ME 3109: Measurement & Instrumentation

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Topic_05: Measurement System Error & Uncertainty[1]	
1	Basic Terminology & Errors
2	Uncertainty Analysis
3	Probability Distribution

[1] Figliola's "***Theory & Design for Mechanical Measurements***", 5th Edition

1. Basic Terminology & Errors

Error → Diff. between the measured value and the true value being measured

True value → Value that would be obtained from a theoretical perfect measurement

Accuracy → Describes how close a set of measurements are to the true value.

Precision → Describes the spread of repeated measurements.

Repeatability → Closeness of agreement betwn repeated measurements of same thing, carried out by same person, at same time, in same way, on same equipments etc.

1. Basic Terminology & Errors

Reproducibility → Closeness of agreement between measurements of same thing carried out in different circumstances.

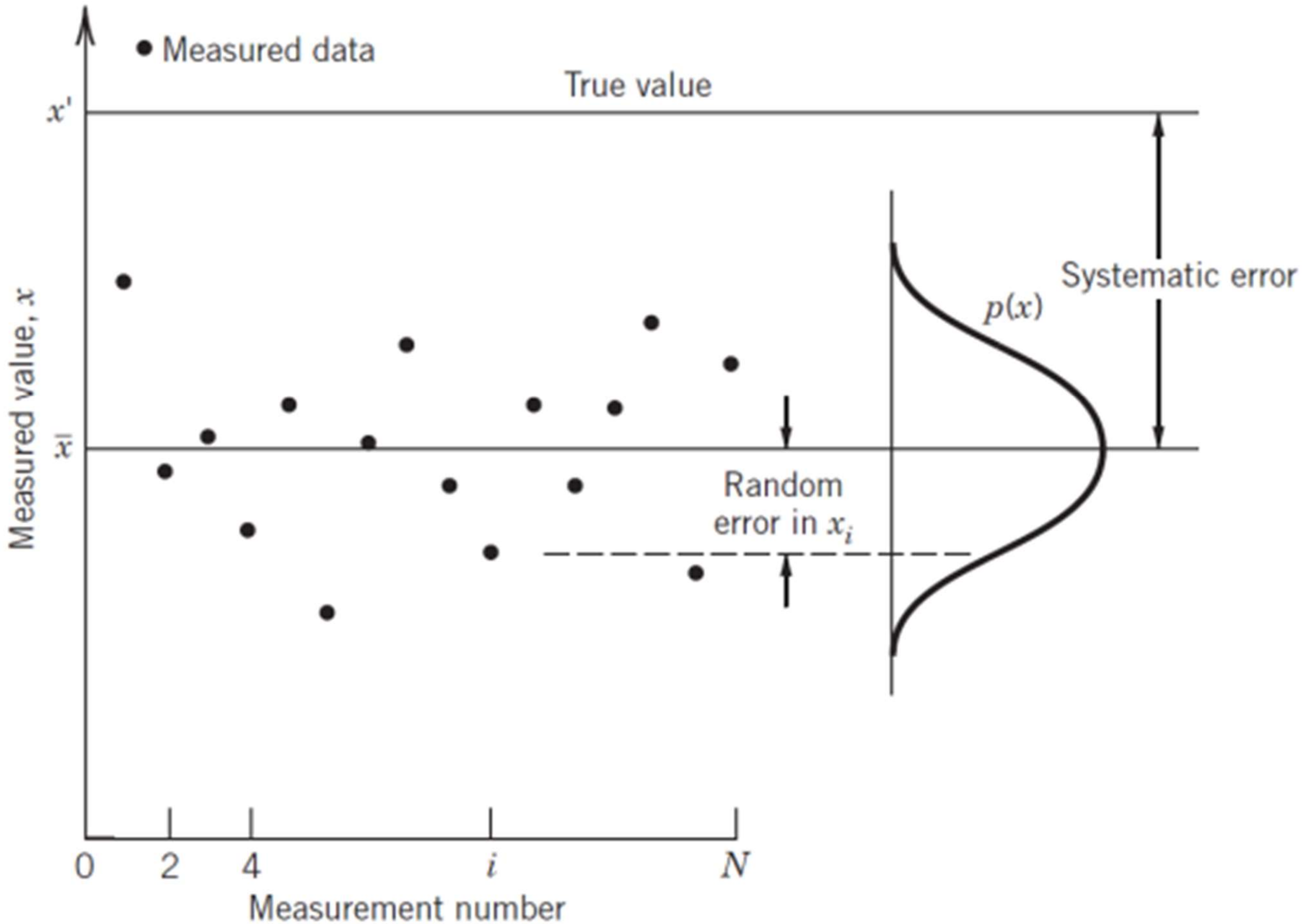
Range → Limits defining the min. and max. values for which the measurement system is to be used.

