In [ ]:
```
import cvxpy
import numpy as np
import matplotlib
import matplotlib.pyplot as plt
%matplotlib inline

import copy
from copy import deepcopy

import math

# from google.colab import drive
# drive.mount('/content/gdrive')
```

Task 1

Optimize function:

$$f(x) = (Qx, x) + (c, x)$$

Task 1.1

Implement gradient descent with

- with step-size γ_k chosen optimally (exact line search):

$$f(x+\gamma p) = \frac{1}{2}(Q(x+\gamma p), (x+\gamma p)) + (c, (x+\gamma p)) \rightarrow \min_{\gamma}$$

$$f(x+\gamma p) = \frac{1}{2}(x+\gamma p)^{\top} Q(x+\gamma p) + c^{\top}(x+\gamma p) \rightarrow \min \implies$$

$$\frac{d}{d\gamma} f(x+\gamma p) = \frac{1}{2}x^{\top}Qp + \frac{1}{2}p^{\top}Qx + \gamma p^{\top}Qp + c^{\top}p = 0$$

$$\gamma = -\frac{(c^{\top} + \frac{1}{2}x^{\top}Q)p + \frac{1}{2}Qx}{p^{\top}Qp} \implies \begin{cases} \gamma = -\dfrac{\alpha p + \beta}{2p^{\top}Qp} \\ \alpha = c^{\top} + x^{\top}Q \\ \beta = Qx \end{cases}$$

> `For further convenience, we can calculate step size with respect to set of equations above.`

- with step-size chosen by golden-ratio search method.

  https://en.wikipedia.org/wiki/Golden-section_search (https://en.wikipedia.org/wiki/Golden-section_search)
- with step-size chosen by Armijo rule.

  http://page.math.tu-berlin.de/~kandler/Hanoi2013/Lecture_NLO/bms-basic-NLP_120609.pdf (http://page.math.tu-berlin.de/~kandler/Hanoi2013/Lecture_NLO/bms-basic-NLP_120609.pdf) p. 16-17

Task 1.2

Implement coordinate descent with interleaving directions

- with step-size chosen optimally (exact line search)

  From previuos subtask
- with step-size chosen by Fibonacci method

  http://web.tecnico.ulisboa.pt/mcasquilho/compute/com/,Fibonacci/pdfHXu_ch1.pdf (http://web.tecnico.ulisboa.pt/mcasquilho/compute/com/,Fibonacci/pdfHXu_ch1.pdf), p.13

In [ ]:
```
class Descent:
    """

    Class implement descent procedure, with
    input function, gradient and step chooser

    """

    arg_losses = []
    fun_losses = []
    grad_sizes = []
    fun_values = []
    x_value = []

    optimal_x, optimal_f = None, None

    def __init__(self, function: callable, gradient: callable,
                 step_chooser=None, hessian=None):
        """

        :param function: optimized function
        :param gradient: gradient
        :param step_chooser: step-size chosing procedure
        :param hessian: optional
        """

        self.function = function
        self.gradient = gradient
        self.step_chooser = step_chooser
        self.hessian = hessian

        self._update()

    def _update(self):
        """

        :return: update self info
        """
        self.arg_losses = []
        self.fun_losses = []
        self.grad_sizes = []
        self.fun_values = []
        self.x_value = []

    def _save_results(self, x, fun_value, grad_size,
                      arg_loss, fun_loss):
        """
          save procedure for plotting

        :param x:
        :param fun_value:
        :param grad_size:
        :param arg_loss:
        :param fun_loss:
        :return:
        """
        self.x_value += [x]
        self.arg_losses += [arg_loss]
        self.fun_losses += [fun_loss]
```

```
        self.grad_sizes += [grad_size]
        self.fun_values += [fun_value]

    def _optimal_step(self, x, *args, default_step=0.1):
        """
         optimal step calculation
        :param x:
        :param default_step:
        :return:
        """

        step = self.step_chooser(x, *args)

        if np.isnan(step):
            return default_step
        else:
            return step

    def _coordinate_descent(self, x_init, tol=1e-3, max_iter=25):
        """
        implementation of coordinate descent
        :param x_init:
        :param tol:
        :param max_iter:
        :return:
        """
        success = False
        x_optimal, f_optimal = None, None
        iters = -1

        steps = []

        dimension, _ = x_init.shape
        E = np.eye(dimension)

        x_next, x_prev = deepcopy(x_init), deepcopy(x_init)

        for iter_num in range(max_iter):

            x_prev = deepcopy(x_next)
            grad = self.gradient(x_prev)

            # START: best direction search
            best_direction_num = iter_num % dimension
            best_direction = np.array([E[best_direction_num]]).T
            best_step = self._optimal_step(x_prev, best_direction, iter_num)

            # for direction_num in range(0, dimension):
            #     direction = np.array([E[direction_num]]).T
            #     step = self._optimal_step(x_prev, direction)
            #
            #     if self.function(x_prev - step * grad[direction_num, 0] * direction) < s
elf.function(
            #             x_prev - best_step * grad[best_direction_num, 0] * best_directio
n):
            #         best_direction = deepcopy(direction)
            #         best_step = deepcopy(step)
            #         best_direction_num = direction_num
            # FINISH: best direction search
```

```
            x_next = x_prev + best_step * best_direction

            arg_loss = np.linalg.norm(x_next - x_prev)
            fun_loss = np.linalg.norm(self.function(x_next) - self.function(x_prev))
            grad_size = np.linalg.norm(grad)
            fun_value = self.function(x_prev)

            steps.append(best_step)

            self._save_results(x_prev, fun_value, grad_size, arg_loss, fun_loss)

            if grad_size < tol:
                x_optimal = x_prev
                success = True
                iters = iter_num
                break

        if not success:
            x_optimal = x_next

        f_optimal = self.function(x_optimal)

        return x_optimal, f_optimal, iters, success

    def _gradient_descent(self, x_init, tol=1e-3, max_iter=25):
        """
        implementation of gradient descent
        :param x_init:
        :param tol:
        :param max_iter:
        :return:
        """

        success = False
        x_optimal, f_optimal = None, None
        iters = -1

        x_next, x_prev = deepcopy(x_init), deepcopy(x_init)

        for iter_num in range(max_iter):

            x_prev = deepcopy(x_next)

            grad = self.gradient(x_prev)
            step = self._optimal_step(x_prev, grad, iter_num)

            x_next = x_prev - step * grad

            arg_loss = np.linalg.norm(x_next - x_prev)
            fun_loss = np.linalg.norm(self.function(x_next) - self.function(x_prev))
            grad_size = np.linalg.norm(grad)
            fun_value = self.function(x_prev)

            self._save_results(x_prev, fun_value, grad_size, arg_loss, fun_loss)

            if grad_size < tol:
                x_optimal = x_prev
```

```
                success = True
                iters = iter_num
                break

        if not success:
            x_optimal = x_next

        f_optimal = self.function(x_optimal)

        return x_optimal, f_optimal, iters, success

    def optimize(self, x_init, tol=1e-3, max_iter=25, method='gradient-descent'):
        """
        main procedure for calculations
        :param x_init:
        :param tol:
        :param max_iter:
        :param method:
        :return:
        """

        self._update()

        x, f, iters, success = -1, -1, -1, False

        if method == 'gradient-descent':
            x, f, iters, success = self._gradient_descent(x_init=x_init, tol=tol, max_iter
=max_iter)
        elif method == 'coordinate-descent':
            x, f, iters, success = self._coordinate_descent(x_init=x_init, tol=tol, max_it
er=max_iter)

        result = {
            'x': x,
            'f': f,
            'iters': iters,
            'success': success,
            'method': method
        }

        self.x_optimal = x
        self.f_optimal = f

        return result
```

