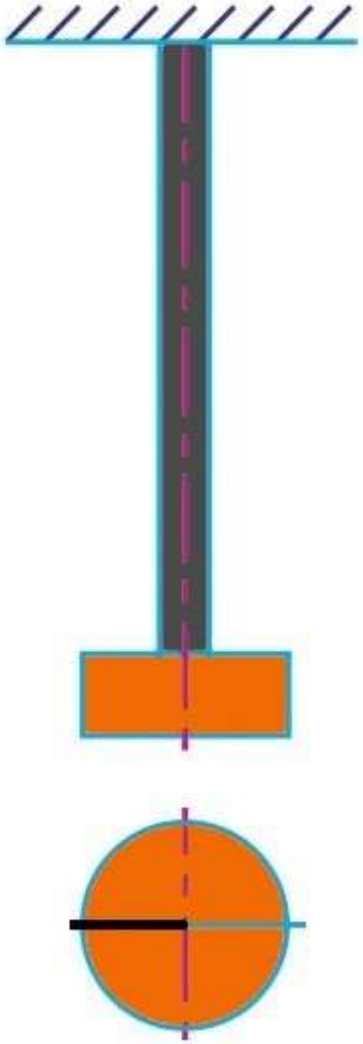
Types of Free Vibrations



Longitudinal vibrations



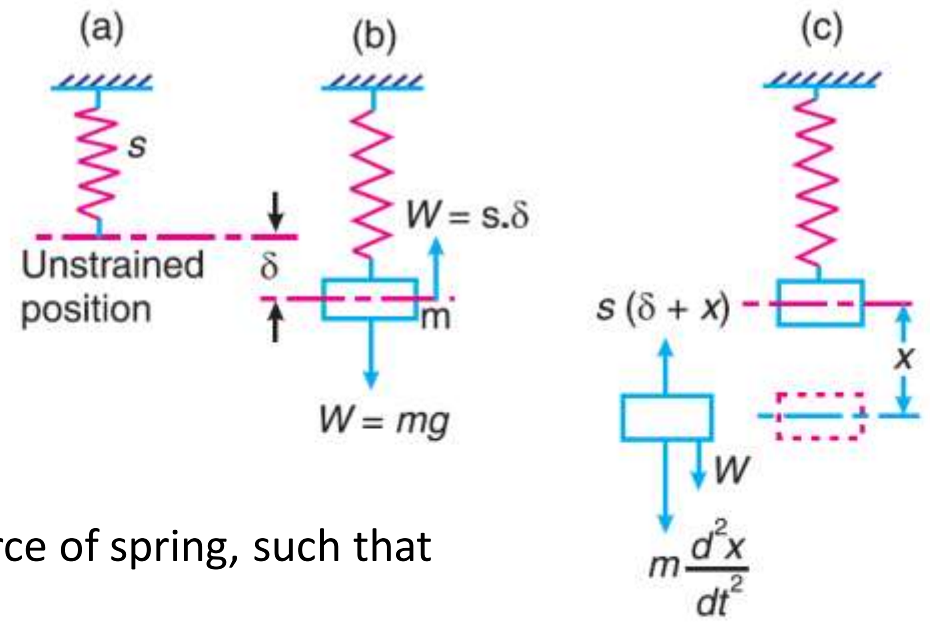
Transverse vibrations



Torsional vibrations

Natural Frequency of Free Longitudinal Vibrations

1. Equilibrium Method: Determines natural frequency by equating static restoring forces to dynamic inertial forces



s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.
m = Mass of the body suspended from the constraint in kg,
W = Weight of the body in newtons = $m.g$,
δ = Static deflection of the spring in metres due to weight W newtons, and
x = Displacement given to the body by the external force, in metres.

In the equilibrium position, the gravitational pull W = m.g, is balanced by a force of spring, such that W = s. δ .

Since the mass is now displaced from its equilibrium position by a distance x, and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x = s.\delta - s.\delta - s.x = -s.x$ . . . $(\because W = s.\delta)$ . . . (*i*)

and Accelerating force = Mass × Acceleration

$$= m \times \frac{d^2x}{dt^2} \quad \text{. . . (Taking downward force as positive) . . . (ii)}$$

Equating equations (*i*) and (*ii*), the equation of motion of the body of mass m after time t is

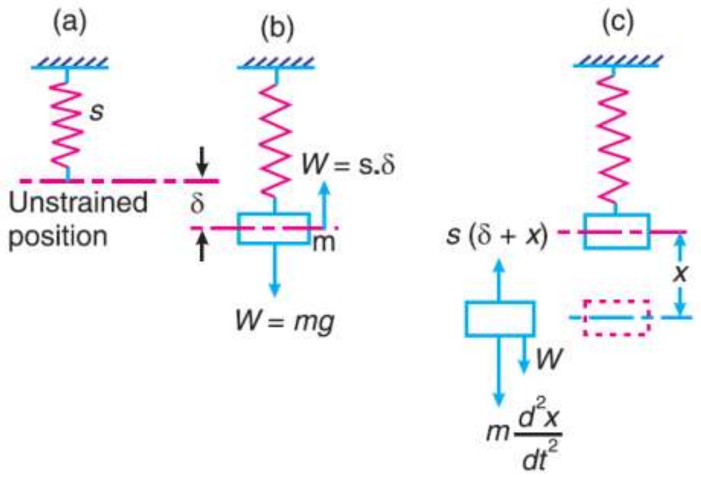
$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

Natural Frequency of Free Longitudinal Vibrations

$$\therefore \qquad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \ldots (iii)$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2 . x = 0 \qquad \ldots (iv)$$



Fundamental Differential Equation:

$$\frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0$$

$$\frac{d^2x}{dt^2} + \omega^2 . x = 0$$

$$\omega = \sqrt{\frac{s}{m}}$$

$$\therefore \quad \text{Time period,} \qquad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

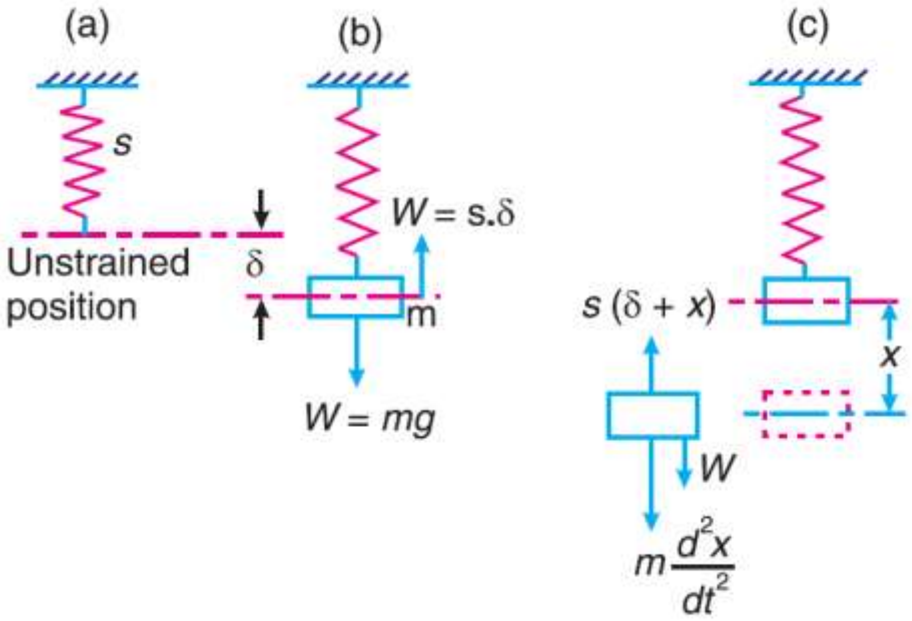
$$\text{natural frequency,} \qquad f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\text{Hz}$$

