7

1

Fourier transform

(Only theory)

$$\boxed{\mathcal{F}\{F(x)\} = f(s) = \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx}$$

OR

$$\boxed{\mathcal{F}\{F(t)\} = f(\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt}$$

Time-domain ⟶ Frequency-domain
for all t i.e. $-\infty < t < \infty$

III

Scanned by CamScanner

Derivation of Fourier transform from Fourier Series:

We know the exponential form of a Fourier Series is as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \quad \text{- - - - (1)}$$

where
$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt \quad \text{- - - (2)}$$

The fundamental frequency is $\omega_0 = \frac{2\pi}{T}$ - - (3) and the spacing between adjacent harmonic is

$$\Delta\omega = (n+1)\omega_0 - n\omega_0 = \omega_0 = \frac{2\pi}{T} \quad \text{- - - - (4)}$$

Substituting eq^n (2) into eq^n (1) gives

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] e^{jn\omega_0 t}$$

$$\Rightarrow f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] e^{jn\omega_0 t} \quad \text{- - (5)}$$

$$\Rightarrow f(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \left[\int_{-T/2}^{T/2} f(t) e^{-jn\omega_0 t} dt\right] \Delta\omega\, e^{jn\omega_0 t}$$

If we let $T \to \infty$, the summation becomes integration, the incremental spacing $\Delta\omega$ becomes the differential separation $d\omega$, and the discrete harmonic frequency $n\omega_0$ becomes a continuous frequency ω. Thus, as $T \to \infty$

$$\left.\begin{array}{l} \sum_{n=-\infty}^{\infty} \Rightarrow \int_{-\infty}^{\infty} \text{ - - - - -} \\ \Delta\omega \Rightarrow d\omega \text{ - - -} \\ n\omega_0 \Rightarrow \omega \text{ - - -} \end{array}\right\} \begin{array}{l} (6) \\ (7) \\ (8) \end{array}$$

So that equation (5) becomes

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt\right] e^{j\omega t} d\omega \quad \text{- - - (7)}$$

The term in the brackets is known as the Fourier transform of f(t) and is represented by F(ω).

Thus $F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$ - - - (8)

where $\mathcal{F}$ is the Fourier transform operator.

\# The Fourier transform is an integral transformation of f(t) from the time domain to the frequency domain.

In general, F(ω) is a complex function and its magnitude is called the amplitude spectrum.

Eqn. (7) can be written in terms of F(ω) and we obtain the inverse Fourier transform as

$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, dt$ - - - (9)

\# Equations (8) and (9) constitute the Fourier transform pair for aperiodic signals that most electrical engineers use. (Some communication engineers prefer to write the frequency variable in hertz rather than rad/s: this can be done by an obvious change of variable). F(ω) is called the Fourier transform of f(t) and plays the same role for aperiodic signals that Cn plays ~~the~~ for periodic signals.

Chapter-02

Introduction:

The Fourier transform, named after Joseph Fourier, is a mathematical transform empolyed to transform signals between time domain and frequency domain. Fourier series enables us to represent a periodic function as a sum of sine and cosine and to obtain the frequency spectrum. The Fourier transform allows us to extend the concept of a frequency spectrum to non-periodic functions. The transform assumes that a non-periodic function is a periodic function with an infinite period. Thus, the Fourier transform is an integral representation of a non-periodic function that is analogous to a Fourier series representation of a periodic function. The Fourier transform is very powerful mathematical tool that is useful in mathematics for solving ~~the solutions of~~ differential equations, in electrical engineering for digital signal processing and communication system, vibration analysis, Noise reduction in audio and video, computer science etc.

N.B: ~~No need to use~~ $\bar{f}(s)$ ~~just~~ $f(s)$

Fourier transform (infinite) or Complex form of Fourier transform:

The infinite Fourier transform (or Complex form of Fourier transform) of a function $F(x)$ of x such that $-\infty < x < \infty$ is denoted by

$$\bar{f}(s) = \mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx \quad \text{---} (1)$$

and the inverse formula for infinite Fourier transform is

$$F(x) = \mathcal{F}^{-1}\{\bar{f}(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(s)\, e^{isx}\, ds \quad \text{---} (2)$$

$\longrightarrow$ See last page

Infinite Fourier sine and Cosine transforms:

The inifinite Fourier sine transform of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $\bar{f}_s(s)$ or $\mathcal{F}_s\{F(x)\}$ and is defined by

$$\bar{f}_s(s) = \mathcal{F}_s\{F(x)\} = \int_{0}^{\infty} F(x) \sin sx\, dx \quad \text{---} (3)$$

and the inverse formula for infinite Fourier sine transform is defined by

$$F(x) = \mathcal{F}^{-1}\{\bar{f}_s(s)\} = \frac{2}{\pi}\int_{0}^{\infty} \bar{f}_s(s) \sin sx\, ds \quad \text{---} (4)$$

The infinite fourier cosine transform of a function $F(x)$ of x such that $0 < x < \infty$ is denoted by $\bar{f}_c(s)$ or $F_c\{F(x)\}$ and is defined by

$$\bar{f}_c(s) = F_c\{F(x)\} = \int_0^\infty F(x)\cos sx\,dx \quad \text{---} \quad (5)$$

and the inverse formula for infinite Fourier cosine transform is defined by

$$F(x) = F_c^{-1}\{\bar{f}_c(s)\} = \frac{2}{\pi}\int_0^\infty \bar{f}_c(s)\cos sx\,ds \quad \text{---} \quad (6)$$

Finite fourier sine and cosine transform:

The finite Fourier sine transform of a function $F(x)$ of x such that $0 < x < l$ is denoted by $\bar{f}_s(s)$ or $F_s\{F(x)\}$ and is defined by

$$\bar{f}_s(s) = F_s\{F(x)\} = \int_0^l F(x)\sin\left(\frac{s\pi x}{l}\right)dx \quad \text{---} \quad (7)$$

and the inverse formula for finite fourier sine transform is defined by

$$F(x) = F^{-1}\{\bar{f}_s(s)\} = \frac{2}{l}\sum_{s=1}^{\infty} \bar{f}_s(s)\sin\left(\frac{s\pi x}{l}\right) \quad \text{---} \quad (8)$$

The finite fourier cosine transform of a function $F(x)$ of x such that $0 < x < l$ is denoted by $\bar{f}_c(s)$ or $F_c\{F(x)\}$ and is defined by

$$\bar{f}_c(s) = F_c\{F(x)\} = \int_0^l F(x)\cos\left(\frac{s\pi x}{l}\right)dx \quad \text{---} \quad (9)$$

and the inverse formula for finite Fourier cosine transform is defined by

$$F(x) = \mathcal{F}_c^{-1}\{\bar{f}_s(s)\} = \frac{1}{l}\bar{f}_c(0) + \frac{2}{l}\sum_{s=1}^{\infty}\bar{f}_c(s)\cos\left(\frac{s\pi x}{l}\right) \quad \text{--- (10)}$$

Properties of Fourier transform:

1/ Linear property:

If $\bar{f}_1(s)$ and $\bar{f}_2(s)$ are the Fourier transform of $F_1(x)$ and $F_2(x)$ respectively, then

$$\mathcal{F}\{a_1 F_1(x) + a_2 F_2(x)\} = a_1 \bar{f}_1(s) + a_2 \bar{f}_2(s),$$

where a_1 and a_2 are arbitrary constants.

Proof: From definition of Fourier transform we ~~know that~~ have

$$\bar{f}_1(s) = \mathcal{F}\{F_1(x)\} = \int_{-\infty}^{\infty} F_1(x)\, e^{-isx}\, dx \quad \text{---- (1)}$$

and $\bar{f}_2(s) = \mathcal{F}\{F_2(x)\}$

$$= \int_{-\infty}^{\infty} F_2(x)\, e^{-isx}\, dx \quad \text{--- (2)}$$

Now
$$\mathcal{F}\{a_1 F_1(x) + a_2 F_2(x)\} = \int_{-\infty}^{\infty} \{a_1 F_1(x) + a_2 F_2(x)\}.\, e^{-isx}\, dx$$

$$= \int_{-\infty}^{\infty} a_1 F_1(x)\, e^{-isx}\, dx + \int_{-\infty}^{\infty} a_2 F_2(x)\, e^{-isx}\, dx$$

$$= a_1 \bar{f}_1(s) + a_2 \bar{f}_2(s) \quad [\text{Using (1) \& (2)}]$$

i.e. $\mathcal{F}\{a_1 F_1(x) + a_2 F_2(x)\} = a_1 \bar{f}_1(s) + a_2 \bar{f}_2(s)$

Change of Scale property:

If $\bar{f}(s) = \mathcal{F}\{F(x)\}$, then $\mathcal{F}\{F(ax)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

In other words, if $\bar{f}(s)$ is the fourier transform of $F(x)$, then $\frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$ is the fourier transform of $F(ax)$.

