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Book: Zill & Wright

12.1 Orthogonal Functions

25. In R^3 the set $\{\mathbf{i}, \mathbf{j}\}$ is not complete since $\mathbf{k}$ is orthogonal to both $\mathbf{i}$ and $\mathbf{j}$. The set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete. To see this suppose that $a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Then

$$0 = (a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, \mathbf{i}) = a(\mathbf{i}, \mathbf{i}) + b(\mathbf{j}, \mathbf{i}) + c(\mathbf{k}, \mathbf{i}) = a(1) + b(0) + c(0) = a.$$

Similarly, $b = 0$ and $c = 0$. Thus, the only vector in R^3 orthogonal to $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ is $\mathbf{0}$, so $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is complete.

EXERCISES 12.2

Fourier Series

1.
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} 1\,dx = 1$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\cos nx\,dx = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\frac{n\pi}{\pi}x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin nx\,dx = \frac{1}{n\pi}(1-\cos n\pi) = \frac{1}{n\pi}[1-(-1)^n]$$
$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

2.
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}(-1)\,dx + \frac{1}{\pi}\int_0^{\pi} 2\,dx = 1$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\cos nx\,dx = 0$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi} 2\sin nx\,dx = \frac{3}{n\pi}[1-(-1)^n]$$
$$f(x) = \frac{1}{2} + \frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin nx$$

3.
$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_0^1 x\,dx = \frac{3}{2}$$
$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_{-1}^{0}\cos n\pi x\,dx + \int_0^1 x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$$
$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_{-1}^{0}\sin n\pi x\,dx + \int_0^1 x\sin n\pi x\,dx = -\frac{1}{n\pi}$$
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x - \frac{1}{n\pi}\sin n\pi x\right]$$

4.
$$a_0 = \int_{-1}^{1} f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$$
$$a_n = \int_{-1}^{1} f(x)\cos n\pi x\,dx = \int_0^1 x\cos n\pi x\,dx = \frac{1}{n^2\pi^2}[(-1)^n - 1]$$
$$b_n = \int_{-1}^{1} f(x)\sin n\pi x\,dx = \int_0^1 x\sin n\pi x\,dx = \frac{(-1)^{n+1}}{n\pi}$$

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi^2}\cos n\pi x + \frac{(-1)^{n+1}}{n\pi}\sin n\pi x\right]$$

5. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\cos nx\,dx = \frac{1}{\pi}\left(\frac{x^2}{\pi}\sin nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{2(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x^2\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x^2}{n}\cos nx\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^{n+1} + \frac{2}{n^3\pi}[(-1)^n - 1]$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos nx + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin nx\right]$$

6. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\pi^2\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi^2 - x^2)\,dx = \frac{5}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\pi^2\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi^2 - x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} x\sin nx\,dx\right) = \frac{2}{n^2}(-1)^{n+1}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{0}\pi^2\sin nx\,dx + \frac{1}{\pi}\int_0^{\pi}(\pi^2 - x^2)\sin nx\,dx$$

$$= \frac{\pi}{n}[(-1)^n - 1] + \frac{1}{\pi}\left(\frac{x^2 - \pi^2}{n}\cos nx\Big|_0^{\pi} - \frac{2}{n}\int_0^{\pi} x\cos nx\,dx\right) = \frac{\pi}{n}(-1)^n + \frac{2}{n^3\pi}[1 - (-1)^n]$$

$$f(x) = \frac{5\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2}(-1)^{n+1}\cos nx + \left(\frac{\pi}{n}(-1)^n + \frac{2[1 - (-1)^n]}{n^3\pi}\right)\sin nx\right]$$

7. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi)\,dx = 2\pi$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x + \pi)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{n}(-1)^{n+1}$$

$$f(x) = \pi + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^{n+1}\sin nx$$

8. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3 - 2x)\,dx = 6$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(3 - 2x)\cos nx\,dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(3 - 2x)\sin nx\,dx = \frac{4}{n}(-1)^n$$

$$f(x) = 3 + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx$$

9. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{2}{\pi}$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\cos nx\,dx = \frac{1}{2\pi}\int_0^{\pi}\left(\sin(1 + n)x + \sin(1 - n)x\right)dx$$

$$= \frac{1+(-1)^n}{\pi(1-n^2)} \quad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{2\pi}\int_0^{\pi} \sin 2x \, dx = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} \Big(\cos(1-n)x - \cos(1+n)x\Big)\, dx = 0 \quad \text{for } n = 2, 3, 4, \ldots$$

$$b_1 = \frac{1}{2\pi}\int_0^{\pi} (1 - \cos 2x)\, dx = \frac{1}{2}$$

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)} \cos nx$$

10. $$a_0 = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x \, dx = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\cos 2nx \, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x \cos 2nx \, dx = \frac{1}{\pi}\int_0^{\pi/2} \Big(\cos(2n-1)x + \cos(2n+1)x\Big)\, dx$$

