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A novel technique to solve the fuzzy system of equations

N. Mikaeilvand[a,*], Z. Noeiaghdam[b,†], S. Noeiaghdam[c,‡]

[a]Department of Mathematics, Ardabil Branch, Islamic Azad University, Ardabil, Iran.
[b]Department of Mathematics and Computer Science, Shahed University, Tehran, Iran.
[c]Department of Mathematics, Central Tehran Branch, Islamic Azad University, Tehran, Iran.

Abstract

The aim of this research is to apply a novel technique based on the embedding method to solve the $n \times n$ fuzzy system of linear equations (FSLEs). By using this method, the strong fuzzy number solutions of FSLEs can be obtained by transforming the $n \times n$ FSLE to the crisp system. In this paper, Ezzati's method to solve the FSLEs is modified and improved. Several theorems are proved to show the number of operations for presented method are less than the methods of Friedman and Ezzati. In order to show the advantages of scheme, two applicable algorithms are presented and several examples are solved by applying them. Also, some graphs of obtained results are demonstrated which show the solutions are in the fuzzy form.

Keywords: Fuzzy linear system, Fuzzy number, Fuzzy number vector, Embedding method.

1 Introduction

The FSLEs have many applications in different fields of science and engineering such as heat transport, fluid flow, electromagnetism and so on. In the recent decades, solving and studying the FSLEs have been appeared in many researches. In 1998, Friedman et al. [18] presented a model to solve the FSLEs and many mathematicians improved and developed this method to find the solution of FSLEs. In last years, Friedman et al. [18, 19, 21], Abbasbany et al. [1, 2, 9], Allahviranloo et al. [3, 4, 5, 6, 7] and others [8, 12, 13, 22, 23, 26, 31, 32] considered the $n \times n$ FSLEs. Also, many authors applied the numerical methods to find the approximate solution of the FSLEs [1, 2, 20, 27, 29]. Furthermore, the CESTAC method [14, 15, 24, 25] based on the stochastic arithmetic has been applied to find the optimal iteration, optimal approximations and the optimal error of numerical methods to solve the FSLEs [16, 17].

In this research, the Ezzati's method [12] to solve the $n \times n$ FSLEs is improved and a novel method is presented. The aim of this paper is to apply the embedding method and substituting the $n \times n$ FSLEs by two $n \times n$ crisp systems. Several theorems and lemmas are proved that show the number of operations in new method is lower than the ezzati's method and the solutions

*E-mail: Mikaeilvand@iauardabil.ac.ir
†E-mail: Z_Noeiaghdam@shahed.ac.ir; zahra.noie@yahoo.com
‡Corresponding author, E-mail: s.noeiaghdam.sci@iauctb.ac.ir; samadnoeiaghdam@gmail.com

of the FSLEs can be obtained by the fast and safe process. In order to show the abilities of method, two algorithms are presented and are applied to solve the examples.

Organization of this paper is in the following form: Section 2 contains several definitions and theorems of fuzzy arithmetic and the FSLEs. Section 3 introduces the new idea to solve the FSLEs. In this section, Ezzati's method is modified and improved. Also, several theorems are proved to show the presented method is better than the methods of Friedman and Ezzati. Furthermore, we will prove the number of operations in presented method is less than the mentioned methods. In Section 4, two applicable and efficient algorithms are presented. Also, several examples are solved by using these algorithms. Graphs of solutions are demonstrated to show the results are in the fuzzy form. Section 5 is the conclusion.

2 Preliminaries

Several definitions and details of fuzzy arithmetic are presented [18, 21]. Also, the methods of Friedman [18] and Ezzati [12] to solve the FSLEs are considered. Ezzati's method had some problems in proving the theorems that we modify and represent them.

Definition 1. *[18, 21] Let $\widetilde{p} = (\underline{p}(z), \overline{p}(z)),\ 0 \leq z \leq 1$ be the arbitrary fuzzy number then the following criteria should be satisfied*

(i) $\underline{p}(z)$ is a bounded monotonic increasing left continuous function.

(ii) $\overline{p}(z)$ is a bounded monotonic decreasing left continuous function.

(iii) $\underline{p}(z) \leq \overline{p}(z),\ 0 \leq z \leq 1.$

The set of all fuzzy numbers is denoted by $\mathbf{E^1}$. The crisp number k is called the singleton when $\overline{p}(z) = \underline{p}(z) = k,\ 0 \leq z \leq 1$.

Let $\widetilde{p} = (\underline{p}(z), \overline{p}(z)),\ \widetilde{q} = (\underline{q}(z), \overline{q}(z))$ be the arbitrary fuzzy functions and k be the scalar value. The operations between two fuzzy functions are defined as follows

$$(\underline{p+q})(z) = \underline{p}(z) + \underline{q}(z), \quad (\overline{p+q})(z) = \overline{p}(z) + \overline{q}(z),$$

$$(\underline{p-q})(z) = \underline{p}(z) - \overline{q}(z), \quad (\overline{p-q})(z) = \overline{p}(z) - \underline{q}(z),$$

$$\widetilde{kp} = \begin{cases} (k\underline{p}(z), k\overline{p}(z)), & k \geq 0, \\ (k\overline{p}(z), k\underline{p}(z)), & k < 0. \end{cases}$$

Also $\widetilde{p} = \widetilde{q}$ if and only if $\underline{p}(z) = \underline{q}(z)$ and $\overline{p}(z) = \overline{q}(z)$.

Remark 1: The triangular fuzzy number $\tilde{p} = (c, \mu, \rho)$ is defined as follows

$$\widetilde{p}(x) = \begin{cases} \dfrac{x - c + \mu}{\mu}, & c - \mu \leq x \leq c, \\ \\ \dfrac{c + \rho - x}{\rho}, & c \leq x \leq c + \rho, \\ \\ 0, & o.w, \end{cases} \tag{1}$$

where $\mu, \rho > 0$. It is clear that $\underline{p}(z) = c - (1 - z)\mu,\ \overline{p}(z) = c + (1 - z)\rho$ and $\overline{p} - \underline{p} = (\mu + \rho)(1 - z)$. The set of all triangular fuzzy numbers is denoted by $\mathbf{TE^1}$.

