POLITECNICO
DI TORINO
Dipartimento
di Automatica e Informatica

Gli Alberi Binari di Ricerca (Binary Search Trees, BST)

Paolo Camurati e Gianpiero Cabodi

Alberi Binari

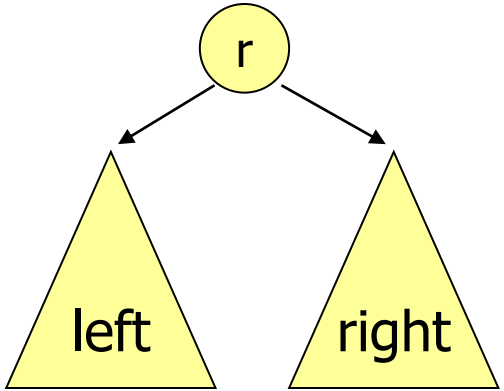
DEFINIZIONI

Alberi binari

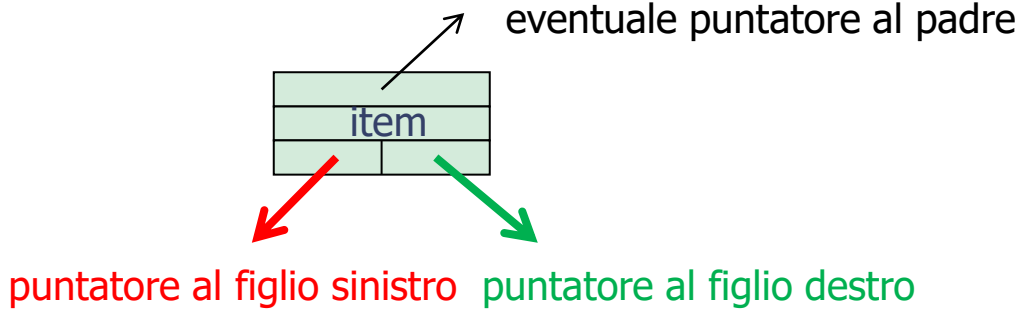
Definizione ricorsiva:

Un albero binario T è:

- un insieme vuoto di nodi vuoto
- una terna formata da una radice, un sottoalbero sinistro e un sottoalbero destro.



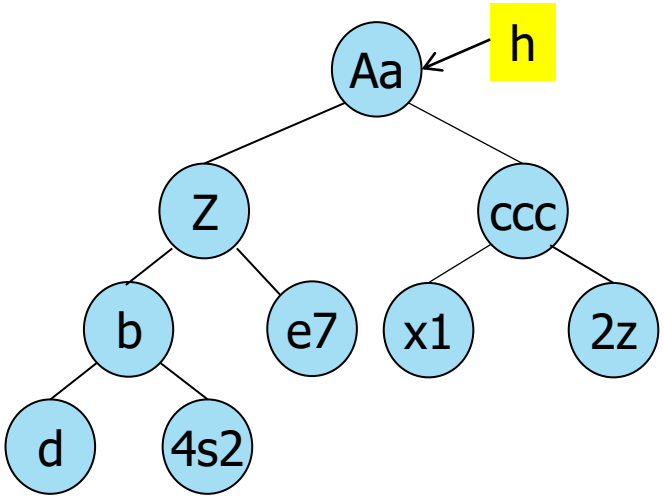
Nodo:




```
typedef struct node* link;
struct node {
  Item item;
  link left;
  link right;
};
```

Accesso

All’albero si accede tramite il puntatore h alla radice:



Calcolo di parametri

numero di nodi

```
int count(link root) {
 if (root == NULL)
   return 0;
 return count(root->left) + count(root->right) + 1;
}

int height(link root) {
 int u, v;
 if (root == NULL)
   return -1;
 u = height(root->left); v = height(root->right);
 if (u>v)
   return u+1;
 return v+1;
}
```

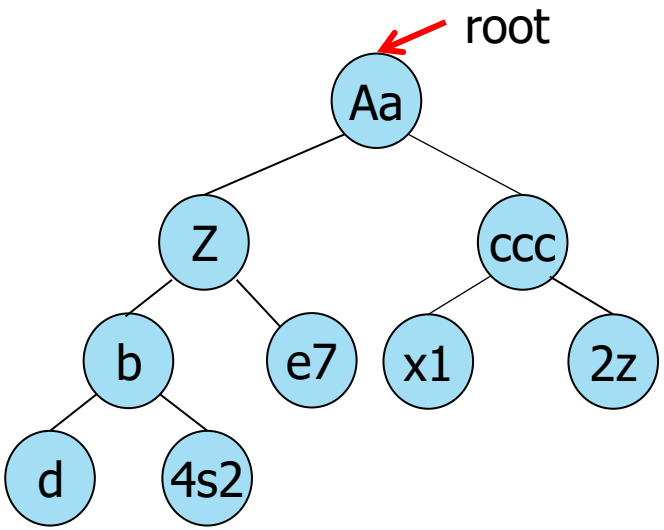
altezza

Visite

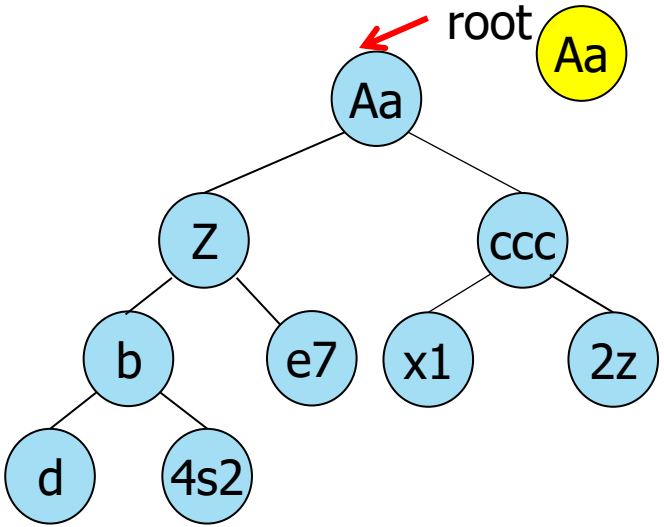
Attraversamento o visita: elenco dei nodi secondo una strategia:

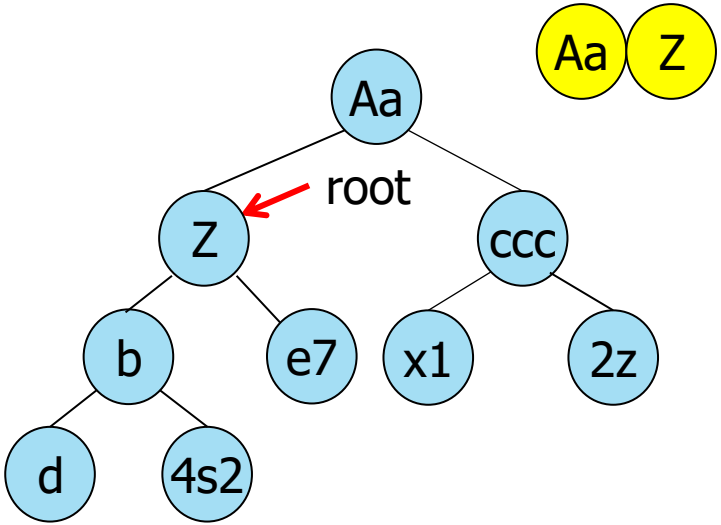
- pre-ordine: root, Left(root), Right(root)
- in-ordine: Left(root), root, Right(root)
- post-ordine: Left(root), Right(root), root

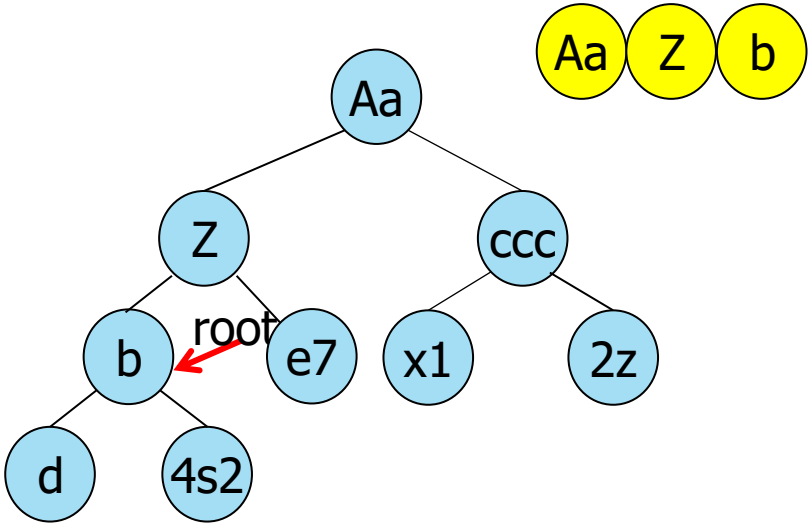
Pre-ordine

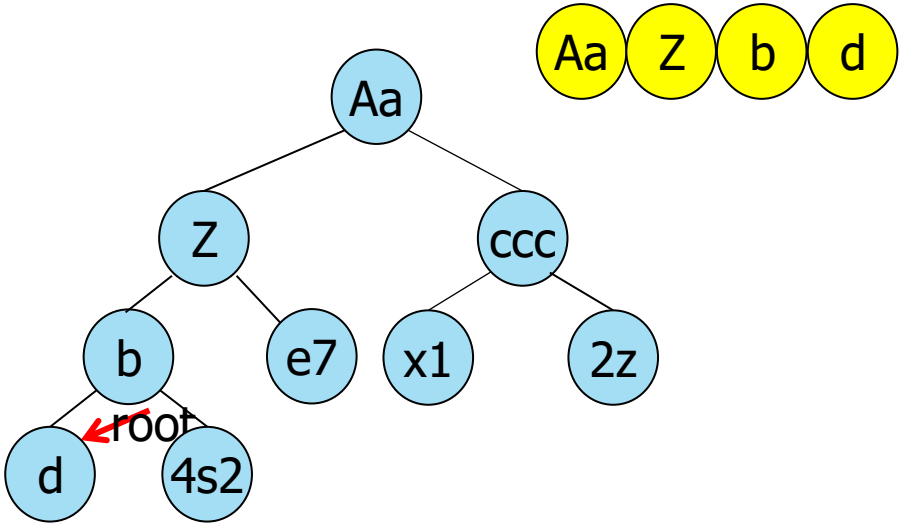



```
void preOrder(link root){
  if (root == NULL)
    return;
  printf("%s ",root->name);
  preOrder(root->left);
  preOrder(root->right);
}
```


root
Aa
Aa
Z
ccc
b
e7
x1
2z
d
4s2


Aa
Z
Aa
root
Z
ccc
b
e7
x1
2z
d
4s2


Aa
Z
b
Aa
Z
ccc
root
b
e7
x1
2z
d
4s2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d
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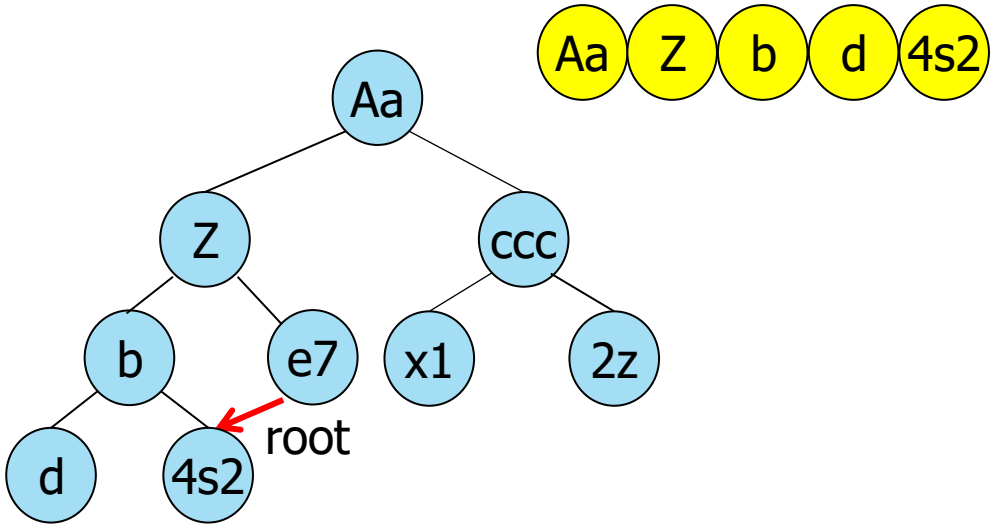
Z
ccc
b
e7
x1
2z
root
d
4s2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Z
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Z
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b
e7
x1
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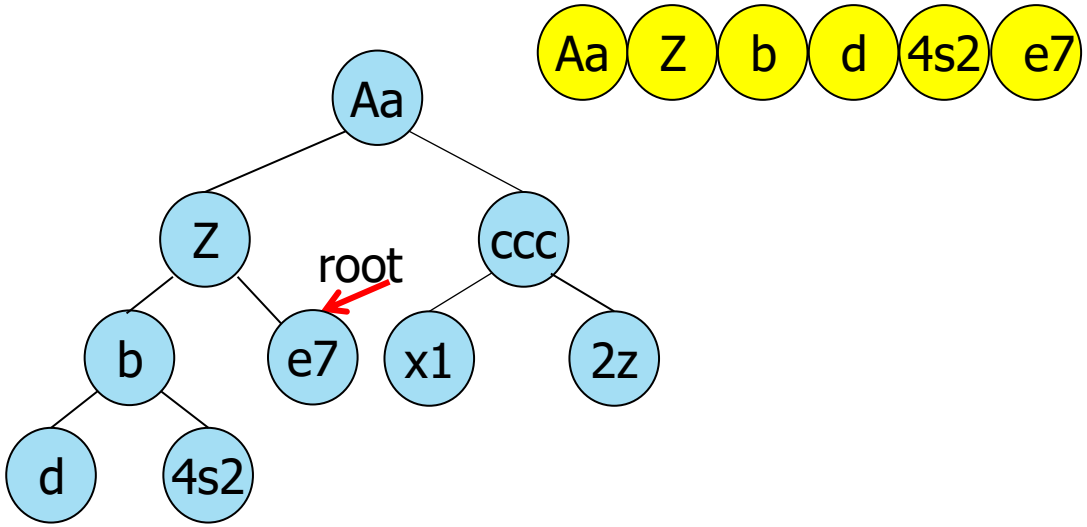
root
d
4s2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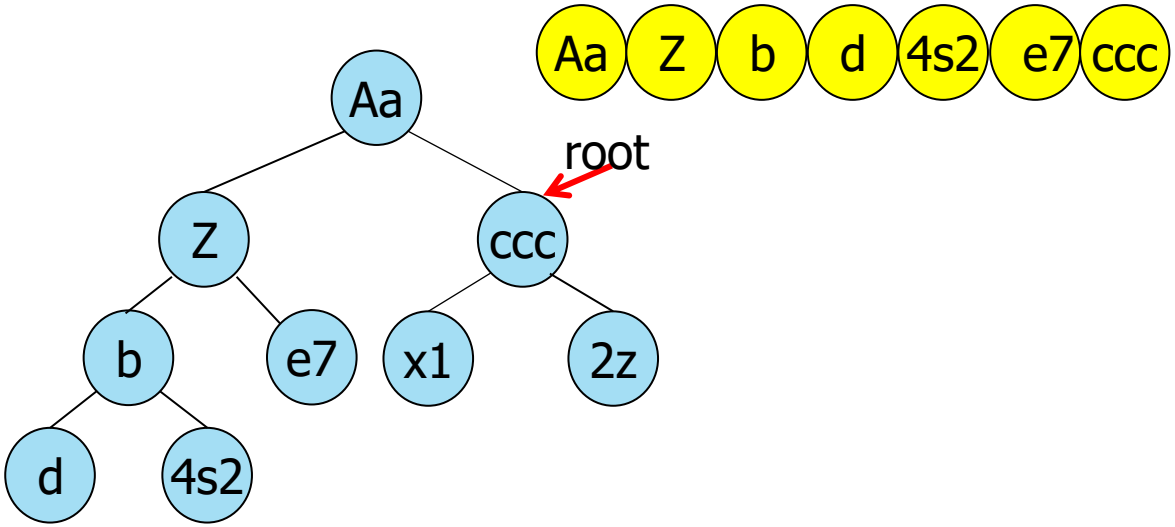
Aa
Z
root
ccc
b
e7
x1
2z
d
4s2


Aa
Z
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4s2
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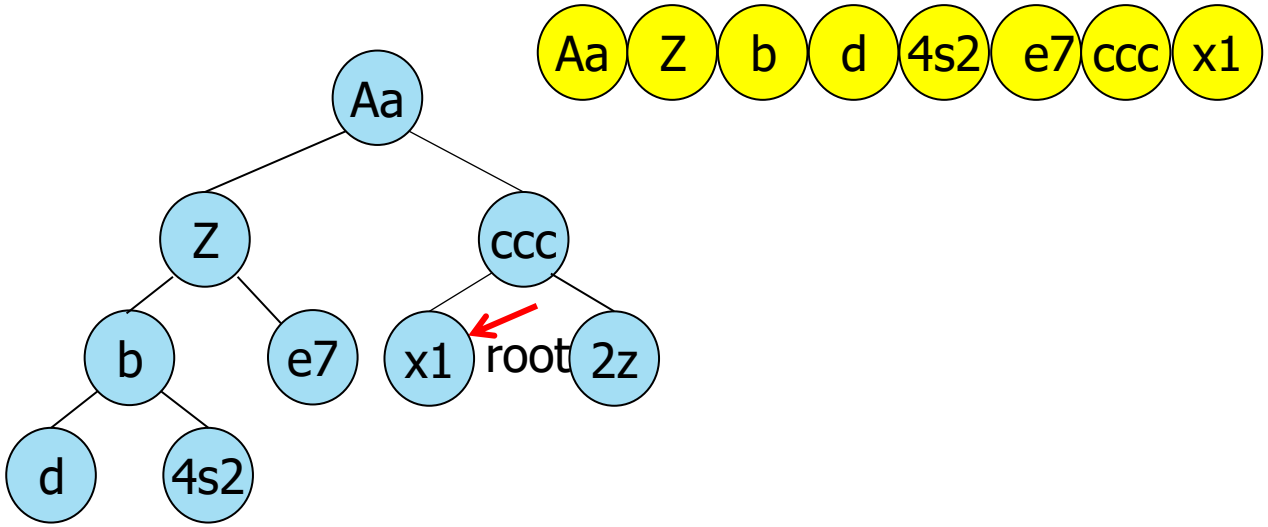
ccc
Aa
root
Z
ccc
b
e7
x1
2z
d
4s2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x1
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Z
ccc
b
e7
x1
root
2z
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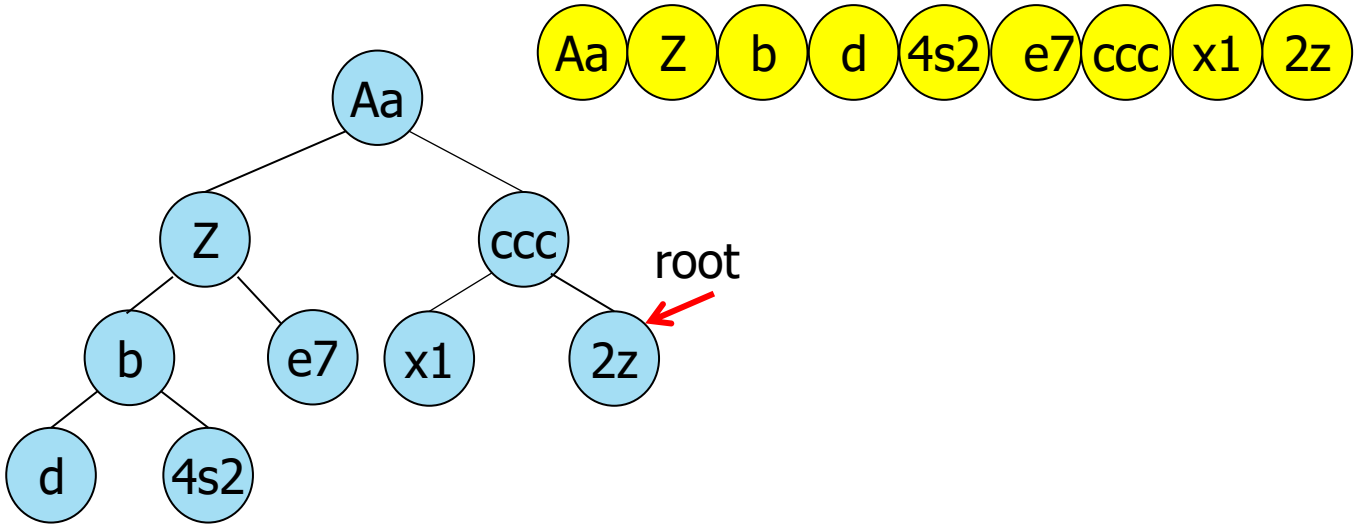
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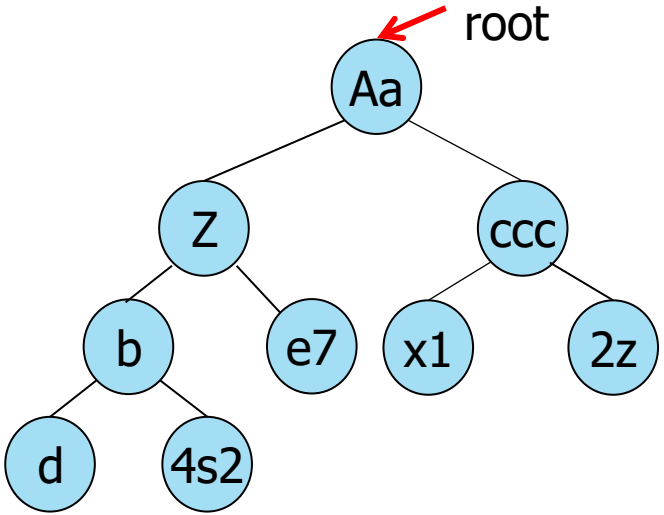
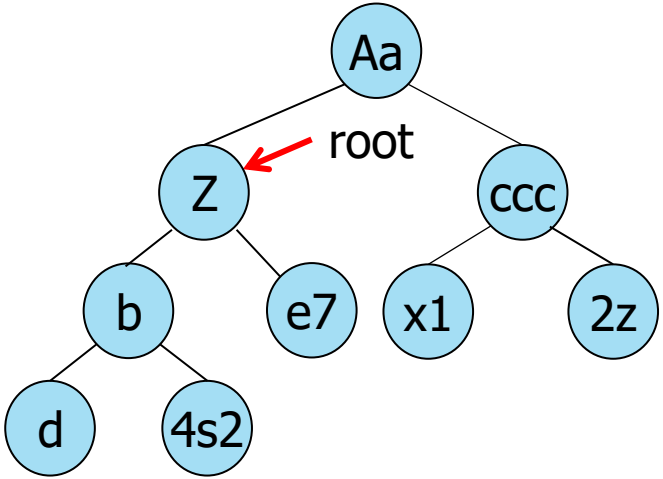
Z
b
d
4s2
e7
ccc
x1
2z
Aa
Z
ccc
root
b
e7
x1
2z
d
4s2