In [ ]:

```
Q, c = np.array([[1.06515021, 1.29457678], [1.29457678, 2.53821185]]), np.array([[0.082337
98, 3.87963941]]).T
x_start = np.array([[-10, 10]]).T
```

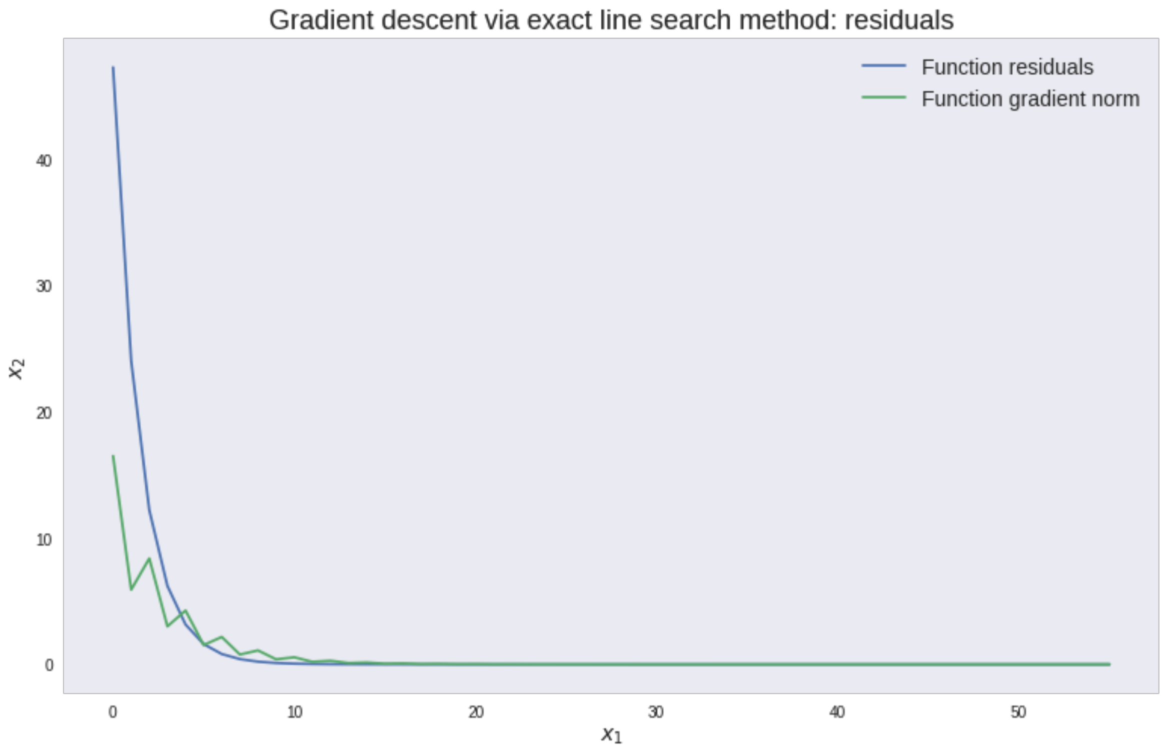
In [ ]:
```
def plot_results(l_b = -10, r_b = 10, n = 100, optimizer = None, title=''):
  l_b, r_b = -10, 10
  n = 100

  x = np.linspace(l_b, r_b, n)
  X, Y = np.meshgrid(x, x)
  Z = np.zeros(X.shape)

  for i in range(n):
    for j in range(n):
      Z[i, j] = function(np.array((X[i, j], Y[i, j])))

  plt.figure(figsize=(13, 8))
  plt.title(title + ' : steps', fontsize=18)
  plt.contour(X, Y, Z, levels=5)

  x_1 = np.array([ p[0] for p in optimizer.x_value ])
  x_2 = np.array([ p[1] for p in optimizer.x_value ])
  fns = np.array([ f for f in optimizer.fun_losses ])
  grd = np.array([ g for g in optimizer.grad_sizes ])

  plt.plot(x_1, x_2, label='Descent steps')

  plt.plot(optimizer.x_optimal[0], optimizer.x_optimal[1], '*g', markersize=20)
  plt.plot(x_start[0], x_start[1], '*r', markersize=20)

  plt.xlim([l_b, r_b])
  plt.ylim([l_b, r_b])

  plt.xlabel(r'$ x_1 $', fontsize=14)
  plt.ylabel(r'$ x_2 $', fontsize=14)

  plt.legend(fontsize=14)
  plt.grid()
  plt.show()

  plt.figure(figsize=(13, 8))
  plt.title(title + ': residuals', fontsize=18)

  plt.plot(fns, label='Function residuals')
  plt.plot(grd, label='Function gradient norm')

  plt.xlabel(r'$ x_1 $', fontsize=14)
  plt.ylabel(r'$ x_2 $', fontsize=14)

  plt.legend(fontsize=14)
  plt.grid()
  plt.show()
```

In [ ]:

```
# Gradient descent via exact line search method
function = lambda x: 0.5 * x.T @ Q @ x + c.T @ x
gradient = lambda x: Q @ x + c

def exact_line_search_opt_step(x, *args):
    p = args[0]
    return (2 * c.T @ p + x.T @ Q @ p + p.T @ Q @ x) / (2 * p.T @ Q @ p)

gradient_descent = {
    'function': function,
    'gradient': gradient,
    'opt_step': exact_line_search_opt_step
}
```

In [635]:

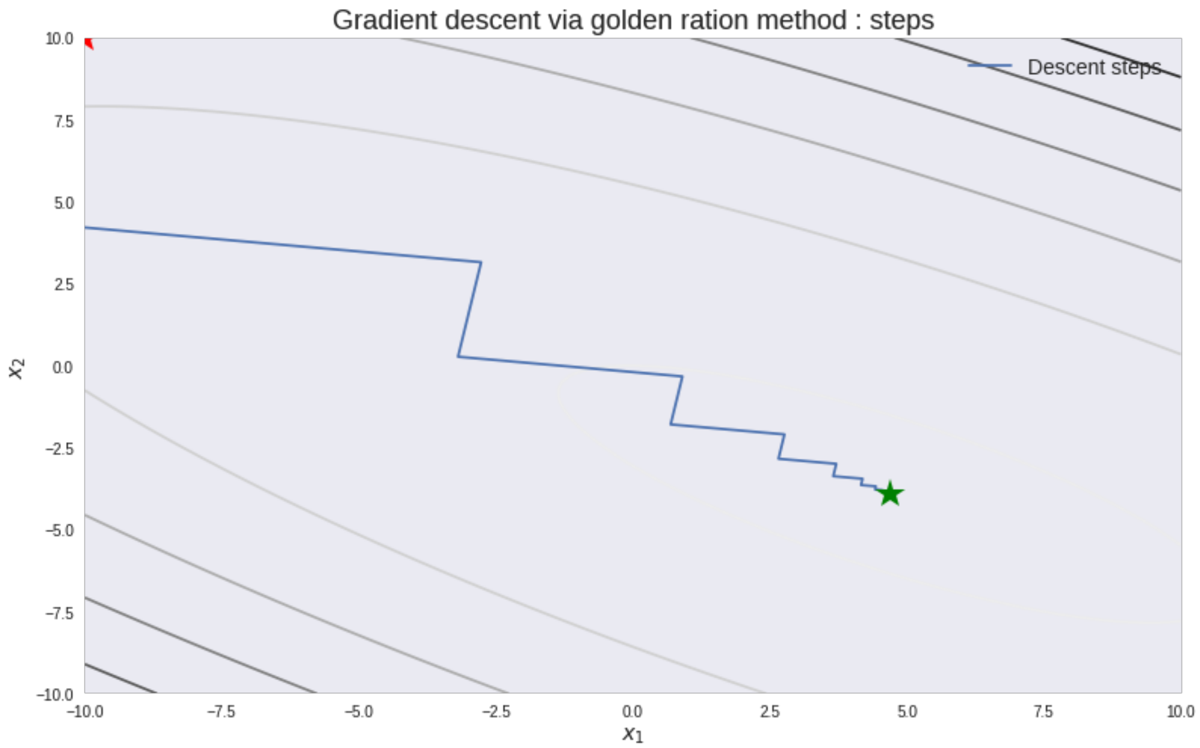
```
gd = Descent(coordinate_descent['function'], coordinate_descent['gradient'], step_chooser=
gradient_descent['opt_step'])
res_gd = gd.optimize(x_start, max_iter=400, tol=1e-7, method='gradient-descent')

plot_results(optimizer=gd, title='Gradient descent via exact line search method')
print(res_gd)
```

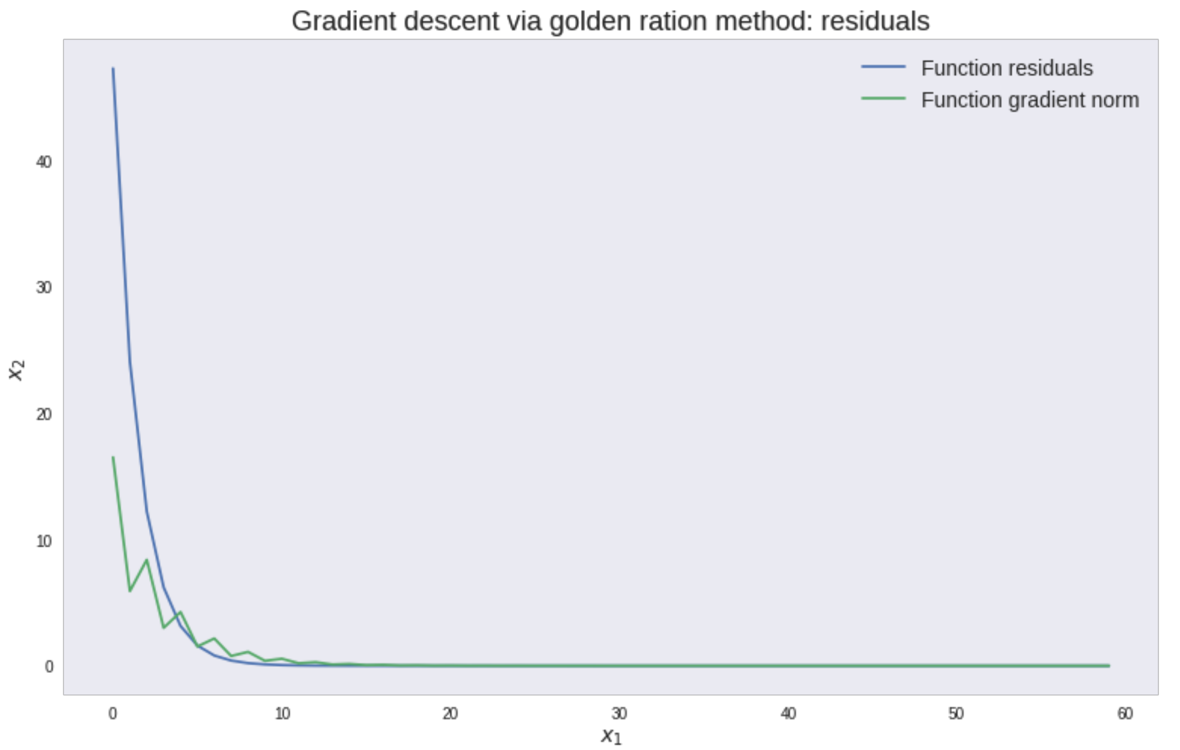



```
{'x': array([[ 4.68399724],
       [-3.91749549]]), 'f': array([[-7.40639968]]), 'iters': 55, 'success': True, 'method': 'gradient-descent'}
```

In [ ]:
```
# Gradient descent via golden ration method
function = lambda x: 0.5 * x.T @ Q @ x + c.T @ x
gradient = lambda x: Q @ x + c

def gss(*args):
    '''
    golden section search
    to find the minimum of f on [a,b]
    f: a strictly unimodal function on [a,b]

    example:
    >>> f = lambda x: (x-2)**2
    >>> x = gss(f, 1, 5)
    >>> x
    2.000009644875678

    '''

    x = args[0]
    grad = args[1]

    gr = (math.sqrt(5) + 1) / 2
    a, b = -100, 100
    tol = 1e-4

    c = b - (b - a) / gr
    d = a + (b - a) / gr
    while np.abs(c - d) > tol:
        if function(x - c * grad) < function(x - d * grad):
            b = d
        else:
            a = c

        # we recompute both c and d here to avoid loss of precision which may lead to inco
rrect results or infinite loop
        c = b - (b - a) / gr
        d = a + (b - a) / gr

    return (b + a) / 2


gradient_descent = {
    'function': function,
    'gradient': gradient,
    'opt_step': gss
}
```

In [645]:
```
gd = Descent(gradient_descent['function'], gradient_descent['gradient'], step_chooser=grad
ient_descent['opt_step'])
res_gd = gd.optimize(x_start, max_iter=400, tol=1e-7, method='gradient-descent')

plot_results(optimizer=gd, title='Gradient descent via golden ration method')
print(res_gd)
```