Definition 2. *[18] Let $(\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_j, \ldots, \tilde{v}_n)^T, \tilde{v}_j = (\underline{v}_j(z), \overline{v}_j(z)),\ 1 \leq j \leq n,\ 0 \leq z \leq 1$ be the fuzzy number vector which is called the solution of FSLEs if and only if*

$$\begin{aligned} &\underline{\sum_{j=1}^{n} a_{ij} v_j(z)} = \sum_{j=1}^{n} \underline{a_{ij} v_j}(z) = \underline{b_i}(z), \\ &\overline{\sum_{j=1}^{n} a_{ij} v_j(z)} = \sum_{j=1}^{n} \overline{a_{ij} v_j}(z) = \overline{b_i}(z), \quad i = 1, 2, \cdots, n. \end{aligned} \tag{2}$$

Finally, the methods of Friedman et al. [18] and Ezzati [12] to solve the FSLEs are reminded.

2.1 Friedman's method

Friedman et al. [18] presented the FSLEs as

$$Sv(z) = w(z) \tag{3}$$

where

$$\begin{aligned} v(z) &= \Big(\underline{v}_1(z), \underline{v}_2(z), \cdots, \underline{v}_n(z), -\overline{v}_1(z), -\overline{v}_2(z), \cdots, -\overline{v}_n(z)\Big)^T, \\ w(z) &= \left(\underline{w}_1(z), \underline{w}_2(z), \cdots, \underline{w}_n(z), -\overline{w}_1(z), -\overline{w}_2(z), \cdots, -\overline{w}_n(z)\right)^T, \end{aligned} \tag{4}$$

and the elements of $S = (s_{ij}),\ 1 \leq i, j \leq 2n$ are obtained based on the following conditions

$$\begin{aligned} a_{ij} \geq 0 &\Rightarrow s_{ij} = s_{i+nj+n} = a_{ij}, \\ a_{ij} < 0 &\Rightarrow s_{ij+n} = s_{i+nj} = -a_{ij}. \end{aligned} \tag{5}$$

We note that for values s_{ij} which are determined by neglecting the criterion (5) we have $s_{ij} = 0$. The matrix S for $s_{ij} \geq 0, 1 \leq i, j \leq 2n$ can be formed as follows

$$S = \begin{pmatrix} B & C \\ C & B \end{pmatrix},$$

where B constructs by the positive entries of A, C constructs by the absolute values of the negative entries of A and $A = B - C$.

For nonsingular matrix S we have $v(z) = S^{-1}w(z)$. But probably the obtained solution has not the proper fuzzy number vector. Therefore, the solution of the FSLEs can be defined in the following form

Definition 3. *[18] Let Eq.* (3) *has the unique solution in the form* $\tilde{v}(z) = \{(\underline{v}_i(z), -\overline{v}_i(z)),\ 1 \leq i \leq n\}$. *We define the fuzzy number vector* $\tilde{P} = \{(\underline{p}_i(z), \overline{p}_i(z)),\ 1 \leq i \leq n\}$ *as*

$$\underline{p}_i(z) = \min\{\underline{v}_i(z), \overline{v}_i(z), \underline{v}_i(1), \overline{v}_i(1)\},$$

$$\overline{p}_i(z) = \max\{\underline{v}_i(z), \overline{v}_i(z), \overline{v}_i(1), \underline{v}_i(1)\},$$

which is called the fuzzy solution of Eq. (3).

If $\underline{p}_i(z) = \underline{v}_i(z)$ and $\overline{p}_i(z) = \overline{v}_i(z),\ 1 \leq i \leq n$ then $\tilde{P}$ is called a strong fuzzy solution. Otherwise, $\tilde{P}$ is called a weak fuzzy solution which it is not the solution of FSLE and it is not always fuzzy number vector. Recently, Allahviranloo et al. [4] showed that the weak solution of a FSLE is not always a fuzzy number vector and it is the main fault of Friedman's method.

2.2 Ezzati's method [12]

Consider the following FSLEs

$$\begin{cases} a_{11}(\underline{v}_1(z) + \overline{v}_1(z)) + \ldots + a_{1n}(\underline{v}_n(z) + \overline{v}_n(z)) & = \underline{w}_1(z) + \overline{w}_1(z), \\ a_{21}(\underline{v}_1(z) + \overline{v}_1(z)) + \ldots + a_{2n}(\underline{v}_n(z) + \overline{v}_n(z)) & = \underline{w}_2(z) + \overline{w}_2(z), \\ \vdots & \vdots \\ a_{n1}(\underline{v}_1(z) + \overline{v}_1(z)) + \ldots + a_{nn}(\underline{v}_n(z) + \overline{v}_n(z)) & = \underline{w}_n(z) + \overline{w}_n(z), \end{cases} \tag{6}$$

where the solution of system (6) is in the following form

$$\mathbf{g}(\mathbf{z}) = \begin{pmatrix} g_1(z) \\ g_2(z) \\ \vdots \\ g_n(z) \end{pmatrix} = \underline{\mathbf{v}}(\mathbf{z}) + \overline{\mathbf{v}}(\mathbf{z}) = \begin{pmatrix} \underline{v}_1(z) + \overline{v}_1(z) \\ \underline{v}_2(z) + \overline{v}_2(z) \\ \vdots \\ \underline{v}_n(z) + \overline{v}_n(z) \end{pmatrix}.$$

Since $(B + C)\underline{v}(z) = \underline{w}(z) + Cg(z)$ and $(B + C)\overline{v}(z) = \overline{w}(z) + Cg(z)$ hence $\underline{v}(z)$ or $\overline{v}(z)$ is determined by solving the following system

$$\begin{aligned} \underline{v}(z) &= (B + C)^{-1}(\underline{w}(z) + Cg(z)), \\ \overline{v}(z) &= (B + C)^{-1}(\overline{w}(z) + Cg(z)). \end{aligned} \tag{7}$$

Therefore, the solution of FSLEs (6) can be obtained by solving system (7) that the vector of solution is unique. But may still not be an appropriate fuzzy number vector.

