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Lecture-04:

Topics include:

1. Taylor series
2. Laurent series

Expansion of Analytic Function as power Series:

$\boxed{\text{Question:}}$ Write down Taylor's and Maclaurin series for an analytic function $f(z)$.

$\boxed{\text{Ans.}}$ Taylor's series: If $f(z)$ is analytic inside ~~and~~ a circle C with center at 'a', then for all z inside C,

$$f(z) = f(a) + \frac{f'(a)}{\lfloor 1}(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \cdots \longrightarrow (1)$$

The above series (1) is known as the Taylor series of $f(z)$ about the point $z = a$.

Now putting $a = 0$ in the Taylor's series (1) we get

$$f(z) = f(0) + \frac{f'(0)}{\lfloor 1} z + \frac{f''(0)}{\lfloor 2} z^2 + \cdots \longrightarrow (2)$$

Which is known as a Maclaurin series.

i.e. The Taylor's series of an analytic function about the origin $(a=0)$ is called a Maclaurin series.

Question Write down Laurent series for an analytic function but has singularity at a point.

Ans. If $f(z)$ is analytic inside and on the boundary of the ring-shaped region R (annulus) bounded by two-concentric circles C_1 and C_2 with centre at 'a' and the respective radii r_1 and r_2 $(r_1 > r_2)$, then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} a_{-n} (z-a)^{-n} \quad \text{---} \quad (1)$$

where $a_n = \dfrac{1}{2\pi i} \oint_{C_1} \dfrac{f(w)\, dw}{(w-a)^{n+1}}$, for $n = 0, 1, 2, \cdots$ with any point w on C_1. ---- (2)

and $a_{-n} = \dfrac{1}{2\pi i} \oint_{C_2} \dfrac{f(w)\, dw}{(w-a)^{-n+1}}$ for $n = 1, 2, \cdots$ with any point w on C_2. ---- (3)

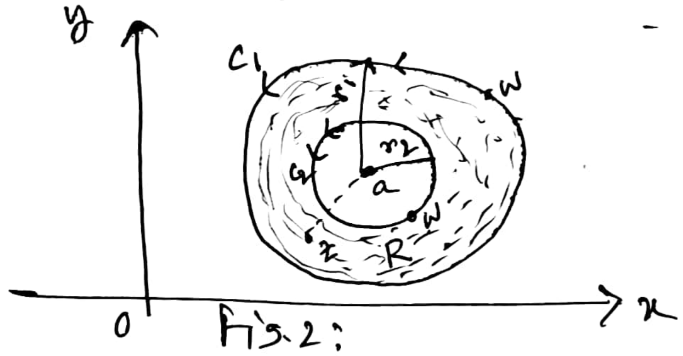

Fig. 2:

The above series (1) with the coefficients (2) & (3) is called Laurent series for $f(z)$ in which the point $z = a$ is the singular point that is $f(z)$ is not analytic at $z = a$.

*Note that Laurent series are particularly useful for representing functions that have singularities, such as poles or branch points.

Question What is analytic part and principal part of the Laurent series?

Ans. The part $\sum_{n=0}^{\infty} a_n (z-a)^n$ is called the analytic part and the part $\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}$ is called principal (singular) part of the Laurent series.

Note that if the principal part of the Laurent series is zero, then the Laurent series reduces to a Taylor series.

Remark-1: Every Taylor series is a Laurent series but the converse is not true.

Remark-2 If $f(z)$ fails to be analytic at a point 'a', then we cannot apply Taylor's theorem at that point. It is often possible to find a Laurent series for $f(z)$ involving both positive and negative powers of $(z-a)$.

* **Remark-3** If $f(z)$ is analytic function inside and on a simple closed curve C, ~~then~~ then by using Taylor's series we can find out the value of the function at any random point inside the circle C with the help of the value at the center i.e. $f(a)$. On the other hand, by using Laurent series, we can find out the value of the function at any ~~point~~ point inside the annular region R ~~except the singular point~~ with the help of value of the function at the centre where the function is singular.

Example-1 Expand $f(z) = \sin z$ in a Taylor series about $z = \pi/4$ and determine the region of convergence of the obtained series.