Finding Static Deflection

The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

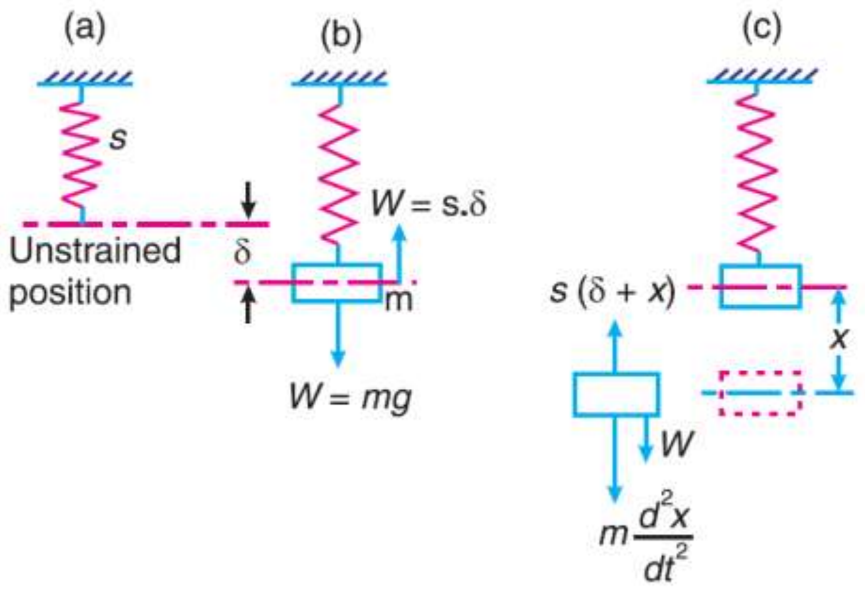


δ = Static deflection *i.e.* extension or compression of the constraint,
W = Load attached to the free end of constraint,
l = Length of the constraint,
E = Young's modulus for the constraint, and
A = Cross-sectional area of the constraint.

Natural Frequency of Free Longitudinal Vibrations

2. Energy method: Finds natural frequency by equating the total kinetic and potential energy over a complete cycle.

3. Rayleigh's method: In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position.



- The time period and the natural frequency may be obtained similarly as discussed before

$$\omega = \sqrt{\frac{s}{m}}$$

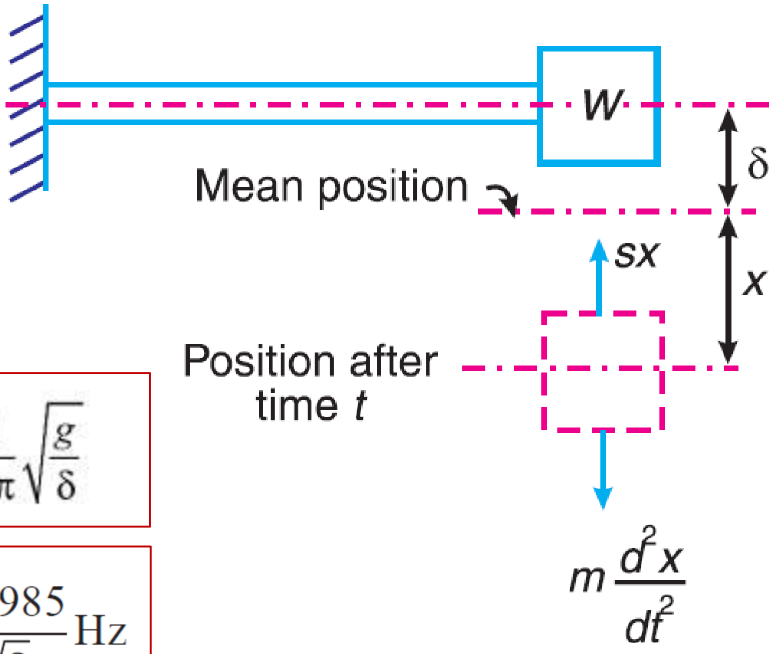
$$\therefore \quad \text{Time period,} \quad t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

$$\text{natural frequency,} \quad f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{s}{m}}$$

- In all the above expressions, ω is known as **natural circular frequency** and is generally denoted by ω_n

Natural Frequency of Free Transverse Vibrations

s = Stiffness of shaft,
δ = Static deflection due to weight of the body,
x = Displacement of body from mean position after time t.
m = Mass of body = W/g



- The time period and the natural frequency may be obtained similarly as discussed before

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $$t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$$

natural frequency, $$f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\,\text{Hz}$$

The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on **Strength of Materials**, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \quad \text{(in metres)}$$

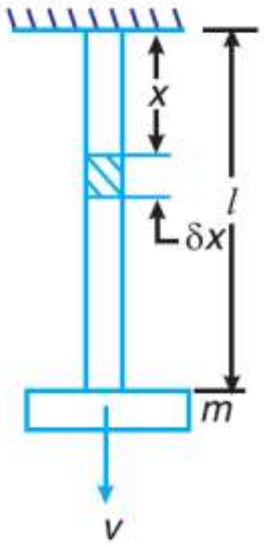
Where,
W = Load at the free end, in newtons,
l = Length of the shaft or beam in metres,
E = Young's modulus for the material of the shaft or beam in N/m^2, and
I = Moment of inertia of the shaft or beam in m^4.

Effect of Inertia of the Constraint/Shaft in Longitudinal and Transverse Vibrations

1. *Longitudinal vibration*

m_1 = Mass of the constraint/shaft per unit length,
l = Length of the constraint,
m_C = Total mass of the constraint = $m_1 . l$, and
v = Longitudinal velocity of the free end.

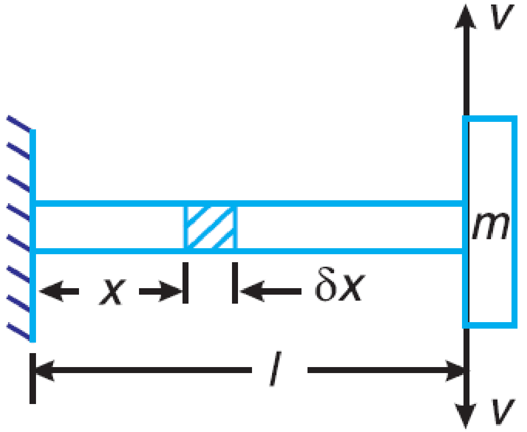
$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{m_C}{3}}}$$



2. *Transverse vibration*

m_1 = Mass of constraint per unit length,
l = Length of the constraint,
m_C = Total mass of the constraint = $m_1 . l$, and
v = Transverse velocity of the free end.

$$f_n = \frac{1}{2\pi}\sqrt{\frac{s}{m + \frac{33\, m_C}{140}}}$$

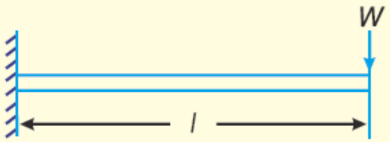
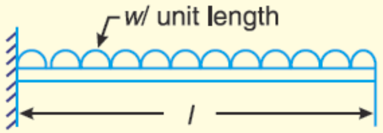
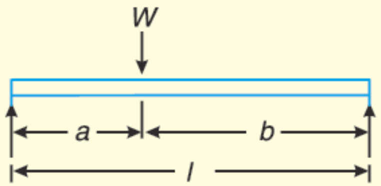
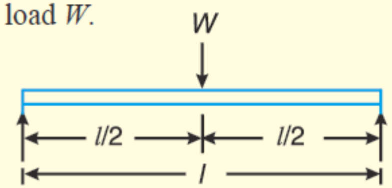