$$= \frac{2(-1)^{n+1}}{\pi(4n^2-1)}$$

$$b_n = \frac{2}{\pi}\int_{-\pi/2}^{\pi/2} f(x)\sin 2nx \, dx = \frac{2}{\pi}\int_0^{\pi/2} \cos x \sin 2nx \, dx = \frac{1}{\pi}\int_0^{\pi/2} \Big(\sin(2n-1)x + \sin(2n+1)x\Big)\, dx$$

$$= \frac{4n}{\pi(4n^2-1)}$$

$$f(x) = \frac{1}{\pi} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx + \frac{4n}{\pi(4n^2-1)}\sin 2nx\right]$$

11. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = \frac{1}{2}\left(\int_{-1}^{0} -2\, dx + \int_0^1 1\, dx\right) = -\frac{1}{2}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_{-1}^{0}(-2)\cos\frac{n\pi}{2}x\, dx + \int_0^1 \cos\frac{n\pi}{2}x\, dx\right) = -\frac{1}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_{-1}^{0}(-2)\sin\frac{n\pi}{2}x\, dx + \int_0^1 \sin\frac{n\pi}{2}x\, dx\right) = \frac{3}{n\pi}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$f(x) = -\frac{1}{4} + \sum_{n=1}^{\infty}\left[-\frac{1}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x + \frac{3}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x\right]$$

12. $$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\, dx = \frac{1}{2}\left(\int_0^1 x\, dx + \int_1^2 1\, dx\right) = \frac{3}{4}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_0^1 x\cos\frac{n\pi}{2}x\, dx + \int_1^2 \cos\frac{n\pi}{2}x\, dx\right) = \frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\, dx = \frac{1}{2}\left(\int_0^1 x\sin\frac{n\pi}{2}x\, dx + \int_1^2 \sin\frac{n\pi}{2}x\, dx\right)$$

$$= \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)$$

$$f(x) = \frac{3}{8} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x + \frac{2}{n^2\pi^2}\left(\sin\frac{n\pi}{2} + \frac{n\pi}{2}(-1)^{n+1}\right)\sin\frac{n\pi}{2}x\right]$$

13. $a_0 = \frac{1}{5}\int_{-5}^{5} f(x)\,dx = \frac{1}{5}\left(\int_{-5}^{0} 1\,dx + \int_{0}^{5}(1+x)\,dx\right) = \frac{9}{2}$

$a_n = \frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\cos\frac{n\pi}{5}x\,dx + \int_{0}^{5}(1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n^2\pi^2}[(-1)^n - 1]$

$b_n = \frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi}{5}x\,dx = \frac{1}{5}\left(\int_{-5}^{0}\sin\frac{n\pi}{5}x\,dx + \int_{0}^{5}(1+x)\cos\frac{n\pi}{5}x\,dx\right) = \frac{5}{n\pi}(-1)^{n+1}$

$$f(x) = \frac{9}{4} + \sum_{n=1}^{\infty}\left[\frac{5}{n^2\pi^2}[(-1)^n - 1]\cos\frac{n\pi}{5}x + \frac{5}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{5}x\right]$$

14. $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\,dx + \int_{0}^{2} 2\,dx\right) = 3$

$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\cos\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\cos\frac{n\pi}{2}x\,dx\right) = \frac{2}{n^2\pi^2}[1 - (-1)^n]$

$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi}{2}x\,dx = \frac{1}{2}\left(\int_{-2}^{0}(2+x)\sin\frac{n\pi}{2}x\,dx + \int_{0}^{2} 2\sin\frac{n\pi}{2}x\,dx\right) = \frac{2}{n\pi}(-1)^{n+1}$

$$f(x) = \frac{3}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n^2\pi^2}[1 - (-1)^n]\cos\frac{n\pi}{2}x + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{2}x\right]$$

15. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\,dx = \frac{1}{\pi}(e^{\pi} - e^{-\pi})$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x\sin nx\,dx = \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}$

$$f(x) = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n(e^{\pi} - e^{-\pi})}{\pi(1+n^2)}\cos nx + \frac{(-1)^n n(e^{-\pi} - e^{\pi})}{\pi(1+n^2)}\sin nx\right]$$

16. $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\,dx = \frac{1}{\pi}(e^{\pi} - \pi - 1)$

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\cos nx\,dx = \frac{[e^{\pi}(-1)^n - 1]}{\pi(1+n^2)}$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{0}^{\pi}(e^x - 1)\sin nx\,dx = \frac{1}{\pi}\left(\frac{ne^{\pi}(-1)^{n+1}}{1+n^2} + \frac{n}{1+n^2} + \frac{(-1)^n}{n} - \frac{1}{n}\right)$

$$f(x) = \frac{e^{\pi} - \pi - 1}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{e^{\pi}(-1)^n - 1}{\pi(1+n^2)}\cos nx + \left(\frac{n}{1+n^2}[e^{\pi}(-1)^{n+1} + 1] + \frac{(-1)^n - 1}{n}\right)\sin nx\right]$$