Solution: Given that $f(z) = \sin z \Rightarrow f(\pi/4) = \sin \pi/4 = \frac{1}{\sqrt{2}}$

$$\therefore f'(z) = \cos z = \sin(\tfrac{\pi}{2}+z) \Rightarrow f'(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z = \sin(2\cdot\tfrac{\pi}{2}+z) \Rightarrow f''(\pi/4) = -\sin(\pi/4) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z = \sin(3\cdot\pi/2+z) \Rightarrow f'''(\pi/4) = -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$$

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$$f^n(z) = \sin(n\cdot\tfrac{\pi}{2}+z) \Rightarrow f^n(\pi/4) = \sin(n\cdot\pi/2+\pi/4)$$

Then the Taylor series for $f(z) = \sin z$ about $z = \pi/4$ is

$$f(z) = f(\pi/4) + \frac{f'(\pi/4)}{\lfloor 1}(z-\pi/4) + \frac{f''(\pi/4)}{\lfloor 2}(z-\pi/4)^2 + \cdots$$

$$\Rightarrow \sin z = \frac{1}{\sqrt{2}} + \frac{1}{\lfloor 1}\cdot\frac{1}{\sqrt{2}}(z-\pi/4) - \frac{1}{\lfloor 2}\frac{1}{\sqrt{2}}(z-\pi/4)^2 \cdots$$

$$\Rightarrow \sin z = \frac{1}{\sqrt{2}}\left[1+(z-\pi/4) - \frac{1}{\lfloor 2}(z-\pi/4)^2 - \cdots\right]$$

Since the above series is convergent for all finite value of z, so the region of convergence is $|z| < \infty$ and radius of convergence is infinite.

Exercise-1 Expand $f(z) = \cos z$ in a Taylor series about $z = \pi/2$ and determine the region of convergence of the obtained series.

Ans. $\cos z = -(z-\pi/2) + \frac{1}{\lfloor 3}(z-\pi/2)^3 - \frac{1}{\lfloor 5}(z-\pi/2)^5 + \cdots$

R.C: $|z| < \infty$.

Example-2 Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in a Laurent series valid for $1 < |z| < 3$.

Solution: Here $f(z) = \frac{1}{(z+1)(z+3)}$

Then $f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$ ---- (1)

Now for $1 < |z| < 3$, we have $1 < |z|$ and $|z| < 3$

$\Rightarrow \left|\frac{1}{z}\right| < 1$ and $\left|\frac{z}{3}\right| < 1$

$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right] = \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{2\cdot 3\left(1+\frac{z}{3}\right)}$

$\Rightarrow f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$ ---- (2)

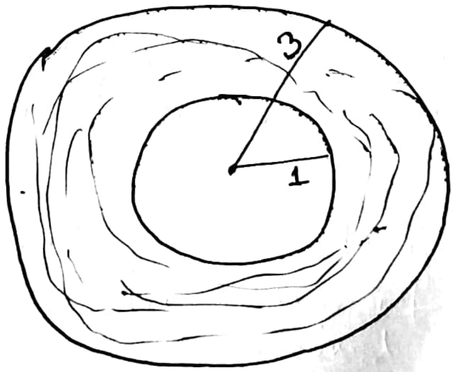
Then the binomial expansion of $f(z)$ is valid for $1 < |z| < 3$ and we have from (2)

$f(z) = \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{3^2} - \frac{z^3}{3^3} + \cdots\right)$

$\Rightarrow f(z) = \left(\frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \cdots\right) - \left(\frac{1}{6} - \frac{z}{18} + \frac{z^2}{54} - \cdots\right)$

$\Rightarrow f(z) = \cdots + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \cdots$

Which is the required Laurent series that is valid for $1 < |z| < 3$
(annulus)



Exercise-1 Obtain the Laurent Series of the following function in the indicated regions:

(a) $f(z) = \frac{z}{(z+1)(z+2)}$ $\left(= \frac{2}{z+2} - \frac{1}{z+1}\right)$ in the region $1 < |z| < 2$.

Ans. $f(z) = \cdots + \frac{1}{z^4} - \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} + 1 - \frac{z}{2} + \frac{z^2}{2^2} \cdots$

(b) $f(z) = \frac{z}{(z-1)(2-z)}$ in the region $1 < |z| < 2$.

$\left(= \frac{1}{z-1} + \frac{2}{2-z}\right)$

Ans. $f(z) = \cdots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots$

Question Why a Laurent series is more advantageous over a Taylor series? Explain this with practical applications.?

Ans: * In Fourier analysis, Laurent series are used to represent and analyze periodic function that have discontinuities or singular points. The negative power terms in the Laurent series allow it to capture the behavior around these singular points. Thus the Laurent Series is very important in areas like signal processing, Communications, and electrical engineering.

* In the study of fluid flows and electromagnetic fields, Laurent series expansions are used to represent the behavior of solution near critical points/stagnation points of the flow ~~near~~ around objects such as airfoils or bodies. It is important for accurately predicting lift, drag and other fluid mechanics properties.

* Also inverse trigonometric function like $f(z) = \frac{1}{\sin z}$, Laurent series is more useful than Taylor series.