Proof: From definition of fourier transform we have

$$\bar{f}(s) = \mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx \quad \text{--- (1)}$$

Now
$$\mathcal{F}\{F(ax)\} = \int_{-\infty}^{\infty} F(ax)\, e^{-isx}\, dx \quad \text{--- (2)}$$

Let $ax = u \Rightarrow dx = \frac{du}{a}$; limit: $x \to -\infty,\ u \to -\infty$; $x \to +\infty,\ u \to +\infty$

and $x = \frac{u}{a}$

So from (2) we get

$$\mathcal{F}\{F(ax)\} = \int_{-\infty}^{\infty} e^{-is\cdot\frac{u}{a}} \cdot F(u)\cdot\frac{1}{a}\, du$$

$$= \frac{1}{a}\int_{-\infty}^{\infty} e^{-i\frac{s}{a}\cdot u} \cdot F(u)\, du$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

i.e. $\mathcal{F}\{F(ax)\} = \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$.

proved.

3. Shifting property:

If $F\{F(x)\} = \bar{f}(s)$, then $F\{F(x-a)\} = e^{-isa}\bar{f}(s)$.

Proof: By definition of Fourier transform

We have $F\{F(x)\} = \int_{-\infty}^{\infty} F(x)\, e^{-isx}\, dx$ ---- ①

$\therefore F\{F(x-a)\} = \int_{-\infty}^{\infty} F(x-a)\, e^{-isx}\, dx$ ---- ②

Let $u = x-a$, $x = u+a$ | Limit: $x \to \infty$, $u \to \infty$

$\Rightarrow du = dx$ | $x \to -\infty$, $u \to -\infty$

Thus from ② we get

$$F\{F(x-a)\} = \int_{-\infty}^{\infty} F(u)\, e^{-is(u+a)}\, du$$

$$= \int_{-\infty}^{\infty} F(u)\, e^{-isu} \cdot e^{-isa} \cdot du$$

$$= e^{-isa} \int_{-\infty}^{\infty} F(u)\, e^{-isu}\, du$$

$$= e^{-isa}\, \bar{f}(s)$$

i.e. $F\{F(x-a)\} = e^{-isa} \cdot \bar{f}(s)$

Proved.

Similarly $F\{F(x+a)\} = e^{isa}\,\bar{f}(s)$.

4. Modulation property:

If $\mathcal{F}\{F(x)\} = \bar{f}(s)$, then $\mathcal{F}\{F(x)\cos(ax)\}$
$= \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right]$.

Proof: By definition of Fourier transform we have

$$\bar{f}(s) = \mathcal{F}\{F(x)\} = \int_{-\infty}^{\infty} e^{-isx} F(x)\, dx \quad \text{--- (1)}$$

$$\therefore \mathcal{F}\{F(x).\cos(ax)\} = \int_{-\infty}^{\infty} e^{-isx} . F(x) \cos(ax)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-isx} . F(x) . \left(\frac{e^{iax} + e^{-iax}}{2}\right) . dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} F(x) . \left\{e^{-i(s-a)x} + e^{-i(s+a)x}\right\}\right] dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} F(x) . e^{-i(s-a)x}\, dx + \int_{-\infty}^{\infty} F(x) . e^{-i(s+a)x} . dx\right]$$

$$= \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right]$$

i.e. $\mathcal{F}\{F(x).\cos(ax)\} = \frac{1}{2}\left[\bar{f}(s-a) + \bar{f}(s+a)\right]$

Similarly

$$\mathcal{F}\{F(x).\sin(ax)\} = \frac{1}{2}\left[\bar{f}(s-a) - \bar{f}(s+a)\right]$$

→ see also last page - Application.