In-ordine




```
void inOrder(link root){
  if (root == NULL)
    return;
  inOrder(root->left);
  printf("%s ",root->name);
  inOrder(root->right);
}
```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ccc
b
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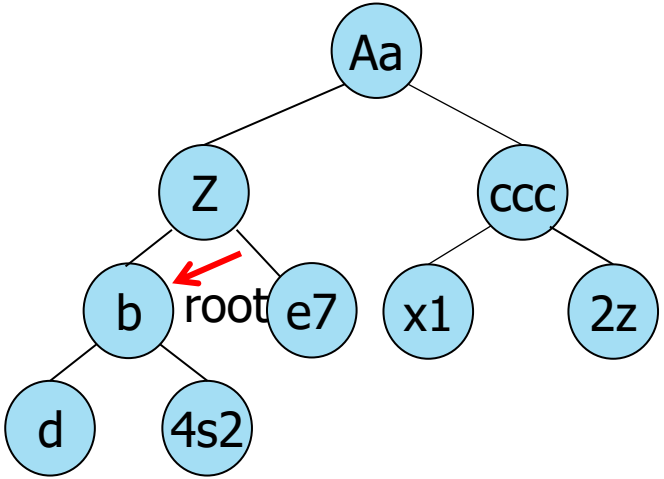
x1
2z
d
4s2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Z
ccc
b
root
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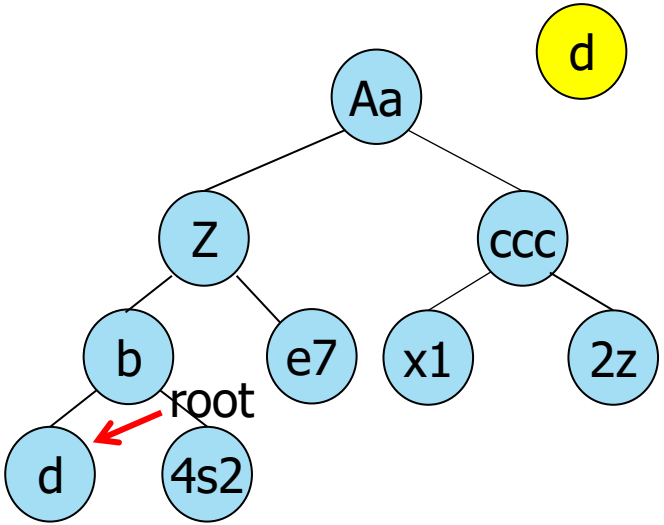
x1
2z
d
4s2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x1
2z
root
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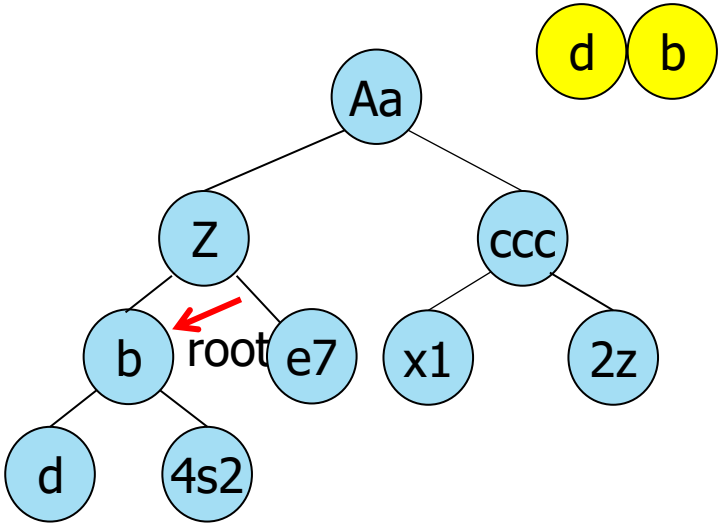
4s2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b
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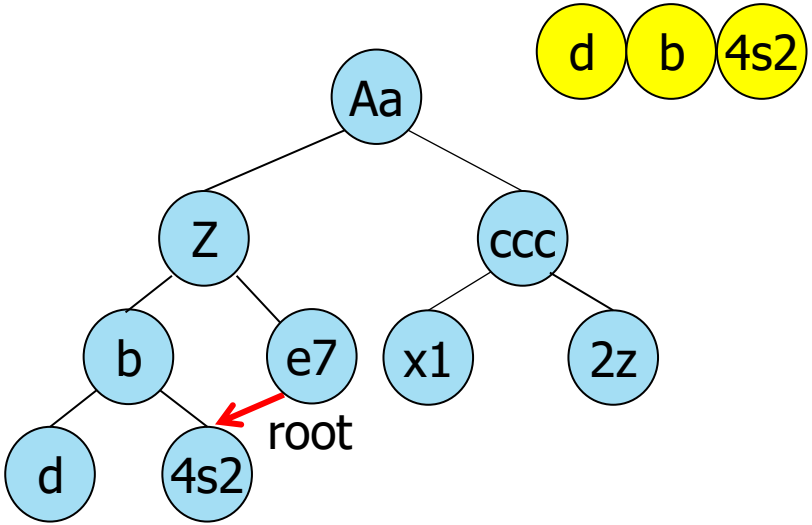
Z
ccc
b
root
e7
x1
2z
d
4s2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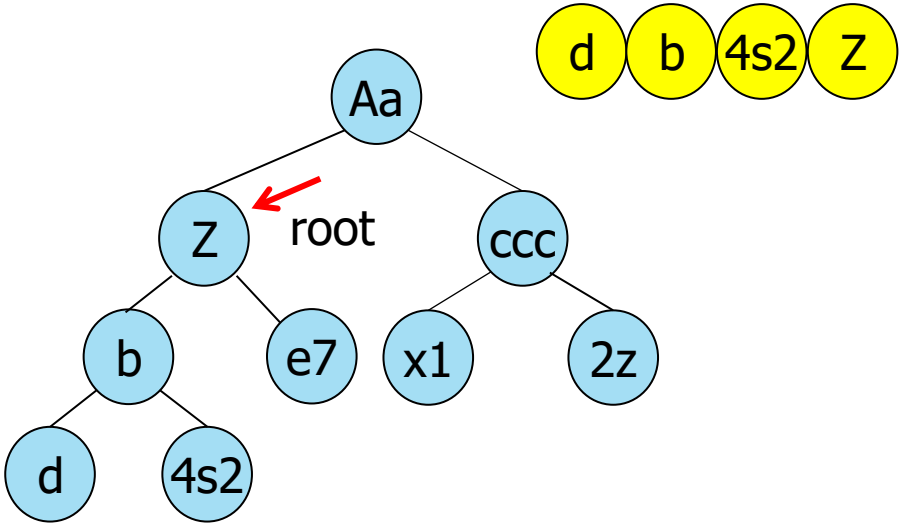
4s2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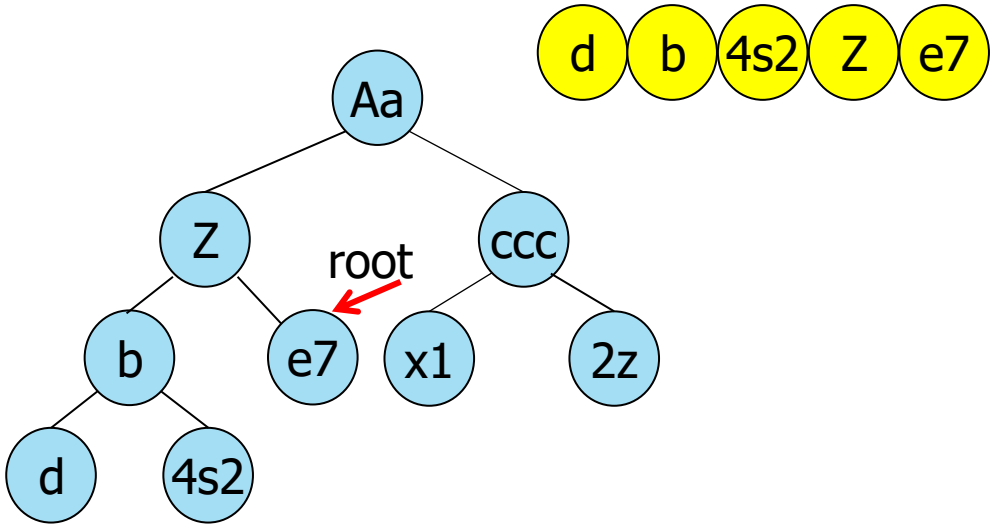
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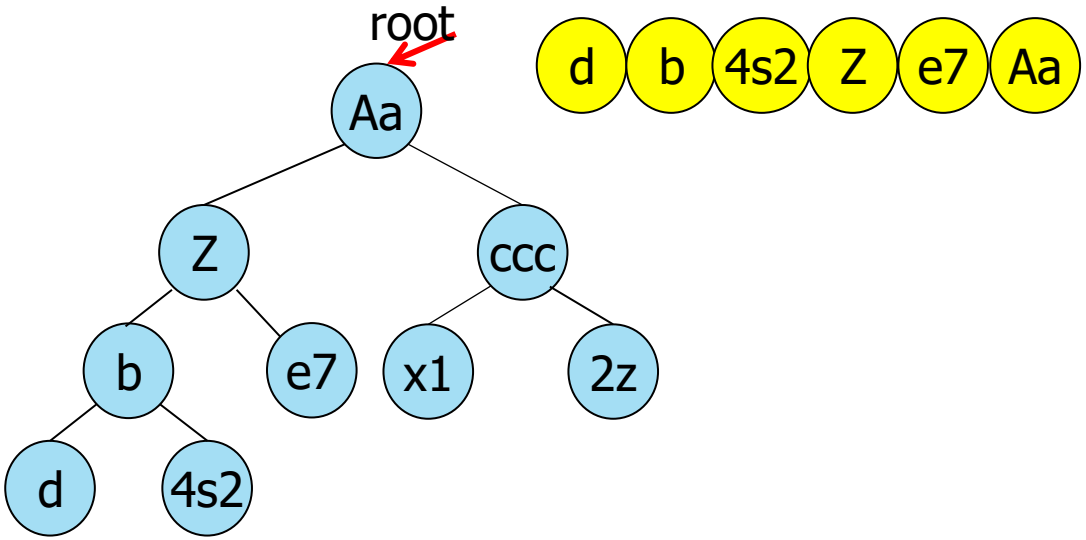
root
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ccc
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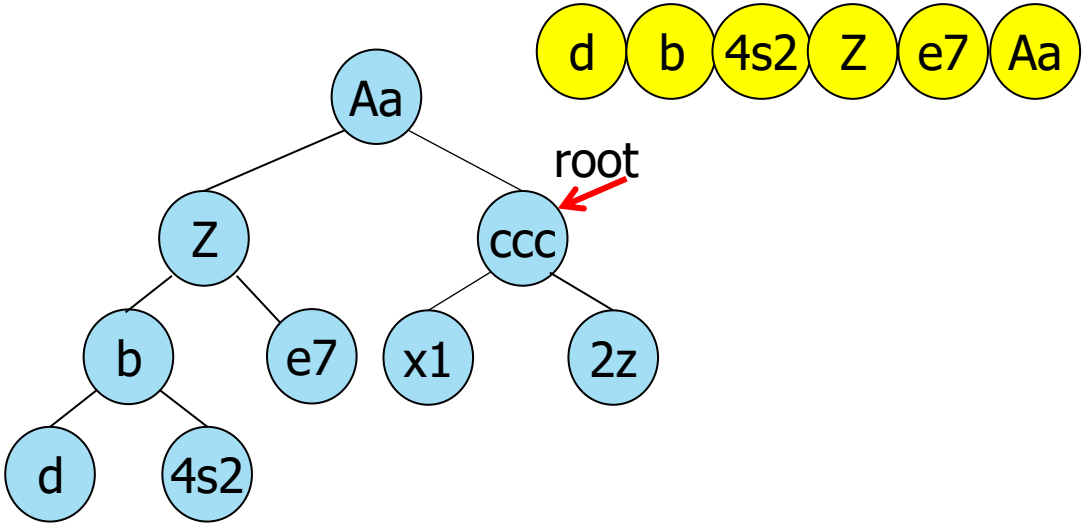
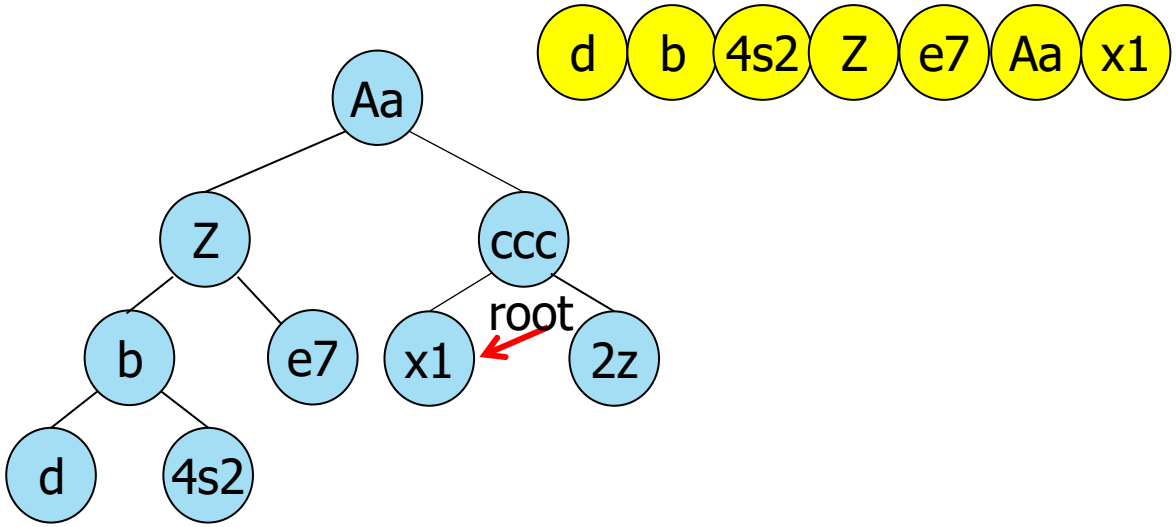
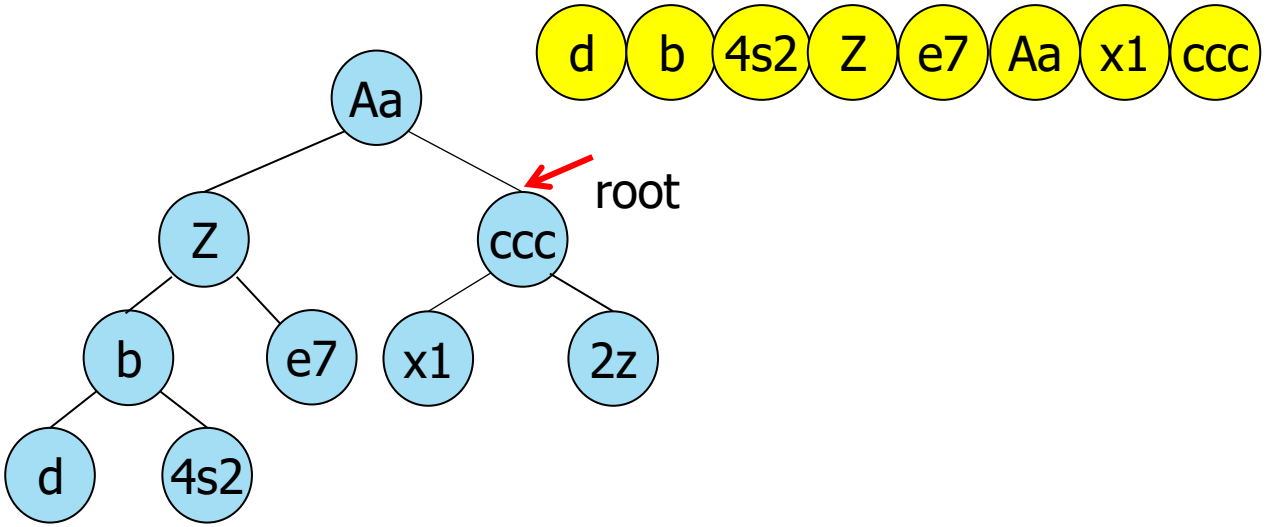
e7
x1
2z
d
4s2
d
b
4s2
Z
e7
Aa


d
b
4s2
Z
e7
Aa
Aa
root
Z
ccc
b
e7
x1
2z
d
4s2


d
b
4s2
Z
e7
Aa
x1
Aa
Z
ccc
b
e7
x1
2z
root
d
4s2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ccc
b
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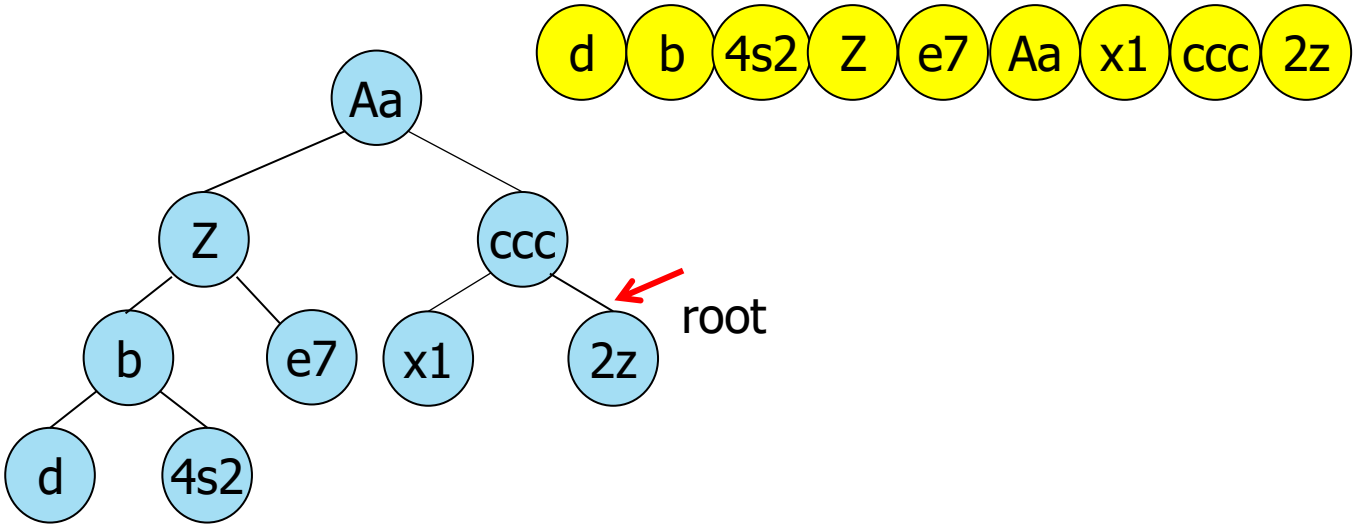
2z
d
4s2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4s2
Z
e7
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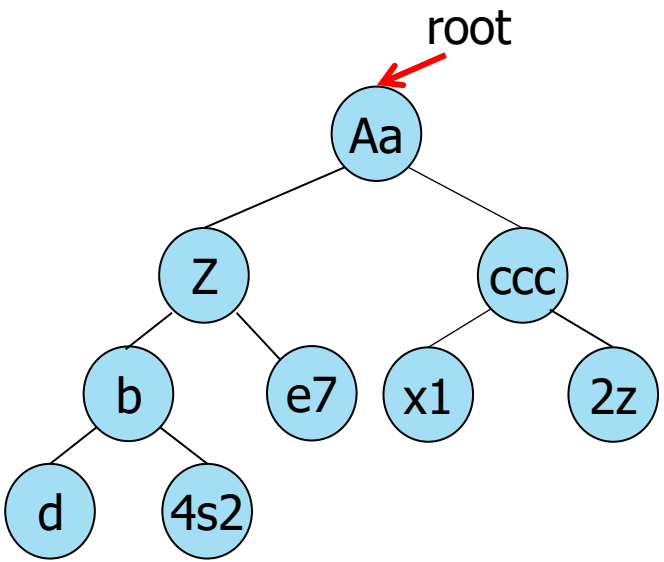
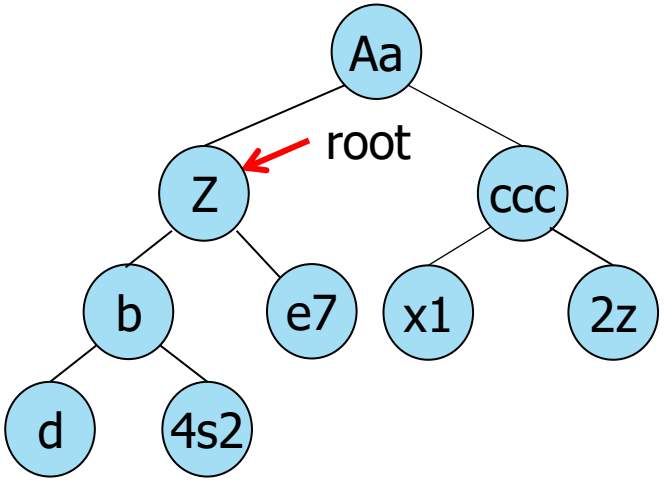
ccc
2z
Aa
Z
ccc
b
e7
x1
2z
d
4s2
root

Post-ordine




```
void postOrder(link root){
  if (root == NULL)
    return;
  postOrder(root->left);
  postOrder(root->right);
  printf("%s ",root->name);
}
```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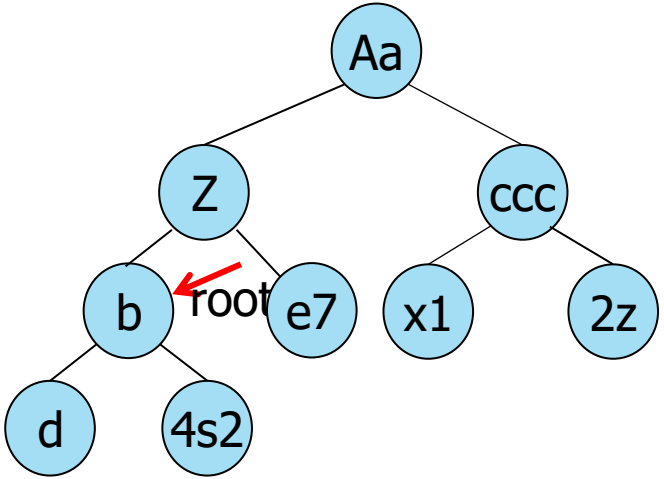
d
4s2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root
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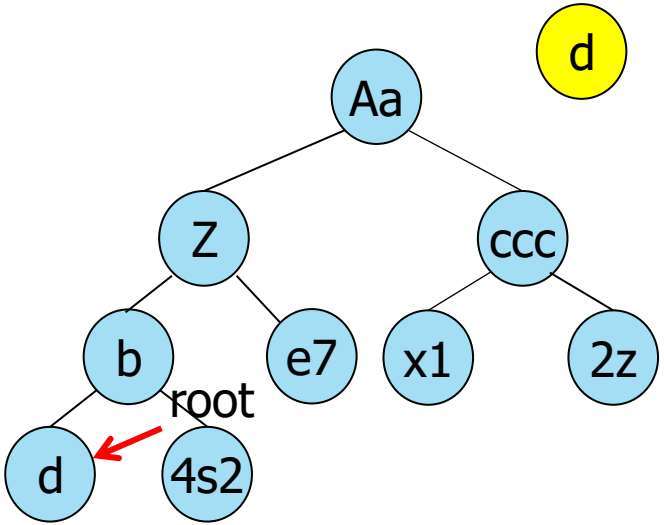
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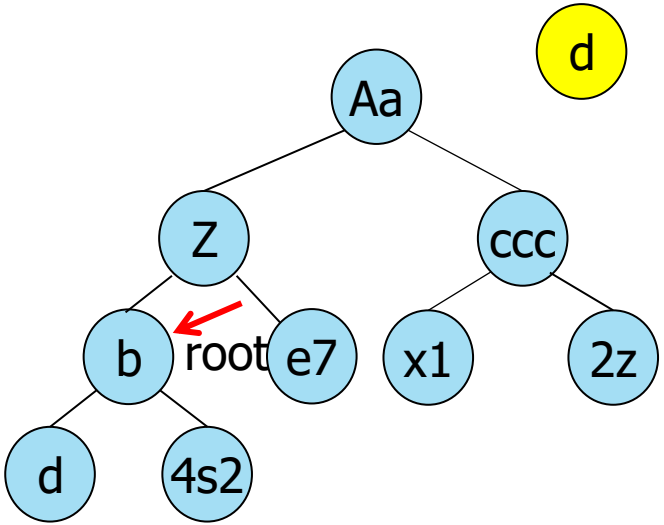
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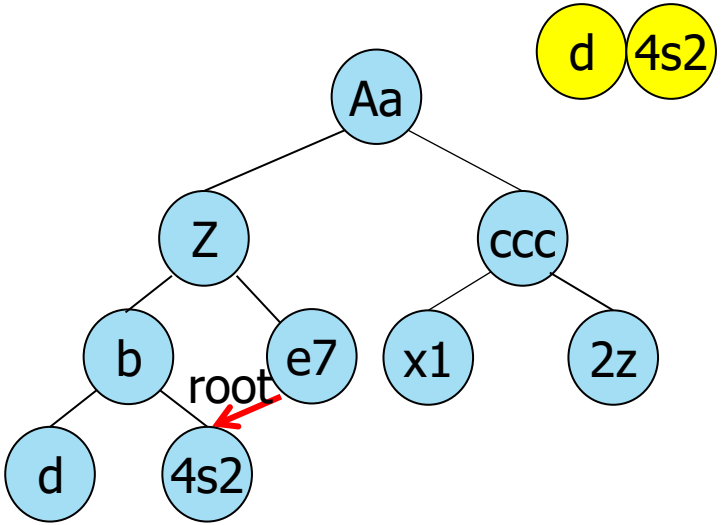
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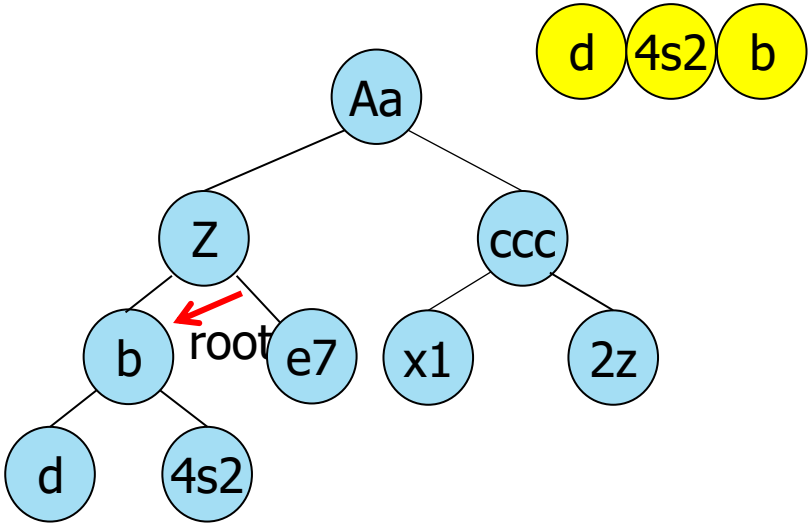
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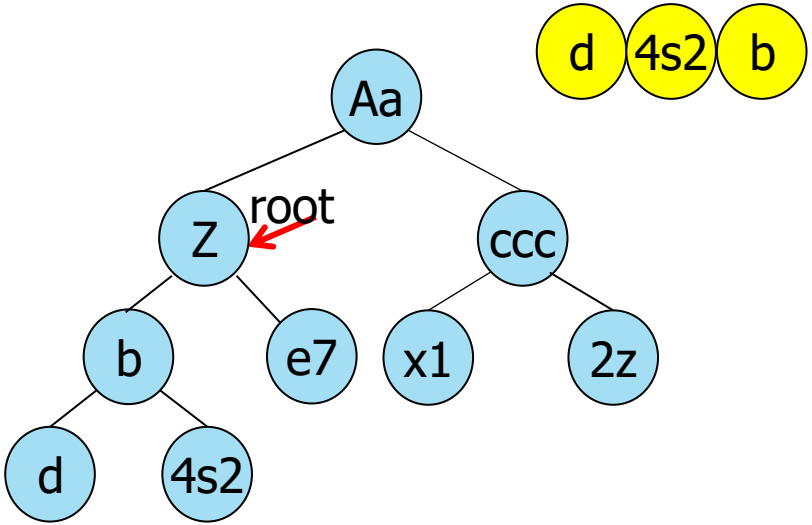
Aa
Z
ccc
b
root
e7
x1
2z
d
4s2


d
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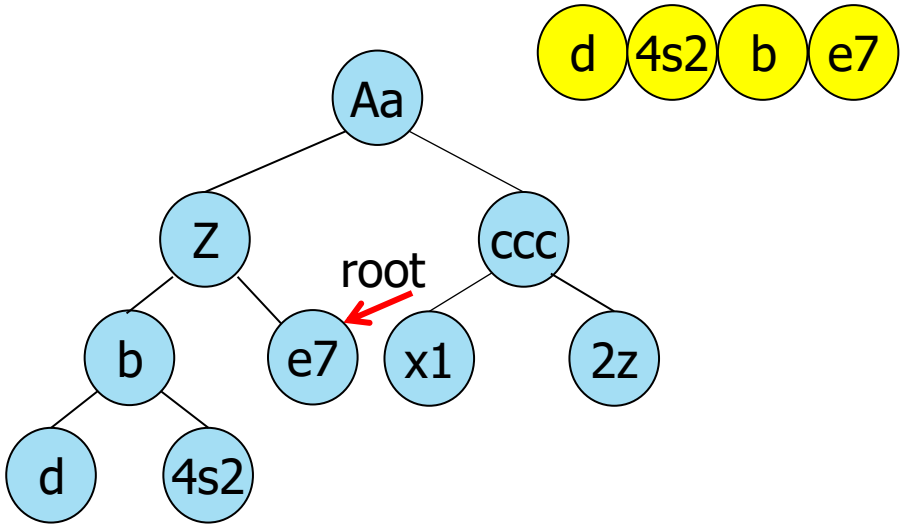
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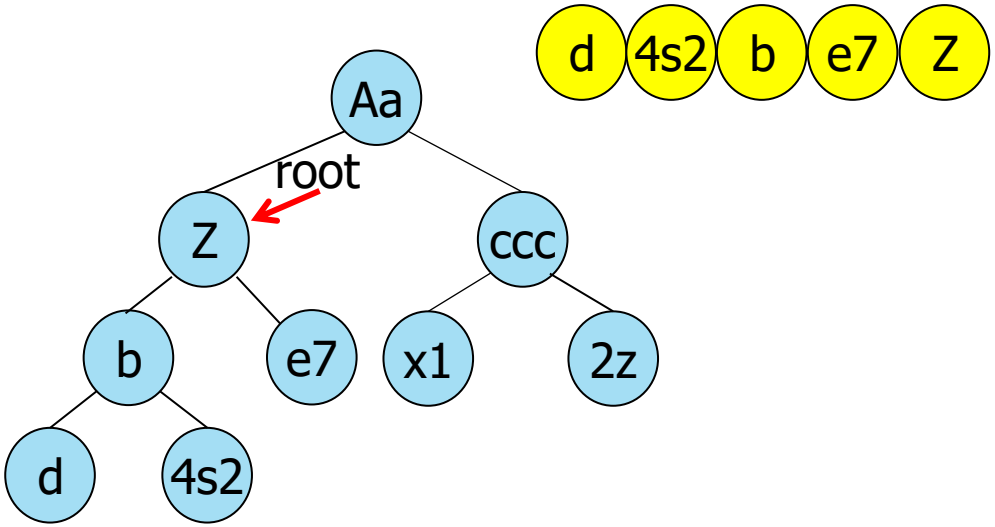
x1
2z
d
4s2