Theorem 1. *Let $\tilde{v}(z) = (\tilde{v}_1(z), \tilde{v}_2(z), \ldots, \tilde{v}_n(z))^T$ be the fuzzy solution of Eq.* (6) *and the matrix $\boldsymbol{A}^{-1}$ exists. Then the solution of system*

$$\boldsymbol{A}(\overline{\boldsymbol{v}}(z) + \underline{\boldsymbol{v}}(z)) = \overline{\boldsymbol{w}}(z) + \underline{\boldsymbol{w}}(z) \ , \tag{8}$$

for $\overline{w}(z) + \underline{w}(z) = (\overline{w}_1(z) + \underline{w}_1(z), \overline{w}_2(z) + \underline{w}_2(z), \ldots, \overline{w}_n(z) + \underline{w}_n(z))^T$ is in the following form

$$\overline{v}(z) + \underline{v}(z) = (\overline{v}_1(z) + \underline{v}_1(z), \overline{v}_2(z) + \underline{v}_2(z), \ldots, \overline{v}_n(z) + \underline{v}_n(z))^T.$$

Since number of operations to solve the $n \times n$ system are less than the $2n \times 2n$ system, thus Ezzati's method is better in comparison with Friedman's method. In Theorem 4 of Ezzati's method [12] the maximum number of multiplication operations (MNMO) were obtained which had some problems. The following theorem is the modified version of Theorem 4 in [12].

Theorem 2. *Assume n is any integer and $n \geq 2$. and denote by F_n and E_n the MNMOs that are required to calculate*

$$v(z) = (\underline{v}_1(z), \underline{v}_2(z), \ldots, \underline{v}_n(z), -\overline{v}_1(z), -\overline{v}_2(z), \ \ldots, -\overline{v}_n(z))^T = S^{-1}w(z)$$

by Friedman's method and

$$v(z) = (\underline{v}_1(z), \underline{v}_2(z), \ldots \underline{v}_n(z), \overline{v}_1(z), \overline{v}_2(z), \ \ldots, \overline{v}_n(z))^T$$

by Ezzati's method, respectively, then

$$F_n \geq E_n,$$

and

$$F_n - E_n = 2n^2.$$

Proof: Suppose $h_n(A)$ is the MNMOs of computing the matrix A^{-1}. Now, we can write

$$\mathbf{S^{-1}} = \begin{pmatrix} D & E \\ E & D \end{pmatrix},$$

where

$$D = \frac{1}{2}[(B + C)^{-1} + (B - C)^{-1}],$$

$$E = \frac{1}{2}[(B + C)^{-1} - (B - C)^{-1}].$$

Therefore, in order to determine S^{-1}, computing matrices $(B + C)^{-1}$ and $(B - C)^{-1}$ are required. It is clear that

$$h_n(S) = h_n(B + C) + h_n(B - C) = 2h_n(A).$$

Since $\tilde{v}(z) \in E^1$; $\underline{v}(z)$ and $\overline{v}(z)$, in the simplest case are lines hence

$$F_n = 2h_n(A) + 8n^2.$$

For computing $\underline{v}(z)+\overline{v}(z)=(\underline{v}_1(z)+\overline{v}_1(z),\underline{v}_2(z)+\overline{v}_2(z),\ldots,\underline{v}_n(z)+\overline{v}_n(z))^T$ from Eq. (6) and $\underline{v}(z)=(\underline{v}_1(z),\underline{v}_2(z),\ldots\underline{v}_n(z))^T$ from Eq. (7) and according to Ezzati's method, the MNMOs are $h_n(A)+2n^2$ and $h_n(B+C)+4n^2$ respectively. Since $h_n(B+C)=h_n(A)$ thus

$$E_n=2h_n(A)+6n^2,$$

and finally $F_n-E_n=2n^2$.$\square$

Definition 4. *[12] Assume $\tilde{v}(z)=\{(\underline{v}_i(z),\overline{v}_i(z)),\ 1\leq i\leq n\}$ is the unique solution of Eqs.* (6), (7). *The fuzzy number vector $\tilde{P}=\{(\underline{p}_i(z),\overline{p}_i(z)),\ 1\leq i\leq n\}$ is defined by*

$$\underline{p}_i(z)=\min\{\underline{v}_i(z),\overline{v}_i(z),\underline{v}_i(1)\},$$

$$\overline{p}_i(z)=\max\{\underline{v}_i(z),\overline{v}_i(z),\overline{v}_i(1)\},$$

which is called a fuzzy vector solution of Eqs. (6) *and* (7).

If $\underline{p}_i(z)=\underline{v}_i(z)$ and $\overline{p}_i(z)=\overline{v}_i(z),\ 1\leq i\leq n$, then $\tilde{P}$ is called a strong fuzzy solution. Otherwise, $\tilde{P}$ is called a weak fuzzy solution which it is not fuzzy linear system's solution and is not always fuzzy number vector.

Remark 2: We know that Friedman et al. [18] and Ezzati [12] find two kinds of solutions, which are called the weak and the strong solutions. The weak solution is not system's solution and it is not always the fuzzy number vector [4]. Hence, we do not interest to find weak fuzzy solution. Also, in these methods the kind of solutions- strong or weak- are determined only in the end of method and it is one of important faults of these methods.

In the next section, a novel method for solving a $n\times n$ FSLEs is presented. It is observed our method can least the computing error, because without carrying out further computation, we can determined that the fuzzy linear system, has no fuzzy number vector solution.

3 Main Idea

In this section, a novel and applicable method to solve the FSLEs is presented. Several theorems and lemmas are illustrated to improve the ezzati's method [12]. By using these theorems we show the number of operations of our method are less than the methods of Ezzati [12] and Friedman [18].

