Fourier transform
(solved problem)

IV

Worked out Examples:

Example-1: Find the Fourier transform of $F(x) = \begin{cases} x : & |x| \le a \\ 0 : & |x| > a. \end{cases}$

Solution: The given function can be written as follows: $F(x) = \begin{cases} x : & -a \le x \le a \\ 0 ; & x > a \text{ or } x < a. \end{cases}$ --- ①

From the definition of Fourier transform we have -

$$f(s) = F\{F(x)\} = \int_{-\infty}^{\infty} F(x) e^{-isx} dx$$

$$= \int_{-\infty}^{-a} F(x) e^{-isx} dx + \int_{-a}^{a} F(x) e^{-isx} dx + \int_{a}^{\infty} F(x) e^{-isx} dx$$

$$= 0 + \int_{-a}^{a} x e^{-isx} dx + 0 \quad [\text{using ①}]$$

$$= \left[x \int e^{-isx} dx \right]_{-a}^{a} - \left[\int \left\{ \frac{d}{dx}(x) \int e^{-isx} dx \right\} dx \right]_{-a}^{a}$$

$$= \left[x . \frac{e^{-isx}}{-is} \right]_{-a}^{a} - \left[\int \frac{e^{-isx}}{-is} dx \right]_{-a}^{a}$$

$$= \frac{1}{-is} \left[x e^{-isx} \right]_{-a}^{a} + \frac{1}{s^2} \left[e^{-isx} \right]_{-a}^{a} \quad [\because i^2 = -1]$$

$$= -\frac{1}{is} \left[a e^{-isa} + a e^{isa} \right] + \frac{1}{s^2} \left[e^{-isa} - e^{isa} \right]$$

$$= \frac{-2a}{is} \left[\frac{e^{-isa} + e^{isa}}{2} \right] + \frac{(-2i)}{s^2} \left[\frac{e^{isa} - e^{-isa}}{2i} \right]$$

$$\Rightarrow F\{F(x)\} = -\frac{2a}{is}\cos(sa) - \frac{2i}{s^2}\sin(sa)$$

$$\Rightarrow F\{F(x)\} = -\frac{2i}{i^2 s}a\cos(sa) - \frac{2i}{s^2}\sin(sa)$$

$$\Rightarrow F\{F(x)\} = \frac{2i}{s^2}\left[a\cos(sa) - \sin(sa)\right], \quad s \neq 0$$

Which is the required Fourier transform of given function.

Example-2: Find the Fourier sine transform of

$$F(x) = \begin{cases} x ; & 0 < x < 1 \\ 2-x ; & 1 < x < 2 \\ 0 ; & x > 2 \end{cases}$$

Solution: Given that

$$F(x) = \begin{cases} x ; & 0 < x < 1 \\ 2-x ; & 1 < x < 2 \\ 0 ; & x > 2 \end{cases} \quad \text{-----} \quad (1)$$

We know from definition that

$$f_s(s) = F_s\{F(x)\} = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^1 F(x)\sin(sx)\,dx + \int_1^2 F(x)\sin(sx)\,dx + \int_2^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^1 x\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx + \int_2^\infty 0.\sin sx\,dx$$

$$= \int_0^1 x\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx + 0.$$

$$= \left[x\int \sin sx\,dx\right]_0^1 - \left[\int\left\{\frac{d}{dx}(x)\int \sin sx\,dx\right\}dx\right]_0^1$$

$$+ 2\int_1^2 \sin sx\,dx - \int_1^2 x\sin sx\,dx$$

$$= \left[\frac{-x\cos sx}{s}\right]_0^1 + \left[\frac{\sin sx}{s^2}\right]_0^1 + 2\left[-\frac{\cos sx}{s}\right]_1^2$$

$$-\left[\frac{-x\cos sx}{s}\right]_1^2 + \left[\frac{\sin sx}{s^2}\right]_1^2$$

$$= \frac{1}{s}\left[-\cos s\right] + \frac{1}{s^2}\sin s - \frac{2}{s}\left[\cos 2s - \cos s\right]$$

$$+ \frac{1}{s}\left[2\cos 2s - \cos s\right] + \frac{1}{s^2}\left[\sin 2s - \sin s\right]$$