5. Parseval's identity for Fourier transform:

If $f(s)$ and $g(s)$ are the complex Fourier transform of $F(x)$ and $G(x)$ respectively, then (i) $\int_{-\infty}^{\infty} F(x)\overline{G}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\overline{g}(s)\,ds$

(ii) $\int_{-\infty}^{\infty} |F(x)|^2\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2\,ds$

Where bar denotes the complex conjugate.

Proof: (i) From definition of inverse Fourier transform, we have

$$G(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(s)\, e^{isx}\,ds \quad \text{-----} \quad (1)$$

Taking the complex conjugate on both sides of (1), we get

$$\overline{G}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(s)\, e^{-isx}\,ds \quad \text{-----} \quad (2)$$

$$\therefore \int_{-\infty}^{\infty} F(x)\,\overline{G}(x)\,dx = \int_{-\infty}^{\infty} F(x).\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(s)\, e^{-isx}\,ds\right]dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(s)\left[\int_{-\infty}^{\infty} F(x)\, e^{-isx}\,dx\right]ds$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{g}(s).\,f(s)\,ds \quad \left[\because f(s) = \int_{-\infty}^{\infty} F(x)\, e^{isx}\,dx\right]$$

→ Similarly for sine & cosine transform Parsevals identity is (5)

$$\int_0^\infty |F(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |f_s(s)|^2 ds \quad \text{--- (2)}$$

$$\& \int_0^\infty |F(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |f_c(s)|^2 ds \quad \text{--- (3)}$$

$$\Rightarrow \int_{-\infty}^{\infty} F(x)\,\bar{G}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\bar{g}(s)\,ds \quad \text{---- (3)}$$

proved.

(ii) Taking $F(x) = G(x)$ and hence $f(s) = g(s)$ we get $\bar{F}(x) = \bar{G}(x)$ and $\bar{f}(s) = \bar{g}(s)$ so that equation (3) becomes

$$\int_{-\infty}^{\infty} F(x)\,\bar{F}(x)\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\,\bar{f}(s)\,ds$$

$$\Rightarrow \int_{-\infty}^{\infty} |F(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2 ds \quad \text{---- (1)}$$

proved.

6. Convolution and convolution theorem for fourier transform:

6.1. Convolution: The convolution of two ~~The function H(x) =~~ integrable functions $F(x)$ and $G(x)$ over the interval $(-\infty, \infty)$ is defined as

$$F(x) * G(x) = \int_{-\infty}^{\infty} F(u)\, G(x-u)\, du \quad \text{--- (1)}$$

[P.T.O.]

6.2: Convolution theorem for Fourier transform:

The Fourier transform of the convolution of $F(x)$ and $G(x)$ is the product of the Fourier transforms of $F(x)$ and $G(x)$.

i.e. $\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\{F(x)\} \cdot \mathcal{F}\{G(x)\}$.

Proof: From definition of convolution

we have $F(x) * G(x) = \int_{-\infty}^{\infty} F(u) \cdot G(x-u)\,du$ --- ①

Again from definition of Fourier transform we have

$$\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} F(u)\, G(x-u)\,du\right\}$$

$$= \int_{-\infty}^{\infty} e^{-isx}\left\{\int_{-\infty}^{\infty} F(u)\, G(x-u)\,du\right\}dx$$

$$= \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{-isx}\, G(x-u)\,dx\right\}du \quad \text{--- ②}$$

Let $x-u = v \Rightarrow dx = dv$ & $x = u+v$, then from ② we get

$$\mathcal{F}\{F(x) * G(x)\} = \int_{-\infty}^{\infty} F(u)\left\{\int_{-\infty}^{\infty} e^{-is(u+v)} \cdot G(v)\,dv\right\}du$$

$$= \int_{-\infty}^{\infty} F(u) \cdot \left\{\int_{-\infty}^{\infty} e^{-isu} \cdot e^{-isv} \cdot G(v)\,dv\right\}du$$

$$\Rightarrow \mathcal{F}\{F(x) * G(x)\} = \int_{-\infty}^{\infty} e^{-isu} \cdot F(u) \left\{ \int_{-\infty}^{\infty} e^{-isv} G(v)\, dv \right\} du$$

$$= \int_{-\infty}^{\infty} e^{-isu} F(u) \left\{ \int_{-\infty}^{\infty} e^{-isx} G(x)\, dx \right\} du \quad \text{[Replacing } v \text{ by } x\text{]}$$

$$= \int_{-\infty}^{\infty} e^{-isu} F(u) . \mathcal{F}\{G(x)\}\, du$$

$$= \left[\int_{-\infty}^{\infty} e^{-isu} \cdot F(u)\, du \right] . \mathcal{F}\{G(x)\}$$

