QUANTUM CHANNELS ON QBITS

1. A little linear algebra

Consider the vector space $\mathcal{H} = \mathbb{C}^n$. An element $\boldsymbol{z} \in \mathcal{H}$ will be denoted in bold, where $\boldsymbol{z} = (z_1, \ldots, z_n)$. We will use the Dirac notation wherein

$$|\boldsymbol{z}\rangle = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}$$

denotes the column vector and

$$\langle \boldsymbol{z}| = \begin{bmatrix} z_1^* & \cdots & z_n^* \end{bmatrix}$$

denotes the complex conjugate row vector. The inner product between two vectors is given by:

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle = \sum_{j=1}^n z_j^* w_j \; .$$

Recall the complex conjugate $(a+ib)^* = a - ib$, for a complex number. Note that

$$\langle \boldsymbol{z}|\boldsymbol{w}\rangle^* = \langle \boldsymbol{w}|\boldsymbol{z}\rangle \; .$$

The *norm* $\|\boldsymbol{z}\|$ of a vector, is it's Euclidean length, given by

$$\|\boldsymbol{z}\|^2 = \sum_{j=1}^n |z_j|^2 = \langle \boldsymbol{z}|\boldsymbol{z}\rangle \; .$$

Let $\boldsymbol{A}$ be an $n \times n$ matrix,

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & \ldots & \ldots & A_{1n} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{n1} & \ldots & \ldots & A_{nn} \end{bmatrix} .$$

The *transpose* $\boldsymbol{A}^T$ of $\boldsymbol{A}$ is the matrix with $\boldsymbol{A}^T_{ij} = A_{ji}$. The complex conjugate of the transpose is called the *adjoint* of $\boldsymbol{A}$ and denoted $\boldsymbol{A}^\dagger$. Thus,

$$A^\dagger_{ij} = A^T_{ji} \, .$$

Note that

$$(\langle \boldsymbol{z}| \, \boldsymbol{A} \, |\boldsymbol{w}\rangle)^* = \langle \boldsymbol{w}| \, \boldsymbol{A}^\dagger \, |\boldsymbol{z}\rangle \; .$$

A matrix $\boldsymbol{A}$ is *self-adjoint* if $\boldsymbol{A}^\dagger = \boldsymbol{A}$, i.e., $A_{ji} = A^*_{ij}$. For a self-adjoint matrix, one has

$$(\langle \boldsymbol{z}| \, \boldsymbol{A} \, |\boldsymbol{w}\rangle)^* = \langle \boldsymbol{w}| \, \boldsymbol{A} \, |\boldsymbol{z}\rangle \tag{1.1}$$

Recall that an *eigenvector* of a matrix A is a non-zero vector $\boldsymbol{z}$ such that

$$A \, |\boldsymbol{z}\rangle = \lambda \boldsymbol{z}$$

with $\lambda \in \mathbb{C}$, called the *eigenvalue* of $\boldsymbol{z}$. Suppose that $\boldsymbol{A}$ is self-adjoint. Then

$$\langle \boldsymbol{z}|\, \boldsymbol{A}\, |\boldsymbol{z}\rangle \;=\; \lambda \,\langle \boldsymbol{z}|\boldsymbol{z}\rangle \;=\; \lambda \|\boldsymbol{z}\|^2 \,.$$

Taking complex conjugates, and using (1.1), we find that

$$\lambda^* \|\boldsymbol{z}\|^2 \;=\; \langle \boldsymbol{z}|\, \boldsymbol{A}\, |\boldsymbol{z}\rangle \;=\; \lambda \|\boldsymbol{z}\|^2 \,.$$

Since $\|\boldsymbol{z}\| \neq 0$, we conclude that $\lambda^* = \lambda$, i.e. that λ is a real number. So every eigenvector of a self-adjoint matrix is real.