d
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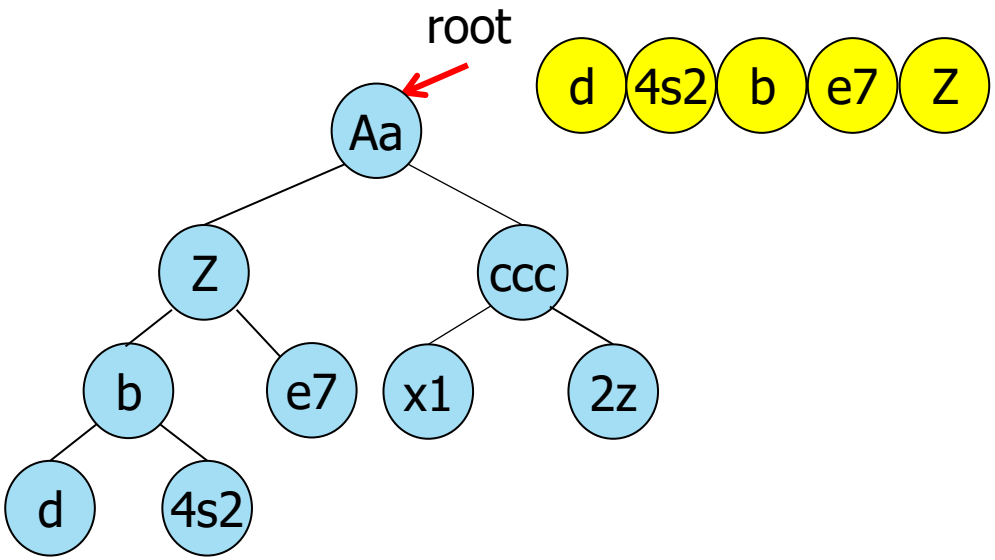
root
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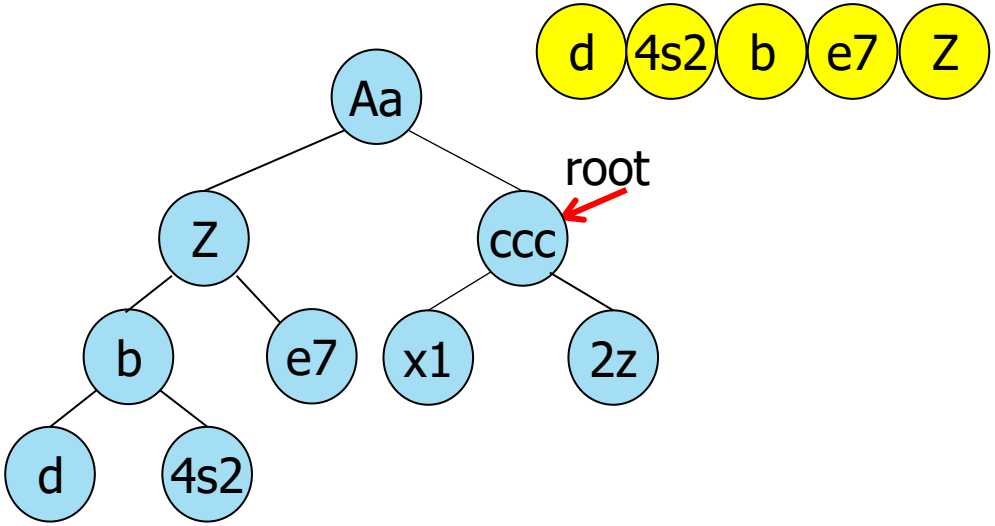
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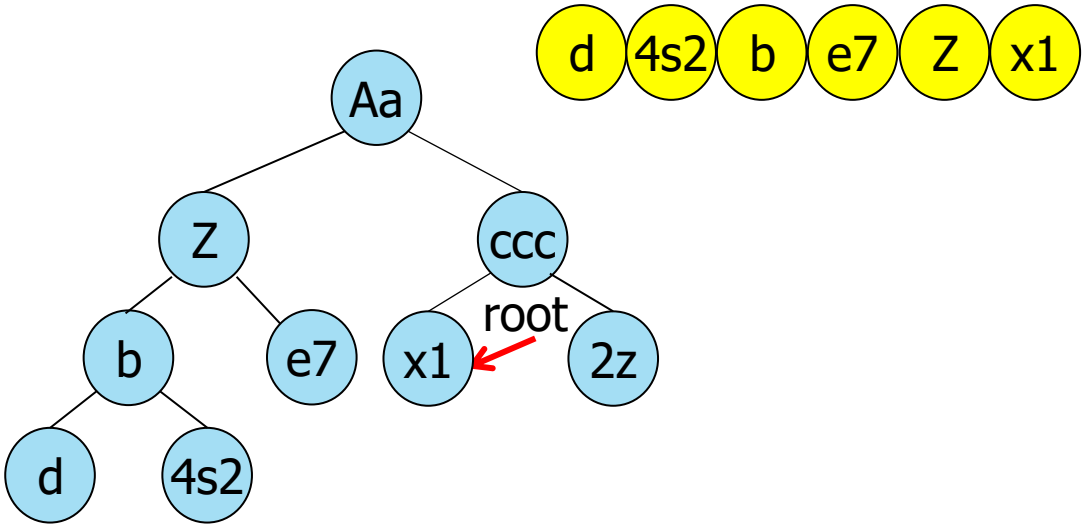
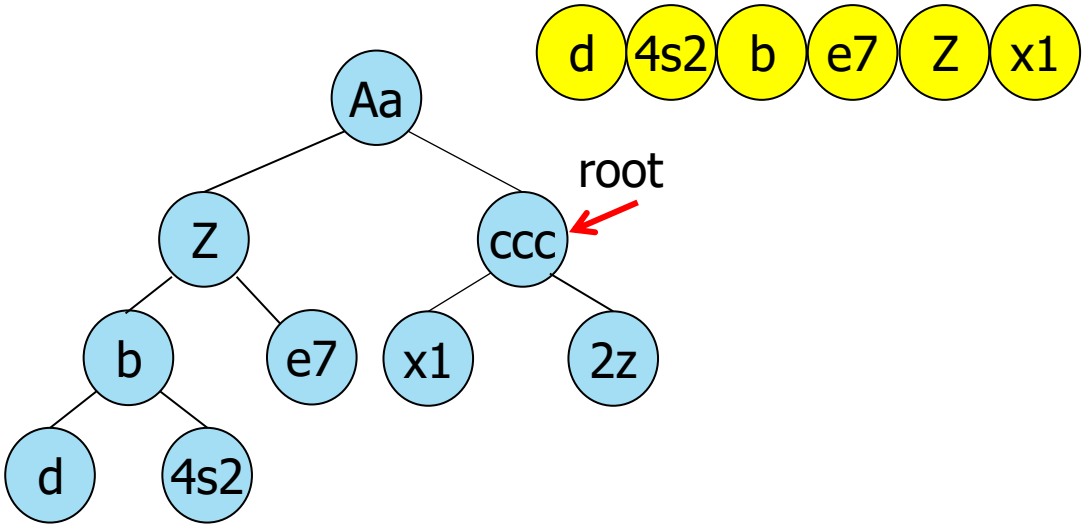
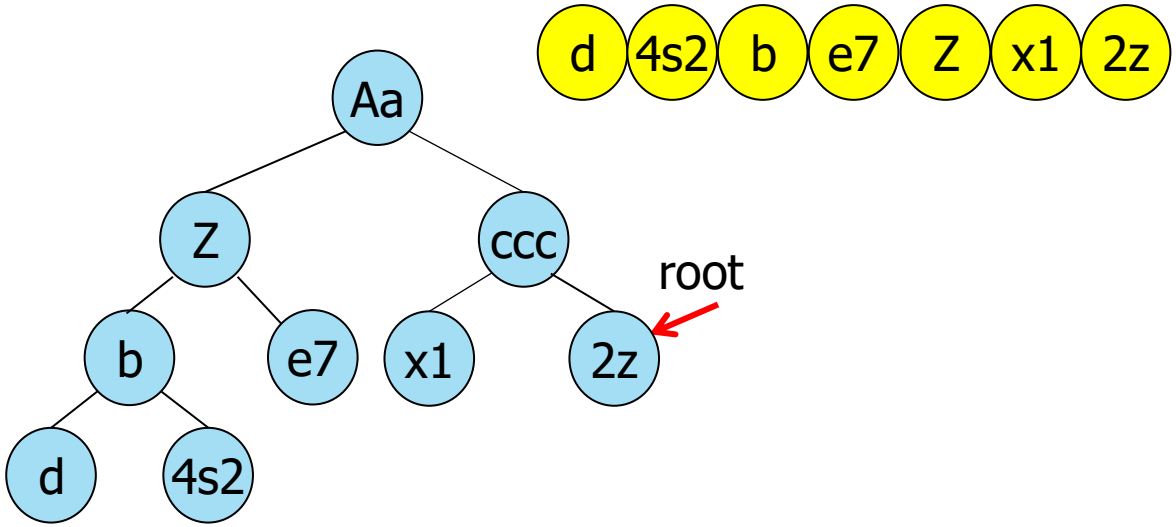
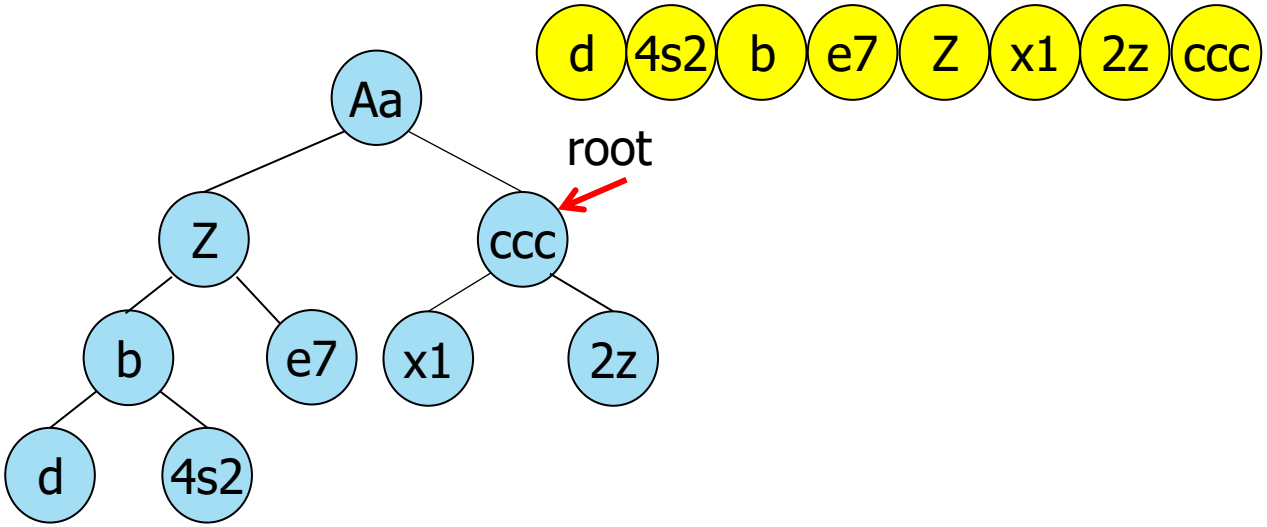
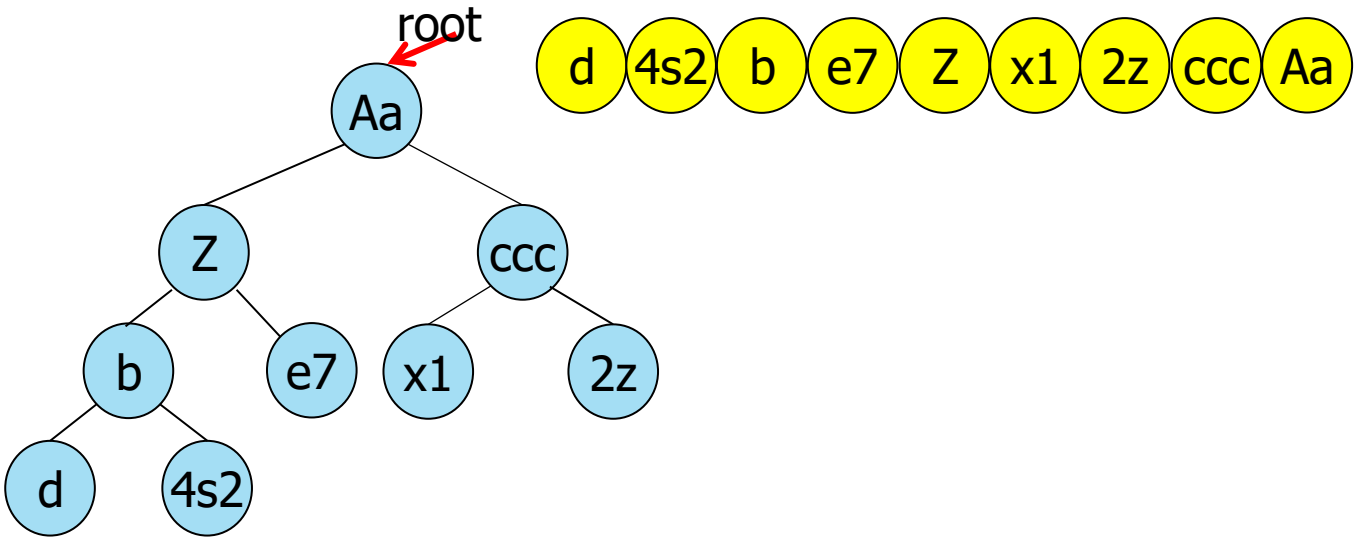

d
4s2
b
e7
Z
x1
Aa
Z
ccc
root
b
e7
x1
2z
d
4s2


d
4s2
b
e7
Z
x1
Aa
root
Z
ccc
b
e7
x1
2z
d
4s2


d
4s2
b
e7
Z
x1
2z
Aa
Z
ccc
root
b
e7
x1
2z
d
4s2


d
4s2
b
e7
Z
x1
2z
ccc
Aa
root
Z
ccc
b
e7
x1
2z
d
4s2


root
Aa
Z
ccc
b
e7
x1
2z
d
4s2
d
4s2
b
e7
Z
x1
2z
ccc
Aa

Analisi di complessità

divide and conquer
a = 2 b = 2

Ipotesi 1: albero completo:

- $D(n) = \Theta(1)$, $C(n) = \Theta(1)$
- a = 2, b = 2 (sottoproblemi di dimensione n-1, approssimati conservativamente a n)

Equazione alla ricorrenze:

$T(n) = 1 + 2T(n/2)$ $\quad n > 1$

$T(1) = 1$

$T(n) = O(n)$

decrease and conquer
$a = 1$ $k_i = 1$

Ipotesi 2: albero totalmente sbilanciato (degenerato in una lista) :

- $D(n) = \Theta(1)$, $C(n) = \Theta(1)$
- $a = 1$, $k_i = 1$

Equazione alla ricorrenze:

$T(n) = 1 + T(n-1)$ $\quad n > 1$

$T(1) = 1$

$T(n) = O(n)$

Alberi Binari

ESEMPI DI USO

Alberi binari ed espressioni

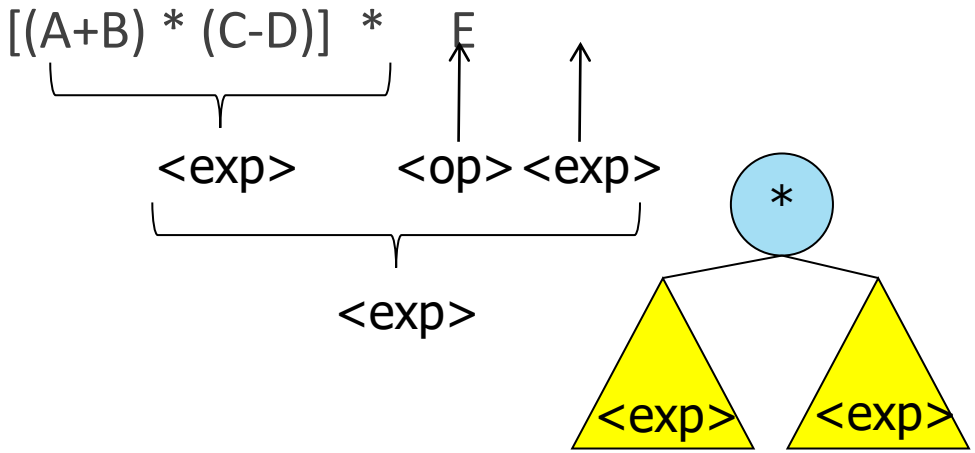
Data un'espressione algebrica in forma infissa (con parentesi per cambiare le precedenze tra operatori), ricostruirne l'albero binario in base alla grammatica semplificata:

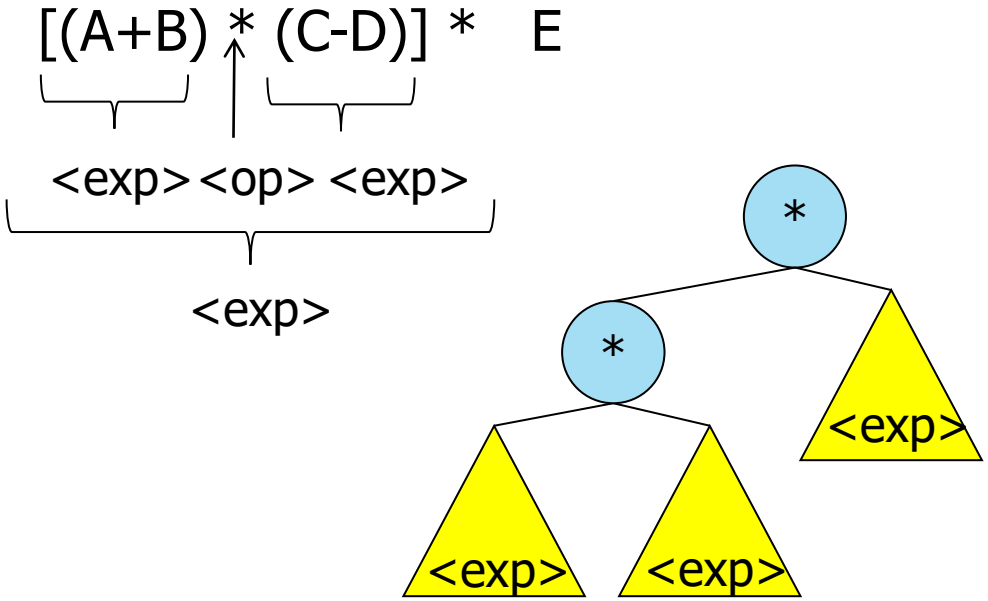
- <exp> = <operand> | <exp> <op> <exp>
- <operand> = A .. Z
- <op> = + | * | - | /

