Lecture Note on Legendre's Polynomials

Topics:

1. Legendre's function of 1st and 2nd kind.
2. Rodrigue's formula for Legendre polynomial
3. Legendre series
4. Generating function for Legendre polynomial
5. Orthogonal properties of Legendre polynomial
6. Recurrence Relation for Legendre polynomials.

Legendre Differential Equation & Legendre polynomials:

Question: What is Legendre differential equation?

Ans: The 2nd order linear ordinary differential equation $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0$, --- (1)

where n is a positive integer, is called Legendre's differential equation named after the French mathematician Adrien-Marie Legendre (1752-1833).

The above eq^n (1) can also written as follows:

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)y = 0 \quad \cdots \cdots (2)$$

This equation is frequently encountered in physics and other technical field. In particular, it occurs when solving Laplace's equation in spherical coordinates. ★

Question: What is Legendre's function of 1st kind and 2nd kind?

Ans. The 1st solution of the Legendre equation can be written as

$$P_n(x) = \frac{1.3.5.\cdots(2n-1)}{\underline{n}}\left[x^n - \frac{n(n-1)}{2\cdot(2n-1)}\cdot x^{n-2} + \cdots\cdots\right]$$

Which is known as Legendre's function of first kind. ~~Here $P_n(x)$ is a terminating series and~~ for different values of n, we get Legendre's polynomials such as $P_0(x)$, $P_1(x)$, $P_2(x)$ --- etc.

★ In engineering, Legendre polynomials find application in signal processing, control system, and image analysis. Also Legendre polynomials have important application in the study of fluid flow in cylindrical and spherical coordinate system.

The 2nd Solution of the Legendre equation Can be written as

$$Q_n(x) = \frac{\lfloor n}{1.3.5.\cdots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2.(2n+3)}x^{-(n+3)} + \cdots\cdots\right]$$

Which is known as Legendre's function of 2nd kind.

 # Question: State and prove Rodrigue's formula for Legendre's polynomials.

Ans: **Statement:** If $P_n(x)$ is Legendre polynomials of 1st kind, then $P_n(x) = \frac{1}{2^n \lfloor n} \cdot \frac{d^n}{dx^n}(x^2-1)^n$

Which is known as Rodrigue's formula for Legendre polynomial $P_n(x)$. It is discovered by French mathematician Olinde Rodrigue in 1816.

proof: Let $y = (x^2-1)^n$ - - - - - (1)

Differentiating (1) w.r. to x we obtain

$$\frac{dy}{dx} = n(x^2-1)^{n-1} \cdot 2x$$

Multiplying both sides by (x^2-1) we get

$$(x^2-1)\frac{dy}{dx} = n(x^2-1)^n \cdot 2x = 2nxy \text{ - - - - - } (2)$$

Now differentiating (2) w.r. to x $(n+1)$ time using Leibnitz theorem we get

$$(x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + {}^{n+1}c_1 \cdot \frac{d^{n+1}y}{dx^{n+1}}(2x) + {}^{n+1}c_2 \frac{d^n y}{dx^n}(2)$$

$$= 2n\left[x\frac{d^{n+1}y}{dx^{n+1}} + {}^{n+1}C_1 \frac{d^n y}{dx^n}\cdot(1)\right]$$

$$= 2nx\frac{d^{n+1}y}{dx^{n+1}} + 2n(n+1)\cdot\frac{d^n y}{dx^n}$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x(\cancel{n}+1-\cancel{n})\cdot\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\cdot\frac{d^n y}{dx^n} = 0$$

$$\Rightarrow (x^2-1)\frac{d^{n+2}y}{dx^{n+2}} + 2x\frac{d^{n+1}y}{dx^{n+1}} - n(n+1)\cdot\frac{d^n y}{dx^n} = 0$$

$$\Rightarrow (1-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0 \quad \text{-----} \ (3)$$

Let $v = \frac{d^n y}{dx^n}$ in (3), then weget

$$(1-x^2)\frac{d^2 v}{dx^2} - 2x\frac{dv}{dx} + n(n+1)v = 0 \quad \text{----} \ (4)$$

Which is exactly Legendre's equation and hence v is a solution of this equation.