$$= -\cancel{\frac{\cos s}{s}} + \frac{\sin s}{s^2} - \cancel{\frac{2\cos 2s}{s}} + \cancel{\frac{2\cos s}{s}}$$

$$+ \cancel{\frac{2\cos 2s}{s}} - \cancel{\frac{\cos s}{s}} - \frac{\sin 2s}{s^2} + \frac{\sin s}{s^2}$$

$$= \frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2}$$

$$= \frac{2\sin s}{s^2} - \frac{2\sin s\cos s}{s^2}$$

$$= \frac{2\sin s}{s^2}\left[1 - \cos s\right]$$ which is the required Fourier sine transform of the given function.

Example-3: (a) Find the fourier sine and cosine transform of $F(x) = e^{-x}$, $x > 0$.

(b) hence evaluate (i) $\int_0^\infty \frac{dx}{(x^2+1)^2}$ & (ii) $\int_0^\infty \frac{x^2 dx}{(x^2+1)^2}$ by use of parseval's identity.

Solution: (a) From the definition of Fourier Cosine transform we have

$$F_c\{F(x)\} = f_c(s) = \int_0^\infty F(x)\cos sx\, dx$$

$$\Rightarrow f_c(s) = \int_0^\infty e^{-x}\cos sx\, dx$$

$$\Rightarrow f_c(s) = \left[\frac{e^{-x}(-1\cos sx + s\sin sx)}{(-1)^2 + s^2}\right]_0^\infty$$

$$\Rightarrow f_c(s) = \frac{0 - e^0(-1+0)}{1+s^2} = \frac{1}{1+s^2}$$

$$\therefore f_c(s) = \frac{1}{1+s^2}$$ ~~or f(s) (less)~~

Now from the parseval's identity of fourier cosine transform we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(s)|^2 ds = \int_{-\infty}^{\infty} |F(x)|^2 dx$$

$$\Rightarrow \frac{2}{2\pi}\int_0^\infty |f(s)|^2 ds = 2\int_0^\infty |F(x)|^2 dx$$

$$\int_0^\infty |F(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |f_c(s)|^2 ds$$

$$\Rightarrow \int_0^\infty |e^{-x}|^2 dx = \frac{2}{\pi}\int_0^\infty \frac{1}{(s^2+1)^2} ds$$

$$\Rightarrow \int_0^\infty e^{-2x} dx = \frac{2}{\pi}\int_0^\infty \frac{ds}{(s^2+1)^2}$$

$$\Rightarrow \left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{ds}{(s^2+1)^2}$$

$$\Rightarrow \frac{0+1}{2} = \frac{2}{\pi}\int_0^\infty \frac{ds}{(s^2+1)^2}$$

$$\Rightarrow \frac{\pi}{4} = \int_0^\infty \frac{dx}{(x^2+1)^2} \quad \text{[Replacing } s \text{ by } x\text{]}$$

i.e. $\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$ Ans.

Again from the definition of Fourier Sine transform we have

$$f_s(s) = F_s\{F(x)\} = \int_0^\infty F(x)\sin sx\, dx$$

$$\Rightarrow f_s(s) = \int_0^\infty e^{-x}\sin sx\, dx$$

$$\Rightarrow f_s(s) = \left[\frac{e^{-x}(-1\sin sx - s\cos sx)}{(-1)^2+s^2}\right]_0^\infty$$

$$\Rightarrow f_s(s) = \frac{1}{1+s^2}\left[0 - e^0(0-s)\right] = \frac{s}{s^2+1}$$

Now from the parseval's idetity of Fourier sine transform we get

$$\int_0^\infty |F(x)|^2 dx = \frac{2}{\pi}\int_0^\infty |f_s(s)|^2 ds$$

$$\Rightarrow \int_0^\infty |e^{-x}|^2 dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+1)^2} ds$$

$$\Rightarrow \int_0^\infty e^{-2x} dx = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+1)^2} ds$$

$$\Rightarrow \left[\frac{e^{-2x}}{-2}\right]_0^\infty = \frac{2}{\pi}\int_0^\infty \frac{s^2}{(s^2+1)^2} ds$$

$$\Rightarrow \frac{1}{2} = \frac{2}{\pi}\int_0^\infty \frac{x^2}{(x^2+1)^2} dx$$

$$\Rightarrow \int_0^\infty \frac{x^2 dx}{(x^2+1)^2} = \frac{\pi}{4}$$ Ans.