Resolution → Smallest increament in the measured value that can be discerned.

Uncertainty → Quantifies the quality of result.

Errors are a property of the measurement & Uncertainty is a property of the result

1. Experimental Errors



$$x' = \bar{x} \pm u_x$$

Sample mean

Uncertainty

2. Uncertainty Analysis

Electrical Power, $P = EI$; where, $E = 50\ V \pm 4\ V$

$I = 5\ A \pm 0.1\ A$

Nominal Power, $P = \overline{E} \times \overline{I} = 50 \times 5 = 250\ W$

Max Power, $P_{max} = E_{max} \times I_{max} = 54 \times 5.1 = 275.4\ W$

Min Power, $P_{min} = E_{min} \times I_{min} = 46 \times 4.9 = 225.4\ W$

Sol Uncertainty in the power is $+10.16\%$, -9.84%

It is quite unlikely that the power would be in error by these amounts because the voltmeter variations would probably not correspond with the ammeter variations

2. Uncertainty Analysis: Propagation of Uncertainty

Result, $R = R(x_1, x_2, x_3, \ldots, x_n)$; where x_i's are independent variable

w_i's $\longrightarrow$ uncertainties in the independent variables

w_R $\longrightarrow$ uncertainty in the result

If the uncertainties in the independent variables are all given with the same odds, then the uncertainty in the result having these odds is given,

$$w_R = \left[\left(\frac{\partial R}{\partial x_1} w_1\right)^2 + \left(\frac{\partial R}{\partial x_2} w_2\right)^2 + \cdots + \left(\frac{\partial R}{\partial x_n} w_n\right)^2\right]^{1/2}$$

$\boxed{\text{Electric Power}}$

$\frac{\partial P}{\partial E} = I = 5A$; $w_E = 4V$

$\frac{\partial P}{\partial I} = E = 50V$; $w_I = 0.1A$

$w_P = \sqrt{(5 \times 4)^2 + (50 \times 0.1)^2}$

$= 20.6\,W \Rightarrow 8.2\%$

2. Uncertainty Analysis: Uncertainties of Additive Functions

Result function, $\longrightarrow$ $R = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \sum a_i x_i$

partial derivatives, $\longrightarrow$ $\dfrac{\partial R}{\partial x_i} = a_i$

The uncertainty, $\longrightarrow$

$$w_R = \left\{ \sum \left[\left(\frac{\partial R}{\partial x_i} \right)^2 w_{x_i}^2 \right] \right\}^{1/2}$$

$$= \left[\sum \left(a_i w_{x_i} \right)^2 \right]^{1/2}$$

2. Uncertainty Analysis : Uncertainties of Product Functions

Result function → $R = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$

Partial derivation → $$\frac{\partial R}{\partial x_i} = x_1^{a_1} x_2^{a_2} \left(a_i x_i^{a_i - 1}\right) \cdots x_n^{a_n}$$

Divide by R → $$\frac{1}{R}\frac{\partial R}{\partial x_i} = \frac{a_i}{x_i}$$

Uncertainty → $$\frac{w_R}{R} = \left[\sum \left(\frac{a_i w_{x_i}}{x_i}\right)^2\right]^{1/2}$$