ricorsione

condizione di terminazione

Dal parsing dell’espressione si ottiene

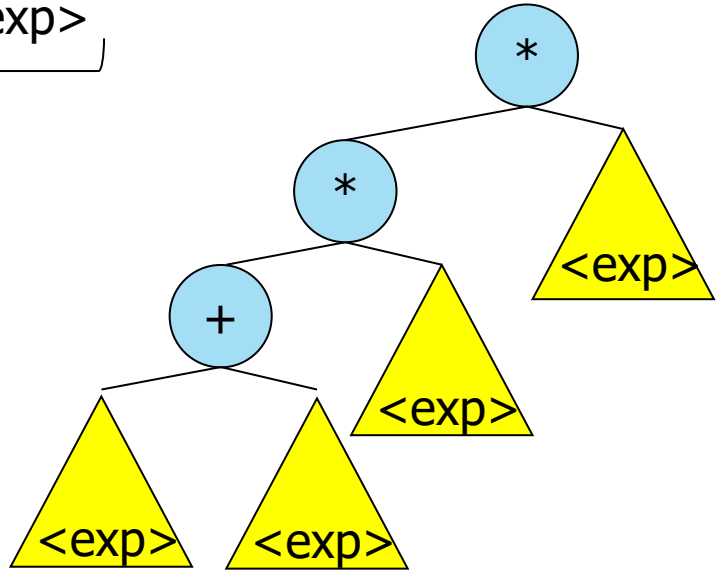



[(A+B) * (C-D)] * E
<exp> <op> <exp>
<exp>
*
*
<exp>
<exp>
<exp>

[(A+B) * (C-D)] * E

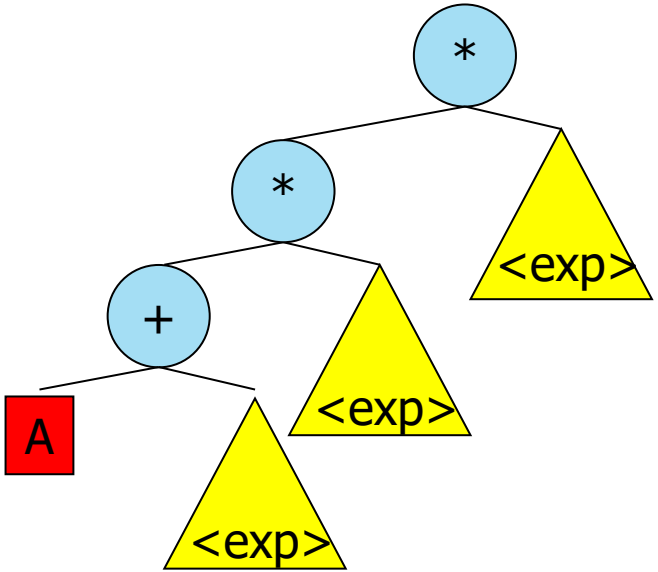
<exp> <op> <exp>

<exp>



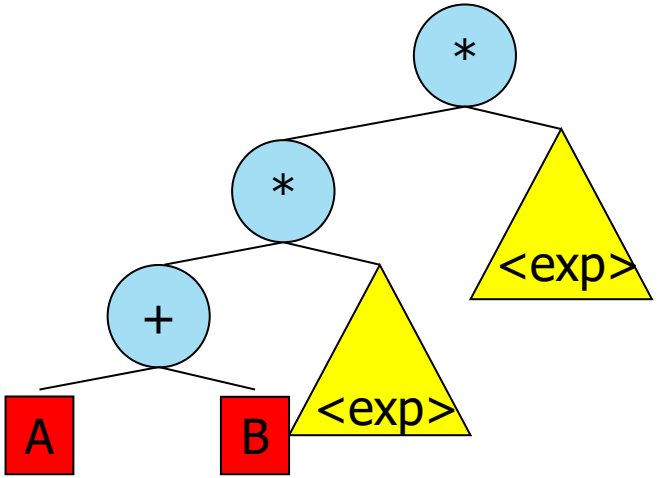
[(A+B) * (C-D)] * E

<operand>

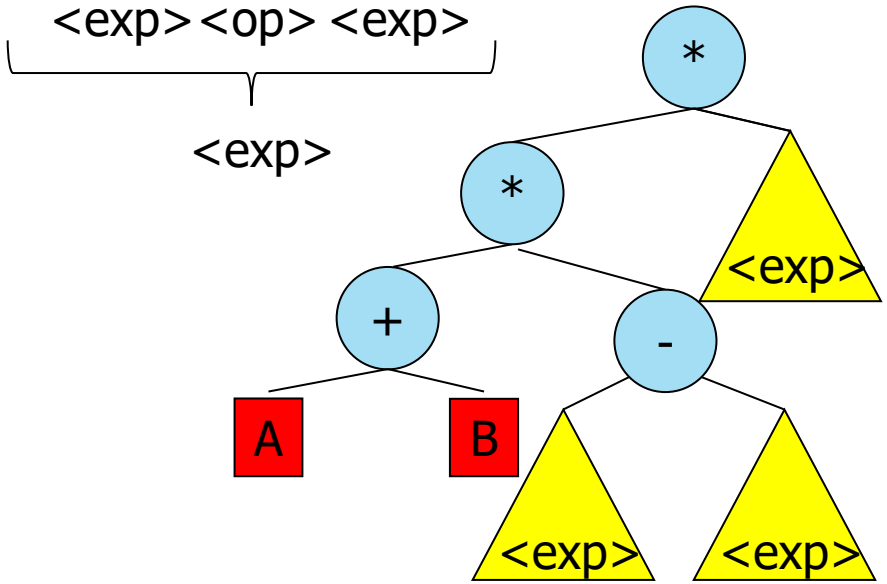


[(A+B) * (C-D)] * E

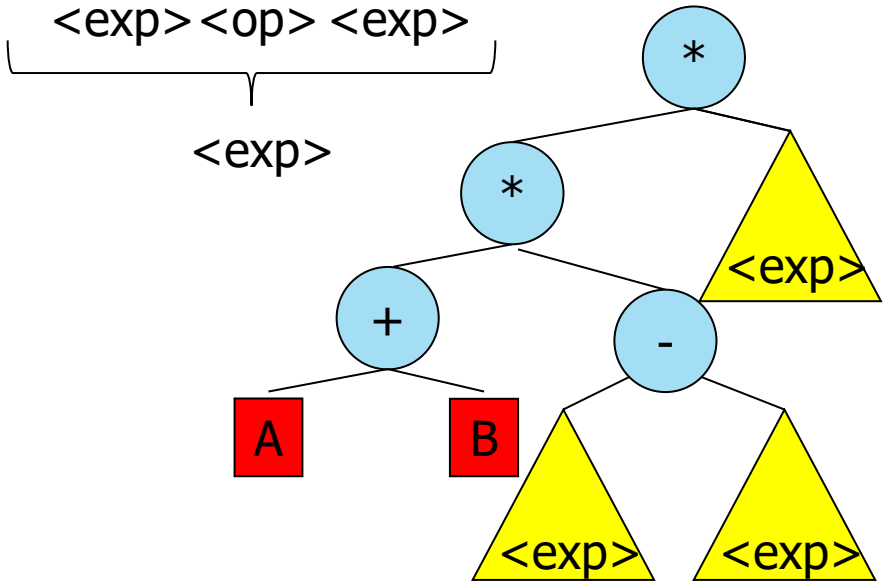
<operand>



[(A+B) * (C-D)] * E

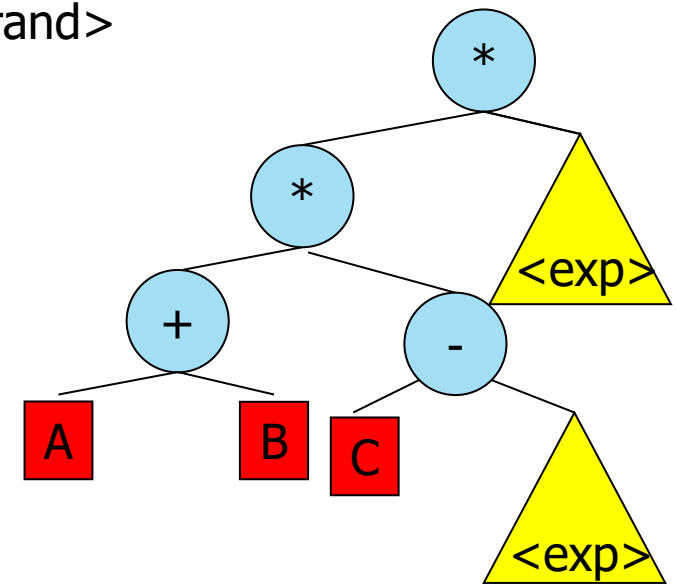


[(A+B) * (C-D)] * E



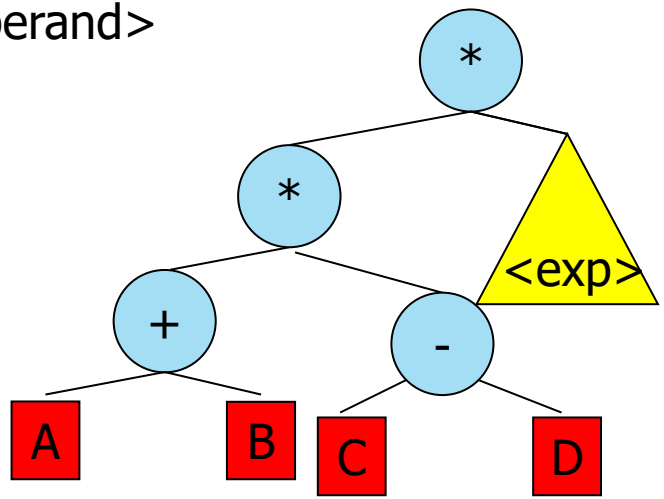
[(A+B) * (C-D)] * E

<operand>

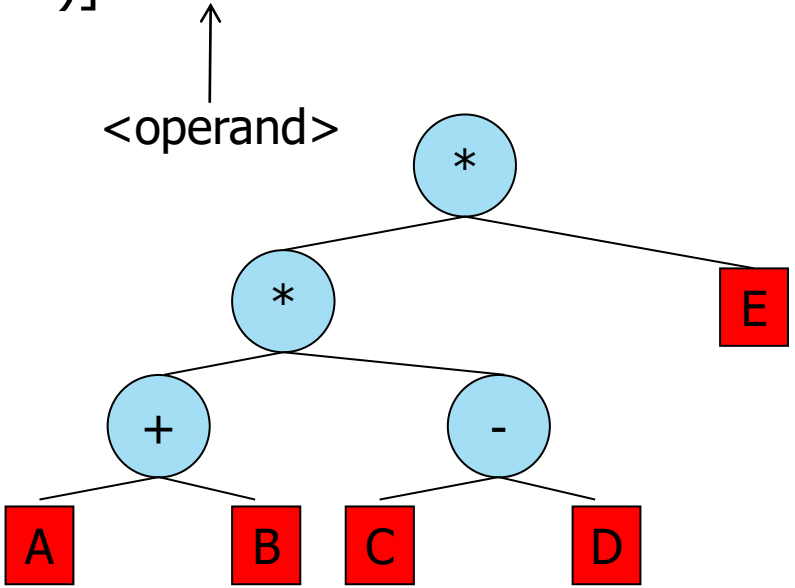


[(A+B) * (C-D)] * E

<operand>



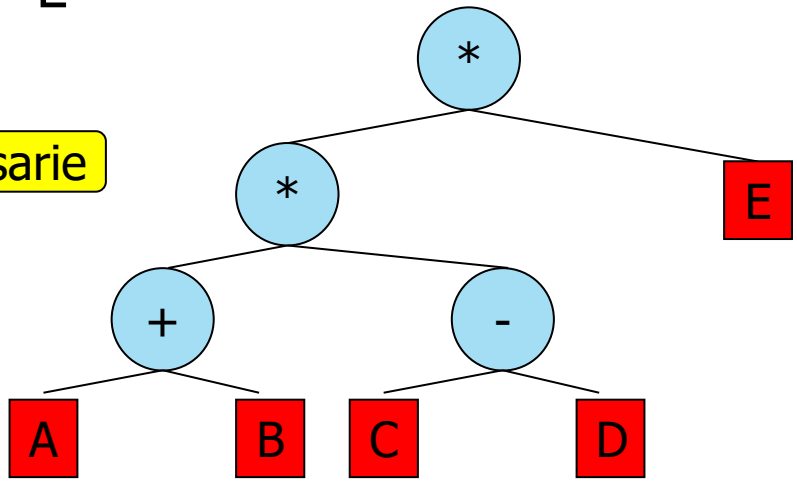
[(A+B) * (C-D)] * E



L’attraversamento in post-ordine dell’albero dà la forma postfissa (Notazione Polacca Inversa o Reverse Polish Notation) dell’espressione

A B + C D - * E *

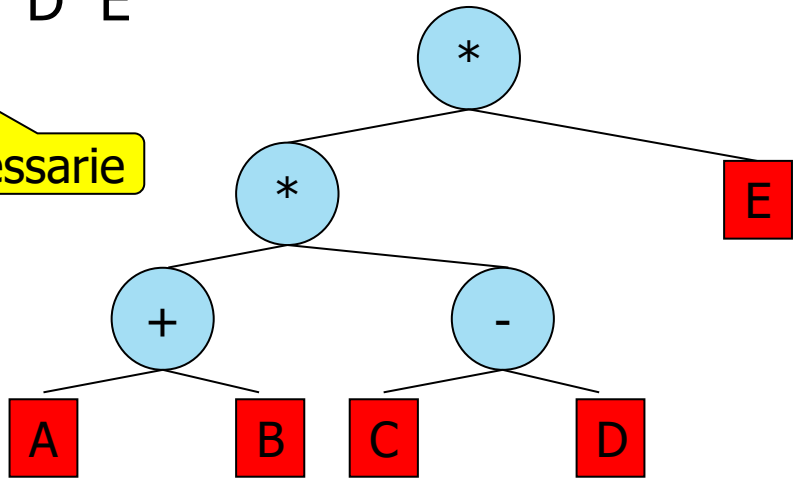
parentesi non più necessarie



L’attraversamento in pre-ordine dell’albero dà la forma prefissa (Notazione Polacca), poco usata in pratica, dell’espressione

* * + A B – C D E

parentesi non più necessarie



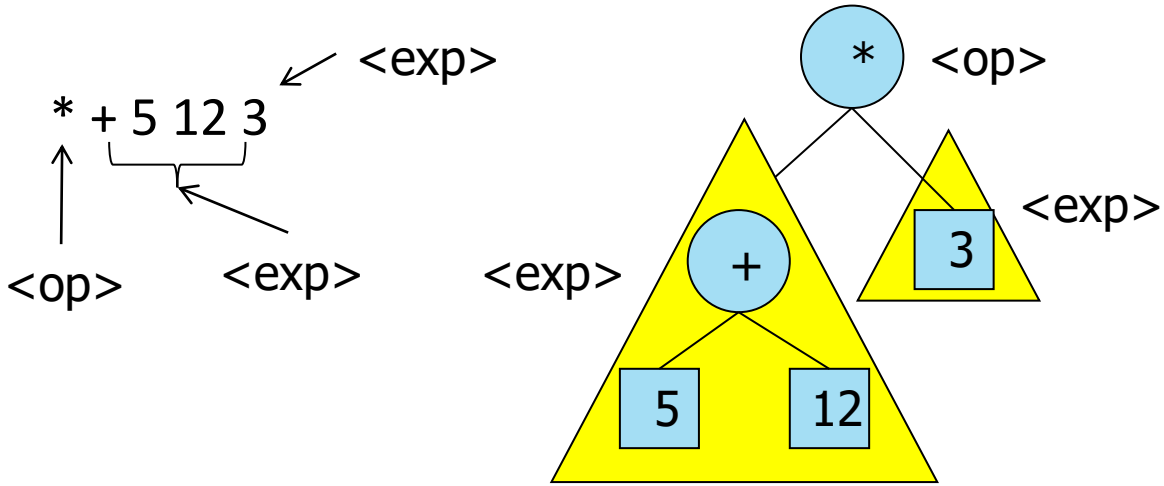
Valutazione di espressione in forma prefissa

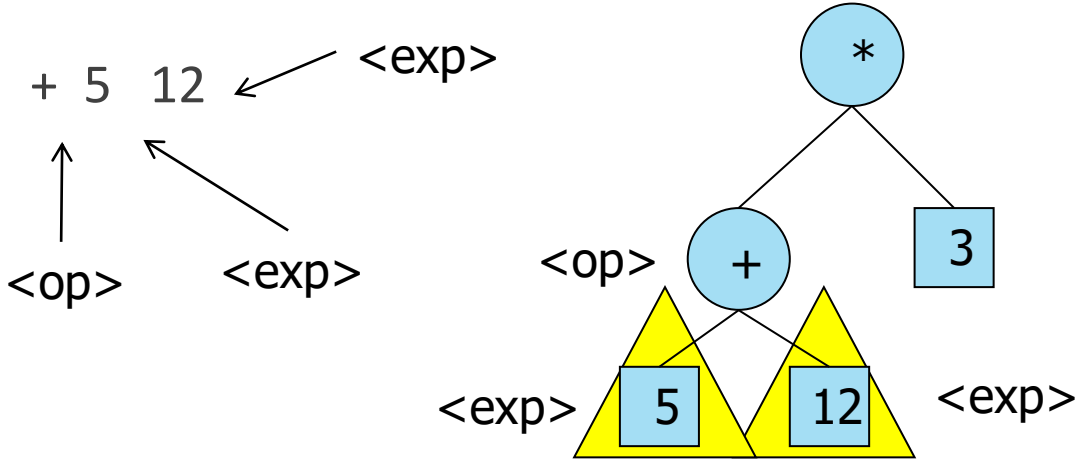
Grammatica per espressioni in forma prefissa (Notazione Polacca) semplificate (solo operatori + e *, operandi interi positivi)

- <exp> = <operand> | <op> <exp> <exp>
- <operand> = digit | digit catenate operand
- <digit> = 0..9
- <op> = + | *

Esempio:

- in forma infissa (5 + 12) * 3
- in forma prefissa * + 5 12 3

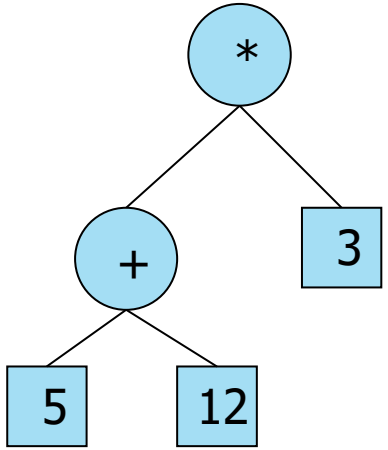

<exp>
* + 5 12 3
<op>
<exp>
*
<op>
<exp>
<exp>
+
3
5
12


+ 5 12
<exp>
<op>
<exp>
*
<op>
+
3
<exp>
5
12
<exp>

Espressione prefissa memorizzata in vettore a di caratteri (spazi per separare):

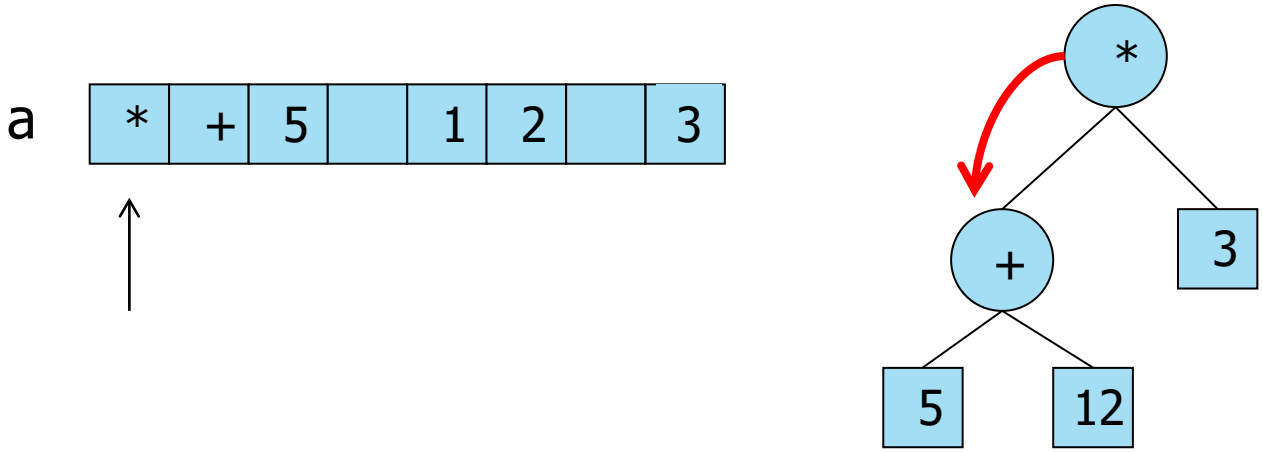
a

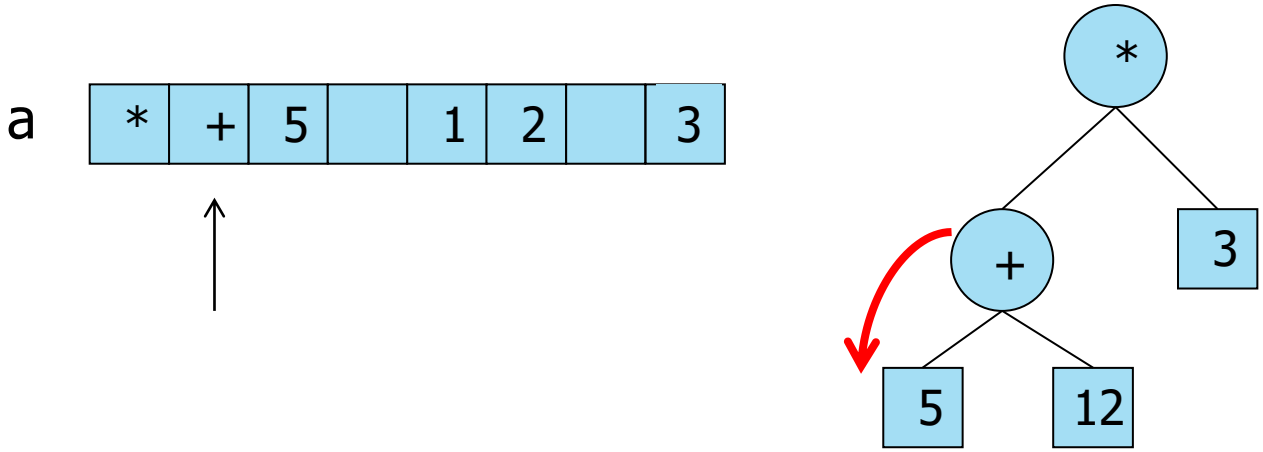
*	+	5		1	2		3

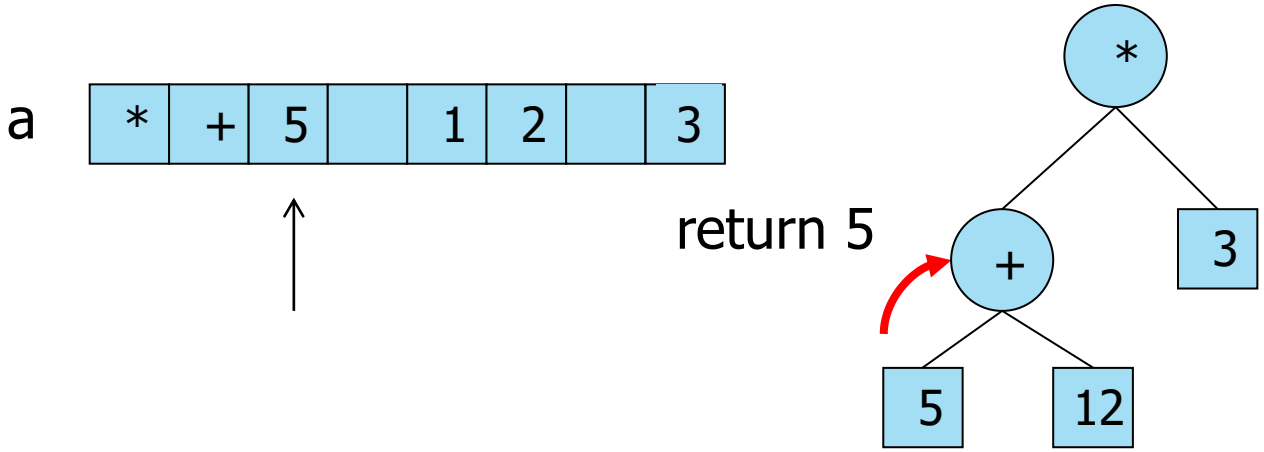


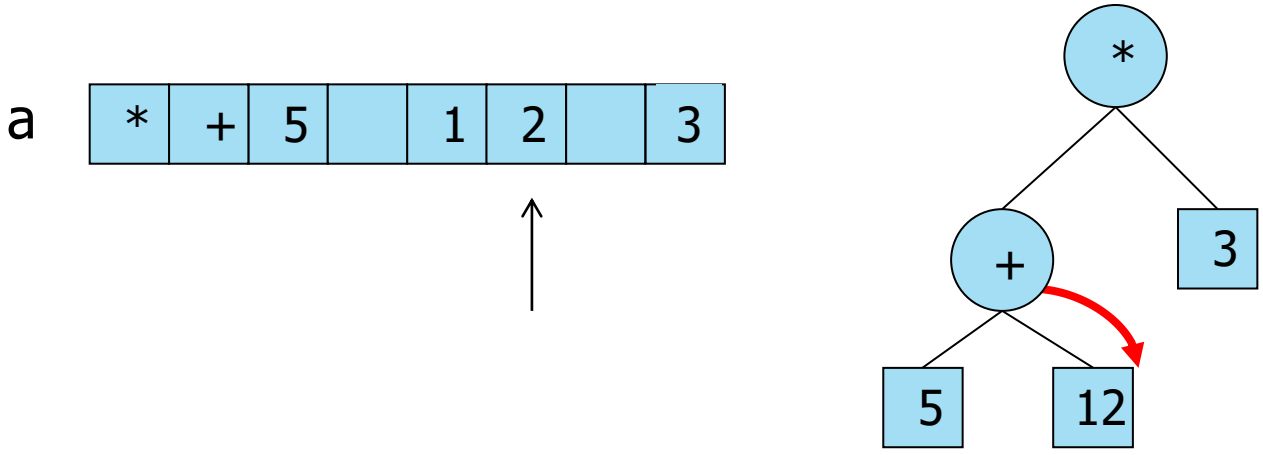
Valutazione: scansione

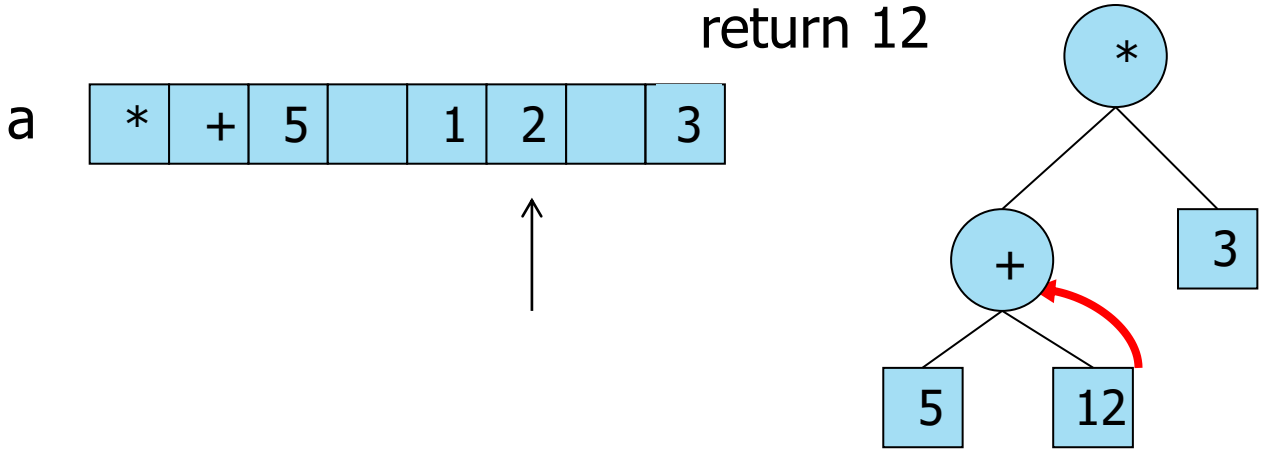
- se a[i] è un operatore op, ritorna eval() op eval()
- terminazione: se a[i] è una cifra, calcola il valore dell'intero formato da cifre fino allo spazio, ritorna l'intero

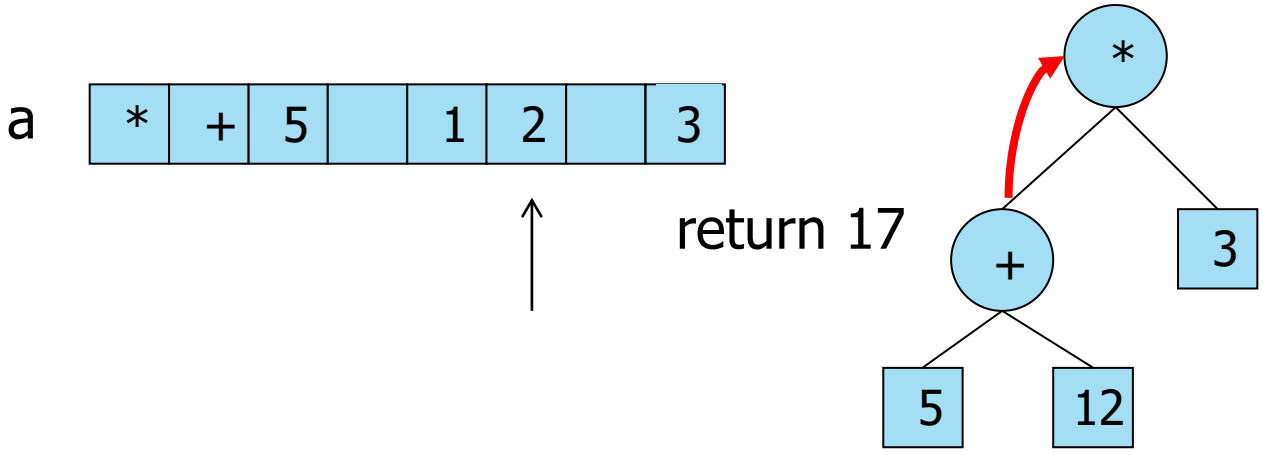

a
*
+
5
1
2
3
*
+
3
5
12

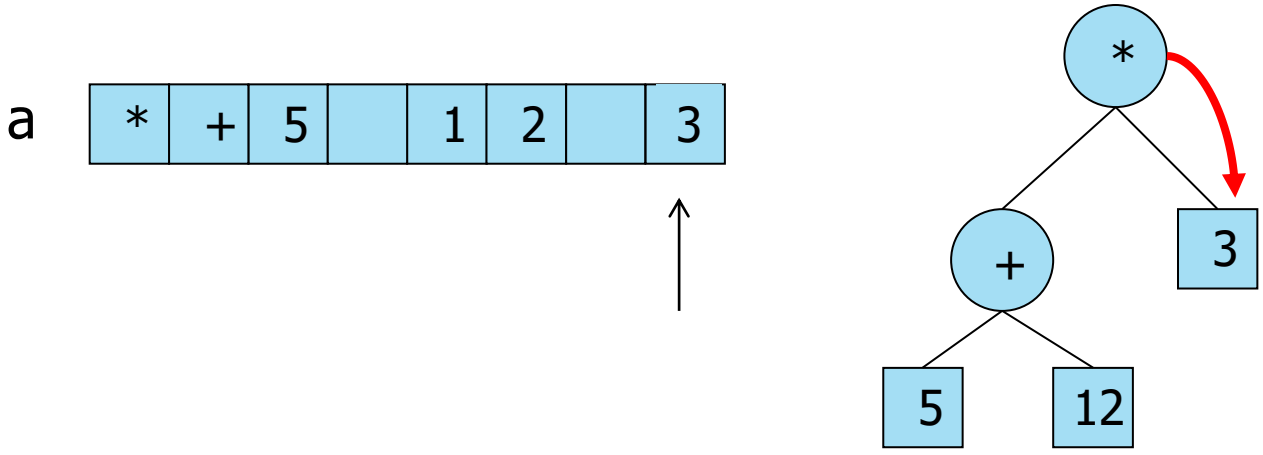

a
*
+
5
1
2
3
*
+
3
5
12

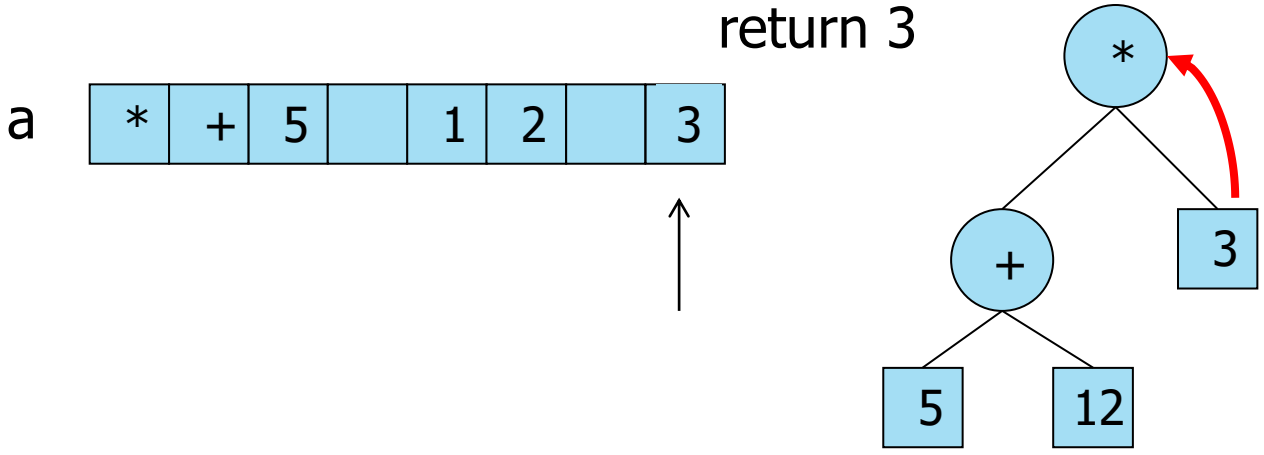

a
*
+
5
1
2
3
return 5
*
+
3
5
12

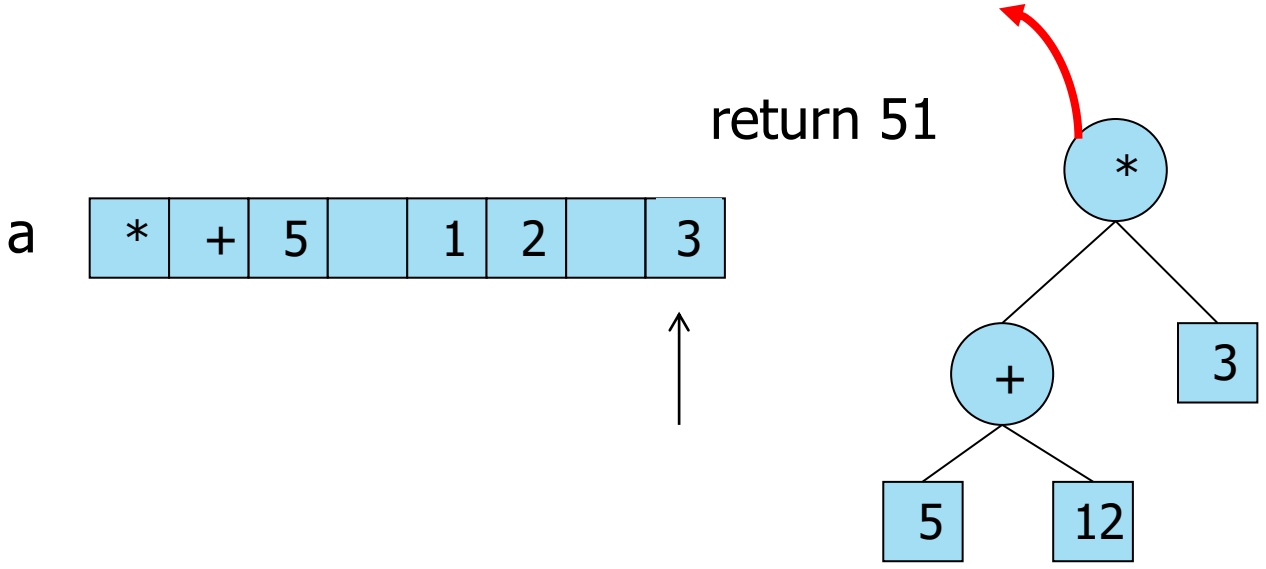

a
*
+
5
1
2
3
*
+
3
5
12


return 12
a
*
+
5
1
2
3
*
+
3
5
12


a
*
+
5
1
2
3
return 17
*
+
3
5
12


a
*
+
5
1
2
3
*
+
3
5
12


return 3
a
*
+
5
1
2
3
*
+
3
5
12


return 51
a
*
+
5
1
2
3
*
+
3
5
12

Nel main si dichiarano variabili globali:

```
static char *a;
static int i;
```

```
int eval() {
  int x = 0;
  while (a[i] == ' ')
    i++;
  if (a[i] == '+') {
    i++;
    return eval() + eval ();
  }
  if (a[i] == '*') {
    i++;
    return eval() * eval ();
  }
  while ((a[i] >= '0') && (a[i] <= '9'))
    x = 10 * x + (a[i++]-'0') ;
  return x;
}
```

Alberi Binari di Ricerca

BST: DEFINIZIONE E IMPLEMENTAZIONE COME ADT

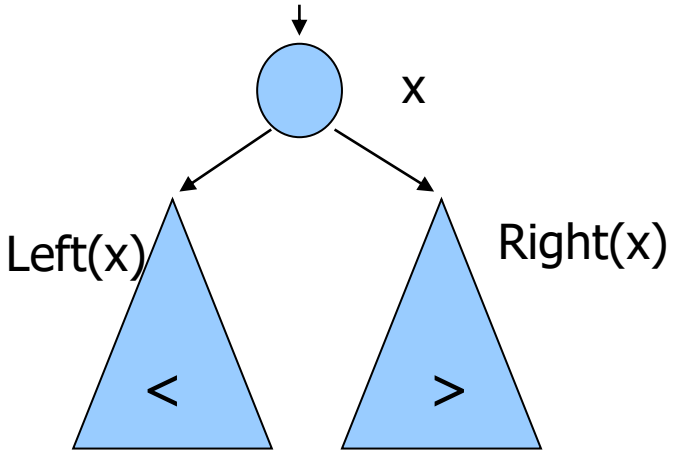
Alberi binari di ricerca (BST)

ADT albero binario con proprietà:

∀nodo x vale che:

- ∀ nodo y ∈Left(x), key[y] < key[x]
- ∀ nodo y ∈Right(x), key[y] > key[x]

chiavi distinte!