Example-4: What is the function $F(x)$ or Find $F(x)$ if its Fourier sine transform is $\frac{e^{-as}}{s}$. Hence deduce $F_s^{-1}\left\{\frac{1}{s}\right\}$.

Solution; Given that $f_s(s) = \frac{e^{-as}}{s}$.

We know $F(x) = F^{-1}\{f_s(s)\} = \frac{2}{\pi}\int_0^\infty f_s(s)\sin sx\,ds$

$$\Rightarrow F(x) = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin(sx)\,ds \text{ - - - } (1)$$

Let $I = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\sin(sx)\,ds$ - - - - (2)

$$\Rightarrow \frac{dI}{dx} = \frac{2}{\pi}\int_0^\infty \frac{e^{-as}}{s}\cdot s\cos(sx)\,ds$$

$$\Rightarrow \frac{dI}{ds} = \frac{2}{\pi}\int_0^\infty e^{-as}\cos(sx)\,ds$$

$$\Rightarrow \frac{dI}{ds} = \frac{2}{\pi}\cdot\frac{a}{x^2+a^2}$$

$$\Rightarrow \frac{dI}{dx} = \frac{2a}{\pi}\cdot\frac{1}{x^2+a^2}$$

Integrating both sides we get

$$I = \frac{2a}{\pi}\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

$$\Rightarrow I = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right) + c \text{ - - - - } (3)$$

Set $x=0$ in (3) we get $I = 0 + c \Rightarrow c = I$

Also set $x=0$ in (2) we get $I = 0 \Rightarrow I = c = 0$

$$\therefore I = \frac{2}{\pi}\tan^{-1}\left(\frac{x}{a}\right)$$

i.e. $F(x) = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right)$.

2nd part:

$$\mathcal{F}^{-1}\{f_s(s)\} = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right)$$

$$\therefore \mathcal{F}^{-1}\left\{\frac{e^{-as}}{s}\right\} = \frac{2}{\pi} \tan^{-1}\left(\frac{x}{a}\right) \quad \text{---- (4)}$$

Set $a=0$, in (4) we get

$$\mathcal{F}^{-1}\left\{\frac{1}{s}\right\} = \frac{2}{\pi} \tan^{-1}(\tan \pi/2) = \frac{2}{\pi} \cdot \frac{\pi}{2} = 1$$

i.e. $\mathcal{F}^{-1}\left\{\frac{1}{s}\right\} = 1$ Ans.

Example-5: Find the (a) finite Fourier sine transform (b) finite Fourier cosine transform of $F(x) = 2x,\ 0<x<4$.

Solution: (a) We know that the finite Fourier sine transform of $F(x)$ is

$$f_s(s) = \mathcal{F}_s\{F(x)\} = \int_0^4 F(x) \sin\left(\frac{s\pi x}{4}\right) dx$$

$$\Rightarrow f_s(s) = \int_0^4 (2x) \sin \frac{s\pi x}{4}\, dx$$

$$= \left[2x \int \sin \frac{s\pi x}{4} dx\right]_0^4 - \int_0^4 \left\{\frac{d}{dx}(2x) \int \sin \frac{s\pi x}{4} dx\right\} dx$$

$$= \left[-2x \cdot \frac{4}{s\pi} \cos \frac{s\pi x}{4}\right]_0^4 + \frac{8}{s\pi}\left[\int \cos\left(\frac{s\pi x}{4}\right) dx\right]_0^4$$

$$= -\frac{32}{s\pi} \cos s\pi + 0 + \frac{32}{s^2\pi^2}\left[\sin \frac{s\pi x}{4}\right]_0^4$$

$$= -\frac{32}{s\pi}\cos s\pi + \frac{32}{s^2\pi^2}[0-0]$$

$$= -\frac{32}{s\pi}\cos s\pi$$ which is the required finite Fourier sine transform of the given function.