17. The function in Problem 5 is discontinuous at $x = \pi$, so the corresponding Fourier series converges to $\pi^2/2$ at $x = \pi$. That is,

$$\frac{\pi^2}{2} = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\left[\frac{2(-1)^n}{n^2}\cos n\pi + \left(\frac{\pi}{n}(-1)^{n+1} + \frac{2[(-1)^n - 1]}{n^3\pi}\right)\sin n\pi\right]$$

$$= \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n^2}(-1)^n = \frac{\pi^2}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2} = \frac{\pi^2}{6} + 2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

and

$$\frac{\pi^2}{6} = \frac{1}{2}\left(\frac{\pi^2}{2} - \frac{\pi^2}{6}\right) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots.$$

At $x = 0$ the series converges to 0 and

$$0 = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n}{n^2} = \frac{\pi^2}{6} + 2\left(-1 + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

so

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots.$$

18. From Problem 17

$$\frac{\pi^2}{8} = \frac{1}{2}\left(\frac{\pi^2}{6} + \frac{\pi^2}{12}\right) = \frac{1}{2}\left(2 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots\right) = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.$$

19. The function in Problem 7 is continuous at $x = \pi/2$ so

$$\frac{3\pi}{2} = f\left(\frac{\pi}{2}\right) = \pi + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^{n+1} \sin\frac{n\pi}{2} = \pi + 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right)$$

and

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots.$$

20. The function in Problem 9 is continuous at $x = \pi/2$ so

$$1 = f\left(\frac{\pi}{2}\right) = \frac{1}{\pi} + \frac{1}{2} + \sum_{n=2}^{\infty} \frac{1+(-1)^n}{\pi(1-n^2)} \cos\frac{n\pi}{2}$$

$$1 = \frac{1}{\pi} + \frac{1}{2} + \frac{2}{3\pi} - \frac{2}{3\cdot 5\pi} + \frac{2}{5\cdot 7\pi} - \cdots$$

and

$$\pi = 1 + \frac{\pi}{2} + \frac{2}{3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

or

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots.$$

21. Writing

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi}{p}x + \cdots + a_n \cos\frac{n\pi}{p}x + \cdots + b_1 \sin\frac{\pi}{p}x + \cdots + b_n \sin\frac{n\pi}{p}x + \cdots$$

we see that $f^2(x)$ consists exclusively of squared terms of the form

$$\frac{a_0^2}{4}, \qquad a_n^2 \cos^2\frac{n\pi}{p}x, \qquad b_n^2 \sin^2\frac{n\pi}{p}x$$

and cross-product terms, with $m \neq n$, of the form

$$a_0 a_n \cos\frac{n\pi}{p}x, \qquad a_0 b_n \sin\frac{n\pi}{p}x, \qquad 2a_m a_n \cos\frac{m\pi}{p}x \cos\frac{n\pi}{p}x,$$

$$2a_m b_n \cos\frac{m\pi}{p}x \sin\frac{n\pi}{p}x, \qquad 2b_m b_n \sin\frac{m\pi}{p}x \sin\frac{n\pi}{p}x.$$

The integral of each cross-product term taken over the interval $(-p, p)$ is zero by orthogonality. For the squared terms we have

$$\frac{a_0^2}{4}\int_{-p}^{p} dx = \frac{a_0^2 p}{2}, \qquad a_n^2 \int_{-p}^{p} \cos^2\frac{n\pi}{p}x\,dx = a_n^2 p, \qquad b_n^2 \int_{-p}^{p} \sin^2\frac{n\pi}{p}x\,dx = b_n^2 p.$$

Thus

$$RMS(f) = \sqrt{\frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)}.$$

EXERCISES 12.3

Fourier Cosine and Sine Series

1. Since $f(-x) = \sin(-3x) = -\sin 3x = -f(x)$, $f(x)$ is an odd function.

2. Since $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$, $f(x)$ is an odd function.

3. Since $f(-x) = (-x)^2 - x = x^2 - x$, $f(x)$ is neither even nor odd.

4. Since $f(-x) = (-x)^3 + 4x = -(x^3 - 4x) = -f(x)$, $f(x)$ is an odd function.

5. Since $f(-x) = e^{|-x|} = e^{|x|} = f(x)$, $f(x)$ is an even function.

6. Since $f(-x) = e^{-x} - e^{x} = -f(x)$, $f(x)$ is an odd function.

7. For $0 < x < 1$, $f(-x) = (-x)^2 = x^2 = -f(x)$, $f(x)$ is an odd function.

8. For $0 \le x < 2$, $f(-x) = -x + 5 = f(x)$, $f(x)$ is an even function.

9. Since $f(x)$ is not defined for $x < 0$, it is neither even nor odd.

10. Since $f(-x) = |(-x)^5| = |x^5| = f(x)$, $f(x)$ is an even function.

11. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx\, dx = \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[1 - (-1)^n\right] \sin nx.$$

12. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_1^2 1\, dx = 1$$

$$a_n = \int_1^2 \cos\frac{n\pi}{2}x\, dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}.$$

Thus

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\frac{n\pi}{2}\cos\frac{n\pi}{2}x.$$

13. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\, dx = \frac{2}{n^2\pi}\left[(-1)^n - 1\right].$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right]\cos nx.$$

12.3 Fourier Cosine and Sine Series

14. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi x\sin nx\,dx = \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^\infty \frac{2}{n}(-1)^{n+1}\sin nx.$$

15. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = 2\int_0^1 x^2\,dx = \frac{2}{3}$$

$$a_n = 2\int_0^1 x^2\cos n\pi x\,dx = 2\left(\frac{x^2}{n\pi}\sin n\pi x\,\Big|_0^1 - \frac{2}{n\pi}\int_0^1 x\sin n\pi x\,dx\right) = \frac{4}{n^2\pi^2}(-1)^n.$$

Thus

$$f(x) = \frac{1}{3} + \sum_{n=1}^\infty \frac{4}{n^2\pi^2}(-1)^n\cos n\pi x.$$

16. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 x^2\sin n\pi x\,dx = 2\left(-\frac{x^2}{n\pi}\cos n\pi x\,\Big|_0^1 + \frac{2}{n\pi}\int_0^1 x\cos n\pi x\,dx\right) = \frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1].$$

Thus

$$f(x) = \sum_{n=1}^\infty\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x.$$

17. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\,dx = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^\pi (\pi^2 - x^2)\cos nx\,dx = \frac{2}{\pi}\left(\frac{\pi^2 - x^2}{n}\sin nx\,\Big|_0^\pi + \frac{2}{n}\int_0^\pi x\sin nx\,dx\right) = \frac{4}{n^2}(-1)^{n+1}.$$

Thus

$$f(x) = \frac{2}{3}\pi^2 + \sum_{n=1}^\infty \frac{4}{n^2}(-1)^{n+1}\cos nx\,dx.$$

18. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi x^3\sin nx\,dx = \frac{2}{\pi}\left(-\frac{x^3}{n}\cos nx\,\Big|_0^\pi + \frac{3}{n}\int_0^\pi x^2\cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\int_0^\pi x\sin nx\,dx$$

$$= \frac{2\pi^2}{n}(-1)^{n+1} - \frac{12}{n^2\pi}\left(-\frac{x}{n}\cos nx\,\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right) = \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n.$$

Thus

$$f(x) = \sum_{n=1}^\infty\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n\right)\sin nx.$$

19. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{2}{\pi}\int_0^\pi (x+1)\sin nx\,dx = \frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{2(\pi+1)}{n\pi}(-1)^{n+1} + \frac{2}{n\pi} \right) \sin nx.$$

20. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = 2\int_0^1 (x-1)\sin n\pi x\,dx = 2\left[\int_0^1 x\sin n\pi x\,dx - \int_0^1 \sin n\pi x\,dx\right]$$

$$= 2\left[\frac{1}{n^2\pi^2}\sin n\pi x - \frac{x}{n\pi}\cos n\pi x + \frac{1}{n\pi}\cos n\pi x\right]_0^1 = -\frac{2}{n\pi}.$$

Thus

$$f(x) = -\sum_{n=1}^{\infty} \frac{2}{n\pi}\sin n\pi x.$$

21. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \int_0^1 x\,dx + \int_1^2 1\,dx = \frac{3}{2}$$

$$a_n = \int_0^1 x\cos\frac{n\pi}{2}x\,dx + \int_1^2 \cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right).$$

Thus

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos\frac{n\pi}{2}x.$$

22. Since $f(x)$ is an odd function, we expand in a sine series:

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin\frac{n}{2}x\,dx + \int_{\pi}^{2\pi} \pi\sin\frac{n}{2}x\,dx = \frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1}.$$

Thus

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{4}{n^2\pi}\sin\frac{n\pi}{2} + \frac{2}{n}(-1)^{n+1} \right)\sin\frac{n}{2}x.$$

23. Since $f(x)$ is an even function, we expand in a cosine series:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \Big(\sin(1+n)x + \sin(1-n)x\Big)\,dx$$

$$= \frac{2}{\pi(1-n^2)}(1+(-1)^n) \quad \text{for } n = 2, 3, 4, \ldots$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} \sin 2x\,dx = 0.$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=2}^{\infty} \frac{2[1+(-1)^n]}{\pi(1-n^2)}\cos nx.$$