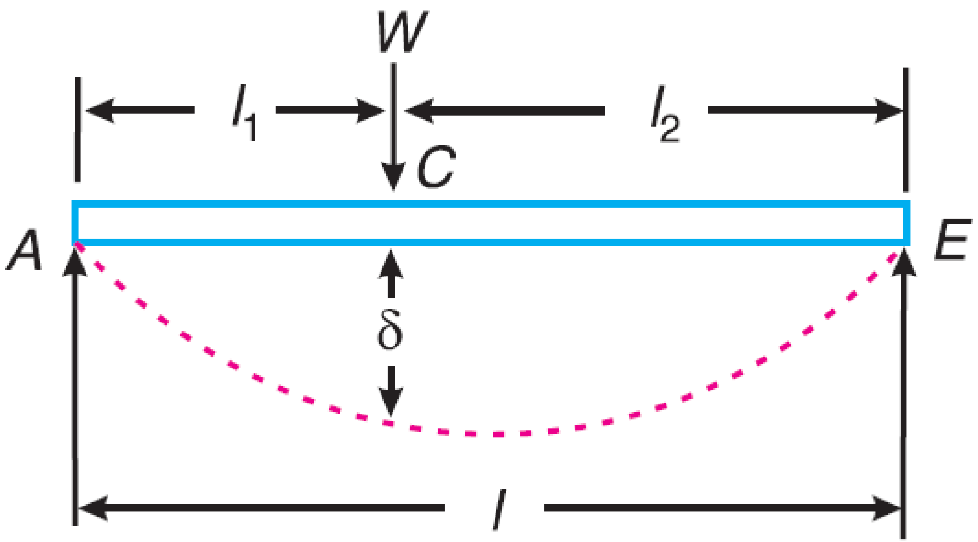
1. If both the ends of the constraint are fixed, and the disc is situated in the middle of it, then proceeding in the similar way as discussed above, we may prove that the inertia of the constraint may be allowed for by adding 13/35 of its mass to the disc.

2. If the constraint is like a simply supported beam, then 17/35 of its mass may be added to the mass of the disc.

f_n of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Table 1. Values of static deflection (δ) for the various types of beams and under various load conditions

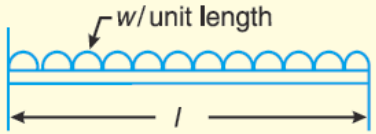
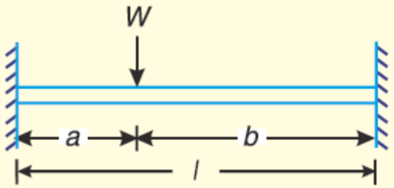
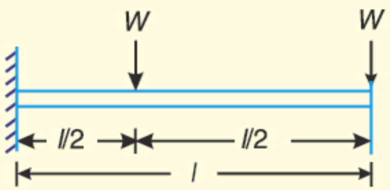
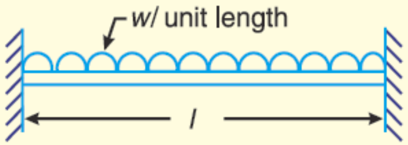
S.No.	Type of beam	Deflection (δ)
1.	Cantilever beam with a point load W at the free end. 	$\delta = \frac{Wl^3}{3EI}$ (at the free end)
2.	Cantilever beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{8EI}$ (at the free end)
3.	Simply supported beam with an eccentric point load W. 	$\delta = \frac{Wa^2b^2}{3EIl}$ (at the point load)
4.	Simply supported beam with a central point load W. 	$\delta = \frac{Wl^3}{48EI}$ (at the centre)

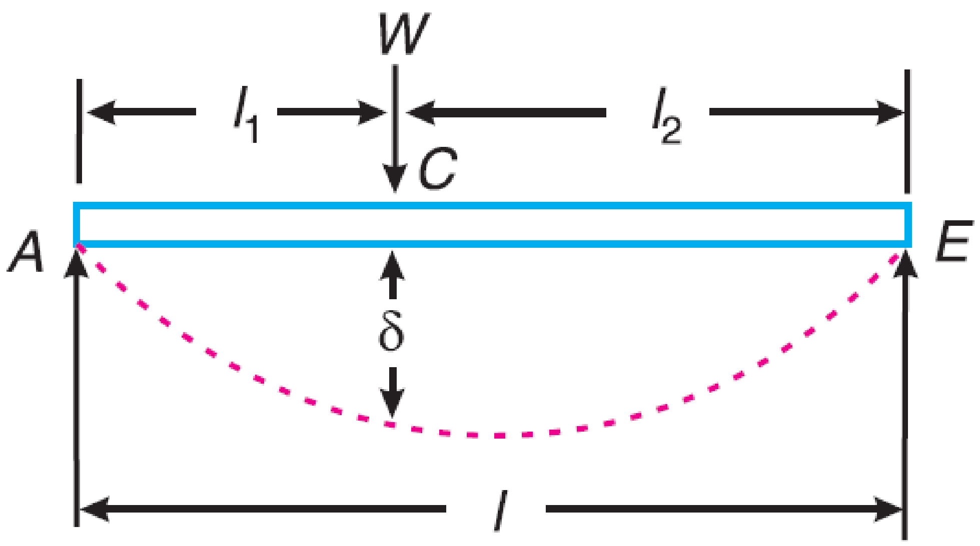


$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\,\text{Hz}$$

f_n of Free Transverse Vibrations Due to a Point Load Acting Over a Simply Supported Shaft

Table 1. Values of static deflection (δ) for the various types of beams and under various load conditions

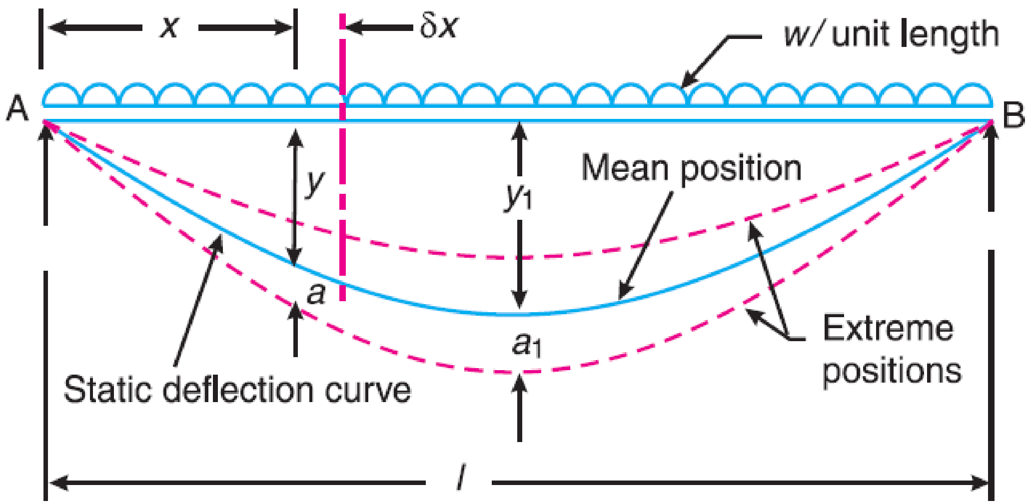
S.No.	*Type of beam*	*Deflection (δ)*
5.	Simply supported beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{5}{384} \times \frac{wl^4}{EI}$ (at the centre)
6.	Fixed beam with an eccentric point load W. 	$\delta = \frac{Wa^3b^3}{3EIl}$ (at the point load)
7.	Fixed beam with a central point load W. 	$\delta = \frac{Wl^3}{192EI}$ (at the centre)
8.	Fixed beam with a uniformly distributed load of w per unit length. 	$\delta = \frac{wl^4}{384EI}$ (at the centre)



$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ Hz}$$

f_n of Free Transverse Vibrations Due to Uniformly Distributed Load Acting Over a Simply Supported Shaft