Theorem 3. *Suppose the inverse matrix of $\boldsymbol{B+C}$ exists and $\tilde{v}(z)=(\tilde{v}_1(z),\tilde{v}_2(z),\ldots,\tilde{v}_n(z))^T$ is a fuzzy solution of Eq* (6). *Then $\overline{v}(z)-\underline{v}(z)=(\overline{v}_1(z)-\underline{v}_1(z),\overline{v}_2(z)-\underline{v}_2(z),\ldots,\overline{v}_n(z)-\underline{v}_n(z))^T$ is the solution of the following system*

$$(\boldsymbol{B+C})(\overline{\boldsymbol{v}}(z)-\underline{\boldsymbol{v}}(z))=\overline{\boldsymbol{w}}(z)-\underline{\boldsymbol{w}}(z)\ ,\tag{9}$$

where $\overline{w}(z)-\underline{w}(z)=(\overline{w}_1(z)-\underline{w}_1(z),\overline{w}_2(z)-\underline{w}_2(z),\ldots,\overline{w}_n(z)-\underline{w}_n(z))^T$.

Proof: Let $\tilde{v}_j(z) = (\underline{v}_j(z), \overline{v}_j(z)), 1 \leq j \leq n$ be the parametric form of $\tilde{v}_j$. For positive values a'_{ij} and a''_{ij} we have

$$a_{ij} = a'_{ij} - a''_{ij},$$

$$a'_{ij}a''_{ij} = 0,$$

where a_{ij}, a'_{ij} and a''_{ij} are the coefficients of matrices A, B and C respectively. By presenting the Eq. (6) to the parametric form, for $i = 1, 2, \ldots, n$ we get

$$(a'_{i1} - a''_{i1})(\underline{v}_1(z), \overline{v}_1(z)) + \ldots + (a'_{in} - a''_{in})(\underline{v}_n(z), \overline{v}_n(z)) = (\underline{w}_i(z), \overline{w}_i(z)).$$

Hence

$$a'_{i1}\underline{v}_1(z) - a''_{i1}\overline{v}_1(z) + a'_{i2}\underline{v}_2(z) - a''_{i2}\overline{v}_2(z) + \ldots + a'_{in}\underline{v}_n(z) - a''_{in}\overline{v}_n(z) = \underline{w}_i(z), \tag{10}$$

and

$$a'_{i1}\overline{v}_1(z) - a''_{i1}\underline{v}_1(z) + a'_{i2}\overline{v}_2(z) - a''_{i2}\underline{v}_2(z) + \ldots + a'_{in}\overline{v}_n(z) - a''_{in}\underline{v}_n(z) = \overline{w}_i(z). \tag{11}$$

Now, we can differentiate Eq. (10) from Eq. (11) as follows

$$(a'_{i1}+a''_{i2})(\overline{v}_1(z)-\underline{v}_1(z))+(a'_{i2}+a''_{i2})(\overline{v}_2(z)-\underline{v}_2(z))+\ldots+(a'_{in}+a''_{in})(\overline{v}_n(z)-\underline{v}_n(z)) = \overline{w}_i(z)-\underline{w}_i(z).$$

Therefore, $d(z) = \overline{v}(z) - \underline{v}(z) = (\overline{v}_1(z) - \underline{v}_1(z), \overline{v}_2(z) - \underline{v}_2(z), \ldots, \overline{v}_n(z) - \underline{v}_n(z))^T$ is the solution of $(B + C)(\overline{v}(z) - \underline{v}(z)) = \overline{w}(z) - \underline{w}(z)$.□

Theorem 4. *Suppose the inverse matrix of* $\boldsymbol{B+C}$ *exists. The equation* (6)*, does not have a fuzzy number vector solution, if the vector solution of the following system is not nonnegative, i.e. at least one of the entries are negative*

$$(\boldsymbol{B+C})(\overline{\boldsymbol{v}}(z) - \underline{\boldsymbol{v}}(z)) = \overline{\boldsymbol{w}}(z) - \underline{\boldsymbol{w}}(z). \tag{12}$$

Proof: We know that, the vector solution of the Eq. (12) is $(\overline{v}(z) - \underline{v}(z))$. Now, suppose that $(\overline{v}(z) - \underline{v}(z))$ is not nonnegative. So, according to the Definition (1), the fuzzy number vector solution is not exist. It is clear that the matrix $(B+C)^{-1}$ is the non positive matrix, i.e. at least one of the entries is positive because $(B + C)$ is the positive matrix.□

Triangular fuzzy numbers are the simple and the popular fuzzy numbers. Also, triangular fuzzy numbers have a special property

$$\overline{w}(z) - \underline{w}(z) = (\rho' + \rho'')(1 - z).$$

Hence when the right hand side vector $\tilde{w}(z)$ is triangular, the parametric linear system (12) can be transformed to the crisp linear system.

Lemma 1 Suppose the inverse matrix of $(B+C)$ exists, and $\tilde{w}(z) \in TE^1$. The Eq. (6), does not have a fuzzy number vector solution, if the vector solution of the following system is not nonnegative, i.e. at least one of the entries is negative

$$(B + C)(\mu' + \mu'') = (\rho' + \rho''), \tag{13}$$

where $(\mu' + \mu'')(1 - z) = \overline{v}(z) - \underline{v}(z),\ (\rho' + \rho'')(1 - z) = \overline{w}(z) - \underline{w}(z)$.