24. Since $f(x)$ is an even function, we expand in a cosine series. [See the solution of Problem 10 in Exercises 12.2 for the computation of the integrals.]

$$a_0 = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x\,dx = \frac{4}{\pi}$$

$$a_n = \frac{2}{\pi/2}\int_0^{\pi/2} \cos x \cos\frac{n\pi}{\pi/2}x\,dx = \frac{4(-1)^{n+1}}{\pi(4n^2-1)}$$

Thus

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{\pi(4n^2-1)}\cos 2nx.$$

25. $a_0 = 2\int_0^{1/2} 1\,dx = 1$

$$a_n = 2\int_0^{1/2} 1\cdot\cos n\pi x\,dx = \frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = 2\int_0^{1/2} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi}{2}\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\sin n\pi x$$

26. $a_0 = 2\int_{1/2}^{1} 1\,dx = 1$

$$a_n = 2\int_{1/2}^{1} 1\cdot\cos n\pi x\,dx = -\frac{2}{n\pi}\sin\frac{n\pi}{2}$$

$$b_n = 2\int_{1/2}^{1} 1\cdot\sin n\pi x\,dx = \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\sin\frac{n\pi}{2}\right)\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{n\pi}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\sin n\pi x$$

27. $a_0 = \frac{4}{\pi}\int_0^{\pi/2}\cos x\,dx = \frac{4}{\pi}$

$$a_n = \frac{4}{\pi}\int_0^{\pi/2}\cos x\cos 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\cos(2n+1)x + \cos(2n-1)x]\,dx = \frac{4(-1)^n}{\pi(1-4n^2)}$$

$$b_n = \frac{4}{\pi}\int_0^{\pi/2}\cos x\sin 2nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}[\sin(2n+1)x + \sin(2n-1)x]\,dx = \frac{8n}{\pi(4n^2-1)}$$

$$f(x) = \frac{2}{\pi} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{\pi(1-4n^2)}\cos 2nx$$

$$f(x) = \sum_{n=1}^{\infty}\frac{8n}{\pi(4n^2-1)}\sin 2nx$$

28. $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x\,dx = \dfrac{4}{\pi}$

$a_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x \cos nx\,dx = \dfrac{1}{\pi}\int_0^{\pi} [\sin(n+1)x - \sin(n-1)x]\,dx = \dfrac{2[(-1)^n + 1]}{\pi(1-n^2)}$ for $n = 2, 3, 4, \ldots$

$b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin x \sin nx\,dx = \dfrac{1}{\pi}\int_0^{\pi} [\cos(n-1)x - \cos(n+1)x]\,dx = 0$ for $n = 2, 3, 4, \ldots$

$a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin 2x\,dx = 0$

$b_1 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} \sin^2 x\,dx = 1$

$f(x) = \sin x$

$f(x) = \dfrac{2}{\pi} + \dfrac{2}{\pi}\displaystyle\sum_{n=2}^{\infty} \dfrac{(-1)^n + 1}{1-n^2}\cos nx$

29. $a_0 = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right) = \dfrac{\pi}{2}$

$a_n = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right) = \dfrac{2}{n^2\pi}\left(2\cos\dfrac{n\pi}{2} + (-1)^{n+1} - 1\right)$

$b_n = \dfrac{2}{\pi}\left(\displaystyle\int_0^{\pi/2} x\sin nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\,dx\right) = \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}$

$f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{2}{n^2\pi}\left(2\cos\dfrac{n\pi}{2} + (-1)^{n+1} - 1\right)\cos nx$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}\sin nx$

30. $a_0 = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\,dx = \dfrac{\pi}{2}$

$a_n = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\cos\dfrac{n}{2}x\,dx = \dfrac{4}{n^2\pi}\left((-1)^n - \cos\dfrac{n\pi}{2}\right)$

$b_n = \dfrac{1}{\pi}\displaystyle\int_{\pi}^{2\pi} (x - \pi)\sin\dfrac{n}{2}x\,dx = \dfrac{2}{n}(-1)^{n+1} - \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}$

$f(x) = \dfrac{\pi}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi}\left((-1)^n - \cos\dfrac{n\pi}{2}\right)\cos\dfrac{n}{2}x$

$f(x) = \displaystyle\sum_{n=1}^{\infty} \left(\dfrac{2}{n}(-1)^{n+1} - \dfrac{4}{n^2\pi}\sin\dfrac{n\pi}{2}\right)\sin\dfrac{n}{2}x$

31. $a_0 = \displaystyle\int_0^1 x\,dx + \int_1^2 1\,dx = \dfrac{3}{2}$

$a_n = \displaystyle\int_0^1 x\cos\dfrac{n\pi}{2}x\,dx = \dfrac{4}{n^2\pi^2}\left(\cos\dfrac{n\pi}{2} - 1\right)$

