Barrier Options

by
Mark Rubinstein and Eric Reiner[1]
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The payoff of a standard European option only depends on the price of the underlying asset on the expiration date. In particular, given the final price of the underlying asset, the payoff will be the same regardless of the path taken by the underlying asset during the life of the option to reach that final price. Whether the underlying asset price reaches a given price by first moving down and then up, or up and then down, matters not to the buyer or seller of the option. It is as if does not matter whether you travel from Paris to London by air or by chunnel, as long as you arrive in London at the appointed time.

The terminology used to describe this feature is "path-independence". More generally, the payoff from an option may depend on some aspect of the price path. For example, the payoff of a lookback option depends on the minimum or maximum price of the underlying asset attained during the life of the option, and the payoff of an "Asian" option depends on the average price. In this article, we will examine a simpler type of path-dependent option where the payoff depends not only on the final price of the underlying asset but also on whether or not the underlying asset has reached some other "barrier" price during the life of the option.

In this essay, our objective is to value a variety of these options in a Black-Scholes environment[2] ; that is,

(1) where the underlying asset return can be assumed to follow a lognormal random walk, and

(2) where arbitrage arguments allow us to use a risk-neutral valuation approach -- discount the expected payoff of the option at expiration by the riskless interest rate, where the underlying asset price is expected to appreciate at the same riskless rate less payouts.

These options are in a sense intermediate between standard European and American options. They are like American options since their value depends on how the underlying asset price behaves through time. But they are simpler to value than American options since the critical boundary of the underlying asset price is determined in advance and specified in the contract. As a result, unlike American options, it will be possible to state "closed form" valuation solutions.

To do this we will need the density of the natural logarithm of the risk-neutral underlying asset return, **u**:

$$f(u) \equiv (1/\sigma\sqrt{2\pi t})e^{-\frac{1}{2}v^2}$$

$$\text{with } \; v \equiv (u - \mu t)/\sigma\sqrt{t}, \quad \mu \equiv \log(r/d) - \tfrac{1}{2}\sigma^2$$

This is just a normal density function. **r** is one plus the rate of interest, **d** is one plus the payout rate of

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[2] To our knowledge, the only published solution for the options covered in this article has been for the down-and-out option (without adjustment for payouts). See, John Cox and Mark Rubinstein, Options Markets, page 410, Prentice-Hall, 1985.

the underlying asset, σ is the volatility of the underlying asset, and t is the time-to-expiration of the option.[3]

We also need another density. Given that the underlying asset price first starts at S above the barrier H, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up below the barrier at expiration is:

$$g(u) \equiv e^{2\mu\alpha/\sigma^2}(1/\sigma\sqrt{2\pi t})e^{-\frac{1}{2}v^2}$$

$$\text{with } v \equiv (u - 2\eta\alpha - \eta\mu t)/\sigma\sqrt{t}, \quad \alpha \equiv \log(H/S)$$

This is a normal density premultiplied by $e^{2\mu\alpha/\sigma^2}$. Here $\eta = 1$. Alternatively, given that the underlying asset price first starts below the barrier, the density of the natural logarithm of the underlying asset return when the underlying asset price breaches the barrier but ends up at expiration below the barrier is the same expression but where $\eta = -1$.

I. "In" Barrier Options

Our first example is a down-and-in call. Although you pay for this option up front, you do not receive the call until the underlying asset price reaches a prespecified level termed the barrier or knock-in boundary, H.[4] If, after elapsed time $\tau \leq t$, the underlying asset price hits the barrier, you then receive a standard European call with striking price K and time-to-expiration $t-\tau$. On the other hand, if through elapsed time t, the barrier is never hit, then instead you receive a rebate R at that time (at expiration). Expressed concisely, the payoff from this option is:

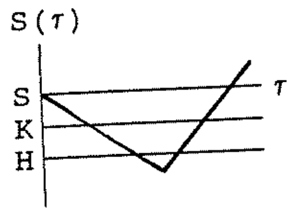
down-and-in call:	$\max[0, S^* - K]$	if for some $\tau \leq t$, $S(\tau) \leq H$
$(S > H)$	R (at expiry)	if for all $\tau \leq t$, $S(\tau) > H$

where $S(\tau)$ is the price of the underlying asset after elapsed time τ, and $S^* \equiv S(t)$ is the price of the underlying asset at expiration. To be interesting, the initial price S of the underlying asset must be greater than the barrier.

[3] Since Black and Scholes are only interested in the price of the underlying asset at expiration, they can allow r, d, and σ to be known functions of time. However, since the options discussed in this article depend in complex ways on the time paths of these variables, to keep matters simple, we assume here that these variables are constant through time.

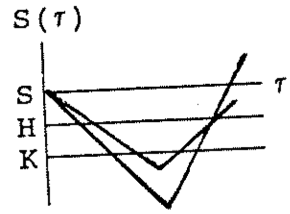
[4] In this case, the standard call is received conditional on the behavior of a random variable (underlying asset price). See, Mark Rubinstein, "Pay Now, Choose Later," RISK, February 1991, for an analysis of a standard call that is received unconditionally but at some prespecified future date.

The graphs below help to visualize what can happen:



call payoff (K>H)

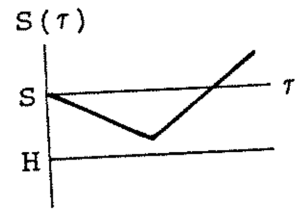
$prob[S^* > K: S(\tau) \leq H]$



call payoff (K<H)

$prob[H \geq S^* > K]$

$+\ prob[S^* > H: S(\tau) \leq H]$



rebate payoff

$prob[S^* > H]$

$-\ prob[S^* > H: S(\tau) \leq H]$

The lines in the first two graphs show paths where the barrier is crossed and the call is received. The line in the third graph shows a path which leads to a rebate payoff at expiry. The statements below state the probability that each outcome will occur. It is necessary to distinguish between two cases: one where **K>H** and one where **K<H**. In the first case, to receive a positive payoff from the call, the underlying asset price must end up above the striking price while having first touched the barrier. On the other hand, the rebate is received only if the underlying price ends up above the barrier without ever having hit the barrier prior to expiration. For this first case, there are thus three types of outcomes:

$S^* > K$ conditional on $S(\tau) \leq H$ for some $\tau \leq t$ ==> payoff = $S^* - K$
$S^* \leq K$ conditional on $S(\tau) \leq H$ for some $\tau \leq t$ ==> payoff = 0
$S(\tau) > H$ for all $\tau \leq t$ ==> payoff = R

In the second case, to receive a positive payoff from the call, the underlying asset price must end up above the barrier while having first touched the barrier, or end up below the barrier but above the striking price. Again, the rebate is only received if the underlying price ends up above the barrier without ever having hit the barrier prior to expiration. Here there are four types of outcomes:

$S^* > H$ conditional on $S(\tau) \leq H$ for some $\tau \leq t$ ==> payoff = $S^* - K$
$S^* \leq H$ and $S^* > K$ ==> payoff = $S^* - K$
$S^* \leq K$ ==> payoff = 0
$S(\tau) > H$ for all $\tau \leq t$ ==> payoff = R

The second and third outcomes are simplified by the fact that since the underlying asset price starts out above the barrier, if the underlying asset price then finishes below the barrier, it must perforce have breached the barrier at some time.

Consider first the **K>H** case. The value of the option is the the sum of two terms, the first (call payoff) corresponding to $prob[S^* > K: S(\tau) \leq H]$ and the second (rebate) corresponding to $prob[S^* > H] - prob[S^* > H: S(\tau) \leq H]$:

$$\textbf{[3]} \equiv r^{-t}\int \phi(Se^u - K)g(u)du = \phi S d^{-t}(H/S)^{2\lambda}N(\eta y) - \phi K r^{-t}(H/S)^{2\lambda-2}N(\eta y - \eta\sigma\sqrt{t})$$

$$\textbf{[5]} \equiv Rr^{-t}\int[f(u) - g(u)]du = Rr^{-t}[N(\eta x_1 - \eta\sigma\sqrt{t}) - (H/S)^{2\lambda-2}N(\eta y_1 - \eta\sigma\sqrt{t})]$$

where the first integral is taken over the region $\log(K/S)$ to $\eta\infty$, the second integral is taken from $\log(H/S)$ to $\eta\infty$,

$$x_1 \equiv [\log(S/H) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$

$$y \equiv [\log(H^2/SK) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$

$$y_1 \equiv [\log(H/S) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$

$$\lambda \equiv 1 + (\mu/\sigma^2)$$

$N(\cdot)$ is the standard normal distribution function

and the binary variables, η and ϕ, are currently both set equal to 1.

The current value of the down-and-in call can then be expressed as

$$C_{di(K>H)} = [3] + [5] \quad \{\eta=1, \phi=1\}$$

If, instead $K<H$, we will need terms corresponding to $\text{prob}[H \geq S^* > K]$ and $\text{prob}[S^* > H: S(\tau) \leq H]$, as well as the rebate term. Since

$$\text{prob}[H \geq S^* > K] = \text{prob}[S^* > K] - \text{prob}[S^* > H]$$

we have the three corresponding integrals:

$$[1] \equiv r^{-t}\int \phi(Se^u - K)f(u)du = \phi S d^{-t}N(\phi x) - \phi K r^{-t}N(\phi x - \phi\sigma\sqrt{t})$$

$$[2] \equiv r^{-t}\int \phi(Se^u - K)f(u)du = \phi S d^{-t}N(\phi x_1) - \phi K r^{-t}N(\phi x_1 - \phi\sigma\sqrt{t})$$

$$[4] \equiv r^{-t}\int \phi(Se^u - K)g(u)du = \phi S d^{-t}(H/S)^{2\lambda}N(\eta y_1) - \phi K r^{-t}(H/S)^{2\lambda-2}N(\eta y_1 - \eta\sigma\sqrt{t})$$

where the first integral is taken over the region $\log(K/S)$ to $\phi\infty$, the second integral is taken over the region $\log(H/S)$ to $\phi\infty$, the third integral is taken over the region $\log(H/S)$ to $\eta\infty$,

$$x \equiv [\log(S/K) \div \sigma\sqrt{t}] + \lambda\sigma\sqrt{t}$$

and the binary variables, η and ϕ, are currently both set equal to 1.

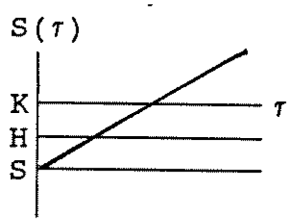
Using this we can write the current value of the down-and-in call as:

$$C_{di(K<H)} = [1] - [2] + [4] + [5] \quad \{\eta=1, \phi=1\}$$

Our next barrier option is an up-and-in call. This option is identical to a down-and-in call except that the underlying asset price starts out below instead of above the barrier. Expressed concisely, the payoff from this option is:

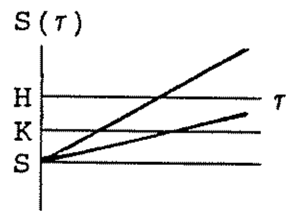
up-and-in call: $\max[0, S^* - K]$ if for some $\tau \leq t$, $S(\tau) \geq H$
$(S < H)$ R (at expiry) if for all $\tau \leq t$, $S(\tau) < H$

The graphs below help to visualize what can happen:



call payoff (K>H)

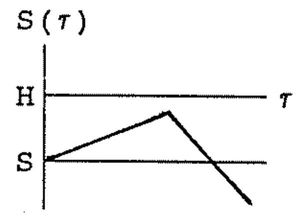
$\text{prob}[S^* > K]$



call payoff (K<H)

$\text{prob}[S^* \geq H]$

$+ \text{prob}[H > S^* > K\text{: } S(\tau) \geq H]$



rebate payoff

$\text{prob}[S^* < H]$

$- \text{prob}[S^* < H\text{: } S(\tau) \geq H]$

For the **K>H** case, the new quantities are $\text{prob}[S^* < H]$ and $\text{prob}[S^* < H\text{: } S(\tau) \geq H]$. The density corresponding to the former is of course $f(u)$ and the density corresponding to the latter is identical to $g(u)$, but with $\eta = -1$. Therefore,

$$\mathbf{C}_{ui(K>H)} = \mathbf{[1] + [5]} \quad \{\eta = -1, \phi = 1\}$$

For the **K<H** case, we first restate

$$\text{prob}[H > S^* > K\text{: } S(\tau) \geq H] = \text{prob}[S^* < H\text{: } S(\tau) \geq H] - \text{prob}[S^* \leq K\text{: } S(\tau) \geq H]$$