A sequence of m-vectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_m$ is called *ortho-normal* if

$$\langle \boldsymbol{z_i}|\boldsymbol{z_j}\rangle \;=\; \delta_{i,j} \,.$$

Here $\delta_{i,j}$ is the *Kronecker δ symbol*; it equals 1 if $i = j$ and 0 if $i \neq j$. One can show that any ortho-normal sequence is linearly independent. If $m = n$, then the sequence is also a basis, and is called an ortho-normal basis. If $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ is an ortho-normal basis and $\boldsymbol{z} \in \mathcal{H}$, then

$$|\boldsymbol{z}\rangle \;=\; \sum_{j=1}^{n} \left(\langle \boldsymbol{z_j}|\boldsymbol{z}\rangle\right) |\boldsymbol{z_j}\rangle \,.$$

Another way to write this expression is to note that the *identity matrix* $\boldsymbol{I}$, with $I_{ij} = \delta_{ij}$, can be written

$$\boldsymbol{I} \;=\; \sum_{j=1}^{n} |\boldsymbol{z_j}\rangle \langle \boldsymbol{z_j}| \,.$$

Theorem 1.1. *Every self-adjoint matrix $\boldsymbol{A}$ has an ortho-normal basis of eigenvectors.*

Proof. First note that every matrix $\boldsymbol{A}$ has at least one eigenvector. This is because $\lambda \in \mathbb{C}$ is an eigenvalue of $\boldsymbol{A}$ if and only if $\det(\lambda \boldsymbol{I} - \boldsymbol{A}) \;=\; 0$. The determinant $\det(\lambda \boldsymbol{I} - \boldsymbol{A})$ is a polynomial of degree n in λ. The Fundamental Theorem of Algebra states that every polynomial has a root in $\mathbb{C}$. For a self-adjoint matrix, the eigenvalue has to be real, of course.

Let $\boldsymbol{z}_1$ be an eigenvector of $\boldsymbol{A}$, with eigenvalue λ_1. Divide $\boldsymbol{z}_1$ by its length if necessary so that it has length one. Consider the set of vectors orthogonal to $\boldsymbol{z}_1$:

$$\mathcal{H}_1 \;=\; \{\boldsymbol{z} \;:\; \langle \boldsymbol{z_1}|\boldsymbol{z}\rangle = 0\} \,.$$

For such vectors,

$$\langle \boldsymbol{z_1}|\, \boldsymbol{A}\, |\boldsymbol{z}\rangle \;=\; \lambda_1 \,\langle \boldsymbol{z_1}|\boldsymbol{z}\rangle \;=\; 0 \,.$$

That is $\boldsymbol{A}$ maps the space $\mathcal{H}_1$ into itself. Choosing an ortho-normal basis for $\mathcal{H}_1$ we can rexpress $\boldsymbol{A}$ on this space as an $n-1 \times n-1$ matrix (because $\dim \mathcal{H}_1 = n-1$). One can show that the matrix is self-adjoint. So it has an eigenvector $\boldsymbol{z}_2$ and eigenvalue $\boldsymbol{\lambda}_2$. We can take $\boldsymbol{z}_2$ to have length one. Now define $\mathcal{H}_2$ to be all the set of all vectors orthogonal to $\boldsymbol{z}_1$ and $\boldsymbol{z}_2$. If $n = 2$ the only such vector is $\mathbf{0}$, but but if not there is a non-trivial subspace. Repeating the above argument we can find an eigenvector among them. Continue this process until you have n eigenvectors, at which point we have a basis which is ortho-normal by construction. $\square$

If $\boldsymbol{A}$ is self-adjoint, with ortho-normal eigenvectors $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$, then

$$\boldsymbol{A} \;=\; \sum_{j=1}^{n} \lambda_j \,|\boldsymbol{z}_j\rangle \langle \boldsymbol{z}_j| \,. \tag{1.2}$$

For any complex valued function f on the eigenvalues, one defines

$$f(\boldsymbol{A}) \;=\; \sum_{j=1}^{n} f(\lambda_j) \left| \boldsymbol{z}_j \right\rangle \left\langle \boldsymbol{z}_j \right| \;. \tag{1.3}$$

2. FINITE DIMENSIONAL QUANTUM SYSTEMS

We will consider *finite dimensional quantum systems*. These are approximations to real systems, which typically need infinite dimensional spaces for their full description. However, in many interesting systems there is a finite number of the dimensions needed to describe the "effective," or important, properties of the system.