$b_n = \displaystyle\int_0^1 x\sin\dfrac{n\pi}{2}x\,dx + \int_1^2 1\cdot\sin\dfrac{n\pi}{2}x\,dx = \dfrac{4}{n^2\pi^2}\sin\dfrac{n\pi}{2} + \dfrac{2}{n\pi}(-1)^{n+1}$

$f(x) = \dfrac{3}{4} + \displaystyle\sum_{n=1}^{\infty} \dfrac{4}{n^2\pi^2}\left(\cos\dfrac{n\pi}{2} - 1\right)\cos\dfrac{n\pi}{2}x$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{2}{n\pi}(-1)^{n+1}\right)\sin\frac{n\pi}{2}x$$

32. $a_0 = \int_0^1 1\,dx + \int_1^2 (2-x)\,dx = \dfrac{3}{2}$

$$a_n = \int_0^1 1\cdot\cos\frac{n\pi}{2}x\,dx + \int_1^2 (2-x)\cos\frac{n\pi}{2}x\,dx = \frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)$$

$$b_n = \int_0^1 1\cdot\sin\frac{n\pi}{2}x\,dx + \int_1^2 (2-x)\sin\frac{n\pi}{2}x\,dx = \frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}$$

$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left(\cos\frac{n\pi}{2} + (-1)^{n+1}\right)\cos\frac{n\pi}{2}x$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi} + \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\sin\frac{n\pi}{2}x$$

33. $a_0 = 2\int_0^1 (x^2+x)\,dx = \dfrac{5}{3}$

$$a_n = 2\int_0^1 (x^2+x)\cos n\pi x\,dx = \frac{2(x^2+x)}{n\pi}\sin n\pi x\Big|_0^1 - \frac{2}{n\pi}\int_0^1 (2x+1)\sin n\pi x\,dx = \frac{2}{n^2\pi^2}[3(-1)^n - 1]$$

$$b_n = 2\int_0^1 (x^2+x)\sin n\pi x\,dx = -\frac{2(x^2+x)}{n\pi}\cos n\pi x\Big|_0^1 + \frac{2}{n\pi}\int_0^1 (2x+1)\cos n\pi x\,dx$$

$$= \frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]$$

$$f(x) = \frac{5}{6} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}[3(-1)^n - 1]\cos n\pi x$$

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{4}{n\pi}(-1)^{n+1} + \frac{4}{n^3\pi^3}[(-1)^n - 1]\right)\sin n\pi x$$

34. $a_0 = \int_0^2 (2x - x^2)\,dx = \dfrac{4}{3}$

$$a_n = \int_0^2 (2x - x^2)\cos\frac{n\pi}{2}x\,dx = \frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]$$

$$b_n = \int_0^2 (2x - x^2)\sin\frac{n\pi}{2}x\,dx = \frac{16}{n^3\pi^3}[1 - (-1)^n]$$

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty}\frac{8}{n^2\pi^2}[(-1)^{n+1} - 1]\cos\frac{n\pi}{2}x$$

$$f(x) = \sum_{n=1}^{\infty}\frac{16}{n^3\pi^3}[1 - (-1)^n]\sin\frac{n\pi}{2}x$$

35. $a_0 = \dfrac{1}{\pi}\int_0^{2\pi} x^2\,dx = \dfrac{8}{3}\pi^2$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx = -\frac{4\pi}{n}$$

$$f(x) = \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right)$$

36. $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos 2nx\,dx = 0$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin 2nx\,dx = -\frac{1}{n}$$

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx$$

37. $a_0 = 2\int_0^1 (x+1)\,dx = 3$

$$a_n = 2\int_0^1 (x+1)\cos 2n\pi x\,dx = 0$$

$$b_n = 2\int_0^1 (x+1)\sin 2n\pi x\,dx = -\frac{1}{n\pi}$$

$$f(x) = \frac{3}{2} - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin 2n\pi x$$

38. $a_0 = \frac{2}{2}\int_0^2 (2-x)\,dx = 2$

$$a_n = \frac{2}{2}\int_0^2 (2-x)\cos n\pi x\,dx = 0$$

$$b_n = \frac{2}{2}\int_0^2 (2-x)\sin n\pi x\,dx = \frac{2}{n\pi}$$

$$f(x) = 1 + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin n\pi x$$

39. We have

$$b_n = \frac{2}{\pi}\int_0^{\pi} 5\sin nt\,dt = \frac{10}{n\pi}[1-(-1)^n]$$

so that

$$f(t) = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin nt$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10-n^2)\sin nt = \sum_{n=1}^{\infty}\frac{10[1-(-1)^n]}{n\pi}\sin nt$$

and so $B_n = 10[1-(-1)^n]/n\pi(10-n^2)$. Thus

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt.$$

40. We have

$$b_n = \frac{2}{\pi}\int_0^1 (1-t)\sin n\pi t\,dt = \frac{2}{n\pi}$$

so that

$$f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t.$$

Substituting the assumption $x_p(t) = \sum_{n=1}^{\infty} B_n \sin n\pi t$ into the differential equation then gives

$$x_p'' + 10x_p = \sum_{n=1}^{\infty} B_n(10 - n^2\pi^2) \sin n\pi t = \sum_{n=1}^{\infty} \frac{2}{n\pi} \sin n\pi t$$

and so $B_n = 2/n\pi(10 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n(10 - n^2\pi^2)} \sin n\pi t.$$