[Replacing u by x]

$$= \left[\int_{-\infty}^{\infty} e^{-isx} F(x)\, dx \right] . \mathcal{F}\{G(x)\}$$

$$= \mathcal{F}\{F(x)\} . \mathcal{F}\{G(x)\}$$

i.e. $\mathcal{F}\{F(x) * G(x)\} = \mathcal{F}\{F(x)\} . \mathcal{F}\{G(x)\}$

proved.

Relation between Fourier and Laplace transform:

Let us consider the function

$$F(t) = \begin{cases} e^{-xt} G(t), & t > 0 \\ 0, & t < 0 \end{cases} \quad \cdots \quad (1)$$

[P.T.O]

Then $\mathcal{F}\{F(t)\} = \int_{-\infty}^{\infty} F(t).\, e^{-iut}\, dt$

$$= \int_{-\infty}^{0} F(t).\, e^{-iut}\, dt + \int_{0}^{\infty} F(t).\, e^{-iut}\, dt$$

$$= \int_{-\infty}^{0} 0.\, e^{-iut}\, dt + \int_{0}^{\infty} e^{-xt}.\, G(t)\, e^{-iut}\, dt$$

[by using (1)]

$$= 0 + \int_{0}^{\infty} G(t)\, e^{-(x+iu)t}\, dt$$

[Put $x + iu = s$]

$$= \int_{0}^{\infty} G(t)\, e^{-st}\, dt$$

$$= \mathcal{L}\{G(t)\}$$

i.e. $\mathcal{F}\{F(t)\} = \mathcal{L}\{G(t)\}$ which is the required relation between Fourier and Laplace transforms. $\boxed{H}$

Q. Write down some merits & demerits of Laplace tranform and Fourier tranform.

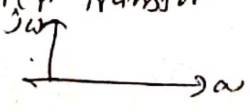
Solⁿ: 1. The Laplace transform is one-sided in that the integral is over $0 < t < \infty$, making it only useful for positive-time function, $f(t);\ t > 0$. On the other hand the Fourier transform is applicable to functions defined for all time.

2. The Laplace transform is applicable to a wider range of functions than the Fourier transform. For example, the function $t\,u(t)$ has a Laplace transform but no Fourier transform. But Fourier transforms exists for signals that are not physically realizable and have no Laplace transforms.

3. The Laplace transform is better suited for the analysis of transient problems involving initial conditions, since it permits the inclusion of the initial condition, whereas the Fourier transform does not. The Fourier transform is especially useful for problems in the steady state.

4. The Fourier transform provides greater insight into the frequency characteristics of signals than does the Laplace transforms.

5. For a function $f(t)$ that is nonzero for positive time only (i.e. $f(t) = 0,\ t < 0$) and $\int_{-\infty}^{\infty} |f(t)|\,dt < \infty$, the two transforms are related by $F(\omega) = F(s)\big|_{s=j\omega}$ ---①. This equation also shows that the Fourier transform can be regarded as a special case of the Laplace transform with $s = j\omega$. Recall that $s = \sigma + j\omega$. Therefore eqn ① shows that the Laplace transform is related to the entire s plane, whereas the Fourier transform is restricted to the $j\omega$ axis.



For engineering purposes:

definition of Fourier transform from time-domain to frequency domain:

The Fourier transform of $F(t)$ is defined as

$$F(\omega) = \mathcal{F}[F(t)] = \int_{-\infty}^{\infty} F(t)\, e^{-i\omega t}\, dt \quad \text{--- (1)}$$

where $\mathcal{F}$ is the Fourier transform operator.

The inverse Fourier transform is defined as

$$F(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, dt \quad \text{--- (2)}$$

Thus Fourier transform is an integral transform of $F(t)$ from the time domain to the frequency domain.