For each system we consider we will take the Hilbert space $\mathcal{H} = \mathbb{C}^n$ for some n. An *observable* of our quantum system is an $n \times n$ self-adjoint matrix. Recall that a matrix ρ is *positive semi-definite* if

$$\left\langle \boldsymbol{z} \right| \rho \left| \boldsymbol{z} \right\rangle \;\geq\; 0 \tag{2.1}$$

for all $\boldsymbol{z} \in \mathcal{H}$. Such a matrix is self-adjoint with all non-negative eigenvalues. Conversely, if ρ is self-adjoint and all of its eigenvalues are non-negative, then ρ is positive semi-definite. A *state* of our system is a positive semi-definite matrix ρ with $\operatorname{tr} \rho = 1$. Here $\operatorname{tr} \boldsymbol{A}$ denotes the *trace* of a matrix, $\operatorname{tr} \boldsymbol{A} \;=\; \sum_{i=1}^{n} A_{ii}$. One can also has

$$\operatorname{tr} \boldsymbol{A} \;=\; \sum_{j=1}^{n} \left\langle \boldsymbol{z}_j \right| \boldsymbol{A} \left| \boldsymbol{z}_j \right\rangle$$

for any ortho-normal basis $\boldsymbol{z}_1, \ldots, \boldsymbol{z}_n$.

According to quantum mechanics, the eigenvalues $\lambda_1, \ldots, \lambda_n$ of an observable $\boldsymbol{A}$ represent the possible outcomes of a measurement of $\boldsymbol{A}$. If our system has state ρ and we measure $\boldsymbol{A}$, quantum mechanics predicts that we obtain the answer λ_j with probability $\left\langle \boldsymbol{z}_j \right| \rho \left| \boldsymbol{z}_j \right\rangle$, where $\boldsymbol{z}_j$ is the corresponding eigenvector. Since $\sum_j \left\langle \boldsymbol{z}_j \right| \rho \left| \boldsymbol{z}_j \right\rangle = \operatorname{tr} \rho = 1$, the probabilities add to one. This all means that if we prepare our system in state ρ many times and measure $\boldsymbol{A}$ each time, we will obtain different answers, but in the long run the fraction of times we get λ_j as an answer will approach $\left\langle \boldsymbol{z}_j \right| \rho \left| \boldsymbol{z}_j \right\rangle$.

3. THE BLOCH SPHERE REPRESENTATION OF QUBITS

A *qubit* is a "two-level" quantum sytem. The Hilbert space of a qbit is $\mathbb{C}^2$, which is the simplest non-trivial Hilbert space. An observable for a qubit is a 2×2 self-adjoint matrix. Any such matrix $\boldsymbol{A}$ is of the form

$$\boldsymbol{A} \;=\; \begin{bmatrix} \alpha & \gamma - i\delta \\ \gamma + i\delta & \beta \end{bmatrix} \tag{3.1}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. The space of observables is a four dimensional vector space with a basis given by the identity

$$\sigma_0 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

together with the *Pauli* matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \;, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \;, \quad \text{and} \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \;. \tag{3.2}$$

Explicitly, if $\boldsymbol{A}$ is given by (3.1), then

$$\boldsymbol{A} \;=\; \frac{\alpha+\beta}{2} I + \gamma\sigma_1 + \delta\sigma_2 + \frac{\alpha-\beta}{2}\sigma_3 \,.$$