(b) From the definition of finite Fourier Cosine transform, we have

$$f_c(s) = F_c\{F(x)\} = \int_0^4 F(x)\cos\left(\frac{s\pi x}{4}\right)dx$$

$$\Rightarrow f_c(s) = \int_0^4 (2x).\cos\frac{s\pi x}{4}\,dx$$

$$\Rightarrow f_c(s) = \left[2x.\frac{4}{s\pi}\sin\frac{s\pi x}{4}\right]_0^4 - \frac{8}{s\pi}\int_0^4 \sin\left(\frac{s\pi x}{4}\right)dx$$

$$\Rightarrow f_c(s) = 0 - \frac{8}{s\pi}\left(-\frac{4}{s\pi}\right).\left[\cos\frac{s\pi x}{4}\right]_0^4$$

$$\Rightarrow f_c(s) = 0 + \frac{32}{s^2\pi^2}[\cos s\pi - 1]$$

i.e. $f_c(s) = \frac{32}{s^2\pi^2}[\cos s\pi - 1]$ which is the required finite Fourier cosine transform of the given function.

Application of fourier transform:

Example-6: Use finite fourier transform to solve $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $U(0,t)=0$, $U(4,t)=0$, $U(x,0)=2x$, where $0<x<4$, $t>0$ and interpret the result physically.

Solution: Given that $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ ---- (1)

with $U(0,t)=0$, $U(4,t)=0$, $U(x,0)=2x$, $0<x<4$

According to given B.C. here FF sine transform is more useful.

Taking the finite fourier sine transform of both sides of (1) with $l=4$, we get

$$\int_0^4 \frac{\partial U}{\partial t} \sin\left(\frac{s\pi x}{4}\right) dx = \int_0^4 \frac{\partial^2 U}{\partial x^2} \sin\left(\frac{s\pi x}{4}\right) dx \text{ --- (2)}$$

Let $u = u(s,t) = \int_0^4 U(x,t) \sin\left(\frac{s\pi x}{4}\right) dx$ $\left(\cancel{= f_s(s)}\right)$ ---- (3)

Then $\frac{du}{dt} = \int_0^4 \frac{\partial U}{\partial t} \sin\left(\frac{s\pi x}{4}\right) dx$

$$= \int_0^4 \frac{\partial^2 U}{\partial x^2} \sin\left(\frac{s\pi x}{4}\right) dx \quad [\text{Using } (2)]$$

$$= \left[\sin\left(\frac{s\pi x}{4}\right) \cdot \frac{\partial U}{\partial x}\right]_0^4 - \frac{s\pi}{4} \int_0^4 \cos\left(\frac{s\pi x}{4}\right) \cdot \frac{\partial U}{\partial x} dx$$

$$= 0 - \frac{s\pi}{4} \int_0^4 \cos\left(\frac{s\pi x}{4}\right) \cdot \frac{\partial U}{\partial x} dx$$

$$= -\frac{s\pi}{4} \left[\cos\left(\frac{s\pi x}{4}\right) \cdot U(x,t)\right]_0^4 - \frac{s^2\pi^2}{16} \int_0^4 U(x,t) \sin\left(\frac{s\pi x}{4}\right) dx$$

$$\Rightarrow \frac{du}{dt} = 0 - \frac{s^2\pi^2}{16}\int_0^4 U(x,t)\sin\left(\frac{s\pi x}{4}\right)dx$$

$$\left[\because U(0,t) = U(4,t) = 0\right]$$

$$\Rightarrow \frac{du}{dt} = -\frac{s^2\pi^2}{16}\cdot u \quad [\text{using } ③]$$

$$\Rightarrow \frac{du}{u} = -\frac{s^2\pi^2}{16}\,dt$$

Integrating both sides we get

$\log u = -\frac{s^2\pi^2}{16}t + \log A$, where A being some constant of integration.