Proof: It is clear.$\square$

Now, a new method to solve the FSLEs is presented. Assume that the inverse matrix of A in Eq. (6) exists. For solving Eq. (6), the following system

$$(\mathbf{B}{+}\mathbf{C})(\overline{\mathbf{v}}(z) - \underline{\mathbf{v}}(z)) = \overline{\mathbf{w}}(z) - \underline{\mathbf{w}}(z), \tag{14}$$

should be solved where the matrices B and C were defined in Subsection 2.2. Let the solution of this system be in the following form

$$\mathbf{d}(\mathbf{z}) = \begin{pmatrix} d_1(z) \\ \\ d_2(z) \\ \\ \vdots \\ \\ d_n(z) \end{pmatrix} = \overline{\mathbf{v}}(\mathbf{z}) - \underline{\mathbf{v}}(\mathbf{z}) = \begin{pmatrix} \overline{v}_1(z) - \underline{v}_1(z) \\ \\ \overline{v}_2(z) - \underline{v}_2(z) \\ \\ \vdots \\ \\ \overline{v}_n(z) - \underline{v}_n(z) \end{pmatrix}.$$

If $d = \overline{v} - \underline{v}$ is not nonnegative, then we do not have the fuzzy number vector solution. Otherwise, in order to show the existence of fuzzy number vector solution for Eq. (6), we continue our idea. At first, we should solve the following system

$$\mathbf{A}(\overline{\mathbf{v}}(z) + \underline{\mathbf{v}}(z)) = \overline{\mathbf{w}}(z) + \underline{\mathbf{w}}(z). \tag{15}$$

According to the Theorem 1, we know that this system has the solution in the following form

$$\mathbf{g}(\mathbf{z}) = \begin{pmatrix} g_1(z) \\ \\ g_2(z) \\ \vdots \\ g_n(z) \end{pmatrix} = \overline{\mathbf{v}}(\mathbf{z}) + \underline{\mathbf{v}}(\mathbf{z}) = \begin{pmatrix} \overline{v}_1(z) + \underline{v}_1(z) \\ \\ \overline{v}_2(z) + \underline{v}_2(z) \\ \vdots \\ \overline{v}_n(z) + \underline{v}_n(z) \end{pmatrix}.$$

Finally, by solving systems (14) and (15) and finding $\mathbf{d}(\mathbf{z})$ and $\mathbf{g}(\mathbf{z})$ we have

$$\begin{cases} \underline{v}(z) = \frac{\mathbf{g}(\mathbf{z}) - \mathbf{d}(\mathbf{z})}{2}, \\ \\ \overline{v}(z) = \frac{\mathbf{g}(\mathbf{z}) + \mathbf{d}(\mathbf{z})}{2}. \end{cases} \tag{16}$$

If the conditions of Definition 1 are true, then the solution of FSLEs (6) can be obtained by solving the crisp linear system of Eqs. (14) and (15) that the solution vector is the fuzzy number vector and unique. Otherwise, if at least one of the conditions do not true, the fuzzy linear system of Eqs. (6) does not have fuzzy number vector solution.

Remark 3 : If $\tilde{w} \in TE^1$, then according to the Lemma 1 the system of Eqs. (14) have the vector solution as $d' = \mu' + \mu''$ where $d(z) = d'(1-z)$. So, the Eqs. (16) can be written in following form

$$\begin{cases} \underline{v}(z) = \frac{\mathbf{g(z) - d'(1-z)}}{2}, \\ \\ \overline{v}(z) = \frac{\mathbf{g(z) + d'(1-z)}}{2}. \end{cases} \tag{17}$$

Theorem 5. *Assume that n is any integer, $n \geq 2$ and denote by E_n and D_n the MNMOs that are required to calculate*

$$v(z) = (\underline{v}_1(z), \underline{v}_2(z), \ldots \underline{v}_n(z), \overline{v}_1(z), \overline{v}_2(z), \ldots, \overline{v}_n(z))^T,$$

in the Ezzati's method [12] and presented method then

$$\begin{cases} E_n - D_n = 2n^2, & \overline{v}(z) - \underline{v}(z) \geq 0, \\ \\ E_n - D_n = h_n(A) + 4n^2, & o.w., \end{cases} \tag{18}$$

where $h_n(A)$ shows the MNMOs that are required to calculate A^{-1}.

Proof : According to the Theorem 2, we have

$$E_n = 2h_n(A) + 6n^2.$$

Assume $d(z) = \overline{v}(z) - \underline{v}(z)$ is the nonnegative matrix then for understanding that whether the fuzzy linear system of Eqs. (6) has the fuzzy number vector solution, we need to solve the system of Eqs. (15). So, for computing

$$\overline{v}(z) - \underline{v}(z) = (\overline{v}_1(z) - \underline{v}_1(z), \overline{v}_2(z) - \underline{v}_2(z), \ldots, \overline{v}_n(z) - \underline{v}_n(z))^T,$$

and

$$\overline{v}(z) + \underline{v}(z) = (\overline{v}_1(z) + \underline{v}_1(z), \overline{v}_2(z) + \underline{v}_2(z), \ldots, \overline{v}_n(z) + \underline{v}_n(z))^T,$$

from Eqs. (14) and (15), the maximum number of multiplication operations are $h_n(B+C) + 2n^2$ and $h_n(A) + 2n^2$, respectively. Clearly $h_n(B+C) = h_n(A)$. Hence

$$D_n = 2h_n(A) + 4n^2,$$

and

$$E_n - D_n = 2n^2.$$

Otherwise, assume that $d(z) = \overline{v}(z) - \underline{v}(z)$ is not the nonnegative matrix. According to the Theorem 4 we do not have the fuzzy number vector solution for solving the FSLEs (6). If we do not have the fuzzy number vector solution, there will be no necessary for computing $\overline{v}(z) + \underline{v}(z)$ from system of Eqs. (15). Thus, we need to compute $d(z) = \overline{v}(z) - \underline{v}(z)$. Therefore, in this case, the maximum number of multiplication operations are $h_n(B+c) + 2n^2$ or $h_n(A) + 2n^2$. Finally, we have

$$E_n \geq D_n, \quad E_n - D_n = h_n(A) + 4n^2.\Box$$

Lemma 2 Let $\tilde{w}(z)$ be the triangular fuzzy number vector, from Eq. (6). Then $\tilde{v}(z)$ is the triangular fuzzy number vector solution, from Eq. (6).