A density matrix of the qubit is a 2×2 positive semi-definite matrix. Such a matrix is the form (3.1) with with $\alpha,\beta \geq 0$, $\alpha+\beta = 1$, and $\det \rho = \alpha\beta - \gamma^2 - \delta^2 \geq 0$. A density matrix ρ represents a *pure state* if ρ is a rank one projection. For qubits, this happens precisely if the determinant $ab - \gamma^2 - \delta^2 = 0$. Then we have $\gamma + i\delta = \mathrm{e}^{i\theta}\sqrt{ab}$ for some $\theta \in [0, 2\pi)$, and so

$$\rho \;=\; \begin{bmatrix} a & e^{-i\theta}\sqrt{ab} \\ \mathrm{e}^{i\theta}\sqrt{ab} & b \end{bmatrix} \;=\; \begin{bmatrix} \sqrt{a} \\ \mathrm{e}^{i\theta}\sqrt{b} \end{bmatrix} \begin{bmatrix} \sqrt{a} & \mathrm{e}^{-i\theta}\sqrt{b} \end{bmatrix} \,.$$

Proposition 3.1 (Bloch sphere representation)**.** *Let ρ be the density matrix of a state of $\mathfrak{A}_2$. Then ρ has the form*

$$\rho \;=\; \tfrac{1}{2}\left[I + \boldsymbol{v}\cdot\boldsymbol{\sigma}\right] \;=\; \tfrac{1}{2}\left[I + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3\right] \,, \tag{3.3}$$

with $\boldsymbol{v} = (v_1, v_2, v_3)$ and $|\boldsymbol{v}|^2 \leq 1$. Furthermore, ρ is a pure state if and only if $|\boldsymbol{v}| = 1$.

Proof. Let ρ have the form (3.1) with $\alpha,\beta \geq 0$, $\alpha+\beta = 1$ and $\alpha\beta - \gamma^2 - \delta^2 \geq 0$. Taking $v_1 = 2\gamma$, $v_2 = 2\delta$ and $v_3 = \alpha - \beta$, we find that

$$\rho \;=\; \tfrac{1}{2}\left[I + v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3\right] \,,$$

with

$$v_1^2 + v_2^2 + v_3^2 \;=\; 4(\gamma^2+\delta^2) + \alpha^2 - 2\alpha\beta + \beta^2 \;\leq\; (\alpha+\beta)^2 \;=\; 1 \,.$$

It also follows that ρ is pure, with $\alpha\beta - \gamma^2 - \delta^2 = 0$, if and only if $|\boldsymbol{v}|^2 = 1$. □

4. THE PAULI MATRICES

The Pauli matrices introduced above have a very nice algebraic relation with one another. First of all, one sees that

$$\sigma_j^2 \;=\; I \quad \text{for } j = 1,2,3 \,.$$

Secondly, one has

$$\sigma_1\sigma_2 \;=\; i\sigma_3 \,,\quad \sigma_2\sigma_3 \;=\; i\sigma_1 \,,\quad \sigma_3\sigma_1 \;=\; i\sigma_2 \,,$$
$$\sigma_3\sigma_2 \;=\; -i\sigma_1 \,,\quad \text{and} \quad \sigma_2\sigma_1 \;=\; -i\sigma_3 \,,\quad \sigma_1\sigma_3 \;=\; -i\sigma_2 \,.$$

These identities can be summarized as

$$\sigma_i\sigma_j \;=\; \delta_{ij} I + i\sum_{k=1}^{3} \varepsilon_{ijk}\sigma_k \tag{4.1}$$

where ε_{ijk} is the *totally antisymmetric symbol*

$$\varepsilon_{ijk} \;=\; \begin{cases} 0 & \text{if any two indices are equal,} \\ 1 & \text{if } (i,j,k) = (1,2,3),\ (2,3,1),\ \text{or } (3,1,2)\ \text{, and} \\ -1 & \text{if } (i,j,k) = (3,2,1),\ (2,1,3),\ \text{or } (1,3,2)\ . \end{cases} \tag{4.2}$$