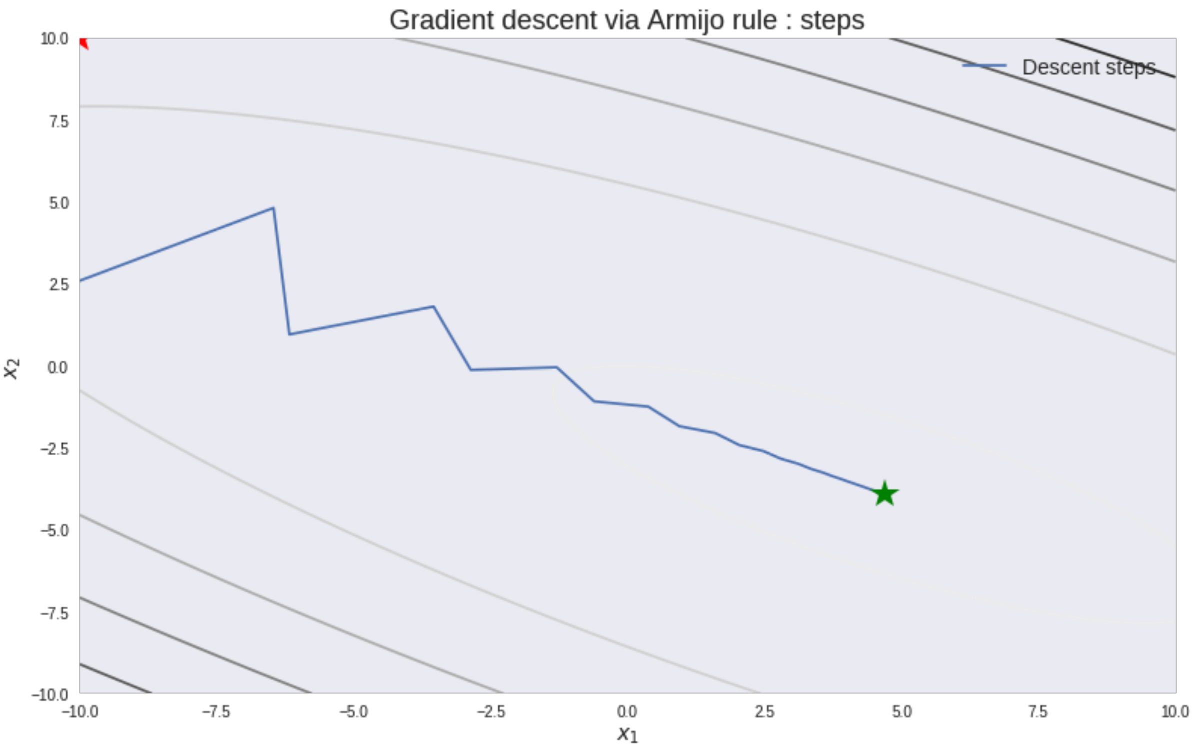


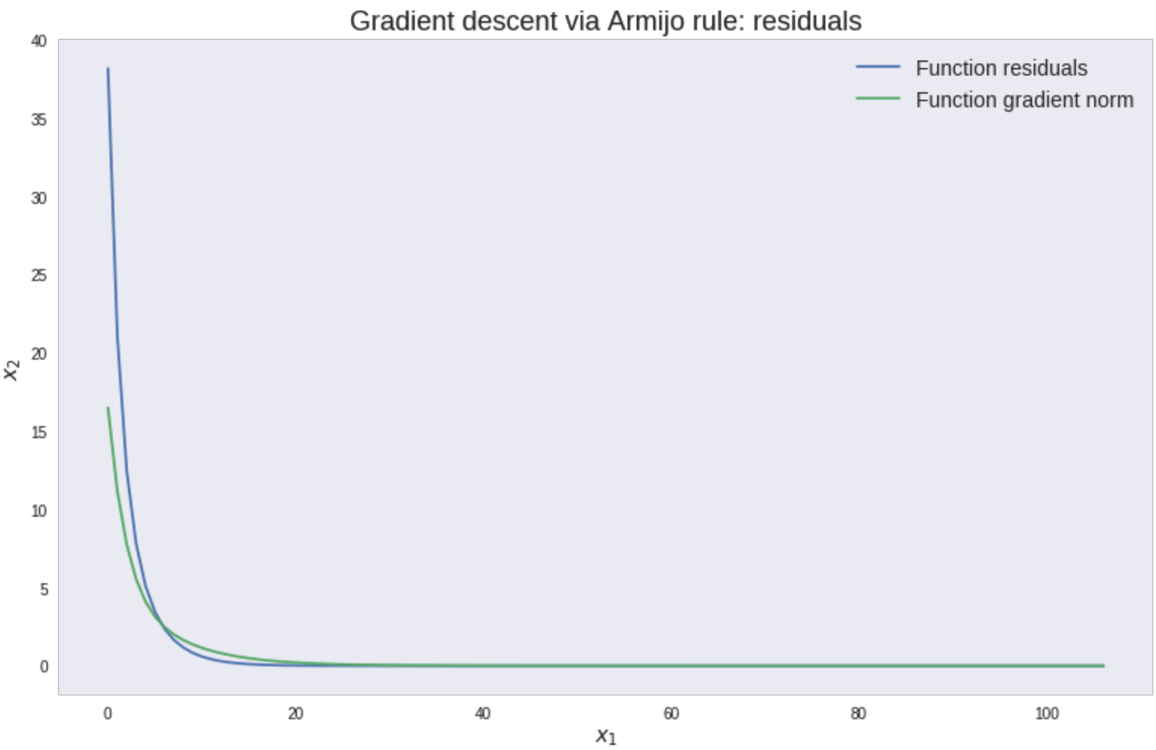



```
{'x': array([[ 4.68399729],
       [-3.91749552]]), 'f': array([[-7.40639968]]), 'iters': 59, 'success': True, 'method': 'gradient-descent'}
```

In [ ]:
```
# Gradient descent via Armijo rule
function = lambda x: 0.5 * x.T @ Q @ x + c.T @ x
gradient = lambda x: Q @ x + c

def armijo_rule_step(*args):

    x = args[0]
    p = args[1]

    beta = 0.5

    t = -beta * p.T @ p

    armijo_rule = lambda alpha: (function(x) - function(x + alpha * p))[0, 0] >= alpha * t

    while armijo_rule(beta):
        beta *= t

    return beta


gradient_descent = {
    'function': function,
    'gradient': gradient,
    'opt_step': armijo_rule_step
}
```

In [647]:
```
gd = Descent(gradient_descent['function'], gradient_descent['gradient'], step_chooser=grad
ient_descent['opt_step'])
res_gd = gd.optimize(x_start, max_iter=400, tol=1e-7, method='gradient-descent')

plot_results(optimizer=gd, title='Gradient descent via Armijo rule')
print(res_gd)
```