$$\Rightarrow \log u = \log e^{-\frac{s^2\pi^2}{16}t} + \log A$$

$$\Rightarrow \log u = \log\left\{A\cdot e^{-\frac{s^2\pi^2}{16}t}\right\}$$

$$\Rightarrow u = Ae^{-\frac{s^2\pi^2}{16}t}$$

$$\Rightarrow u(s,t) = Ae^{-\frac{s^2\pi^2}{16}t} \quad \cdots\cdots ④$$

When $t=0$, then $u(s,0) = Ae^0 = A$

$$\Rightarrow A = u(s,0) \quad \cdots\cdots ⑤$$

Now from ③ we get

$$u(s,t) = \int_0^4 U(x,t)\sin\left(\frac{s\pi x}{4}\right)dx$$

$$\Rightarrow u(s,0) = \int_0^4 U(x,0)\sin\left(\frac{s\pi x}{4}\right)dx \quad [\because t=0]$$

$$\Rightarrow u(s,0) = \int_0^4 (2x)\cdot\sin\frac{s\pi x}{4}\,dx$$

$$\Rightarrow u(s,0) = \left[-2x \cdot \frac{4}{s\pi} \cos\left(\frac{s\pi x}{4}\right)\right]_0^4 + \frac{8}{s\pi}\int_0^4 \cos\left(\frac{s\pi x}{4}\right) dx$$

$$\Rightarrow u(s,0) = -\frac{32}{s\pi}\cos(s\pi) + 0 + \frac{32}{s^2\pi^2}\left[\sin\left(\frac{s\pi x}{4}\right)\right]_0^4$$

$$\Rightarrow u(s,0) = -\frac{32}{s\pi}\cos(s\pi) + 0$$

$$\Rightarrow u(s,0) = -\frac{32}{s\pi}\cos(s\pi)$$

$$\Rightarrow A = -\frac{32}{s\pi}\cos(s\pi) \quad [\text{using } 5]$$

Now putting the values of A in 4

$$u(s,t) = -\frac{32}{s\pi}\cos(s\pi) \cdot e^{-\frac{s^2\pi^2}{16}t} \quad (= f_s(s) \text{ say})$$

Now applying the inversion formula for finite Fourier sine transform, we get

$$U(x,t) = \frac{2}{4} \sum_{s=1}^{\infty}\left[\left(-\frac{32}{s\pi}\right)\cos(s\pi)\, e^{-\frac{s^2\pi^2}{16}t} \cdot \sin\left(\frac{s\pi x}{4}\right)\right]$$

Which is the required solution of the given partial differential equation.

$$\because \cos s\pi = (-1)^s$$

[P.T.O]

$$U(x,t) = \frac{16}{\pi}\sum_{s=1}^{\infty}\left[\frac{(-1)^{s+1}}{s} e^{-\frac{s^2\pi^2 t}{16}} \cdot \sin\left(\frac{s\pi x}{4}\right)\right]$$

2nd part: physical interpretation:

Physically, $U(x,t)$ represents the temperature at any point x at any time t in solid bounded by the planes $x=0$ and $x=4$. The Conditions $U(0,t)=0$ and $U(4,t)=0$ implies that the ends are kept at zero temperature while $U(x,0)=2x$ implies that the initial temperature is a function of x.

Example-7: Using Fourier transform (finite) to solve $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$; $U_x(0,t)=0$; $U_x(6,t)=0$, $U(x,0)=2x$, $0<x<6$, $t>0$.

solution: Given that $\frac{\partial U}{\partial t}=\frac{\partial^2 U}{\partial x^2}$ --- ①

with $U_x(0,t)=0$, $U_x(6,t)=0$ & $U(x,0)=2x$ --- ②

According to the given boundary conditions, here the finite Fourier cosine transform is more useful.

Now taking the finite Fourier Cosine transform (with $l=6$) of both sides of ①,

we get -
$$\int_0^6 \frac{\partial U}{\partial t}\cos\frac{s\pi x}{6}dx=\int_0^6\frac{\partial^2 U}{\partial x^2}\cos\frac{s\pi x}{6}dx \quad \text{--- ③}$$

Let $u = u(s,t) = \int_0^6 U(x,t) \cos \frac{s\pi x}{6}\, dx$ ... (4)

Then $\frac{du}{dt} = \int_0^6 \frac{\partial U}{\partial t} \cos \frac{s\pi x}{6}\, dx$

$= \int_0^6 \frac{\partial^2 U}{\partial x^2} \cos \frac{s\pi x}{6}\, dx$ [using (3)]