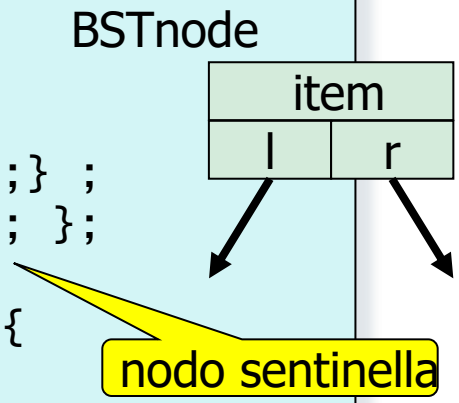
ADT di I classe BST

BST.h

```
typedef struct  binarysearchtree *BST;

BST   BSTinit() ;
void  BSTfree(BST bst);
int   BSTcount(BST bst);
int   BSTempty(BST bst);
Item  BSTsearch(BST bst, Key k);
void  BSTinsert_leafI(BST bst, Item x);
void  BSTinsert_leafR(BST bst, Item x);
void  BSTinsert_root(BST bst, Item x);
Item  BSTmin(BST bst);
Item  BSTmax(BST bst);
void  BSTvisit(BST bst, int strategy);
```

BST.c

```
#include <stdlib.h>
#include "Item.h"
#include "BST.h"

typedef struct BSTnode* link;
struct BSTnode { Item item; link l; link  r;} ;
struct binarysearchtree { link root; link z; };

static link NEW(Item item, link l, link r) {
  link x = malloc(sizeof *x);
  x->item = item; x->l = l; x->r = r;
  return x;
}
BST BSTinit( ) {
  BST bst = malloc(sizeof *bst) ;
  bst->root= ( bst->z = NEW(ITEMsetNull(), NULL, NULL));
  return bst;
}
```



```
void BSTfree(BST bst) {
  if (bst == NULL)
    return;
  treeFree(bst->root, bst->z);
  free(bst->z);
  free(bst);
}

static void treeFree(link h, link z) {
  if (h == z)
    return;
  treeFree(h->l, z);
  treeFree(h->r, z);
  free(h);
}
```

```
static int countR(link h, link z) {
 if (h == z)
   return 0;
 return countR(h->l, z) + countR(h->r, z) +1;
}

int BSTcount(BST bst) {
  return countR(bst->root, bst->z);
}

int BSTempty(BST bst) {
  if (BSTcount(bst) == 0)
    return 1;
  return 0;
}
```

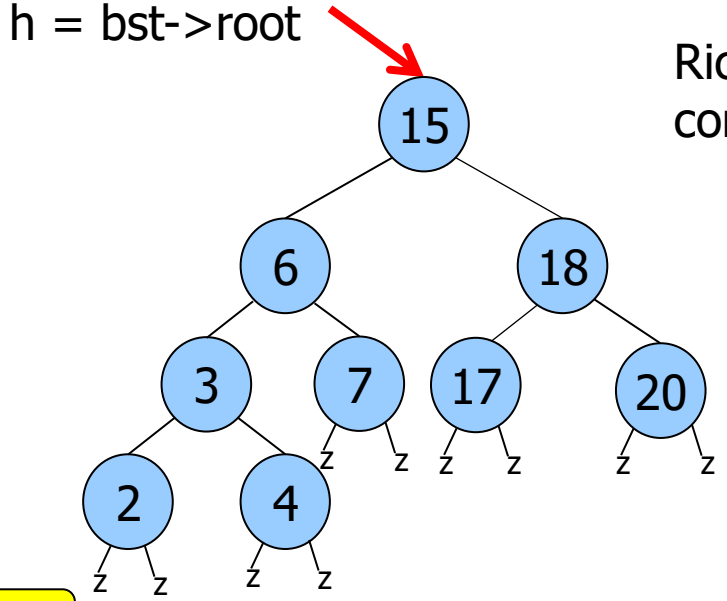
BSTsearch

Ricerca ricorsiva di un nodo che contiene un item con una chiave data:

- percorrimento dell'albero dalla radice
- terminazione: la chiave dell'item cercato è uguale alla chiave del nodo corrente (search hit) oppure si è giunti ad un albero vuoto (search miss)
- ricorsione: dal nodo corrente
  - su sottoalbero sinistro se la chiave dell'item cercato è < della chiave del nodo corrente
  - su sottoalbero destro altrimenti

```
Item searchR(link h, Key k, link z) {
  int cmp;
  if (h == z)
    return ITEMsetNull();
  cmp = KEYcmp(k, KEYget(h->item));
  if (cmp == 0)
    return h->item;
  if (cmp == -1)
    return searchR(h->l, k, z);
  return searchR(h->r, k, z);
  }

Item BSTsearch(BST bst, Key k) {
  return searchR(bst->root, k, bst->z);
}
```

Esempio

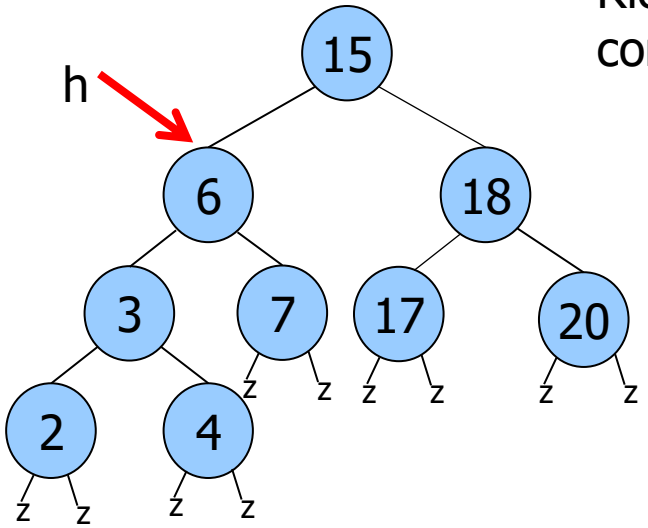


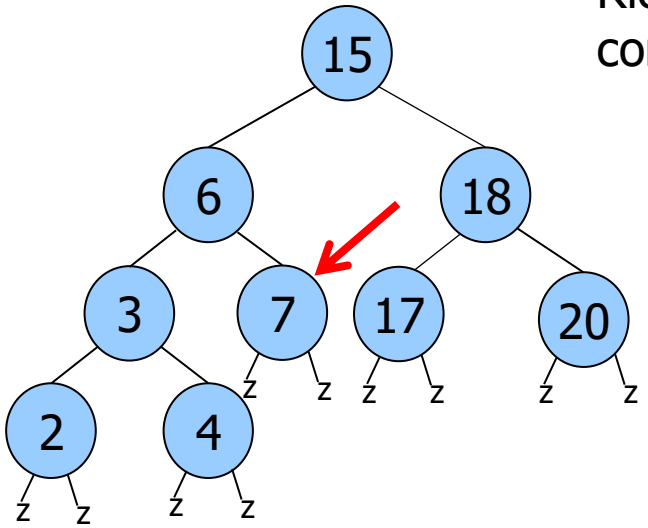
Ricerca dell'item con chiave 7

Sono visualizzate solo le chiavi intere, non gli item

z: nodo sentinella

Ricerca dell'item con chiave 7

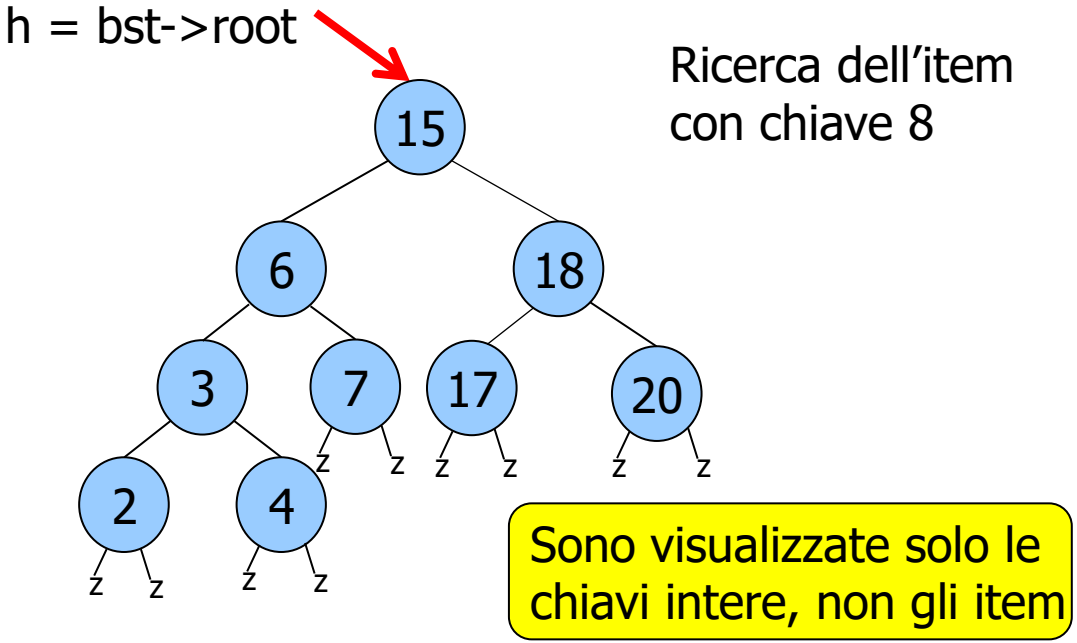


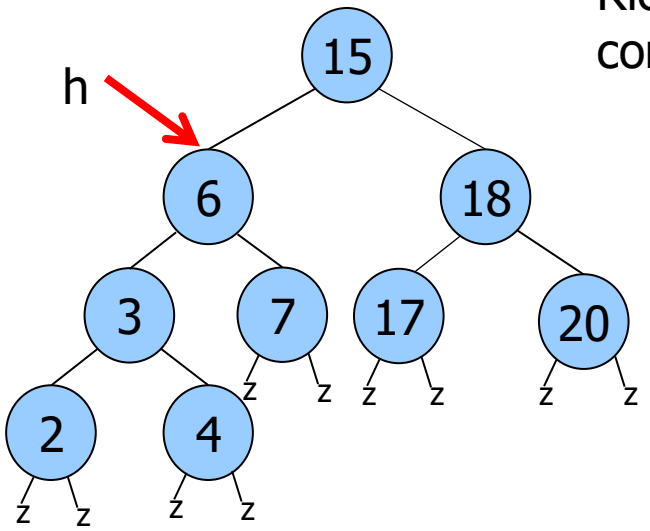


Ricerca dell'item con chiave 7

Hit!

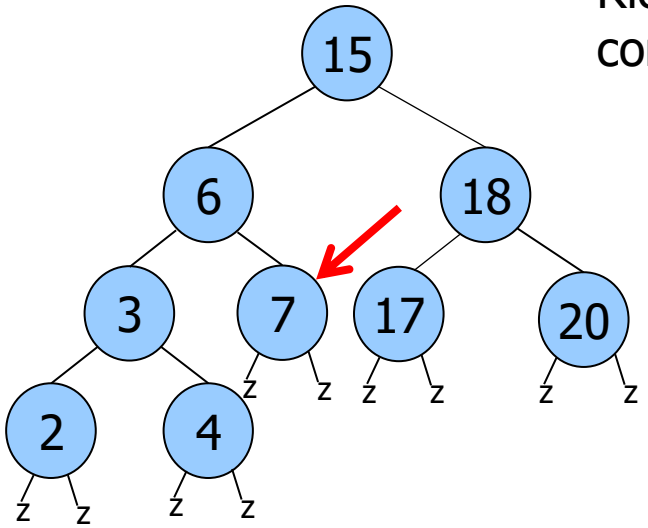
Esempio

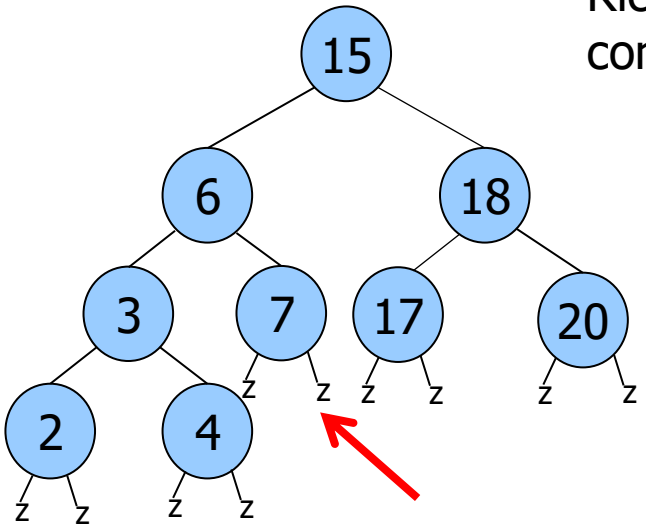

h = bst->root
15
6
18
3
7
17
20
2
4
z
Ricerca dell'item
con chiave 8
Sono visualizzate solo le
chiavi intere, non gli item



Ricerca dell'item con chiave 8

Ricerca dell'item con chiave 8





Ricerca dell'item con chiave 8

Miss!

BSTmin

```
Item minR(link h, link z) {
  if (h == z)
    return ITEMsetNull();
  if (h->l == z)
    return (h->item);
  return minR(h->l, z);
}

Item BSTmin(BST bst) {
  return minR(bst->root, bst->z);
}
```

Seguire il puntatore al sottoalbero sinistro finché si arriva al nodo sentinella z

BSTmax

```
Item maxR(link h, link z) {
  if (h == z)
    return ITEMsetNull();
  if (h->r == z)
    return (h->item);
  return maxR(h->r, z);
}

Item BSTmax(BST bst) {
  return maxR(bst->root, bst->z);
}
```

Seguire il puntatore al sottoalbero destro finché si arriva al nodo sentinella z

BSTinsert (in foglia)

Inserire in un albero binario di ricerca un nodo che contiene un item $\Rightarrow$ mantenimento della proprietà:

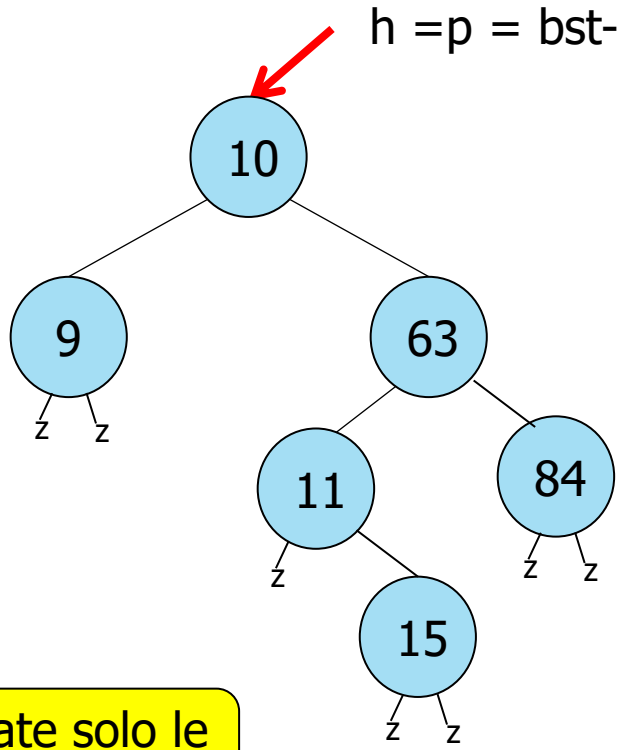
- se il BST è vuoto, creazione del nuovo albero
- inserimento ricorsivo nel sottoalbero sinistro o destro a seconda del confronto tra la chiave dell'item e quella del nodo corrente
- inserimento iterativo: prima si ricerca la posizione, poi si appende il nuovo nodo.

RICORSIVO

```
static link insertR(link h, Item x, link z) {
  if (h == z)
    return NEW(x, z, z);
  if (KEYcmp(KEYget(x), KEYget(h->item))==-1)
    h->l = insertR(h->l, x, z);
  else
    h->r = insertR(h->r, x, z);
  return h;
}

void BSTinsert_leafR(BST bst, Item x) {
  bst->root = insertR(bst->root, x, bst->z);
}
```

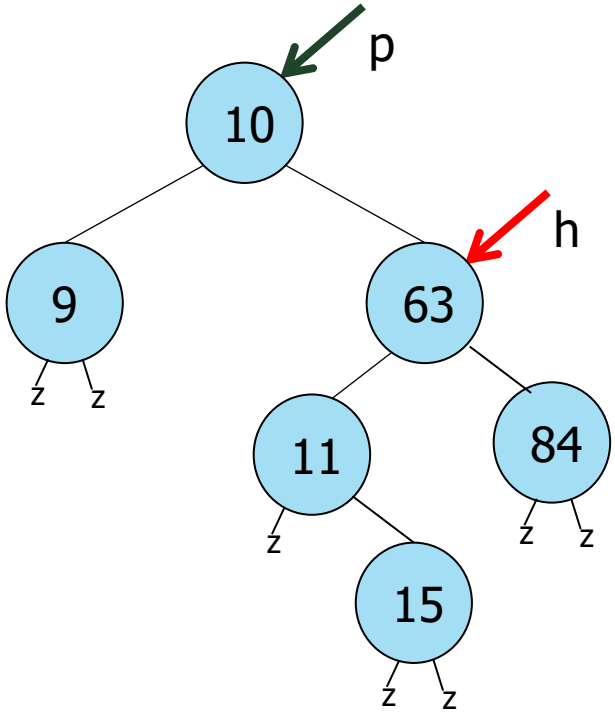
```
void BSTinsert_leafI(BST bst, Item x) {
  link p = bst->root, h = p;
  if (bst->root == bst->z) {
    bst->root = NEW(x, bst->z, bst->z);
    return;
  }
  while (h != bst->z) {
    p = h;
    h=(KEYcmp(KEYget(x),KEYget(h->item))==-1) ? h->l : h->r;
  }
  h = NEW(x, bst->z, bst->z);
  if (KEYcmp(KEYget(x), KEYget(p->item))==-1)
    p->l = h;
  else
    p->r = h;
}
```

Esempio

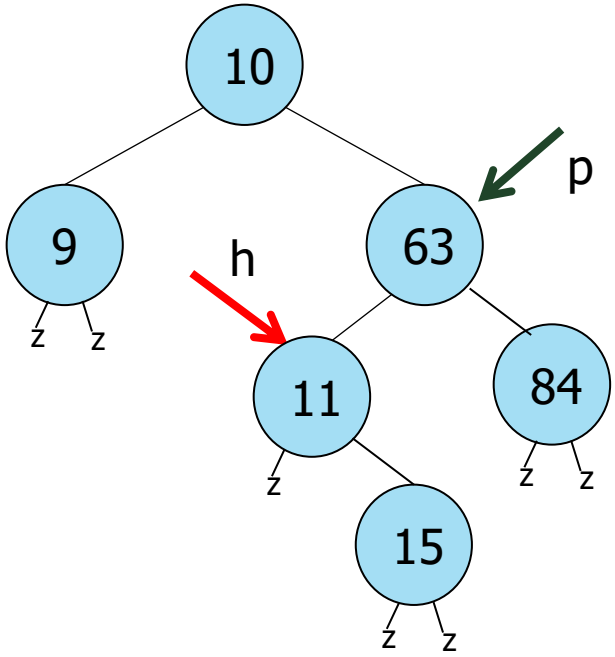


inserimento dell'item con chiave 62

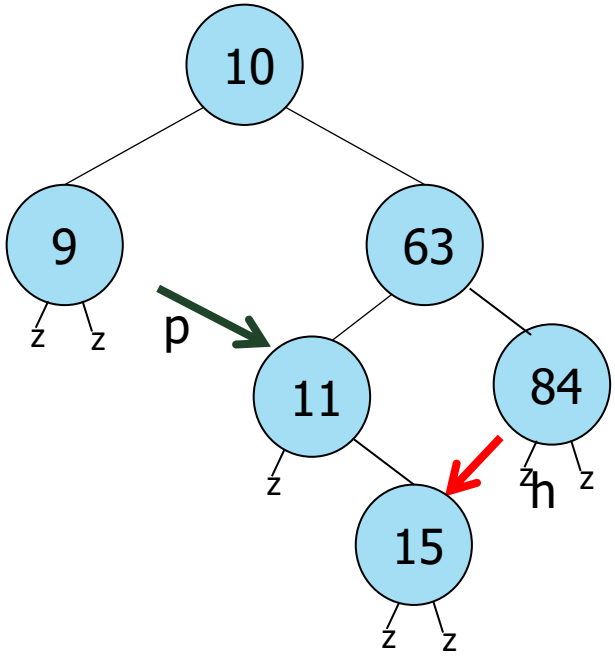
Sono visualizzate solo le chiavi intere, non gli item



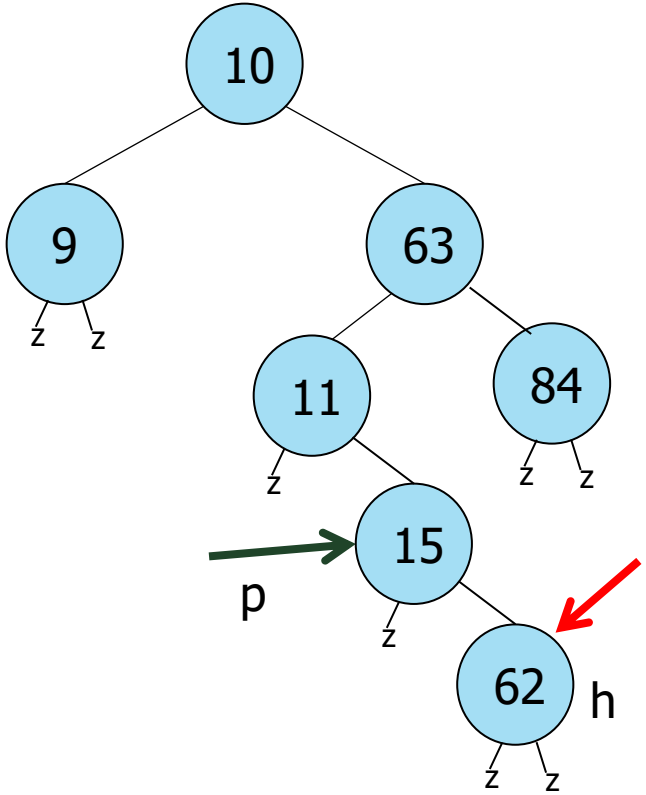
inserimento dell'item con chiave 62



inserimento dell'item con chiave 62



inserimento dell'item con chiave 62



inserimento dell'item con chiave 62

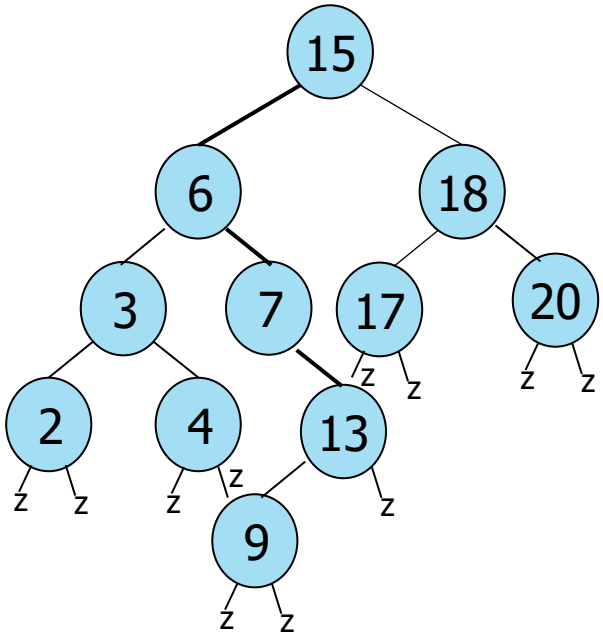
Complessità

Le operazioni hanno complessità T(n) = O(h):

- albero con n nodi completamente bilanciato
  - altezza h = $\alpha(\log_2 n)$
- albero con n nodi completamente sbilanciato ha
  - altezza h = $\alpha(n)$
- $O(\log n) \leq T(n) \leq O(n)$

BSTvisit

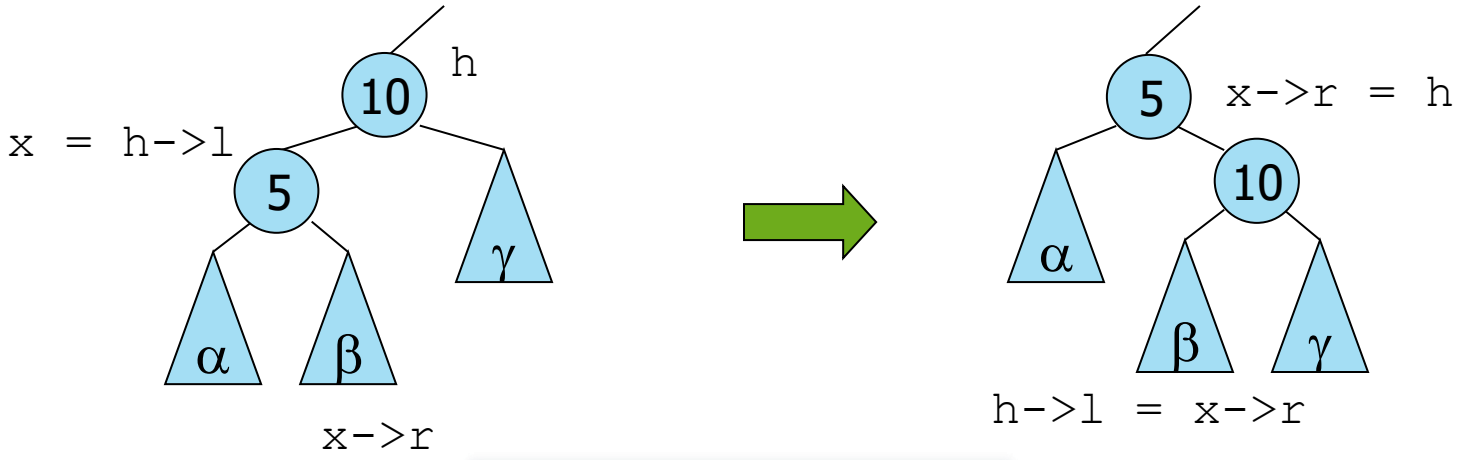
Attraversamento in-ordine: ordinamento crescente delle chiavi.