In general, $F(\omega)$ is complex function, its magnitude is the amplitude spectrum.

parseval's identity:

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega. \rightarrow *$$

In signal theory a square integrable function is also called a signal with finite energy-content or energy signal for short. The value $\int_{-\infty}^{\infty} |f(t)|^2\, dt$ is then called the energy-content of the signal $f(t)$.

* Therefore: parseval's identity can be used
  - to find the energy content of the signal $f(t)$.
  - to calculate certain definite integrals.

* The $\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$ can be [illegible] as the total energy of the physical system.

parseval's identity from time-domain to frequency domain and calculation of total energy:

If $p(t)$ is the power associated with the signal, the energy carried by the signal is

$$W = \int_{-\infty}^{\infty} p(t)\, dt \quad \text{------} \quad (3)$$

For a 1-Ω resistor, $p(t) = v^2(t) = i^2(t) = f^2(t)$, where $f(t)$ stands for either voltage or current. Then the energy delivered to the 1-Ω resistor is

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt \quad \text{----} \quad (4)$$

parseval's theorem states that this same energy can be calculated in the frequency domain as

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega \quad \text{---} \quad (5)$$

Parseval's theorem states that the total energy delivered to a 1-Ω resistor equals the total area under the square of $f(t)$ or $\frac{1}{2\pi}$ times the total area under the square of the magnitude of the Fourier transform of $f(t)$.

To prove (5) we start with (4)

$$W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} f(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega) e^{j\omega t}\, d\omega\right] dt \quad \text{--} \quad (6)$$

[using (2)]

The function $f(t)$ can be moved inside the integral with the brackets, since the integral does not involve time:

$$W_{1\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t) F(\omega) e^{j\omega t}\, d\omega\, dt$$

Reversing the order of integration,

$$W_{1\Omega} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\left[\int_{-\infty}^{\infty} f(t)\, e^{i(-\omega)t}\, dt\right] d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, F(-\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, F^*(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$$

Hence $W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega$ .. (7)

Equation (7) indicates that the energy carried by a signal can be found by integrating either the square of $f(t)$ in the time domain or $\frac{1}{2\pi}$ times the square of $F(\omega)$ in the frequency domain.

Notice that parseval's theorem as stated here applies to nonperiodic functions. On the other hand parseval's theorem for periodic functions was presented in fourier series analysis.

<u>Example-1:</u> (a) Calculate the total energy absorbed by a $1\text{-}\Omega$ resistor with $i(t) = 10e^{-2|t|}$ A in the time domain, (b) Repeat (a) in the frequency domain.

<u>Soln:</u> $$i(t) = \begin{cases} 10e^{-2t} & \text{for } t > 0 \\ 10e^{2t} & \text{for } t < 0 \end{cases}$$

$$\therefore W_{1\Omega} = \int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} i^2(t)\,dt = \int_{-\infty}^{0} i^2(t)\,dt + \int_{0}^{\infty} i^2(t)\,dt$$

$$= \int_{-\infty}^{0} (10e^{2t})^2\,dt + \int_{0}^{\infty} (10e^{-2t})^2\,dt$$

$$= \int_{-\infty}^{0} 100 e^{4t} dt + \int_{0}^{\infty} 100 e^{-4t} dt$$

$$= 100 \left[\frac{e^{4t}}{4} \right]_{-\infty}^{0} + 100 \left[\frac{e^{-4t}}{-4} \right]_{0}^{\infty}$$

$$= \frac{100}{4} [1-0] + \frac{100}{-4} [0-1]$$

$= 25 + 25 = 50J$ in time domain.

Again in the frequency domain

$$F(\omega) = \cancel{V(\omega)} \cdot i(\omega) = \frac{10}{2+j\omega}$$

So that $|F(\omega)|^2 = F(\omega) F^*(\omega) = \frac{100}{4+\omega^2}$

Hence the ~~to~~ energy dissipated by the resistor is

$$W_{1\Omega} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{100}{4+\omega^2} d\omega$$

$$= \frac{100}{2\pi} \left(\frac{1}{2} \tan^{-1} \frac{\omega}{2} \right) \Big|_{-\infty}^{\infty}$$

$$= \frac{100}{4\pi} \left[\frac{\pi}{2} + \frac{\pi}{2} \right] = ?$$