$= \left[\cos \frac{s\pi x}{6} \cdot \frac{\partial U}{\partial x} \right]_0^6 + \frac{s\pi}{6} \int_0^6 \sin \frac{s\pi x}{6} \frac{\partial U}{\partial x}\, dx$

$= \left[\cos \frac{s\pi x}{6} U_x(x,t) \right]_0^6 + \frac{s\pi}{6} \left\{ \left[\sin \frac{s\pi x}{6} . U(x,t) \right]_0^6 \right.$

$\left. - \frac{s\pi}{6} \int_0^6 \cos \frac{s\pi x}{6} U(x,t)\, dx \right\}$

$= \cos s\pi . U_x(6,t) - 1.U_x(0,t) + \frac{s\pi}{6} \left\{ \sin s\pi . U(6.t) - 0 \right.$

$\left. - \frac{s\pi}{6} u \right\}$ [using (4)]

$= \cos s\pi . 0 - 1.0 + \frac{s\pi}{6} \left\{ 0 . U(6.t) - \frac{s\pi}{6} u \right\}$

[using (2) & $\sin s\pi = 0$]

$\Rightarrow \frac{du}{dt} = - \frac{s^2 \pi^2}{36} u$

$\Rightarrow \frac{du}{u} = - \frac{s^2 \pi^2}{36} dt$

Integrating both sides we get

$\log u = - \frac{s^2 \pi^2}{36} t + \log A$, A being the const.

$\Rightarrow \log u = \log e^{- \frac{s^2 \pi^2}{36} t} + \log A$

$\Rightarrow \log u = \log \left\{ A . e^{- \frac{s^2 \pi^2}{36} t} \right\}$

$$\Rightarrow u = Ae^{-\frac{s^2\pi^2}{36}t}$$

$$\Rightarrow u(s,t) = Ae^{-\frac{s^2\pi^2}{36}t} \quad \text{-----} (5)$$

putting $t=0$ in (5) we get

$$u(s,0) = Ae^0 = A \quad \Rightarrow A = u(s,0) \quad \text{-----} (6)$$

Again putting $t=0$ in (4) we get

$$u(s,0) = \int_0^6 u(x,0)\cos\frac{s\pi x}{6}\,dx$$

$$\Rightarrow A = \int_0^6 2x.\cos\frac{s\pi x}{6}\,dx \quad [\text{using } (2) \,\&\, (6)]$$

$$\Rightarrow A = 2\left[x.\frac{\sin(\frac{s\pi x}{6})}{s\pi/6} - (1).\frac{-\cos(s\pi x/6)}{(s\pi/6)^2}\right]_0^6$$

$$\Rightarrow A = 2.\left[\frac{6x}{s\pi}\sin\left(\frac{s\pi x}{6}\right) + \frac{36}{s^2\pi^2}\cos\left(\frac{s\pi x}{6}\right)\right]_0^6$$

$$\Rightarrow A = 2.\left[\frac{36}{s\pi}\sin s\pi - 0 + \frac{36}{s^2\pi^2}\cos s\pi - \frac{36}{s^2\pi^2}.1\right]$$

$$\Rightarrow A = 2.\left[\frac{36}{s\pi}.0 - 0 + \frac{36}{s^2\pi^2}\cos s\pi - \frac{36}{s^2\pi^2}\right]$$

$$\Rightarrow A = \frac{72}{s^2\pi^2}\left[\cos s\pi - 1\right]$$

$\therefore$ Putting the value of A in (5) we obtain

$$u(s,t) = \frac{72}{s^2\pi^2}\left[\cos s\pi - 1\right].e^{-\frac{s^2\pi^2}{36}t} \quad \text{---} (7)$$

Now taking the inverse finite Fourier Cosine transform on both sides of (7) we get

$$U(x,t) = \frac{1}{6}f_c(0) + \frac{2}{6}\sum_{s=1}^{\infty}\frac{72}{s^2\pi^2}(\cos s\pi - 1)e^{-\frac{s^2\pi^2}{36}t}.\cos\frac{s\pi x}{6}$$

$$\text{---} (8)$$

Since $f_c(s) = \int_0^6 \cancel{U(x,t)}\, F(x) \cos \frac{s\pi x}{6}\, dx$

So $f_c(0) = \int_0^6 F(x)\, dx = \int_0^6 U(x,0)\, dx$

$\Rightarrow f_c(0) = \int_0^6 2x\, dx = [x^2]_0^6 = 36$

Then from (6) we obtain

$$U(x,t) = 6 + \frac{24}{\pi^2} \sum_{s=1}^{\infty} \frac{(\cos s\pi - 1)}{s^2} e^{-\frac{s^2\pi^2}{36}t} \cdot \cos \frac{s\pi x}{6}$$

Which is the required solution of the given partial differential equation.