Proof : It is clear. $\Box$

Lemma 3 Suppose that in the Theorem 5, $\tilde{w}(z)$ is the triangular fuzzy number vector from Eq. (6), then $E_n \geq D_n$ and

$$\begin{cases} E_n - D_n = 3n^2 - n, & \overline{v}(z) - \underline{v}(z) \geq 0, \\ E_n - D_n = h_n(A) + 5n^2, & o.w. \end{cases} \tag{19}$$

Proof : If $\tilde{w}(z)$ is the triangular fuzzy number vector from Eq. (6) i.e., since $\tilde{w}(z) \in TE^1$; $\underline{w}(z)$ and $\overline{w}(z)$ in the simplest case is the line. So clearly, according to the Theorem 2, we have

$$E_n = 2h_n(A) + 6n^2,$$

and according to the Remark 1, we get

$$\overline{v}(z) = c + (1-z)\mu', \qquad \underline{v}(z) = c - (1-z)\mu''.$$

So,

$$\overline{v}(z) - \underline{v}(z) = (\mu' + \mu'')(1-z),$$

$$\overline{w}(z) - \underline{w}(z) = (\rho' + \rho'')(1-z),$$

and from the system of Eqs. (14), we have

$$(B+C)(\mu' + \mu'')(1-z) = (\rho' + \rho'')(1-z).$$

If $r \neq 1$, the following relation can be obtained as

$$(B+C)(\mu' + \mu'') = \rho' + \rho'', \tag{20}$$

where it is the crisp linear system. It is clear that for $r = 1$ the FSLE replaced by crisp linear system.

Now, we assume that $d' = (\mu' + \mu'')$ is nonnegative. Then for understanding that whether the FSLEs (6) has the fuzzy number vector solution, we need to solve the system of Eqs. (15). So, for computing $(\mu' + \mu'') = (\mu'_1 + \mu''_1, \mu'_2 + \mu''_2, \ldots, \mu'_n + \mu''_n)$ from Eq. (20) and $\overline{v}(z) + \underline{v}(z) = (\overline{v}_1(z) + \underline{v}_1(z), \overline{v}_2(z) + \underline{v}_2(z), \ldots, \overline{v}_n(z) + \underline{v}_n(z))^T$ from Eq. (15) and $d(z) = (1-z)d'$ for final solution in Eq. (17) the maximum number of multiplication operations are $h_n(B+C) + n^2$, $h_n(A) + 2n^2$ and n respectively. Clearly $h_n(B+C) = h_n(A)$. So

$$D_n = 2h_n(A) + 3n^2 + n,$$

and

$$E_n \geq D_n, \quad E_n - D_n = 3n^2 - n.$$

Otherwise, assume that $d\,' = (\mu' + \mu'')$ is not nonnegative. Then, according to the Lemma 1 we do not have a fuzzy number vector solution for solving the fuzzy linear system of Eqs. (6). We know that if we do not have a fuzzy number vector solution, there will be no necessary for computing $\underline{v}(z) + \overline{v}(z)$ from Eq. (15). Thus, we need to compute $d\,' = (\mu' + \mu'')$. Therefore, in this case the maximum number of multiplication operations are

$$h_n(B+C) + n^2 \ or\ h_n(A) + n^2.$$

Then, we have

$$E_n \geq D_n, \quad E_n - D_n = h_n(A) + 5n^2.\square$$

4 Numerical illustrations

In this section, some examples of the FSLEs are presented [18]. Also, two algorithms are applied to solve the problems. Furthermore, several graphs are demonstrated that show the fuzzy form of solutions.

Algorithm 1: Let **A** be the nonsingular matrix.

Step 1: Input matrix $A = [a_{ij}] \in R^{n\times n}$ and $\tilde{v}(z) = (\underline{v}(z), \overline{v}(z)), \tilde{w}(z) = (\underline{w}(z), \overline{w}(z)) \in E^1$.

Step 2: Calculate $B = [b_{ij}]$ and $C = [c_{ij}]$.

$$\begin{cases} If\ \ a_{ij} > 0 \ \Rightarrow\ b_{ij} = a_{ij};\ \ else\ \ b_{ij} = 0, \\ \\ If\ \ a_{ij} < 0 \ \Rightarrow\ c_{ij} = -a_{ij};\ \ else\ \ c_{ij} = 0. \end{cases}$$

Step 3: Calculate $M = (B + C)^{-1}$.

Step 4: Calculate $d_1(z) = \overline{w}(z) - \underline{w}(z)$ and $d(z) = M.d_1(z)$. If $d(z)$ is not nonnegative, go to step 8.

Step 5: Calculate $k = A^{-1}, g_1(z) = \overline{w}(z) + \underline{w}(z)$ and $g(z) = k.g_1(z)$.

Step 6: Calculate $\underline{v}(z) = \frac{g(z)+d(z)}{2}$ and $\overline{v}(z) = \frac{g(z)-d(z)}{2}$.

Step 7: If conditions of Definition 1 are true then $\tilde{v}(z) = (\underline{v}(z), \overline{v}(z))$ and go to step 9. Else go to step 8.

Step 8: Show the message "The system does not have fuzzy number vector solution".

Step 9: End.

The following algorithm is presented to triangular fuzzy linear system.

Algorithm 2:

Step 1: Input matrix $A = [a_{ij}] \in R^{n\times n}$ and $\tilde{v}(z) = (\underline{v}(z), \overline{v}(z)), \tilde{w}(z) = (\underline{w}(z), \overline{w}(z)) \in TE^1$.

Step 2: Calculate $B=[b_{ij}]$ and $C=[c_{ij}]$.

$$\begin{cases} If \;\; a_{ij}>0 \;\Rightarrow\; b_{ij}=a_{ij}; \;\; else \;\; b_{ij}=0, \\ \\ If \;\; a_{ij}<0 \;\Rightarrow\; c_{ij}=-a_{ij}; \;\; else \;\; c_{ij}=0. \end{cases}$$

Step 3: Calculate $M=(B+C)^{-1}$.
Step 4: Calculate $d_1=\rho'+\rho''$ and $d'=M.d_1$. If d' is not nonnegative, go to step 8.
Step 5: Calculate $k=A^{-1}, g_1(z)=\overline{w}(z)+\underline{w}(z)$ and $g(z)=k.g_1(z)$.
Step 6: Calculate $\underline{v}(z)=\frac{g(z)+d'(1-z)}{2}$; and $\overline{v}(z)=\frac{g(z)-d'(1-z)}{2}$.
Step 7: If conditions of Definition 1 are true then $\tilde{v}(z)=(\underline{v}(z),\overline{v}(z))$ and go to step 9. Else go to step 8.
Step 8: Show the message "The system does not have fuzzy number vector solution".
Step 9: End.