Then, we can write immediately that

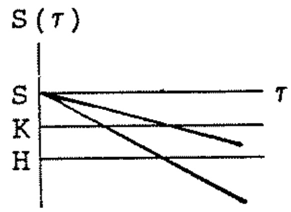
$$\mathbf{C}_{ui(K<H)} = \mathbf{[2] - [3] + [4] + [5]} \quad \{\eta = -1, \phi = 1\}$$

For our next options, down-and-in puts and up-and-in puts, we simply provide graphs from which the stated results can easily be inferred:

<u>down-and-in put</u>: $(S > H)$

$\max[0, K - S^*]$ if for some $\tau \leq t$, $S(\tau) \leq H$

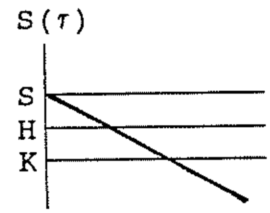
R (at expiry) if for all $\tau \leq t$, $S(\tau) > H$



put payoff (K>H)

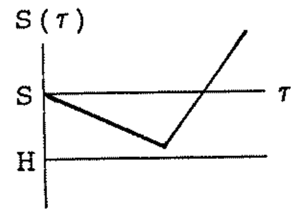
$\text{prob}[S^* \leq H]$

$+ \text{prob}[H < S^* < K\text{: } S(\tau) \leq H]$



put payoff (K<H)

$\text{prob}[S^* < K]$



rebate payoff

$\text{prob}[S^* > H]$

$- \text{prob}[S^* > H\text{: } S(\tau) \leq H]$

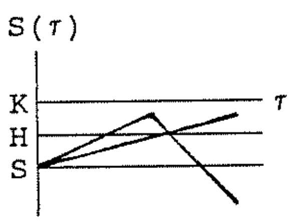
$$P_{di(K>H)} = [2] - [3] + [4] + [5] \quad \{\eta = 1, \phi = -1\}$$

$$P_{di(K<H)} = [1] + [5] \quad \{\eta = 1, \phi = -1\}$$

<u>up-and-in put</u>: $(S < H)$

$\max[0, K - S^*]$ if for some $\tau \leq t$, $S(\tau) \geq H$

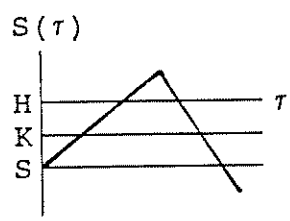
R (at expiry) if for all $\tau \leq t$, $S(\tau) < H$



put payoff (K>H)

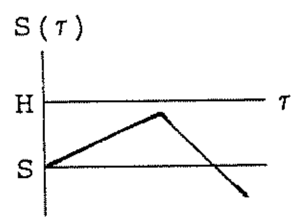
$\text{prob}[H \leq S^* < K]$

$+ \text{prob}[S^* < H\text{: } S(\tau) \geq H]$



put payoff (K<H)

$\text{prob}[S^* < K\text{: } S(\tau) \geq H]$



rebate payoff

$\text{prob}[S^* < H]$

$- \text{prob}[S^* < H\text{: } S(\tau) \geq H]$

$$P_{ui(K>H)} = [1] - [2] + [4] + [5] \quad \{\eta = -1, \phi = -1\}$$

$$P_{ui(K<H)} = [3] + [5] \quad \{\eta = -1, \phi = -1\}$$

II. "Out" Barrier Options

Corresponding to each of these four "in" barrier options are four "out" options. For example, in a down-and-out call, a standard call comes into existence when the down-and-out is issued, but the standard call is extinguished prior to expiration if the underlying asset price ever drops below the knock-out boundary, **H**. In that case the buyer of the option may be paid a fixed rebate, **R**. Otherwise, if the underlying asset price never drops below **H**, the down-and-out call will have the same payoff as a standard call. Expressed concisely, the payoff from this option is:

<u>down-and-out call</u>:	$\max[0, S^* - K]$	if for all $\tau \leq t$, $S(\tau) > H$
$(S > H)$	R (at hit)	if for some $\tau \leq t$, $S(\tau) \leq H$

Here is one possible use of a down-and-out call. Suppose you are holding a covered call but will be forced to liquidate the underlying asset if its price falls sharply. If you sold a down-and-out call in place of a standard call, you could arrange to have the call liquidated automatically at the same time.

If the rebate $\mathbf{R = 0}$, the following parity relationship makes it easy to write down the values of down-and-out calls:

payoff from standard option = payoff from down-and-out option + payoff from down-and-in option

To see this, suppose you own otherwise identical down-and-out and down-and-in options with no rebates. If the common barrier is never hit, then you receive the payoff from a standard option; if the common barrier is hit, as the down-and-out option is extinguished, the down-and-in option delivers you a standard option identical to the one you lost when the down-and-out option was cancelled. Thus, even in this case, you end up receiving the payoff from a standard option.