In terms of the *commutators* $[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i$, (4.1) implies the following *commutation relations*

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \varepsilon_{ijk}\sigma_k \,. \tag{4.3}$$

That is

$$[\sigma_1, \sigma_2] = -[\sigma_2, \sigma_1] = 2i\sigma_3 \,, \quad [\sigma_2, \sigma_3] = -[\sigma_3, \sigma_2] = 2i\sigma_1 \,,$$
$$\text{and} \quad [\sigma_3, \sigma_1] = -[\sigma_1, \sigma_3] = 2i\sigma_2 \,.$$

5. Evolution of an isolated Qubit

To do some interesting physics we need to understand how the state of our system evolves in time. Let us first consider a qubit all by itself. In that case the state $\rho(t)$ of the system as a function of time t satisfies the *Liouville-Schrödinger* equation

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] \,; \tag{5.1}$$

on the right hand side we have the commutator $[H, \rho(t)] = H\rho(t) - \rho(t)H$. Here H is a special observable, called the *Hamiltonian*, which corresponds to the energy of the system. The equation should have a factor of $\frac{1}{\hbar} = \frac{2\pi}{h}$ where h is Planck's constant; by choosing the right units we can take $\hbar = 1$.

To solve (5.1) we can use the commutation relations (4.3):

Exercise 1. Suppose that $H = cI + \boldsymbol{w} \cdot \boldsymbol{\sigma}$, where $\boldsymbol{w} \in \mathbb{R}^3$ and that $\rho = \frac{1}{2}[I + \boldsymbol{v} \cdot \boldsymbol{\sigma}]$, where $|\boldsymbol{v}| \le 1$. Show that

$$[H, \rho] = i(\boldsymbol{w} \times \boldsymbol{v}) \cdot \boldsymbol{\sigma} \,,$$

where

$$\boldsymbol{w} \times \boldsymbol{v} = (w_2v_3 - w_3v_2,\ w_3v_1 - w_1v_3,\ w_1v_2 - w_2v_1)$$

is the cross product.

Note that the constant term cI in H drops out of the commutator. This is a usual fact from physics: the origin of the energy doesn't matter; it is energy *differences* that are physical. So we can assume that $H = \boldsymbol{w} \cdot \boldsymbol{\sigma}$. If $\rho(t) = \frac{1}{2}[I + \boldsymbol{v}(t) \cdot \boldsymbol{\sigma}]$, then the Liouville-Schrödinger equation (5.1) becomes

$$\frac{d}{dt}\boldsymbol{v}(t) = \boldsymbol{w} \times \boldsymbol{v}(t) \,. \tag{5.2}$$

Eq. (5.2) is a a linear differential equation for $\boldsymbol{v}(t)$; given an initial value $\boldsymbol{v}(0)$ it is well known that there is a unique solution to (5.2) for all $t \in \mathbb{R}$. This equation describes the rotation of $\boldsymbol{v}(t)$ around the axis $\hat{\boldsymbol{w}} = \frac{1}{|\boldsymbol{w}|}\boldsymbol{w}$, with angular speed $\omega = |\boldsymbol{w}|$.

Exercise 2. Let $\boldsymbol{v}_0$ be a point in the Bloch sphere $\{|\boldsymbol{v}| \le 1\}$ and consider the solution $\boldsymbol{v}(t)$ to (5.2) with $\boldsymbol{v}(0) = \boldsymbol{v}_0$. Show that

$$\boldsymbol{v}(t) = \boldsymbol{v}_0 + (\cos(\omega t) - 1)\boldsymbol{u} + \sin(\omega t)\boldsymbol{u}' \,,$$

where $\boldsymbol{u} = \boldsymbol{v}_0 - (\hat{\boldsymbol{w}} \cdot \boldsymbol{v})\hat{\boldsymbol{w}}$ is the projection of $\boldsymbol{v}_0$ onto the plane orthogonal to $\boldsymbol{w}$, and $\boldsymbol{u}' = \hat{\boldsymbol{w}} \times \boldsymbol{u}$.