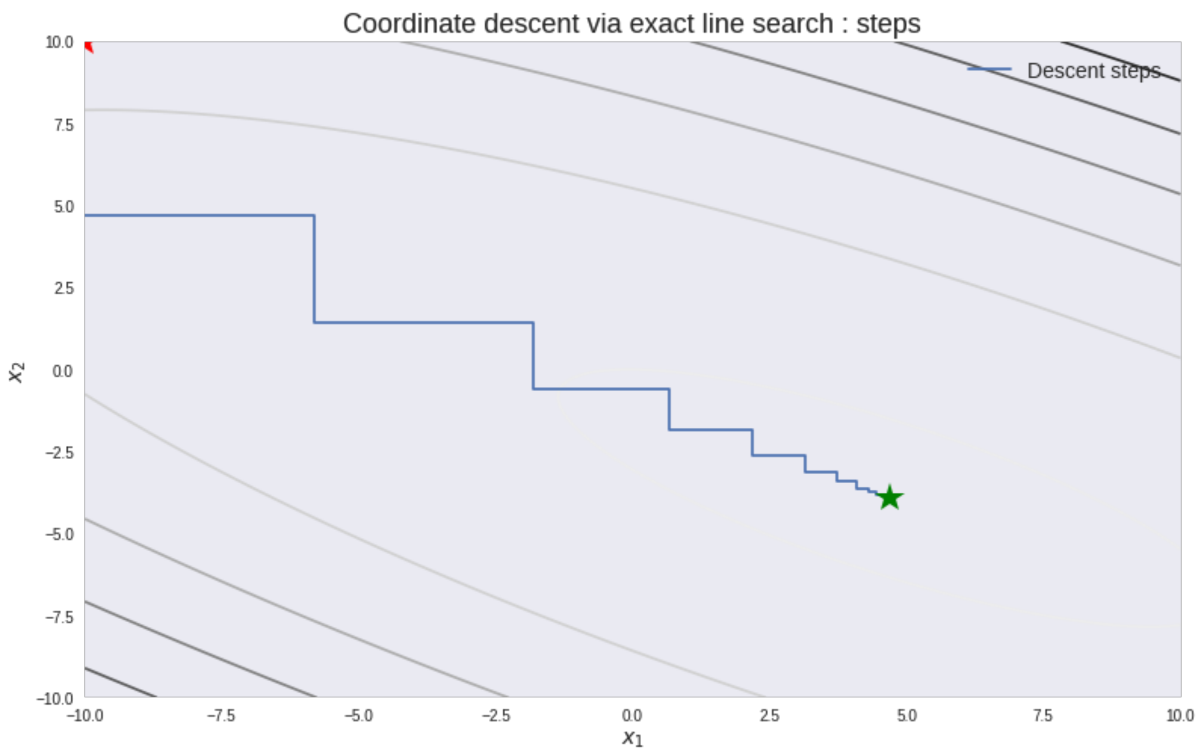


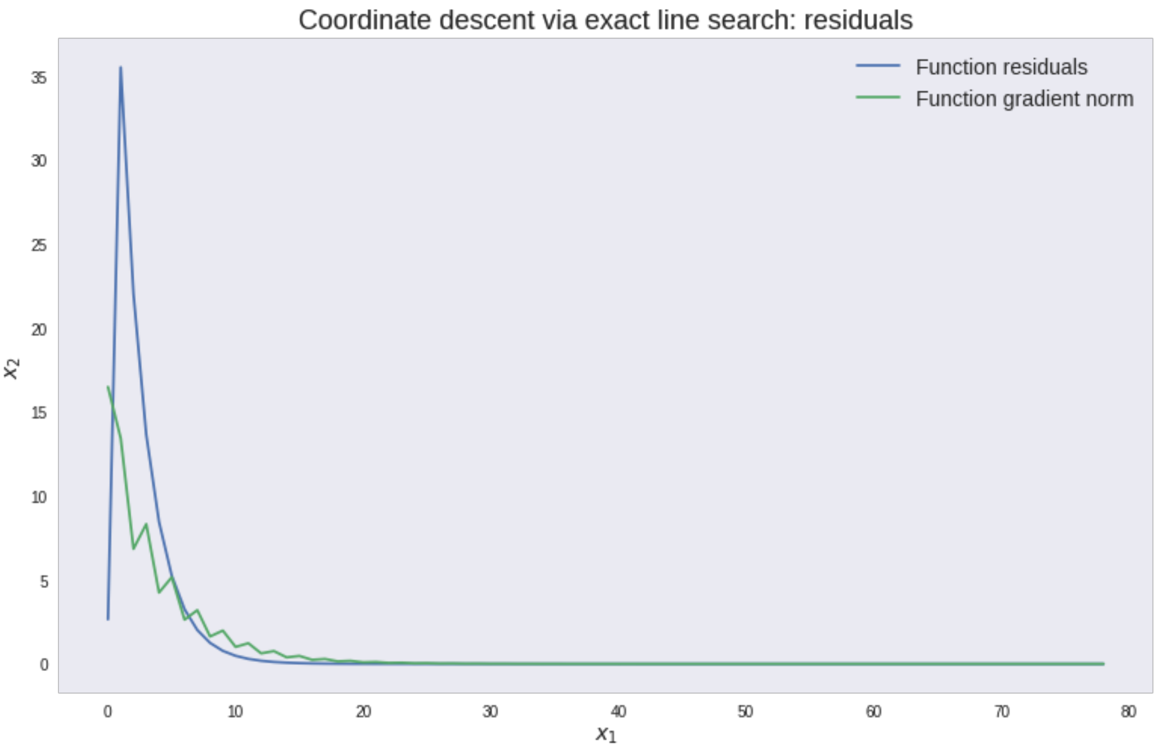



```
{'x': array([[ 4.68399716],
       [-3.91749543]]), 'f': array([[-7.40639968]]), 'iters': 106, 'success': True, 'method': 'gradient-descent'}
```

In [ ]:
```
# Coordinate descent via exact line search
function = lambda x: 0.5 * x.T @ Q @ x + c.T @ x
gradient = lambda x: Q @ x + c

def exact_line_search_opt_step(x, *args):
    p = args[0]
    return -(2 * c.T @ p + x.T @ Q @ p + p.T @ Q @ x) / (2 * p.T @ Q @ p)

coordinate_descent = {
    'function': function,
    'gradient': gradient,
    'opt_step': exact_line_search_opt_step
}
```

In [649]:
```
cd = Descent(coordinate_descent['function'], coordinate_descent['gradient'], step_chooser=
coordinate_descent['opt_step'])
res_cd = cd.optimize(x_start, max_iter=400, tol=1e-7, method='coordinate-descent')

plot_results(optimizer=cd, title='Coordinate descent via exact line search')
print(res_cd)
```