The only difficulty comes from the rebate. For "in" options, it is not possible to receive the rebate prior to expiration, since one continues to remain in doubt about whether or not the barrier will never be hit. However, for an "out" option, it is possible as well as customary for the rebate to be paid the moment the barrier is hit. This complicates the risk-neutral valuation problem since the rebate may now be received at a random rather than prespecified time. Thus, we need an additional density of the first passage time (τ) for the underlying asset price to hit the barrier:[5]

$$h(\tau) \equiv (-\eta\alpha/\sigma\tau\sqrt{2\pi\tau})e^{-\frac{1}{2}v^2}$$

$$\text{with } v \equiv (-\eta\alpha + \eta\mu\tau)/\sigma\sqrt{\tau}$$

Here, $\eta = 1$ if the barrier is being approached from above and $\eta = -1$ if the barrier is being approached from below. The present value of the rebate is then the expected rebate discounted by the interest rate raised to the power of the first passage time:

$$\textbf{[6]} \equiv R\int r^{-\tau}h(\tau)d\tau = R[(H/S)^{a+b}N(\eta z) + (H/S)^{a-b}N(\eta z - 2\eta b\sigma\sqrt{t})]$$

where the region of integration is from 0 to t, and

$$z \equiv [\log(H/S) \div \sigma\sqrt{t}] + b\sigma\sqrt{t}$$

[5] This is not actually a new density since it can be derived by differentiating the integral of $g(u)$ with respect to t.

$$a \equiv \mu/\sigma^2, \qquad b \equiv [\sqrt{(\mu^2 + 2(\log r)\sigma^2)}]/\sigma^2$$

Using these relationships, we can now write down the valuation solutions for the down-and-out call and the three remaining "out" options:

$$C_{do(K>H)} = [1] - [3] + [6] \quad \{\eta=1, \phi=1\}$$

$$C_{do(K<H)} = [2] - [4] + [6] \quad \{\eta=1, \phi=1\}$$

up-and-out call: (S < H)	max[0, S^* - K]	if for all $\tau \leq t$, $S(\tau) < H$
	R (at hit)	if for some $\tau \leq t$, $S(\tau) \geq H$

$$C_{uo(K>H)} = [6] \quad \{\eta=-1, \phi=1\}$$

$$C_{uo(K<H)} = [1] - [2] + [3] - [4] + [6] \quad \{\eta=-1, \phi=1\}$$

down-and-out put: (S > H)	max[0, K - S^*]	if for all $\tau \leq t$, $S(\tau) > H$
	R (at hit)	if for some $\tau \leq t$, $S(\tau) \leq H$

$$P_{do(K>H)} = [1] - [2] + [3] - [4] + [6] \quad \{\eta=1, \phi=-1\}$$

$$P_{do(K<H)} = [6] \quad \{\eta=1, \phi=-1\}$$

up-and-out put: (S < H)	max[0, K - S^*]	if for all $\tau \leq t$, $S(\tau) < H$
	R (at hit)	if for some $\tau \leq t$, $S(\tau) \geq H$

$$P_{uo(K>H)} = [2] - [4] + [6] \quad \{\eta=-1, \phi=-1\}$$

$$P_{uo(K<H)} = [1] - [3] + [6] \quad \{\eta=-1, \phi=-1\}$$

At first, it may be surprising that the rebate provides the only contribution to the value of an up-and-out call when the striking price is greater than the barrier. But it is easy to see why. Since **S < H < K**, in order for the underlying asset price to end up above the striking price it must first breach the barrier, but in this event, the call is extinguished. Similarly, a down-and-out put will also only be valued for the rebate when the striking price is less than the barrier.

61.00 BAR: BARRIER EUROPEAN OPTIONS (LOGNORMAL)

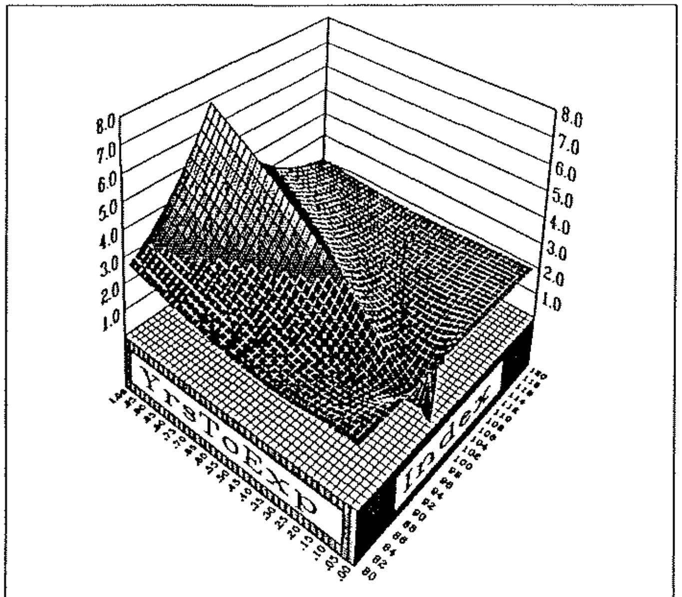
Call/Put =Call Index = 100
IntRate =.100 Vol =.200 DivYld =.050
Rebate = 2

Value		Out/In = Out					Out/In = In				
		YrsToExp									
Barrie	Strike	.50	.75	1.00	1.25	1.50	.50	.75	1.00	1.25	1.50
97	90	6.48	6.45	6.45	6.46	6.47	8.55	9.97	11.20	12.30	13.30
	100	4.69	4.98	5.18	5.33	5.45	3.91	5.30	6.55	7.69	8.73
	110	3.16	3.66	4.00	4.27	4.48	1.60	2.58	3.58	4.55	5.49
100	90	2.00	2.00	2.00	2.00	2.00	13.05	14.45	15.69	16.81	17.83
	100	2.00	2.00	2.00	2.00	2.00	6.63	8.32	9.78	11.07	12.24
	110	2.00	2.00	2.00	2.00	2.00	2.78	4.27	5.62	6.87	8.02
103	90	1.94	1.89	1.87	1.87	1.87	13.09	14.53	15.78	16.90	17.91
	100	1.69	1.75	1.78	1.80	1.82	6.92	8.54	9.96	11.22	12.37
	110	1.69	1.75	1.78	1.80	1.82	3.07	4.49	5.81	7.03	8.16