<u>Example- 8:</u> Use the complex form of the Fourier transform to solve the boundary value problem $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$, $U(x,0) = f(x)$, $|U(x,t)| < M$ where $-\infty < x < \infty$ and also give a physical interpretation.

<u>Solution:</u> Given that $\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}$ - - - (1)

$U(x,0) = f(x)$ - - - - (2)

since $|U(x,t)| < M$

So $\lim_{x \to \infty} U(x,t) = U(\infty, t) = 0$ - - - (3)

& $\lim_{x \to \infty} U_x(x,t) = U_x(\infty, t) = 0$ - - - (4)

Taking the Fourier transform of both sides of (1) we get

$$\mathcal{F}\left\{\frac{\partial U}{\partial t}\right\} = \mathcal{F}\left\{\frac{\partial^2 U}{\partial x^2}\right\}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{-isx} dx = \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{-isx} dx \quad \text{--- (5)}$$

Let $u = u(s,t) = \int_{-\infty}^{\infty} U(x,t) e^{-isx} dx$ --- (6)

$$\therefore \frac{du}{dt} = \int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{-isx} dx$$

$$\Rightarrow \frac{du}{dt} = \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{-isx} dx \quad [\text{Using (5)}]$$

$$\Rightarrow \frac{du}{dt} = \left[e^{-isx} \frac{\partial U}{\partial x} \right]_{-\infty}^{\infty} + is \int_{-\infty}^{\infty} e^{-isx} \frac{\partial U}{\partial x} dx$$

$$\Rightarrow \frac{du}{dt} = \left[e^{-isx} U_x(x,t) \right]_{-\infty}^{\infty} + is \left\{ \left[e^{-isx} U(x,t) \right]_{-\infty}^{\infty} + is \int_{-\infty}^{\infty} e^{-isx} U(x,t) dx \right\}$$

$$\Rightarrow \frac{du}{dt} = 0 + is \cdot 0 + i^2 s^2 u \quad [\text{by using (3), (4) \& (6)}]$$

$$\Rightarrow \frac{du}{dt} = -s^2 u$$

$\Rightarrow \frac{du}{u} = -s^2 dt$, integrating both sides we get

$\Rightarrow \log u = -s^2 t + \log A$, A being constant

$$\Rightarrow \log u = \log e^{-s^2 t} + \log A = \log(A e^{-s^2 t})$$

$$\Rightarrow u = A e^{-s^2 t}$$

$$\therefore u(s,t) = A e^{-s^2 t} \quad \cdots \quad (7)$$

$$\Rightarrow u(s,0) = A e^0 = A$$

$$\Rightarrow A = u(s,0) \quad \cdots \quad (8)$$

Putting $t=0$ in (6) we get

$$u(s,0) = \int_{-\infty}^{\infty} u(x,0)\, e^{-isx}\, dx$$

$$\Rightarrow A = \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx = F\{f(x)\} = f(s) \quad \cdots \quad (9)$$

where $f(s)$ is the Fourier transform of $f(x)$.

$\therefore$ From (7) we get

$$u(s,t) = f(s) \cdot e^{-s^2 t}$$

Now taking the inverse Fourier transform on both sides we obtain

$$U(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ e^{-s^2 t} \cdot f(s) \} \cdot e^{isx}\, ds$$

$$\Rightarrow U(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(s) \cdot e^{-s^2 t + isx} \cdot ds$$

Which is the required solution, where $f(s)$ is the Fourier transform of $f(x)$.

Ans.

Physical interpretation:

The problem is that of determining the temperature in a thin infinite bar whose surface is insulated and whose initial temperature is $f(x)$.