Statistical Analysis of Experimental Data

Arithmetic mean, $x_m = \frac{1}{n}\sum_{i=1}^{n} x_i$

population standard deviation, $\sigma = \left[\frac{1}{n}\sum_{i=1}^{n}(x_i - x_m)^2\right]^{1/2}$

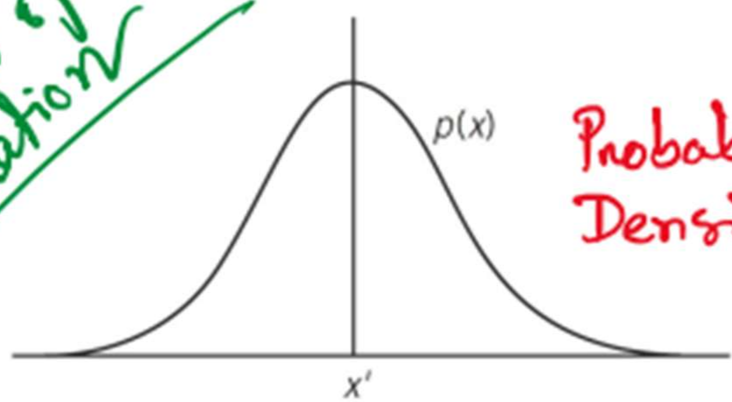
sample standard deviation, $\sigma = \left[\frac{\sum_{i=1}^{n}(x_i - x_m)^2}{n-1}\right]^{1/2}$

standard deviation of mean, $\sigma_m = \frac{\sigma}{\sqrt{n}}$

finite-sized Data set

Normal Distribution

Behavior of Population



Probability Density → $P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(x - x_m)^2}{\sigma^2}\right]$

True mean

Variance

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1$$

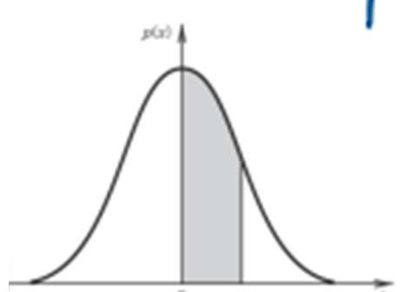
Gaussian or normal distribution

Define, $z_1 = \frac{x_1 - x_m}{\sigma}$

$\beta = \frac{x - x_m}{\sigma}$

$$\Rightarrow P(-z_1 \le \beta \le z_1) = \frac{1}{\sqrt{2\pi}} \int_{-z_1}^{z_1} e^{-\beta^2/2}\, d\beta = \left[\frac{1}{\sqrt{2\pi}} \int_{0}^{z_1} e^{-\beta^2/2}\, d\beta\right] \times 2$$