Example 1. *[18] Consider the following 2×2 FSLEs*

$$\begin{cases} \tilde{v}_1-\tilde{v}_2=(z,2-z), \\ \\ \tilde{v}_1+3\tilde{v}_2=(4+z,7-2z), \end{cases}$$

where $\tilde{w}$ is a triangular vector of fuzzy numbers hence the Algorithm 2 is applied. By using this algorithm we have:
Step1. *Input matrix*

$$A=\begin{pmatrix} 1 & -1 \\ \\ 1 & 3 \end{pmatrix},$$

$$\tilde{v}(z)=(\underline{v}(z),\overline{v}(z))\in TE^1,$$

$$\tilde{w}(z)=(\underline{w}(z),\overline{w}(z))=\begin{pmatrix} (z,2-z) \\ \\ (4+z,7-2z) \end{pmatrix}.$$

Step 2. *Calculate*

$$B=[b_{ij}]=\begin{pmatrix} 1 & 0 \\ \\ 1 & 3 \end{pmatrix},$$

$$C=[c_{ij}]=\begin{pmatrix} 0 & 1 \\ \\ 0 & 0 \end{pmatrix}.$$

Step 3. *Compute*

$$M = (B + C)^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

Step 4. *Calculate*

$$d_1 = \rho' + \rho'' = \begin{pmatrix} \rho'_1 + \rho''_1 \\ \rho'_2 + \rho''_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$

$$d' = M.d_1 = \begin{pmatrix} \frac{3}{2} \\ \frac{1}{2} \end{pmatrix}.$$

It is clear that $d' = \mu' + \mu''$ *is the nonnegative matrix, therefore go to the step 5.*

Step 5. *Calculate*

$$k = A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & \frac{1}{4} \end{pmatrix},$$

$$g_1(z) = \overline{w}(z) + \underline{w}(z) = \begin{pmatrix} 2 \\ 11 - z \end{pmatrix},$$

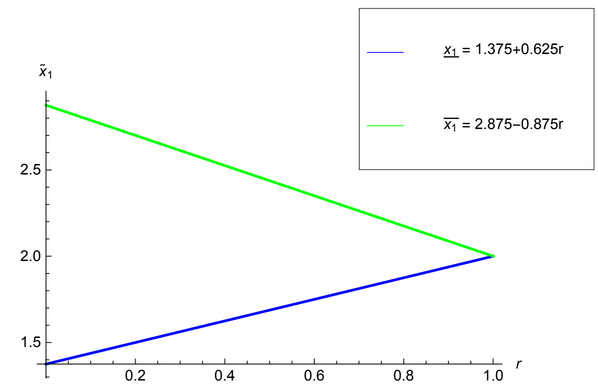
$$g(z) = k.g_1(z) = \begin{pmatrix} \frac{17 - z}{4} \\ \frac{9 - z}{4} \end{pmatrix}.$$

Step 6. *Compute*

$$\underline{v}(z) = \frac{g(z) - d(z)}{2} = \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix} = \begin{pmatrix} 1.375 + 0.625z \\ 0.875 + 0.125z \end{pmatrix},$$

$$\overline{v}(z) = \frac{g(z) + d(z)}{2} = \begin{pmatrix} \overline{v}_1 \\ \overline{v}_2 \end{pmatrix} = \begin{pmatrix} 2.875 - 0.875z \\ 1.375 - 0.375z \end{pmatrix}.$$

Since the conditions of Definition 1 are true, the FSLEs has the fuzzy number vector solution. Fig. 1 shows the obtained solution is in the fuzzy form.



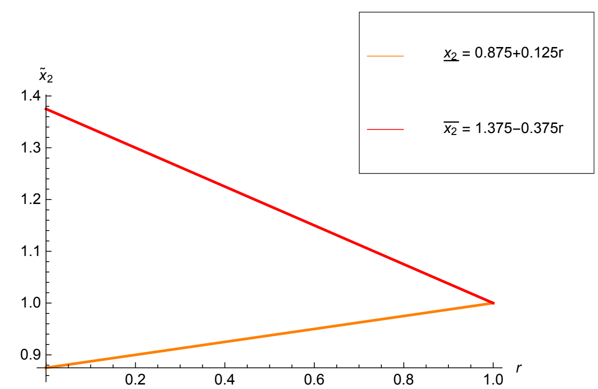


Figure 1: The solutions of Example 1.

Example 2. *[18] Consider the* 3×3 *FSLEs*

$$\begin{cases} \tilde{v}_1 + \tilde{v}_2 - \tilde{v}_3 = (z, 2-z), \\ \\ \tilde{v}_1 - 2\tilde{v}_2 + \tilde{v}_3 = (2+z, 3), \\ \\ 2\tilde{v}_1 + \tilde{v}_2 + 3\tilde{v}_3 = (-2, -1-z), \end{cases}$$

where $\tilde{Y}$ *is a triangular fuzzy number vector. By using Algorithm 2 we have*
Step 1. *Input matrix*

$$A = \begin{pmatrix} 1 & 1 & -1 \\ \\ 1 & -2 & 1 \\ \\ 2 & 1 & 3 \end{pmatrix},$$

$$\tilde{w}(z) = (\underline{w}(z), \overline{w}(z)) = \begin{pmatrix} (z, 2-z) \\ \\ (2+z, 3) \\ \\ (-2, -1-z) \end{pmatrix}.$$

Step 2. *Compute*

$$B = [b_{ij}] = \begin{pmatrix} 1 & 1 & 0 \\ \\ 1 & 0 & 1 \\ \\ 2 & 1 & 3 \end{pmatrix},$$

$$C = [c_{ij}] = \begin{pmatrix} 0 & 0 & 1 \\ \\ 0 & 2 & 0 \\ \\ 0 & 0 & 0 \end{pmatrix}.$$