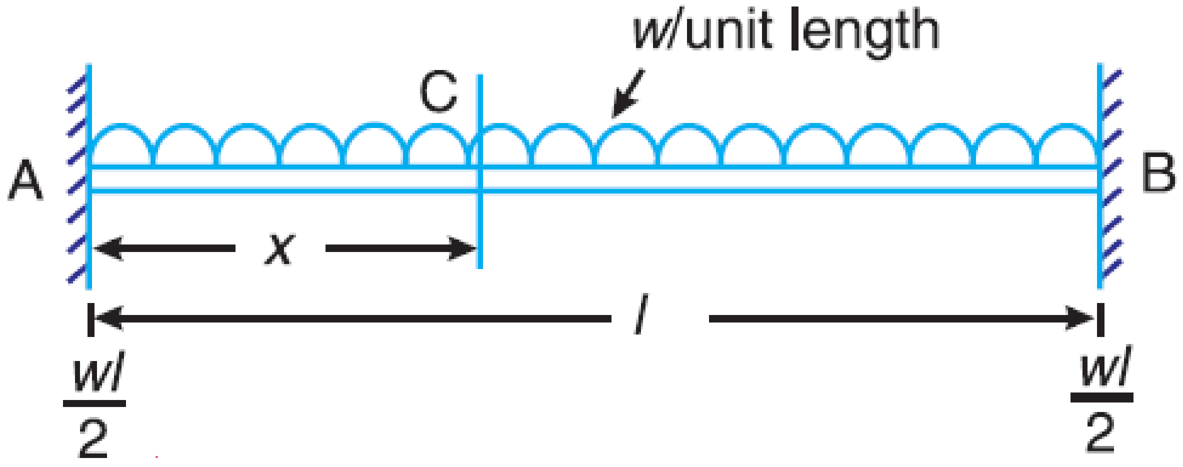
y_1 = Static deflection at the middle of the shaft,
a_1= Amplitude of vibration at the middle of the shaft, and
w_1 = Uniformly distributed load per unit static deflection at the middle of the shaft = w/y_1.
δ_s =static deflection of a simply supported shaft due to uniformly distributed load of w per unit length



$$f_n = \frac{\pi}{2}\sqrt{\frac{5\,g}{384\,\delta_S}} = \frac{0.5615}{\sqrt{\delta_S}} \ \text{Hz}$$

f_n of Free Transverse Vibrations of a Shaft Fixed at Both Ends Carrying a Uniformly Distributed Load

- Consider a shaft *AB* fixed at both ends and carrying a uniformly distributed load of *w* per unit length.



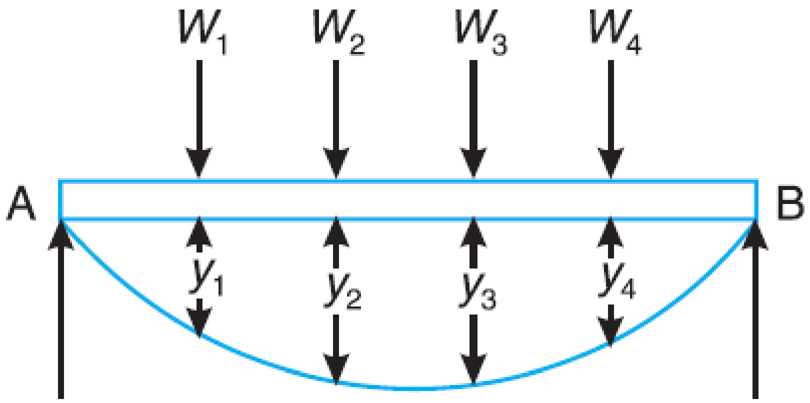
$$f_n = 3.573\sqrt{\frac{g}{384\,\delta_S}} = \frac{0.571}{\sqrt{\delta_S}} \text{ Hz}$$

f_n of Free Transverse Vibrations For a Shaft Subjected to a Number of Point Loads

1. *Energy (or Rayleigh's) method*

Natural frequency of transverse vibration,

$$f_n = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{g\Sigma m.y}{\Sigma m.y^2}}$$



2. *Dunkerley's method*

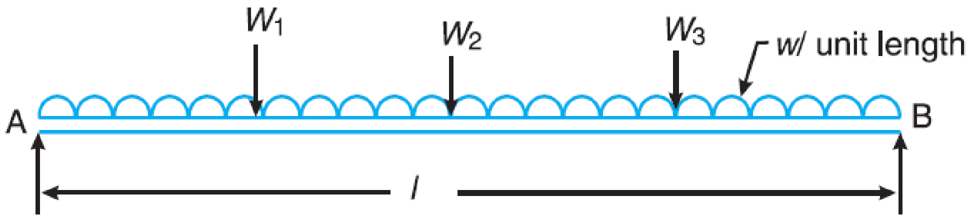
The natural frequency of transverse vibration for a shaft carrying a number of point loads and uniformly distributed load is obtained from Dunkerley's empirical formula. According to this

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^2} + \ldots + \frac{1}{(f_{ns})^2}$$

f_n = Natural frequency of transverse vibration of the shaft carrying point loads and uniformly distributed load
f_{n1}, f_{n2}, f_{n3}, etc. = Natural frequency of transverse vibration of each point load
f_{ns} = Natural frequency of transverse vibration of the uniformly distributed load (or due to the mass of the shaft)

Dunkerley’s Method (cntd.)

Now, consider a shaft *AB* loaded as shown.



Let $\delta_1, \delta_2, \delta_3$, etc. = Static deflection due to the load W_1, W_2, W_3 etc. when considered separately.

δ_S = Static deflection due to the uniformly distributed load or due to the mass of the shaft.

We know that natural frequency of transverse vibration due to load W_1,

$$f_{n_1} = \frac{0.4985}{\sqrt{\delta_1}} \text{ Hz}$$

Similarly, natural frequency of transverse vibration due to load W_2,

$$f_{n_2} = \frac{0.4985}{\sqrt{\delta_2}} \text{ Hz}$$

and, natural frequency of transverse vibration due to load W_3,

$$f_{n_3} = \frac{0.4985}{\sqrt{\delta_3}} \text{ Hz}$$

Also natural frequency of transverse vibration due to uniformly distributed load or weight of the shaft,

$$f_{ns} = \frac{0.5615}{\sqrt{\delta_S}} \text{ Hz}$$

Therefore, according to Dunkerley’s empirical formula, the natural frequency of the whole system,

$$\frac{1}{(f_n)^2} = \frac{1}{(f_{n1})^2} + \frac{1}{(f_{n2})^2} + \frac{1}{(f_{n3})^3} + \ldots + \frac{1}{(f_{ns})^2}$$

$$= \frac{\delta_1}{(0.4985)^2} + \frac{\delta_2}{(0.4985)^2} + \frac{\delta_3}{(0.4985)^2} + \ldots + \frac{\delta_S}{(0.5615)^2}$$

$$= \frac{1}{(0.4985)^2}\left[\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}\right]$$

$$\boxed{f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \ldots + \frac{\delta_S}{1.27}}} \text{ Hz}}$$

Dunkerley's Method (cntd.)