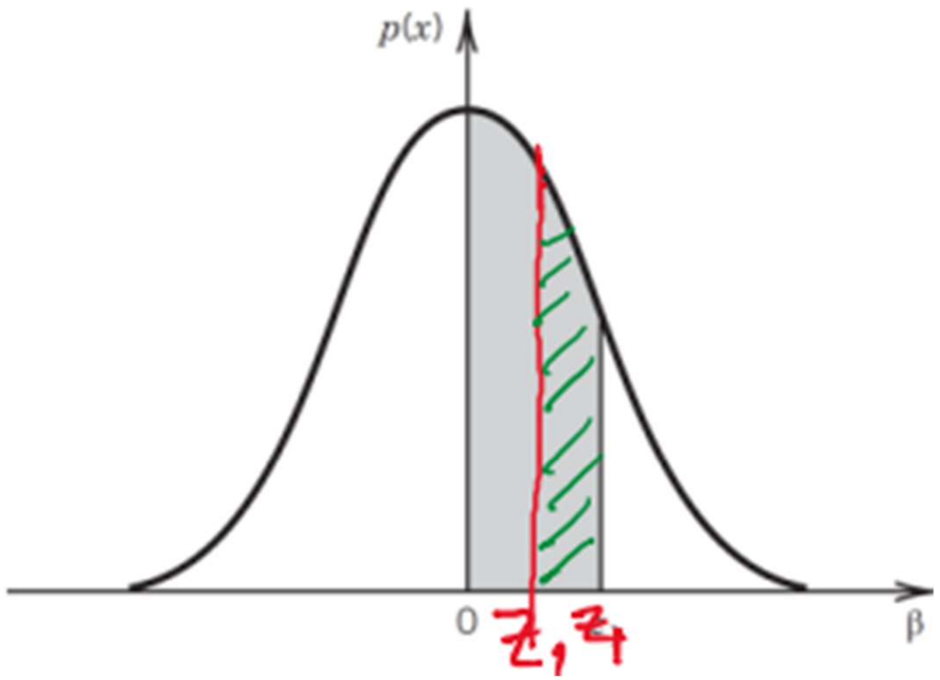
Normal Error Function



3. Probability Distribution

Problem 01:

The population of a well-defined varying voltage signal are described by x_m = 8.5 V and σ^2 =2.25 V^2. If a single measurement of the voltage signal is made, determine the probability that this measured value will have a value of between 10.0 and 11.5 V. Signal has a normal distribution about x_m.



Soln: $\sigma = \sqrt{\sigma^2} = 1.5\ V$

$$z_1 = \frac{11.5 - 8.5}{1.5} = 2$$

$$z_1 = \frac{10 - 8.5}{1.5} = 1$$

$$P(10 \le x \le 11.5) = P(8.5 \le x \le 11.5) - P(8.5 \le x \le 10)$$

$$= 0.4772 - 0.3413$$

$$= 0.1359$$

So, there is a 13.59% probability that the measurement will yield a value between 10.0 and 11.5 V

3. Probability Distribution

Table 4.3 Probability Values for Normal Error Function: One-Sided Integral Solutions for $P(z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-\beta^2/2} d\beta$

$z_1 = \frac{x_1 - x'}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0199	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1809	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2794	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2967	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3531	0.3554	0.3577	0.3599	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4799	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.49865	0.4987	0.4987	0.4988	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990

Confidence Interval & Level of Significance

The **confidence interval** expresses the probability that the mean value will lie within a certain number of σ values and is given by the symbol z,

$$\bar{x}_t = \bar{x} \pm z\sigma = \bar{x} \pm \Delta \text{ (\% confidence level)}$$

for small data samples, $z\sigma/\sqrt{n} = \Delta$

The *level of significance* is 1 minus the confidence level

Confidence Interval	Confidence Level, %	Level of Significance, %
3.30	99.9	0.1
3.0	99.7	0.3
2.57	99.0	1.0
2.0	95.4	4.6
1.96	95.0	5.0
1.65	90.0	10.0
1.0	68.3	31.7

Student's t-Distribution

True mean value, x_t, from the finite sized data set can be estimated →

$$x_t = x_m \pm t_{(\nu,P)}\,\sigma_m = x_m \pm \Delta \quad (\%P)$$

here, $\Delta = t_{(\nu,P)}\,\sigma_m = t_{(\nu,P)}\dfrac{\sigma}{\sqrt{n}}$

ν = degree of freedom, $\nu = n-1$

$\%P$ = given confidence level

$$\Delta = \begin{cases} z\dfrac{\sigma}{\sqrt{n}} & ;\ \text{for sample size} \geq 20 \\ t_{(\nu,P)}\dfrac{\sigma}{\sqrt{n}} & ;\ \text{for sample size} < 20 \end{cases}$$

3. Probability Distribution

Subscript designates percent confidence level.