So a qubit by itself has a vector $\boldsymbol{v}(t)$ in the Bloch sphere which just rotates around a given axis at a certain speed.

6. Quantum Channels

Things can get more interesting if we allow the qubit to interact with its environment. In general it is quite a hard problem to solve for the evolution of a large quantum system, such as a qubit together with its environment. However, there is a useful approximation, called the *Markov approximation*, which roughly speaking applies if the environment is so big, and complicated, that we can neglect the effect of the system on its environment. You don't really need to worry about what that all means. In the end it comes down to the following: *the evolution of a system interacting with an environment over a discrete interval of time is given by a* quantum channel*, which is a* completely positive, trace preserving *map on density matrices.*

We should define the terms that appear in the above assertion.

Definition 6.1. A *completely positive map* on the states of a quantum system is a map

$$\Phi(\rho) \;=\; \sum_{j=1}^{m} A_j \rho A_j^{\dagger} \tag{6.1}$$

where $A_1, \ldots, A_m$ are certain $n \times n$ matrices, called the *Kraus operators of the map* .

In fact, the correct mathematical definition of a completely positive map is more abstract; the fact that any such map (on a finite dimensional system) has the form (6.1) is called *Kraus' Theorem*.

Definition 6.2. A map $\rho \mapsto T(\rho)$ defined on matrices is *trace preserving* if $\operatorname{tr} \rho = \operatorname{tr} T(\rho)$.

Putting together the two definitions we have

Definition 6.3. A *quantum channel* is a trace preserving, completely positive map on the states of a quantum system.

Recall that the trace $\operatorname{tr} \rho = \sum_i \rho_{ii}$. If A and B are two matrices, it follows that

$$\operatorname{tr} AB \;= \sum_i \left(\sum_j A_{i,j} B_{j,i} \right) \;=\; \sum_j \left(\sum_i B_{j,i} A_{i,j} \right) \;= \operatorname{tr} BA \;. \tag{6.2}$$

This fact is called *cyclicity of the trace*. For a quantum channel Φ and a density matrix ρ, we have

$$1 = \operatorname{tr} \Phi(\rho) \;=\; \sum_{j=1}^{m} \operatorname{tr} A_j \rho A_j^{\dagger} \;=\; \sum_{j=1}^{m} \operatorname{tr} A_j^{\dagger} A_j \rho \;=\; \operatorname{tr} \left(\sum_{j=1}^{m} A_j^{\dagger} A_j \right) \rho \;.$$

Since ρ could be any density matrix, this holds if and only if

$$\sum_{j=1}^{m} A_j^{\dagger} A_j \;=\; I.$$

So *a quantum channel is a completely positive map with Kraus operators that satisfy* $\sum_j A_j^{\dagger} A_j$.

7. Amplitude Damping Channel of a Qubit

A simple *amplitude damping* of a qubit has the form

$$\Phi_\gamma(\rho) \;=\; A_0\rho A_0^\dagger + A_1\rho A_1^\dagger \tag{7.1}$$

where

$$A_0 \;=\; \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \quad \text{and} \quad A_1 \;=\; \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix} . \tag{7.2}$$

It represents the "relaxation" of a qubit to a ground state given by $\rho_{GS} \;=\; \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Exercise 3. For $\rho = \frac{1}{2}[I + \boldsymbol{v}\cdot\boldsymbol{\sigma}]$, show that $\Phi_\gamma(\rho) \;=\; \frac{1}{2}[I + \boldsymbol{v}'\cdot\boldsymbol{\sigma}]$ with

$$\begin{aligned}\boldsymbol{v}' \;&=\; \left(\sqrt{1-\gamma}v_1, \sqrt{1-\gamma}v_2, \gamma + (1-\gamma)v_3\right) \\ &=\; \sqrt{1-\gamma}(\boldsymbol{v} - (\boldsymbol{e}_3\cdot\boldsymbol{v})\boldsymbol{e}_3) + (\gamma + (1-\gamma)\boldsymbol{e}_3\cdot\boldsymbol{v})\boldsymbol{e}_3 ,\end{aligned}$$

where $\boldsymbol{e}_3 = (0,0,1)$.