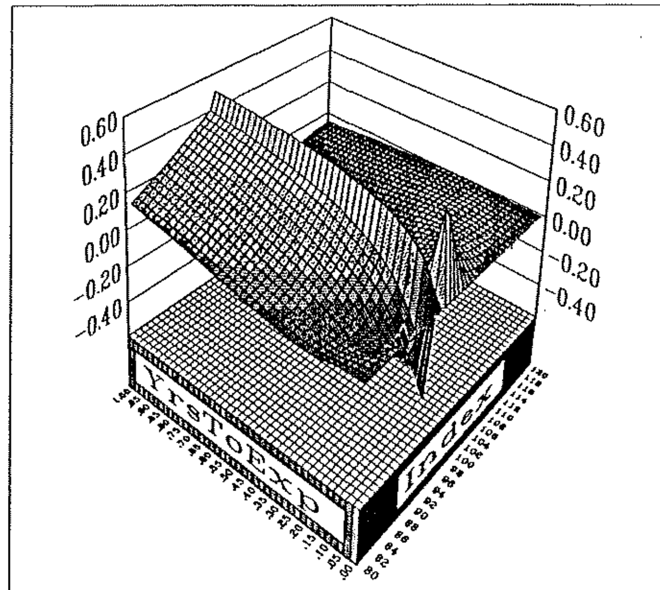
61.00 BAR: BARRIER EUROPEAN OPTIONS (LOGNORMAL)

Call/Put = Put Index = 100
IntRate =.100 Vol =.200 DivYld =.050
Rebate = 2

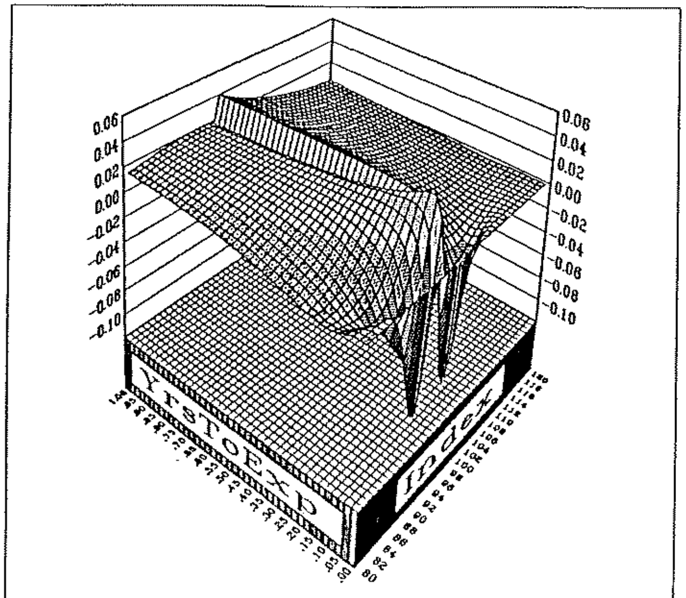
Value		Out/In = Out					Out/In = In				
		YrsToExp									
Barrie	Strike	.50	.75	1.00	1.25	1.50	.50	.75	1.00	1.25	1.50
97	90	1.61	1.67	1.70	1.72	1.74	1.63	2.13	2.53	2.84	3.10
	100	1.62	1.67	1.70	1.73	1.74	4.74	5.31	5.70	5.98	6.18
	110	1.87	1.81	1.79	1.79	1.79	10.18	10.43	10.55	10.59	10.58
100	90	2.00	2.00	2.00	2.00	2.00	1.28	1.84	2.27	2.62	2.89
	100	2.00	2.00	2.00	2.00	2.00	4.39	5.02	5.45	5.75	5.97
	110	2.00	2.00	2.00	2.00	2.00	10.08	10.27	10.39	10.43	10.43
103	90	2.37	2.55	2.64	2.68	2.70	.88	1.25	1.60	1.89	2.15
	100	3.55	3.52	3.47	3.40	3.34	2.82	3.47	3.95	4.31	4.59
	110	4.97	4.63	4.38	4.19	4.02	7.08	7.62	7.97	8.20	8.36



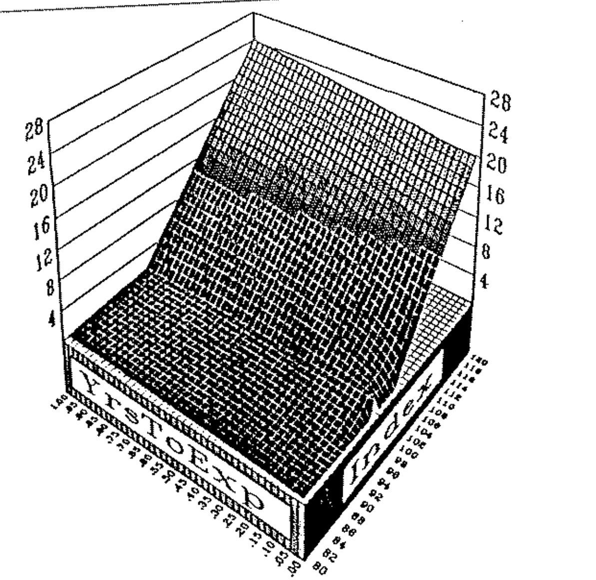
Value: Down-and-In, Up-and-In European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2



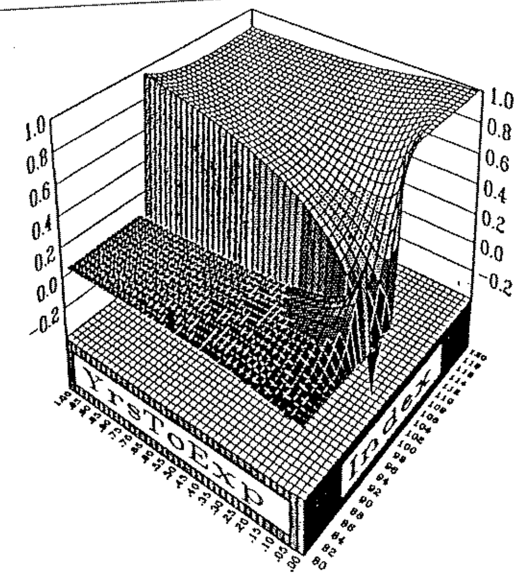
Delta: Down-and-In, Up-and-In European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2