41. We have

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2)\, dt = \frac{4}{3}\pi^2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (2\pi t - t^2) \cos nt\, dt = -\frac{4}{n^2}$$

so that

$$f(t) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos nt$$

into the differential equation then gives

$$\frac{1}{4} x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n \left(-\frac{1}{4}n^2 + 12\right) \cos nt = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nt$$

and $A_0 = \pi^2/9$, $A_n = 16/n^2(n^2 - 48)$. Thus

$$x_p(t) = \frac{\pi^2}{18} + 16 \sum_{n=1}^{\infty} \frac{1}{n^2(n^2 - 48)} \cos nt.$$

42. We have

$$a_0 = \frac{2}{1/2} \int_0^{1/2} t\, dt = \frac{1}{2}$$

$$a_n = \frac{2}{1/2} \int_0^{1/2} t \cos 2n\pi t\, dt = \frac{1}{n^2\pi^2} [(-1)^n - 1]$$

so that

$$f(t) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t.$$

Substituting the assumption

$$x_p(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos 2n\pi t$$

into the differential equation then gives

$$\frac{1}{4} x_p'' + 12x_p = 6A_0 + \sum_{n=1}^{\infty} A_n(12 - n^2\pi^2) \cos 2n\pi t = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2\pi^2} \cos 2n\pi t$$

and $A_0 = 1/24$, $A_n = [(-1)^n - 1]/n^2\pi^2(12 - n^2\pi^2)$. Thus

$$x_p(t) = \frac{1}{18} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2(12 - n^2\pi^2)}\cos 2n\pi t.$$

43. (a) The general solution is $x(t) = c_1\cos\sqrt{10}t + c_2\sin\sqrt{10}t + x_p(t)$, where

$$x_p(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n(10-n^2)}\sin nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 0$. Since $x_p(0) = 0$, we have $c_1 = 0$ and $x(t) = c_2\sin\sqrt{10}t + x_p(t)$. Then $x'(t) = c_2\sqrt{10}\cos\sqrt{10}t + x_p'(t)$ and $x'(0) = 0$ implies

$$c_2\sqrt{10} + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{10-n^2}\cos 0 = 0.$$

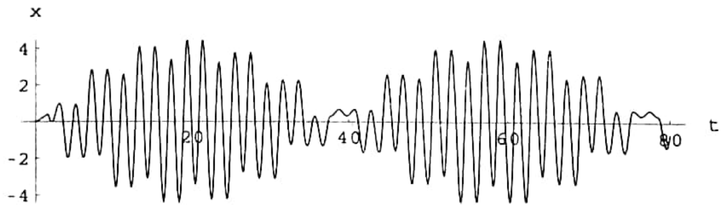
Thus

$$c_2 = -\frac{\sqrt{10}}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{10-n^2}$$

and

$$x(t) = \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{10-n^2}\left[\frac{1}{n}\sin nt - \frac{1}{\sqrt{10}}\sin\sqrt{10}t\right].$$

(b) The graph is plotted using eight nonzero terms in the series expansion of $x(t)$.



44. (a) The general solution is $x(t) = c_1\cos 4\sqrt{3}t + c_2\sin 4\sqrt{3}t + x_p(t)$, where

$$x_p(t) = \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}\cos nt.$$

The initial condition $x(0) = 0$ implies $c_1 + x_p(0) = 1$ or

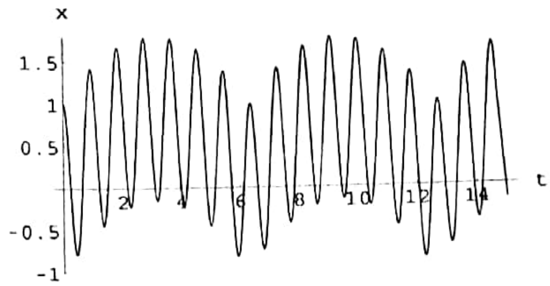
$$c_1 = 1 - x_p(0) = 1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}.$$