Hence $P_n(x) = c\cdot v = c\cdot\frac{d^n y}{dx^n}$ ---- (5)

Where c is a constant.

Now put $x=1$ in (5) weget

$$P_n(1) = c\cdot\left.\frac{d^n y}{dx^n}\right]_{x=1}$$

$$\Rightarrow c\cdot\left.\frac{d^n y}{dx^n}\right]_{x=1} = 1 \qquad [\because P_n(1) = 1] \quad \text{----} \ (6)$$

Again $y = (x^2-1)^n = (x-1)^n \cdot (x+1)^n$

Now differentiating w. r. to x, n times using Leibnitz's theorem we get

$$\frac{d^n y}{dx^n} = (x-1)^n \cdot \frac{d^n (x+1)^n}{dx^n} + n \cdot \frac{d^{n-1}(x+1)^n}{dx^{n-1}} \left\{ n(x-1)^{n-1} \right\}$$

$$+ \cdots\cdots + \underset{\sim\sim\sim}{(x+1)^n} \cdot \frac{d^n (x-1)^n}{dx^n}$$

$$\Rightarrow \left.\frac{d^n y}{dx^n}\right]_{x=1} = (1+1)^n \cdot \frac{d^n (x-1)^n}{dx^n} = 2^n \cdot \lfloor n$$

$$\left[\because \frac{d^n (x-1)^n}{dx^n} = \lfloor n \right]$$

$$\Rightarrow \left.\frac{d^n y}{dx^n}\right]_{x=1} = 2^n \lfloor n \quad \cdots\cdots \quad (7)$$

From (6) and (7) we get

$$c \cdot 2^n \lfloor n = 1 \quad \Rightarrow \quad c = \frac{1}{2^n \lfloor n}$$

Now putting the value of c in eqn (5) we get

$$P_n(x) = c \cdot \frac{d^n y}{dx^n} = \frac{1}{2^n \lfloor n} \cdot \frac{d^n (x^2-1)^n}{dx^n}$$

Hence proved.

Question: Using Rodrigue's formula to find the values of $P_0(x)$, $P_1(x)$, $P_2(x)$, $P_3(x)$ .... etc.

Ans. From Rodrigue's formula we know that $P_n(x) = \frac{1}{2^n \cdot \lfloor n} \cdot \frac{d^n}{dx^n}(x^2-1)^n$ .... (1)

Now putting $n=0$ in (1) we get

$$P_0(x) = \frac{1}{2^0 \cdot \lfloor 0} \cdot \frac{d^0}{dx^0}(x^2-1)^0 = \frac{1}{1 \cdot 1} \cdot 1 = 1$$

i.e. $\boxed{P_0(x) = 1}$

Again putting $n=1$ in (1) we get

$$P_1(x) = \frac{1}{2^1 \cdot \lfloor 1} \cdot \frac{d}{dx}(x^2-1) = \frac{1}{2} \cdot 2x = x$$

i.e. $\boxed{P_1(x) = x}$

Also putting $n=2$ in (1) we get

$$P_2(x) = \frac{1}{2^2 \cdot \lfloor 2} \frac{d^2}{dx^2}(x^2-1)^2$$

$$\Rightarrow P_2(x) = \frac{1}{4 \cdot 2} \frac{d}{dx}\left\{\frac{d}{dx}(x^2-1)^2\right\}$$

$$\Rightarrow P_2(x) = \frac{1}{8} \cdot \frac{d}{dx}\left\{2(x^2-1) \cdot 2x\right\}$$

$$\Rightarrow P_2(x) = \frac{1}{\cancel{8}_2} \cdot \cancel{4} \cdot \frac{d}{dx}(x^3 - x) = \frac{1}{2}(3x^2-1)$$

$$\Rightarrow \boxed{P_2(x) = \frac{1}{2}(3x^2-1)}$$

Again for $n=3$, we get from (1)