Step 3. *Calculate*

$$M = (B+C)^{-1} = \begin{pmatrix} 5 & -1 & -3 \\ \\ -2 & 1 & 1 \\ \\ -1 & 0 & 1 \end{pmatrix}.$$

Step 4. *Compute*

$$d_1 = \rho' + \rho'' = \begin{pmatrix} \rho'_1 + \rho''_1 \\ \\ \rho'_2 + \rho''_2 \end{pmatrix} = \begin{pmatrix} 2 \\ \\ 1 \\ \\ 1 \end{pmatrix},$$

$$d' = M.d_1 = \begin{pmatrix} 6 \\ \\ -2 \\ \\ -1 \end{pmatrix}.$$

Since $d' = \mu' + \mu''$ *is not nonnegative, therefore the system has not a fuzzy number vector solution.*

Example 3. *Consider the following* 2×2 *FSLEs*

$$\begin{cases} \tilde{v}_1 + \tilde{v}_2 = (4z, 6 - 2z), \\ \\ \tilde{v}_1 + 2\tilde{v}_2 = (5z, 8 - 3z). \end{cases} \tag{21}$$

In this example, by applying the Algorithm 1 we have
Step 1. *Input matrix*

$$A = \begin{pmatrix} 1 & 1 \\ \\ 1 & 2 \end{pmatrix},$$

$$\tilde{w}(z) = (\underline{w}(z), \overline{w}(z)) = \begin{pmatrix} (4z, 6 - 2z) \\ \\ (5z, 8 - 3z) \end{pmatrix}.$$

Step 2. *Compute*

$$B = [b_{ij}] = \begin{pmatrix} 1 & 1 \\ \\ 1 & 2 \end{pmatrix},$$

$$C = [c_{ij}] = \begin{pmatrix} 0 & 0 \\ \\ 0 & 0 \end{pmatrix}.$$

Step 3. *Calculate*

$$M = (B + C)^{-1} = \begin{pmatrix} 2 & -1 \\ \\ -1 & 1 \end{pmatrix}.$$

Step 4. *Compute*

$$d_1(z) = \overline{w}(z) - \underline{w}(z) = \begin{pmatrix} 6-6z \\ \\ 8-8z \end{pmatrix},$$

$$d(z) = M.d_1 = \begin{pmatrix} 4-4z \\ \\ 2-2z \end{pmatrix}.$$

Since $d(z) = \overline{v}(z) - \underline{v}(z)$ *is nonnegative for* $0 \leq z \leq 1$ *,therefore, in order to find that whether the fuzzy system has the fuzzy number vector solution we need to go to step 5.*

Step 5. *Calculate*

$$k = A^{-1} = \begin{pmatrix} 2 & -1 \\ \\ -1 & 1 \end{pmatrix},$$

$$g_1(z) = \overline{w}(z) + \underline{w}(z) = \begin{pmatrix} 6+2z \\ \\ 8+2z \end{pmatrix},$$

$$g(z) = k.g_1(z) = \begin{pmatrix} 4+2z \\ \\ 2 \end{pmatrix}.$$

Step 6. *Compute*

$$\underline{v}(z) = \frac{g(z) - d(z)}{2} = \begin{pmatrix} 3z \\ \\ z \end{pmatrix},$$

$$\overline{v}(z) = \frac{g(z) + d(z)}{2} = \begin{pmatrix} 4-z \\ \\ 2-z \end{pmatrix}.$$

Since the conditions of Definition 1 are connected hence the vector solution is the fuzzy number vector solution. Thus, the FSLEs (21) has the fuzzy number vector solution. Fig. 2 shows the fuzzy solutions of this example.

5 Conclusion

Weakly fuzzy solution was introduced by Friedman et al. [18]. This solution is not always fuzzy number vector and it is not fuzzy linear system's solution. Also, the kind of solution is only determined in the end of solving problem. Hence, it is important to introduce a novel method for solving the FSLEs and find its fuzzy number vector solution. In the novel proposed method, the original fuzzy system is replaced by two $n \times n$ crisp linear system. By proving several

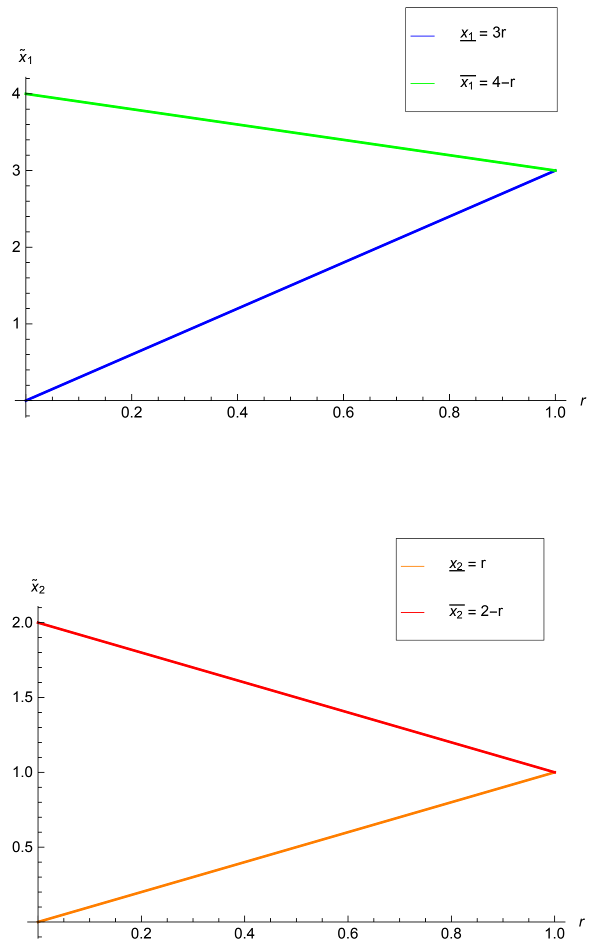


Figure 2: The solutions of Example 2.

theorems we showed the number of operations for presented method are less than the methods of Friedman and Ezzati. Presented algorithms show the accuracy and efficiency of method to solve the examples.

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