Notes : 1. When there is no uniformly distributed load or mass of the shaft is negligible, then $\delta_S = 0$.

$$\therefore \quad f_n = \frac{0.4985}{\sqrt{\delta_1 + \delta_2 + \delta_3 + \ldots}} \text{ Hz}$$

2. The value of δ_1, δ_2, δ_3 etc. for a simply supported shaft may be obtained from the relation

$$\delta = \frac{W a^2 b^2}{3\,EIl}$$

where δ = Static deflection due to load W,

a and b = Distances of the load from the ends,

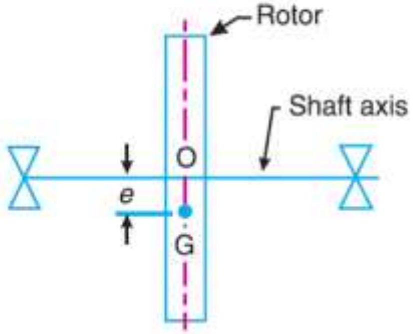
E = Young's modulus for the material of the shaft,

I = Moment of inertia of the shaft, and

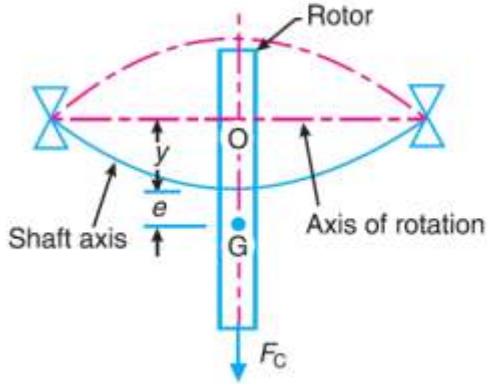
l = Total length of the shaft.

Critical or Whirling Speed of a Shaft

- The speed at which the shaft runs so that the additional deflection of the shaft from the axis of rotation becomes infinite, is known as ***critical or whirling speed***
- **Cause of Whirling:**
  - Mountings like gears or pulleys may have their center of gravity (CG) offset from the shaft's axis of rotation, leading to eccentricity
  - As the shaft rotates, centrifugal force acts on the offset CG, causing bending in the shaft
  - The bending increases the CG's eccentricity, further amplifying the centrifugal force, leading to greater bending in a self-reinforcing cycle
  - This cumulative bending can result in shaft failure if the deflection becomes uncontrollable.
- **Factors Influencing Whirling:**
  - **Eccentricity:** Distance between the CG and the axis of rotation
  - **Speed:** Higher speeds exacerbate the bending effect
- **Critical Speed Condition:**
  - At a specific rotational speed, the additional deflection grows without limit, marking the critical speed of the shaft



(*a*) When shaft is stationary.

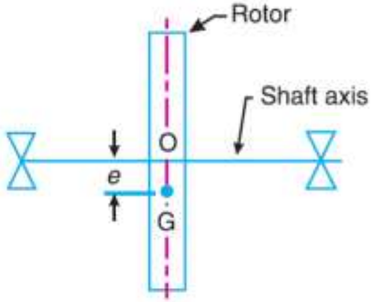


(*b*) When shaft is rotating.

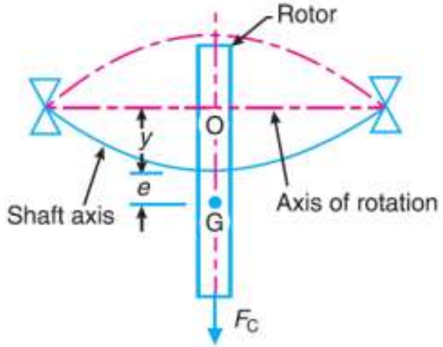
Critical or Whirling Speed of a Shaft

Consider a shaft of negligible mass carrying a rotor, as shown in (*a*). The point *O* is on the shaft axis and *G* is the centre of gravity of the rotor. When the shaft is stationary, the centre line of the bearing and the axis of the shaft coincides. Fig. (*b*) shows the shaft when rotating about the axis of rotation at a uniform speed of ω rad/s.

m = Mass of the rotor,
e = Initial distance of centre of gravity of the rotor from the centre line of the bearing or shaft axis, when the shaft is stationary,
y = Additional deflection of centre of gravity of the rotor when the shaft starts rotating at ω rad/s, and
s = Stiffness of the shaft *i.e.* the load required per unit deflection of the shaft.



(*a*) When shaft is stationary.



(*b*) When shaft is rotating.

Since the shaft is rotating at ω rad/s, therefore centrifugal force acting radially outwards through G causing the shaft to deflect is given by

$$F_C = m.\omega^2 (y+e)$$

The shaft behaves like a spring. Therefore the force resisting the deflection y,

$$= s.y$$

For the equilibrium position,

$$m.\omega^2 (y+e) = s.y$$

or $$m.\omega^2.y + m.\omega^2.e = s.y \quad \text{or} \quad y(s - m.\omega^2) = m.\omega^2.e$$

$$\therefore \qquad y = \frac{m.\omega^2.e}{s - m.\omega^2} = \frac{\omega^2.e}{s/m - \omega^2} \qquad \ldots (i)$$

We know that circular frequency,

$$\omega_n = \sqrt{\frac{s}{m}} \quad \text{or} \quad y = \frac{\omega^2.e}{(\omega_n)^2 - \omega^2} \qquad \ldots \text{[From equation } (i) \text{]}$$

A little consideration will show that when $\omega > \omega_n$, the value of y will be negative and the shaft deflects is the opposite direction as shown dotted in Fig .

In order to have the value of y always positive, both *plus* and *minus* signs are taken.

$$\therefore \qquad y = \pm\frac{\omega^2 e}{(\omega_n)^2 - \omega^2} = \frac{\pm e}{\left(\frac{\omega_n}{\omega}\right)^2 - 1} = \frac{\pm e}{\left(\frac{\omega_c}{\omega}\right)^2 - 1}$$

... (Substituting $\omega_n = \omega_c$)

Critical or Whirling Speed of a Shaft

We see from the above expression that when $\omega_n = \omega_c$, the value of y becomes infinite.

Therefore ω_c is the **critical or whirling speed.**

$\therefore$ Critical or whirling speed,

$$\omega_c = \omega_n = \sqrt{\frac{s}{m}} = \sqrt{\frac{g}{\delta}} \text{ Hz} \quad \ldots \left(\because \delta = \frac{m.g}{s}\right)$$

If N_c is the critical or whirling speed in r.p.s., then

$$2\pi N_c = \sqrt{\frac{g}{\delta}} \quad \text{or} \quad N_c = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}} = \frac{0.4985}{\sqrt{\delta}} \text{ r.p.s.}$$

where δ = Static deflection of the shaft in metres.

Hence the **critical or whirling speed is the same as the natural frequency of transverse vibration but its unit will be revolutions per second.**

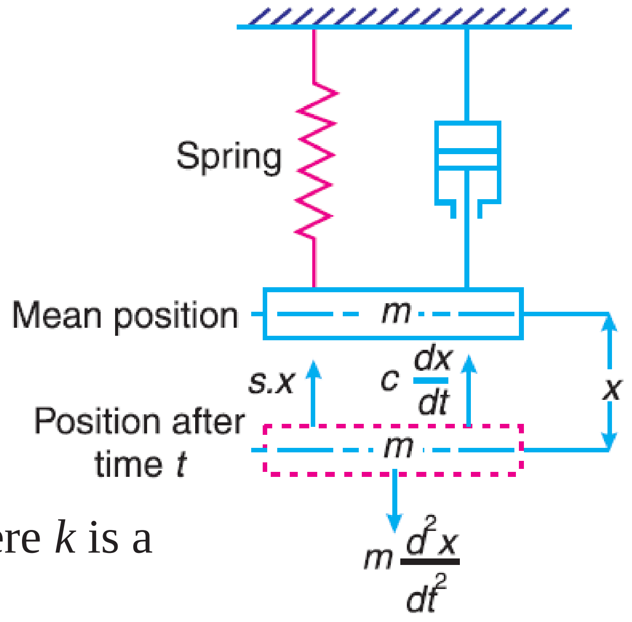
Notes : 1. When the centre of gravity of the rotor lies between the centre line of the shaft and the centre line of the bearing, e is taken negative. On the other hand, if the centre of gravity of the rotor does not lie between the centre line of the shaft and the centre line of the bearing (as in the above article) the value of e is taken positive.