$$P_3(x) = \frac{1}{2^3 \lfloor 3} \frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}\left\{3(x^2-1)^2 . 2x\right\}$$

$$= \frac{1}{8}\frac{d^2}{dx^2}(x^5-2x^3+x) = \frac{1}{8}\frac{d}{dx}(5x^4-6x^2-1)$$

$$= \frac{1}{8}(20x^3-12x) = \frac{5x^3-3x}{2}$$

i.e. $\boxed{P_3(x) = \frac{1}{2}(5x^3-3x)}$

proceeding as above, we can show that

$$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$$

$$P_5(x) = \frac{1}{8}(63x^5-70x^3-15x)$$

$$P_6(x) = \frac{1}{16}(231x^6-315x^4+105x^2-5)$$ and so on.

The graphs of these above polynomials are as follows

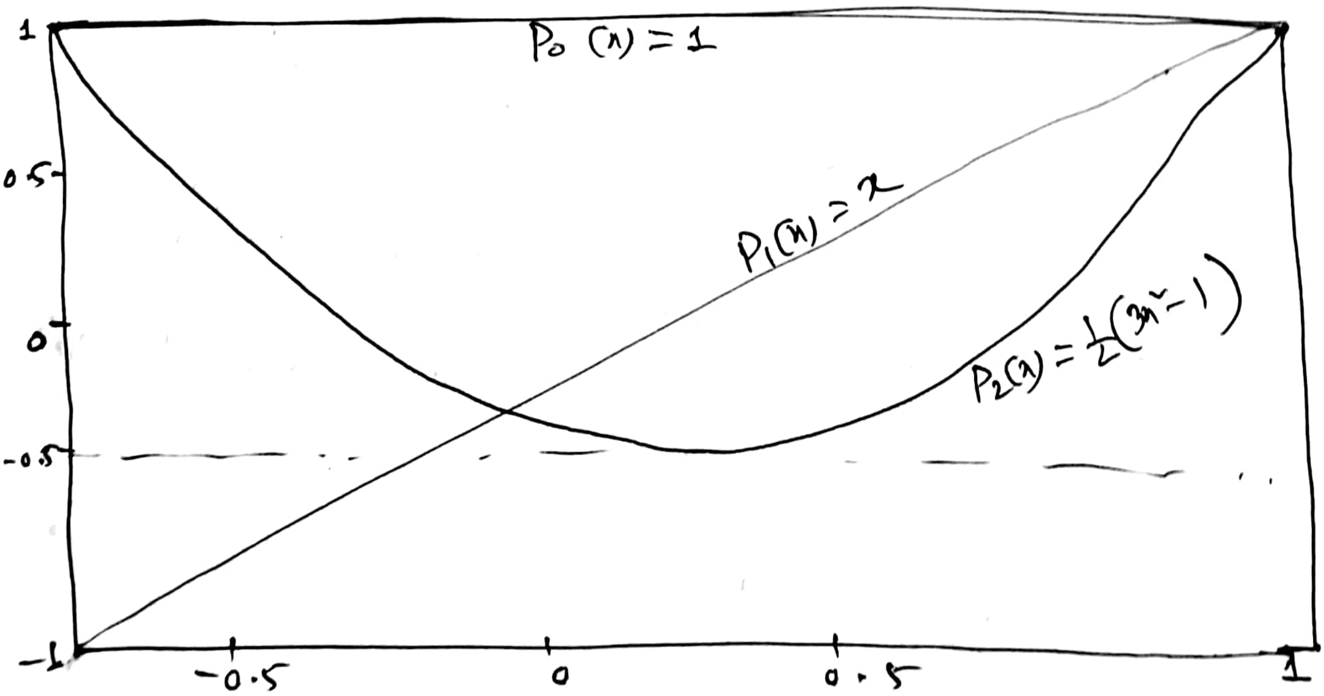

Fig.1: Graphs of Legendre polynomials.

Question: Define Legendre series.