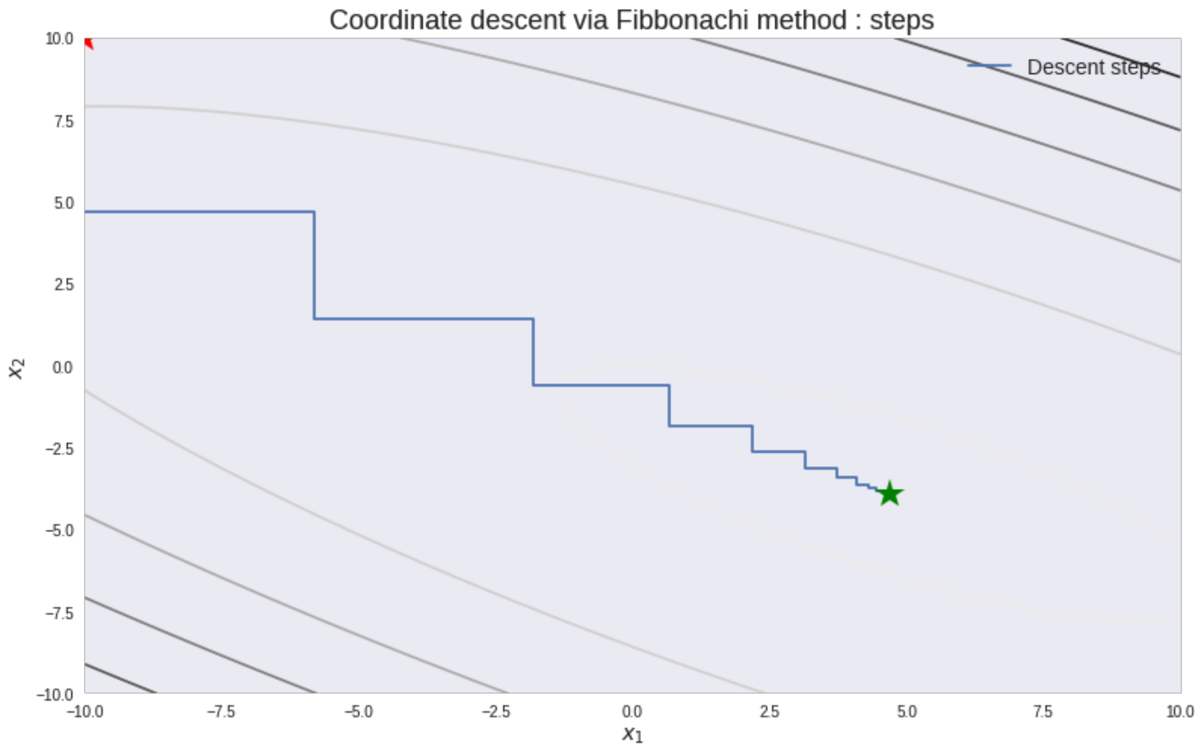

```
{'x': array([[ 4.6839972 ],
       [-3.91749547]]), 'f': array([[-7.40639968]]), 'iters': 78, 'success': True, 'method': 'coordinate-descent'}
```

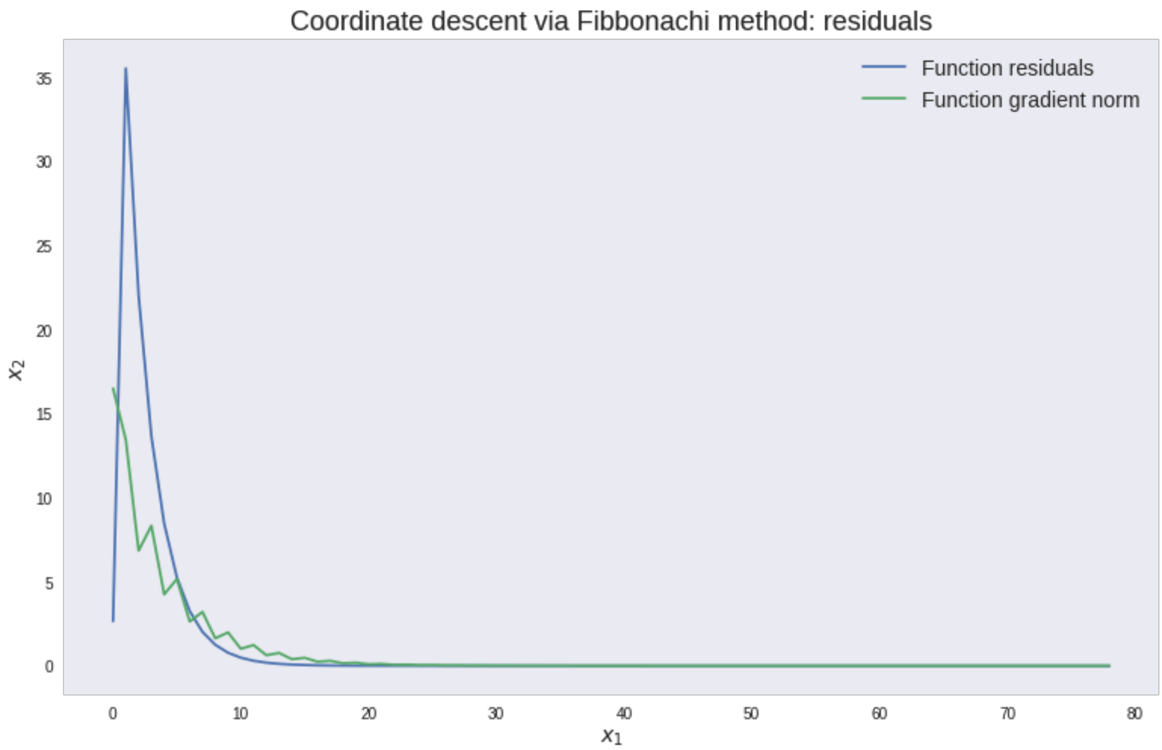
In [ ]:

```
# Coordinate descent via Fibbonachi method
function = lambda x: 0.5 * x.T @ Q @ x + c.T @ x
gradient = lambda x: Q @ x + c


def fibonacci_method(*args):

    N = 50
    a, b = -100, 100

    x = args[0]
    p = args[1]

    def fibonacci(n):
        x, y = 0.0, 1.0
        if n < 1.0:
            return 0.0
        if n == 1.0:
            return 1.0

        for _ in range(2, n):

            t = x
            x = y
            y = t + y

        return y

    for i in range(N):
        x_1 = fibonacci(N - i - 1) / fibonacci(N - i + 1) * (b - a) + a
        x_2 = fibonacci(N - i) / fibonacci(N - i + 1) * (b - a) + a

        if function(x + x_2 * p) > function(x + x_1 * p):
            b = x_2

        else:
            a = x_1

    return (a + b) / 2


coordinate_descent = {
    'function': function,
    'gradient': gradient,
    'opt_step': fibonacci_method
}
```

In [651]:

```
cd = Descent(coordinate_descent['function'], coordinate_descent['gradient'], step_chooser=
coordinate_descent['opt_step'])
res_cd = cd.optimize(x_start, max_iter=400, tol=1e-7, method='coordinate-descent')

plot_results(optimizer=cd, title='Coordinate descent via Fibbonachi method')
print(res_cd)
```






```
{'x': array([[ 4.6839972 ],
       [-3.91749545]]), 'f': array([[-7.40639968]]), 'iters': 78, 'success': True, 'method': 'coordinate-descent'}
```