2. To determine the critical speed of a shaft which may be subjected to point loads, uniformly distributed load or combination of both, find the frequency of transverse vibration which is equal to critical speed of a shaft in r.p.s. The Dunkerley's method may be used for calculating the frequency.

3. A shaft supported with short bearings (or ball bearings) is assumed to be a simply supported shaft while the shaft supported in long bearings (or journal bearings) is assumed to have both ends fixed.

Frequency of Free Damped Vibrations (Viscous Damping)

In vibrating systems, the effect of friction is referred to as damping. The damping provided by fluid resistance is known as ***viscous damping*.**



m = Mass suspended from the spring,
s = Stiffness of the spring,
x = Displacement of the mass from the mean position at time t,
δ = Static deflection of the spring = $m.g/s$, and
c = Damping coefficient or the damping force per unit velocity.

Fundamental Differential Equation:

$$\frac{d^2x}{dt^2}+\frac{c}{m}\times\frac{dx}{dt}+\frac{s}{m}\times x=0$$

This is a differential equation of the second order. Assuming a solution of the form $x = e^{kt}$ where k is a constant to be determined.

The two roots of the equation are

$$k_1=-\frac{c}{2m}+\sqrt{\left(\frac{c}{2m}\right)^2-\frac{s}{m}}$$

$$k_2=-\frac{c}{2m}-\sqrt{\left(\frac{c}{2m}\right)^2-\frac{s}{m}}$$

It may be noted that the roots k_1 and k_2 may be real, complex conjugate (imaginary) or equal. We shall now discuss these three cases.

Frequency of Free Damped Vibrations (Viscous Damping)

1. *When the roots are real (overdamping)*

If $\left(\frac{c}{2m}\right)^2 > \frac{s}{m}$, then the roots k_1 and k_2 are real but negative. This is a case of ***overdamping*** or ***large damping*** and the mass moves slowly to the equilibrium position. This motion is known as ***aperiodic.***

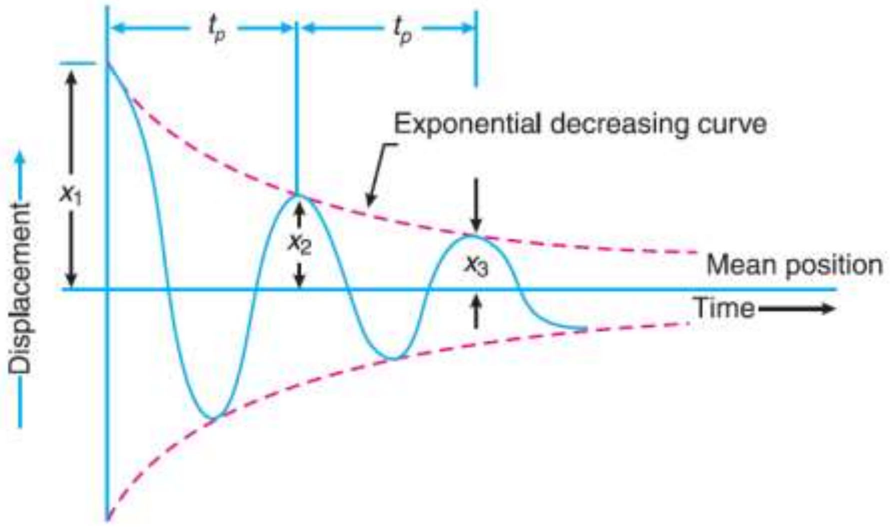
Note : In actual practice, the overdamped vibrations are avoided.

2. *When the roots are complex conjugate (underdamping)*

If $\left(\frac{c}{2m}\right)^2 < \frac{s}{m}$, then the radical (*i.e.* the term under the square root) becomes negative. The two roots k_1 and k_2 are then known as complex conjugate. This is a most practical case of damping and it is known as ***underdamping*** or ***small damping.***

$$t_p = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}} = \frac{2\pi}{\sqrt{(\omega_n)^2 - a^2}}$$

$$f_d = \frac{1}{t_p} = \frac{\omega_d}{2\pi} = \frac{1}{2\pi}\sqrt{(\omega_n)^2 - a^2} = \frac{1}{2\pi}\sqrt{\frac{s}{m} - \left(\frac{c}{2m}\right)^2}$$



Frequency of Free Damped Vibrations (Viscous Damping)

3. When the roots are equal (critical damping)

- If $\left(\frac{c}{2m}\right)^2 = \frac{s}{m}$, then the radical becomes zero and the two roots k_1 and k_2 are equal. This is a case of **critical damping**
- In other words, the critical damping is said to occur when frequency of **damped vibration (f_d) is zero** (*i.e.* motion is aperiodic)
- This type of damping is also **avoided** because the mass **moves back rapidly to its equilibrium position**, in the shortest possible time
- The critical damping coefficient is the amount of damping required for a system to be critically damped.

$$\boxed{c_c = 2m\sqrt{\frac{s}{m}} = 2m \times \omega_n}$$

Damping Factor or Damping Ratio

- The ratio of the actual damping coefficient (c) to the critical damping coefficient (cc) is known as ***damping factor or damping ratio***. Mathematically,

$$\text{Damping factor} = \frac{c}{c_c} = \frac{c}{2m.\omega_n} \qquad \ldots (\because\ c_c = 2\pi.\omega_n)$$

- The damping factor is the measure of the relative amount of damping in the existing system with that necessary for the critical damped system

Logarithmic Decrement

It is defined as the natural logarithm of the amplitude reduction factor. The amplitude reduction factor is the ratio of any two successive amplitudes on the same side of the mean position. If x_1 and x_2 are successive values of the amplitude on the same side of the mean position,

In general, amplitude reduction factor,

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \ldots = \frac{x_n}{x_{n+1}} = e^{at_p} = \text{constant}$$

Damping factor $= \dfrac{c}{c_c} = \dfrac{c}{2m.\omega_n}$ . . . $(\because \; c_c = 2\pi.\omega_n)$

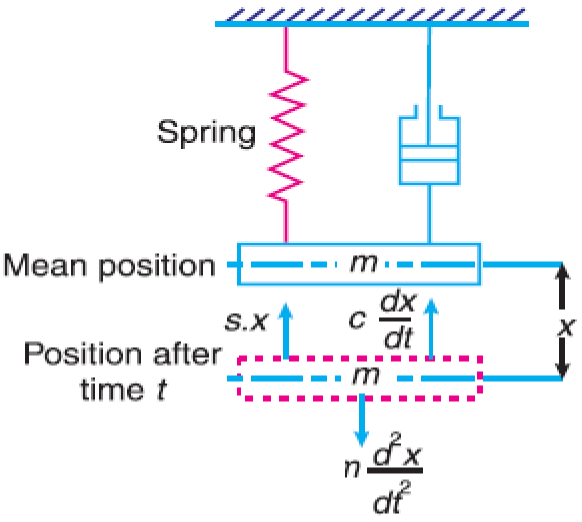
Frequency of Under Damped Forced Vibrations

Consider a system consisting of spring, mass and damper as shown in Fig. Let the system is acted upon by an external periodic (i.e. simple harmonic) disturbing force,

$$F_x = F \cos\omega.t$$

where F = Static force, and ω = Angular velocity of the periodic disturbing force.

