AHSANULLAH UNIVERSITY OF SCIENCE AND TECHNOLOGY (AUST)

ME-3105: FLUID MECHANICS

(LC-7: Problem on Rotation of Fluid Particle and Stream Function)

BY

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‘ϕ (phi)‘ AND Ψ (psi)’ FORMULA FOR PROBLEM SOLUTION (Note)

Velocity Potential ‘ϕ ‘ : The velocity potential $\phi = f(x, y, z)$ for steady flow ,

$$u = -\frac{\partial \phi}{\partial x} \;;\; v = -\frac{\partial \phi}{\partial y} \;;\; w = -\frac{\partial \phi}{\partial z}$$

Continuity Equation: For incompressible fluid if the flow is steady then equation of continuity is given by,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(-\frac{\partial \phi}{\partial z}\right) = 0 \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi = 0$$

if the velocity potential satisfies the Laplace equation it represents the possible steady, incompressible, irrotational flow. Often an irrotational flow is known as **Potential flow**.

‘ϕ (phi) ‘ AND ‘ Ψ (psi)’ FORMULA FOR PROBLEM SOLUTION (Note)

For roatational flow, the rotation components are given by,

$$\omega_x = \frac{1}{2}\left(\frac{\delta w}{\delta y} - \frac{\delta v}{\delta z}\right) \quad \omega_y = \frac{1}{2}\left(\frac{\delta u}{\delta z} - \frac{\delta w}{\delta x}\right) \quad \omega_z = \frac{1}{2}\left(\frac{\delta v}{\delta x} - \frac{\delta u}{\delta y}\right)$$

Substituting the value of u, v and w interms of ϕ,

$$\omega_x = \frac{1}{2}\left(-\frac{\delta^2 \phi}{\delta y \delta z} + \frac{\delta^2 \phi}{\delta z \delta y}\right) \quad \omega_y = \frac{1}{2}\left(-\frac{\delta^2 \phi}{\delta z \delta x} + \frac{\delta^2 \phi}{\delta x \delta z}\right) \quad \omega_z = \frac{1}{2}\left(-\frac{\delta^2 \phi}{\delta x \delta y} + \frac{\delta^2 \phi}{\delta y \delta x}\right)$$

However ϕ is a continuous function then,

$$\frac{\delta^2 \phi}{\delta y \delta z} = \frac{\delta^2 \phi}{\delta z \delta y} \qquad \frac{\delta^2 \phi}{\delta y \delta x} = \frac{\delta^2 \phi}{\delta x \delta z} \qquad \frac{\delta^2 \phi}{\delta x \delta y} = \frac{\delta^2 \phi}{\delta y \delta x}$$

so $\omega_z = 0$, $\omega_y = 0$ and $\omega_x = 0$ i.e the flow is irrotational.

‘ϕ (phi)‘ AND Ψ (psi)’ FORMULA FOR PROBLEM SOLUTION (Note)

Stream Function ‘ Ψ ‘ : ψ = f (x, y) for steady flow such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

The Continuity equation for 2D incompressible flow,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

By substituting the value of u, v and w in terms of ψ,

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} \equiv 0$$

Difference in ψ between streamlines is equal to volume flow rate between streamlines.

‘ϕ (phi) ‘ AND ‘ Ψ (psi)’ FORMULA FOR PROBLEM SOLUTION (Note)

For two dimensional flow,

$$u = \frac{\delta\psi}{\delta y} \; ; \; v = -\frac{\delta\psi}{\delta x} \text{ (for stream line) and } \psi = \psi(x,y)$$

$$u = -\frac{\delta\phi}{\delta x} \; ; \; v = -\frac{\delta\phi}{\delta y} \text{ (for velocity potential) and } \phi = \phi(x,y)$$

$$d\psi = \frac{\delta\psi}{\delta x}\cdot dx + \frac{\delta\psi}{\delta y}\cdot dy = -v\cdot dx + u\cdot dy \text{ and}$$

$$d\phi = \frac{\delta\phi}{\delta x}\cdot dx + \frac{\delta\phi}{\delta y}\cdot dy = -u\cdot dx - v\cdot dy$$

The line of constant ψ are streamline i.e $d\psi = 0$ $\left.\frac{dy}{dx}\right|_{\psi_{constant}} = \frac{v}{u}$

Along a line of constant ϕ, $d\phi = 0$ $\left.\frac{dy}{dx}\right|_{\phi_{constant}} = -\left(\frac{u}{v}\right)$

The product of slope of ψ = constant line and velocity potential line ϕ = constant line is negative 1 i.e. both lines are perpendicular where they intersect.