Task 2

$$x(t) \in R^n, t = 0, \dots N, u(t) \in R$$
$$\text{The dynamics of the system :}$$
$$x(t+1) = Ax(t) + bu(t), t = 0, \dots, N-1 \text{Task - choose the inputs } u(0), \dots,$$
$$u(N-1) \text{ in order to minimize the total fuel consumed :}$$
$$F = \sum_{t=0}^{N-1} f(u(t))$$
$$\text{subject to the constraint that } x(N) = x_{des}.$$
$$\text{Where } : f(a) = \begin{cases} |a|, |a| \leq 1, \\ 2|a| - 1, |a| \geq 1 \end{cases}$$

Task 2.1

Formulate the minimum fuel optimal control problem as an LP.

$$x(N) = x_{des}$$
$$x(1) = bu(0)$$
$$x(2) = Abu(0) + bu(1)$$
$$x(3) = A^2bu(0) + Abu(1) + bu(0)$$
$$\vdots$$
$$x(N) = A^{N-1}bu(0) + A^{N-2}bu(1) + \dots + Abu(N-1) + bu(N-1)$$
$$\text{We can define controllability matrix :}$$
$$C = \left[A^{N-1}b, A^{N-2}b, \dots mAb, b\right] \text{ and } u^\top = [u(0), \dots u(N-1)]$$
$$\text{Problem :}$$
$$Cu = x_{des}$$
$$\text{Now, turn attention back to } f(a). \text{ First, add single variable } t, togetherwithconstraints$$
$$|a| \leq t$$
$$2|a| - 1 \leq t \implies$$
$$-t \leq a \leq t$$
$$-\frac{t+1}{2} \leq a \leq \frac{t+1}{2}$$
$$\text{Introduce vector } t = [t_0, \dots, t_{N-1}] \text{ LP is :}$$
$$\mathcal{P} = \begin{cases} min 1^\top t \\ \text{ subject to :} \\ -t \leq u \leq t \\ -\frac{t+1}{2} \leq u \leq \frac{t+1}{2} \\ Cu = x_{des} \end{cases}$$

Task 2.2

Calculation and visualization of the results

In [ ]:
```
import cvxpy
from cvxpy import Variable, Minimize, maximum, abs, sum, Problem
```

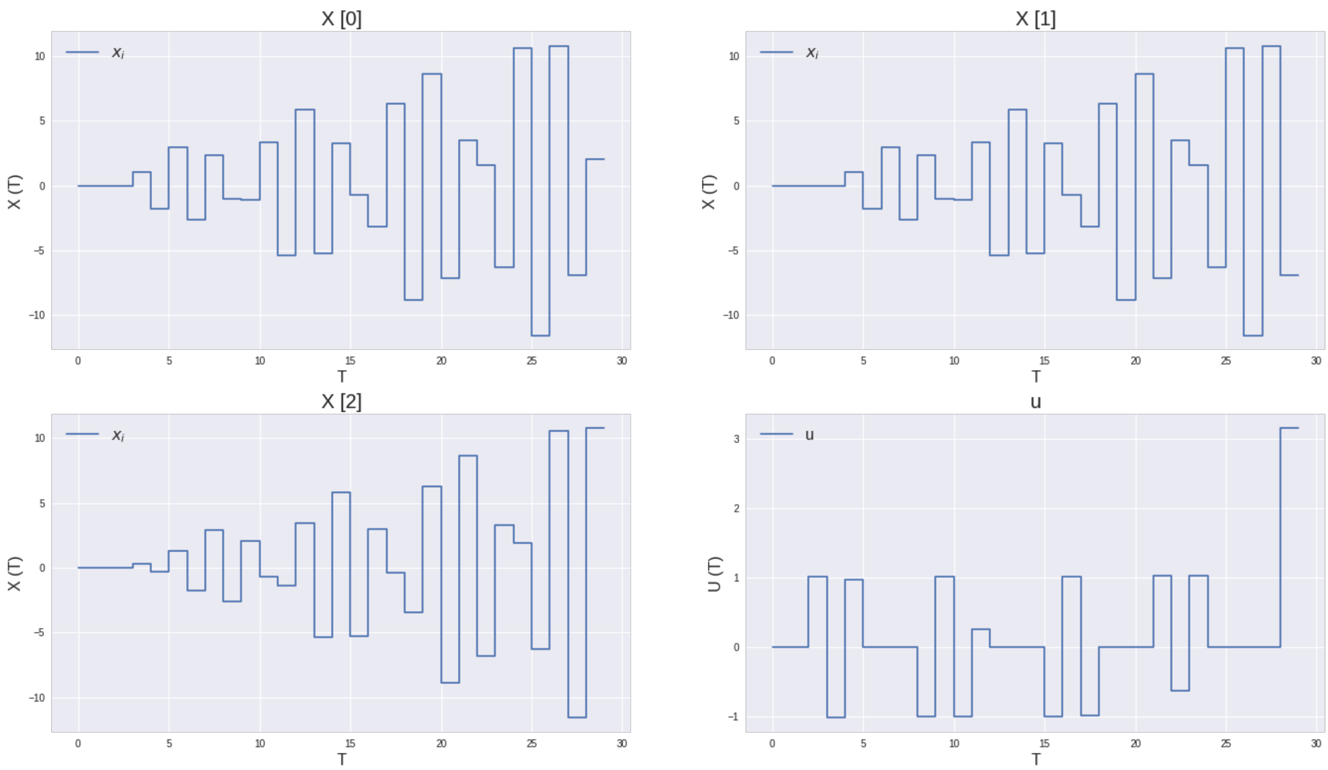
In [ ]:
```
n = 3
N = 30
A = np.array([[-1, 0.4, 0.8], [1, 0, 0], [0, 1, 0]])
b = np.array([[1, 0, 0.3]]).T
x_des = np.array([7, 2, -6])
```

In [ ]:
```
def optimal_control(A, b, x_des, n, N):

  X = Variable((n, N+1))
  u = Variable((1, N))

  objective = Minimize(sum(maximum(abs(u), 2 * abs(u) - 1)))

  constraints = [
      X[:, 0] == 0,
      X[:, 1:] == A * X[:, :N] + b * u,
      X[:, -1] == x_des
  ]

  prob = Problem(objective=objective, constraints=constraints)
  optimal_f = prob.solve()

  return optimal_f, X.value, u.value
```

In [ ]:
```
optimal_f, x, u = optimal_control(A, b, x_des, n, N)
```

In [ ]:
```
def plot_results(x, u, title):

  fig, axs = plt.subplots(2, 2)

  fig.set_size_inches(23, 13)

  axs = axs.flatten()

  for i, (ax) in enumerate(axs[:-1]):

    ax.step(np.arange(N), x[i, :-1], label=r'$ x_i $')
    ax.set_title('X [{}]'.format(i), fontsize=20)
    ax.set_xlabel('T', fontsize=16)
    ax.set_ylabel('X (T)', fontsize=16)
    ax.legend(fontsize=16)


  ax = axs[3]
  ax.step(np.arange(N), u[0, :], label='u')
  ax.set_title('u', fontsize=20)
  ax.set_xlabel('T', fontsize=16)
  ax.set_ylabel('U (T)', fontsize=16)
  ax.legend(fontsize=16)

  plt.show()
```

In [657]:
```
plot_results(x, u, 'asd')
```



Task 2.3

Perturbing of x_{des}

Validate this fact numerically by perturbing xdes and analytically (based on largange multiplier values).