Now $x'(t) = -4\sqrt{3}c_1\sin 4\sqrt{3}t + 4\sqrt{3}c_2\cos 4\sqrt{3}t + x_p'(t)$, so $x'(0) = 0$ implies $4\sqrt{3}c_2 + x_p'(0) = 0$. Since $x_p'(0) = 0$, we have $c_2 = 0$ and

$$x(t) = \left(1 - \frac{\pi^2}{18} - 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}\right)\cos 4\sqrt{3}t + \frac{\pi^2}{18} + 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}\cos nt$$

$$= \frac{\pi^2}{18} + \left(1 - \frac{\pi^2}{18}\right)\cos 4\sqrt{3}t + 16\sum_{n=1}^{\infty}\frac{1}{n^2(n^2-48)}\left[\cos nt - \cos 4\sqrt{3}t\right].$$

(b) The graph is plotted using five nonzero terms in the series expansion of $x(t)$.



45. **(a)** We have

$$b_n = \frac{2}{L}\int_0^L \frac{w_0 x}{L}\sin\frac{n\pi}{L}x\,dx = \frac{2w_0}{n\pi}(-1)^{n+1}$$

so that

$$w(x) = \sum_{n=1}^{\infty} \frac{2w_0}{n\pi}(-1)^{n+1}\sin\frac{n\pi}{L}x.$$

(b) If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)} = \sum_{n=1}^{\infty} \frac{n^4\pi^4}{L^4} B_n \sin\frac{n\pi}{L}x$$

and so the differential equation $EIy_p^{(4)} = w(x)$ gives

$$B_n = \frac{2w_0(-1)^{n+1}L^4}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\sin\frac{n\pi}{L}x.$$

46. We have

$$b_n = \frac{2}{L}\int_{L/3}^{2L/3} w_0 \sin\frac{n\pi}{L}x\,dx = \frac{2w_0}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)$$

so that

$$w(x) = \sum_{n=1}^{\infty}\frac{2w_0}{n\pi}\left(\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}\right)\sin\frac{n\pi}{L}x.$$

If we assume $y_p(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x/L)$ then

$$y_p^{(4)}(x) = \sum_{n=1}^{\infty}\frac{n^4\pi^4}{L^4}B_n \sin\frac{n\pi}{L}x$$

and so the differential equation $EIy_p^{(4)}(x) = w(x)$ gives

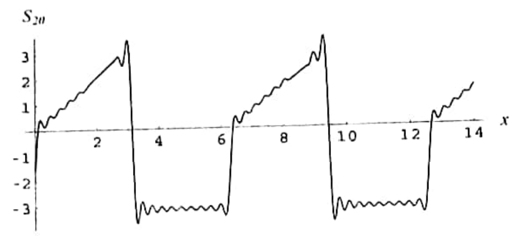
$$B_n = 2w_0L^4\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{EIn^5\pi^5}.$$

Thus

$$y_p(x) = \frac{2w_0L^4}{EI\pi^5}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi}{3} - \cos\frac{2n\pi}{3}}{n^5}\sin\frac{n\pi}{L}x.$$

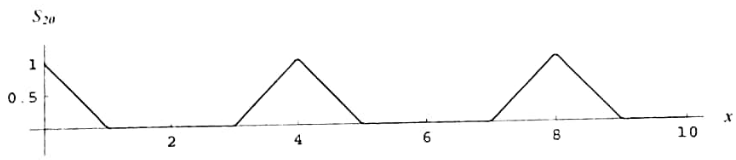
47. The graph is obtained by summing the series from $n = 1$ to 20. It appears that

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$



48. The graph is obtained by summing the series from $n = 1$ to 10. It appears that

$$f(x) = \begin{cases} 1 - x, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$



49. The function in Problem 47 is not unique; it could also be defined as

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ 1, & x = \pi \\ -\pi, & \pi < x < 2\pi. \end{cases}$$

The function in Problem 48 is not unique; it could also be defined as

$$f(x) = \begin{cases} 0, & -2 < x < -1 \\ x + 1, & -1 < x < 0 \\ -x + 1, & 0 < x < 1 \\ 0, & 1 < x < 2. \end{cases}$$

50. The cosine series converges to an even extension of the function on the interval $(-\pi, 0)$. Since the even extension of $f(x)$ is $f(-x)$, in this case $f(-x) = e^{-x}$ on $(-\pi, 0)$.

51. No, it is not a full Fourier series. A full Fourier series of $f(x) = e^x, 0 < x < \pi$, would converge to the π-periodic extension of f. The cosine and sine series converge to a 2π-periodic extension (even and odd, respectively). The average of the two series converges to a 2π-periodic extension of

$$f(x) = \begin{cases} e^x, & 0 < x < \pi \\ 0, & -\pi < x < 0. \end{cases}$$

52. **(a)** If f and g are even and $h(x) = f(x)g(x)$ then

$$h(-x) = f(-x)g(-x) = f(x)g(x) = h(x)$$

and h is even.