La chiave mediana (inferiore) di un insieme di n elementi è l'elemento che si trova in posizione $\lfloor (n + 1)/2 \rfloor$ nella sequenza ordinata degli elementi dell'insieme

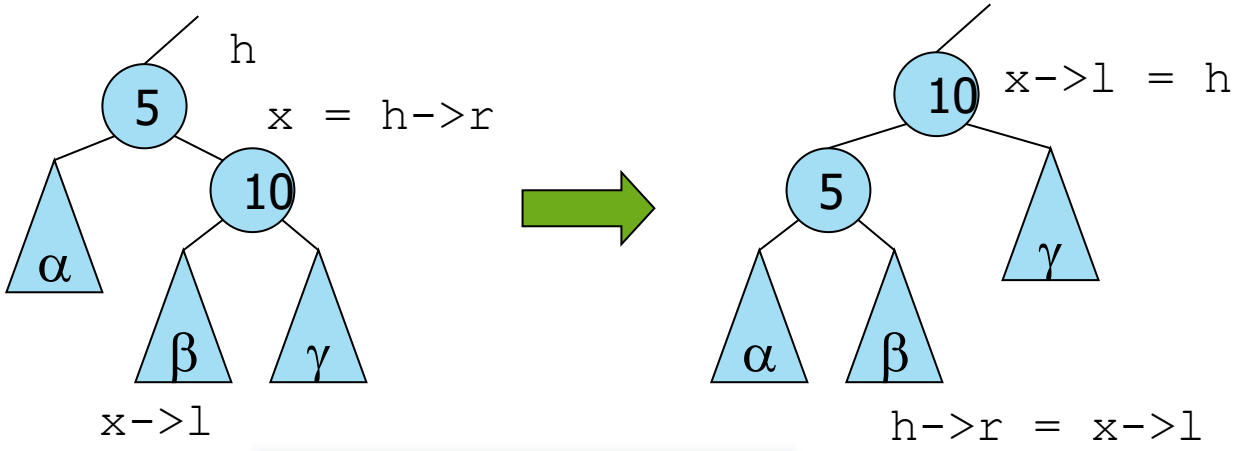
```
static void treePrintR(link h, link z, int strategy) {
  if (h == z)
    return;
  if (strategy == PREORDER)
    ITEMstore(h->item);
  treePrintR(h->l, z, strategy);
  if (strategy == INORDER)
    ITEMstore(h->item);
  treePrintR(h->r, z, strategy);
  if (strategy == POSTORDER)
    ITEMstore(h->item);
}
void BSTvisit(BST bst, int strategy) {
  if (BSTempty(bst))
    return;
  treePrintR(bst->root, bst->z, strategy);
}
```

Rotazione a destra di BST




```
link rotR(link h) {
  link x = h->l;
  h->l = x->r;
  x->r = h;
  return x;
}
```

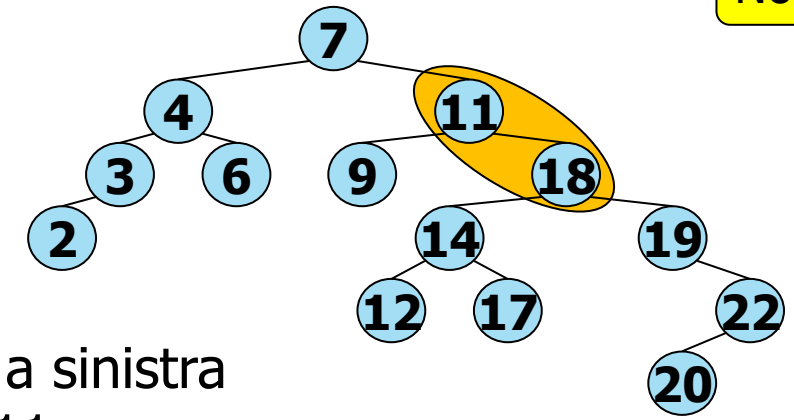
Rotazione a sinistra di BST



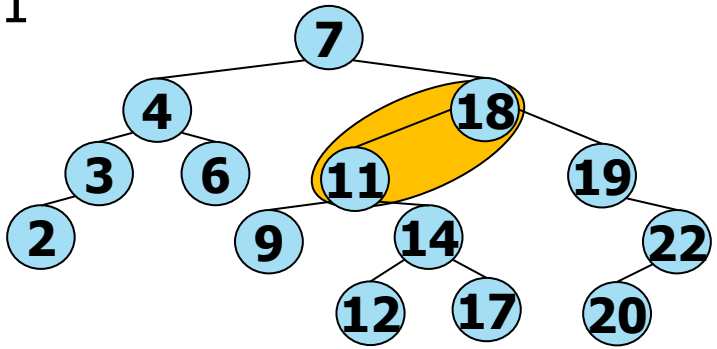

```
link rotL(link h) {
  link x = h->r;
  h->r = x->l;
  x->l = h;
  return x;
}
```

Esempio

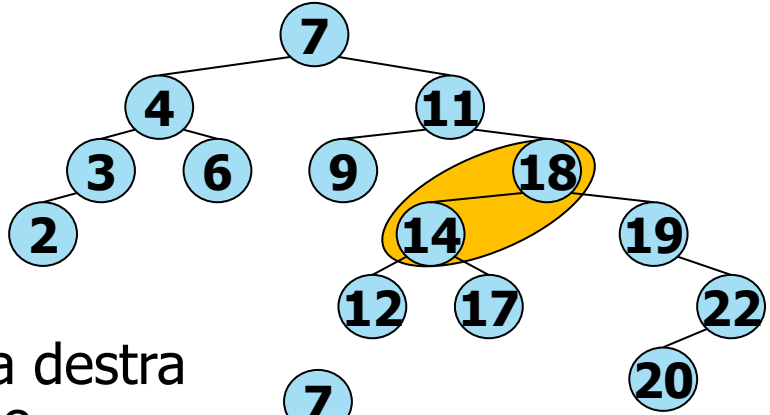
Nodi sentinella z non riportati



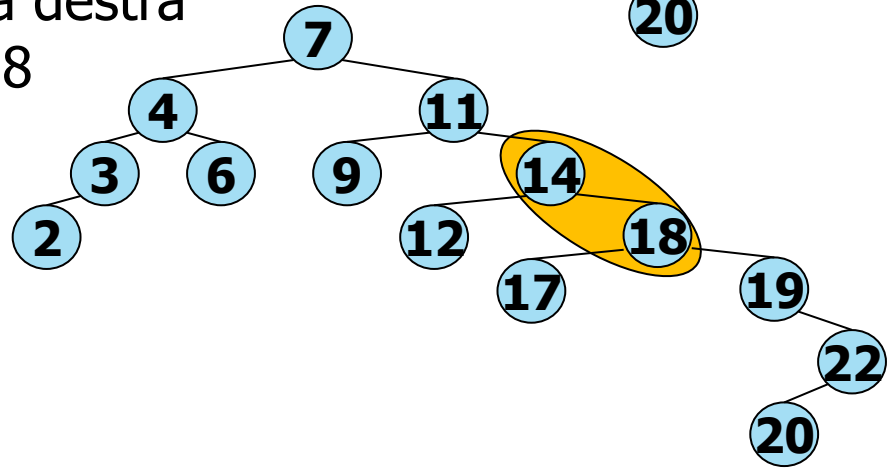
Rotazione a sinistra attorno a 11



Esempio



Rotazione a destra attorno a 18

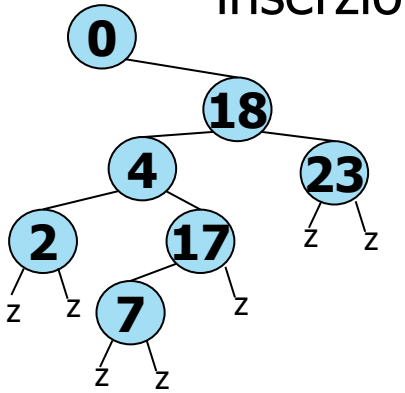


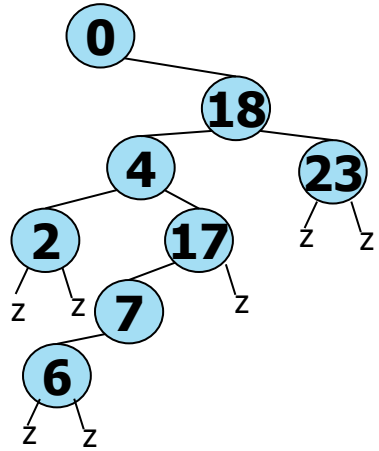
BSTinsert (in radice)

- Inserimento dalle foglie a scelta e non obbligatorio
- Nodi più recenti nella parte alta del BST
- Inserimento ricorsivo alla radice:
  - inserimento nel sottoalbero appropriato
  - rotazione per farlo diventare radice dell'albero principale.

Esempio

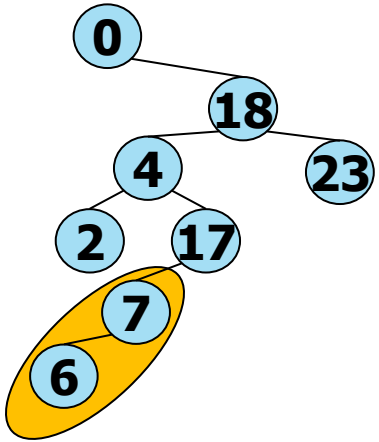
inserzione di 6



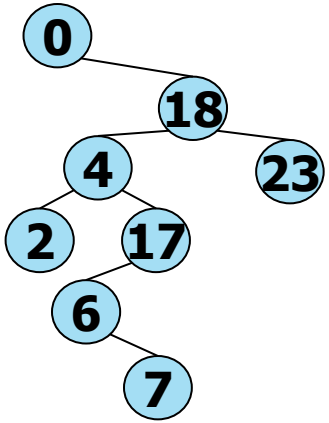


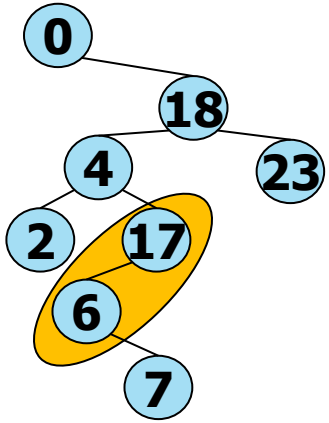
inserzione di 6 alla radice del sottoalbero opportuno

Nodi sentinella z non più riportati

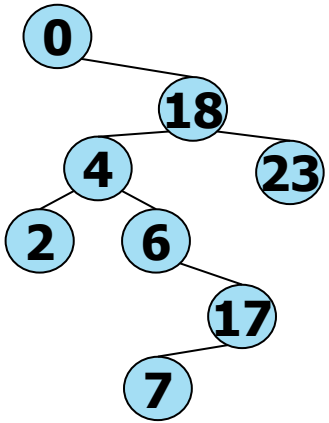


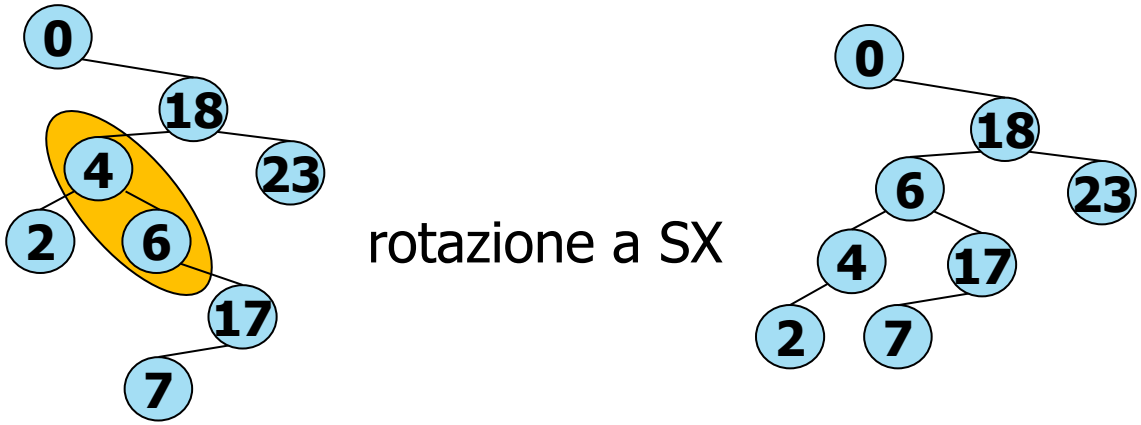
rotazione a DX

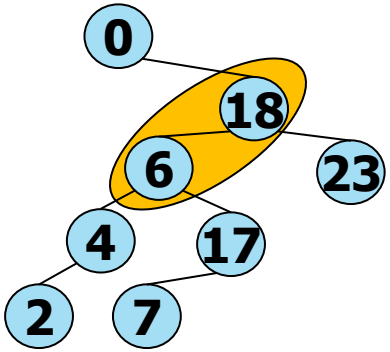




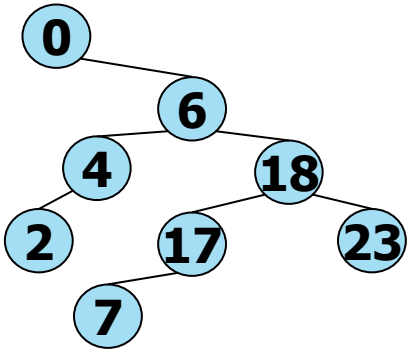
rotazione a DX

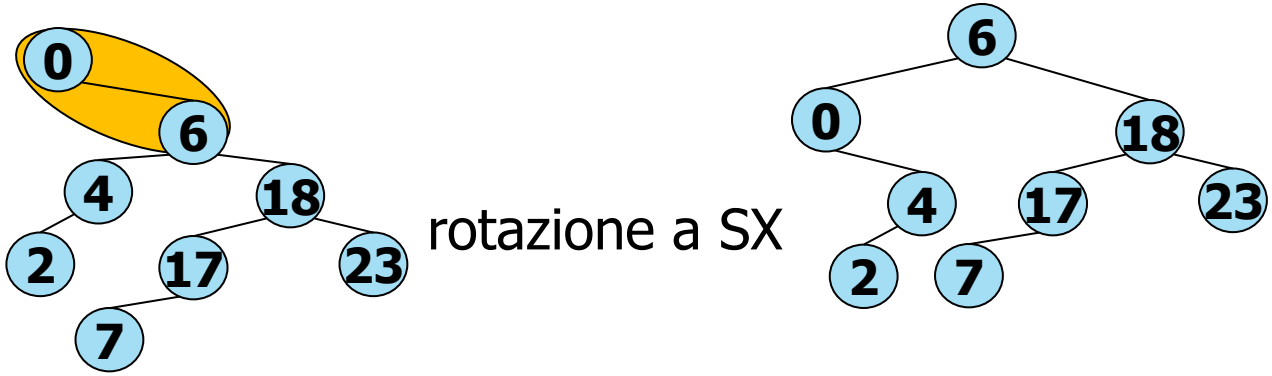



0
18
4
23
2
6
17
7
rotazione a SX
0
18
6
23
4
17
2
7



rotazione a DX




0
6
4
18
2
17
23
7
rotazione a SX
6
0
18
4
17
23
2
7

```
static link insertT(link h, Item x, link z) {
  if (h == z)
    return NEW(x, z, z);
  if (KEYcmp(KEYget(x), KEYget(h->item))==-1) {
    h->l = insertT(h->l, x, z);
    h = rotR(h);
  }
  else {
    h->r = insertT(h->r, x, z);
    h = rotL(h);
  }
  return h;
}

void BSTinsert_root(BST bst, Item x) {
  bst->root = insertT(bst->root, x, bst->z);
}
```

Estensioni dei BST elementari

PUNTATORE AL PADRE, NUMERO DI NODI NEL SOTTO-ALBERO

Estensione dei BST elementari

Al nodo elementare si possono aggiungere informazioni che permettono lo sviluppo semplice di nuove funzioni:

- puntatore al padre
- numero di nodi dell'albero radicato nel nodo corrente.

Queste informazioni devono ovviamente essere gestite (quando necessario) da tutte le funzioni già viste.

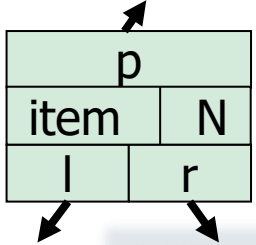
ADT di I classe BST

nuove funzioni
funzioni modificate

BST.h

```
typedef struct  binarysearchtree *BST;

BST     BSTinit();
void    BSTfree(BST bst); int  BSTcount(BST bst);
int     BSTempty(BST bst);
Item    BSTmin(BST bst); Item  BSTmax(BST bst);
void    BSTinsert_leafI(BST bst, Item x);
void    BSTinsert_leafR(BST bst, Item x);
void    BSTinsert_root(BST bst, Item x);
Item    BSTsearch(BST bst, Key k);
void    BSTdelete(BST bst, Key k);
Item    BSTselect(BST bst, int r);
void    BSTvisit(BST bst, int strategy);
Item    BSTsucc(BST bst, Key k);
Item    BSTpred(BST bst, Key k);
void    BSTbalance(BST bst);
```

Order-Statistic BST



BSTnode

item | name | value ⟸ chiave

BST.c

```
#include <stdlib.h>
#include "Item.h"
#include "BST.h"

typedef struct BSTnode* link;
struct BSTnode {Item item;link p;link l;link  r;int N;};
struct binarysearchtree { link root; link z; };

static link NEW(Item item, link p, link l, link r, int N){
  link x = malloc(sizeof *x);
  x->item = item;
  x->p = p; x->l = l; x->r = r; x->N = N;
  return x;
}
```

puntatore al padre

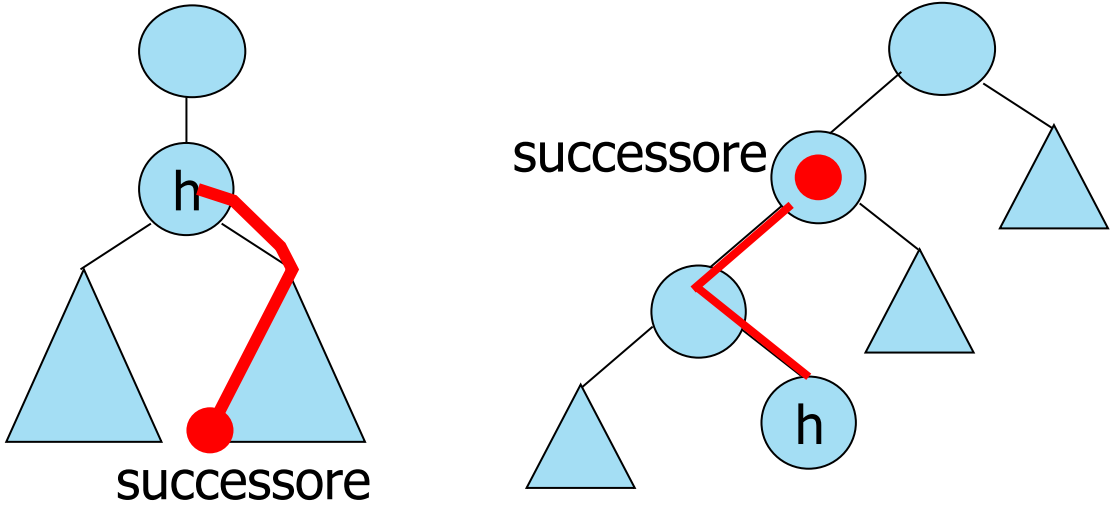
dimensione sottoalbero

```
BST BSTinit( ) {
  BST bst = malloc(sizeof *bst) ;
  bst->root=(bst->z=NEW(ITEMsetNull(), NULL, NULL, NULL, 0));
  return bst;
}
```

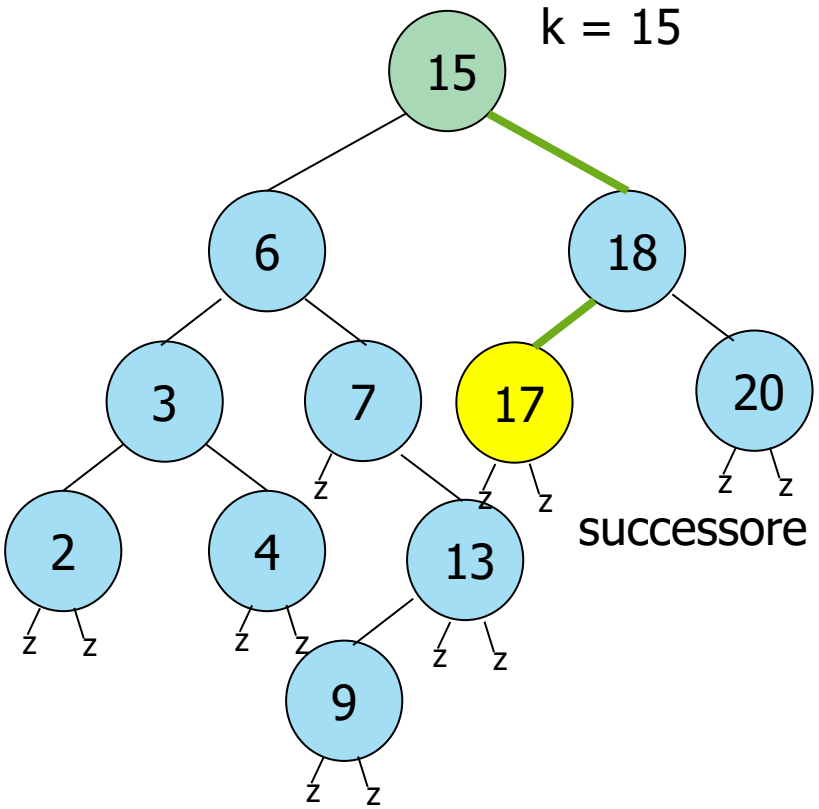
Successore

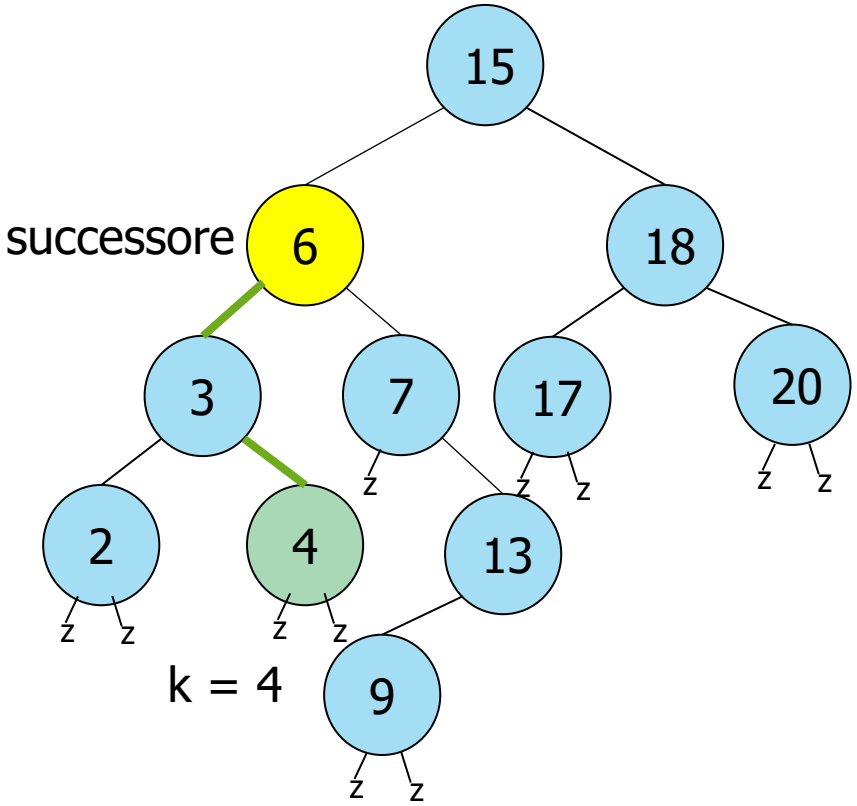
Successore di un item: se esiste, nodo h con un item con la più piccola chiave > della chiave di item, altrimenti item vuoto. Due casi:

- $\exists$ Right(h): succ(key(h)) = min(Right(h))
- $\nexists$ Right(h): succ(key(h)) = primo antenato di h il cui figlio sinistro è anche un antenato di h.