Degrees of freedom ν	t_{50}	t_{80}	t_{90}	t_{95}	t_{98}	t_{99}	$t_{99.9}$
1	1.000	3.078	6.314	12.706	31.821	63.657	636.619
2	0.816	1.886	2.920	4.303	6.965	9.925	31.598
3	0.765	1.638	2.353	3.182	4.541	5.841	12.941
4	0.741	1.533	2.132	2.776	3.747	4.604	8.610
5	0.727	1.476	2.015	2.571	3.365	4.032	6.859
6	0.718	1.440	1.943	2.447	3.143	3.707	5.959
7	0.711	1.415	1.895	2.365	2.998	3.499	5.405
8	0.706	1.397	1.860	2.306	2.896	3.355	5.041
9	0.703	1.383	1.833	2.262	2.821	3.250	4.781
10	0.700	1.372	1.812	2.228	2.764	3.169	4.587
11	0.697	1.363	1.796	2.201	2.718	3.106	4.437
12	0.695	1.356	1.782	2.179	2.681	3.055	4.318
13	0.694	1.350	1.771	2.160	2.650	3.012	4.221
14	0.692	1.345	1.761	2.145	2.624	2.977	4.140
15	0.691	1.341	1.753	2.131	2.602	2.947	4.073
16	0.690	1.337	1.746	2.120	2.583	2.921	4.015
17	0.689	1.333	1.740	2.110	2.567	2.898	3.965
18	0.688	1.330	1.734	2.101	2.552	2.878	3.922
19	0.688	1.328	1.729	2.093	2.539	2.861	3.883
20	0.687	1.325	1.725	2.086	2.528	2.845	3.850
21	0.686	1.323	1.721	2.080	2.518	2.831	3.819
22	0.686	1.321	1.717	2.074	2.508	2.819	3.792
23	0.685	1.319	1.714	2.069	2.500	2.807	3.767
24	0.685	1.318	1.711	2.064	2.492	2.797	3.745
25	0.684	1.316	1.708	2.060	2.485	2.787	3.725

Problems: Homework

1. Two resistors R_1 and R_2 are connected in series and parallel. The values of the resistances are $R_1 = 100.0 \pm 0.1\ \Omega$ and $R_2 = 50.0 \pm 0.03\ \Omega$. Calculate the uncertainty in the combined resistance for both the series and the parallel arrangements.

2. A capacitor discharges through a resistor according to the relation $E/E_0 = e^{-t/RC}$; where E_0 = voltage at time zero, R = resistance, C = capacitance. The value of the capacitance is to be measured by recording the time necessary for the voltage to drop to a value E_1. Assuming that the resistance is known accurately, derive an expression for the percent uncertainty in the capacitance as a function of the uncertainty in the measurements of E_1 and t.

3. A certain steel bar is measured with a device which has a known precision of ±0.5 mm when a large number of measurements is taken. How many measurements are necessary to establish the mean length $\bar{x}$ with a 5% level of significance such that $\overline{x_t} = \bar{x} \pm 0.2\ mm$.

Problems: Homework

4. A resistor is measured with a device which has a known precision of ±0.1 kΩ when a large number of measurements is taken. How many measurements are necessary to ensure that the resistance is within ±0.05 kΩ with a 5% level of significance? Make the calculation both with and without the *t*-distribution. Repeat for a 5% level of significance of ±0.1 kΩ.

5. Ten measurements are made of the thickness of a metal plate which give 3.61, 3.62, 3.60, 3.63, 3.61, 3.62, 3.60, 3.62, 3.64, and 3.62 mm. Determine the mean value and the tolerance limits for a 90% confidence level. Use *t*-distribution.

Reading

Experimental Methods for Engineers, 8th Edition, *J. P. Holman*

- ***Chapter 3***
  - ✓ *3.1 – 3.4*
  - ✓ *3.6 – 3.8*
  - ✓ *3.15*
  - ✓ *Example problem: 3.1-3.9, 3.12, 3.22-3.26*

Theory and Design for Mechanical Measurements, 7th Edition, *Richard S. Figliola & Donald E. Beasley*

- ***Chapter 4***
  - ✓ *4.3*
  - ✓ *Example problem: 4.3, 4.4*

Thank you