Ans: If $f(x)$ is a polynomial of degree n, then

$$f(x) = \sum_{r=0}^{n} c_r P_r(x) \quad \text{-------} (1)$$

Where

$$c_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_r(x)\, dx \quad \text{----} (2)$$

Example-1 Expand $f(x) = x^2$ in the form of Legendre polynomials.

Solution: Since $f(x) = x^2$ is a polynomials of degree 2, so from Legendre's series we get

$$f(x) = x^2 = \sum_{r=0}^{2} c_r P_r(x)$$

$$\Rightarrow x^2 = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) \quad \text{---} (1)$$

Where

$$c_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} x^2 P_r(x)\, dx \quad \text{---} (2)$$

But we know that $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}(3x^2-1)$

Now putting $r = 0, 1, 2$ successively in (2) and using the values of $P_0(x)$, $P_1(x)$ & $P_2(x)$ we get

$$c_0 = \frac{1}{2}\int_{-1}^{1} x^2 \cdot P_0(x)\, dx = \frac{1}{2}\int_{-1}^{1} x^2 \cdot 1 \cdot dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{+1}$$

$$= \frac{1}{6}[1+1] = \frac{1}{3}$$

$$c_1 = \left(1+\frac{1}{2}\right)\int_{-1}^{1} f(x) P_1(x)\, dx = \frac{3}{2}\int_{-1}^{1} x^2 \cdot x\, dx = \frac{3}{2}\int_{-1}^{1} x^3 dx$$

$$\Rightarrow c_1 = \frac{3}{2}\cdot\frac{1}{4}\cdot\left[x^4\right]_{-1}^{1} = 0$$

Also $c_2 = \left(2+\frac{1}{2}\right)\int_{-1}^{1} f(x)\, P_2(x)\, dx = \frac{5}{2}\int_{-1}^{1} x^2 \cdot \frac{1}{2}(3x^2-1)\,dx$

$$\Rightarrow c_2 = \frac{5}{4}\int_{-1}^{1}(3x^4 - x^2)\,dx = \frac{5}{4}\left[\frac{3x^5}{5} - \frac{x^3}{3}\right]_{-1}^{1}$$

$$\Rightarrow c_2 = \frac{5}{4}\left[\left(\frac{3}{5}-\frac{1}{3}\right) - \left(-\frac{3}{5}+\frac{1}{3}\right)\right]$$

$$\Rightarrow c_2 = \frac{\cancel{5}}{\cancel{4}_{2}} \times \frac{\cancel{8}^{\,2}}{\cancel{15}_{3}} = \frac{2}{3} \Rightarrow \boxed{c_2 = \frac{2}{3}}$$

Hence $f(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x)$

$$\Rightarrow f(x) = \frac{1}{3} P_0(x) + 0 \cdot P_1(x) + \frac{2}{3} P_2(x)$$

Thus $x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_2(x)$ which is the required Legendre series. Ans.

Exercise-: (1) Expand $f(x) = x^3$ in a series of Legendre polynomials.

Ans. $x^3 = \frac{3}{5} P_1(x) + \frac{2}{5} P_3(x)$

(2) Expand $f(x) = x^4$ in a series of Legendre polynomials.

Ans. $x^4 = \frac{1}{5} P_0(x) + \frac{4}{7} P_2(x) + \frac{8}{35} P_4(x)$

(3) Expand $f(x) = x^5$ in a series of Legendre polynomials.

Ans. $x^5 = \frac{3}{7} P_1(x) + \frac{4}{9} P_3(x) + \frac{8}{63} P_5(x)$

Fourier-Legendre Expansion of $f(x)$:

If $f(x)$ be a function defined from $x=-1$ to $x=1$, then $f(x)=\sum_{r=0}^{\infty} C_r P_r(x)$, where

$$C_r=\left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)\,P_r(x)\,dx.$$

Example: 1 Expand $f(x)=\begin{cases} 0; & -1<x<0 \\ 1: & 0<x<1 \end{cases}$

in a series of Legendre polynomials.