Esempio




15
successore
6
18
3
7
17
20
2
4
13
9
k = 4
z

```
Item searchSucc(link h, Key k, link z) {
  link p;
  if (h == z) return ITEMsetNull();
  if (KEYcmp(k, KEYget(h->item))==0) {
    if (h->r != z) return minR(h->r, z);
    else {
      p = h->p;
      while (p != z && h == p->r) {
        h = p; p = p->p;
      }
      return p->item;
    }
  }
  if (KEYcmp(k, KEYget(h->item))==-1)
    return searchSucc(h->l, k, z);
  return searchSucc(h->r, k, z);
}
Item BSTsucc(BST bst, Key k) {
  return searchSucc(bst->root, k, bst->z);
}
```

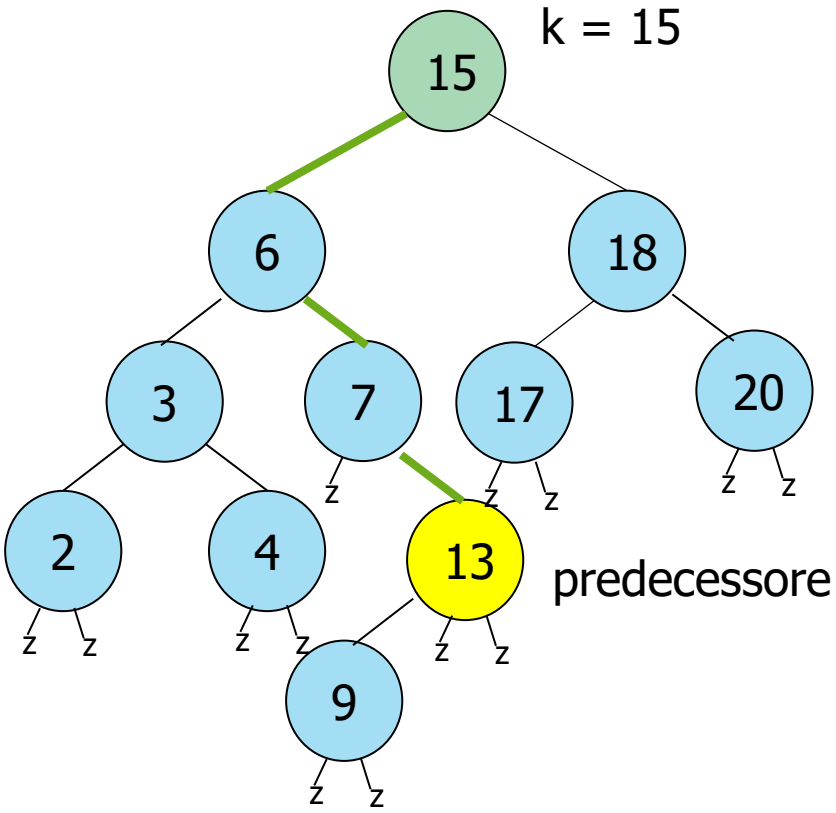
Predecessore

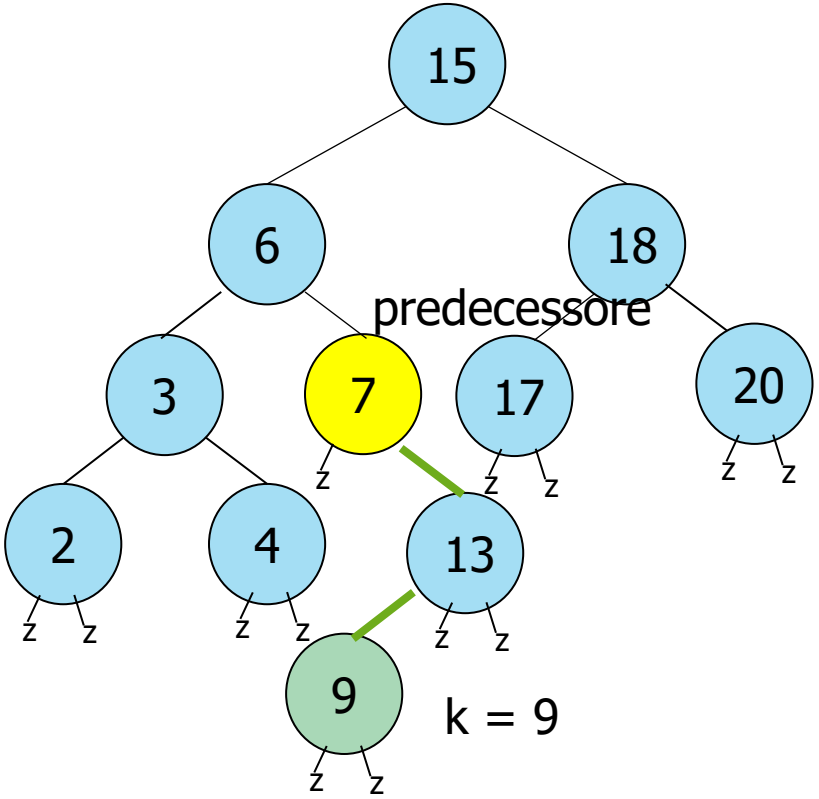
Predecessore di un item: nodo h con item con la più grande chiave < della chiave di item.

Due casi:

- ∃ Left(h): pred(key(h)) = max(Left(h))
- ~~∃~~ Left(h): pred(key(h)) = primo antenato di h il cui figlio destro è anche un antenato di h.

Esempio




15
6
18
predecessore
3
7
17
20
2
4
13
9
k = 9

```
Item searchPred(link h, Key k, link z) {
  link p;
  if (h == z) return ITEMsetNull();
  if (KEYcmp(k, KEYget(h->item))==0) {
    if (h->l != z) return maxR(h->l, z);
    else {
      p = h->p;
      while (p != z && h == p->l) {h = p; p = p->p;}
      return p->item;
    }
  }
  if (KEYcmp(k, KEYget(h->item))==-1)
    return searchPred(h->l, k, z);
  return searchPred(h->r, k, z);
}
Item BSTpred(BST bst, Key k) {
  return searchPred(bst->root, k, bst->z);
}
```

BSTinsert (in foglia)

RICORSIVO

```
link insertR(link h, Item x, link z) {
  if (h == z)
    return NEW(x, z, z, z, 1);
  if (KEYcmp(KEYget(x), KEYget(h->item))==-1) {
    h->l = insertR(h->l, x, z); h->l->p = h;
  }
  else {
    h->r = insertR(h->r, x, z); h->r->p = h;
  }
  (h->N)++;
  return h;
}
void BSTinsert_leafR(BST bst, Item x) {
  bst->root = insertR(bst->root, x, bst->z);
}
```

BSTinsert (in foglia)

ITERATIVO

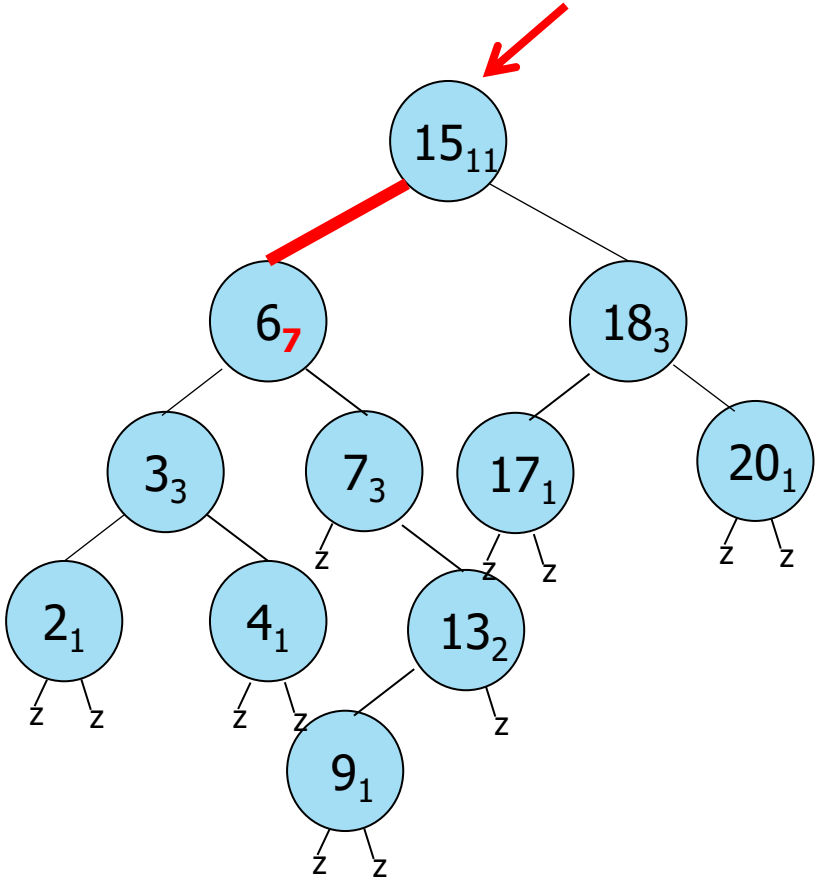
```
void BSTinsert_leafI(BST bst, Item x) {
  link p = bst->root, h = p;
  if (bst->root == bst->z) {
    bst->root = NEW(x, bst->z, bst->z, bst->z, 1);
    return;
  }
  while (h != bst->z) {
    p = h; h->N++;
    h=(KEYcmp(KEYget(x),KEYget(h->item))==-1) ? h->l : h->r;
  }
  h = NEW(x, p, bst->z, bst->z, 1);
  if (KEYcmp(KEYget(x), KEYget(p->item))==-1)
    p->l = h;
  else
    p->r = h;
}
```

BSTselect

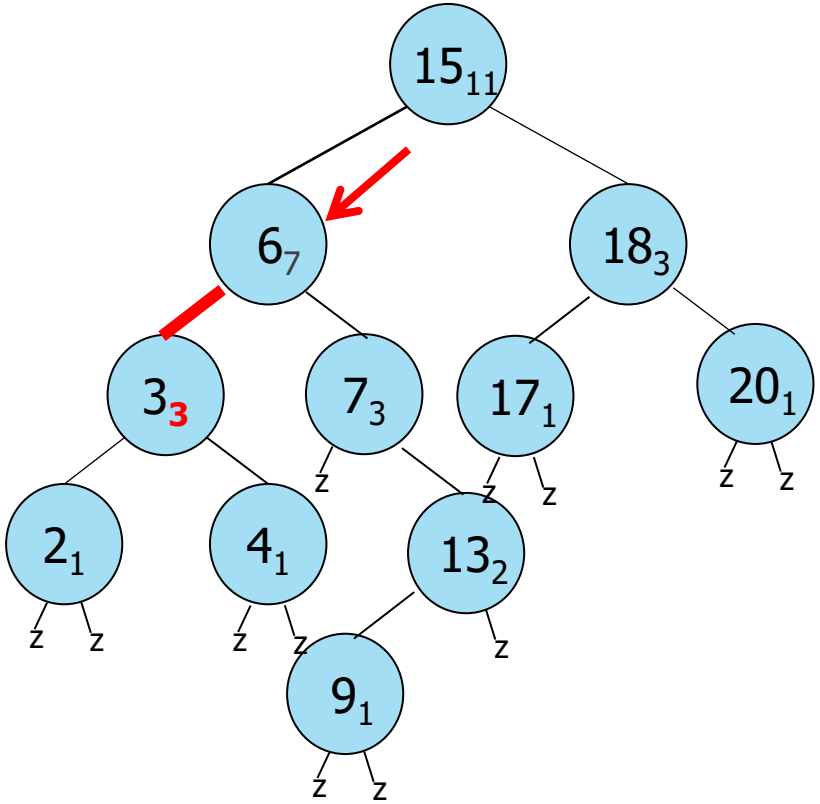
Selezione dell'item con la r-esima chiave più piccola (rango r = chiave in posizione r nell'ordinamento, ad esempio se r=0 item con chiave minima): t è il numero di nodi del sottoalbero sinistro:

- t = r: ritorno la radice del sottoalbero
- t > r: ricorsione nel sottoalbero sinistro alla ricerca della k-esima chiave più piccola
- t < r: ricorsione nel sottoalbero destro alla ricerca della (r-t-1)-esima chiave più piccola

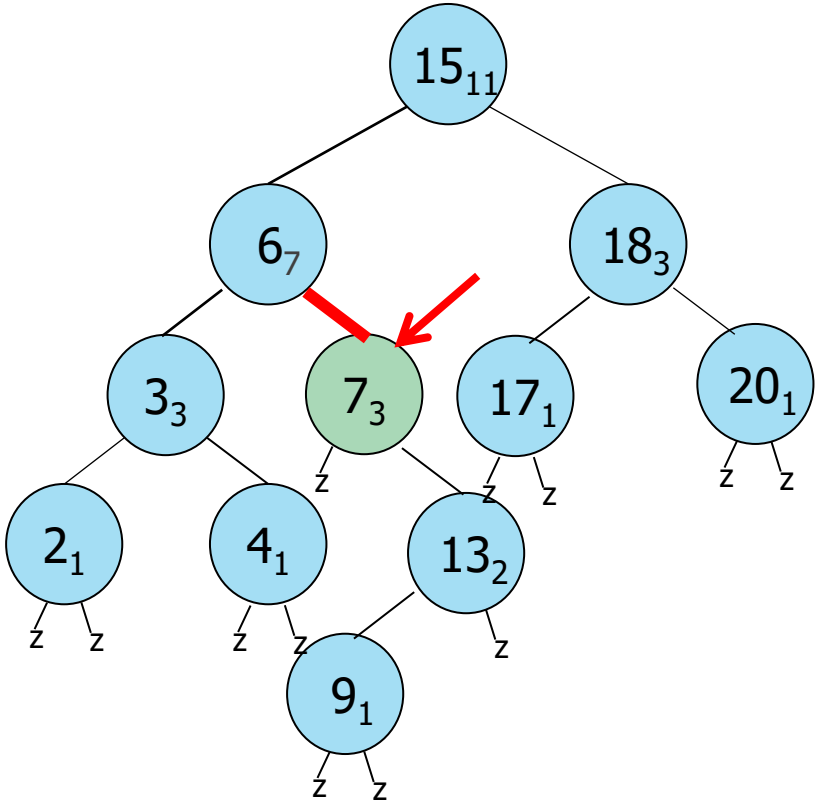
Esempio



rango = r = 4
quinta più piccola chiave
t=7
7>4 scendo a SX



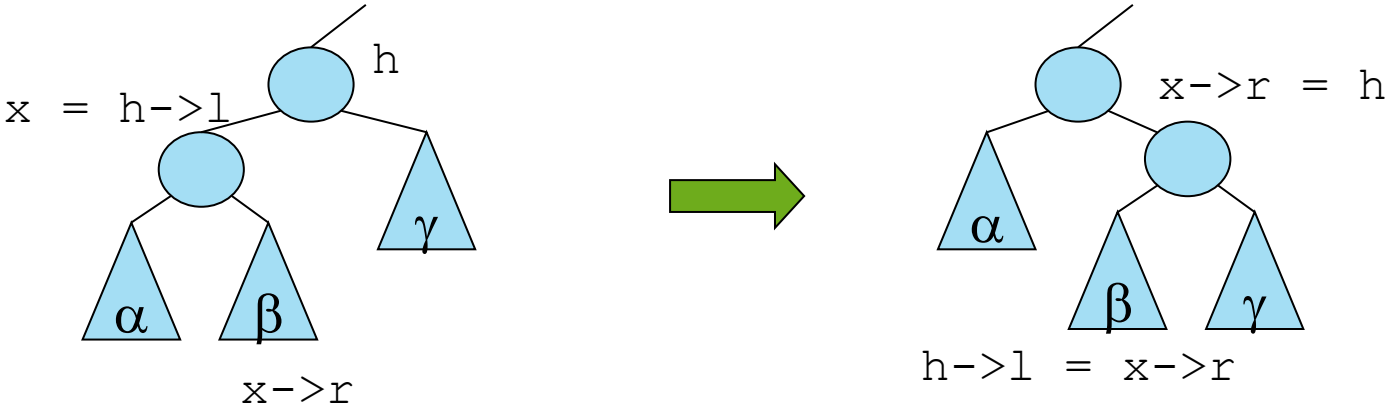
r = 4
t=3
3<4 scendo a DX
cerco r=4-3-1=0



r = 0
t=0
return 7

```
Item selectR(link h, int r, link z) {
  int t;
  if (h == z)
    return ITEMsetNull();
  t = h->l->N;
  if (t > r)
    return selectR(h->l, r, z);
  if (t < r)
    return selectR(h->r, r-t-1, z);
  return h->item;
}

Item BSTselect(BST bst, int r) {
  return selectR(bst->root, r, bst->z);
}
```

Rotazione a destra di BST

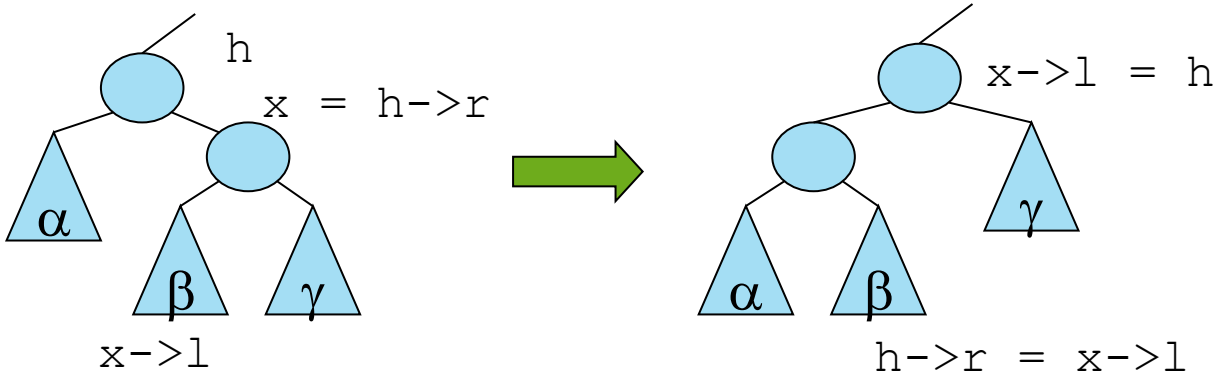


```
link rotR(link h) {
  link x = h->l;
  h->l = x->r;
  x->r->p = h;
  x->r = h;
  x->p = h->p;
  h->p = x;
  x->N = h->N;
  h->N = 1;
  h->N += (h->l) ? h->l->N : 0;
  h->N += (h->r) ? h->r->N : 0;
  return x;
}
```

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

Rotazione a sinistra di BST



```
link rotL(link h) {
  link x = h->r;
  h->r = x->l;
  x->l->p = h;
  x->l = h;
  x->p = h->p;
  h->p = x;
  x->N = h->N;
  h->N = 1;
  h->N += (h->l) ? h->l->N : 0;
  h->N += (h->r) ? h->r->N : 0;
  return x;
}
```

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

BSTinsert (in radice)

```
link insertT(link h, Item x, link z) {
  if ( h == z)
    return NEW(x, z, z, z, 1);
  if (KEYcmp(KEYget(x), KEYget(h->item))==-1) {
    h->l = insertT(h->l, x, z); h = rotR(h); h->N++;
  }
  else {
    h->r = insertT(h->r, x, z); h = rotL(h); h->N++;
  }
  return h;
}

void BSTinsert_root(BST bst, Item x) {
  bst->root = insertT(bst->root, x, bst->z);
}
```

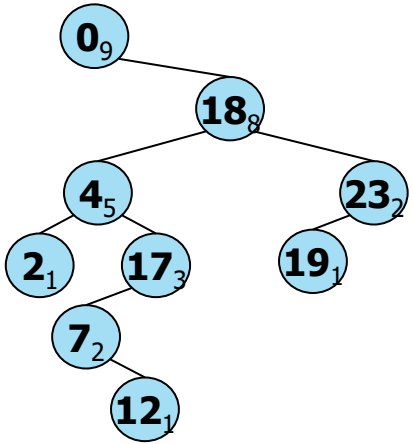
BSTpartition

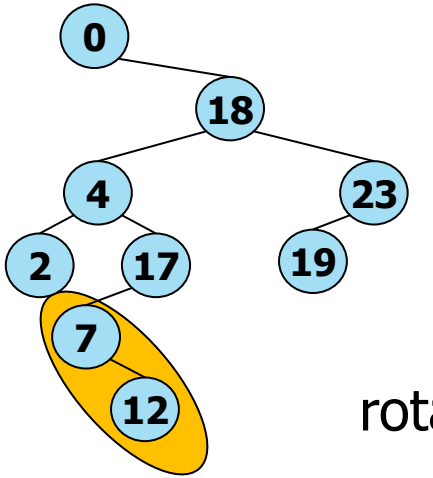
- Riorganizza l'albero avendo l'item con la k-esima chiave più piccola nella radice:
  - porre il nodo come radice di un sottoalbero:
    - t > k: ricorsione nel sottoalbero sinistro, partizionamento rispetto alla k-esima chiave più piccola, al termine rotazione a destra
    - t < k: ricorsione nel sottoalbero destro, partizionamento rispetto alla (k-t-1)-esima chiave più piccola , al termine rotazione a sinistra
- Sovente il partizionamento si fa attorno alla chiave mediana

Esempio

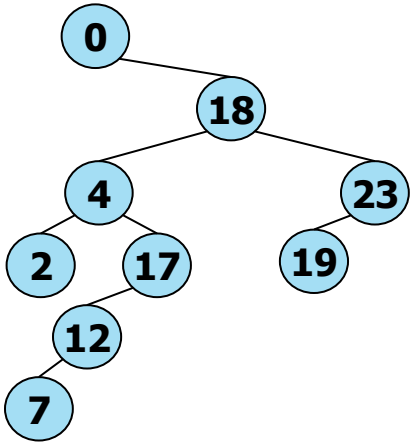
Nodi sentinella z non riportati

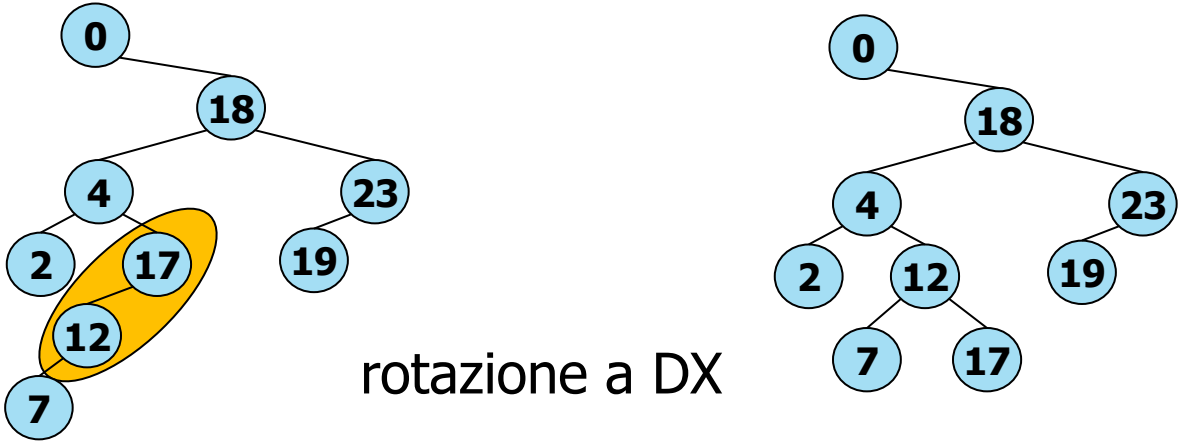
Partizionamento rispetto alla
5a chiave più piccola (12, k=4)

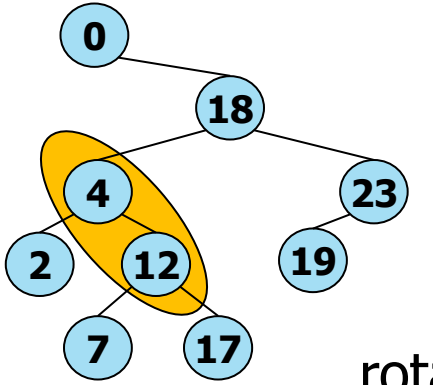




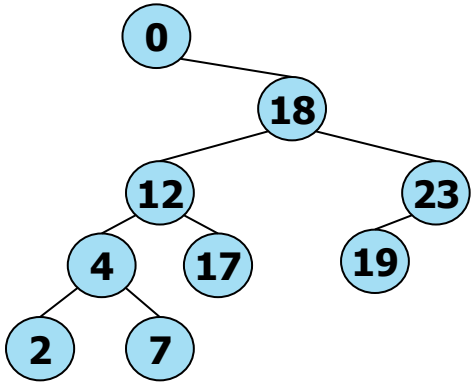
rotazione a SX

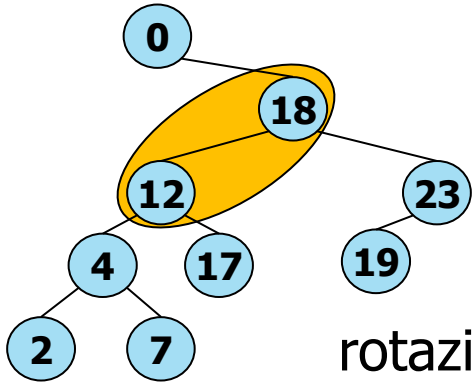



0
18
4
23
2
17
19
12
7
rotazione a DX
0
18
4
23
2
12
19
7
17

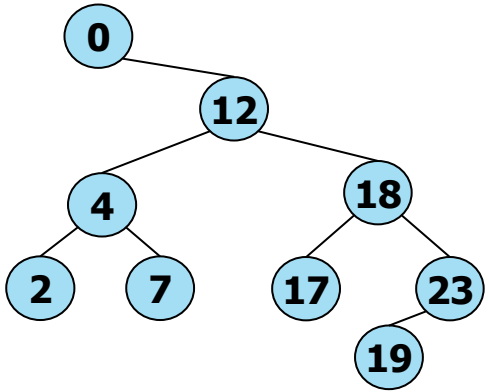


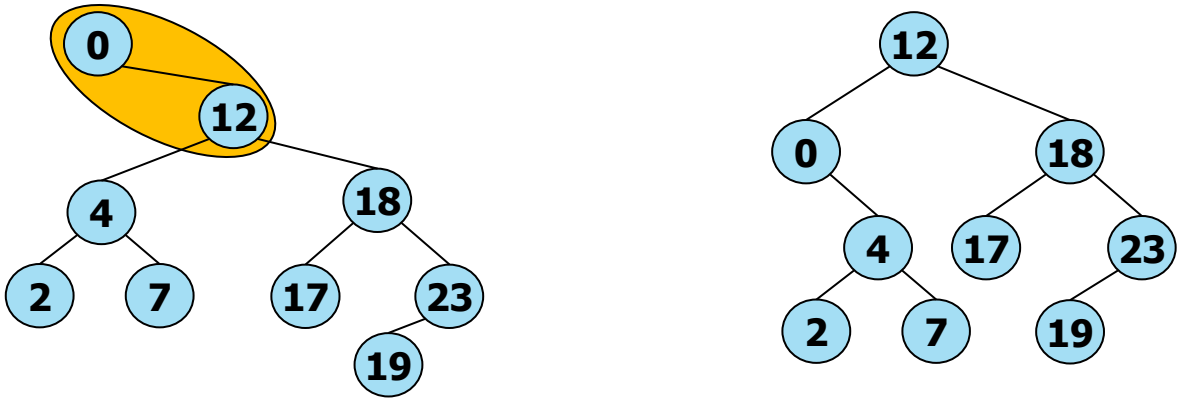
rotazione a LX





rotazione a DX





rotazione a SX

```
link partR(link h, int r) {
  int t  = h->l->N;
  if (t > r) {
    h->l = partR(h->l, r);
    h = rotR(h);
  }
  if (t < r) {
    h->r = partR(h->r, r-t-1);
    h = rotL(h);
  }
  return h;
}
```