When the system is constrained to move in vertical guides, it has only one degree of freedom. Let at sometime t, the mass is displaced downwards through a distance x from its mean position. Using the symbols as discussed in the previous article, the equation of motion may be written as,



$$\therefore \qquad x = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}} \times \cos(\omega.t - \phi)$$

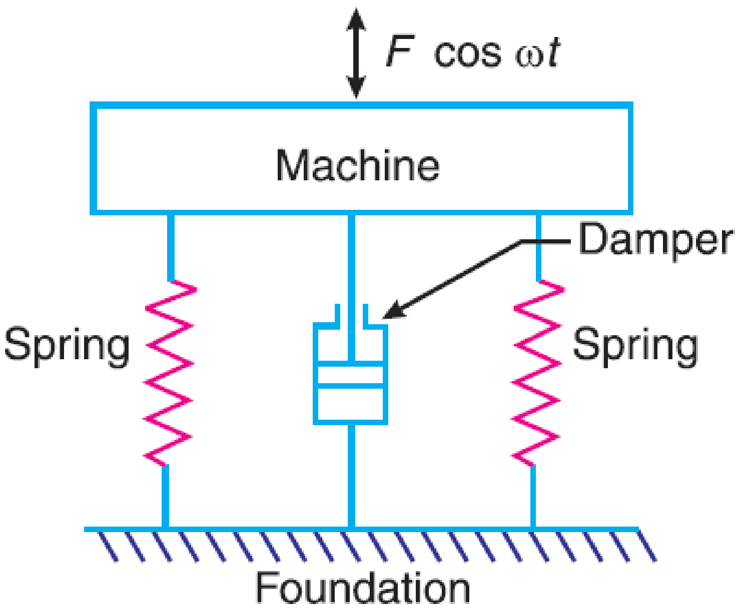
This equation shows that motion is simple harmonic whose circular frequency is ω and the amplitude is $\dfrac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$.

Maximum displacement or the amplitude of forced vibration,

$$x_{max} = \frac{F}{\sqrt{c^2.\omega^2 + (s - m.\omega^2)^2}}$$

Vibration Isolation and Transmissibility

A little consideration will show that when an unbalanced machine is installed on the foundation, it produces vibration in the foundation. In order to prevent these vibrations or to minimise the transmission of forces to the foundation, the machines are mounted on springs and dampers or on some vibration isolating material, as shown in Fig. The arrangement is assumed to have one degree of freedom, *i.e.* it can move up and down only. It may be noted that when a periodic (*i.e.* simple harmonic) disturbing force $F \cos \omega t$ is applied to a machine of mass m supported by a spring of stiffness s, then the force is transmitted by means of the spring and the damper or dashpot to the fixed support or foundation.



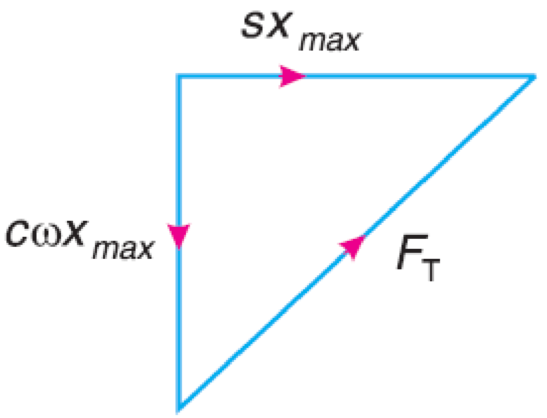
The ratio of the force transmitted (F_T) to the force applied (F) is known as the ***isolation factor*** or ***transmissibility ratio*** of the spring support.

We have discussed above that the force transmitted to the foundation consists of the following two forces :

1. Spring force or elastic force which is equal to $s.\,x_{max}$, and

2. Damping force which is equal to $c.\omega.x_{max}$.

Since these two forces are perpendicular to one another, as shown in Fig., therefore the force transmitted,



$$F_T = \sqrt{(s.x_{max})^2 + (c.\omega.x_{max})^2}$$

$$= x_{max}\sqrt{s^2 + c^2.\omega^2}$$

Vibration Isolation and Transmissibility

$\therefore$ Transmissibility ratio,

$$\varepsilon = \frac{F_T}{F} = \frac{x_{max}\sqrt{s^2 + c^2.\omega^2}}{F}$$

We know that

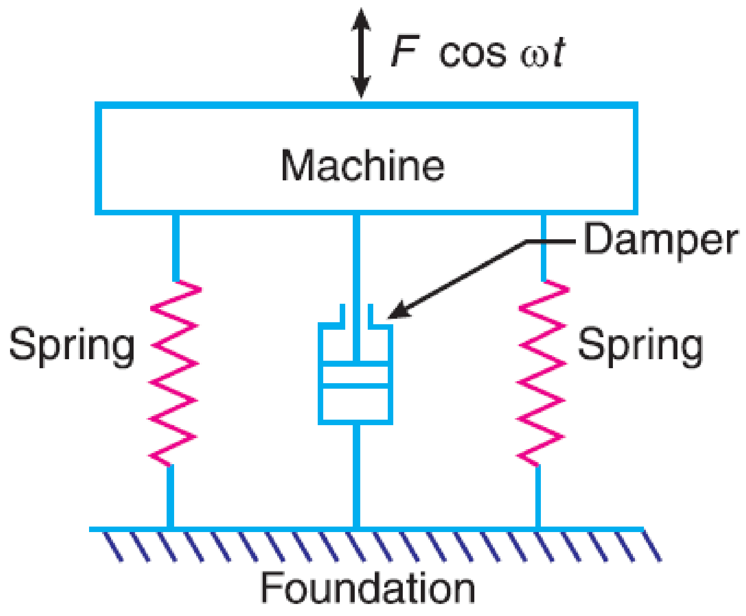
$$x_{max} = x_o \times D = \frac{F}{s} \times D \qquad \ldots \left(\because\ x_o = \frac{F}{s}\right)$$

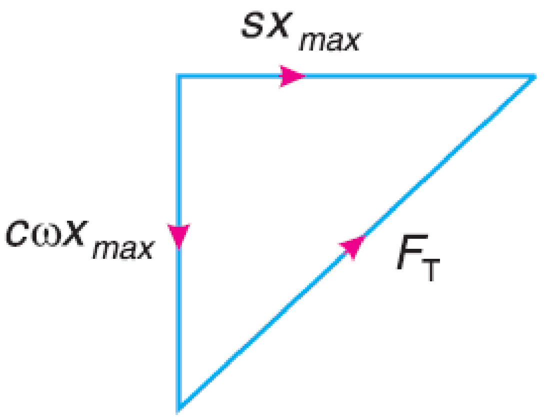
$$\therefore \qquad \varepsilon = \frac{D}{s}\sqrt{s^2 + c^2.\omega^2} = D\sqrt{1 + \frac{c^2.\omega^2}{s^2}}$$

$$= D\sqrt{1 + \left(\frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)^2} \qquad \ldots \left(\because\ \frac{c.\omega}{s} = \frac{2c}{c_c} \times \frac{\omega}{\omega_n}\right)$$

the magnification factor,

$$D = \frac{1}{\sqrt{\left(\frac{2c.\omega}{c_c.\omega_n}\right)^2 + \left(1 - \frac{\omega^2}{(\omega_n)^2}\right)^2}}$$