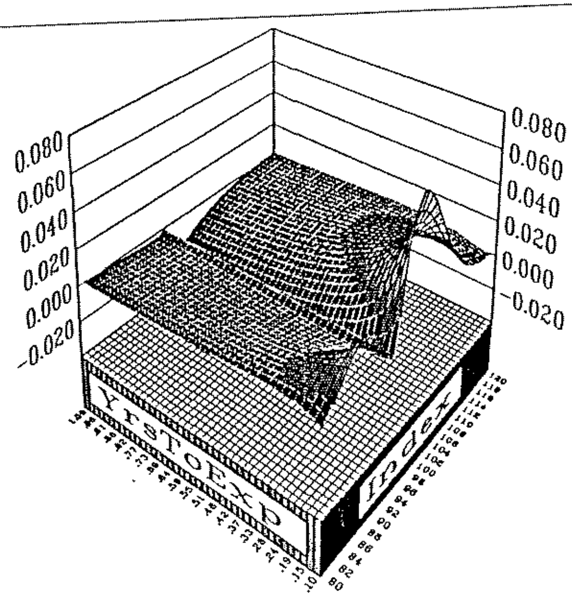
Gamma: Down-and-In, Up-and-In European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2



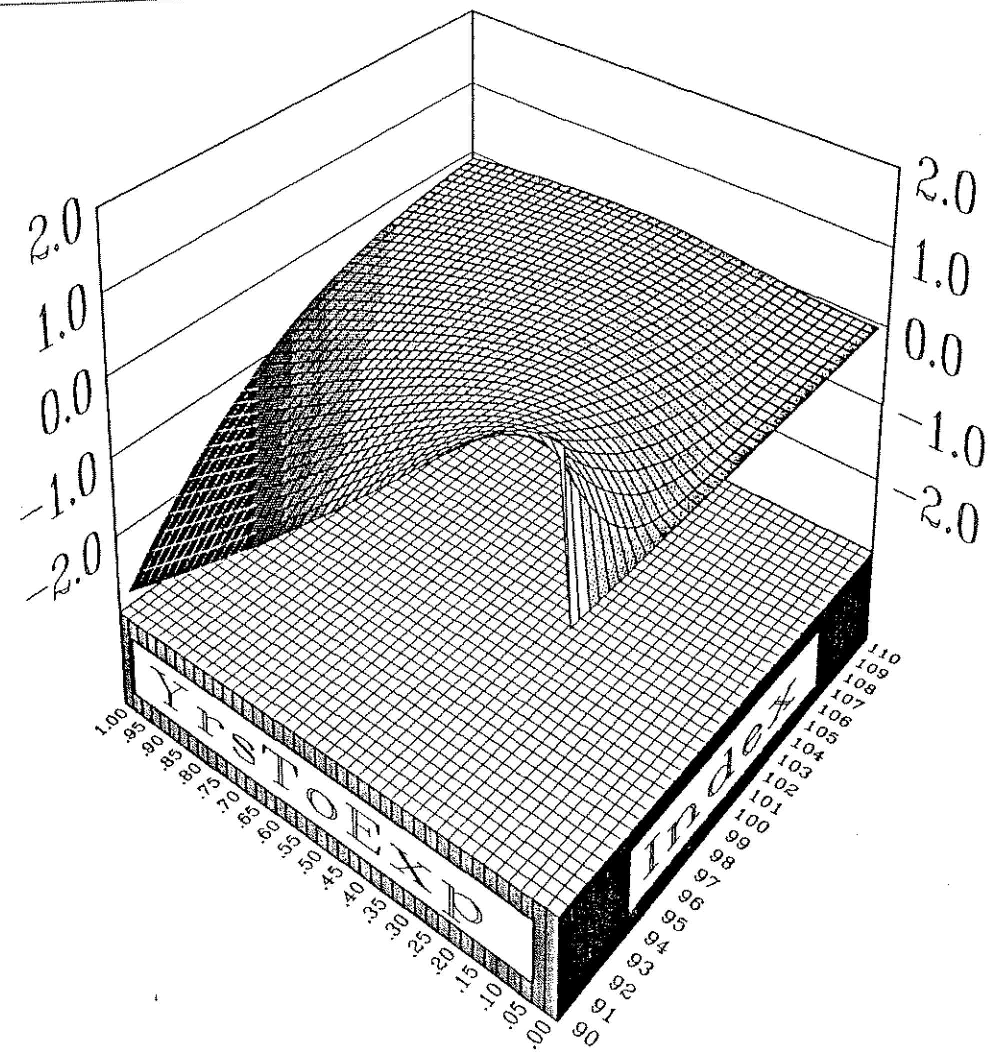
Value: Down-and-Out, Up-and-Out European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2



Delta: Down-and-Out, Up-and-Out European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2