BSTdelete

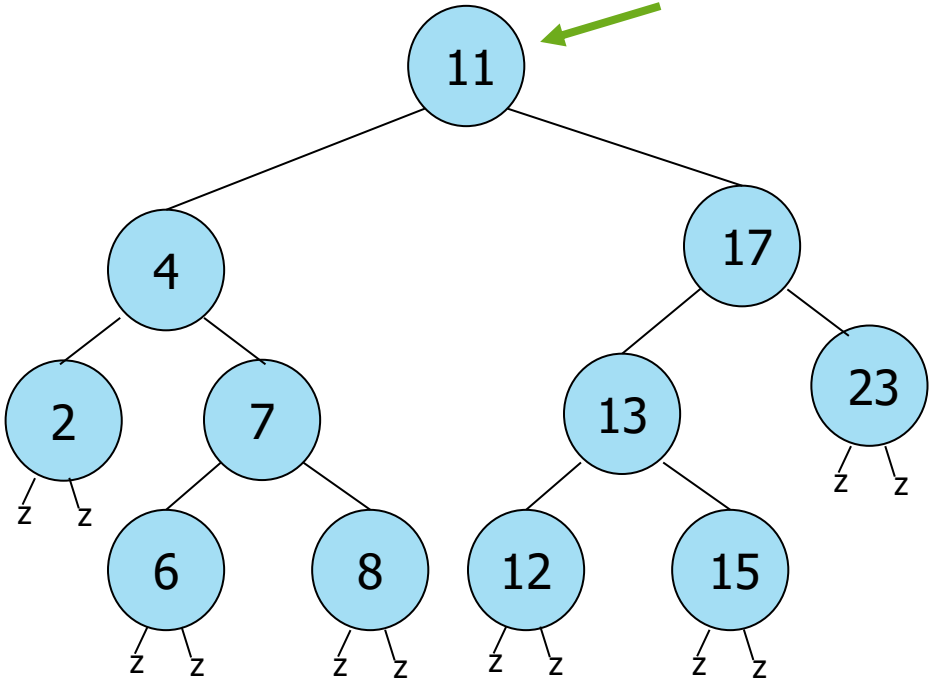
Per cancellare da un albero binario di ricerca un nodo con item con chiave k bisogna mantenere:

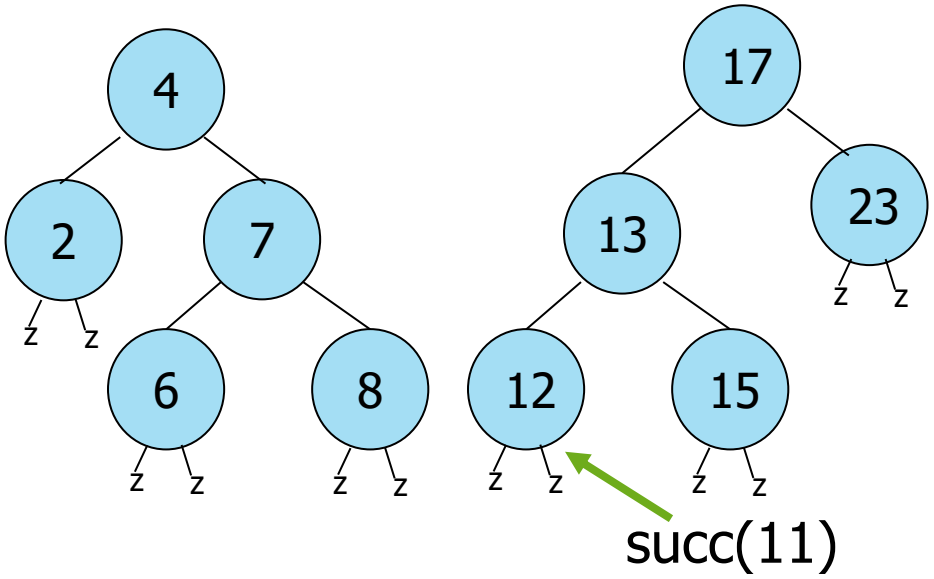
- la proprietà dei BST
- la struttura ad albero binario

Passi:

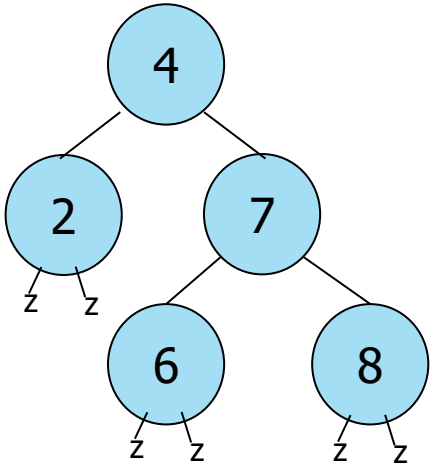
- controllare se il nodo con l'item da cancellare è in uno dei sottoalberi. Se sì, cancellazione ricorsiva nel sottoalbero
- se è la radice, eliminarlo e ricombinare i 2 sottoalberi. La nuova radice è il successore o il predecessore dell'item cancellato.

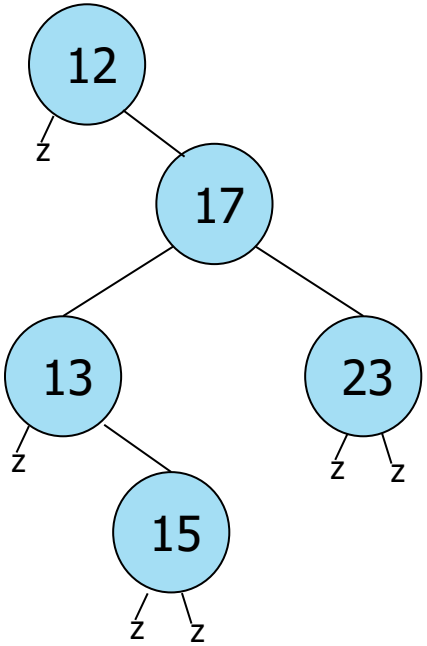
Cancellazione di una radice



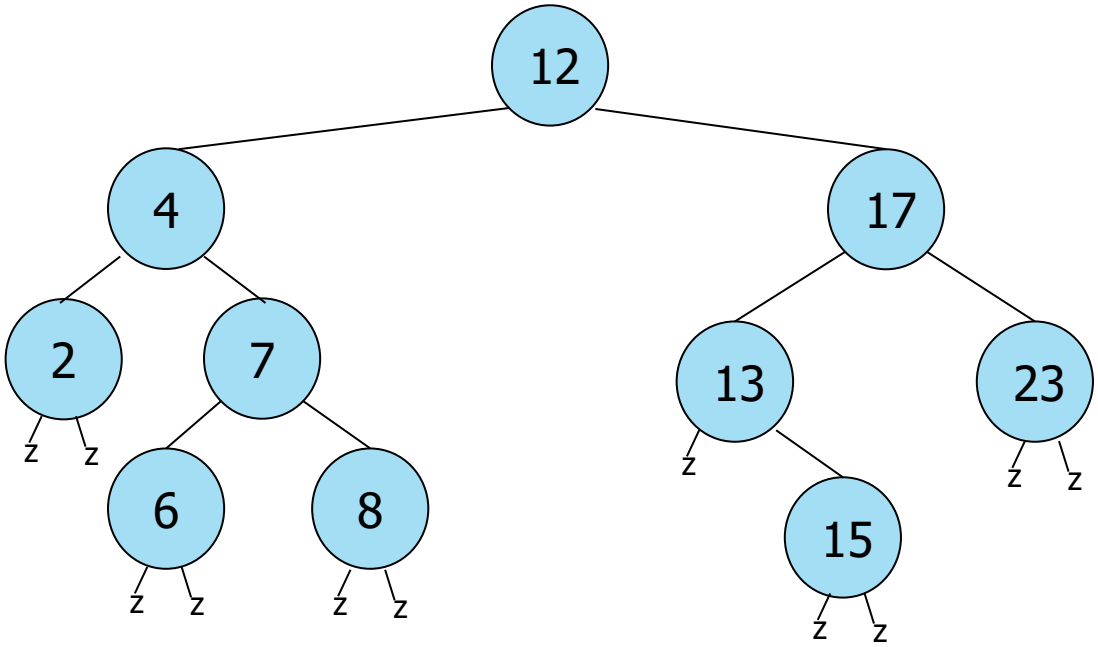

4
2
7
z
z
6
8
z
z
z
17
13
23
z
z
12
15
z
z
z
z
succ(11)

partition rispetto a 12





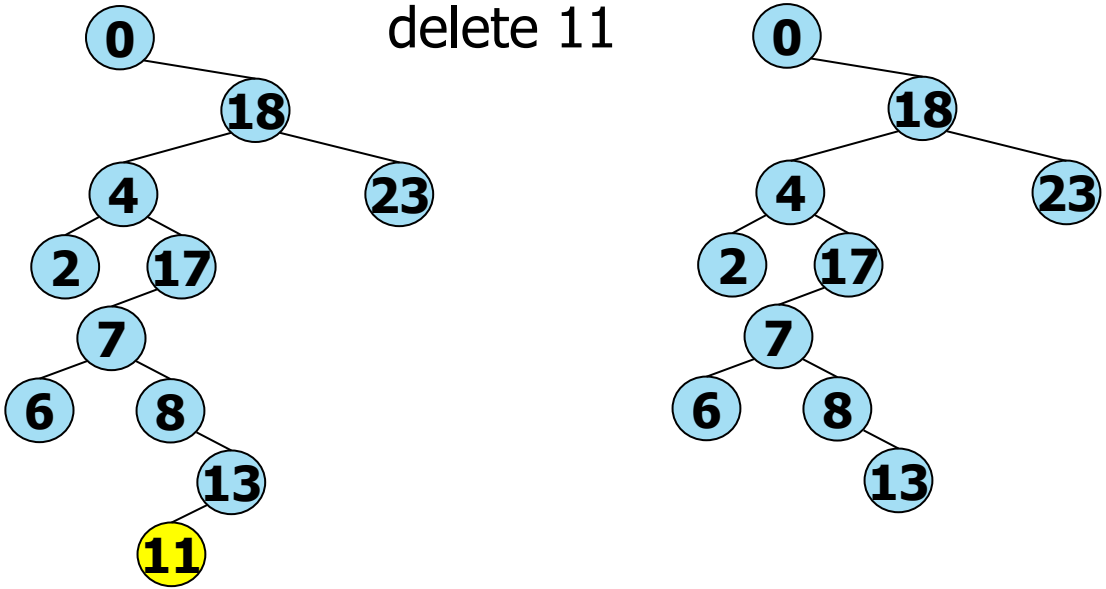
ricostruzione dell'albero con radice 12



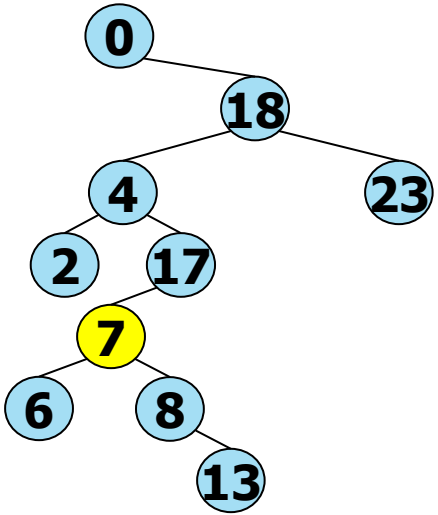
Esempio

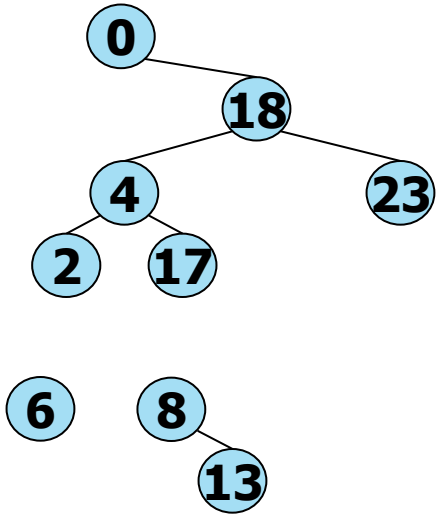
Nodi sentinella z non riportati

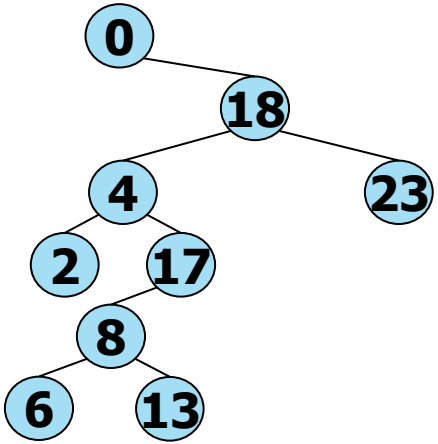
cancellazione in sequenza di 11, 7, 4



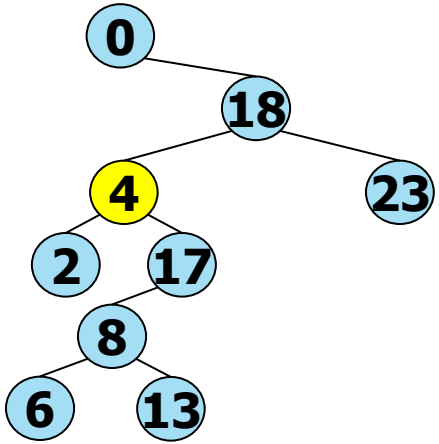
delete 7

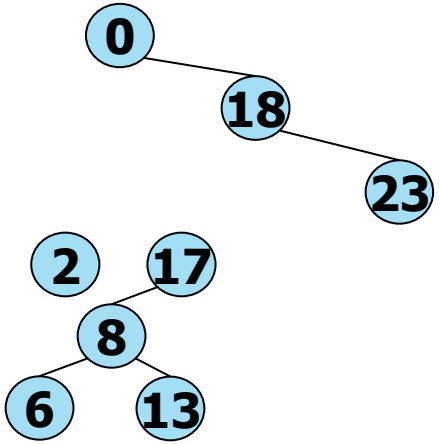


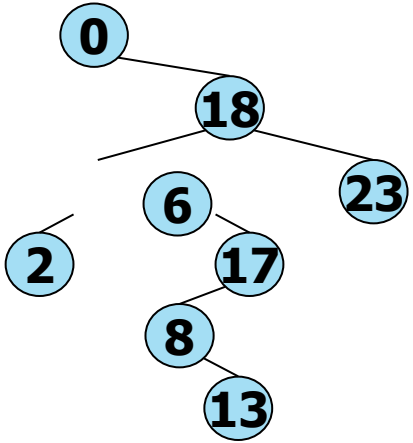


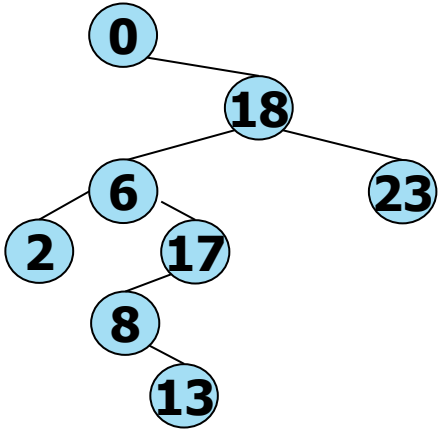

0
18
4
23
2
17
8
6
13

delete 4






0
18
6
23
2
17
8
13


0
18
6
23
2
17
8
13

```
link joinLR(link a, link b, link z) {
  if (b == z)
    return a;
  b = partR(b, 0);
  b->l = a;
  a->p = b;
  b->N = a->N + b->r->N +1;
  return b;
}
```

aggiornamento puntatore al padre

aggiornamento dimensione sottoalberi

```
link deleteR(link h, Key k, link z) {
  link y, p;
  if (h == z) return z;
  if (KEYcmp(k, KEYget(h->item))==-1)
    h->l = deleteR(h->l, k, z);
  if (KEYcmp(k, KEYget(h->item))==1)
    h->r = deleteR(h->r, k, z);
  (h->N)--;
  if (KEYcmp(k, KEYget(h->item))==0) {
    y = h;   p = h->p;
    h = joinLR(h->l, h->r, z);  h->p = p;
    free(y);
  }
  return h;
}
void BSTdelete(BST bst, Key k) {
  bst->root= deleteR(bst->root, k, bst->z);
}
```

aggiornamento numero di nodi

aggiornamento puntatore al padre

Bilanciamento

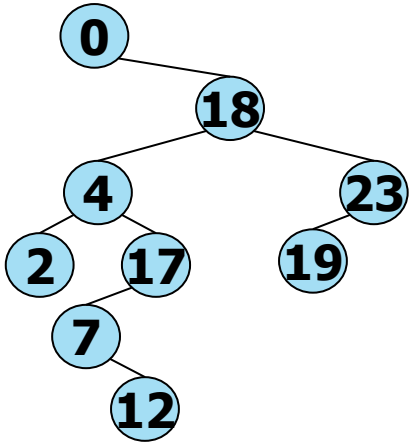
- A priori: vincoli che garantiscono un albero perfettamente bilanciato (B-tree) o con sbilanciamento limitato (alberi 2-3-4, RB-tree)
- Ribilanciamento a richiesta:
  - partizionamento ricorsivo attorno alla chiave mediana inferiore (algoritmo semplice)
  - algoritmo di Day, Stout e Warren, di complessità O(n) costruisce un albero quasi completo (tutti i livelli completi, tranne l'ultimo riempito da sinistra a destra, cfr heap)

Bisogna valutare il rapporto tra costo e frequenza di ribilanciamento.

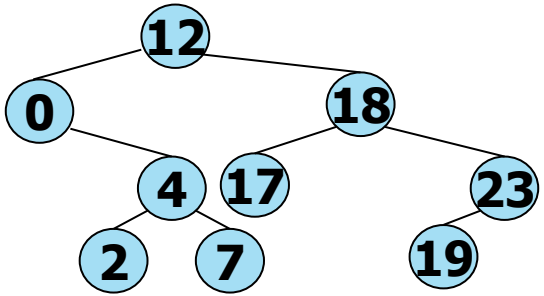
```
static link balanceR(link h, link z) {
  int r;
  if (h == z)
    return z;
  r = (h->N+1)/2-1;
  h = partR(h, r);
  h->l = balanceR(h->l, z);
  h->r = balanceR(h->r, z);
  return h;
}

void BSTbalance(BST bst) {
  bst->root = balanceR(bst->root, bst->z);
}
```

Esempio

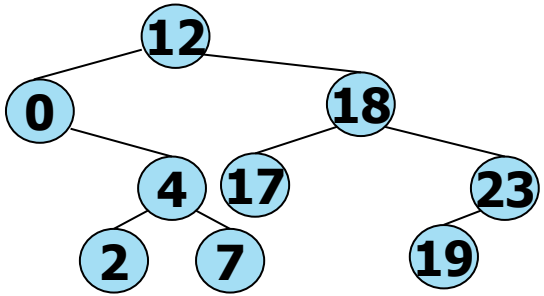
Nodi sentinella z non riportati

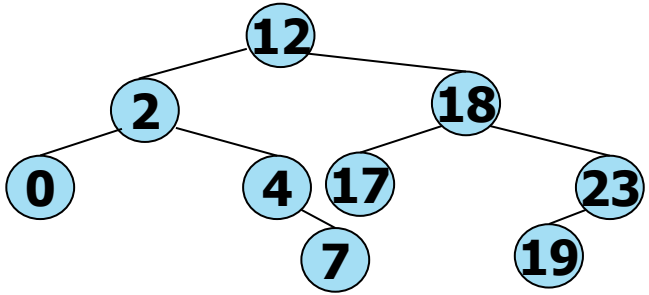


La mediana inferiore è 12. `partR` con `r`=4 dà:

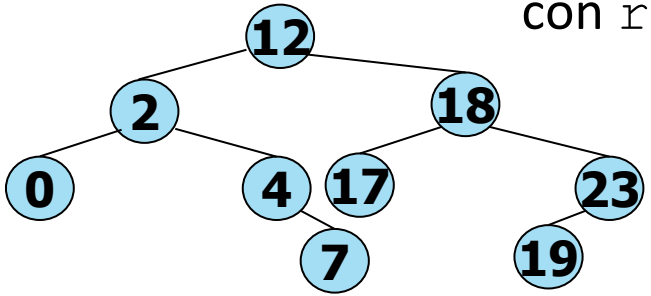


Ricorsione sul sottoalbero sinistro di 12. La mediana inferiore è 2. `partR` con `r`=1 dà:

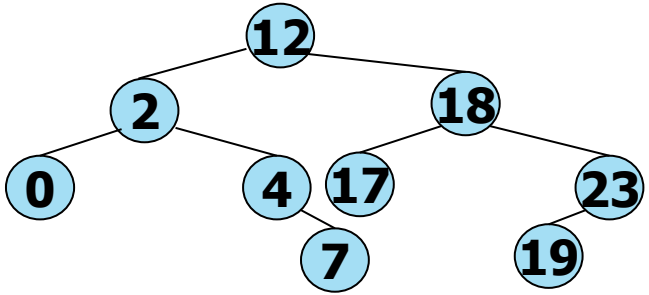




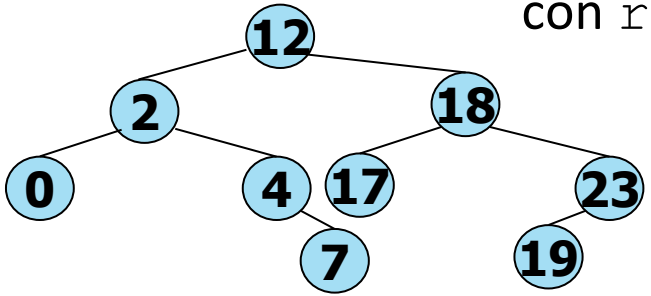
Sui sottoalberi radicati in 0, 4 e 7 la ricorsione non produce modifiche.



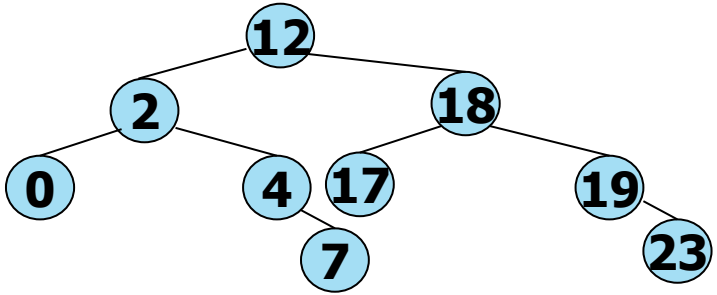
Ricorsione sul sottoalbero destro di 12. La mediana inferiore è 18. `partR` con `r`=1 non produce modifiche.



Sul sottoalbero radicato in 17 la ricorsione non produce modifiche.



Ricorsione sul sottoalbero destro di 18. La mediana inferiore è 19. `partR` con `r`=0 dà:



Sul sottoalbero radicato in 23 la ricorsione non produce modifiche.

Estensioni delle strutture dati

Prima di sviluppare una nuova struttura dati è bene valutare se si possano «estendere» strutture esistenti con informazioni opportune.

Procedura:

1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

Esempio: Order-Statistic BST

Procedura:

1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

BST

dimensione del sottoalbero

O(1)

```
Item BSTselect(BST, int);
```

Interval BST

BST la cui chiave è un intervallo chiuso: coppia ordinata di reali $[t_1, t_2]$, dove $t_1 \leq t_2$ e $[t_1, t_2] = \{t \in \Re: t_1 \leq t \leq t_2\}$.
L'item intervallo $[t_1, t_2]$ può essere realizzato da una `struct` con campi `low` $= t_1$ e `high` $= t_2$.

BST con inserzione secondo l'estremo inferiore

`max`: massimo `high` del sottoalbero

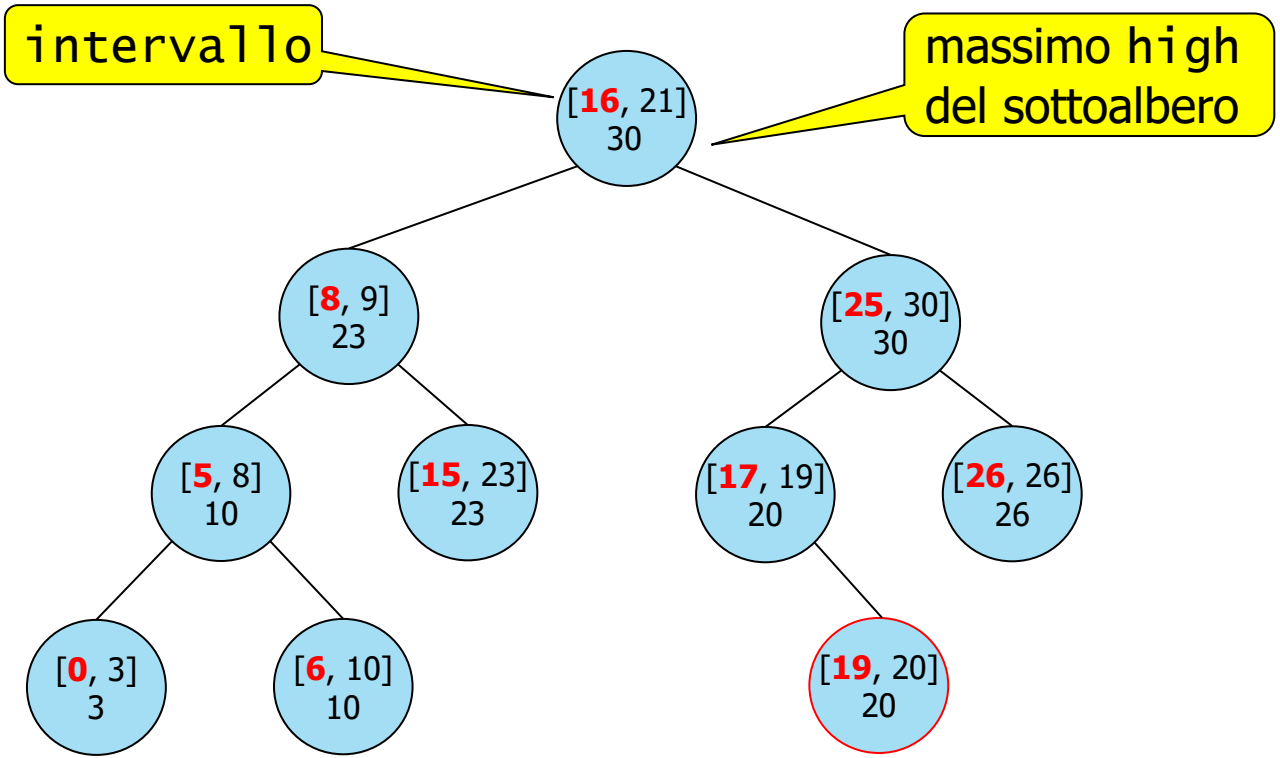
Procedura:

1. identificare la struttura dati candidata
2. identificare le informazioni supplementari
3. verificare di poter mantenere le informazioni supplementari senza alterare la complessità delle operazioni esistenti
4. sviluppare nuove operazioni.

O(1)

```
Item IBSTsearch(BST, Item);
```

Esempio


intervallo
massimo high del sottoalbero
[16, 21]
30
[8, 9]
23
[25, 30]
30
[5, 8]
10
[15, 23]
23
[17, 19]
20
[26, 26]
26
[0, 3]
3
[6, 10]
10
[19, 20]
20

Riferimenti

- Alberi binari
  - Sedgewick 5.6, 5.7
- Binary Search Trees
  - Cormen 13.1, 13.2, 13.3
  - Sedgewick 12.5, 12.8, 12.9
- Order-statistic BST
  - Cormen 15.1
- Interval BST
  - Cormen 15.3
  - Lucidi di approfondimento