FORMULA FOR ACCELERATION OF FLUID PERTICLE PROBLEM (Note)

$$u = \frac{dx_{particle}}{dt} \qquad v = \frac{dy_{particle}}{dt} \qquad w = \frac{dz_{particle}}{dt}$$

$$\overrightarrow{a_{particle}}(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + u\frac{\delta \vec{V}}{\delta x} + v\frac{\delta \vec{V}}{\delta y} + w\frac{\delta \vec{V}}{\delta z}$$

$$\vec{\nabla} = \vec{i}\cdot\frac{\delta}{\delta x} + \vec{j}\cdot\frac{\delta}{\delta y} + \vec{k}\cdot\frac{\delta}{\delta z} = \left(\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}\right) \qquad \vec{V} = \vec{i}\cdot u + \vec{j}\cdot v + \vec{k}\cdot w$$

$$\vec{a}\cdot(x,y,z,t) = \frac{d\vec{V}}{dt} = \frac{\delta \vec{V}}{\delta t} + \left(\vec{V}\cdot\vec{\nabla}\right)\vec{V}$$

$$a_x = \frac{\delta u}{\delta t} + u\frac{\delta u}{\delta x} + v\frac{\delta u}{\delta y} + w\frac{\delta u}{\delta z}$$

$$a_y = \frac{\delta v}{\delta t} + u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta y} + w\frac{\delta v}{\delta z}$$

$$a_z = \frac{\delta w}{\delta t} + u\frac{\delta w}{\delta x} + v\frac{\delta w}{\delta y} + w\frac{\delta w}{\delta z}$$

Material Derivative,

$$\frac{D}{Dt} = \frac{d}{dt} = \frac{\delta}{\delta t} + \left(\vec{V}\cdot\vec{\nabla}\right)$$

PROBLEM ON VELOCITY POTENTIAL ‘ϕ ‘ AND STREAM FUNCTION ‘ Ψ ’

Problem-1: In a two dimensional incompressible flow the fluid velocity components are given by $u = x - 4y$ and $v = -y - 4x$. Show that the flow satisfies the continuity equation and obtain the expression for stream function. If the flow is potential. Obtain the expression for the velocity potential.

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PROBLEM ON VELOCITY POTENTIAL ‘ϕ ‘ AND STREAM FUNCTION ‘ Ψ ’

Problem-2: The velocity components in a two-dimensional flow field are expressed by

$$u = \frac{y^3}{3} + 2x - x^2 y$$

$$v = x y^2 - 2y - \frac{x^3}{3}$$

Show that the flow is (a). Incompressible and irrotational; (b). Obtain expression for stream function ψ and (c). Obtain expression for velocity potential φ.

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PROBLEM ON VELOCITY POTENTIAL 'ϕ' AND STREAM FUNCTION 'Ψ'

Problem 3: The velocity components in a two-dimensional flow field are expressed by

$$u = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad v = -\frac{2xy}{(x^2 + y^2)^2}$$

Show that (a). Flow is incompressible and irrotational;

(b). The points (2,2) and $(1, 2-\sqrt{3})$ are located in the same streamline.

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(c). Determine the discharge across a line joining points (1,1) and (2,2) given that thickness of the fluid stream normal to the xy plane is 't'.

PROBLEM ON FLUID ACCELERATION

Problem-4: The velocity of a fluid flow is given by the following vector,

$$\vec{V} = x^3 y \vec{i} + y^2 z \vec{j} - \left(3x^2 y z + y z^2\right) \vec{k}$$

(a). Show that the flow is steady and incompressible.

(b). Find the velocity and acceleration of fluid particle at point (1, 2, 3).

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