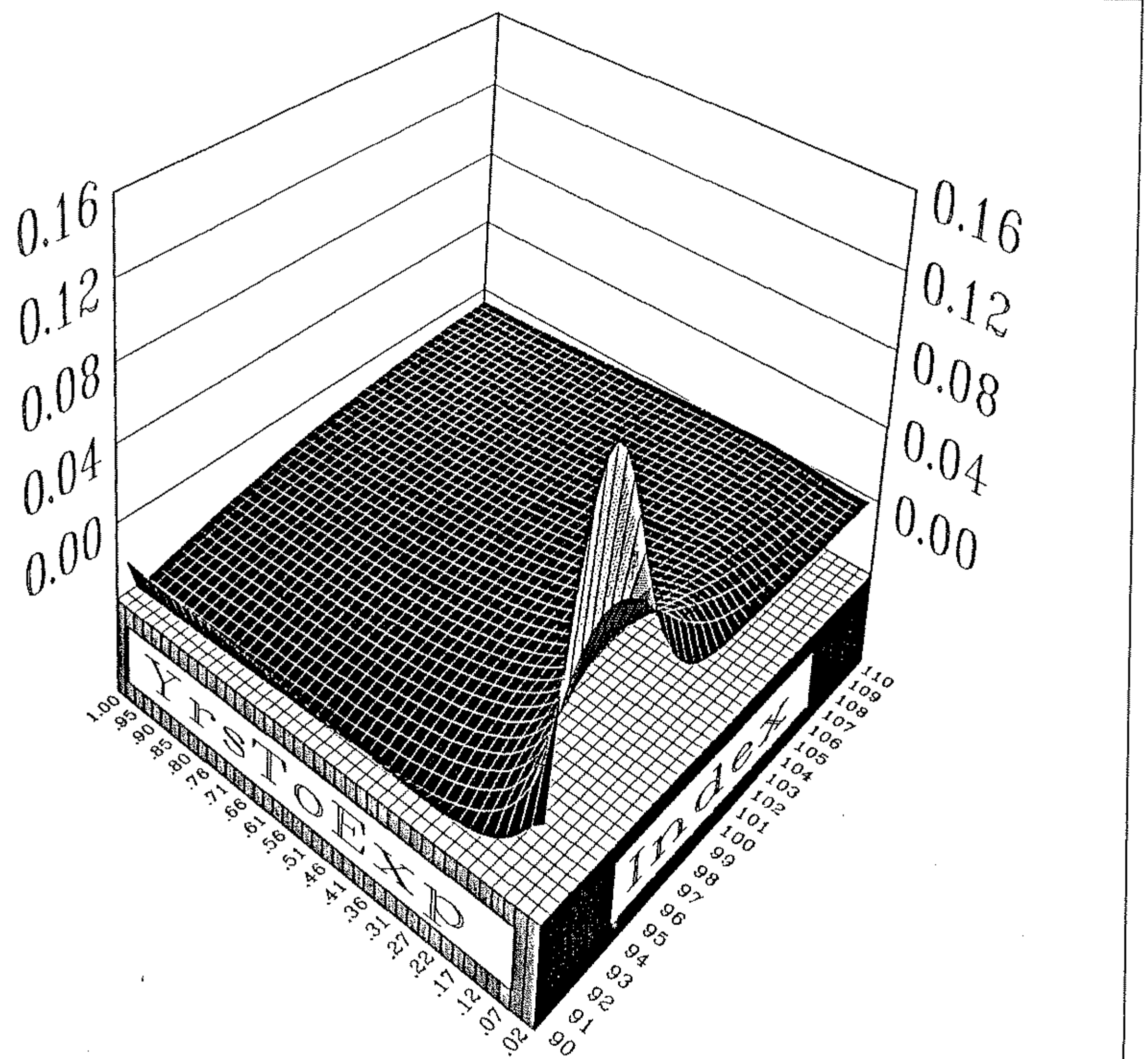


Gamma: Down-and-Out, Up-and-Out European Call (Lognormal)
Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=95, Rebate=2



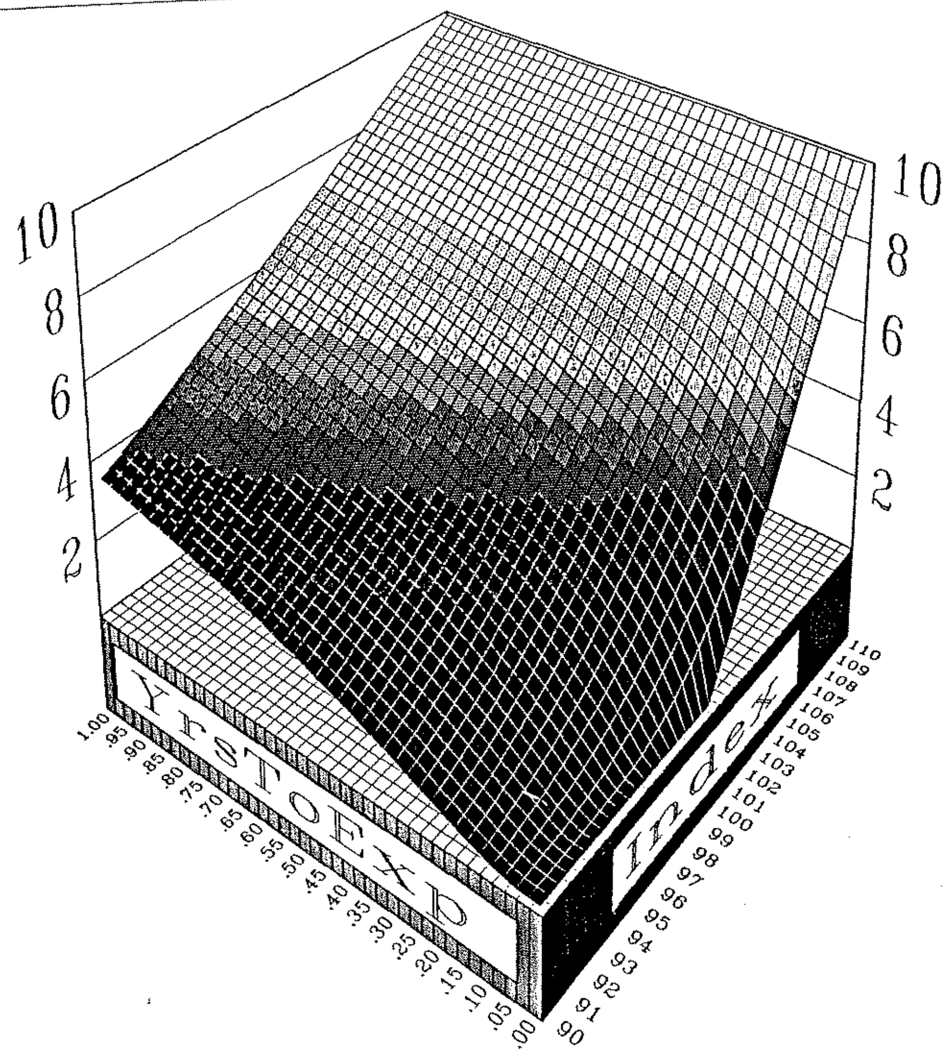
Down-and-out Minus Standard Call Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=90, Rebate=2