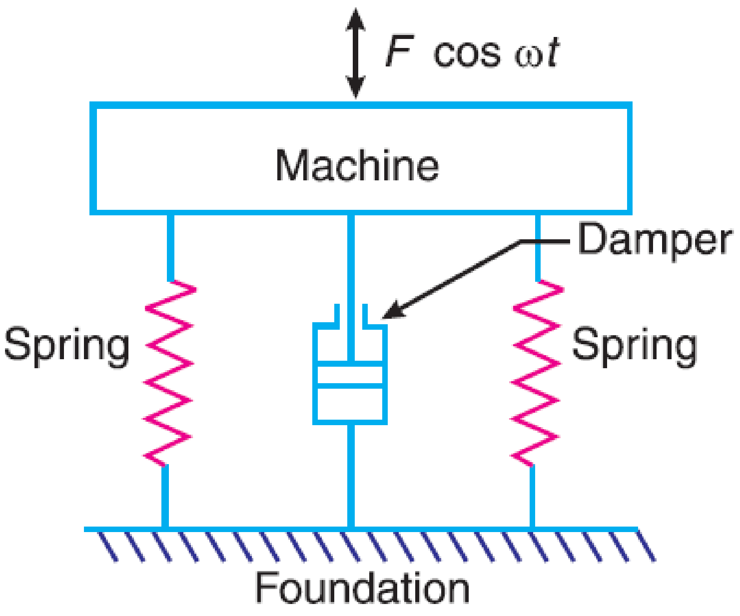


Vibration Isolation and Transmissibility

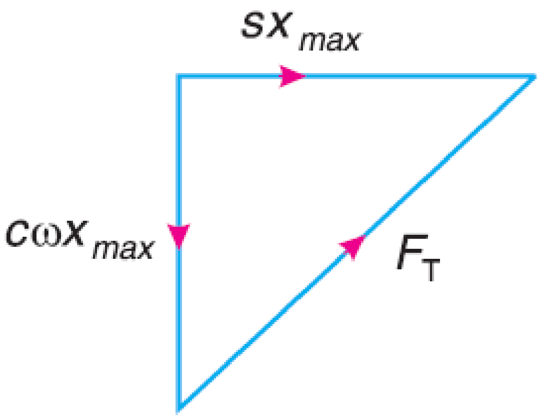
$$\therefore \qquad \varepsilon = \frac{\sqrt{1+\left(\dfrac{2c.\omega}{c_c.\omega_n}\right)^2}}{\sqrt{\left(\dfrac{2c.\omega}{c_c.\omega_n}\right)^2+\left(1-\dfrac{\omega^2}{(\omega_n)^2}\right)^2}} \qquad \ldots (i)$$

When the damper is not provided, then $c = 0$, and

$$\varepsilon = \frac{1}{1-(\omega/\omega_n)^2} \qquad \ldots (ii)$$



From above, we see that when $\omega/\omega_n > 1$, ε is negative. This means that there is a phase difference of 180° between the transmitted force and the disturbing force (F cos$\omega.t$) . The value of ω/ω_n must be greater than 2 if ε is to be less than 1 and it is the numerical value of ε, independent of any phase difference between the forces that may exist which is important. It is therefore, more convenient to use equation **(*ii*)** in the following form, *i.e.*



$$\varepsilon = \frac{1}{(\omega/\omega_n)^2-1} \qquad \ldots (iii)$$

Vibration Isolation and Transmissibility

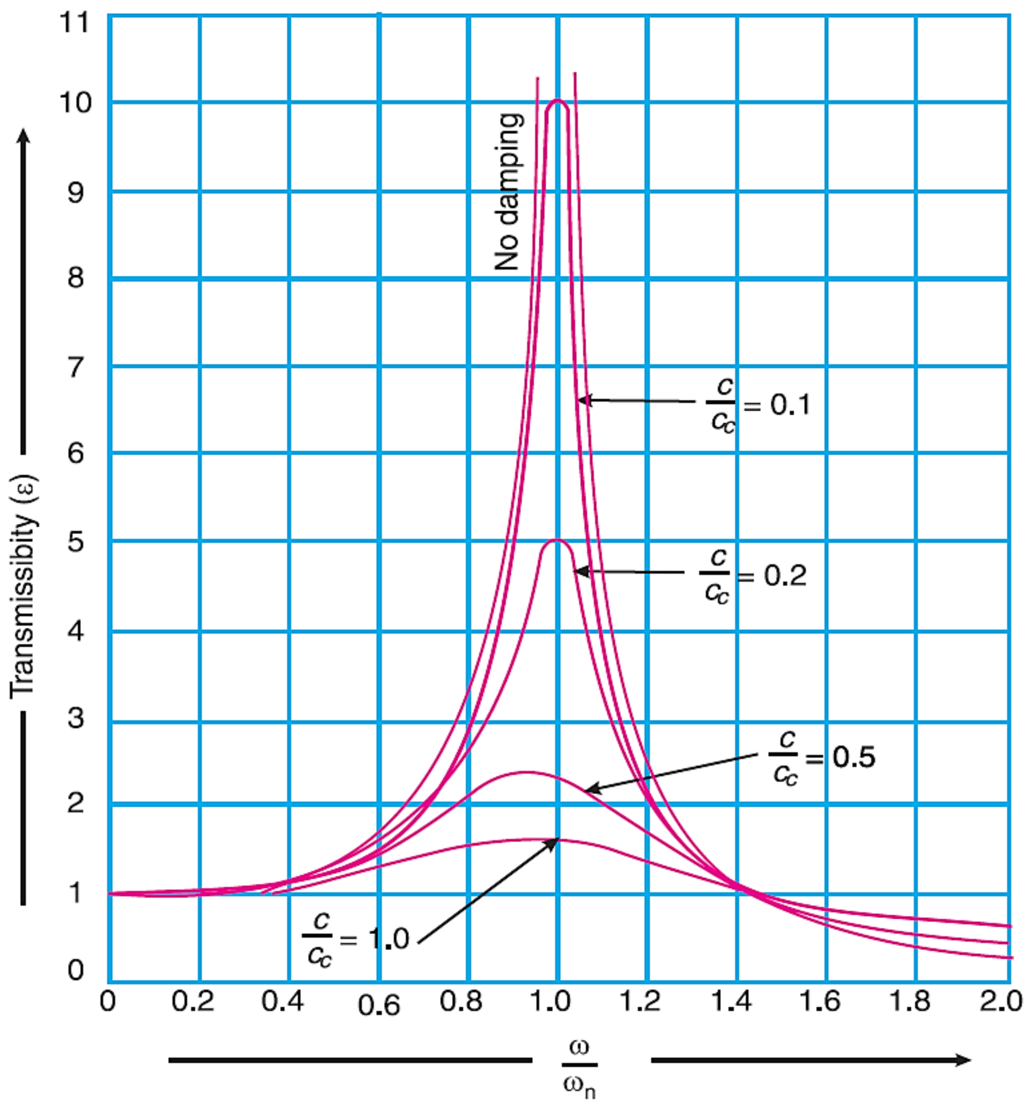
Here is the graph for different values of damping factor c/c_c to show the variation of transmissibility ratio (ε) against the ratio ω/ω_n.

1. When $\omega/\omega_n = 2$, then all the curves pass through the point $\varepsilon = 1$ for all values of damping factor c/c_c.

2. When $\omega/\omega_n < 2$, then $\varepsilon > 1$ for all values of damping factor c/c_c. This means that the force transmitted to the foundation through elastic support is greater than the force applied.

3. When $\omega/\omega_n > 2$, then $\varepsilon < 1$ for all values of damping factor c/c_c. This shows that the force transmitted through elastic support is less than the applied force. Thus, vibration isolation is possible only in the range of $\omega/\ \omega_n > 2$.

We also see from the curves in graph that the damping is detrimental beyond $\omega/\ \omega_n > 2$ and advantageous only in the region $\omega/\ \omega_n < 2$. It is thus concluded that for the vibration isolation, dampers need not to be provided but in order to limit resonance amplitude, stops may be provided.



Solve by Yourself

Book: Mechanics of Machines: Advanced Theory and Examples

Chapter 13

Exercise: 7,8

Book: Mechanics of Machines: Advanced Theory and Examples

Chapter 14

Example: 4

Book: Theory of Machines by RS Khurmi

Chapter 23

Example: 23.1,23.2,23.3,23.4,23.6,23.7,23.8,23.10,23.11,23.13,23.16,23.17,23.23,23.24


Aim for the Moon
If you miss,
you may hit a star.