Solution: We know that $f(x)=\sum_{r=0}^{\infty} C_r P_r(x)$ ---- ①

where $C_r=\left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)\,P_r(x)\,dx$ --- ②

Now $C_r=\left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)\,P_r(x)\,dx$

$$\Rightarrow C_r=\left(r+\frac{1}{2}\right)\left[\int_{-1}^{0} f(x)\,P_r(x)\,dx+\int_{0}^{1} f(x)\,P_r(x)\,dx\right]$$

$$\Rightarrow C_r=\left(r+\frac{1}{2}\right).\left[\int_{-1}^{0} 0.\,P_r(x)\,dx+\int_{0}^{1} 1.\,P_r(x)\,dx\right]$$

$$\Rightarrow C_r=\left(r+\frac{1}{2}\right)\int_{0}^{1} P_r(x)\,dx \quad \text{----} \; ③$$

Now putting $r=0,1,2,3,\ldots$ successively in ③, we get

$$C_0 = \left(0+\frac{1}{2}\right)\cdot\int_0^1 P_0(x)\,dx = \frac{1}{2}\int_0^1 1\,dx = \frac{1}{2}[x]_0^1 = \frac{1}{2}$$

$$C_1 = \left(1+\frac{1}{2}\right)\int_0^1 P_1(x)\,dx = \frac{3}{2}\int_0^1 x\,dx = \frac{3}{2}\left[\frac{x^2}{2}\right]_0^1 = \frac{3}{4}$$

$$C_2 = \left(2+\frac{1}{2}\right)\int_0^1 P_2(x)\,dx = \frac{5}{2}\int_0^1 \frac{1}{2}(3x^2-1)\,dx$$

$$\Rightarrow C_2 = \frac{5}{4}\int_0^1 (3x^2-1)\,dx = \frac{5}{4}[x^3-x]_0^1 = 0$$

$$C_3 = \left(3+\frac{1}{2}\right)\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{1}{2}(5x^3-3x)\,dx$$

$$\Rightarrow C_3 = \frac{7}{4}\left[\frac{5x^4}{4}-\frac{3x^2}{2}\right]_{x=0}^1 = \frac{7}{4}\left[\frac{5}{4}-\frac{3}{2}\right] = -7/16$$

and so on.

Now putting the values of $C_0, C_1, C_2 \ldots$ in ①
we get

$$f(x) = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x) + C_3 P_3(x) + \cdots$$
$$\cdots + C_r P_r(x) + \cdots$$

$$\Rightarrow f(x) = \frac{1}{2}P_0(x) + \frac{3}{4}P_1(x) - \frac{7}{16}P_3(x) + \cdots$$
$$\cdots + C_r P_r(x) + \cdots$$

Where $C_r = \left(r+\frac{1}{2}\right)\int_0^1 P_r(x)\,dx$

Ans.

$\boxed{\text{Exercise-1:}}$ Expand $f(x)$ in a series of Legendre polynomials if $f(x) = \begin{cases} 0 : -1 < x < 0 \\ x : 0 < x < 1 \end{cases}$

Ans. $f(x) = \frac{1}{4} P_0(x) + \frac{1}{2} P_1(x) + \frac{5}{16} P_2(x) - \frac{3}{32} P_4(x) + \cdots$

$\boxed{\text{Exercise-2:}}$ Expand $f(x)$ in a series of Legendre polynomials if $f(x) = \begin{cases} -1 ; & -1 < x < 0 \\ 1 : & 0 < x < 1 \end{cases}$.

Ans: $f(x) = \frac{3}{2} P_1(x) - \frac{7}{8} P_3(x) + \frac{11}{16} P_5(x) - \cdots$

Generating Function for Legendre polymials:

$\boxed{\text{Question:}}$ prove that $(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n \cdot P_n(x)$

Note that the left hand side of the equation is known as generating function for Legendre. Polynomials. $\longrightarrow$ proof see book.

orthogonal property of Legendre polynomials $P_n(x)$:

$\boxed{\text{Question:}}$ prove that $\int_{-1}^{1} P_m(x) P_n(x)\, dx = \begin{cases} 0 : \text{if } m \neq n \\ \frac{2}{2n+1} \text{ if } m = n \end{cases}$

proof see any book.