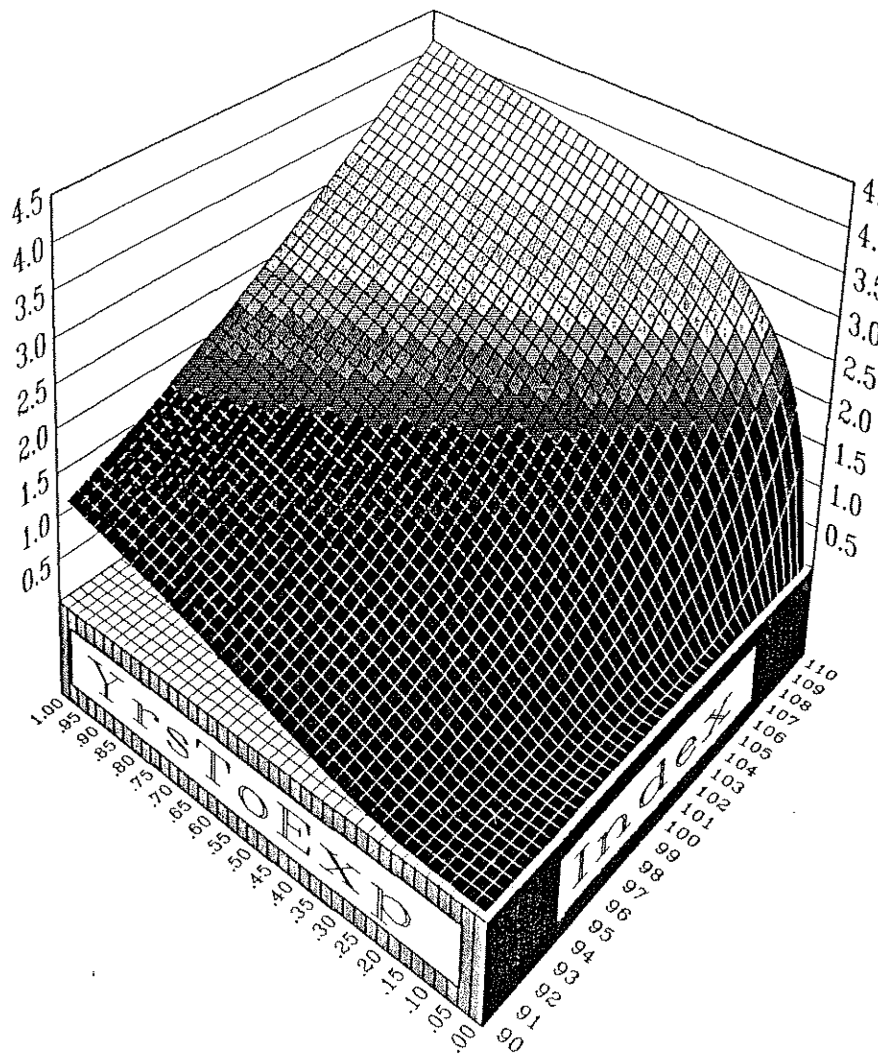
Down-and-out Minus Standard Call Lognormal European Gammas

Strike=100, IntRate=.1, Vol=.2, DivYld=.05, Barrier=90, Rebate=2



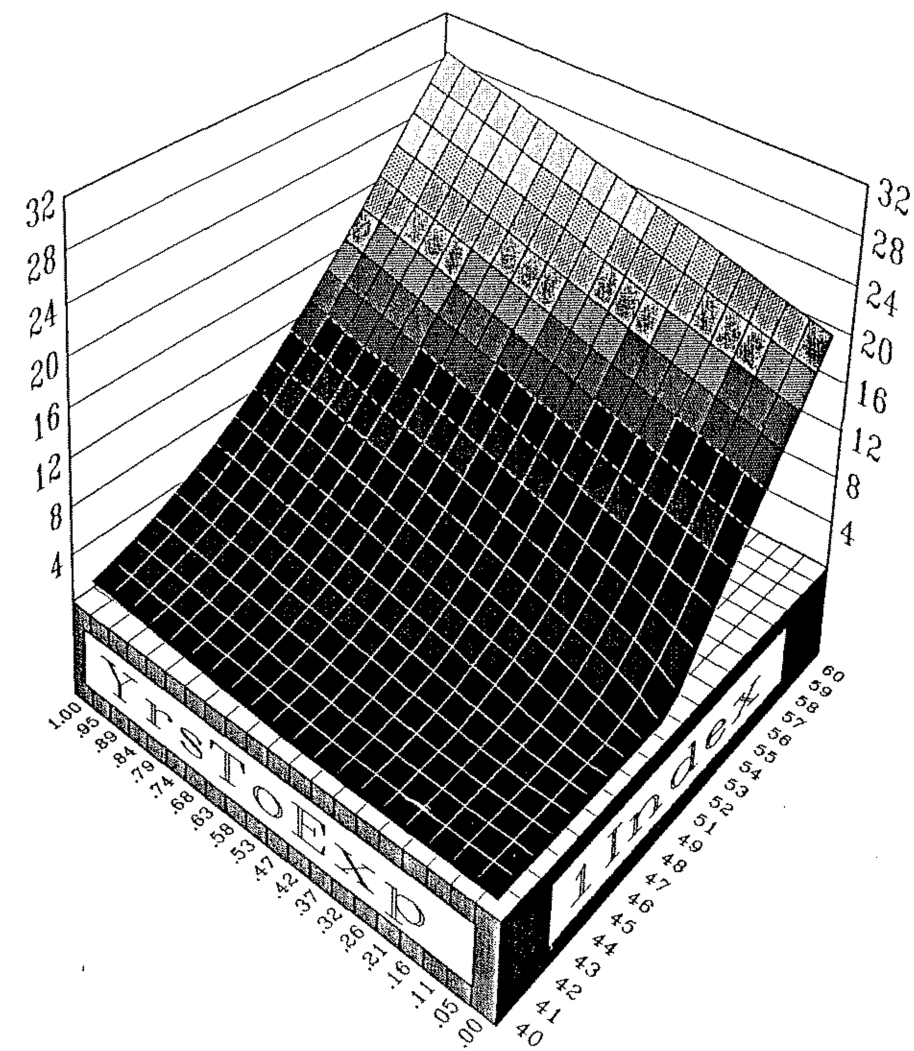
Up-and-out Call Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, Yld=.05, Barrier=110, Rebate=10



Up-and-out Call Minus Vertical Spread Lognormal European Values

Strike=100, IntRate=.1, Vol=.2, Yld=.05, Barrier=110, Rebate=10



2-Asset Portfolio Binomial European Call Values

1Index=2Index, Strike=100, IntRate=.1, 1Vol=.1, 2Vol=.3, Corr=0