Note that, for A_0 and A_1 as in (7.2), we have

$$A_0 \;=\; \frac{1}{2}\left[(1+\sqrt{1-\gamma})I + (1-\sqrt{1-\gamma})\sigma_3\right] \quad \text{and} \quad A_1 \;=\; \frac{\sqrt{\gamma}}{2}\left[\sigma_1 + i\sigma_2\right] .$$

Using this observation, one can define an amplitude damping channel with respect to any axis $\hat{\boldsymbol{w}}$ (in place of $\boldsymbol{e}_3$).

Exercise 4. Consider the map on states given by $\frac{1}{2}[I + \boldsymbol{v}\cdot\boldsymbol{\sigma}] \mapsto [I + \boldsymbol{v}'\cdot\boldsymbol{\sigma}]$ where

$$\boldsymbol{v}' \;=\; \sqrt{1-\gamma}(\boldsymbol{v} - (\hat{\boldsymbol{w}}\cdot\boldsymbol{v})\hat{\boldsymbol{w}}) + (\gamma + (1-\gamma)\hat{\boldsymbol{w}}\cdot\boldsymbol{v})\hat{\boldsymbol{w}} .$$

Show that this map corresponds to the channel $\Phi_{\gamma,\hat{\boldsymbol{w}}}$ given by (7.1) with

$$A_0 \;=\; \frac{1}{2}\left[(1+\sqrt{1-\gamma})I + (1-\sqrt{1-\gamma})\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}\right] \quad \text{and} \quad A_1 \;=\; \frac{\sqrt{\gamma}}{2}\left[\boldsymbol{w}_1^\perp\cdot\boldsymbol{\sigma} + i\boldsymbol{w}_2^\perp\cdot\boldsymbol{\sigma}\right] ,$$

where $\hat{\boldsymbol{w}}_1^\perp$ and $\hat{\boldsymbol{w}}_2^\perp$ are two vectors such that

(1) $\hat{\boldsymbol{w}}_j^\perp$, $j = 1,2$, are orthogonal to $\hat{\boldsymbol{w}}$, i.e $\hat{\boldsymbol{w}}_j^\perp\cdot\hat{\boldsymbol{w}} = 0$, $j = 1,2$, and
(2) $\hat{\boldsymbol{w}}_1^\perp \times \hat{\boldsymbol{w}}_2^\perp = \hat{\boldsymbol{w}}$.

Returning to the simple amplitude damping channel with A_0, A_1 as in (7.2), note that if we write $\boldsymbol{v} = (v_1, v_2, v_3)$ as a row vector, then the channel amounts to the map

$$\boldsymbol{v} \;\mapsto\; \gamma\boldsymbol{e}_3 + \boldsymbol{v}\begin{bmatrix} 1-\gamma & 0 & 0 \\ 0 & \sqrt{1-\gamma} & 0 \\ 0 & 0 & \sqrt{1-\gamma} \end{bmatrix} .$$

Exercise 5. Show that the result $\Phi_\gamma^n(\rho)$ of n repeated applications of Φ_γ to $\rho = \frac{1}{2}[I + \boldsymbol{v}\cdot\boldsymbol{\sigma}]$ is given by

$$\boldsymbol{v} \;\mapsto\; \gamma\left(\sum_{j=0}^{n-1}(1-\gamma)^j\right)\boldsymbol{e}_3 + \boldsymbol{v}\begin{bmatrix} (1-\gamma)^n & 0 & 0 \\ 0 & (1-\gamma)^{\frac{n}{2}} & 0 \\ 0 & 0 & (1-\gamma)^{\frac{n}{2}} \end{bmatrix} .$$