Recurrence relation for Legendre Polynomials P_n

1. $n P_n(x) = (2n-1) x P_{n-1}(x) - (n-1) P_{n-2}(x)$.

2. $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$.

x3. $n P_n(x) = x P_n'(x) - P_{n-1}'(x)$.

4. $(2n+1) P_n(x) = P_{n+1}'(x) - P_{n-1}'(x)$.

5. $P_{n+1}'(x) - x P_n'(x) = (n+1) P_n(x)$.

6. $(2n-1) x P_{n-1}(x) = n P_n(x) + (n-1) P_{n-2}(x)$

$\boxed{1}$ $n P_n(x) = (2n-1) x P_{n-1}(x) - (n-1) P_{n-2}(x)$.

Proof: We know from generating function

$$(1-2xt+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{-----} \; (1)$$

Now differentiating (1) w.r.to t we get

$$-\frac{1}{2}(1-2xt+t^2)^{-\frac{3}{2}} \cdot (-2x+2t) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t)(1-2xt+t^2)^{-\frac{1}{2}} \cdot (1-2xt+t^2)^{-1} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow (x-t) \cdot \sum_{n=0}^{\infty} t^n P_n(x) = (1-2xt+t^2) \sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow x \sum_{n=0}^{\infty} t^n P_n(x) - \sum_{n=0}^{\infty} t^{n+1} P_n(x) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) - 2x \sum_{n=0}^{\infty} n t^n P_n(x) + \sum_{n=0}^{\infty} n t^{n+1} P_n(x) \quad \text{--} (2)$$

Now equating the coefficients of t^{n-1} from both sides of (2) we get

$$x P_{n-1}(x) - P_{n-2}(x) = n P_n(x) - 2x(n-1) P_{n-1}(x) + (n-2) P_{n-2}(x)$$

$$\Rightarrow x\{(2n-2)+1\} P_{n-1}(x) - \{1+(n-2)\} P_{n-2}(x) = n P_n(x)$$

$$\Rightarrow n P_n(x) = x(2n-1) P_{n-1}(x) - (n-1) P_{n-2}(x)$$

[proved]

$\boxed{2.}$ $(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$.

Proof: Equating the coefficient of t^n from both sides of (2) we obtain

$$x P_n(x) - P_{n-1}(x) = (n+1) P_{n+1}(x) - 2xn P_n(x) + (n-1) P_{n-1}(x)$$

$$\Rightarrow (2n+1) x P_n(x) - n P_{n-1}(x) = (n+1) P_{n+1}(x)$$

$$\Rightarrow (n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)$$

[proved]

3. $n P_n(x) = x P_n'(x) - P_{n-1}'(x)$

proof: We know from generating function

$$(1-2xt+t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \quad \text{----- (1)}$$

Now differentiating (1) w.r. to t we get

$$(x-t)(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} n t^{n-1} P_n(x) \quad \text{-- (2)}$$

Again differentiating (1) w.r. to x we obtain

$$t(1-2xt+t^2)^{-3/2} = \sum_{n=0}^{\infty} t^n P_n'(x) \quad \text{--- (3)}$$

Now dividing (2) by (3) we get

$$\frac{(x-t)}{t} = \frac{\sum_{n=0}^{\infty} n t^{n-1} P_n(x)}{\sum_{n=0}^{\infty} t^n P_n'(x)}$$

$$\Rightarrow (x-t)\sum_{n=0}^{\infty} t^n P_n'(x) = t\sum_{n=0}^{\infty} n t^{n-1} P_n(x)$$

$$\Rightarrow x\sum_{n=0}^{\infty} t^n P_n'(x) - \sum_{n=0}^{\infty} t^{n+1} P_n'(x) = \sum_{n=0}^{\infty} n t^n P_n(x)$$

On equating the coefficient of t^n we get

$$x P_n'(x) - P_{n-1}'(x) = n P_n(x)$$

$$\Rightarrow \boxed{n P_n(x) = x P_n'(x) - P_{n-1}'(x)}$$

Proved.

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