Optimization Methods

Homework # 3

Deadline: February 17, 2019 23:00

Max: 40 points

1. (1 point) Prove that if functions $f(\cdot), g(\cdot)$ are convex, then $h(x) = \max\{f(x), g(x)\}$ is convex as well.

2. (1 point) Derive gradient and Hessian matrix (both in vector form) for the quadratic form $f(x) = (Ax, x)$. Matrix $A \in \mathbb{R}^{n\times n}$ may be non-symmetric.

3. (2 points) Prove composition rule "$f(x) = h(g(x))$ is convex if $\overline{h}$ is convex non-increasing, and g is concave" (here $h : \mathbb{R} \to \mathbb{R}$, while $g : \mathbb{R}^n \to \mathbb{R}$).

4. (3 points) Prove that all sub-level sets $Q(c) = \{x : f(x) \leq c\}$ of a strongly convex function $f(\cdot)$ are bounded.
   Hint: assume that there exists unbounded $Q(c)$ for some c, and show contradiction.

5. (1 point) Assume a convex function $f : \mathbb{R}^k \to \mathbb{R}$ has non-empty epigraph. A support hyperplane to the epigraph set has equation $(c, x) + b = 0$. What are the dimensions of c and b?

6. (2 points) Find conjugate function $f^*(y)$ for $f(x) = (Qx, x)$ with $Q \succ 0$.

7. (1 point) Solve one-dimensional problem (approximation by constant)

$$\min_{z\in\mathbb{R}} \frac{1}{m}\sum_{i=1}^{m}(z - x_i)^2$$

8. (3 points) Best approximating line (coefficients $a, b \in \mathbb{R}$).
   For points on plane $(x_i, y_i) \in \mathbb{R}^2, i = 1, ..., m$ there is a best matching line $ax + b$, which minimizes mean of residuals:

$$\min_{a,b} \frac{1}{m}\sum_{i=1}^{m}(ax_i + b - y_i)^2$$

   Solve this problem explicitly with respect to a, b.
   Hint: it is an unconstrained optimization problem.

9. (3 + 3 points) Check that BGFS update formulae for a) B_{k+1} and b) H_{k+1} satisfy quasi-Newton conditions (cf. Lecture 5, e.g. $H_{k+1}d = s$, $d = B_{k+1}s$).

Writing down derivation of dual function and solving dual problem explicitly is necessary for the next two problems.

10. (4 points) Solve the following "projection on a Euclidean ball" problem via dual function. The $c, z \in \mathbb{R}^n$ and $r > 0$ are parameters.

$$\min_{x\in\mathbb{R}^n:\|x-c\|\leq r} \|z - x\|_2$$

   Hint: make the target function differentiable and rewrite constraints in more convenient form.

11. (4 points) Find projection of a point to hyperplane $Q = \{(a, x) = b\}, \quad a \neq 0$ via solving dual problem, with projection defined as a closest point on the set

$$\mathbf{Proj}_Q(z) = \min_{x \in Q} \|x - z\|_2$$

i.e. solve

$$\min_{(a,x)=b} \|x - z\|_2$$

12. (4 points) Write down dual function and its domain for the constrained optimization problem ($f : \mathbb{R}^n \to \mathbb{R}, A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m$)

$$\min_{Ax=b} f(x)$$

via conjugate function $f^*(y) = \sup_{x \in R^n} \big((x, y) - f(x)\big)$.

13. (4+4 points) a) Derive (Lagrange) dual function $d(\cdot)$, and write down dual problem for the following primal problem:

$$\min_{(Ax,x) \leq 1} (c, x),$$

where $x, c \in \mathbb{R}^n, c \neq 0, A \in S_n, A \succ 0$.
b) solve the dual problem and find solution to the primal one.

Extra tasks

1. (0 points)
Is composition rule of Problem 3 still valid with non-extended-valued function h used instead of $\overline{h}$? If not, demonstrate a counter-example of non-convex $f(x)$.

2. (0 points)
Find out subdifferential for $\|x\|$ (take special attention to $x = 0$).