Lagrange form:

$$L(t, u, \alpha, \beta, \gamma, \Delta, \theta, \delta) =$$

$$= 1^\top t - \alpha^\top (u + t) + \beta^\top (u - t) - \gamma^\top (2u + 1 + t) + \Delta^\top (2u - 1 - t) + \theta(Cu - x_{des}) + \delta_{R_+^{4N}} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \Delta \end{bmatrix} =$$

$$= (-\alpha + \beta - 2\gamma + 2\Delta + C^\top \theta)^\top u + (1 - \alpha - \beta - \gamma - \Delta)^\top t - (\gamma^\top 1 + \Delta^\top 1 + \theta^\top x_{des}) + \delta_{R_+^{4N}} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \Delta \end{bmatrix}$$

By definition,

$$g(\alpha,\beta,\gamma,\Delta,\theta,\delta) = \inf_{u,t} =$$

$$= \inf_{u,t}\left[(-\alpha+\beta-2\gamma+2\Delta+C^\top\theta)^\top u + (1-\alpha-\beta-\gamma-\Delta)^\top t - (\gamma^\top 1+\Delta^\top 1+\theta^\top x_{des}) + \delta_{R_+^{4N}}\begin{bmatrix}\alpha\\\beta\\\gamma\\\Delta\end{bmatrix}\right] =$$

$$= \inf_{u,t}\left[(-\alpha+\beta-2\gamma+2\Delta+C^\top\theta)^\top u + (1-\alpha-\beta-\gamma-\Delta)^\top t - (\gamma^\top 1+\Delta^\top 1+\theta^\top x_{des}) + \delta_{R_+^{4N}}\begin{bmatrix}\alpha\\\beta\\\gamma\\\Delta\end{bmatrix}\right] =$$

$$= \inf_{u}\left[(-\alpha+\beta-2\gamma+2\Delta+C^\top\theta)^\top u\right] + \inf_{t}\left[(1-\alpha-\beta-\gamma-\Delta)^\top t\right] - (\gamma^\top 1+\Delta^\top 1+\theta^\top x_{des})$$

$$+ \delta_{R_+^{4N}}\begin{bmatrix}\alpha\\\beta\\\gamma\\\Delta\end{bmatrix}$$

At the part of t part of the expression, we see, that $(1-\alpha-\beta-\gamma-\Delta)^\top t = 1^\top t - (\alpha+\beta+\gamma+\Delta)^\top t$. In order for g to be finite, clear, that: $\alpha+\beta+\gamma+\Delta \leq 1$. By the same logic, implies, that $-\alpha+\beta-2\gamma+2\Delta+C^\top\theta = 0$ With this condition implies, that:

$$g(\alpha,\beta,\gamma,\Delta,\theta,\delta) = -(\gamma^\top 1+\Delta^\top 1+\theta^\top x_{des}) + \delta_{R_+^{4N}}\begin{bmatrix}\alpha\\\beta\\\gamma\\\Delta\end{bmatrix}$$

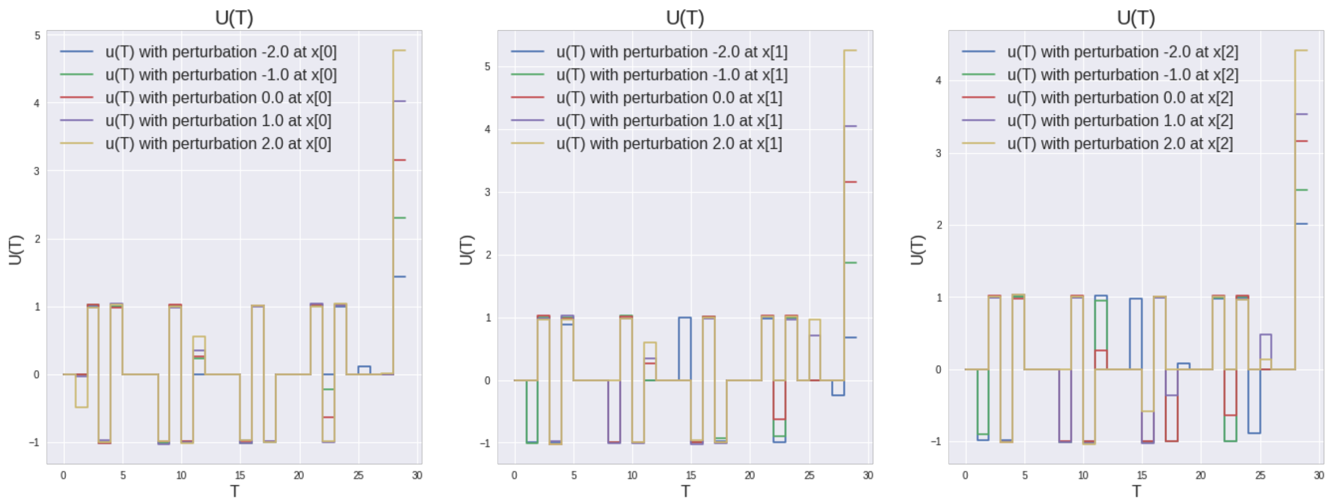
Then dual problem is:

$$\mathcal{P} \hat{=} \begin{cases} max\left[\gamma^\top 1+\Delta^\top 1+\theta^\top x_{des}\right] \\ 0 \leq \alpha,\beta,\gamma,\Delta \\ -\alpha+\beta-2\gamma+2\Delta+C^\top\theta = 0 \\ \alpha+\beta+\gamma+\Delta \leq 1 \end{cases}$$

From the formulation of dual problem we see, that perturbing of x_{des} influences directly on behavior of u. This infuence, we can

In [ ]:
```
def perturbations(component = 0, l_b = -2, r_b = 2, _n = 4):

  xs = []
  us = []

  dxs = np.linspace(l_b, r_b, _n)

  for dx in dxs:
    _x_des = deepcopy(x_des)
    _x_des[component] += dx

    optimal_f, x, u = optimal_control(A, b, _x_des, n, N)

    xs += [x]
    us += [u]

  return xs, us, dxs
```

In [659]:
```
fig, axs = plt.subplots(1, 3)
fig.set_size_inches(23, 8)
axs = axs.flatten()

for i, (ax) in enumerate(axs):

  xs, us, dxs = perturbations(component=i, _n = 5)

  for j in range(len(xs)):

    ax.step(np.arange(N), us[j][0, :], label='u(T) with perturbation {} at x[{}]'.format(d
xs[j], i))
    ax.set_title('U(T)'.format(i), fontsize=20)
    ax.set_xlabel('T', fontsize=16)
    ax.set_ylabel('U(T)', fontsize=16)
    ax.legend(fontsize=16)

plt.show()
```



Some information and hint are from :

- **https://sites.math.washington.edu/~burke/crs/cvx08/projects/4-16-sasha.pdf (https://sites.math.washington.edu/~burke/crs/cvx08/projects/4-16-sasha.pdf)**