Recall the geometric series

$$\sum_{j=0}^{\infty}(1-\gamma)^j \;=\; \frac{1}{\gamma}\,,$$

for $0<\gamma\leq 1$. In particular,

$$\sum_{j=0}^{n-1}(1-\gamma)^j \;=\; \frac{1}{\gamma}-\sum_{j=n}^{\infty}(1-\gamma)^j \;=\; \frac{1}{\gamma}-\frac{(1-\gamma)^n}{\gamma}\,.$$

Exercise 6. For a general amplitude damping channel $\Phi_{\gamma,\hat{\boldsymbol{w}}}$ show that $\Phi_{\gamma,\hat{\boldsymbol{w}}}(\rho) \;=\; \frac{1}{2}[I+\boldsymbol{v}_n\cdot\boldsymbol{\sigma}]$ with

$$\boldsymbol{v}_n \;=\; \hat{\boldsymbol{w}}+(1-\gamma)^n(\hat{\boldsymbol{w}}\cdot\boldsymbol{v}-1)\hat{\boldsymbol{w}}+(1-\gamma)^{\frac{n}{2}}(\boldsymbol{v}-(\hat{\boldsymbol{w}}\cdot\boldsymbol{v})\boldsymbol{w})\,.$$

Note that $\boldsymbol{v}_n=\hat{\boldsymbol{w}}+O((1-\gamma)^{\frac{n}{2}})$, so the state converges exponentially fast to the pure state $\frac{1}{2}[I+\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}]$.

Finally let us introduce a combination of the free dynamics of the qubit and an amplitude damping channel. Let $\Phi_{\gamma,\hat{\boldsymbol{w}},\delta}$ be the channel of the form (7.1) with

$$A_0 \;=\; \frac{1}{2}\left[(1+\sqrt{1-\gamma})I+(1-\sqrt{1-\gamma})\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}\right]\mathrm{e}^{i\delta\hat{\boldsymbol{w}}\cdot\sigma}$$

$$\text{and}\quad A_1 \;=\; \frac{\sqrt{\gamma}}{2}\left[\boldsymbol{w}_1^{\perp}\cdot\boldsymbol{\sigma}+i\boldsymbol{w}_2^{\perp}\cdot\boldsymbol{\sigma}\right]\mathrm{e}^{i\delta\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}}\,,$$

with $\boldsymbol{w}_j^{\perp}$, $j=1,2$, as above. This represents the evolution of a qubit subject to amplitude damping and a Hamiltonian $-\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}$ over a time interval of length δ. The equilibrium state is still $\frac{1}{2}[I+\hat{\boldsymbol{w}}\cdot\boldsymbol{\sigma}]$, but the evolution is different in that the component perpendicular to $\hat{\boldsymbol{w}}$ rotates as it shrinks to zero.

Exercise 7. Consider the generalized amplitude damping channel $\Phi_{\gamma,\hat{\boldsymbol{w}},\delta}$. Show that for $\rho=\frac{1}{2}[I+\boldsymbol{v}\cdot\sigma]$, we have $\Phi^n_{\gamma,\hat{\boldsymbol{w}},\delta}(\rho)=\frac{1}{2}[I+\boldsymbol{v}_n\cdot\sigma]$ with

$$\boldsymbol{v}_n \;=\; (1+(1-\gamma)^n(\hat{\boldsymbol{w}}\cdot\boldsymbol{v}-1))\hat{\boldsymbol{w}}+(1-\gamma)^{\frac{n}{2}}(\cos(n\delta)\boldsymbol{u}+\sin(n\delta)\boldsymbol{u}')\,,$$

where $\boldsymbol{u} \;=\; \boldsymbol{v}-(\hat{\boldsymbol{w}}\cdot\boldsymbol{v})\hat{\boldsymbol{w}}$ and $\boldsymbol{u}'=\hat{\boldsymbol{w}}\times\boldsymbol{u}$.

8. Ergodic sequences of quantum channels